# Convex nonsmooth optimization
# Part II: Proximity operator

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# Collaboration

This course is a direct adaptation of the course built by
Jean-Christophe Pesquet (CentraleSupélec) and Nelly Pustelnik (LPENSL)

# Motivation

Let $\mathcal{H}$ be a real Hilbert space. Let $f \in \Gamma_0(\mathcal{H})$ have a Lipschitz gradient with Lipschitz constant $\beta > 0$.
Find

$$\widehat{x} \in \underset{x\in\mathcal{H}}{\mathrm{Argmin}}\, f(x).$$

- **Gradient descent algorithm**

  Set $\gamma \in \, ]0, +\infty[$ and $x_0 \in \mathcal{H}$.
  For $n = 0, 1 \ldots$
  $\lfloor\; x_{n+1} = x_n - \gamma \nabla f(x_n).$

The sequence $(x_n)_{n\in\mathbb{N}}$ generated by this *explicit* scheme converges to a minimizer of $f$ provided that such a minimizer exists and $\gamma \in ]0, 2/\beta[$.

# Motivation

Let $\mathcal{H}$ be a real Hilbert space. Let $f \in \Gamma_0(\mathcal{H})$ have a Lipschitz gradient with Lipschitz constant $\beta > 0$.
Find

$$\widehat{x} \in \underset{x \in \mathcal{H}}{\mathrm{Argmin}}\, f(x).$$

- **Alternative algorithm**

  Set $\gamma \in \,]0, +\infty[$ and $x_0 \in \mathcal{H}$.
  For $n = 0, 1 \ldots$
  $\lfloor\; x_{n+1} = x_n - \gamma \nabla f(x_{n+1}).$

Questions:

- How to determine $x_{n+1}$ at each iteration $n$ of this *implicit* scheme ?
- Which values of $\gamma$ guarantee the convergence of $(x_n)_{n \in \mathbb{N}}$ ?
- What to do if $f$ is nonsmooth ?

# Proximity operator: definition

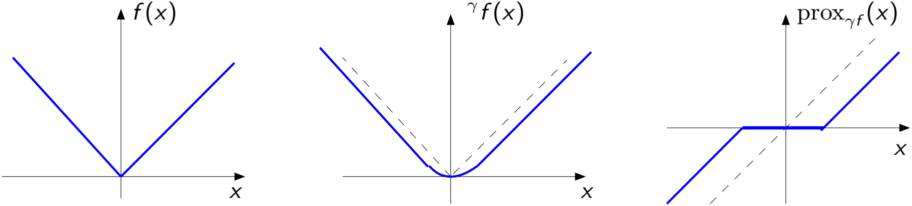
Let $\mathcal{H}$ be a Hilbert space. Let $f \in \Gamma_0(\mathcal{H})$.

- The **Moreau envelope** of $f$ of parameter $\gamma \in \,]0, +\infty[$ is

$$^{\gamma}f\colon \mathcal{H} \to \mathbb{R}\colon x \mapsto \inf_{y \in \mathcal{H}} f(y) + \frac{1}{2\gamma}\|y - x\|^2.$$

- The **proximity operator** of $f$ is

$$\operatorname{prox}_{\gamma f}\colon \mathcal{H} \to \mathcal{H}\colon x \mapsto \underset{y \in \mathcal{H}}{\operatorname{argmin}}\; f(y) + \frac{1}{2\gamma}\|y - x\|^2.$$



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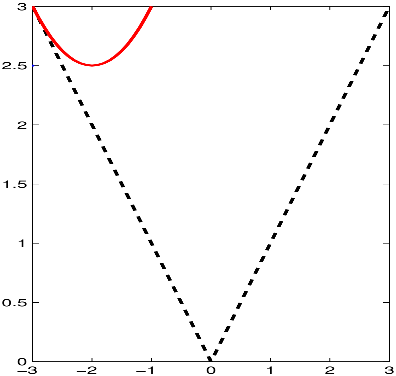
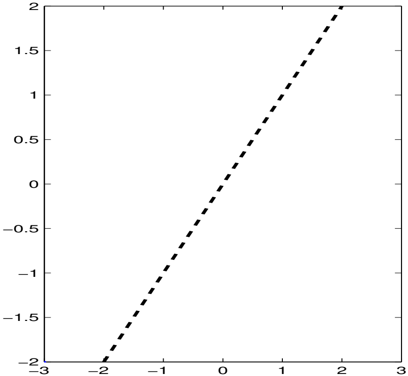



