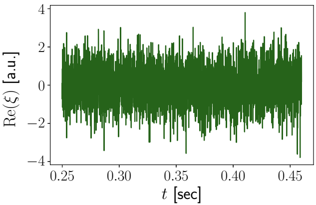
Point processes and spatial statistics in time-frequency analysis

Stochastic Geometry Days, 9th Edition, *November 2021*

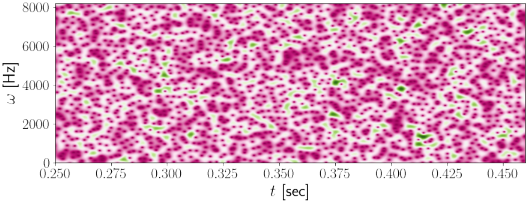
Barbara Pascal Rémi Bardenet

Teaser

Pure noise



Time-frequency representation



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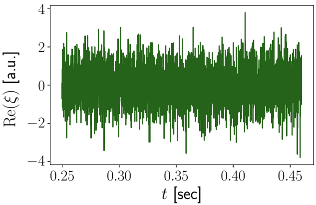

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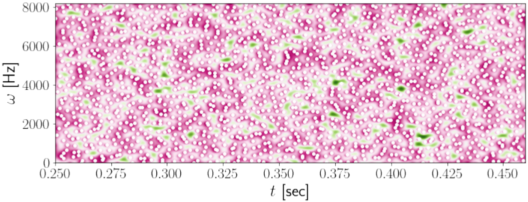

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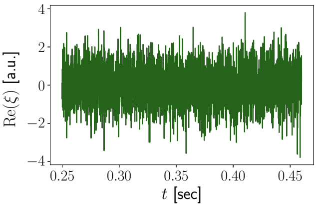

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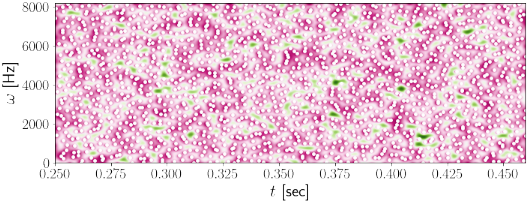

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[Bardenet & Hardy, 2021]

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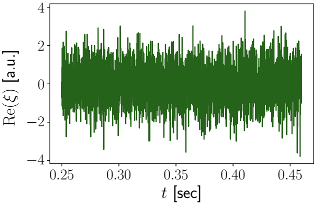
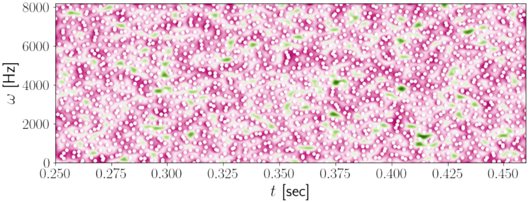


Time-frequency representation



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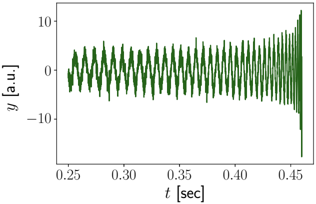
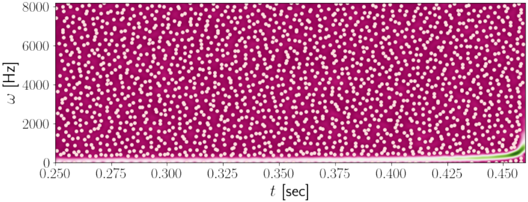




Part I: A scientific journey from signal processing to stochastic geometry

B. Pascal

(i) Signal processing in a nutshell

(ii) Example of the detection of gravitational waves

(iii) Time-frequency representations of nonstationary processes

(iv) Mathematical definition of the noise

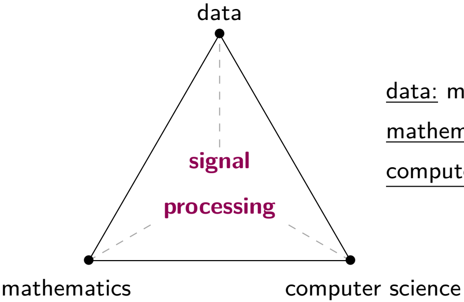
(v) Zeros of Gaussian spectrograms

Signal processing aims to extract **information** from *real* data.

