





The Kravchuk transform:

A novel covariant representation for discrete signals amenable to zero-based detection tests

December, 4, 2024

Barbara Pascal, Rémi Bardenet

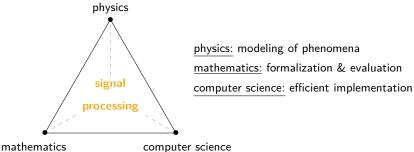
Sigma seminar CRIStAL, Lille

Signal processing aims to extract **information** from **data**.

Data of very diverse types:

- measurements of a physical quantity,
- biological or epidemiological indicators,
- data produced by human activities.

The Golden triangle of signal processing



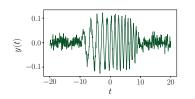
inspired from P. Flandrin

Outline of the presentation

- Signal detection: the role of representations
- Time-frequency analysis: the Short-Time Fourier Transform
- Signal detection based on the spectrogram zeros I
- Covariance principle and stationary point processes
- The Kravchuk transform and its zeros
- Numerical implementation of the Kravchuk transform
- Signal detection based on the spectrogram zeros II

Time and frequency: two dual descriptions of temporal signals

A continuous finite energy **signal** is a function of time y(t) with $y \in L^2_{\mathbb{C}}(\mathbb{R})$.



- electrical cardiac activity,
- audio recording,
- · seismic activity,
- light intensity on a photosensor
- ...

Information of interest:

- time events, e.g., an earthquake and its replica
- frequency content, e.g., monitoring of the heart beating rate

time

ever-changing world marker of events and evolutions

frequency

waves, oscillations, rhythms intrinsic mechanisms

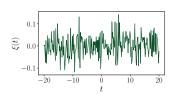
Signal-plus-noise observation model

A **chirp** is a transient waveform modulated in amplitude and frequency:

$$x(t) = A_{\nu}(t) \sin \left(2\pi \left(f_1 + (f_2 - f_1)\frac{t + \nu}{2\nu}\right)t\right)$$

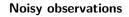
White noise is a random variable $\xi(t)$ such that

$$\mathbb{E}[\xi(t)] = 0$$
 and $\mathbb{E}[\overline{\xi(t)}\xi(t')] = \delta(t - t')$

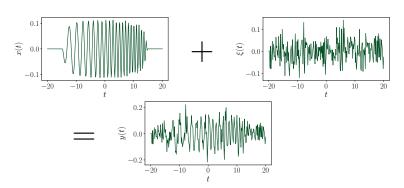


P. Flandrin: 'A signal is characterized by a structured organization.'

Signal-plus-noise observation model



$$y(t) = \operatorname{snr} \times x(t) + \xi(t)$$



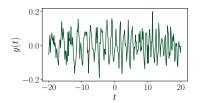
Signal processing task:

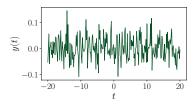
Given an observation y(t)

detection decides whether there is an underlying signal or only noise.

The role of representations in signal processing

Direct observation

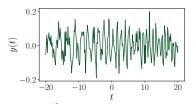


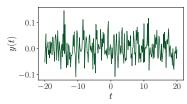


X hard to distinguish between oscillations and fluctuations

The role of representations in signal processing

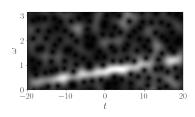
Direct observation

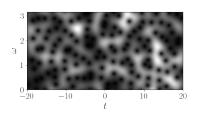




🗡 hard to distinguish between oscillations and fluctuations

Time-frequency representation





✓ lines of local maxima: structures are simpler to capture

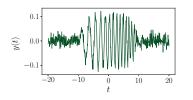
Outline of the presentation

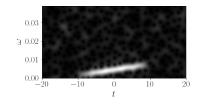
- Signal detection: the role of representations
- Time-frequency analysis: the Short-Time Fourier Transform
- Signal detection based on the spectrogram zeros I
- Covariance principle and stationary point processes
- The Kravchuk transform and its zeros
- Numerical implementation of the Kravchuk transform
- Signal detection based on the spectrogram zeros II

Time-frequency analysis

Time and frequency Short-Time Fourier Transform with window h:

$$V_h y(t,\omega) \triangleq \int_{-\infty}^{\infty} \overline{y(u)} h(u-t) \exp(-\mathrm{i}\omega u) \,\mathrm{d}u$$



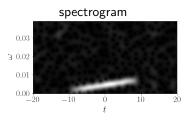


Energy density interpretation

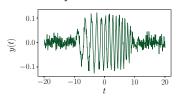
$$S_h y(t,\omega) = \left|V_h y(t,\omega)\right|^2$$
 the spectrogram

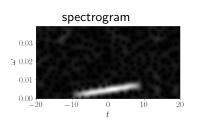
$$\int \int_{-\infty}^{+\infty} S_h y(t,\omega) dt \frac{d\omega}{2\pi} = \int_{-\infty}^{+\infty} |x(t)|^2 dt \quad \text{if} \quad ||h||_2^2 = 1$$

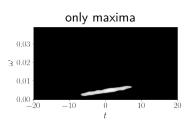
Signal, i.e., information of interest: regions of maximal energy.

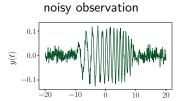


noisy observation

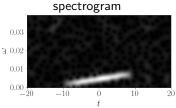


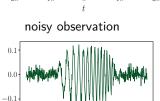






Inversion formula
$$y(t) = \int \int_{-\infty}^{+\infty} \overline{V_h y(u,\omega)} h(t-u) \exp(\mathrm{i}\omega u) \,\mathrm{d}u \frac{\mathrm{d}\omega}{2\pi}$$



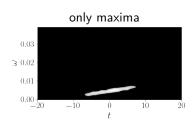


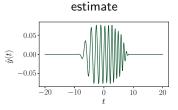
10

20

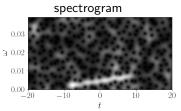
-20

-10

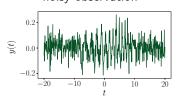


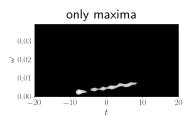


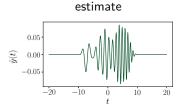
Inversion formula
$$y(t) = \int \int_{-\infty}^{+\infty} \overline{V_h y(u,\omega)} h(t-u) \exp(\mathrm{i}\omega u) \,\mathrm{d}u \frac{\mathrm{d}\omega}{2\pi}$$



noisy observation

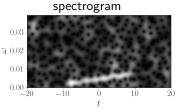




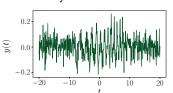


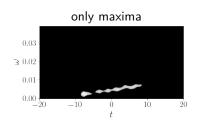
Denoising in the time-frequency plane: $y = \operatorname{snr} \times x + \xi$, $\operatorname{snr} = 0.5$