Moreau envelope

$^{\gamma}f(x) = \inf_{y\in\mathcal{H}} f(y) + \frac{1}{2\gamma}\|y-x\|^2$




Proximity operator

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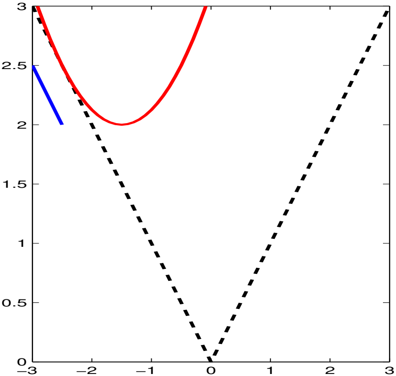


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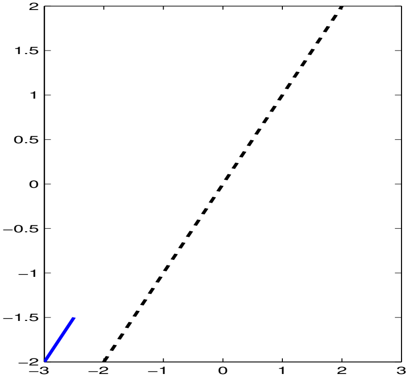


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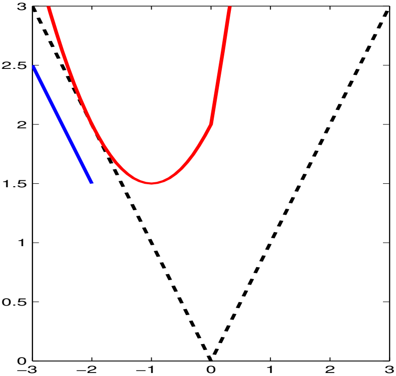


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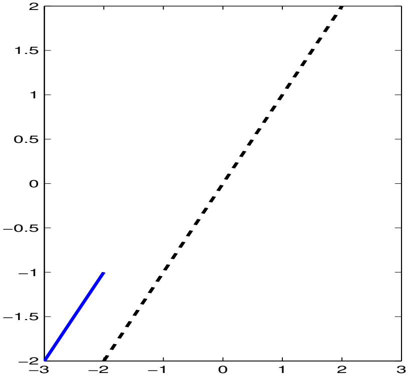


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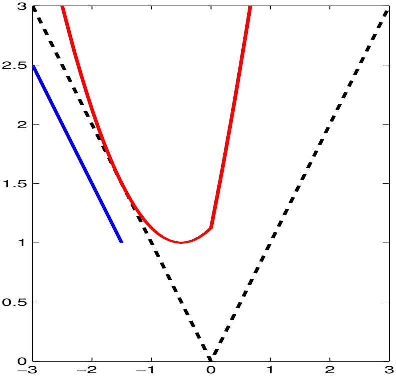


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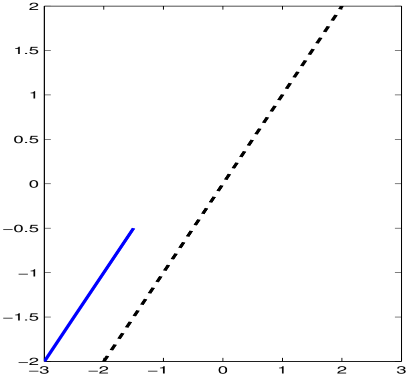


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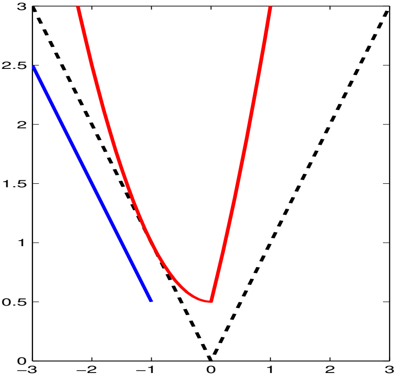


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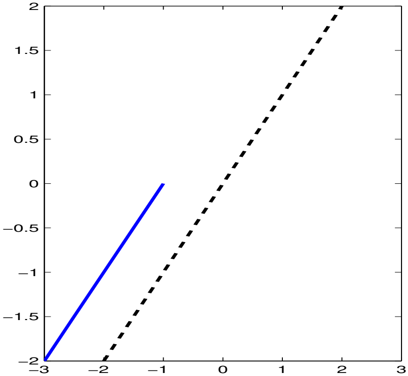


