Convex nonsmooth optimization
Part I: Moreau subdifferential

Barbara Pascal

LS2N, CNRS, Centrale Nantes, Nantes University, Nantes, France
barbara.pascal@cnrs.fr

Collaboration

This course is a direct adaptation of the course built by
Jean-Christophe Pesquet (CentraleSupélec) and Nelly Pustelnik (LPENSL)

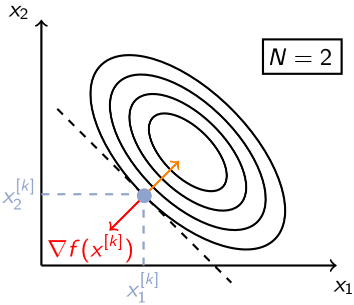
Gradient descent in dimension N

Gradient descent

Let $f : \mathbb{R}^N \to \mathbb{R}$ be convex, continuously differentiable on $\mathbb{R}^N$ and with a β-Lipschitz gradient.
Let $x_0 \in \mathbb{R}^N$ and $\gamma_n \in]0, 2/\beta[$

$$(\forall n \in \mathbb{N}) \quad x_{n+1} = x_n - \gamma_n \nabla f(x_n).$$

$(x_n)_{n\in\mathbb{N}}$ converges to a minimizer of f.



β-Lipschitz gradient Let $f : \mathbb{R}^N \to \mathbb{R}$ be convex, continuously differentiable on $\mathbb{R}^N$. f is gradient β-Lipschitz with $\beta > 0$ if

$$(\forall (u, v) \in \mathbb{R}^N \times \mathbb{R}^N) \quad \|\nabla f(u) - \nabla f(v)\| \le \beta \|u - v\|$$

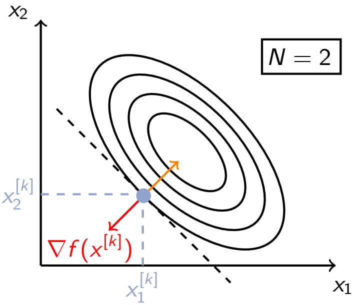
Gradient descent in dimension N

Gradient descent

Let $f : \mathbb{R}^N \to \mathbb{R}$ be convex, continuously differentiable on $\mathbb{R}^N$ and with a β-Lipschitz gradient.
Let $x_0 \in \mathbb{R}^N$ and $\gamma_n \in]0, 2/\beta[$

$$(\forall n \in \mathbb{N}) \quad x_{n+1} = x_n - \gamma_n \nabla f(x_n).$$

$(x_n)_{n\in\mathbb{N}}$ converges to a minimizer of f.

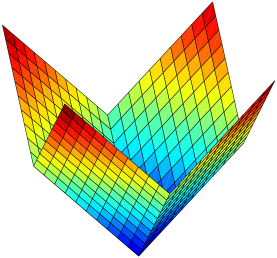


- Iterative method: build a sequence $(x_n)_{n\in\mathbb{N}}$ s.t., at each iteration n

$$f(x_{n+1}) < f(x_n)$$

- Choose γ_n for fast convergence: Newton method, ...
- Convergence proof: fixed point theorem.

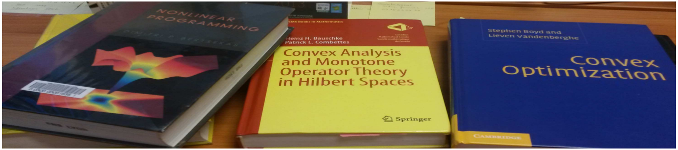
Non-smooth convex optimization



$$\|\cdot\|_1 : \left\{ \begin{array}{ccc} \mathbb{R}^2 & \to & \mathbb{R} \\ (x, y) & \mapsto & |x| + |y| \end{array} \right.$$

not differentiable on
$\{0\} \times \mathbb{R} \cup \mathbb{R} \times \{0\}$

Reference books



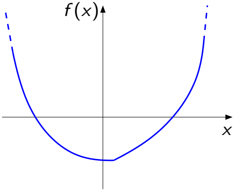
- **D. Bertsekas**, Nonlinear programming, Athena Scientic, Belmont, Massachussets, 1995.
- **Y. Nesterov**, Introductory Lectures on Convex Optimization: A Basic Course, Springer, 2004.
- **S. Boyd and L. Vandenberghe**, Convex optimization, Cambridge University Press, 2004.
- **H. H. Bauschke and P. L. Combettes**, Convex Analysis and Monotone Operator Theory in Hilbert Spaces, Springer, New York, 2011.

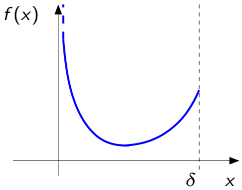
Functional analysis: definitions

Let $f : \mathcal{H} \to \,]-\infty, +\infty]$ where $\mathcal{H}$ is a Hilbert space.

- The domain of f is $\operatorname{dom} f = \{x \in \mathcal{H} \mid f(x) < +\infty\}$.
- The function f is proper if $\operatorname{dom} f \neq \varnothing$.

Domains of the functions ?



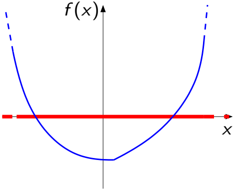


Functional analysis: definitions

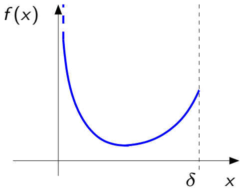
Let $f : \mathcal{H} \to]-\infty, +\infty]$ where $\mathcal{H}$ is a Hilbert space.

