

# Multiscale analysis in image processing

## Preliminaries on wavelets

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Barbara Pascal<sup>†</sup> and Nelly Pustelnik<sup>‡</sup>

[bpascal-fr.github.io/talks](http://bpascal-fr.github.io/talks)

June 2025

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INSTITUT DE FRANCE

# Multiple scales in real data



**Albert Marquet**, *Paysage, baie méditerranéenne, vue d'Agay*, 1905

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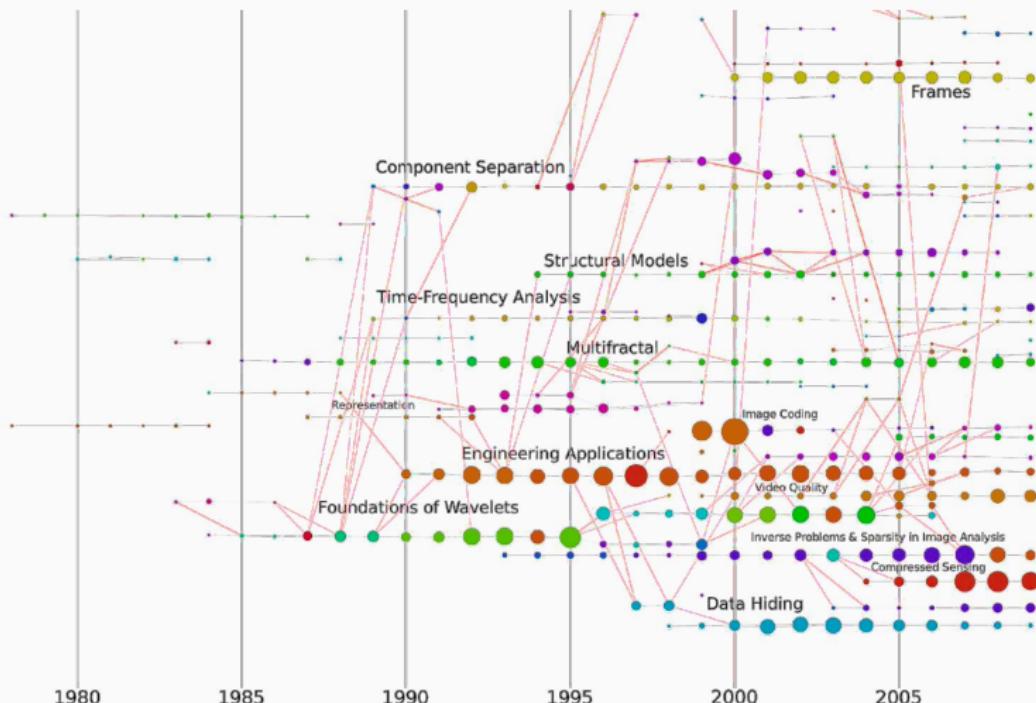


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# Overview of the history of wavelets

## Revealing evolutions in dynamical networks

Matteo Morini, Patrick Flandrin, Eric Fleury, Tommaso Venturini, Pablo Jensen<sup>1</sup>  
IXXI, ENS de Lyon, INRIA, CNRS, LIP UMR 5668, LP UMR 5672



# Wavelet Transforms

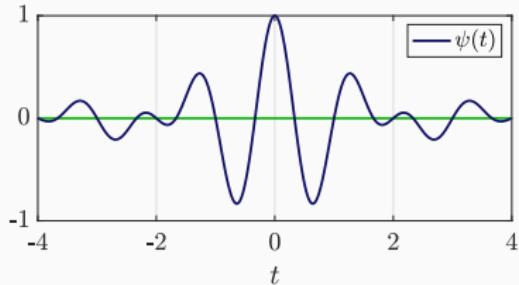
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# Continuous Wavelet Transform