Data of very diverse types:

- measurements of a physical quantity,
- biological or epidemiological indicators,
- data produced by human activities.

The Golden triangle of signal processing



data: modeling of phenomena

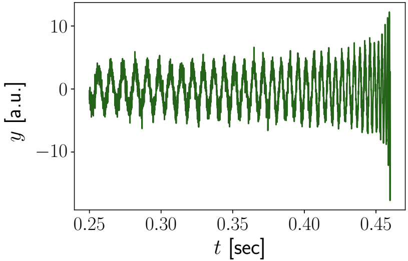
mathematics: formalization & evaluation

computer science: efficient implementation

inspired from P. Flandrin

Time and frequency as a natural language

Standard case: data consists in *complex* functions of the *real* variable $y : \mathbb{R} \to \mathbb{C}$.



- electrical cardiac activity,
- audio recording,
- seismic activity,
- light intensity on a photosensor
- ...

Information of interest:

- time events, e.g., an earthquake and its replica
- frequency content, e.g., monitoring of the heart beating rate

time	**frequency**
ever-changing world	waves, oscillations, rhythms
marker of events and evolutions	intrinsic mechanisms

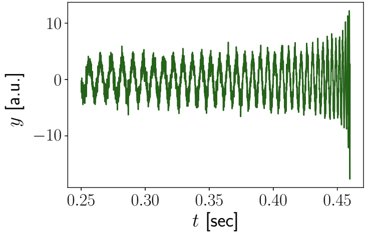
From time to frequency: the Fourier transform

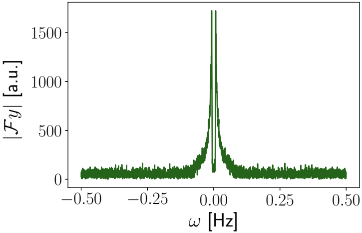
Definition $$\forall f \in L^1(\mathbb{R}), \quad \mathcal{F}f(\omega) \triangleq \int_{\mathbb{R}} \overline{f(t)} \exp(-\mathrm{i}\omega t)\,\mathrm{d}t$$

Properties

- $\mathcal{F}$ extends to $L^2(\mathbb{R})$ and to distributions.
- $\mathcal{F} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is an isometry, *i.e.*, perserves the *energy* $\|f\|_2 = \|\mathcal{F}f\|_2$.
- $\mathcal{F} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is invertible with:

$$\forall \widehat{f} \in L^1(\mathbb{R}), \quad \mathcal{F}^{-1}\widehat{f}(t) = \frac{1}{2\pi} \int_{\mathbb{R}} \overline{\widehat{f}(\omega)} \exp(\mathrm{i}\omega t)\,\mathrm{d}\omega.$$





A bit of history

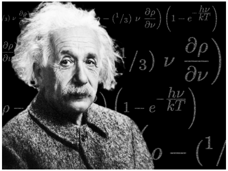
1905: Special Relativity by Einstein, Lorentz and Poincaré

1906: Poincaré proposed the 'gravific waves'

1915: Publication of General Relativity Einstein,

1916: Modern formulation of 'gravitational waves' as

'*The propagation of a space-time deformation.*'



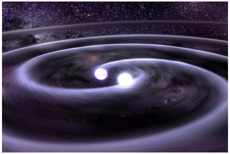
Formidable challenge

- not much empirical assessment of General Relativity exists,
- extremely tiny effect induced on matter,
- result from very rare events involving heavy object: neutrons stars, black holes, ...

Einstein & Rosen (1937) : *'Together with a young collaborator, I arrived at the interesting result that gravitational waves do not exist, though they had been assumed a certainty to the first approximation.'*

Recent high impact success: the detection of gravitational waves

What mechanism gives rise to a gravitational wave?



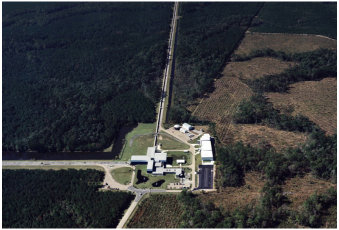
Two heavy objects revolving around each other

- radiation of a gravitational wave: loss of energy
- the objects get closer and closer,
- increase revolution pace to preserve angular momentum.

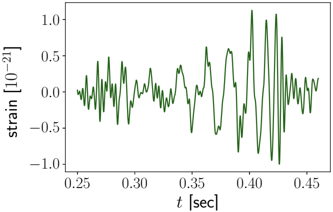
60s-70s astronomers observed binary systems and pulsars **loosing energy**: *indirect* evidence of the existence of gravitational waves.