- The **domain** of f is $\operatorname{dom} f = \{x \in \mathcal{H} \mid f(x) < +\infty\}$.
- The function f is **proper** if $\operatorname{dom} f \neq \varnothing$.

<u>Domains of the functions ?</u>



$\operatorname{dom} f = \mathbb{R}$
(proper)

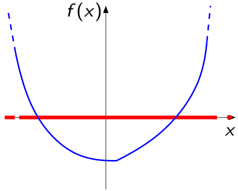


Functional analysis: definitions

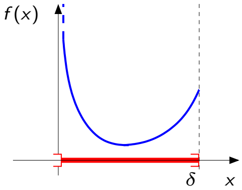
Let $f : \mathcal{H} \to \,]-\infty, +\infty]$ where $\mathcal{H}$ is a Hilbert space.

- The domain of f is $\operatorname{dom} f = \{x \in \mathcal{H} \,|\, f(x) < +\infty\}$.
- The function f is proper if $\operatorname{dom} f \neq \varnothing$.

Domains of the functions ?

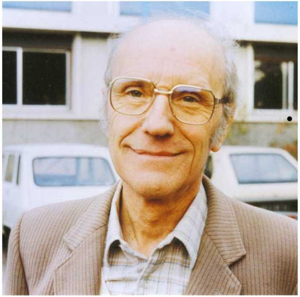


$\operatorname{dom} f = \mathbb{R}$
(proper)



$\operatorname{dom} f =]0, \delta]$
(proper)

A pioneer



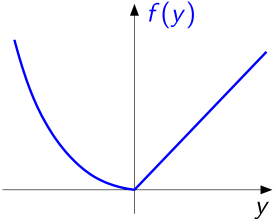
Jean-Jacques Moreau
(1923–2014)

Subdifferential of function: definition

The (Moreau) subdifferential of f, denoted by ∂f,

Subdifferential of function: definition

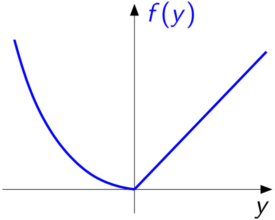
Let $f : \mathcal{H} \to \,]-\infty, +\infty]$ be a proper function.
The (Moreau) subdifferential of f, denoted by ∂f,

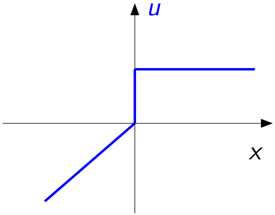


Subdifferential of function: definition

Let $f : \mathcal{H} \to \,]-\infty, +\infty]$ be a proper function.
The (Moreau) subdifferential of f, denoted by ∂f, is such that

$$\begin{aligned} \partial f : \mathcal{H} &\to 2^{\mathcal{H}} \\ x &\to \{u \in \mathcal{H} \,|\, (\forall y \in \mathcal{H})\ \langle y - x | u \rangle + f(x) \le f(y)\} \end{aligned}$$

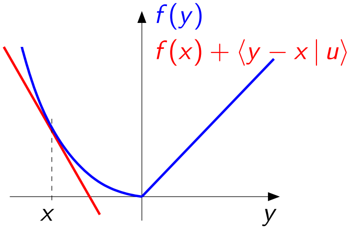


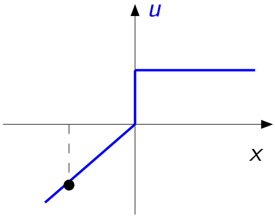


Subdifferential of function: definition

Let $f : \mathcal{H} \to \,]-\infty, +\infty]$ be a proper function.
The (Moreau) subdifferential of f, denoted by ∂f, is such that

$$\begin{aligned} \partial f : \mathcal{H} &\to 2^{\mathcal{H}} \\ x &\to \{u \in \mathcal{H} \,|\, (\forall y \in \mathcal{H})\ \langle y - x | u\rangle + f(x) \leq f(y)\} \end{aligned}$$

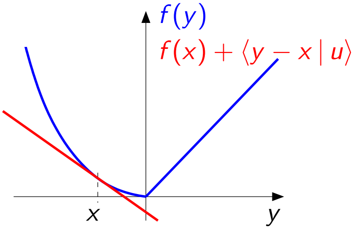


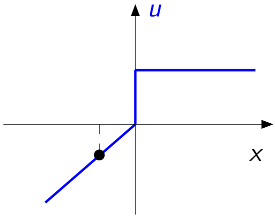


Subdifferential of function: definition

Let $f : \mathcal{H} \to]-\infty, +\infty]$ be a proper function.
The (Moreau) subdifferential of f, denoted by ∂f, is such that

$$\begin{aligned} \partial f : \mathcal{H} &\to 2^{\mathcal{H}} \\ x &\to \{u \in \mathcal{H} \mid (\forall y \in \mathcal{H})\ \langle y - x | u \rangle + f(x) \leq f(y)\} \end{aligned}$$