**Wavelet:**  $\psi \in L^2(\mathbb{R})$  locally oscillating, integrable with  $\int_{\mathbb{R}} \psi(s) \, ds = 0$

**Example:** real Shannon wavelet

$$\psi(t) = \frac{\sin(2\pi t) - \sin(\pi t)}{\pi t}$$

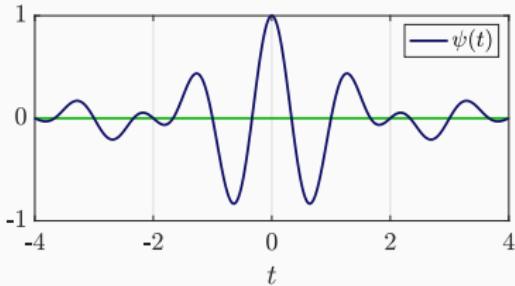


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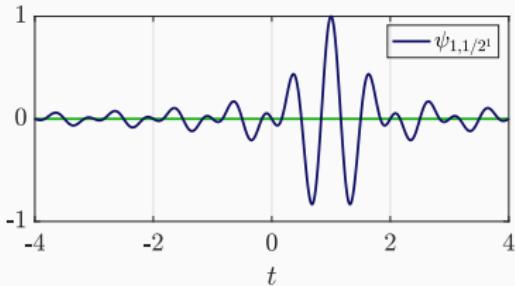
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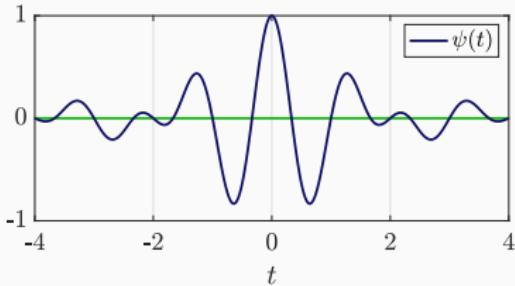
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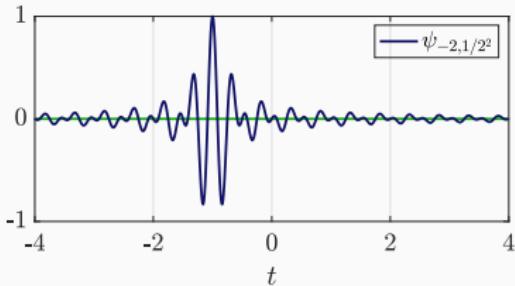
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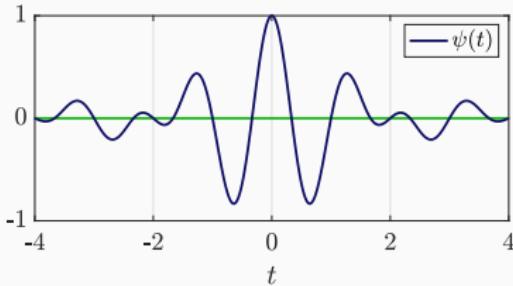
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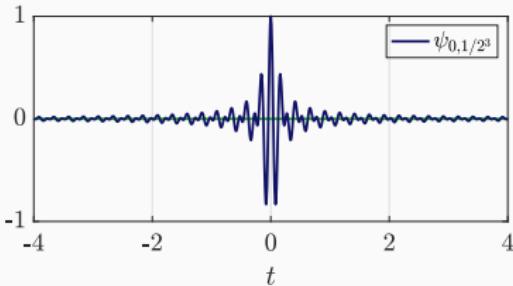
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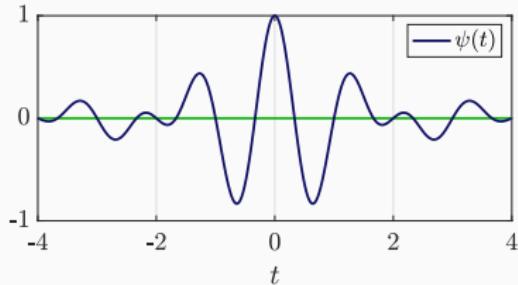
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**Continuous Wavelet Transform** of a finite-energy signal  $f \in L^2(\mathbb{R})$

$$\mathcal{W}_f(t, a) = \langle f, \psi_{t,a} \rangle = \int_{\mathbb{R}} f(s) \frac{1}{\sqrt{a}} \overline{\psi\left(\frac{s-t}{a}\right)} \, ds$$

$\langle \cdot, \cdot \rangle$  scalar product in  $L^2(\mathbb{R})$ ,  $\bar{\cdot}$  complex conjugate

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## Example of a gravitational wave

**Physics:** two objects  $M = m_1 + m_2$  at distance  $R$ ,  $\mu^{-1} = m_1^{-1} + m_2^{-1}$

$$f(t) = A(t_0 - t)^{-1/4} \cos(d(t_0 - t)^{5/8} + \varphi) \mathbf{1}_{(-\infty; t_0]}(t)$$

**chirp:** amplitude  $a(t) = A(t_0 - t)^{-\frac{1}{4}}$ , frequency  $\omega(t) = \frac{10\pi d}{8}(t_0 - t)^{-\frac{3}{8}}$