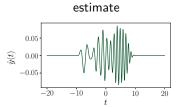
Inversion formula
$$y(t) = \int \int_{-\infty}^{+\infty} \overline{V_h y(u,\omega)} h(t-u) \exp(\mathrm{i}\omega u) \,\mathrm{d}u \frac{\mathrm{d}\omega}{2\pi}$$



noisy observation





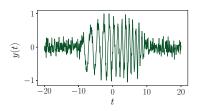


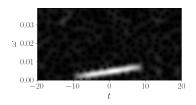
Maxima extraction: reassignment, synchrosqueezing, ridge extraction (Meignen et al., 2017)

Outline of the presentation

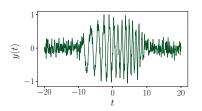
- Signal detection: the role of representations
- Time-frequency analysis: the Short-Time Fourier Transform
- Signal detection based on the spectrogram zeros I
- Covariance principle and stationary point processes
- The Kravchuk transform and its zeros
- Numerical implementation of the Kravchuk transform
- Signal detection based on the spectrogram zeros II

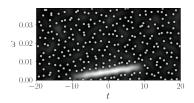
Look for the zeros, i.e., the points (t_i, ω_i) such that $|V_g y(t_i, \omega_i)|^2 = 0$.



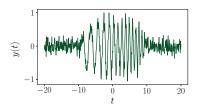


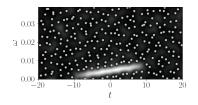
Look for the zeros, i.e., the points (t_i, ω_i) such that $|V_g y(t_i, \omega_i)|^2 = 0$.





Look for the **zeros**, i.e., the points (t_i, ω_i) such that $|V_g y(t_i, \omega_i)|^2 = 0$.

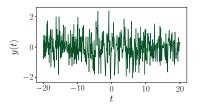


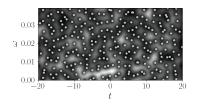


Observations: (Gardner & Magnasco, 2006), (Flandrin, 2015)

- Zeros are repelled by the signal.
- In the noise region zeros are evenly spread.
- There exists a short-range repulsion between zeros.

Look for the **zeros**, i.e., the points (t_i, ω_i) such that $|V_{\mathcal{E}}y(t_i, \omega_i)|^2 = 0$.

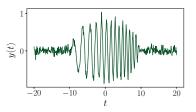


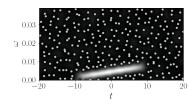


Observations: (Gardner & Magnasco, 2006), (Flandrin, 2015)

- Zeros are repelled by the signal.
- In the noise region zeros are evenly spread.
- There exists a short-range repulsion between zeros.

Look for the **zeros**, i.e., the points (t_i, ω_i) such that $|V_g y(t_i, \omega_i)|^2 = 0$.

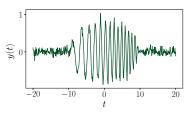


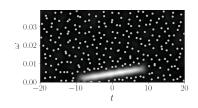


Observations: (Gardner & Magnasco, 2006), (Flandrin, 2015)

- Zeros are repelled by the signal.
- In the noise region zeros are evenly spread.
- There exists a short-range repulsion between zeros.

Look for the **zeros**, i.e., the points (t_i, ω_i) such that $|V_g y(t_i, \omega_i)|^2 = 0$.





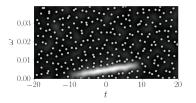
Observations: (Gardner & Magnasco, 2006), (Flandrin, 2015)

- Zeros are repelled by the signal.
- In the noise region zeros are evenly spread.
- There exists a short-range repulsion between zeros.

What can be said theoretically about the zeros of the spectrogram?

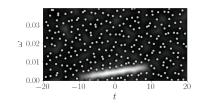
Unorthodox time-frequency analysis: spectrogram zeros

Idea assimilate the time-frequency plane with $\mathbb C$ through $z=(\omega+\mathrm{i} t)/\sqrt{2}$



Unorthodox time-frequency analysis: spectrogram zeros

Idea assimilate the time-frequency plane with $\mathbb C$ through $z=(\omega+\mathrm{i} t)/\sqrt{2}$



Bargmann factorization

$$V_g y(t,\omega) = \mathrm{e}^{-|z|^2/2} \mathrm{e}^{-\mathrm{i}\omega t/2} B y(z)$$

 $\ensuremath{\textit{g}}$ the circular Gaussian window

Bargmann transform of the signal y

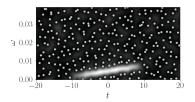
$$By(z) \triangleq \pi^{-1/4} e^{-z^2/2} \int_{\mathbb{R}} \overline{y(u)} \exp\left(\sqrt{2}uz - u^2/2\right) du,$$

By is an **entire** function, almost characterized by its infinitely many zeros:

$$By(z) = z^m e^{C_0 + C_1 z + C_2 z^2} \prod_{n \in \mathbb{N}} \left(1 - \frac{z}{z_n} \right) \exp\left(\frac{z}{z_n} + \frac{1}{2} \left(\frac{z}{z_n} \right)^2 \right).$$

Unorthodox time-frequency analysis: spectrogram zeros

Idea assimilate the time-frequency plane with $\mathbb C$ through $z=(\omega+\mathrm{i} t)/\sqrt{2}$



Bargmann factorization

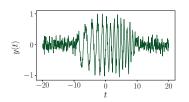
$$V_g y(t,\omega) = e^{-|z|^2/2} e^{-i\omega t/2} By(z)$$

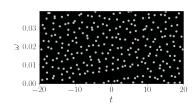
 $\it g$ the circular Gaussian window

Theorem The zeros of the Gaussian spectrogram $V_g y(t, \omega)$

- coincide with the zeros of the **entire** function By,
- hence are isolated and constitute a Point Process,
- which almost completely characterizes the spectrogram.

(Flandrin, 2015)





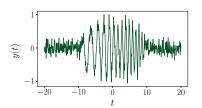
Advantages of working with the zeros

- Easy to find compared to relative maxima.
- Form a robust pattern in the time-frequency plane.
- Require little memory space for storage.
- Efficient tools were recently developed in **stochastic geometry**.