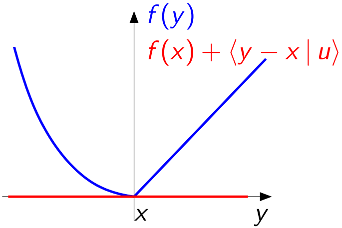


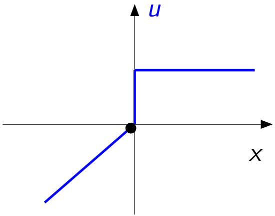


Subdifferential of function: definition

Let $f : \mathcal{H} \to]-\infty, +\infty]$ be a proper function.
The (Moreau) subdifferential of f, denoted by ∂f, is such that

$$\begin{aligned} \partial f : \mathcal{H} &\to 2^{\mathcal{H}} \\ x &\to \{u \in \mathcal{H} \mid (\forall y \in \mathcal{H})\ \langle y - x | u \rangle + f(x) \leq f(y)\} \end{aligned}$$

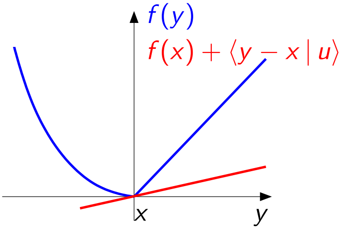


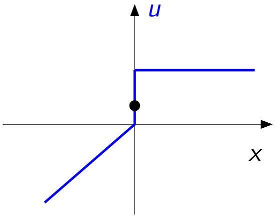


Subdifferential of function: definition

Let $f : \mathcal{H} \to \,]-\infty, +\infty]$ be a proper function.
The (Moreau) subdifferential of f, denoted by ∂f, is such that

$$\begin{aligned} \partial f : \mathcal{H} &\to 2^{\mathcal{H}} \\ x &\to \{u \in \mathcal{H} \mid (\forall y \in \mathcal{H})\ \langle y - x | u \rangle + f(x) \leq f(y)\} \end{aligned}$$

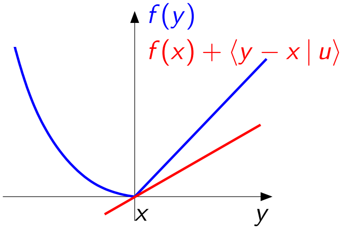


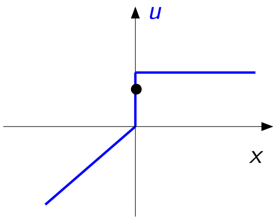


Subdifferential of function: definition

Let $f : \mathcal{H} \to \left]-\infty, +\infty\right]$ be a proper function.
The (Moreau) subdifferential of f, denoted by ∂f, is such that

$$\begin{aligned} \partial f : \mathcal{H} &\to 2^{\mathcal{H}} \\ x &\to \{u \in \mathcal{H} \mid (\forall y \in \mathcal{H})\ \langle y - x | u \rangle + f(x) \leq f(y)\} \end{aligned}$$

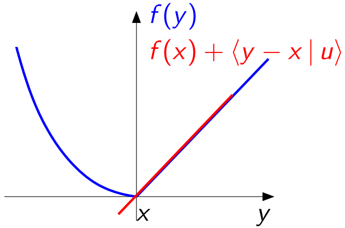


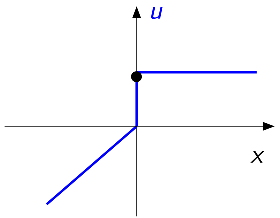


Subdifferential of function: definition

Let $f : \mathcal{H} \to]-\infty, +\infty]$ be a proper function.
The (Moreau) subdifferential of f, denoted by ∂f, is such that

$$\begin{aligned} \partial f : \mathcal{H} &\to 2^{\mathcal{H}} \\ x &\to \{u \in \mathcal{H} \mid (\forall y \in \mathcal{H})\ \langle y - x | u \rangle + f(x) \leq f(y)\} \end{aligned}$$

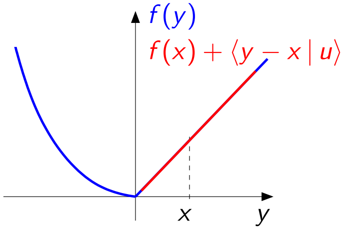


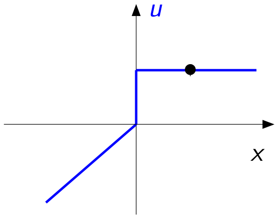


Subdifferential of function: definition

Let $f : \mathcal{H} \to]-\infty, +\infty]$ be a proper function.
The (Moreau) subdifferential of f, denoted by ∂f, is such that

$$\begin{aligned} \partial f : \mathcal{H} &\to 2^{\mathcal{H}} \\ x &\to \{u \in \mathcal{H} \mid (\forall y \in \mathcal{H})\ \langle y - x | u \rangle + f(x) \leq f(y)\} \end{aligned}$$

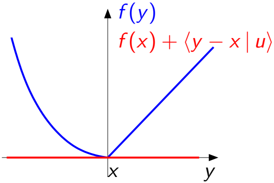


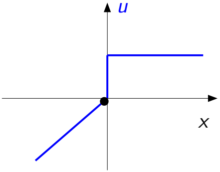


Subdifferential of a function: properties

Let $f : \mathcal{H} \to]-\infty, +\infty]$ be a proper function.
The (Moreau) subdifferential of f, denoted by ∂f, is such that

$$\begin{aligned} \partial f : \mathcal{H} &\to 2^{\mathcal{H}} \\ x &\to \{u \in \mathcal{H} \,|\, (\forall y \in \mathcal{H})\ \langle y - x | u \rangle + f(x) \le f(y)\} \end{aligned}$$