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–  $d$ : instantaneous frequency parameter

$$d \simeq 241 \mathcal{M}_\odot^{-5/8},$$

–  $A$ : amplitude reference

$$A \simeq 3.37 \times 10^{-21} \mathcal{M}_\odot^{5/4} / R.$$

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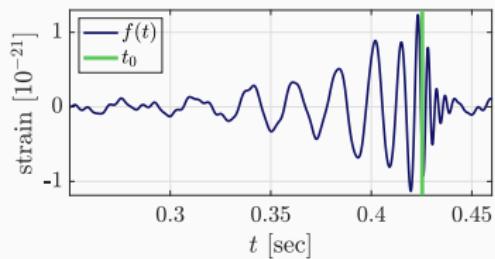
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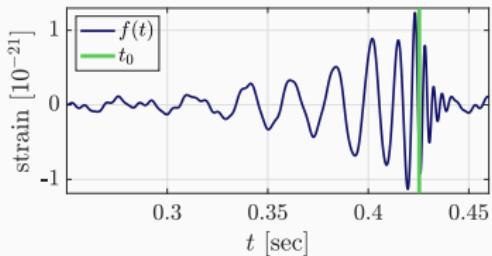
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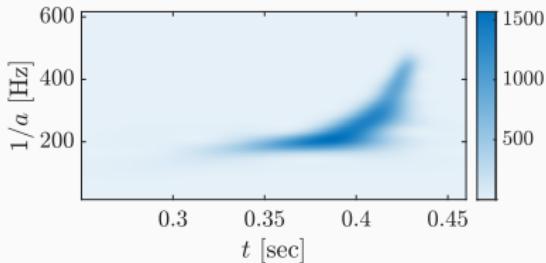
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Morlet scalogram  $|\mathcal{W}_f(t, a)|^2$   
[tftb.nongnu.org](http://tftb.nongnu.org)

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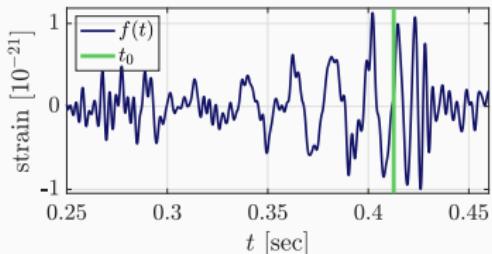
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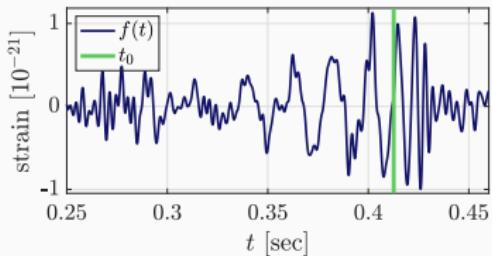
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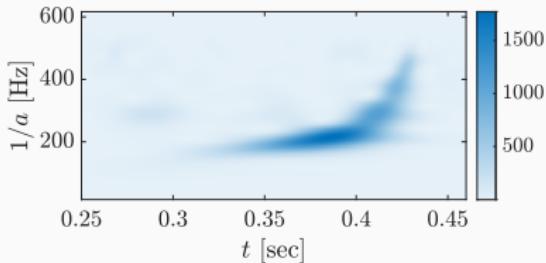
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# Continuous Wavelet Transform

**Reconstruction formula** For  $\tilde{\psi}$  the Fourier transform of  $\psi$ ,

if  $C_\psi = \int_{\mathbb{R}} \frac{|\tilde{\psi}(\omega)|^2}{|\omega|} d\omega < \infty$  then  $\psi$  is **admissible** and for  $f \in L^2(\mathbb{R})$

$$f(t) = \frac{1}{C_\psi} \int_{\mathbb{R} \times \mathbb{R}_+} \mathcal{W}_f(s, a) \frac{1}{\sqrt{a}} \psi\left(\frac{s-t}{a}\right) ds \frac{da}{a^2}$$

with  $\mathcal{W}_f(s, a) = \langle f, \psi_{s,a} \rangle$

[A. P. Calderón, 1964, *Stud. Math.*;

