

Multiscale analysis in image processing

Preliminaries on wavelets

Barbara Pascal[†] and Nelly Pustelnik[‡]

bpascal-fr.github.io/talks

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FONDATION
SIMONE ET CINO
DEL DUCA
INSTITUT DE FRANCE

Multiple scales in real data



Albert Marquet, *Paysage, baie méditerranéenne, vue d'Agay*, 1905

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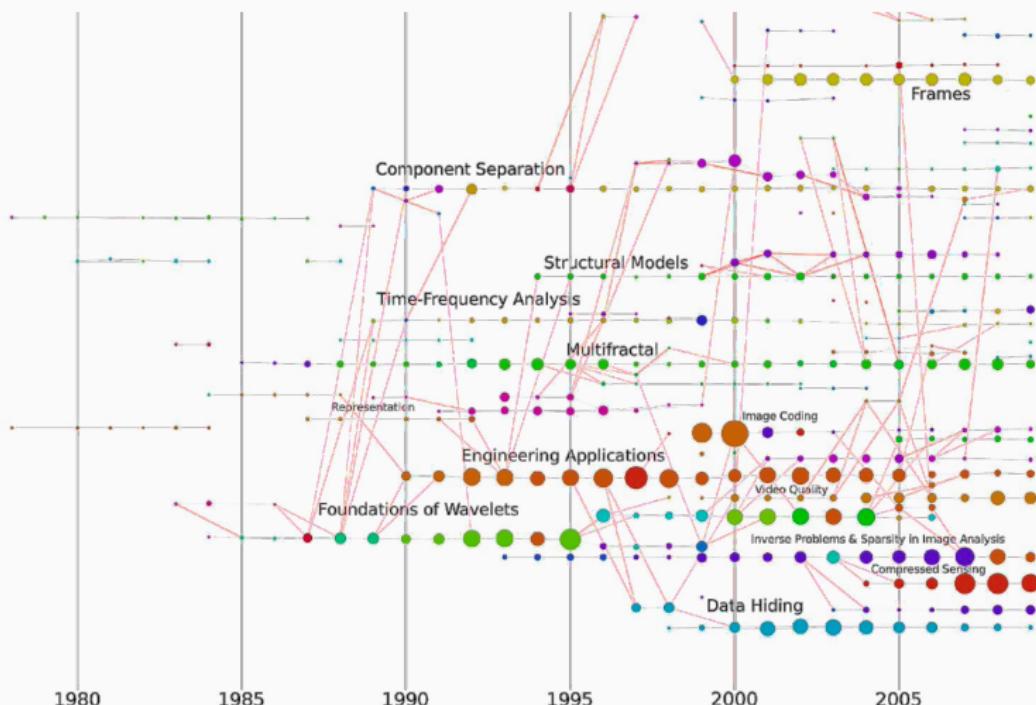
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Overview of the history of wavelets

Revealing evolutions in dynamical networks

Matteo Morini, Patrick Flandrin, Eric Fleury, Tommaso Venturini, Pablo Jensen¹

IXXI, ENS de Lyon, INRIA, CNRS, LIP UMR 5668, LP UMR 5672



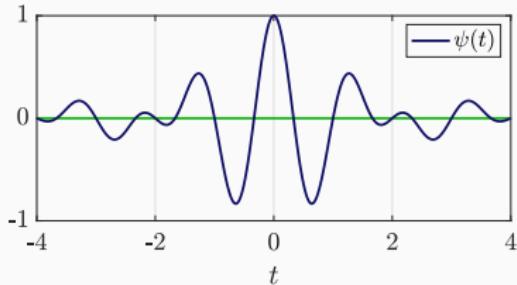
Wavelet Transforms

Continuous Wavelet Transform

Wavelet: $\psi \in L^2(\mathbb{R})$ locally oscillating, integrable with $\int_{\mathbb{R}} \psi(s) \, ds = 0$