Fermat's rule: $0 \in \partial f(x) \Leftrightarrow x \in \mathrm{Argmin}\ f$

Subdifferential of a function: properties

Let $f : \mathcal{H} \to \,]-\infty, +\infty]$ be a proper function.
The (Moreau) subdifferential of f, denoted ∂f, is such that

$$\begin{aligned} \partial f : \mathcal{H} &\to 2^{\mathcal{H}} \\ x &\to \{u \in \mathcal{H} \,|\, (\forall y \in \mathcal{H})\ \langle y - x | u \rangle + f(x) \le f(y)\} \end{aligned}$$

- $u \in \partial f(x)$ is a subgradient of f at x.

Subdifferential of a convex function: properties

If $f : \mathcal{H} \to \,]-\infty, +\infty]$ is convex and it is Gâteaux differentiable at x, then

$$\partial f(x) = \{\nabla f(x)\}$$

Subdifferential of a convex function: properties

If $f\colon \mathcal{H} \to \left]-\infty, +\infty\right]$ is convex and it is Gâteaux differentiable at x, then

$$\partial f(x) = \{\nabla f(x)\}$$

$$(\forall y \in \mathcal{H}) \qquad \langle \nabla f(x) \mid y \rangle = \lim_{\substack{\alpha \to 0 \\ \alpha \neq 0}} \frac{f(x+\alpha y) - f(x)}{\alpha}.$$

Proof:

For every $\alpha \in [0,1]$ and $y \in \mathcal{H}$,

$$f\big(x+\alpha(y-x)\big) \leq (1-\alpha) f(x) + \alpha f(y)$$

$$\Rightarrow \quad \langle \nabla f(x) \mid y - x \rangle = \lim_{\substack{\alpha \to 0 \\ \alpha \neq 0}} \frac{f\big(x+\alpha(y-x)\big) - f(x)}{\alpha} \leq f(y) - f(x)$$

Then $\nabla f(x) \in \partial f(x)$.

Subdifferential of a convex function: properties

If $f\colon \mathcal{H} \to \,]-\infty,+\infty]$ is convex and it is Gâteaux differentiable at x, then

$$\partial f(x) = \{\nabla f(x)\}$$

$$(\forall y \in \mathcal{H}) \qquad \langle \nabla f(x) \mid y\rangle = \lim_{\substack{\alpha\to 0\\ \alpha\neq 0}} \frac{f(x+\alpha y)-f(x)}{\alpha}.$$

Proof:
Conversely, if $u \in \partial f(x)$, then, for every $\alpha \in [0,+\infty[$ and $y \in \mathcal{H}$,

$$\begin{aligned} & f(x+\alpha y) \geq f(x) + \langle u \mid x+\alpha y - x\rangle \\ \Rightarrow \quad & \langle \nabla f(x) \mid y\rangle = \lim_{\substack{\alpha\to 0\\ \alpha\neq 0}} \frac{f(x+\alpha y)-f(x)}{\alpha} \geq \langle u \mid y\rangle \end{aligned}$$

By selecting $y = u - \nabla f(x)$, it results that $\|u-\nabla f(x)\|^2 \leq 0$ and then $u = \nabla f(x)$.

Subdifferential of a convex function: properties

Let $f\colon \mathcal{H} \to\ ]-\infty,+\infty]$ be Gâteaux differentiable on $\operatorname{dom} f$, with $\operatorname{dom} f$ a convex subset of $\mathcal{H}$.
Then, f is convex if and only if

$$(\forall (x,y) \in (\operatorname{dom} f)^2) \quad f(y) \geq f(x) + \langle \nabla f(x) \mid y - x \rangle \,.$$

Subdifferential of a convex function: properties

Let $f\colon \mathcal{H} \to \left]-\infty, +\infty\right]$ be Gâteaux differentiable on $\operatorname{dom} f$, with $\operatorname{dom} f$ a convex subset of $\mathcal{H}$.
Then, f is convex if and only if

$$(\forall (x,y) \in (\operatorname{dom} f)^2) \quad f(y) \geq f(x) + \langle \nabla f(x) \mid y - x \rangle \, .$$

Proof:
We have already seen that the gradient inequality holds when f is convex and differentiable at $x \in \mathcal{H}$.

Subdifferential of a convex function: properties

Let $f\colon \mathcal{H} \to \,]-\infty,+\infty]$ be Gâteaux differentiable on $\operatorname{dom} f$, with $\operatorname{dom} f$ a convex subset of $\mathcal{H}$.
Then, f is convex if and only if

$$(\forall (x,y) \in (\operatorname{dom} f)^2) \quad f(y) \geq f(x) + \langle \nabla f(x) \mid y - x \rangle \,.$$

Proof:
Conversely, if the gradient inequality is satisfied, we have, for every $(x,y) \in (\operatorname{dom} f)^2$ and $\alpha \in [0,1]$, $\alpha x + (1-\alpha) y \in \operatorname{dom} f$, and

$$f(x) \geq f(\alpha x + (1-\alpha)y) + (1-\alpha)\,\langle \nabla f(\alpha x + (1-\alpha)y) \mid x - y \rangle$$
$$f(y) \geq f(\alpha x + (1-\alpha)y) + \alpha\,\langle \nabla f(\alpha x + (1-\alpha)y) \mid y - x \rangle \,.$$

