Week 3 Monday: Boolean Algebra & Karnaugh Maps

Focus Question: If we already know boolean algebra, why are kmaps useful? How are the two similar?

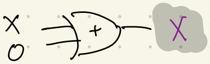
The Basics:

AND	0	1
0	0	0
1	0	1

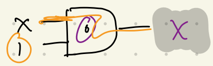
OR	0	1
0	0	1
1	1	1

identity:

1.) $x + 0 = x$

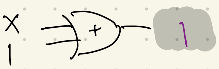


1D.) $x \cdot 1 = x$

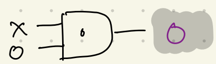


identity absorption

2.) $x + 1 = 1$

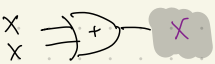


2D.) $x \cdot 0 = 0$

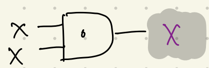


idempotency

3.) $x + x = x$



3D.) $x \cdot x = x$

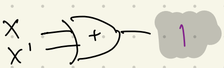


involution

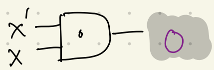
4.) $(x')' = x$

complements

5.) $x + x' = 1$



5D.) $x \cdot x' = 0$



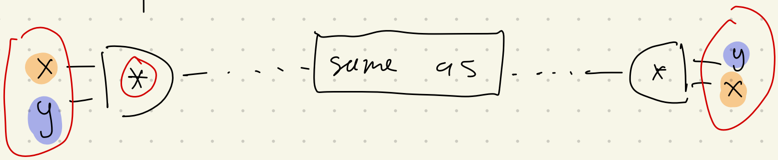
Duality?
swap +, ·
0, 1

Axioms:

$*$:= a generic gate, you can also think of this as a generic type of operation (like $+, \cdot$).

commutativity: $x * y = y * x$

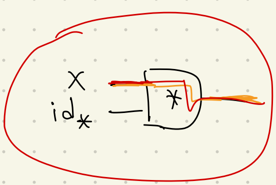
- gate input order <u>should not</u> matter, so we expect:



(1) $x + y = y + x$ (2) $x \cdot y = y \cdot x$

identity: $x * id = x = id * x$

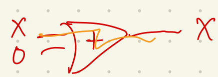
- For a given gate, we expect there to exist some <u>unique</u> input (element) which when operated on any other input, x, outputs x. We call this input / element the <u>identity</u> <u>with respect to $*$</u>.



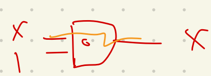
$id * x = x$, so the identity allows x to pass through the gate <u>unaffected</u>!

We have 2 identities with $+, \cdot$ boolean algebra:

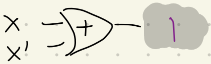
(1) 0 := additive identity $x + 0 = x$



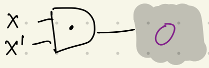
(2) 1 := multiplicative identity $x \cdot 1 = x$



complement: · $x + x'$ = multiplicative id **not 0**



· $x \cdot x'$ = additive id **not 1**



- note: $x * x' = id$

just like the id, must be unique!

Practice:

- $x * y = y * x$
- $x * id = x = id * x$
- $x + x' = 1$
- $x \cdot x' = 0$

ex 1:

*	a	b	c	d
a	d	a	b	c
b	a	b	c	d
c	b	c	a	d
d	c	d	d	b

x	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	2	4	6
3	0	3	6	9

*	_	_	_
_	A	D	E
_	D	B	F
_	E	F	C

ex 2:

*	a	b	c	d
a	a	b	c	d
b	b	c	d	a
c	c	d	d	b
d	d	a	b	a

ex 1:

- commutative? yes
- identity? b
- a'? c
- b'? b
- c'? a
- d'? d
- valid? yes!

complements

$x * x' = id$
$= b$

ex 2:

- commutative? yes
- identity? a
- a'? a
- b'? d
- c'? x none exist
- d'? d, b x $\underline{x'}$ must be unique
- valid? No!

Focus Question :

(1) What about closure, cardinality, and distributivity?

more axioms :

Distributivity :

kinda weird !

(1) $x + yz = (x+y)(x+z)$

(2) $x(y+z) = xy + xz$

just like regular algebra!

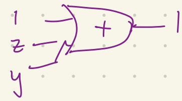
proof of (1) :

$x \cdot x = x$

$$(x+y)(x+z) = \underline{xx + xz + xy + yz}$$

$$= x + xz + xy + yz \quad \text{(idempotency on } x \text{)}$$

$$= x(\underline{1 + z + y}) + yz$$



3 $= x(1) + yz$

$$= x + yz$$

means you can distribute over + as well as over · !

Visualizing Boolean Algebra w/ kmaps:

uniting: $xy + xy' = x$

$x(y+y') = x \cdot 1 = x$

ex1: $f(xyz) = xy + xy'$

yz \ x	0	1
00		1
01		1
11		1
10		1

yz \ x	0	1
00	0	4
01	1	5
11	3	7
10	2	6

minimal cover: x

$xy + xy' = x$

pf

$= \Sigma m(4,5,7,6)$

$= xy'z' + xy'z + xyz' + xyz$

$x_1 = xy'$, $y' = z'$ $x_1 = xy'$, $y = z$

$= xy' + xy$ $\leftarrow$ (uniting)

" $xy + xy' = x$

$= x(y'+y)$

$= x$

absorption: $x + xy = x$

$x(1+y) = x$

ex 2: $f(wxyz) = (w)(w+z')$

How does AND gate eval. to 0

yz \ wx	00	01	11	10
00	0	0		
01	0	0		
11	0	0		
10	0	0		

yz \ wx	00	01	11	10
00	0	4	12	8
01	1	5	13	9
11	3	7	15	11
10	2	6	14	10

minimal cover: w P.O.S

w S.O.P ?

pf

$= \Pi M(0,1,2,3,4,5,6,7)$

$= (w)(w+z')$

$= w \cdot w + w \cdot z'$

$= w + wz'$ (absorption)

$= w$

let $x = w$

let $y = z'$

$x + xy = x$

ex 3: $f(x,y,z) = x'y' + y'z + xz$

yz \ x	0	1
00	1	
01	1	1
11		1
10		

yz \ x	0	1
00	0	4
01	1	5
11	3	7
10	2	6

minimal cover: $x'y' + xz$

⋮

PF

$= \sum m(0,1,5,7)$

consensus: $xy + x'z = xy + x'z + yz$

$= x'y'z' + x'y'z + xy'z + xyz$

$+\ x'y'$ $\qquad$ xz

(consensus)

$x + xy = x$

$= x'y' + xz + x'\cancel{y'}z + x'\cancel{y'}z + x\cancel{y'}z + x\cancel{y}z$

(absorption)

$= \boxed{x'y' + xz}$