Example: real Shannon wavelet

$$\psi(t) = \frac{\sin(2\pi t) - \sin(\pi t)}{\pi t}$$

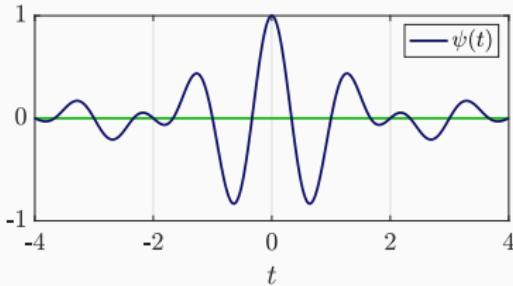


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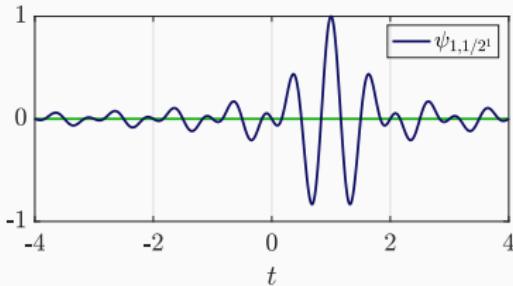
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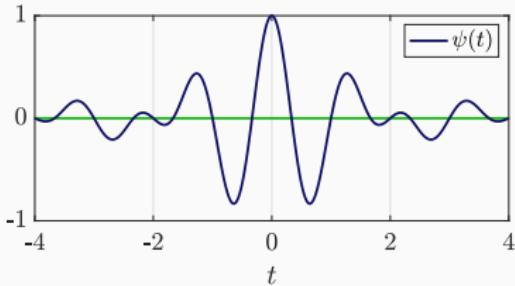
$\implies \{\psi_{t,a}, (t, a) \in \mathbb{R} \times \mathbb{R}_+\}$ atoms with **different time supports**

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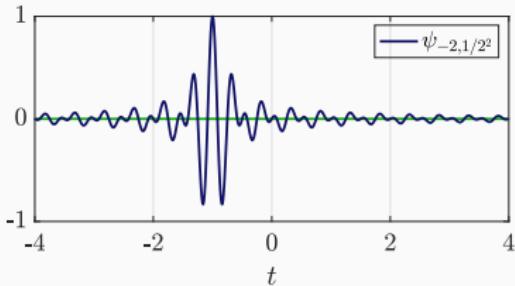
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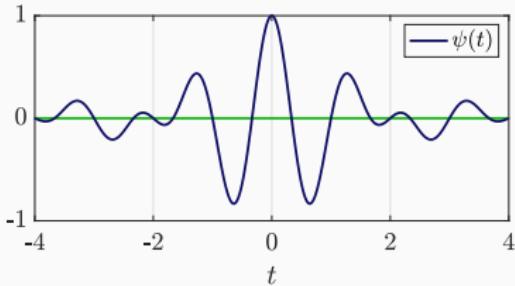
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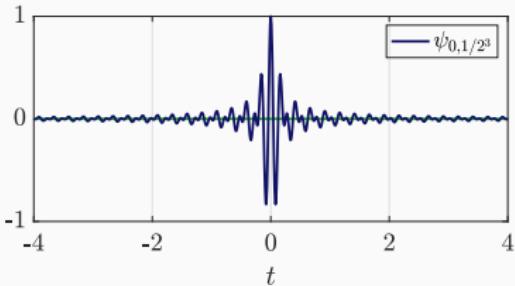
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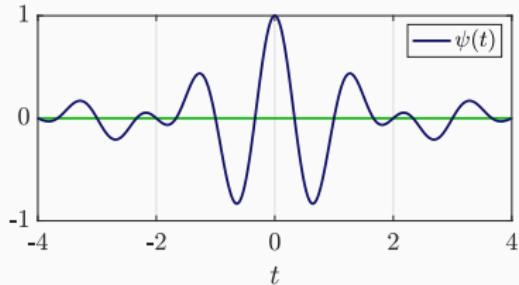
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Continuous Wavelet Transform of a finite-energy signal $f \in L^2(\mathbb{R})$

$$\mathcal{W}_f(t, a) = \langle f, \psi_{t,a} \rangle = \int_{\mathbb{R}} f(s) \frac{1}{\sqrt{a}} \overline{\psi\left(\frac{s-t}{a}\right)} \, ds$$

$\langle \cdot, \cdot \rangle$ scalar product in $L^2(\mathbb{R})$, $\bar{\cdot}$ complex conjugate

$\implies \{\psi_{t,a}, (t, a) \in \mathbb{R} \times \mathbb{R}_+\}$ atoms with **different time supports**

Example of a gravitational wave

Physics: two objects $M = m_1 + m_2$ at distance R , $\mu^{-1} = m_1^{-1} + m_2^{-1}$

$$f(t) = A(t_0 - t)^{-1/4} \cos(d(t_0 - t)^{5/8} + \varphi) \mathbf{1}_{(-\infty; t_0]}(t)$$

chirp: amplitude $a(t) = A(t_0 - t)^{-\frac{1}{4}}$, frequency $\omega(t) = \frac{10\pi d}{8}(t_0 - t)^{-\frac{3}{8}}$

– t_0 : time of coalescence,

– d : instantaneous frequency parameter

$$d \simeq 241 \mathcal{M}_\odot^{-5/8},$$

– A : amplitude reference

$$A \simeq 3.37 \times 10^{-21} \mathcal{M}_\odot^{5/4} / R.$$

Unknown: $\mathcal{M}_\odot = \mu^{3/5} M^{2/5} / M_\odot$: chirp mass in solar mass unit M_\odot