By multiplying the first inequality by α and the second one by $1-\alpha$ and summing them, we get

$$\alpha f(x) + (1-\alpha) f(y) \geq f(\alpha x + (1-\alpha) y).$$

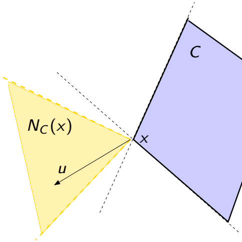
Subdifferential of a convex function: example

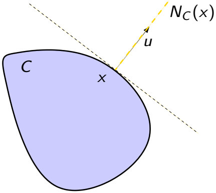
Let C be a nonempty subset of $\mathcal{H}$ with indicator function defined as

$$(\forall x \in \mathcal{H}) \qquad \iota_C(x) = \begin{cases} 0 & \text{if } x \in C \\ +\infty & \text{otherwise.} \end{cases}$$

For every $x \in \mathcal{H}$, $\partial \iota_C(x)$ is the normal cone to C at x defined by

$$N_C(x) = \begin{cases} \{u \in \mathcal{H} \mid (\forall y \in C) \;\; \langle u \mid y - x \rangle \leq 0\} & \text{if } x \in C \\ \varnothing & \text{otherwise.} \end{cases}$$





Subdifferential calculus

Let $\mathcal{H}$ and $\mathcal{G}$ be two real Hilbert spaces.

- Let $f\colon \mathcal{H} \to \,]-\infty, +\infty]$ be proper, then $\forall \lambda \in \,]0, +\infty[\; \partial(\lambda f) = \lambda \partial f$.
- Let $f\colon \mathcal{H} \to \,]-\infty, +\infty]$, $g\colon \mathcal{G} \to \,]-\infty, +\infty]$, and $L \in \mathcal{B}(\mathcal{H}, \mathcal{G})$.

  Define $g \circ L(x) := g(Lx)$ and L^* the *adjoint* operator of L:

  $$(\forall (x, y) \in \mathcal{H} \times \mathcal{G}) \quad \langle y \mid Lx \rangle = \langle L^* y \mid x \rangle \,.$$

  If $\operatorname{dom} g \cap L(\operatorname{dom} f) \neq \varnothing$, then

  $$(\forall x \in \mathcal{H}) \qquad \partial f(x) + L^* \partial g(Lx) \subset \partial (f + g \circ L)(x).$$

Subdifferential calculus

Let $\mathcal{H}$ and $\mathcal{G}$ be two real Hilbert spaces.

- Let $f\colon \mathcal{H} \to \,]-\infty,+\infty]$ be proper, then $\forall \lambda \in \,]0,+\infty[\; \partial(\lambda f) = \lambda \partial f$.
- Let $f\colon \mathcal{H} \to \,]-\infty,+\infty]$, $g\colon \mathcal{G} \to \,]-\infty,+\infty]$, and $L \in \mathcal{B}(\mathcal{H},\mathcal{G})$.

  Define $g \circ L(x) := g(Lx)$ and L^* the *adjoint* operator of L:

  $$(\forall (x,y) \in \mathcal{H} \times \mathcal{G}) \quad \langle y \mid Lx \rangle = \langle L^* y \mid x \rangle\,.$$

  If $\operatorname{dom} g \cap L(\operatorname{dom} f) \neq \varnothing$, then

  $$(\forall x \in \mathcal{H}) \qquad \partial f(x) + L^* \partial g(Lx) \subset \partial (f + g \circ L)(x).$$

Proof: Let $x \in \mathcal{H}$, $u \in \partial f(x)$ and $v \in \partial g(Lx)$. We have:
$u + L^* v \in \partial f(x) + L^* \partial g(Lx)$ and

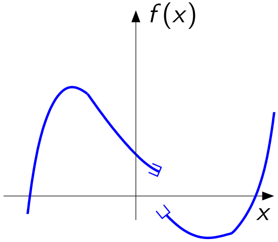
$$(\forall y \in \mathcal{H}) \qquad \begin{aligned} f(y) &\geq f(x) + \langle y - x \mid u \rangle \\ g(Ly) &\geq g(Lx) + \langle L(y-x) \mid v \rangle\,. \end{aligned}$$

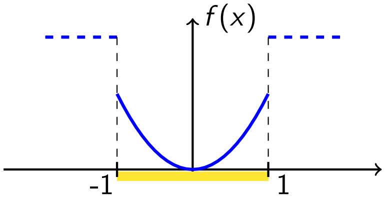
Therefore, by summing,

$$f(y) + g(Ly) \geq f(x) + g(Lx) + \langle y - x \mid u + L^* v \rangle\,.$$

We deduce that $u + L^* v \in \partial (f + g \circ L)(x)$.