September, 14th 2015 LIGO-Virgo interferometers detected `GW150914` collision and merging of two black holes **1.3 billions years** ago.

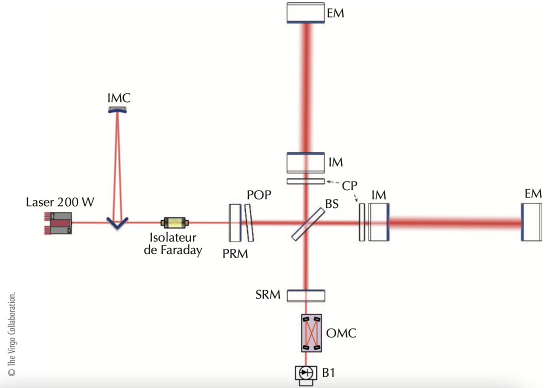
Recent high impact success: the detection of gravitational waves



Oscillation of the fringe pattern



preprocessed



Recent high impact success: the detection of gravitational waves

Physical model: two objects $M = m_1 + m_2$ at distance R, $\mu^{-1} = m_1^{-1} + m_2^{-1}$.

A first order *Newtonian* approximation yields

$$x(t) = A(t_0 - t)^{-1/4} \cos(d(t_0 - t)^{5/8} + \varphi)\mathbf{1}_{(-\infty;t_0[}(t)$$

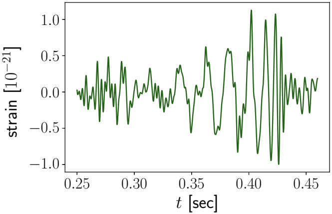
– t_0: time of coalescence,

– d: instantaneous frequency parameter $\qquad d \simeq 241\mathcal{M}_\odot^{-5/8}$,

– A: amplitude reference $\qquad A \simeq 3.37 \times 10^{-21}\mathcal{M}_\odot^{5/4}/R$.

Unknown: $\mathcal{M}_\odot = \mu^{3/5}M^{2/5}/M_\odot$: chirp mass in solar mass unit.

'Chirp' of amplitude $a(t) = A(t_0 - t)^{-1/4}$ and frequency $\omega(t) = 10\pi d/8(t_0 - t)^{-3/8}$.



From LIGO Open Science Center.

Where signal processing enters the game:

- clean the observation,
- verify whether the power laws are correct,
- estimate A, d and t_0,
- deduce $\mathcal{M}_\odot$, M and R.

***Musical score* paradigm**

intrinsically *joint* time-frequency representation

Time-frequency representation combines

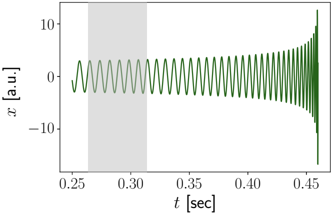
- instantaneous frequency spectrum
- global temporal behavior

Localized Fourier transform

- Choose a window, e.g., $g(t) = \pi^{-1/4} \exp\left(-t^2/2\right)$,
- compute the Fourier transform of x on $\operatorname{supp}(g)$,
- repeat for translated copies of g.

At time t and frequency ω

$$\int_{\mathbb{R}} \overline{x(u)} g(u-t) \exp(-\mathrm{i}\omega u) \, \mathrm{d}u$$



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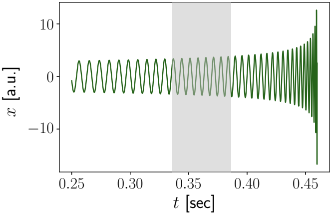

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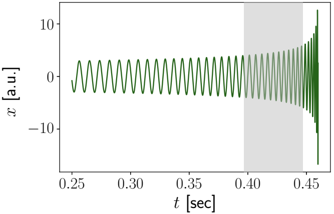

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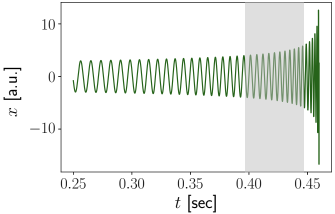
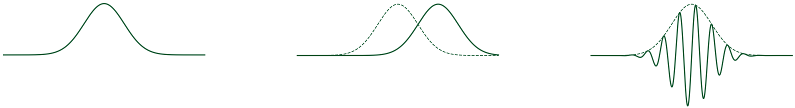