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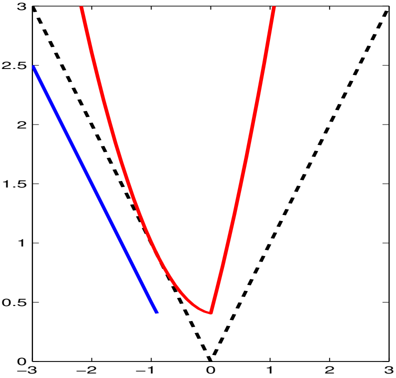


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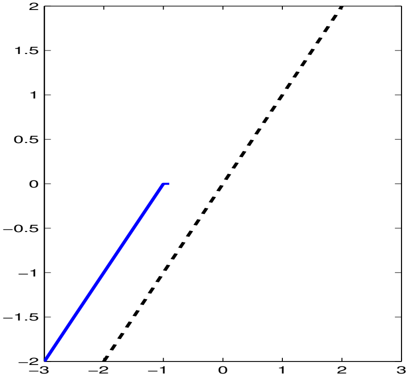


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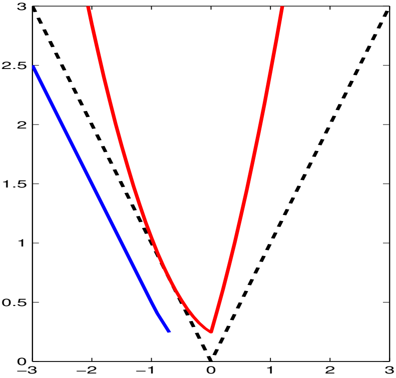


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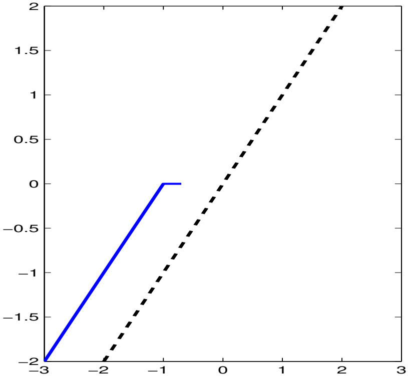


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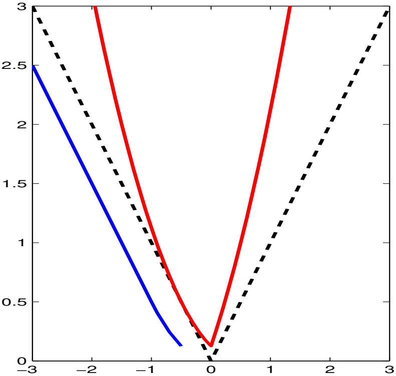


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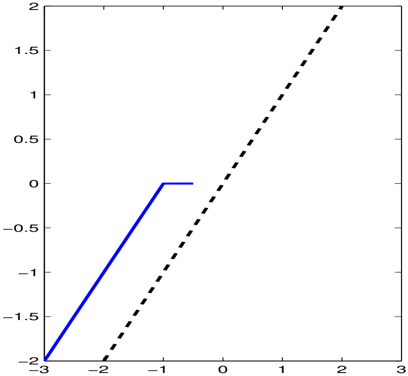


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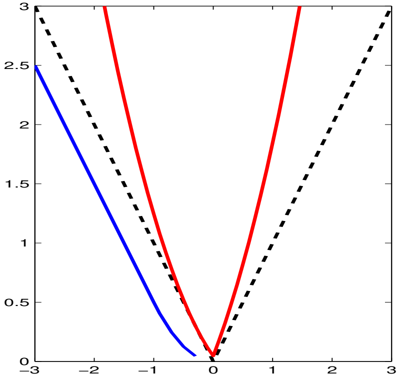


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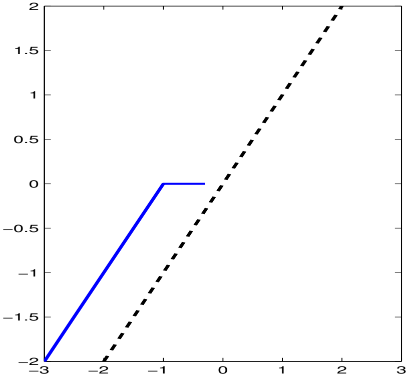