A. Grossmann & J. Morlet, 1984, *SIAM J. Math. Anal.*]

**Reproducing kernel**  $\mathcal{W}_f(t, a)$  **redundant** representation of  $f$

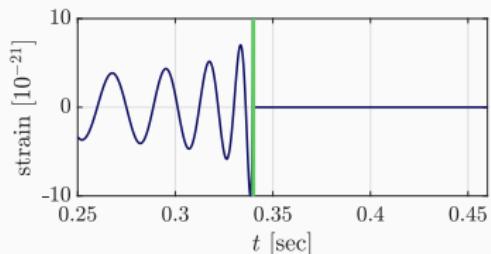
$$\mathcal{W}_f(t', a') = \frac{1}{C_\psi} \int_{\mathbb{R} \times \mathbb{R}_+} \mathcal{K}(t', t; a', a) \mathcal{W}_f(t, a) dt \frac{da}{a^2}$$

with  $\mathcal{K}(t', t; a', a) = \langle \psi_{t,a}, \psi_{t',a'} \rangle$  correlations between wavelets

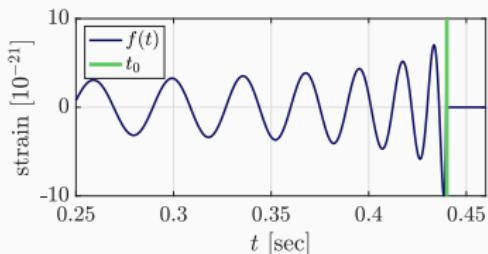
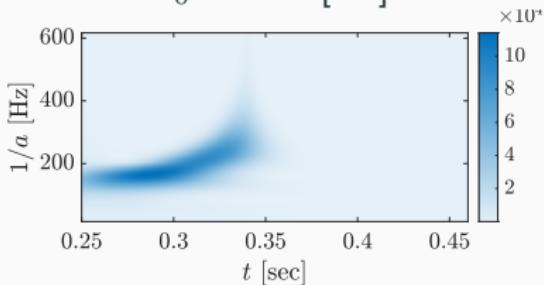
# Continuous Wavelet Transform

## Translation invariance

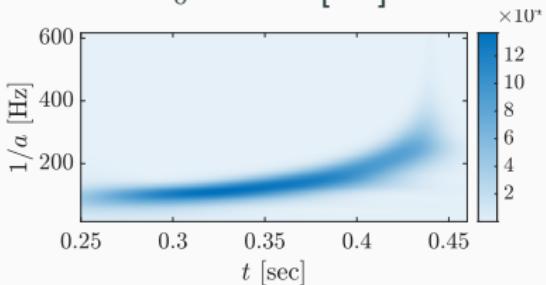
Let  $\tilde{f}(t) = f(t - \Delta t)$ , then  $\mathcal{W}_{\tilde{f}}(t, a) = \mathcal{W}_f(t - \Delta t, a)$ .



$$t_0 = 0.44 \text{ [sec]}$$



$$t_0 = 0.34 \text{ [sec]}$$



# Discrete Signals and Wavelets

## From continuous signals to discrete vectors

- $f$  continuous on  $[0, 1]$ , discretized in  $z_n = f\left(\frac{n}{N}\right)$ ,  $n = 0, 1, \dots, N$   
**discrete wavelet transform** can be computed at scales  $N^{-1} < a^j < 1$
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## Practical implementation

Integral representation

$$\mathcal{W}_f(t, a) = \int_{\mathbb{R}} f(s) \frac{1}{\sqrt{a}} \overline{\psi\left(\frac{s-t}{a}\right)} ds$$

Discrete convolution  $z * \overline{\psi_{t,a}(-\cdot)}$

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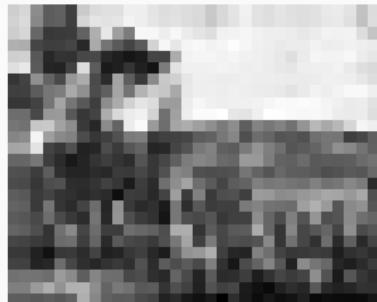
**complexity**  $\mathcal{O}(vN (\log_2 N)^2)$

## Multiresolution Analysis

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# Aims and principles of multiresolution analysis

**Motivation:** process only details at relevant **discrete** resolutions



[P. J. Burt & E. H. Adelson, 1983, *Proc. IEEE Int. Conf. Commun.*]

## Multiresolution analysis

**Definition:** A multiresolution analysis of  $L^2(\mathbb{R})$  is a subspaces sequence  
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- **regularity**:  $\varphi$  **father wavelet** or **scaling function** such that

$\{\varphi(t - k), \quad k \in \mathbb{Z}\}$  is an orthonormal basis of  $V_0$

- **completeness**:  $\cup_{j \in \mathbb{Z}} V_j$  is dense in  $L^2(\mathbb{R})$  and  $\cap_{j \in \mathbb{Z}} V_j = \{0\}$

$\implies V_j$  **approximation space** at *scale*  $2^j$ , i.e., *resolution*  $2^{-j}$ .