Weyl-Heisenberg group of elementary operations on signals



$g(t) = \pi^{-1/4} \exp\left(-t^2/2\right)$ $\quad$ $\mathbf{T}_u g(t) = g(t-u)$ $\quad$ $\mathbf{M}_\omega g(t) = g(t) \exp\left(-\mathrm{i}\omega t\right)$

Repository on GitHub

`https://github.com/bpascal-fr/GeoSto-PP-for-TF`

- `.pdf` of the presentation,
- `synthetic.ipynb`: processing synthetic signal
- `gravitational-wave.ipynb`: real observation, with `data/gw.mat`.

colab

`https://colab.research.google.com/`

Definition $V_h x(t,\omega) \triangleq \int_{-\infty}^{\infty} \overline{x(u)} h(u-t) \exp(-\mathrm{i}\omega u)\,\mathrm{d}u = \langle x, \mathbf{T}_t \mathbf{M}_\omega h \rangle$

Properties

(i) V_h is an **isometry** $\int_{\mathbb{R}} \int_{\mathbb{R}} V_h x(t,\omega) \overline{V_h y(t,\omega)}\,\mathrm{d}t \frac{\mathrm{d}\omega}{2\pi} = \|h\|_2^2 \int_{\mathbb{R}} \overline{x(t)} y(t)\,\mathrm{d}t.$

(ii) $\left\{ V_h x \in L^2(\mathbb{R}^2, \mathbb{C}),\, x \in L^2(\mathbb{R}, \mathbb{C}) \right\}$ is a **Reproducing Kernel Hilbert Space**, i.e.,

$$V_h x(t,' \omega') = \int_{\mathbb{R}} \int_{\mathbb{R}} K(t', \omega'; t, \omega) V_h x(t,\omega)\,\mathrm{d}t \frac{\mathrm{d}\omega}{2\pi},$$

with kernel $K(t', \omega'; t, \omega) = V_h h(t'-t, \omega'-\omega)$.

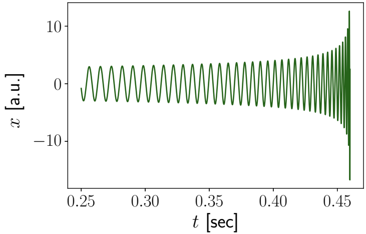
(iii) V_{h_1} is **invertible** and if $\langle h_1, h_2 \rangle \neq 0$

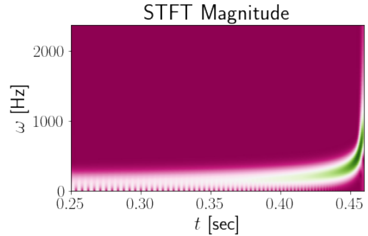
$$\forall x \in L^2(\mathbb{R}, \mathbb{C}), \quad x(t') = \frac{1}{\langle h_1, h_2 \rangle} \int_{\mathbb{R}} \int_{\mathbb{R}} \overline{V_{h_1} x(t,\omega)} \mathbf{M}_\omega \mathbf{T}_t h_2(t)\,\mathrm{d}t \frac{\mathrm{d}\omega}{2\pi}$$

Definition $V_h x(t,\omega) \triangleq \int_{-\infty}^{\infty} \overline{x(u)} h(u-t) \exp(-\mathrm{i}\omega u)\,\mathrm{d}u = \langle x, \mathbf{T}_t \mathbf{M}_\omega h \rangle$

Ideal situation

$$x(t) = A(t_0 - t)^{-1/4} \cos(d(t_0 - t)^{5/8} + \varphi) \mathbf{1}_{(-\infty; t_0[}(t)$$





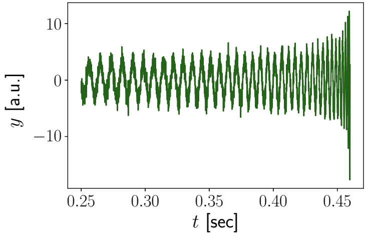
$$a(t) = A(t_0 - t)^{-1/4}$$

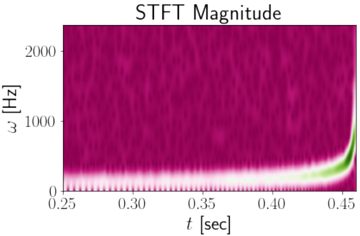
$$\omega(t) = 10\pi d/8 (t_0 - t)^{-3/8}$$