Subdifferential: the case of discontinuous functions





Epigraph

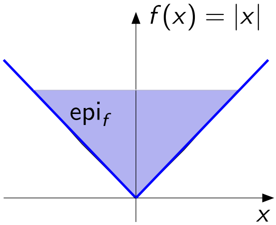
Let $f : \mathcal{H} \to \left]-\infty, +\infty\right]$. The epigraph of f is

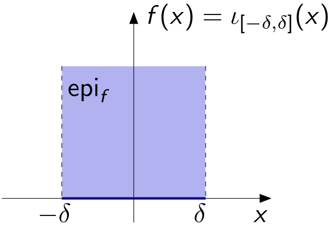
$$\operatorname{epi} f = \left\{ (x, \zeta) \in \operatorname{dom} f \times \mathbb{R} \;\middle|\; f(x) \leq \zeta \right\}$$

Epigraph

Let $f : \mathcal{H} \to \,]-\infty, +\infty]$. The epigraph of f is

$$\text{epi}\, f = \left\{(x, \zeta) \in \operatorname{dom} f \times \mathbb{R} \;\middle|\; f(x) \le \zeta\right\}$$

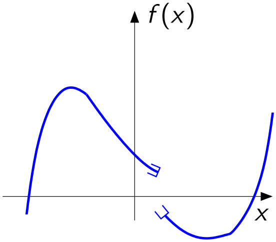


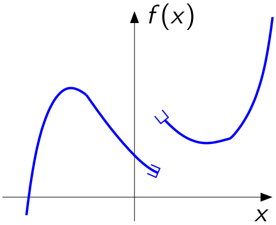


Lower semi-continuity

Let $f : \mathcal{H} \to \,]-\infty, +\infty]$.
f is a lower semi-continuous function on $\mathcal{H}$ if and only if $\operatorname{epi} f$ is closed

- l.s.c. functions ?

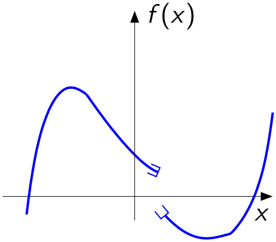


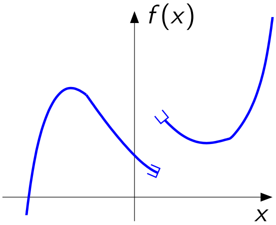


Lower semi-continuity

Let $f : \mathcal{H} \to \,]-\infty, +\infty]$.
f is a lower semi-continuous function on $\mathcal{H}$ if and only if $\operatorname{epi} f$ is closed

- l.s.c. functions ?

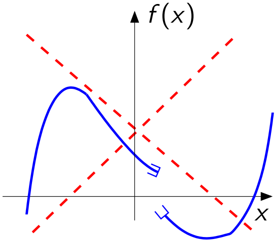


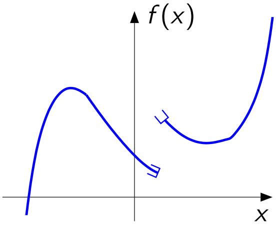


Lower semi-continuity

Let $f : \mathcal{H} \to \,]-\infty, +\infty]$.
f is a lower semi-continuous function on $\mathcal{H}$ if and only if epi f is closed

- <u>l.s.c. functions</u> ?

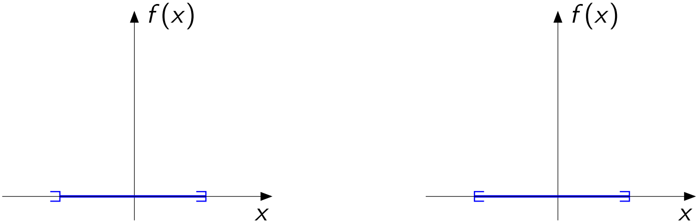




Lower semi-continuity

Let $f : \mathcal{H} \to \,]-\infty, +\infty]$.
f is a lower semi-continuous function on $\mathcal{H}$ if and only if $\operatorname{epi} f$ is closed

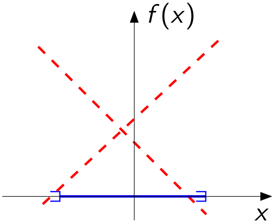
- l.s.c. functions ?

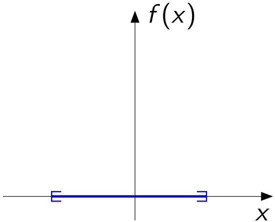


Lower semi-continuity

Let $f : \mathcal{H} \to \,]-\infty, +\infty]$.
f is a lower semi-continuous function on $\mathcal{H}$ if and only if epi f is closed

- l.s.c. functions ?

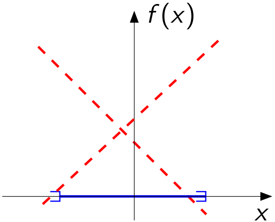


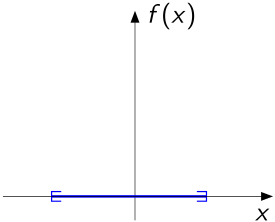


Lower semi-continuity

Let $f : \mathcal{H} \to]-\infty, +\infty]$.
f is a lower semi-continuous function on $\mathcal{H}$ if and only if epi f is closed

- l.s.c. functions ?





Lower semi-continuity

- Every continuous function on $\mathcal{H}$ is l.s.c.
- Every finite sum of l.s.c. functions is l.s.c.
- Let $(f_i)_{i\in I}$ be a family of l.s.c functions.
  Then, $\sup_{i\in I} f_i$ is l.s.c.