[P. J. Burt & E. H. Adelson, 1983, *Proc. IEEE Int. Conf. Commun.*;

S. Mallat, 1989, *Trans. Amer. Math. Soc.*;

S. Mallat, 1989, *IEEE Trans. Pattern Anal. Mach. Intell.*;

Y. Meyer, 1992, *Cambridge University Press*]

# Aims and principles of multiresolution analysis

**Motivation:** process only details at relevant **discrete** resolutions



scale 4  $\iff$  resolution 1/4



scale 2  $\iff$  resolution 1/2



scale 1  $\iff$  resolution 1

$$\text{scale } 2^j \iff \text{resolution } 2^{-j}$$

[P. J. Burt & E. H. Adelson, 1983, *Proc. IEEE Int. Conf. Commun.*]

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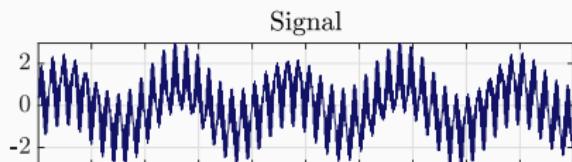
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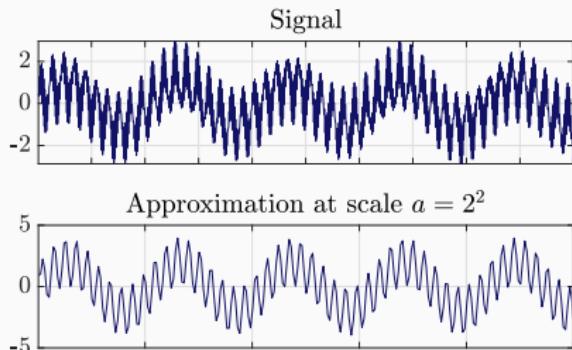
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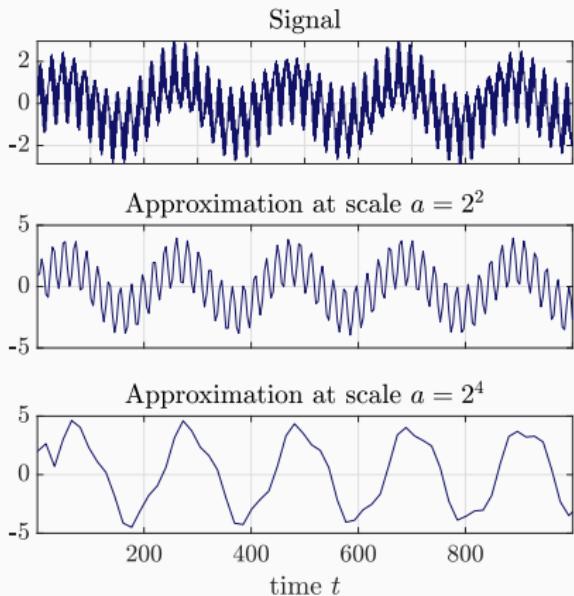
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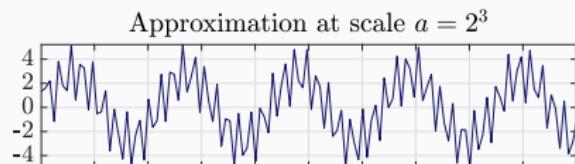
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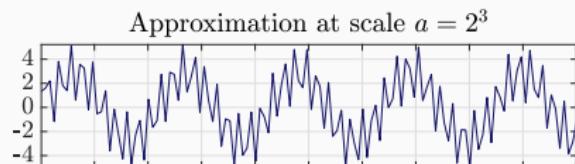
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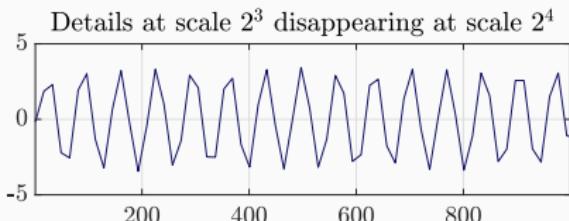
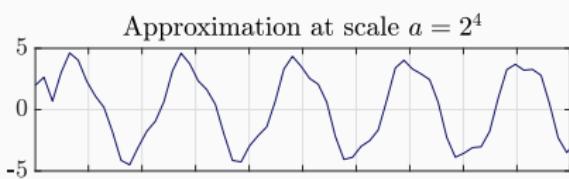
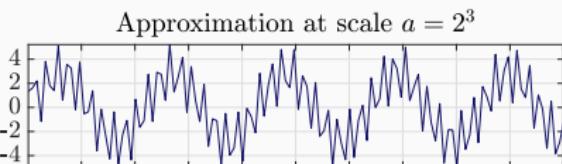
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## Orthonormal wavelet basis