Definition $V_h x(t,\omega) \triangleq \int_{-\infty}^{\infty} \overline{x(u)} h(u-t) \exp(-\mathrm{i}\omega u)\, \mathrm{d}u = \langle x, \mathbf{T}_t \mathbf{M}_\omega h \rangle$

Realistic measurements

$$y(t) = A(t_0 - t)^{-1/4} \cos(d(t_0 - t)^{5/8} + \varphi)\mathbf{1}_{(-\infty;t_0[}(t) + n(t)$$





$\widehat{a(t)}$ $\widehat{\omega(t)}$

Mathematical definition of '*noise*'

Noisy measurements *or* **additive noise** model

$$y(t) = x(t) + n(t)$$

with
- $x(t)$: the signal, meaningful to the receiver, **contains information**
- $n(t)$: the noise, disturbance, **characteristic of the sensor**.

Signal processing task: given $y(t)$

- detect whether there is a signal, '$y(t) = x(t) + n(t)$' or '$y(t) = n(t)$',
- disentangle the signal and the noise, i.e., estimating $\widehat{x(t)}$ from $y(t)$.

Reminder

$$V_h x(t,\omega) = \int_{-\infty}^{\infty} \overline{x(u)} h(u-t) \exp(-\mathrm{i}\omega u) \, \mathrm{d}u$$

is **linear** and **invertible**. Then $V_h y(t,\omega) = V_h x(t,\omega) + V_h n(t,\omega)$.

Thus an estimate $\widehat{V_h x(t,\omega)}$, provides an estimate of the signal

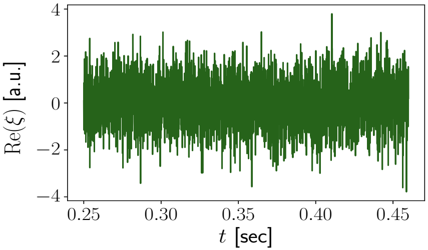
$$\widehat{x(t)} = V_h^{-1} \widehat{V_h x(t,\omega)}$$

Mathematical definition of ‘*noise*’

But ... what do ‘signal’ and ‘noise’ mean? How to distinguish them?

But ... what do 'signal' and 'noise' mean? How to distinguish them?

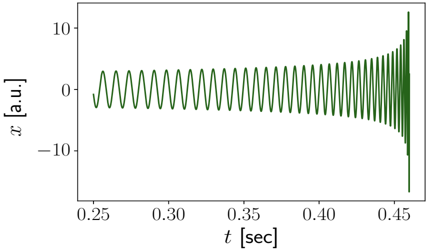
Pure noise $n(t)$



random/disordered

Stochastic

Clean signal $s(t)$



deterministic/ordered

Geometry

P. Flandrin: 'A signal is characterized by some well-defined structured organization.'

Mathematical definition of '*noise*'

Definition of white noise: $\xi \sim \mu$ with

$$\mathbb{E}_\mu \left[\exp\left(\mathrm{i}\langle x, \xi \rangle\right)\right] = \mathrm{e}^{-\|x\|_2^2/2}, \quad \forall x \in \mathcal{H},$$

i.e., μ is the standard Gaussian measure on the Hilbert space of signal $\mathcal{H}$.

Properties If such standard Gaussian measure μ exists, then

- ξ is *isometry invariant*, thus does not depend on a basis of $\mathcal{H}$,
- $\forall x \in \mathcal{H}$, $\langle x, \xi \rangle \sim \mathcal{N}_\mathbb{C}(0, \|x\|_2^2)$.

The finite-dimensional case $\mathcal{H} = \mathbb{C}^N$ with canonical basis $\{\mathrm{e}_k,\ k = 1, \ldots, N\}$

$$\xi = \sum_{k=1}^{N} \xi_k \mathrm{e}_k, \quad n_k \sim \mathcal{N}_\mathbb{C}(0, 1) \quad \text{independent random variables,}$$

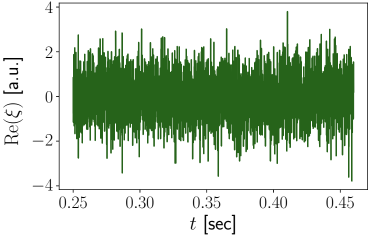
realizes a complex white noise of dimension N.