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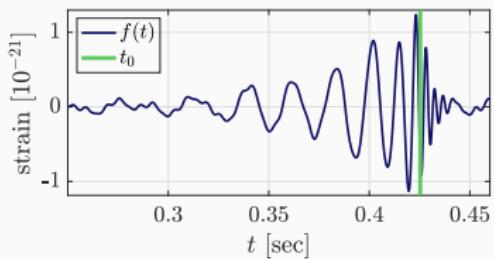
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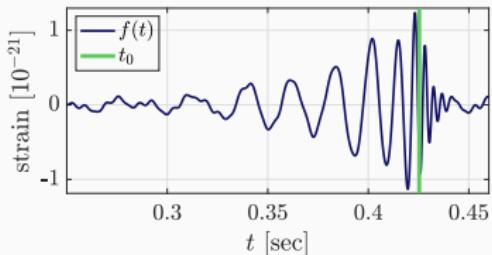
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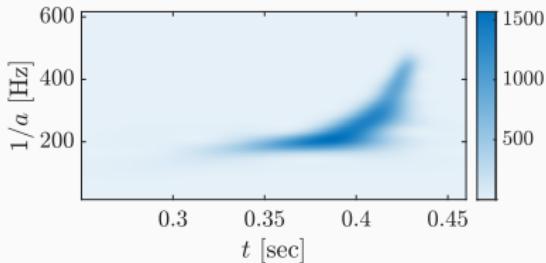
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Morlet scalogram $|\mathcal{W}_f(t, a)|^2$
tftb.nongnu.org

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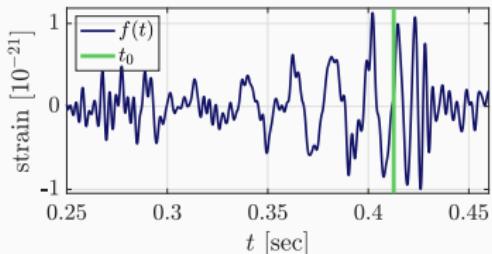
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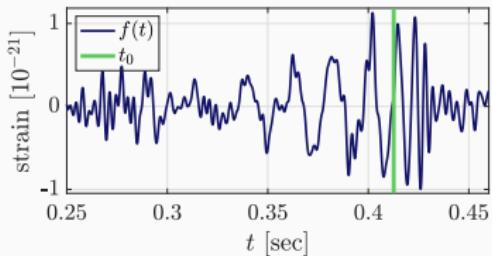
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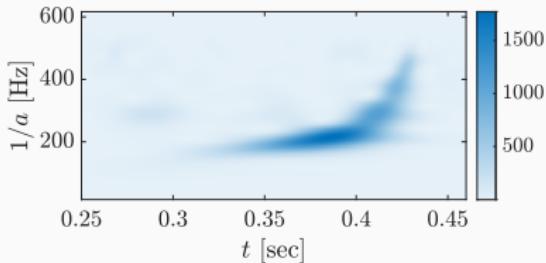
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Continuous Wavelet Transform

Reconstruction formula For $\tilde{\psi}$ the Fourier transform of ψ ,

if $C_\psi = \int_{\mathbb{R}} \frac{|\tilde{\psi}(\omega)|^2}{|\omega|} d\omega < \infty$ then ψ is **admissible** and for $f \in L^2(\mathbb{R})$

$$f(t) = \frac{1}{C_\psi} \int_{\mathbb{R} \times \mathbb{R}_+} \mathcal{W}_f(s, a) \frac{1}{\sqrt{a}} \psi\left(\frac{s-t}{a}\right) ds \frac{da}{a^2}$$

with $\mathcal{W}_f(s, a) = \langle f, \psi_{s,a} \rangle$

[A. P. Calderón, 1964, *Stud. Math.*;

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Reproducing kernel $\mathcal{W}_f(t, a)$ **redundant** representation of f

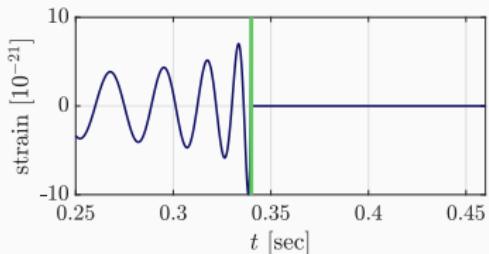
$$\mathcal{W}_f(t', a') = \frac{1}{C_\psi} \int_{\mathbb{R} \times \mathbb{R}_+} \mathcal{K}(t', t; a', a) \mathcal{W}_f(t, a) dt \frac{da}{a^2}$$

with $\mathcal{K}(t', t; a', a) = \langle \psi_{t,a}, \psi_{t',a'} \rangle$ correlations between wavelets