A class of convex functions

- $\Gamma_0(\mathcal{H})$: class of convex, l.s.c., and proper functions from $\mathcal{H}$ to $]-\infty, +\infty]$.

- $\iota_C \in \Gamma_0(\mathcal{H}) \Leftrightarrow C$ is a nonempty closed convex set.

  Proof: $\mathrm{epi}_{\iota_C} = C \times [0, +\infty[$.

Subdifferential calculus

Let $\mathcal{H}$ and $\mathcal{G}$ be two real Hilbert spaces.
Let $f \in \Gamma_0(\mathcal{H})$, $g \in \Gamma_0(\mathcal{G})$, and $L \in \mathcal{B}(\mathcal{H}, \mathcal{G})$.
If $\operatorname{int}(\operatorname{dom} g) \cap L(\operatorname{dom} f) \neq \varnothing$ or $\operatorname{dom} g \cap \operatorname{int}\big(L(\operatorname{dom} f)\big) \neq \varnothing$, then

$$\partial f + L^* \partial g \, L = \partial(f + g \circ L) .$$

<u>Particular case</u>:

- If $f \in \Gamma_0(\mathcal{H})$, $g \in \Gamma_0(\mathcal{G})$, and f is finite valued, then $\partial f + \partial g = \partial(f + g)$.
- If $g \in \Gamma_0(\mathcal{G})$, $L \in \mathcal{B}(\mathcal{G}, \mathcal{H})$, and $\operatorname{int}(\operatorname{dom} g) \cap \operatorname{ran} L \neq \varnothing$, then $L^* \partial g \, L = \partial(g \circ L)$.

Subdifferential calculus

Let $(\mathcal{H})_{i\in I}$ where $I \subset \mathbb{N}$ be Hilbert spaces and let $\mathcal{H} = \bigoplus_{i\in I} \mathcal{H}_i$.
For every $i \in I$, let $f_i\colon \mathcal{H}_i \to \left]-\infty, +\infty\right]$ be a proper function. Let

$$f\colon \mathcal{H} \to \left]-\infty, +\infty\right] : x = (x_i)_{i\in I} \mapsto \sum_{i\in I} f_i(x_i)$$

Then,

$$\big(\forall x = (x_i)_{i\in I} \in \mathcal{H}\big) \qquad \partial f(x) = \mathop{\times}_{i\in I} \partial f_i(x_i).$$

Subdifferential calculus

Let $(\mathcal{H})_{i\in I}$ where $I \subset \mathbb{N}$ be Hilbert spaces and let $\mathcal{H} = \bigoplus_{i\in I} \mathcal{H}_i$.
For every $i \in I$, let $f_i \colon \mathcal{H}_i \to \,]-\infty,+\infty]$ be a proper function. Let

$$f \colon \mathcal{H} \to \,]-\infty,+\infty] : x = (x_i)_{i\in I} \mapsto \sum_{i\in I} f_i(x_i)$$

Then,

$$(\forall x = (x_i)_{i\in I} \in \mathcal{H}) \qquad \partial f(x) = \bigtimes_{i\in I} \partial f_i(x_i).$$

<u>Proof</u>: Let $x = (x_i)_{i\in I} \in \mathcal{H}$. We have

$$\begin{aligned}
& t = (t_i)_{i\in I} \in \bigtimes_{i\in I} \partial f_i(x_i) \\
\Leftrightarrow\quad & (\forall i \in I)(\forall y_i \in \mathcal{H}_i)\;\; f_i(y_i) \geq f_i(x_i) + \langle t_i \mid y_i - x_i \rangle \\
\Rightarrow\quad & (\forall y = (y_i)_{i\in I} \in \mathcal{H})\;\; \sum_{i\in I} f_i(y_i) \geq \sum_{i\in I} f_i(x_i) + \sum_{i\in I} \langle t_i \mid y_i - x_i \rangle \\
\Leftrightarrow\quad & (\forall y \in \mathcal{H})\;\; f(y) \geq f(x) + \langle t \mid y - x \rangle .
\end{aligned}$$

Subdifferential calculus

Let $(\mathcal{H})_{i\in I}$ where $I \subset \mathbb{N}$ be Hilbert spaces and let $\mathcal{H} = \bigoplus_{i\in I} \mathcal{H}_i$.
For every $i \in I$, let $f_i : \mathcal{H}_i \to \,]-\infty, +\infty]$ be a proper function. Let

$$f : \mathcal{H} \to \,]-\infty, +\infty] : x = (x_i)_{i\in I} \mapsto \sum_{i\in I} f_i(x_i)$$

Then,

$$(\forall x = (x_i)_{i\in I} \in \mathcal{H}) \qquad \partial f(x) = \bigtimes_{i\in I} \partial f_i(x_i).$$

<u>Proof</u>: Conversely,

$$\begin{aligned} & t = (t_i)_{i\in I} \in \partial f(x) \\ \Leftrightarrow \quad & (\forall y = (y_i)_{i\in I} \in \mathcal{H}) \ \ \sum_{i\in I} f_i(y_i) \geq \sum_{i\in I} f_i(x_i) + \sum_{i\in I} \langle t_i \mid y_i - x_i \rangle . \end{aligned}$$

Let $j \in I$. By setting $(\forall i \in I \setminus \{j\})\ y_i = x_i \in \operatorname{dom} f_i$, we get

$$(\forall y_j \in \mathcal{H}_j) \ \ f_j(y_j) \geq f_j(x_j) + \langle t_j \mid y_j - x_j \rangle .$$