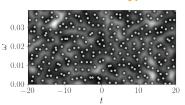
Signal detection based on the spectrogram zeros

(Bardenet, Flamant & Chainais, 2020)

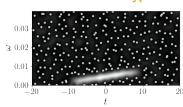
- \mathbf{H}_0 white noisy only, i.e., $y(t) = \xi(t)$
- \mathbf{H}_1 presence of a signal, i.e., $y(t) = \operatorname{snr} \times x(t) + \xi(t)$, $\operatorname{snr} > 0$



null hypothesis



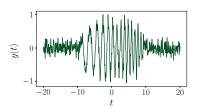
alternative hypothesis



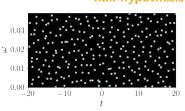
Signal detection based on the spectrogram zeros

(Bardenet, Flamant & Chainais, 2020)

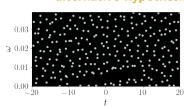
- \mathbf{H}_0 white noisy only, i.e., $y(t) = \xi(t)$
- \mathbf{H}_1 presence of a signal, i.e., $y(t) = \operatorname{snr} \times x(t) + \xi(t)$, $\operatorname{snr} > 0$



null hypothesis



alternative hypothesis

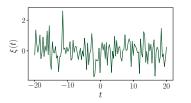


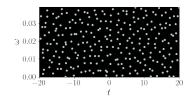
The zeros of the spectrogram of white noise

Continuous complex white Gaussian noise

(Bardenet et al., 2020), (Bardenet & Hardy, 2021)

$$\xi(t) = \sum_{n=0}^{\infty} \xi[n] h_n(t), \ \xi[n] \sim \mathcal{N}_{\mathbb{C}}(0,1), \quad \{h_n, k=0,1,\ldots\}$$
 Hermite functions



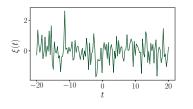


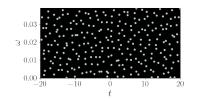
The zeros of the spectrogram of white noise

Continuous complex white Gaussian noise

(Bardenet et al., 2020), (Bardenet & Hardy, 2021)

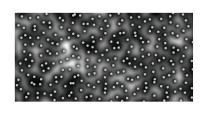
$$\xi(t) = \sum_{n=0}^{\infty} \xi[n] h_n(t), \; \xi[n] \sim \mathcal{N}_{\mathbb{C}}(0,1), \quad \{h_n, k=0,1,\ldots\} \; \text{Hermite functions}$$





Theorem
$$V_g\xi(t,\omega)=\mathrm{e}^{-|z|^2/4}\mathrm{e}^{-\mathrm{i}\omega t/2}\,\mathrm{GAF}_{\mathbb{C}}(z)$$
 (Bardenet & Hardy, 2021) $\mathrm{GAF}_{\mathbb{C}}(z)=\sum_{n=0}^{\infty}\xi[n]\frac{z^n}{\sqrt{n!}}$ the planar Gaussian Analytic Function and $z=\frac{\omega+\mathrm{i}t}{\sqrt{2}}$.

The zeros of the planar Gaussian Analytic Function



$$V_g \xi(t,\omega) \stackrel{ ext{non-vanishing}}{\propto} \mathsf{GAF}_{\mathbb{C}}(z)$$
 $z = (\omega + \mathrm{i} t)/\sqrt{2}$

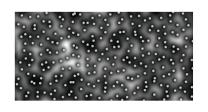
Zeros of $GAF_{\mathbb{C}}$: random set of points forming a **Point Process** characterized by a probability distribution on point configurations

Properties of the Point Process of the zeros of $\mathsf{GAF}_\mathbb{C}$:

- invariant under the isometries of \mathbb{C} , i.e., **stationary**,
- has a uniform density $\rho^{(1)}(z) = \rho^{(1)} = 1/\pi$,
- explicit two-point correlation function $\rho^{(2)}(z,z') = \rho^{(2)}(|z-z'|)$,
- scaling of the hole probability: $r^{-4} \log p_r \to -3e^2/4$, as $r \to \infty$

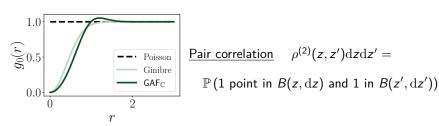
$$p_r = \mathbb{P}$$
 (no point in the disk of center 0 and radius r)

The zeros of the planar Gaussian Analytic Function

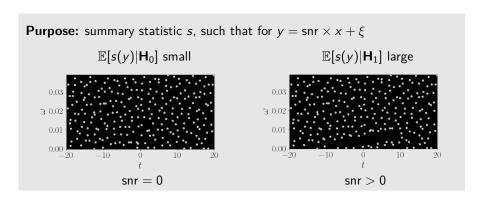


$$V_g \xi(t,\omega) \stackrel{ ext{non-vanishing}}{\propto} \mathsf{GAF}_\mathbb{C}(z)$$
 $z = (\omega + \mathrm{i} t)/\sqrt{2}$

Zeros of $GAF_{\mathbb{C}}$: random set of points forming a **Point Process** characterized by a probability distribution on point configurations



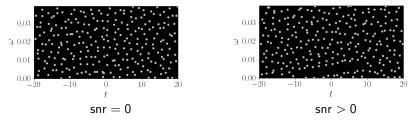
The point process of the zeros of the spectrogram is not **determinantal**.



'Large value of s(y) is a strong indication that there is a signal.'

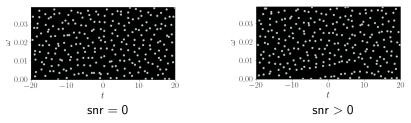
Tools from stochastic geometry to capture spatial statistics of the zeros.

Unorthodox path: zeros of Gaussian Analytic Functions



The signal creates **holes** in the zeros pattern: **sedond order** statistics.

Unorthodox path: zeros of Gaussian Analytic Functions



The signal creates **holes** in the zeros pattern: **sedond order** statistics.

A functional statistic: the empty space function

Z a stationary point process, z_0 any reference point

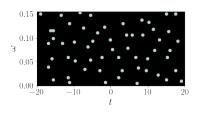
$$F(r) = \mathbb{P}\left(\inf_{z_i \in \mathcal{Z}} \mathrm{d}(z_0, z_i) < r\right)$$

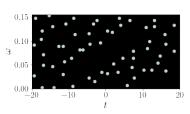
 \rightarrow probability to find a zero at distance less than r from z_0

Estimation of the F-function of a **stationary** Point Process

(Møller, 2007)

$$F(r) = \mathbb{P}\left(\inf_{z_i \in \mathcal{Z}} \mathrm{d}(z_0, z_i) < r
ight)$$
: empty space function

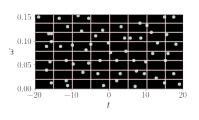


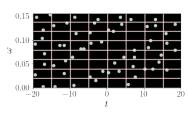


Estimation of the *F*-function of a **stationary** Point Process

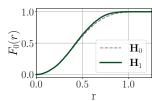
(Møller, 2007)

$$F(r) = \mathbb{P}\left(\inf_{z_i \in \mathcal{Z}} \mathrm{d}(z_0, z_i) < r \right)$$
: empty space function





$$\widehat{F}(r) = \frac{1}{N_{\#}} \sum_{j=1}^{N_{\#}} \mathbf{1} \left(\inf_{z \in \mathsf{Zeros}} \mathsf{d}(z_j, z) < r \right) \qquad \widehat{\mathbb{Q}}_{0.5}$$

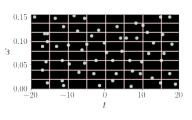


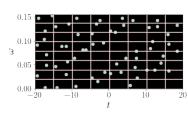
Signal detection based on the spectrogram zeros

Estimation of the *F*-function of a **stationary** Point Process

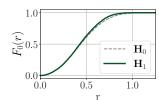
(Møller, 2007)

$$F(r) = \mathbb{P}\left(\inf_{z_i \in \mathcal{Z}} \mathrm{d}(z_0, z_i) < r\right)$$
: empty space function