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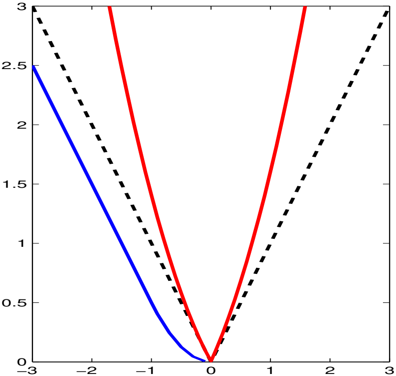


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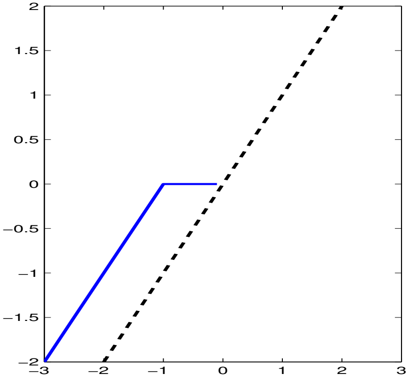


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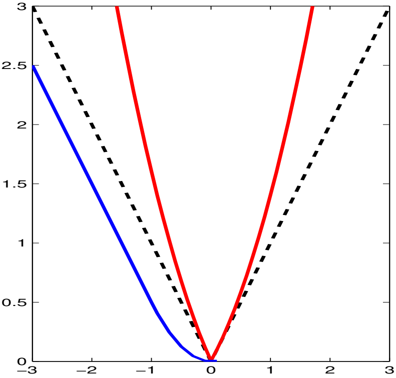


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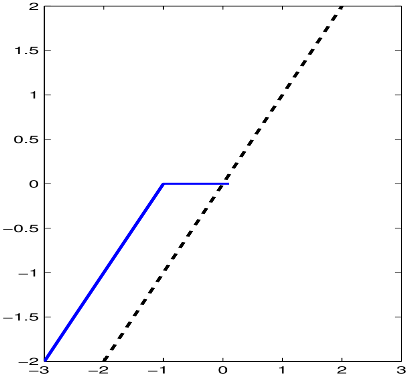


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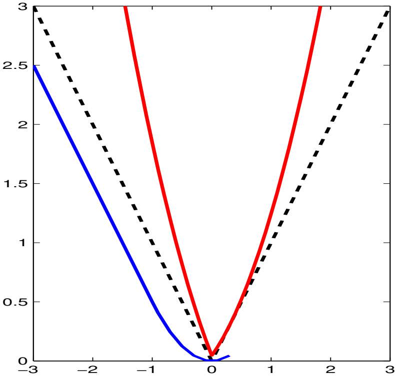


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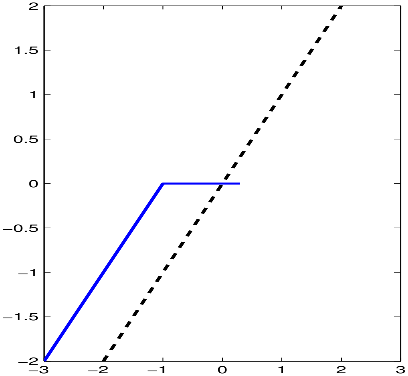


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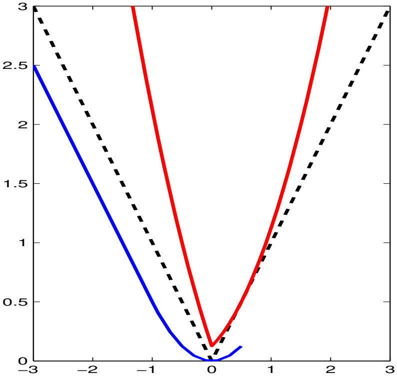


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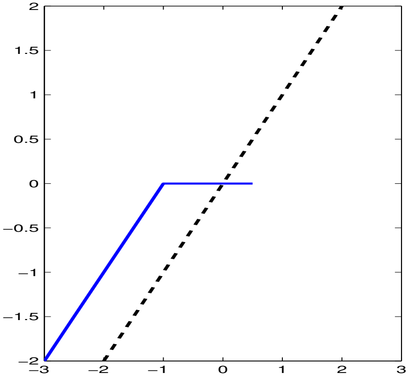