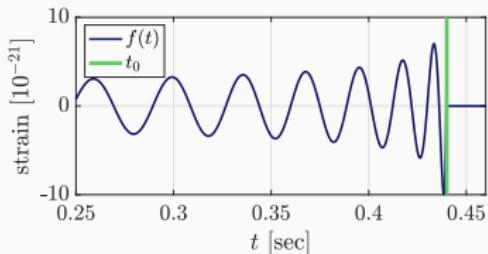
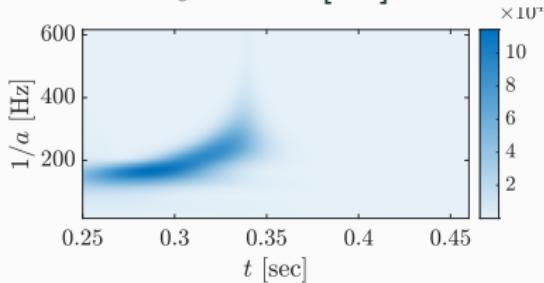
Continuous Wavelet Transform

Translation invariance

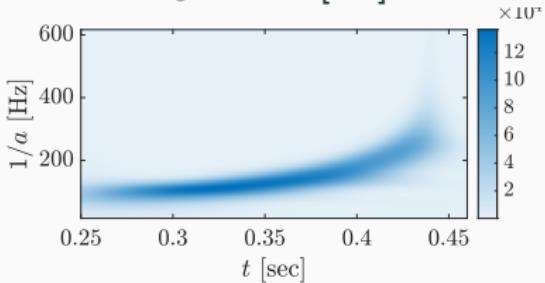
Let $f^{\Delta t}(t) = f(t - \Delta t)$, then $\mathcal{W}_{f^{\Delta t}}(t, a) = \mathcal{W}_f(t - \Delta t, a)$.



$$t_0 = 0.44 \text{ [sec]}$$



$$t_0 = 0.34 \text{ [sec]}$$



Discrete Signals and Wavelets

From continuous signals to discrete vectors

- f continuous on $[0, 1]$, discretized in $z_n = f\left(\frac{n}{N}\right)$, $n = 0, 1, \dots, N$
discrete wavelet transform can be computed at scales $N^{-1} < a^j < 1$
- **discrete scales:** $a = 2^{1/v} \implies v$ intermediate scales in octave $[2^j, 2^{j+1})$

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Practical implementation

Integral representation

$$\mathcal{W}_f(t, a) = \int_{\mathbb{R}} f(s) \frac{1}{\sqrt{a}} \overline{\psi\left(\frac{s-t}{a}\right)} ds$$

Discrete convolution $z * \overline{\psi_{t,a}(-\cdot)}$

$$\mathcal{W}_z[n, j] = \sum_{m=0}^{N-1} z_m \frac{1}{\sqrt{a^j}} \overline{\psi\left(\frac{m-n}{a^j}\right)}$$

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complexity $\mathcal{O}(vN (\log_2 N)^2)$

Multiresolution Analysis

Aims and principles of multiresolution analysis

Motivation: process only details at relevant **discrete** resolutions



[P. J. Burt & E. H. Adelson, 1983, *Proc. IEEE Int. Conf. Commun.*]

Multiresolution analysis

Definition: A multiresolution analysis of $L^2(\mathbb{R})$ is a subspaces sequence
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- **regularity**: φ **father wavelet** or **scaling function** such that

$\{\varphi(t - k), \quad k \in \mathbb{Z}\}$ is an orthonormal basis of V_0

- **completeness**: $\cup_{j \in \mathbb{Z}} V_j$ is dense in $L^2(\mathbb{R})$ and $\cap_{j \in \mathbb{Z}} V_j = \{0\}$
 $\implies V_j$ **approximation space** at **scale** 2^j , i.e., **resolution** 2^{-j} .

[P. J. Burt & E. H. Adelson, 1983, *Proc. IEEE Int. Conf. Commun.*;

S. Mallat, 1989, *Trans. Amer. Math. Soc.*;

S. Mallat, 1989, *IEEE Trans. Pattern Anal. Mach. Intell.*;

Y. Meyer, 1992, *Cambridge University Press*]

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scale 4 \iff resolution 1/4



scale 2 \iff resolution 1/2



scale 1 \iff resolution 1

$$\text{scale } 2^j \iff \text{resolution } 2^{-j}$$

[P. J. Burt & E. H. Adelson, 1983, *Proc. IEEE Int. Conf. Commun.*]

Multiresolution analysis – Daubechies 2 wavelets

From **time** and **scale** invariance and **regularity** condition:

$$\{\sqrt{2^{-j}}\varphi(t/2^j - k), \quad k \in \mathbb{Z}\} \text{ is an orthonormal basis of } V_j$$