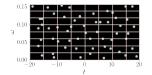


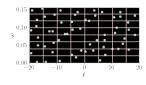
$$\widehat{F}(r) = \frac{1}{N_{\#}} \sum_{j=1}^{N_{\#}} \mathbf{1} \left(\inf_{z \in \mathsf{Zeros}} \mathsf{d}(z_j, z) < r \right) \qquad \widehat{\mathbb{Q}}_{0.5}^{\circ}$$

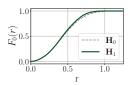


lacktriangle Monte Carlo envelope test based on the discrepancy between \widehat{F} and F_0

Monte Carlo envelope test







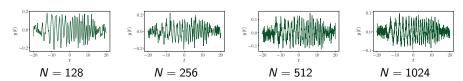
$$s(\mathbf{y}) = \sqrt{\int_0^{r_{\text{max}}} \left| \widehat{F}_{\mathbf{y}}(r) - F_0(r) \right|^2} dr,$$

Test settings: α level of significance, m number of samples under \mathbf{H}_0

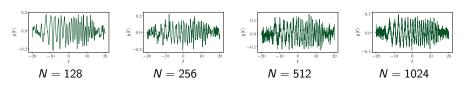
Index
$$k$$
, chosen so that $\alpha = k/(m+1)$

- (i) generate *m* independent samples of complex white Gaussian noise;
- (ii) compute their summary statistics $s_1 \geq s_2 \geq \ldots \geq s_m$;
- (iii) compute the summary statistic of the observation $m{y}$ under concern;
- (iv) if $s(y) \ge s_k$, then reject the null hypothesis with confidence 1α .

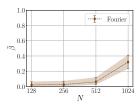
Detection of a noisy chirp of duration $2\nu=30~\mathrm{s}$



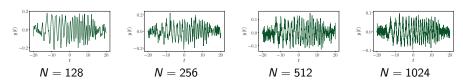
Detection of a noisy chirp of duration $2\nu=30~\mathrm{s}$



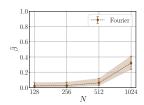
Performance: power of the test computed over 200 samples



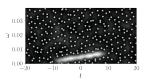
Detection of a noisy chirp of duration $2\nu = 30 \text{ s}$



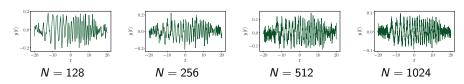
Performance: power of the test computed over 200 samples



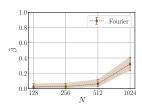
- ✓ Fast Fourier Transform;
- X Low detection power ;
- X Requires large number of samples



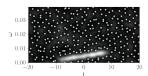
Detection of a noisy chirp of duration $2\nu=30~\mathrm{s}$



Performance: power of the test computed over 200 samples



- ✓ Fast Fourier Transform;
- X Low detection power ;
- X Requires large number of samples



Limitations:

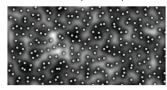
- Necessary discretization of the STFT: arbitrary resolution;
- Observe only a bounded window: edge corrections to compute $\widehat{F}(r)$.

Other Gaussian Analytic Functions, other transforms?

Short-Time Fourier Transform

$$V_g \xi(t,\omega) \propto \mathsf{GAF}_{\mathbb{C}}(z) = \sum_{n=0}^{\infty} \xi[n] \frac{z^n}{\sqrt{n!}}$$

Unbounded phase space $\mathbb C$



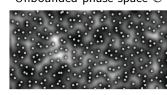
 $\rightarrow \, \mathsf{edge} \,\, \mathsf{corrections}$

Other Gaussian Analytic Functions, other transforms?

Short-Time Fourier Transform

$$V_g \xi(t,\omega) \propto \mathsf{GAF}_{\mathbb{C}}(z) = \sum_{n=0}^{\infty} \xi[n] \frac{z^n}{\sqrt{n!}}$$

Unbounded phase space \mathbb{C}



 \rightarrow edge corrections

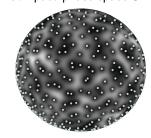
New transform?

?
$$\propto \mathsf{GAF}_{\mathbb{S}}(z) = \sum_{n=0}^{N} \xi[n] \sqrt{\binom{N}{n}} z^n$$

stereographic projection $z = \cot(\vartheta/2)e^{i\varphi}$

 \rightarrow spherical coordinates $(\vartheta, \varphi) \in S^2$

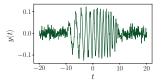
Compact phase space S^2 ?



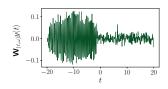
 \rightarrow no border!

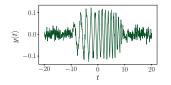
Outline of the presentation

- Signal detection: the role of representations
- Time-frequency analysis: the Short-Time Fourier Transform
- Signal detection based on the spectrogram zeros I
- Covariance principle and stationary point processes
- The Kravchuk transform and its zeros
- Numerical implementation of the Kravchuk transform
- Signal detection based on the spectrogram zeros II

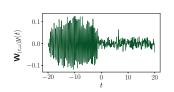


$$\mathbf{W}_{(t,\omega)}y(u) = e^{-\mathrm{i}\omega u}y(u-t)$$

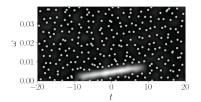


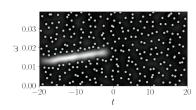


$$\mathbf{W}_{(t,\omega)}y(u)=e^{-\mathrm{i}\omega u}y(u-t)$$



$$V_h[\mathbf{W}_{(t,\omega)}y](t',\omega') \stackrel{(covariance)}{=} e^{-i(\omega'-\omega)t}V_hy(t'-t,\omega'-\omega),$$





$$\mathbf{W}_{(t,\omega)}y(u)=e^{-\mathrm{i}\omega u}y(u-t)$$

$$\left| V_h[\boldsymbol{W}_{\!(t,\omega)} \boldsymbol{y}](t',\omega') \right|^2 \overset{\text{(covariance)}}{=} \left| V_h \boldsymbol{y}(t'-t,\omega'-\omega) \right|^2,$$

Time and frequency shifts

$$\mathbf{W}_{(t,\omega)}y(u)=e^{-\mathrm{i}\omega u}y(u-t)$$

$$\left|V_h[\mathbf{W}_{(t,\omega)}y](t',\omega')\right|^2 \stackrel{\text{(covariance)}}{=} \left|V_hy(t'-t,\omega'-\omega)\right|^2$$
,

Complex white Gaussian noise

$$\widetilde{\xi} = extbf{W}_{(t,\omega)} \xi$$

•
$$\mathbb{E}[\widetilde{\xi}(u)] = e^{-i\omega u} \mathbb{E}[\xi(u-t)] = 0$$

Time and frequency shifts

$$\mathbf{W}_{(t,\omega)}y(u) = e^{-\mathrm{i}\omega u}y(u-t)$$

$$\left|V_h[\mathbf{W}_{(t,\omega)}y](t',\omega')\right|^2 \stackrel{\text{(covariance)}}{=} \left|V_hy(t'-t,\omega'-\omega)\right|^2,$$