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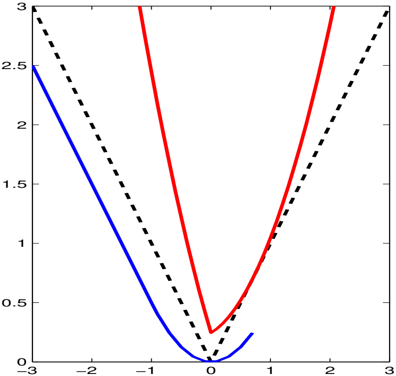


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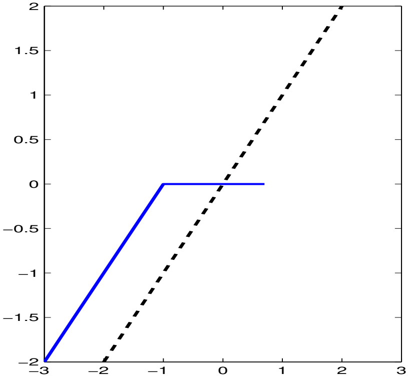


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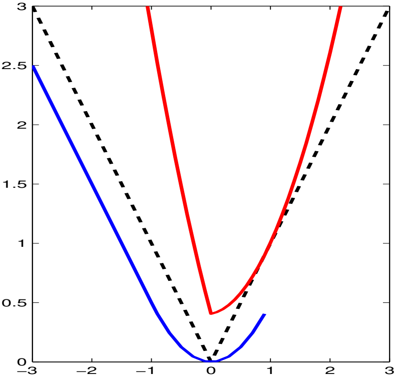


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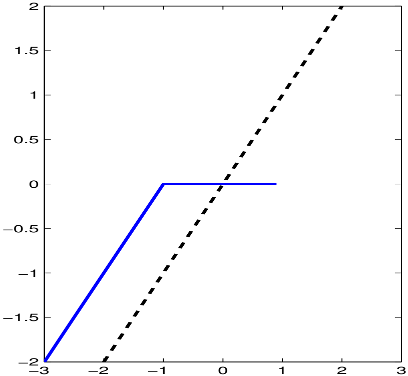


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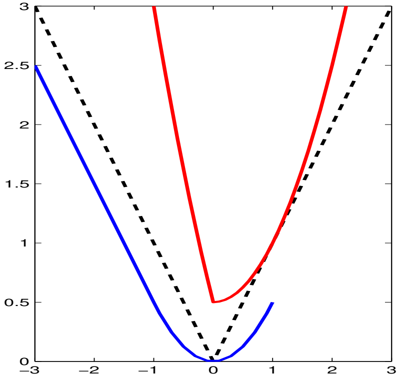


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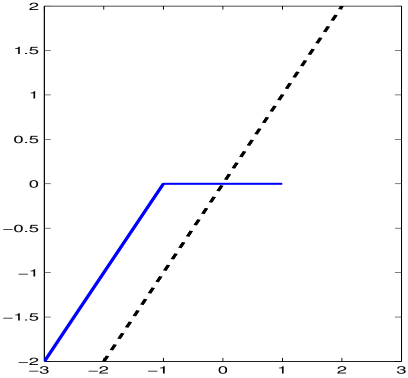


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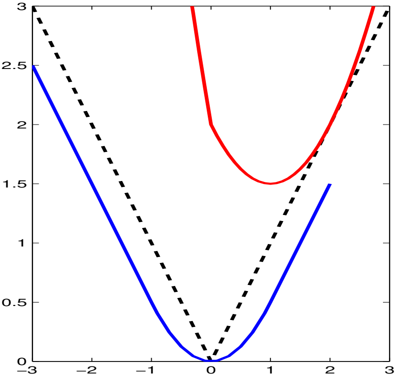


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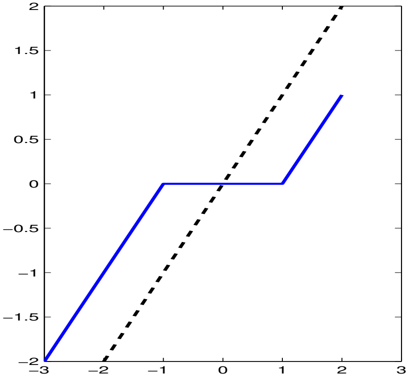