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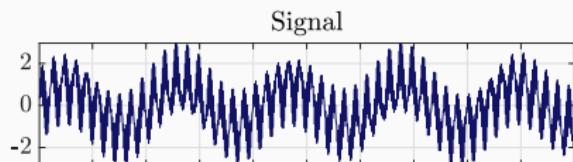
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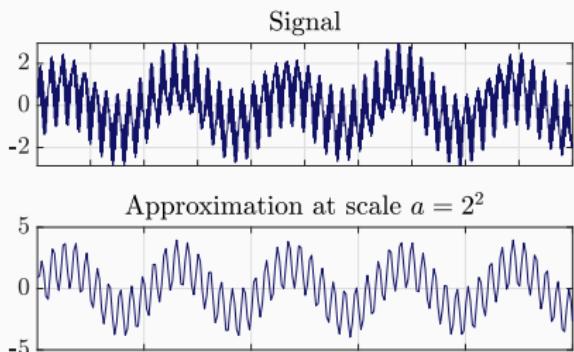
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Multiresolution analysis – Daubechies 2 wavelets

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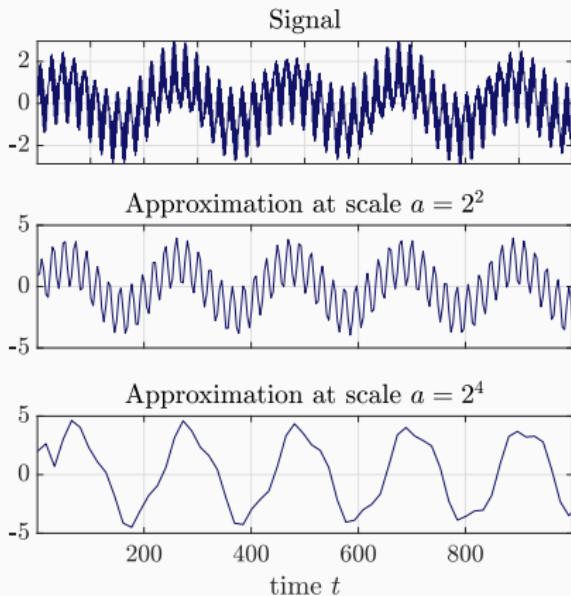
$$\{\sqrt{2^{-j}}\varphi(t/2^j - k), \quad k \in \mathbb{Z}\} \text{ is an orthonormal basis of } V_j$$

Orthogonal projection of f onto V_j

$$f|_{V_j} = \sum_{k \in \mathbb{Z}} \phi_{j,k} \sqrt{2^{-j}} \varphi(t/2^j - k)$$

with *approximation coefficients*

$$\phi_{j,k} = \int_{\mathbb{R}} f(s) \frac{1}{\sqrt{2^j}} \overline{\varphi(s/2^j - k)} ds$$



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Approximation and details: $V_{j-1} = V_j \oplus W_j$, W_j : lost information between $f|_{V_{j-1}}$ at resolution $2^{-(j-1)}$ and $f|_{V_j}$ at resolution 2^{-j}

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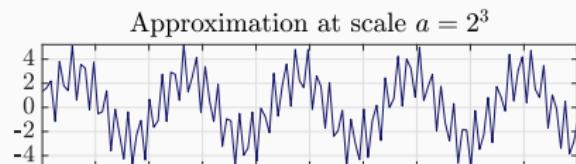
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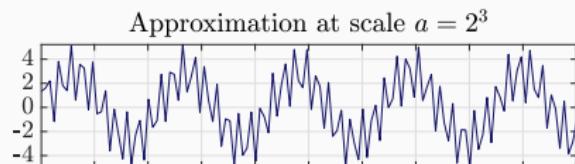
Orthogonal decomposition of $f|_{V_{j-1}}$

$$f|_{V_{j-1}} = f|_{V_j} + f|_{W_j}$$

$$\text{with } f|_{W_j} = \sum_{k \in \mathbb{Z}} \zeta_{j,k} \sqrt{2^{-j}} \psi(t/2^j - k)$$

involving **wavelet coefficients**

$$\zeta_{j,k} = \int_{\mathbb{R}} f(s) \frac{1}{\sqrt{2^j}} \overline{\psi(s/2^j - k)} \, ds$$



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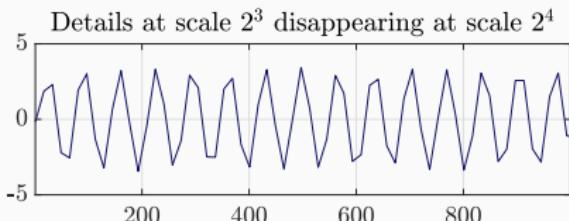
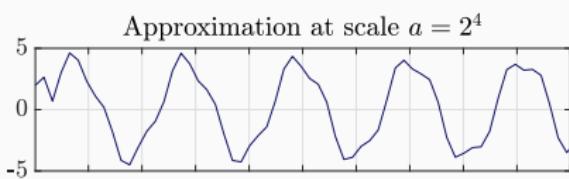
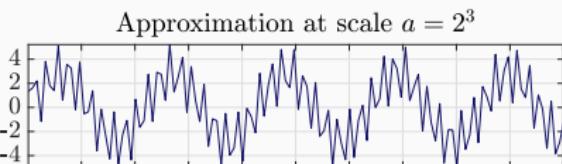
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Orthonormal wavelet basis

By the **completeness** property of the multiresolution analysis

$$\{\sqrt{2^{-j}}\psi(t/2^j - k), \quad (j, k) \in \mathbb{Z}^2\}$$

is an orthonormal wavelet basis of $L^2(\mathbb{R})$.