By the **completeness** property of the multiresolution analysis

$$\{\sqrt{2^{-j}}\psi(t/2^j - k), \quad (j, k) \in \mathbb{Z}^2\}$$

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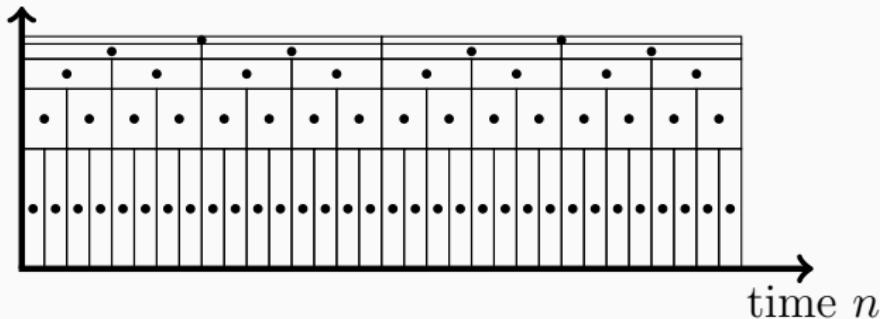
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## Tiling of the time-scale half-plane

scale  $2^j$



# Vanishing Moments of Wavelets

**Theorem:** Let  $\varphi$  scaling function,  $\psi$  mother wavelet and  $\tilde{\psi}$  its Fourier transform s.t.

$$|\varphi(t)| \underset{|t| \rightarrow \infty}{=} \mathcal{O}\left((1+t^2)^{-n_\psi/2-1}\right), \quad |\psi(t)| \underset{|t| \rightarrow \infty}{=} \mathcal{O}\left((1+t^2)^{-n_\psi/2-1}\right)$$

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## Interpretation:

- mother wavelet **orthogonal** to polynomials of degree at most  $n_\psi - 1$
- if signal  $f$  is  $C^k$ ,  $k < n_\psi$

wavelet coefficients  $\zeta_{j,n} = \langle f, \psi_{j,n} \rangle$  **small at fine scales**

## Support size and number vanishing moments trade-off

If  $f$  has an **isolated singularity** at  $t_0$  contained in the support of  $\psi_{j,n}$  then  
the wavelet coefficient  $\langle f, \psi_{j,n} \rangle$  is **large**.

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If  $\psi$  has a compact support of size  $\Delta \in \mathbb{N}^*$ , at scale  $2^j$

$\Delta$  wavelet coefficients  $\zeta_{j,n} = \langle f, \psi_{j,n} \rangle$  are **large**

$\implies$  to reduce the number of significant coefficients, **reduce support size** of  $\psi$ .

## Support size and number vanishing moments trade-off

**Theorem:** Let  $\psi$  a wavelet with  $n_\psi$  vanishing moments generating an orthonormal basis of  $L^2(\mathbb{R})$ , then its support is of size at least  $2n_\psi - 1$ .

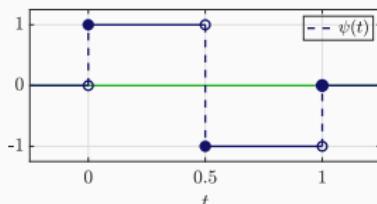
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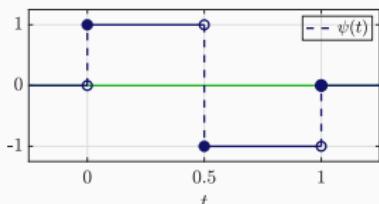
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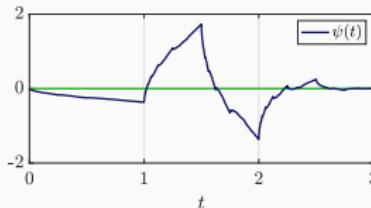
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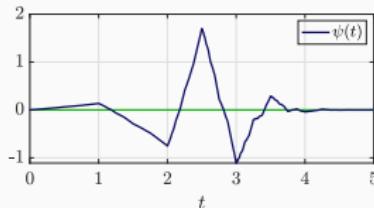