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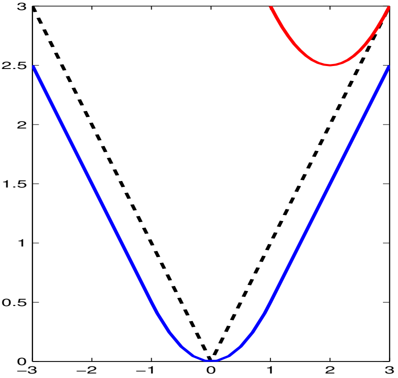


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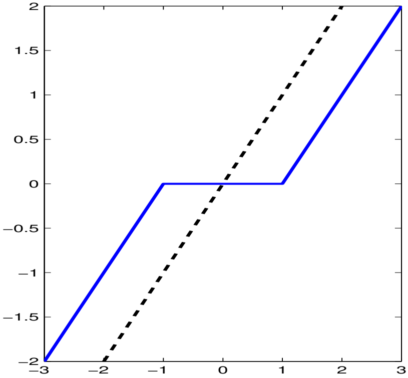


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# Proximity operator: characterization

Let $\mathcal{H}$ be a Hilbert space and $f \in \Gamma_0(\mathcal{H})$.

$$(\forall x \in \mathcal{H}) \quad \boxed{p = \mathrm{prox}_f(x) \quad \Leftrightarrow \quad x - p \in \partial f(p)}.$$

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Proof: By using Fermat's rule, for every $x \in \mathcal{H}$, $p = \mathrm{prox}_f(x)$ if and only if

$$\begin{aligned}
& p = \underset{y \in \mathcal{H}}{\arg\min}\; f(y) + \frac{1}{2}\|y - x\|^2 \\
\Leftrightarrow \quad & 0 \in \partial\Big(f + \frac{1}{2}\|\cdot - x\|^2\Big)(p) \\
\Leftrightarrow \quad & 0 \in \partial f(p) + p - x \\
\Leftrightarrow \quad & x \in (\mathrm{Id} + \partial f)(p).
\end{aligned}$$

# Proximity operator: examples

Let $\mathcal{H}$ be a Hilbert space and $C \subset \mathcal{H}$.
The indicator function of $C$ is

$$(\forall x \in \mathcal{H}) \qquad \iota_C(x) = \begin{cases} 0 & \text{if } x \in C \\ +\infty & \text{otherwise.} \end{cases}$$

Projection :
If $C$ be a nonempty closed convex subset of $\mathcal{H}$, then

$$(\forall x \in \mathcal{H}) \qquad \mathrm{prox}_{\iota_C}(x) = \underset{y \in C}{\operatorname{argmin}} \frac{1}{2}\|y - x\|^2 = P_C(x).$$

# Proximity operator: examples

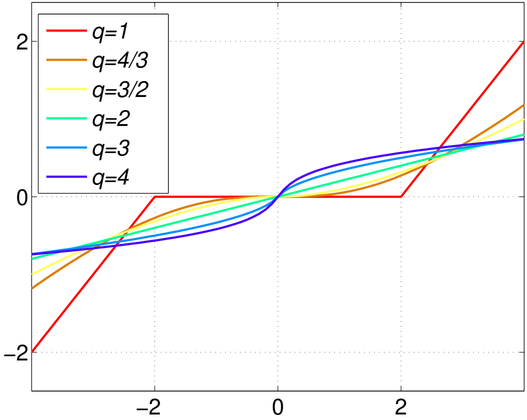
Power $q$ function with $q \geq 1$ :

Let $\chi > 0$, $q \in [1, +\infty[$ and $\varphi\colon \mathbb{R} \to ]-\infty, +\infty] : \eta \mapsto \chi|\xi|^q$.
Then, for every $\xi \in \mathbb{R}$,

$$\operatorname{prox}_\varphi \xi = \begin{cases} \operatorname{sign}(\xi) \max\{|\xi| - \chi, 0\} & \text{if } q = 1 \\ \xi + \frac{4\chi}{3\,.\,2^{1/3}}\left((\epsilon - \xi)^{1/3} - (\epsilon + \xi)^{1/3}\right) & \\ \qquad\qquad \text{where} \quad \epsilon = \sqrt{\xi^2 + 256\chi^3/729} & \text{if } q = \frac{4}{3} \\ \xi + \frac{9\chi^2 \operatorname{sign}(\xi)}{8}\left(1 - \sqrt{1 + \frac{16|\xi|}{9\chi^2}}\right) & \text{if } q = \frac{3}{2} \\ \frac{\xi}{1+2\chi} & \text{if } q = 2 \\ \operatorname{sign}(\xi) \frac{\sqrt{1+12\chi|\xi|}-1}{6\chi} & \text{if } q = 3 \\ \left(\frac{\epsilon+\xi}{8\chi}\right)^{1/3} - \left(\frac{\epsilon-\xi}{8\chi}\right)^{1/3} \quad \text{where} \quad \epsilon = \sqrt{\xi^2 + 1/(27\chi)} & \text{if } q = 4 \end{cases}$$