Exercise 1: Huber function

Let $\rho > 0$ and set

$$f : \mathbb{R} \to \mathbb{R} : \ \mapsto \begin{cases} \frac{x^2}{2}, & \text{if } |x| \leq \rho \\ \rho|x| - \frac{\rho^2}{2}, & \text{otherwise.} \end{cases}$$

1. What is the domain of f ?
2. Plot the subdifferential of f.
3. Is f differentiable ? Prove that f is convex.

Exercise 2

Let $\mathcal{H}$ be a Hilbert space. Let $f\colon \mathcal{H} \to \left]-\infty, +\infty\right]$ and let $C \subset \mathcal{H}$ such that $\operatorname{dom} f \cap C \neq \varnothing$. Give a sufficient condition for $x \in \mathcal{H}$ to be a global minimizer of $f + \iota_C$.

Exercice 3: Monotony of the subdifferential of a function

Let $f\colon \mathcal{H} \to \,]-\infty, +\infty]$ be a proper function.
Its subdifferential is a monotone operator, i.e.

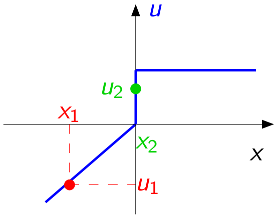
$$\big(\forall (x_1, x_2) \in \mathcal{H}^2\big)\big(\forall u_1 \in \partial f(x_1)\big)\big(\forall u_2 \in \partial f(x_2)\big) \quad \langle u_1 - u_2 \mid x_1 - x_2 \rangle \geq 0.$$

Exercice 3: Monotony of the subdifferential of a function

Let $f : \mathcal{H} \to \,]-\infty, +\infty]$ be a proper function.
Its subdifferential is a monotone operator, i.e.

$$\big(\forall (x_1, x_2) \in \mathcal{H}^2\big)\big(\forall u_1 \in \partial f(x_1)\big)\big(\forall u_2 \in \partial f(x_2)\big) \quad \langle u_1 - u_2 \mid x_1 - x_2 \rangle \geq 0.$$

- Proof:
  By definition:
  $$\langle x_2 - x_1 | u_1 \rangle + f(x_1) \leq f(x_2)$$
  $$\langle x_1 - x_2 | u_2 \rangle + f(x_2) \leq f(x_1)$$
- It results that $\boxed{\langle x_1 - x_2 \mid u_1 - u_2 \rangle \geq 0}$.



Exercice 4: Convexity and monotony

Let $f\colon \mathcal{H} \to\ ]-\infty, +\infty]$ be Gâteaux differentiable on $\operatorname{dom} f$, which is convex.
Then, f is convex if and only if ∇f is monotone on $\operatorname{dom} f$, i.e.

$$(\forall (x, y) \in (\operatorname{dom} f)^2) \quad \langle \nabla f(y) - \nabla f(x) \mid y - x \rangle \geq 0.$$

Exercice 4: Convexity and monotony

Let $f: \mathcal{H} \to]-\infty, +\infty]$ be Gâteaux differentiable on $\operatorname{dom} f$, which is convex.
Then, f is convex if and only if ∇f is monotone on $\operatorname{dom} f$, i.e.

$$(\forall (x,y) \in (\operatorname{dom} f)^2) \quad \langle \nabla f(y) - \nabla f(x) \mid y - x \rangle \geq 0.$$

Proof:
When f is convex, we have seen that its subdifferential is monotone and, for every $x \in \operatorname{dom} f$, $\partial f(x) = \{\nabla f(x)\}$.

Exercice 4: Convexity and monotony

Let $f\colon \mathcal{H} \to \,]-\infty, +\infty]$ be Gâteaux differentiable on $\operatorname{dom} f$, which is convex.
Then, f is convex if and only if ∇f is monotone on $\operatorname{dom} f$, i.e.

$$(\forall (x, y) \in (\operatorname{dom} f)^2) \quad \langle \nabla f(y) - \nabla f(x) \mid y - x \rangle \geq 0.$$

Proof:
Conversely, assume that ∇f is monotone on $\operatorname{dom} f$. For every $(x, y) \in (\operatorname{dom} f)^2$, let $\varphi\colon [0, 1] \to \mathbb{R}\colon \alpha \mapsto f(x + \alpha(y - x))$.
φ is differentiable on $[0, 1]$ and

$$(\forall \alpha \in [0, 1]) \qquad \varphi'(\alpha) = \langle \nabla f(x + \alpha(y - x)) \mid y - x \rangle \,.$$

On the other hand, for every $\alpha \in]0, 1]$

$$\begin{aligned}
& \langle \nabla f(x + \alpha(y - x)) - \nabla f(x) \mid y - x \rangle \geq 0 \\
\Leftrightarrow \quad & \varphi'(\alpha) \geq \langle \nabla f(x) \mid y - x \rangle \\
\Rightarrow \quad & \varphi(1) - \varphi(0) = \int_0^1 \varphi'(\alpha) d\alpha \geq \langle \nabla f(x) \mid y - x \rangle \\
\Leftrightarrow \quad & f(y) - f(x) \geq \langle \nabla f(x) \mid y - x \rangle \,.
\end{aligned}$$