Complex white Gaussian noise

$$\widetilde{\xi} = W_{(t,\omega)} \xi$$

•
$$\mathbb{E}[\xi(u)]$$
 = $e^{-i\omega u}\mathbb{E}[\xi(u-t)]$ = 0

•
$$\mathbb{E}[\overline{\widetilde{\xi}(u)}\widetilde{\xi}(u')] = e^{i\omega(u-u')}\mathbb{E}[\overline{\xi(u)}\xi(u')] = \delta(u-u')$$

Time and frequency shifts

$$\mathbf{W}_{(t,\omega)}y(u)=e^{-\mathrm{i}\omega u}y(u-t)$$

$$\left|V_h[\mathbf{W}_{(t,\omega)}y](t',\omega')\right|^2 \stackrel{\text{(covariance)}}{=} \left|V_hy(t'-t,\omega'-\omega)\right|^2$$

Complex white Gaussian noise

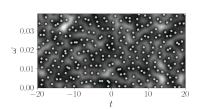
$$\widetilde{\xi} = W_{(t,\omega)} \xi$$

•
$$\mathbb{E}[\widetilde{\xi}(u)] = e^{-i\omega u} \mathbb{E}[\xi(u-t)] = 0$$

•
$$\mathbb{E}[\widetilde{\xi}(u)\widetilde{\xi}(u')] = e^{i\omega(u-u')}\mathbb{E}[\overline{\xi}(u)\xi(u')] = \delta(u-u')$$

Invariance under time-frequency shifts:

$$\widetilde{\xi} = extbf{ extit{W}}_{(t,\omega)} \xi \overset{(\mathsf{law})}{=} \xi$$



Time and frequency shifts

$$\mathbf{W}_{(t,\omega)}y(u)=e^{-\mathrm{i}\omega u}y(u-t)$$

$$\left|V_h[\boldsymbol{W}_{(t,\omega)}y](t',\omega')\right|^2 \stackrel{\text{(covariance)}}{=} \left|V_hy(t'-t,\omega'-\omega)\right|^2,$$

Complex white Gaussian noise

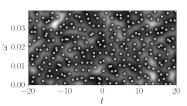
$$\widetilde{\xi} = W_{(t,\omega)} \xi$$

•
$$\mathbb{E}[\widetilde{\xi}(u)] = e^{-i\omega u} \mathbb{E}[\xi(u-t)] = 0$$

•
$$\mathbb{E}[\overline{\widetilde{\xi}(u)}\widetilde{\xi}(u')] = e^{i\omega(u-u')}\mathbb{E}[\overline{\xi(u)}\xi(u')] = \delta(u-u')$$

Invariance under time-frequency shifts:

$$\widetilde{\xi} = \mathbf{W}_{(t,\omega)} \xi \overset{(\mathsf{law})}{=} \xi$$



Covariance is the key to get stationarity: how to get covariant transforms?

$$\mathbf{W}_{(t,\omega)}y(u) = e^{-\mathrm{i}\omega u}y(u-t)$$

$$\left| \left. V_h [\textbf{\textit{W}}_{(t,\omega)} y](t',\omega') \right|^2 \overset{\text{(covariance)}}{=} \left| V_h y(t'-t,\omega'-\omega) \right|^2,$$

$$\mathbf{W}_{(t,\omega)}y(u)=e^{-\mathrm{i}\omega u}y(u-t)$$

$$\left| V_h[\textbf{\textit{W}}_{(t,\omega)} y](t',\omega') \right|^2 \overset{\text{(covariance)}}{=} \left| V_h y(t'-t,\omega'-\omega) \right|^2,$$

Weyl-Heisenberg group
$$\{e^{i\gamma} \textit{W}_{(t,\omega)},\, (\gamma,t,\omega)\in [0,2\pi]\times \mathbb{R}^2\}$$

$$\mathbf{W}_{(t',\omega')}\mathbf{W}_{(t,\omega)} = e^{\mathrm{i}\omega t'}\mathbf{W}_{(t+t',\omega+\omega')}.$$

Time and frequency shifts

$$\mathbf{W}_{(t,\omega)}y(u) = e^{-\mathrm{i}\omega u}y(u-t)$$

$$\left|V_h[\mathbf{W}_{(t,\omega)}y](t',\omega')\right|^2 \stackrel{\text{(covariance)}}{=} \left|V_hy(t'-t,\omega'-\omega)\right|^2$$
,

 $\text{Weyl-Heisenberg group} \quad \{ \mathrm{e}^{\mathrm{i} \gamma} \textit{W}_{(t,\omega)}, \, (\gamma,t,\omega) \in [0,2\pi] \times \mathbb{R}^2 \}$

$$\mathbf{W}_{(t',\omega')}\mathbf{W}_{(t,\omega)} = e^{\mathrm{i}\omega t'}\mathbf{W}_{(t+t',\omega+\omega')}.$$







$$g(t) = \pi^{-1/4} \exp\left(-t^2/2\right)$$
 $\mathbf{T}_u g(t) = g(t-u)$ $\mathbf{M}_\omega g(t) = g(t) \exp\left(-\mathrm{i}\omega t\right)$

Time and frequency shifts

$$\mathbf{W}_{(t,\omega)}y(u) = e^{-\mathrm{i}\omega u}y(u-t)$$

$$\left|V_h[\boldsymbol{W}_{(t,\omega)}y](t',\omega')\right|^2 \stackrel{\text{(covariance)}}{=} \left|V_hy(t'-t,\omega'-\omega)\right|^2$$
,

Weyl-Heisenberg group $\{e^{i\gamma}W_{(t,\omega)}, (\gamma, t, \omega) \in [0, 2\pi] \times \mathbb{R}^2\}$

$$\mathbf{W}_{(t',\omega')}\mathbf{W}_{(t,\omega)} = e^{\mathrm{i}\omega t'}\mathbf{W}_{(t+t',\omega+\omega')}.$$

Coherent state interpretation $\{ \mathbf{W}_{(t,\omega)} h, t, \omega \in \mathbb{R} \}$ covariant family

$$V_h y(t,\omega) = \int_{-\infty}^{\infty} \overline{y(u)} h(u-t) \exp(-\mathrm{i}\omega u) \, \mathrm{d}u = \langle y, \mathbf{W}_{(t,\omega)} h \rangle$$







$$g(t) = \pi^{-1/4} \exp\left(-t^2/2\right)$$
 $\mathbf{T}_u g(t) = g(t-u)$ $\mathbf{M}_{\omega} g(t) = g(t) \exp\left(-\mathrm{i}\omega t\right)$

$$\mathbf{M}_{\omega}g(t)=g(t)\exp\left(-\mathrm{i}\omega t
ight)$$

Outline of the presentation

- Signal detection: the role of representations
- Time-frequency analysis: the Short-Time Fourier Transform
- Signal detection based on the spectrogram zeros I
- Covariance principle and stationary point processes
- The Kravchuk transform and its zeros
- Numerical implementation of the Kravchuk transform
- Signal detection based on the spectrogram zeros II