# Proximity operator: examples

Power $q$ function with $q \geq 1$ and $\chi = 2$.



# Proximity operator: examples

Quadratic function :

Let $\mathcal{H}$ and $\mathcal{G}$ be two Hilbert spaces.

Let $L \in \mathcal{B}(\mathcal{G}, \mathcal{H})$, $\gamma \in \, ]0, +\infty[$ and $z \in \mathcal{G}$.

$$f = \gamma \left\| L \cdot - z \right\|^2 / 2 \quad \Rightarrow \quad \mathrm{prox}_f = (\mathrm{Id} + \gamma L^* L)^{-1}(\cdot + \gamma L^* z).$$

# Proximity operator: properties

Let $\mathcal{H}$ be a Hilbert space, $x \in \mathcal{H}$ and $f \in \Gamma_0(\mathcal{H})$.

| Properties | $g(x)$ | $\mathrm{prox}_g x$ |
|---|---|---|
| Translation | $f(x-z), z \in \mathcal{H}$ | $z + \mathrm{prox}_f(x-z)$ |
| Quadratic perturbation | $f(x) + \alpha \parallel x \parallel^2 /2 + \langle z \mid x \rangle + \gamma$ <br> $z \in \mathcal{H}, \alpha > 0, \gamma \in \mathbb{R}$ | $\mathrm{prox}_{\frac{f}{\alpha+1}}(\frac{x-z}{\alpha+1})$ |
| Scaling | $f(\rho x), \rho \in \mathbb{R}^*$ | $\frac{1}{\rho}\mathrm{prox}_{\rho^2 f}(\rho x)$ |
| Reflexion | $f(-x)$ | $-\mathrm{prox}_f(-x)$ |
| Moreau enveloppe | ${}^{\gamma}f(x) = \inf_{y \in \mathcal{H}} f(y) + \frac{1}{2\gamma}\|x-y\|^2$ <br> $\gamma > 0$ | $\frac{1}{1+\gamma}\left(\gamma x + \mathrm{prox}_{(1+\gamma)f}(x)\right)$ |

# Proximity operator: properties

For every $i \in \{1, \ldots, n\}$, let $\mathcal{H}_i$ be a Hilbert space and let $f_i \in \Gamma_0(\mathcal{H}_i)$.
If

$$\big(\forall x = (x_1, \ldots, x_n) \in \mathcal{H}_1 \times \cdots \times \mathcal{H}_n\big) \quad f(x) = \sum_{i=1}^{n} f_i(x_i),$$

then

$$\big(\forall x = (x_1, \ldots, x_n) \in \mathcal{H}_1 \times \cdots \times \mathcal{H}_n\big) \quad \operatorname{prox}_f(x) = \big(\operatorname{prox}_{f_i}(x_i)\big)_{1 \le i \le n}.$$

## Proximity operator: properties

Let $\mathcal{H}$ be a separable Hilbert space.
Let $(b_i)_{i\in I}$ be an orthonormal basis of $\mathcal{H}$.
For every $i \in I$, let $\varphi_i \in \Gamma_0(\mathbb{R})$ such that $\varphi_i \geq 0$. For every $x \in \mathcal{H}$, if

$$f(x) = \sum_{i\in I} \varphi_i(\langle x \mid b_i \rangle)$$

then

$$\mathrm{prox}_f(x) = \textstyle\sum_{i\in I} \mathrm{prox}_{\varphi_i}(\langle x \mid b_i \rangle) b_i.$$

Remark: The assumption $(\forall i \in I)\ \varphi_i \geq 0$ can be relaxed if $\mathcal{H}$ is finite dimensional.