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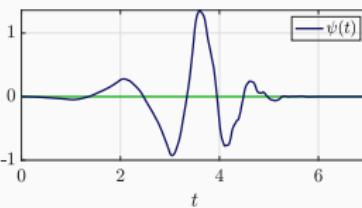
**Daubechies wavelet:**



$$n_\psi = 2$$



$$n_\psi = 3$$



$$n_\psi = 4$$

## **Decompositions on Wavelet Frames**

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# Theory of Frames

$\mathcal{H}$ : Hilbert space, e.g.,  $L^2(\mathbb{R})$  or subspace of  $L^2(\mathbb{R})$ ;  $\mathbb{I} \subset \mathbb{N}$ : set of indices

**Definition** A family of elements of  $\mathcal{H}$ ,  $\{\mathbf{e}_n, n \in \mathbb{I}\}$ , s.t.

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## Stable analysis and synthesis

$f \in \mathcal{H} \mapsto (\langle f, \mathbf{e}_n \rangle)_{n \in \mathbb{N}}$   $\ell^2(\mathbb{I})$  bounded linear operator

$(f_n)_{n \in \mathbb{N}} \in \ell^2(\mathbb{I}) \mapsto \sum_{n \in \mathbb{Z}} f_n \mathbf{e}_n \in \mathcal{H}$  bounded linear operator

# Theory of Frames

**Initially:** reconstruction of irregularly sampled band-limited signals

[Duffin & Schaeffer, 1952, *Trans. Amer. Math. Soc.*]

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**Wavelet frames:**  $\tilde{\psi}_{j,n} = \left\{ \frac{1}{\sqrt{\gamma^j}} \tilde{\psi} \left( \frac{t - bn\gamma^j}{\gamma^j} \right), (j, n) \in \mathbb{Z}^2 \right\}, \gamma > 1, b > 0$

⇒ more freedom in the design of the wavelet  $\tilde{\psi}$

# Wavelet Decomposition of Images

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# Mathematical representation of images



**Real-valued** square-integrable **field**

$$X : \mathbb{R}^2 \rightarrow \mathbb{R}$$

restricted to a **rectangular** domain  $\Omega = [0, n_1 - 1] \times [0, n_2 - 1]$

## Two-dimensional wavelet decomposition

### Separable wavelet bases:

$\varphi$  and  $\psi$  the scaling function and mother wavelet of a **1D multiresolution analysis**

Define

$$\begin{cases} \psi^{(0)}(\underline{x}) = \varphi(x_1)\varphi(x_2), & \psi^{(1)}(\underline{x}) = \psi(x_1)\varphi(x_2) \\ \psi^{(2)}(\underline{x}) = \varphi(x_1)\psi(x_2), & \psi^{(3)}(\underline{x}) = \psi(x_1)\psi(x_2). \end{cases}$$

Then, the family

$$\left\{ 2^{-j}\psi^{(m)}(\underline{x}2^{-j} - \underline{n}), m \in \{1, 2, 3\}, \underline{x} = (x_1, x_2) \in \mathbb{R}^2, \underline{n} = (n_1, n_2) \in \mathbb{Z}^2 \right\}$$

defines an **orthonormal wavelet basis** of  $L^2(\mathbb{R})$ .

The **wavelet coefficients** of a 2D field  $X \in L^2(\mathbb{R}^2)$  are defined as

$$\zeta_{j,\underline{n}}^{(m)} = \langle X, \psi_{j,\underline{n}}^{(m)} \rangle, \quad \psi_{j,\underline{n}}^{(m)}(\underline{x}) = 2^{-j}\psi^{(m)}(\underline{x}2^{-j} - \underline{n})$$