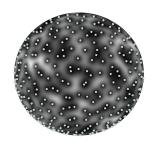


Coherent state interpretation

$$extbf{ extit{y}} \in \mathbb{C}^{ extit{ extit{N}}+1}$$

$$T\mathbf{y}(\vartheta, \varphi) = \langle \mathbf{y}, \Psi_{(\vartheta, \varphi)} \rangle$$

$$\vartheta \in [0,\pi], \varphi \in [0,2\pi]$$

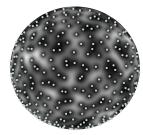


Coherent state interpretation

$$extbf{ extit{y}} \in \mathbb{C}^{ extit{N}+1}$$

$$T \mathbf{y}(\vartheta, arphi) = \langle \mathbf{y}, \mathbf{\Psi}_{(\vartheta, arphi)}
angle$$

$$\vartheta \in [0,\pi], \varphi \in [0,2\pi]$$



SO(3) coherent states (Gazeau, 2009)

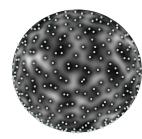
$$\boldsymbol{\Psi}_{\vartheta,\varphi} = \sum_{n=0}^{N} \sqrt{\binom{N}{n}} \left(\cos\frac{\vartheta}{2}\right)^{n} \left(\sin\frac{\vartheta}{2}\right)^{N-n} \mathrm{e}^{\mathrm{i}n\varphi} \boldsymbol{q}_{n} = \boldsymbol{R}_{\boldsymbol{u}(\vartheta,\varphi)} \boldsymbol{\Psi}_{(0,0)},$$

Coherent state interpretation

$$\mathbf{y} \in \mathbb{C}^{N+1}$$

$$\mathcal{T} extbf{ extit{y}}(artheta,arphi) = \langle extbf{ extit{y}}, extbf{ extit{\Psi}}_{(artheta,arphi)}
angle$$

$$\vartheta \in [0,\pi], \varphi \in [0,2\pi]$$



SO(3) coherent states (Gazeau, 2009)

$$\Psi_{\vartheta,\varphi} = \sum_{n=0}^{N} \sqrt{\binom{N}{n}} \left(\cos \frac{\vartheta}{2}\right)^{n} \left(\sin \frac{\vartheta}{2}\right)^{N-n} e^{in\varphi} \boldsymbol{q}_{n} = \boldsymbol{R}_{\boldsymbol{u}(\vartheta,\varphi)} \Psi_{(0,0)},$$

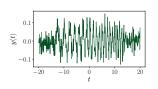
$$\{\boldsymbol{q}_n, n=0,1,...,N\}$$
 the Kravchuk functions

$$T \mathbf{y}(z) = rac{1}{\sqrt{(1+|z|^2)^N}} \sum_{n=0}^N \langle \mathbf{y}, \mathbf{q}_n \rangle \sqrt{\binom{N}{n}} z^n, \quad z = \cot(\vartheta/2) \mathrm{e}^{\mathrm{i}\varphi}$$

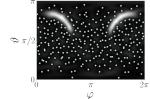
Kravchuk transform

 $\{\boldsymbol{q}_{n}, n=0,1,...,N\}$ the Kravchuk functions

$$T \mathbf{y}(z) = \frac{1}{\sqrt{(1+|z|^2)^N}} \sum_{n=0}^N \langle \mathbf{y}, \mathbf{q}_n \rangle \sqrt{\binom{N}{n}} z^n, \quad z = \cot(\vartheta/2) \mathrm{e}^{\mathrm{i}\varphi}$$



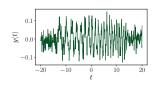




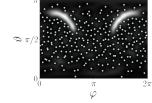
Kravchuk transform

$$\{\boldsymbol{q}_n, n=0,1,...,N\}$$
 the Kravchuk functions

$$T \mathbf{y}(z) = rac{1}{\sqrt{(1+|z|^2)^N}} \sum_{n=0}^N \langle \mathbf{y}, \mathbf{q}_n \rangle \sqrt{\binom{N}{n}} z^n, \quad z = \cot(\vartheta/2) \mathrm{e}^{\mathrm{i}\varphi}$$







Theorem
$$T\xi(\vartheta,\varphi) = \sqrt{(1+|z|^2)}^{-N} \operatorname{GAF}_{\mathbb{S}}(z), \qquad z = \cot(\vartheta/2)\mathrm{e}^{\mathrm{i}\varphi}$$
 $\operatorname{GAF}_{\mathbb{S}}(z) = \sum_{n=0}^{N} \xi[n] \sqrt{\binom{N}{n}} z^n$ the spherical Gaussian Analytic Function (Pascal & Bardenet, 2022)

Outline of the presentation

- Signal detection: the role of representations
- Time-frequency analysis: the Short-Time Fourier Transform
- Signal detection based on the spectrogram zeros I
- Covariance principle and stationary point processes
- The Kravchuk transform and its zeros
- Numerical implementation of the Kravchuk transform
- Signal detection based on the spectrogram zeros II

Practical computation of the Kravchuk transform

Kravchuk transform

$$\{\boldsymbol{q}_n, n=0,1,...,N\}$$
 the Kravchuk basis

$$Toldsymbol{y}(z) = rac{1}{\sqrt{(1+|z|^2)^N}} \sum_{n=0}^N \langle oldsymbol{y}, oldsymbol{q}_n
angle \sqrt{inom{N}{n}} z^n, \quad z = \cot(artheta/2) \mathrm{e}^{\mathrm{i}arphi}$$

$$\rightarrow$$
 first: basis change, i.e., computation of $\langle \boldsymbol{y}, \boldsymbol{q}_n \rangle = \sum_{\ell=0}^{N} \overline{\boldsymbol{y}[\ell]} q_n(\ell; N)$

Practical computation of the Kravchuk transform

Kravchuk transform

$$\{\boldsymbol{q}_n, n=0,1,...,N\}$$
 the Kravchuk basis

$$T oldsymbol{y}(z) = rac{1}{\sqrt{(1+|z|^2)^N}} \sum_{n=0}^N \langle oldsymbol{y}, oldsymbol{q}_n
angle \sqrt{inom{N}{n}} z^n, \quad z = \cot(artheta/2) \mathrm{e}^{\mathrm{i}arphi}$$

$$\rightarrow$$
 first: basis change, i.e., computation of $\langle \boldsymbol{y}, \boldsymbol{q}_n \rangle = \sum_{\ell=0}^N \overline{\boldsymbol{y}[\ell]} q_n(\ell; N)$