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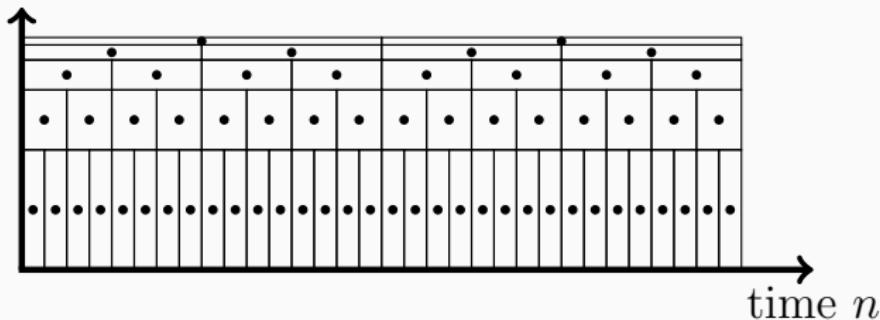
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Tiling of the time-scale half-plane

scale 2^j



Vanishing Moments of Wavelets

Theorem: Let φ scaling function, ψ mother wavelet and $\tilde{\psi}$ its Fourier transform s.t.

$$|\varphi(t)| \underset{|t| \rightarrow \infty}{=} \mathcal{O}\left((1+t^2)^{-n_\psi/2-1}\right), \quad |\psi(t)| \underset{|t| \rightarrow \infty}{=} \mathcal{O}\left((1+t^2)^{-n_\psi/2-1}\right)$$

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Interpretation:

- mother wavelet **orthogonal** to polynomials of degree at most $n_\psi - 1$
- if signal f is C^k , $k < n_\psi$

wavelet coefficients $\zeta_{j,n} = \langle f, \psi_{j,n} \rangle$ **small at fine scales**

Support size and number vanishing moments trade-off

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the wavelet coefficient $\langle f, \psi_{j,n} \rangle$ is **large**.

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If ψ has a compact support of size $\Delta \in \mathbb{N}^*$, at scale 2^j

Δ wavelet coefficients $\zeta_{j,n} = \langle f, \psi_{j,n} \rangle$ are **large**

\implies to reduce the number of significant coefficients, **reduce support size** of ψ .

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Theorem: Let ψ a wavelet with n_ψ vanishing moments generating an orthonormal basis of $L^2(\mathbb{R})$, then its support is of size at least $2n_\psi - 1$.

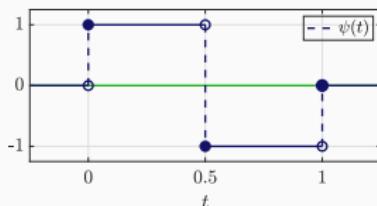
A **Daubechies** wavelet has a minimum-size support $[-n_\psi + 1, n_\psi]$ and the support of the corresponding scaling function is $[0, 2n_\psi - 1]$.

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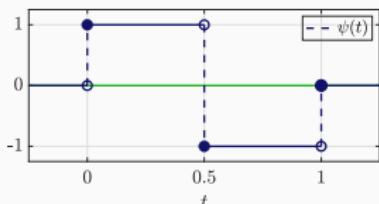
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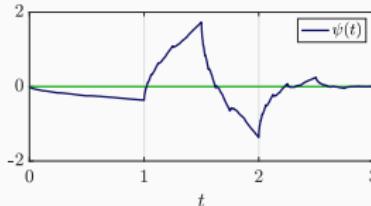
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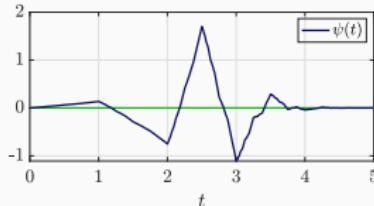


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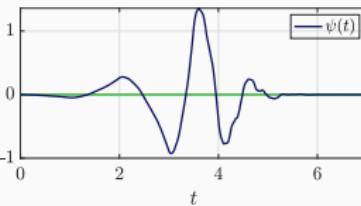
Daubechies wavelet:



$$n_\psi = 2$$



$$n_\psi = 3$$



$$n_\psi = 4$$

Decompositions on Wavelet Frames

Theory of Frames

\mathcal{H} : Hilbert space, e.g., $L^2(\mathbb{R})$ or subspace of $L^2(\mathbb{R})$; $\mathbb{I} \subset \mathbb{N}$: set of indices

Definition A family of elements of \mathcal{H} , $\{\mathbf{e}_n, n \in \mathbb{I}\}$, s.t.

$$(\forall f \in \mathcal{H}) \quad \underline{\mu} \|f\|^2 \leq \sum_{n \in \mathbb{Z}} |\langle f, \mathbf{e}_n \rangle|^2 \leq \bar{\mu} \|f\|^2$$

for some bounds $0 < \underline{\mu} \leq \bar{\mu}$, is a **frame**.