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Example: $\mathcal{H} = \mathbb{R}^N$, $(b_i)_{1\leq i\leq N}$ canonical basis of $\mathbb{R}^N$, $f = \lambda\|\cdot\|_1$ with $\lambda \in [0, +\infty[$.

$$(\forall x = (x^{(i)})_{1\leq i\leq N}) \in \mathbb{R}^N) \qquad \mathrm{prox}_{\lambda\|\cdot\|_1}(x) = \big(\mathrm{prox}_{\lambda|\cdot|}(x^{(i)})\big)_{1\leq i\leq N}$$

# Proximity operator: properties

Let $\mathcal{H}$ and $\mathcal{G}$ be two Hilbert spaces. Let $f \in \Gamma_0(\mathcal{H})$ and $L \in \mathcal{B}(\mathcal{G}, \mathcal{H})$ such that $LL^* = \mu \mathrm{Id}$ where $\mu \in \,]0, +\infty[$. Then

$$\mathrm{prox}_{f \circ L} = \mathrm{Id} - \mu^{-1} L^* \circ (\mathrm{Id} - \mathrm{prox}_{\mu f}) \circ L.$$

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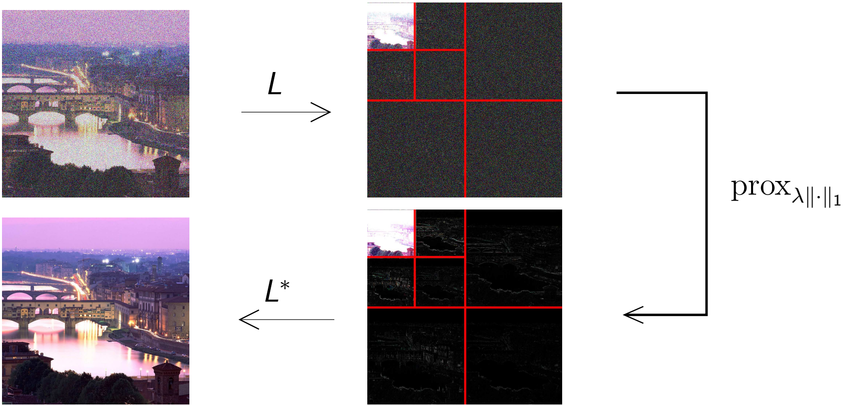
Proof: $LL^* = \mu \mathrm{Id} \Rightarrow \mathrm{ran}\, L = \mathcal{H}$ is closed, hence $V = \mathrm{ran}\,(L^*) = (\ker L)^\perp$ is closed. The orthogonal projection onto $V$ is $P_V = L^*(LL^*)^{-1}L = \mu^{-1}L^*L$.
For every $x \in \mathcal{H}$, $p = \mathrm{prox}_{f \circ L} x \Leftrightarrow x - p \in= \partial(f \circ L)(p) = L^* \partial f(Lp)$ (since $\mathrm{ran}\, L = \mathcal{H}$). Thus, $x - p \in V$.
It can be deduced that $P_{V^\perp} p = P_{V^\perp} x = x - P_V x = x - \mu^{-1} L^* L x$.
Furthermore,
$x - p \in L^* \partial(Lp) \Rightarrow Lx - Lp \in \mu \partial f(Lp) \Leftrightarrow Lp = \mathrm{prox}_{\mu f}(Lx)$.
We have thus $P_V p = \mu^{-1} L^* L p = \mu^{-1} L^* \mathrm{prox}_{\mu f}(Lx)$ and
$p = P_V p + P_{V^\perp} p = x - \mu^{-1} L^* (\mathrm{Id} - \mathrm{prox}_{\mu f})(Lx)$.

# Proximity operator: properties

Particular case : $L \in \mathcal{B}(\mathcal{H}, \mathcal{H})$ unitary, $\mathrm{prox}_{f \circ L} = L^* \mathrm{prox}_f L$.

- Illustration: denoising using an $\ell_1$ penalty on the coefficients resulting from an orthogonal wavelet transform $L$.



# Useful websites

- Exhaustive list of proximity operators, Matlab and Python codes:
  `http://proximity-operator.net/`
  authors: Chierchia, Chouzenoux, Combettes, Pesquet

- On Github: `https://github.com/cvxgrp/proximal`
  authors: Parikh, Chu, Boyd

- SPAMS: `http://spams-devel.gforge.inria.fr/`
  authors: Mairal, Bach, Ponce, Sapiro, Jenatton, Obozinski

- Fast implementation:
  `https://www.gipsa-lab.grenoble-inp.fr/~laurent.condat/software.html`
  author: Condat