Evaluation of Kravchuk functions
$$q_n(\ell; N) = \frac{1}{\sqrt{2^N}} \sqrt{\binom{N}{n}} Q_n(\ell; N) \sqrt{\binom{N}{\ell}}$$

$$(N-n)Q_{n+1}(t;N) = (N-2t)Q_n(t;N) - nQ_{n-1}(t;N),$$

 $\{\mathit{Q}_{\mathit{n}}(\mathit{t};\mathit{N}),\mathit{n}=0,1,\ldots,\mathit{N}\}$ orthogonal family of Kravchuk polynomials

$$\sum_{\ell=0}^{N} \binom{N}{\ell} Q_n(\ell; N) Q_{n'}(\ell; N) = 2^{N} \binom{N}{n}^{-1} \delta_{n,n'}$$

Evaluation of Kravchuk functions

(i) recursion to compute the Kravchuk polynomials

$$(N-n)Q_{n+1}(t;N) = (N-2t)Q_n(t;N) - nQ_{n-1}(t;N),$$

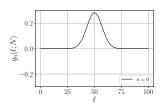
$$q_n(\ell; N) = \frac{1}{\sqrt{2^N}} \sqrt{\binom{N}{n}} Q_n(\ell; N) \sqrt{\binom{N}{\ell}}$$

Evaluation of Kravchuk functions

(i) recursion to compute the Kravchuk polynomials

$$(N-n)Q_{n+1}(t;N) = (N-2t)Q_n(t;N) - nQ_{n-1}(t;N),$$

$$q_n(\ell; N) = \frac{1}{\sqrt{2^N}} \sqrt{\binom{N}{n}} Q_n(\ell; N) \sqrt{\binom{N}{\ell}}$$

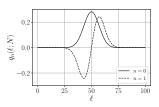


Evaluation of Kravchuk functions

(i) recursion to compute the Kravchuk polynomials

$$(N-n)Q_{n+1}(t;N) = (N-2t)Q_n(t;N) - nQ_{n-1}(t;N),$$

$$q_n(\ell; N) = \frac{1}{\sqrt{2^N}} \sqrt{\binom{N}{n}} Q_n(\ell; N) \sqrt{\binom{N}{\ell}}$$

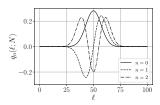


Evaluation of Kravchuk functions

(i) recursion to compute the Kravchuk polynomials

$$(N-n)Q_{n+1}(t;N) = (N-2t)Q_n(t;N) - nQ_{n-1}(t;N),$$

$$q_n(\ell; N) = \frac{1}{\sqrt{2^N}} \sqrt{\binom{N}{n}} Q_n(\ell; N) \sqrt{\binom{N}{\ell}}$$

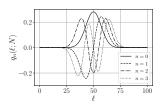


Evaluation of Kravchuk functions

(i) recursion to compute the Kravchuk polynomials

$$(N-n)Q_{n+1}(t;N) = (N-2t)Q_n(t;N) - nQ_{n-1}(t;N),$$

$$q_n(\ell; N) = \frac{1}{\sqrt{2^N}} \sqrt{\binom{N}{n}} Q_n(\ell; N) \sqrt{\binom{N}{\ell}}$$



Instability of the computation of Kravchuk polynomials

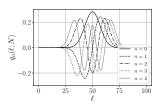
Evaluation of Kravchuk functions

(i) recursion to compute the Kravchuk polynomials

$$(N-n)Q_{n+1}(t;N) = (N-2t)Q_n(t;N) - nQ_{n-1}(t;N),$$

(ii) multiplication by the binomial coefficients

$$q_n(\ell; N) = \frac{1}{\sqrt{2^N}} \sqrt{\binom{N}{n}} Q_n(\ell; N) \sqrt{\binom{N}{\ell}}$$



Instability of the computation of Kravchuk polynomials

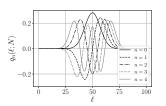
Evaluation of Kravchuk functions

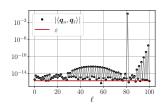
(i) recursion to compute the Kravchuk polynomials

$$(N-n)Q_{n+1}(t;N) = (N-2t)Q_n(t;N) - nQ_{n-1}(t;N),$$

(ii) multiplication by the binomial coefficients

$$q_n(\ell; N) = \frac{1}{\sqrt{2^N}} \sqrt{\binom{N}{n}} Q_n(\ell; N) \sqrt{\binom{N}{\ell}}$$





Instability of the computation of Kravchuk polynomials

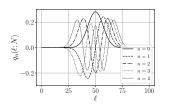
Evaluation of Kravchuk functions

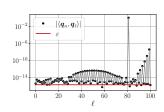
(i) recursion to compute the Kravchuk polynomials

$$(N-n)Q_{n+1}(t;N) = (N-2t)Q_n(t;N) - nQ_{n-1}(t;N),$$

(ii) multiplication by the binomial coefficients

$$q_n(\ell; N) = \frac{1}{\sqrt{2^N}} \sqrt{\binom{N}{n}} Q_n(\ell; N) \sqrt{\binom{N}{\ell}}$$





 \rightarrow estimated basis is **not orthogonal**! Not possible to compute $\langle \mathbf{y}, \mathbf{q}_n \rangle$.

Rewritting of the Kravchuk transform

Kravchuk transform

$$\{\boldsymbol{q}_n, n = 0, 1, ..., N\}$$
 the Kravchuk basis

$$T y(z) = \frac{1}{\sqrt{(1+|z|^2)^N}} \sum_{n=0}^{N} \left(\sum_{\ell=0}^{N} \overline{y[\ell]} q_n(\ell; N) \right) \sqrt{\binom{N}{n}} z^n \rightarrow \text{intractable}$$

A generative function for Kravchuk polynomials

$$\sum_{n=0}^{N} {N \choose n} Q_n(\ell; N) z^n = (1-z)^{\ell} (1+z)^{N-\ell}$$

$$\implies \sum_{n=0}^{N} \sqrt{{N \choose n}} q_n(\ell; N) z^n = \sqrt{{N \choose \ell}} \frac{(1-z)^{\ell} (1+z)^{N-\ell}}{\sqrt{2^N}}$$

$$T\mathbf{y}(z) = \frac{1}{\sqrt{(1+|z^2|)^N}} \sum_{\ell=0}^{N} \sqrt{\binom{N}{\ell}} \overline{\mathbf{y}[\ell]} \frac{(1-z)^{\ell} (1+z)^{N-\ell}}{\sqrt{2^N}}$$

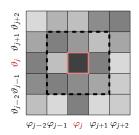
X no more Fast Fourier Transform algorithm using $z^n = \cot(\vartheta/2)^n e^{in\varphi}$



Advantage compared to Fourier: can tune the resolution of phase space.



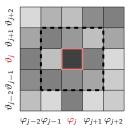
Advantage compared to Fourier: can tune the resolution of phase space.



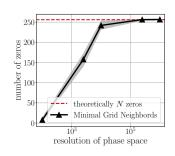
Minimal Grid Neighbors



Advantage compared to Fourier: can tune the resolution of phase space.