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Stable analysis and synthesis

$f \in \mathcal{H} \mapsto (\langle f, \mathbf{e}_n \rangle)_{n \in \mathbb{N}}$ $\ell^2(\mathbb{I})$ bounded linear operator

$(f_n)_{n \in \mathbb{N}} \in \ell^2(\mathbb{I}) \mapsto \sum_{n \in \mathbb{Z}} f_n \mathbf{e}_n \in \mathcal{H}$ bounded linear operator

Theory of Frames

Initially: reconstruction of irregularly sampled band-limited signals

[Duffin & Schaeffer, 1952, *Trans. Amer. Math. Soc.*]

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Motivations in multiresolution analysis:

Continuous Wavelet Transform: highly redundant, computationally costly

$$\mathcal{W}_f(t, a) = \int_{\mathbb{R}} f(s) \frac{1}{\sqrt{a}} \overline{\psi\left(\frac{s-t}{a}\right)} ds$$

Orthonormal wavelet basis decomposition: not invariant under translations

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Wavelet frames: $\tilde{\psi}_{j,n} = \left\{ \frac{1}{\sqrt{\gamma^j}} \tilde{\psi}\left(\frac{t - bn\gamma^j}{\gamma^j}\right), (j, n) \in \mathbb{Z}^2 \right\}, \gamma > 1, b > 0$

⇒ more freedom in the design of the wavelet $\tilde{\psi}$

Wavelet Decomposition of Images

Mathematical representation of images



Real-valued square-integrable **field**

$$X : \mathbb{R}^2 \rightarrow \mathbb{R}$$

restricted to a **rectangular** domain $\Omega = [0, n_1 - 1] \times [0, n_2 - 1]$

Two-dimensional wavelet decomposition

Separable wavelet bases:

φ and ψ the scaling function and mother wavelet of a **1D multiresolution analysis**

Define

$$\begin{cases} \psi^{(0)}(\underline{x}) = \varphi(x_1)\varphi(x_2), & \psi^{(1)}(\underline{x}) = \psi(x_1)\varphi(x_2) \\ \psi^{(2)}(\underline{x}) = \varphi(x_1)\psi(x_2), & \psi^{(3)}(\underline{x}) = \psi(x_1)\psi(x_2). \end{cases}$$

Then, the family

$$\left\{ 2^{-j}\psi^{(m)}(\underline{x}2^{-j} - \underline{n}), m \in \{1, 2, 3\}, \underline{x} = (x_1, x_2) \in \mathbb{R}^2, \underline{n} = (n_1, n_2) \in \mathbb{Z}^2 \right\}$$

defines an **orthonormal wavelet basis** of $L^2(\mathbb{R})$.

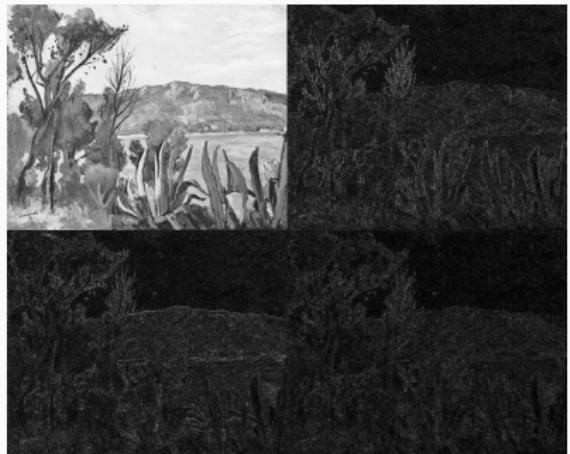
The **wavelet coefficients** of a 2D field $X \in L^2(\mathbb{R}^2)$ are defined as

$$\zeta_{j,\underline{n}}^{(m)} = \langle X, \psi_{j,\underline{n}}^{(m)} \rangle, \quad \psi_{j,\underline{n}}^{(m)}(\underline{x}) = 2^{-j}\psi^{(m)}(\underline{x}2^{-j} - \underline{n})$$