# Wavelet transform of images



**Albert Marquet**, *Paysage, baie méditerranéenne, vue d'Agay*, 1905

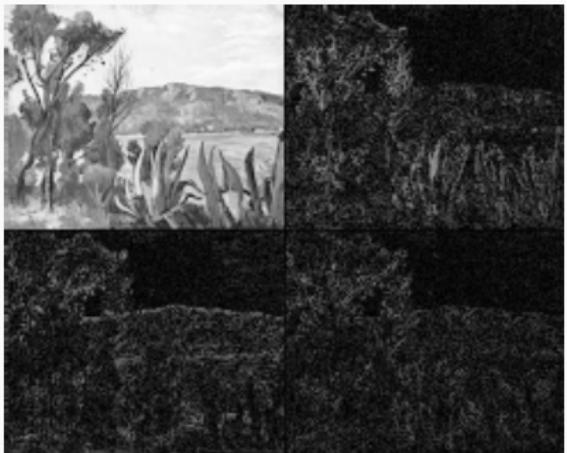


**Daubechies wavelet transform**  
with  $n_\psi = 2$  vanishing moments  
at scale  $2^1$

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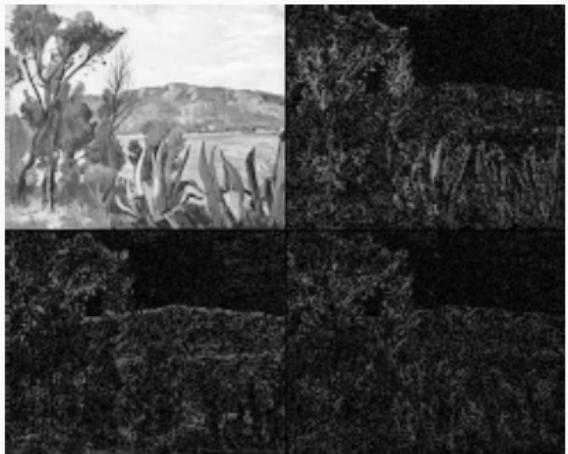


**Daubechies wavelet transform**  
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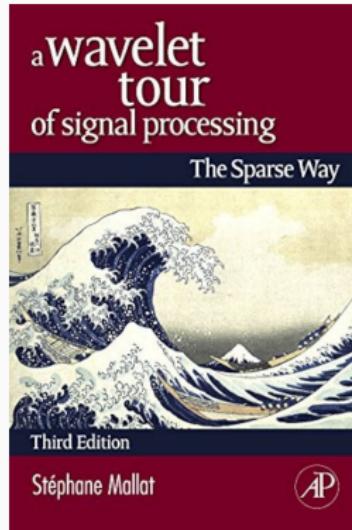
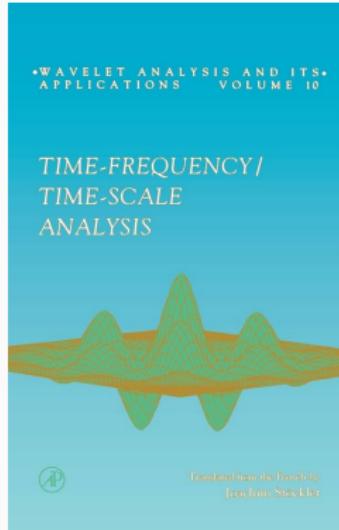
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**Application:** compression of images and videos: **JPEG2000, MPEG-4**

# References and further readings



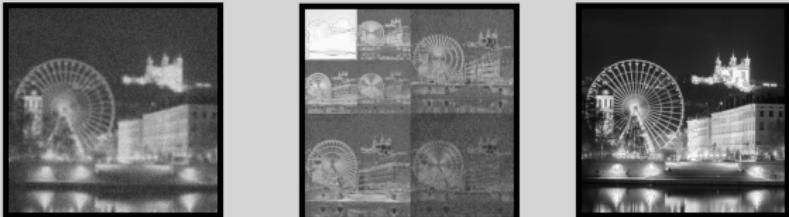
Pesquet-Popescu, B., & Pesquet, J.-C. (2001). "Ondelettes et applications", *Techniques de l'ingénieur*, Réf. TE, 5, 215.

Chaux, C. (2006). "Analyse en ondelettes M-bandes en arbre dual; application à la restauration d'images", *Université de Marne la Vallée*.

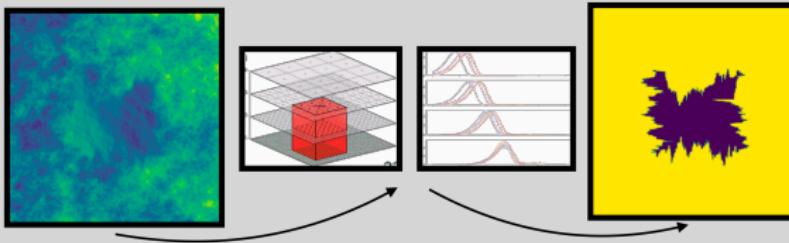
Pustelnik, N. (2010). "Proximal methods for the resolution of inverse problems: application to positron emission tomography", *Université Paris-Est*.

# Multiresolution/multilevel

Multiresolution  
to perform  
image restoration  
(~2000–2015)



Multiresolution  
to perform  
texture  
segmentation  
(~2014- now)



Multiresolution  
to accelerate  
algorithms  
(~2016- now)