Mathematical definition of '*noise*'

Finite-dimensional white noise $\xi = \sum_{k=1}^{N} \xi_k \mathrm{e}_k,\ \xi_k \sim \mathcal{N}_{\mathbb{C}}(0,1)$ independent

Signal processing interpretation i.i.d. Gaussian measurements $y_k \triangleq y(t_k)$:

$$y_k \sim \mathcal{N}_{\mathbb{C}}(x_k, 1/\mathrm{SNR}^2), \quad \text{i.e.,} \quad y_k = x_k + n_k, \quad n_k = \frac{1}{\mathrm{SNR}}\xi_k$$



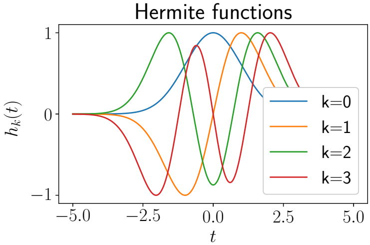
Definition A random process $\{\xi(t),\ t \in \mathbb{R}\}$ is an infinite-dimensional white noise if

$$\forall N \in \mathbb{N}, \quad \forall t_1, \ldots, t_N \in \mathbb{R}, \quad \xi \triangleq (\xi(t_k))_{k=1}^{N} \in \mathbb{C}^N$$

is a finite-dimensional white noise on $\mathbb{C}^N$.

Finite-dimensional white noise $\quad \xi = \sum_{k=1}^{N} \xi_k \mathrm{e}_k, \ \xi_k \sim \mathcal{N}_{\mathbb{C}}(0,1)$ independent

Heuristic extension to infinite-dimensional $L^2(\mathbb{R},\mathbb{C})$ with basis $\{h_k,\ k = 0, 1, ...\}$



$$\xi(t) \triangleq \sum_{k=0}^{\infty} \xi_k h_k(t),$$

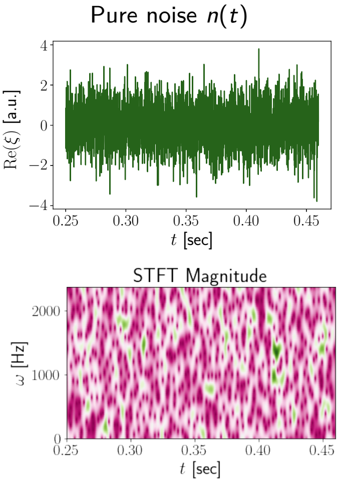
$\xi_k \sim \mathcal{N}_{\mathbb{C}}(0,1)$ independent random variables

Observations and limitations *(see* **Part II** *by Rémi Bardenet)*

- Depends on the Hilbertian basis.
- The series diverges a.s. in $L^2(\mathbb{R},\mathbb{C})$!
- Further work to interpret it as a standard Gaussian measure on a space Θ.

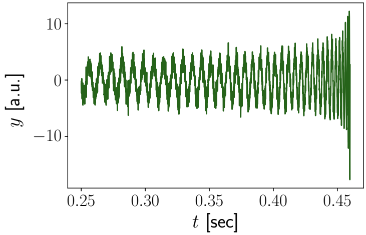
Spectrogram of signal and white noise

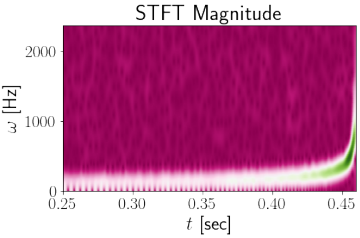
Definition $V_h y(t,\omega) \triangleq \int_{-\infty}^{\infty} \overline{y(u)} h(u-t) \exp(-\mathrm{i}\omega u)\,\mathrm{d}u = \langle y, \mathbf{T}_t \mathbf{M}_\omega h \rangle$



Realistic noisy measurements

$$y(t) = A(t_0 - t)^{-1/4} \cos(d(t_0 - t)^{5/8} + \varphi)\mathbf{1}_{(-\infty;t_0[}(t) + n(t) = x(t) + n(t)$$