Wavelet transform of images



Albert Marquet, *Paysage, baie méditerranéenne, vue d'Agay*, 1905

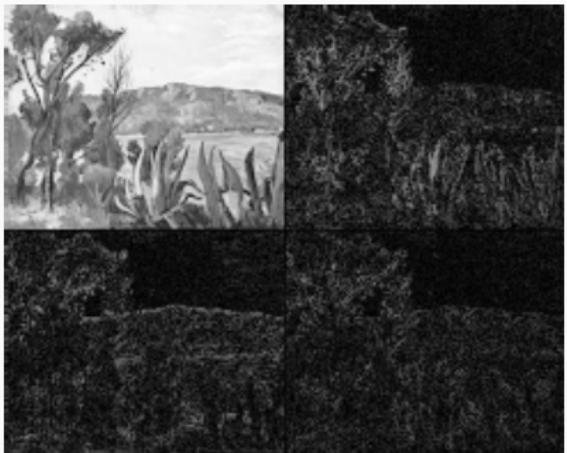


Daubechies wavelet transform
with $n_\psi = 2$ vanishing moments
at scale 2^1

Wavelet transform of images



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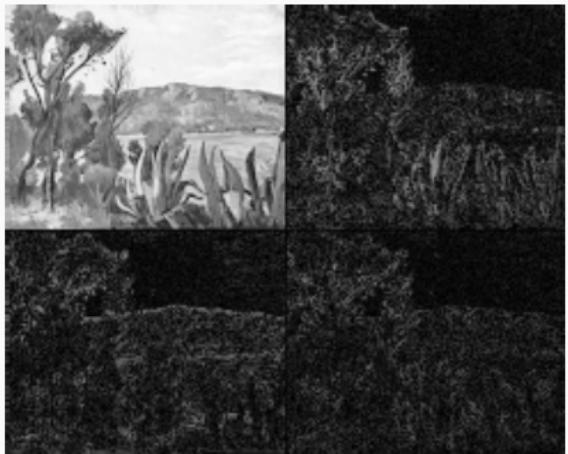


Daubechies wavelet transform
with $n_\psi = 2$ vanishing moments
at scale 2^3

Wavelet transform of images



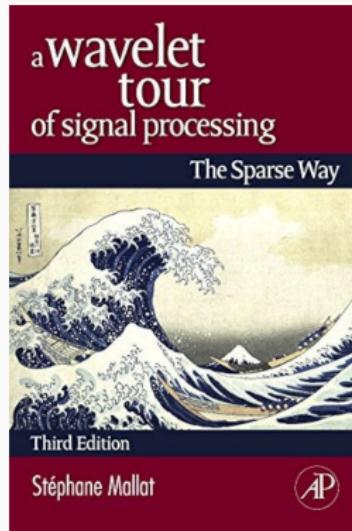
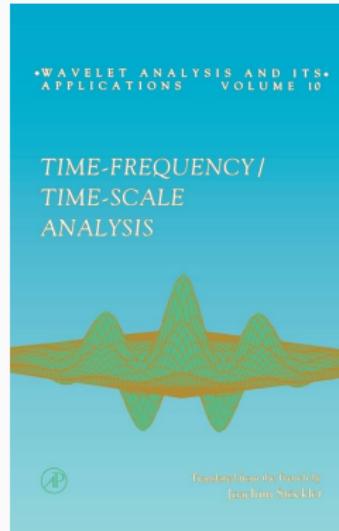
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Daubechies wavelet transform
with $n_\psi = 2$ vanishing moments
at scale 2^3

Application: compression of images and videos: **JPEG2000, MPEG-4**

References and further readings



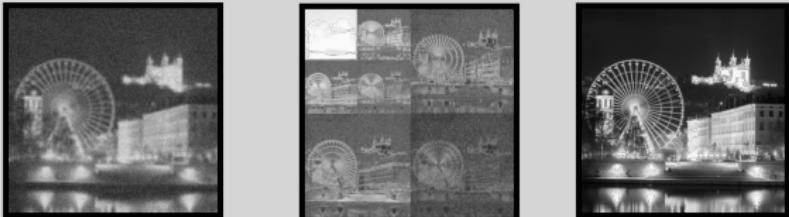
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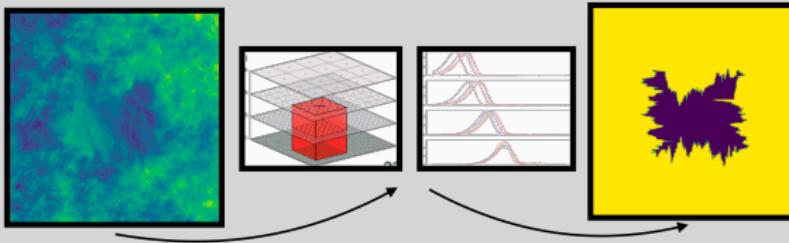
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Multiresolution/multilevel

Multiresolution
to perform
image restoration
(~2000–2015)



Multiresolution
to perform
texture
segmentation
(~2014- now)



Multiresolution
to accelerate
algorithms
(~2016- now)