Minimal Grid Neighbors

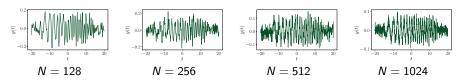


Proposition: all local minima of $|Ty(z)|^2$ are zeros.

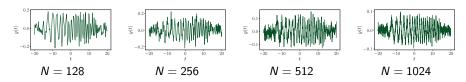
Outline of the presentation

- Signal detection: the role of representations
- Time-frequency analysis: the Short-Time Fourier Transform
- Signal detection based on the spectrogram zeros I
- Covariance principle and stationary point processes
- The Kravchuk transform and its zeros
- Numerical implementation of the Kravchuk transform
- Signal detection based on the spectrogram zeros II

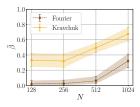
Detection of a noisy chirp of duration $2\nu=30~\mathrm{s}$



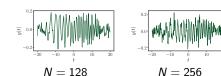
Detection of a noisy chirp of duration $2\nu=30~\mathrm{s}$

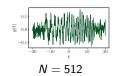


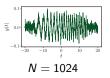
Performance: power of the test computed over 200 samples



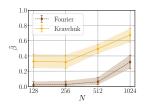
Detection of a noisy chirp of duration $2\nu=30~\mathrm{s}$





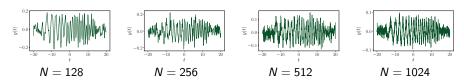


Performance: power of the test computed over 200 samples

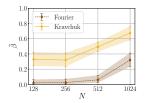


- √ higher detection power
- ✓ more robust to small N
- no fast algorithm yet

Detection of a noisy chirp of duration $2\nu=30~\text{s}$



Performance: power of the test computed over 200 samples



- ✓ higher detection power
- ✓ more robust to small N
- no fast algorithm yet

Advantages of using Kravchuk vs. Fourier spectrogram

- intrinsically encoded resolution: no need for prior knowledge
- compact phase space: no edge correction

Point Processes in time-frequency analysis

Take home messages

- Novel covariant discrete Kravchuk transform $T\mathbf{y}(\vartheta,\varphi)$
 - * Interpreted as a coherent state decomposition,
 - * Representation on a compact phase space,
 - * Zeros of the Kravchuk spectrogram of white noise fully characterized.
- Signal detection based on spectrogram zeros
 - * Preliminary work using the zeros of the Fourier spectrogram,
 - * Significant improvement using the Kravchuk spectrogram.

Pascal & Bardenet, 2022: arxiv:2202.03835

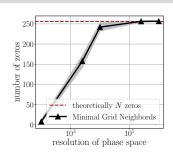
 ${\tt GitHub: bpascal-fr/kravchuk-transform-and-its-zeros}$

Work in progress and perspectives

- Interpretation of the action of SO(3) on \mathbb{C}^{N+1} ;
- Implementation of the inversion formula: denoising based on zeros ;
- Design of a Kravchuk FFT counterpart;
- \bullet Convergence of Kravchuk toward the Fourier spectrogram as $\textit{N} \rightarrow \infty.$



Opening: can the Kravchuk spectrogram have multiple zeros?



Spherical Gaussian Analytic Function

$$\mathsf{GAF}_{\mathbb{S}}(z) = \sum_{n=0}^{N} \xi[n] \sqrt{\binom{N}{n}} z^{n}$$

with $\boldsymbol{\xi}[n] \sim \mathcal{N}_{\mathbb{C}}(0,1)$ i.i.d.

 \rightarrow only simple zeros

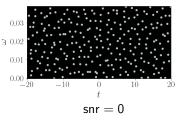
General case
$$Ty(z) = \sqrt{(1+|z|^2)}^{-N} \sum_{n=0}^{N} \sqrt{\binom{N}{n}} (\mathbf{Q}y) [n] z^n$$

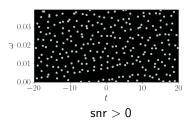
If \mathbf{y} deterministic, such that $(\mathbf{Q}\mathbf{y})[n] = \sqrt{\binom{N}{n}} a^{N-n} b^n, a \in \mathbb{C}, b \in \mathbb{C}^*,$

$$\sqrt{(1+|z|^2)}^{-N}\sum_{n=0}^{N}\sqrt{\binom{N}{n}}\left(\mathbf{Q}\mathbf{y}\right)\left[n\right]z^n=\left(a+bz\right)^N$$

 $\rightarrow -a/b$ multiple root of order of degeneracy N

Unorthodox path: zeros of Gaussian Analytic Functions





The signal creates holes in the zeros pattern: sedond order statistics.

Functional statistics:

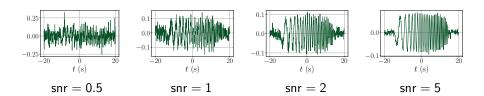
• the empty space function

$$F(r) = \mathbb{P}\left(\inf_{z_i \in Z} \mathrm{d}(z_0, z_i) < r\right)$$
: probability to find a zero at less than r

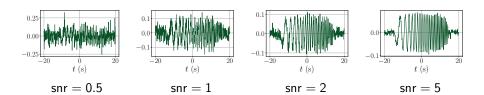
• Ripley's K-function

$$K(r) = 2\pi \int_0^r sg_0(s) ds$$
: expected # of pairs at distance less than r

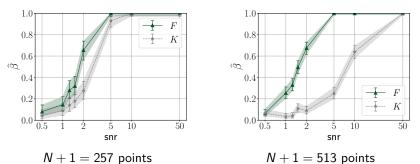
Detection test: choice of the functional statistic



Detection test: choice of the functional statistic



Ripley's K functional vs. empty space functional F



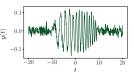
Detection test: snr and relative duration of the signal

Fixed observation window of 40 s

long time event $\stackrel{0.1}{\underset{-0.1}{\rightleftharpoons}} \stackrel{0.1}{\underset{-0.1}{\longleftarrow}} \stackrel{0.1}{\underset{-20}{\longleftarrow}} \stackrel{0.1}{\underset{-10}{\longleftarrow}} \stackrel{0.1}{\underset{0}{\longleftarrow}} \stackrel$

duration $2\nu=30~\mathrm{s}$

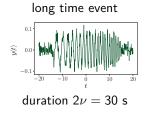
short time event



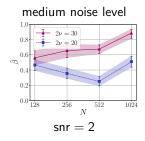
duration $2\nu=20~\mathrm{s}$

Detection test: snr and relative duration of the signal

Fixed observation window of 40 s



Robustness to small number of samples and short duration.



Detection test: snr and relative duration of the signal

Fixed observation window of 40 s

long time event $\stackrel{0.1}{=} \underbrace{\stackrel{0.0}{=} \stackrel{0.0}{=} \stackrel{0.0$

short time event

duration $2\nu = 20 \text{ s}$

Robustness to small number of samples and short duration.