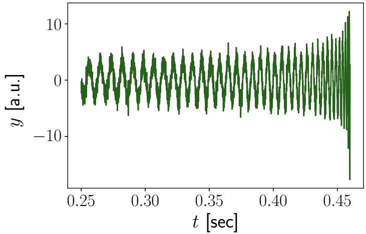


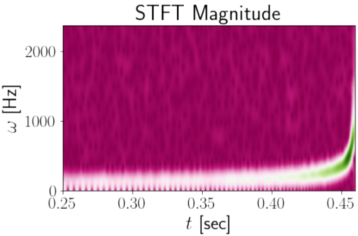
$$V_h y(t,\omega) = V_h x(t,\omega) + V_h n(t,\omega)$$

Purpose Estimation of

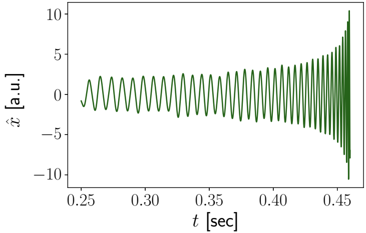
- time-evolving amplitude $a(t) = A(t_0 - t)^{-1/4}$, then deduce A,
- instantaneous frequency $\omega(t) = 10\pi d/8(t_0 - t)^{-3/8}$, infer d,
- reconstruction of the clean signal $\widehat{x(t)}$.

Realistic noisy measurements

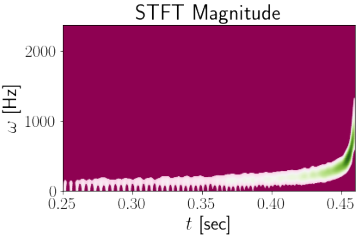




Denoising based on maxima

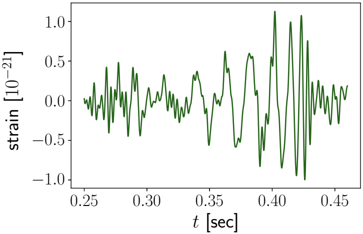


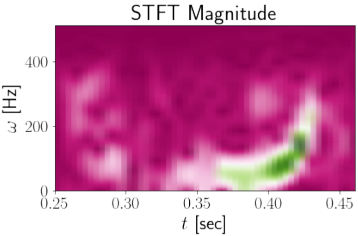
reconstruction



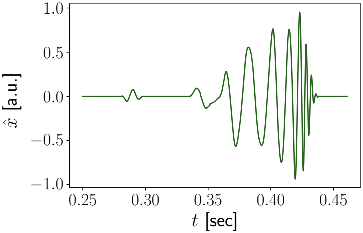
thresholding at 35%

Experimental measurements

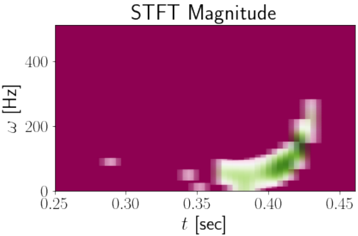




Denoising based on maxima



reconstruction



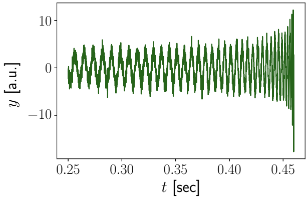
thresholding at 35%

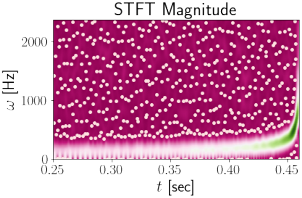
Zeros of *Gaussian* spectrograms

Gaussian spectrogram $g(t) = \pi^{-1/4} \exp(-t^2/2)$

$$V_g y(t,\omega) \triangleq \int_{-\infty}^{\infty} \overline{y(u)} g(u-t) \exp(-\mathrm{i}\omega u)\,\mathrm{d}u = \langle y, \mathbf{T}_t \mathbf{M}_\omega g \rangle$$

Realistic noisy measurements looking at the zeros of $|V_g y(t,\omega)|^2$





Claim The zeros of a Gaussian spectrogram

- are located where '*there is no signal*',
- are evenly spread,
- tend not to clutter.

[Flandrin, 2015]

Idea assimilate the time-frequency plane with $\mathbb{C}$.

Property [Gaussian STFT factorization]

Let $g(t) = \pi^{-1/4} \exp(-t^2/2)$ the circular Gaussian window, then $\forall x \in L^2(\mathbb{R}, \mathbb{C})$

$$V_g x(t, \omega) = \exp(-|z|^2/4) \exp(-\mathrm{i}\omega t/2) \mathcal{B}x(z/\sqrt{2}), \quad z = \omega + \mathrm{i}t$$

where the **Bargmann transform** of x is defined as

$$\mathcal{B}x(z) \triangleq \pi^{-1/4} \exp(-z^2/2) \int_{\mathbb{R}} \overline{x(u)} \exp\left(\sqrt{2}uz - u^2/2\right) \mathrm{d}u,$$

Further $\mathcal{B}x$ is entire on $\mathbb{C}$. [Gröchenig, 2001; Flandrin, 2015]

Theorem The zeros of the Gaussian spectrogram of $x \in L^2(\mathbb{R}, \mathbb{C})$

- coincide with the zeros of $\mathcal{B}x(\cdot/\sqrt{2})$,
- hence constitute a **point process**,
- which almost completely characterizes the spectrogram. [Flandrin, 2015]

Gaussian spectrogram of white noise $\xi(t) = \sum_{k=0}^{\infty} \xi_k h_k(t)$, $\xi_k \sim \mathcal{N}_{\mathbb{C}}(0,1)$

$$V_g\xi(t,\omega) = \int_{-\infty}^{\infty} \overline{\xi(u)} g(u-t) \exp(-\mathrm{i}\omega u)\,\mathrm{d}u = \exp(-|z|^2/4)\exp(-\mathrm{i}\omega t/2)\mathcal{B}\xi(z/\sqrt{2}),$$

Theorem The zeros of the spectrogram of $\xi(t)$ coincides with the zeros of

$$\mathrm{GAF}_{\mathbb{C}}(z) = \sum_{k=0}^{\infty} \xi_k \frac{z^k}{\sqrt{k!}}, \quad \xi_k \sim \mathcal{N}_{\mathbb{C}}(0,1)$$

which is the Planar Gaussian Analytic Function. [Bardenet & Hardy, 2021]

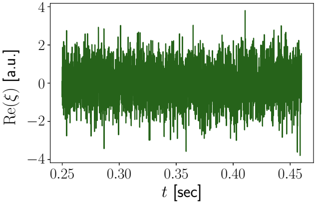
Principle of the proof $\forall k \in \mathbb{N}, \quad \mathcal{B}h_k(z) = \frac{z^k}{\sqrt{k!}}$ *[Property of Bargmann transform]*

Thus

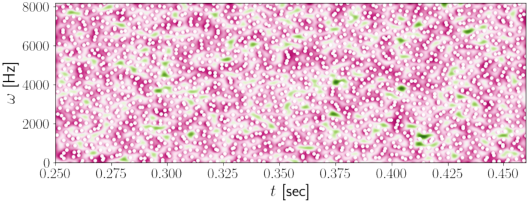
$$\mathcal{B}\xi(z) = \sum_{k\in\mathbb{N}} \xi \frac{z^k}{\sqrt{k!}}$$

which defines the **Planar Gaussian Analytic Function**.

White noise



Gaussian spectrogram $|V_g\xi(t,\omega)|^2$



$$\xi(t) = \sum_{k=0}^{\infty} \xi_k h_k(t),$$
$$\xi_k \sim \mathcal{N}_{\mathbb{C}}(0,1)$$

$$V_g\xi(t,\omega) = \int_{-\infty}^{\infty} \overline{\xi(u)} g(u-t) \exp(-\mathrm{i}\omega u)\,\mathrm{d}u$$

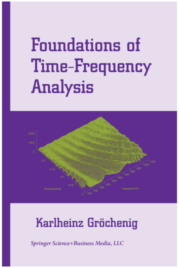
Connection with the Planar Gaussian Analytic Function

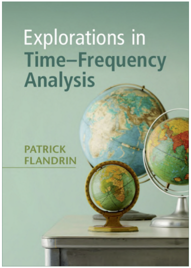
$$V_g\xi(\sqrt{2}t, \sqrt{2}\omega) \propto \mathcal{B}\xi(z) \overset{\text{law}}{=} \mathrm{GAF}_{\mathbb{C}}(z)$$

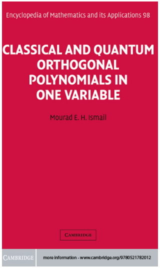
[Bardenet & Hardy, 2021]

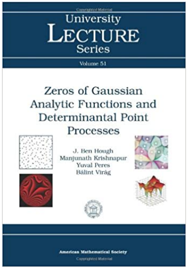
References

Books









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