#1.

$$\dot{x} = rx + x^3 - x^5$$
$$= x(r + x^2 - x^4)$$
$$\Rightarrow r = x^4 - x^2$$

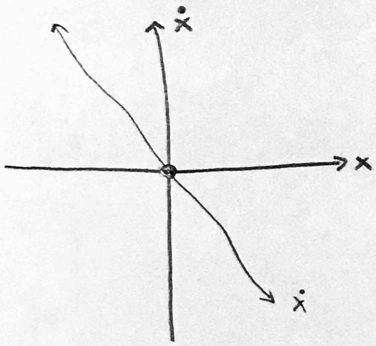
(looked up quartic solution)

if $x = -\infty$
and $r = -\infty$
$\dot{x} = \infty$

$$X_c = 0, \pm \frac{\sqrt{1-\sqrt{4r+1}}}{\sqrt{2}}, \pm \frac{\sqrt{1+\sqrt{4r+1}}}{\sqrt{2}}$$

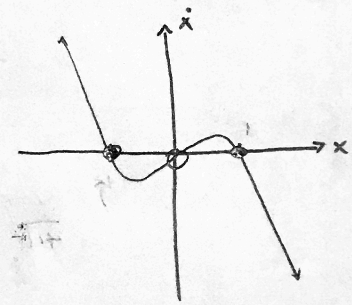
5 fixed points, matches 5th degree $\dot{x}$, according to Fundamental Theorem of Algebra

for $r \to -\infty$



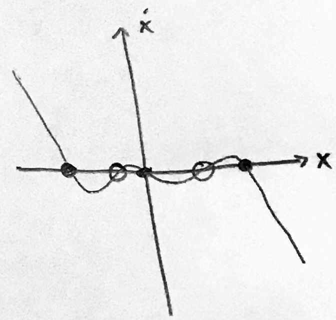
one critical point $X_c = 0$

for $r \geq 0$



$X_c = \pm \frac{\sqrt{1+\sqrt{4r+1}}}{\sqrt{2}}$

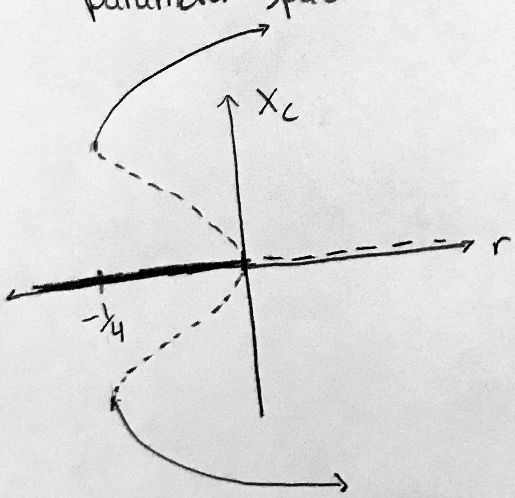
for $-\frac{1}{4} < r < 0$



all 5 fixed points appear in this region of r

The stability can be seen from the direction of the slope

parameter space



Transcritical Bifurcation

the fixed point at $X_c = 0$ exists for $-\infty < r < \infty$, however it changes stability at $r = 0$, but continues to exist for all values. The continual existence and exchange of stability is characteristic of a transcritical bifurcation

#2

$\dot{X} = ax - bxy$ $\qquad \dot{x} = \frac{\partial x}{\partial t} \quad \frac{[m]}{[s]}$ $\qquad$ variables: $[x, y, t]$ $\quad$ params $[a,b,c,d]$ $\quad$ p2

$\dot{y} = -cy + dxy$ $\qquad \dot{y} = \frac{\partial y}{\partial t} \quad \frac{[g]}{[s]}$ $\qquad$ params - dimensions = # of Dimensionless

$a = by$

$c = dx$

$4 - 3 = 1$ (found to be $\left(\frac{c}{a}\right)$)

$\dot{x} = \frac{[m]}{s} + \frac{[m][g]}{s\,g}$ $\qquad a\left[\frac{1}{s}\right]$ $\quad b\left[\frac{1}{gs}\right]$

$\dot{y} = \frac{[g]}{s} + \frac{[g][m]}{m\,s}$ $\qquad c\left[\frac{1}{s}\right]$ $\quad d\left[\frac{1}{ms}\right]$

$\tilde{x} = \frac{[m]}{(ms)}\left(\frac{s}{1}\right) = \frac{xd}{a} \Rightarrow x = \tilde{x}\left(\frac{a}{d}\right)$

$\tilde{y} = \frac{[g]}{(gs)}\left(\frac{s}{1}\right) = \frac{yb}{a} \Rightarrow y = \tilde{y}\left(\frac{a}{b}\right)$

$\tilde{t} = [s]\left(\frac{1}{s}\right) = ta$

$\boxed{\begin{matrix} \text{let } I = x \\ J = y \end{matrix}}$

$\frac{\partial \tilde{x}}{\partial \tilde{t}} = \left(\frac{\partial \tilde{x}}{\partial x}\right)\left(\frac{\partial t}{\partial \tilde{t}}\right)\left(\frac{\partial x}{\partial t}\right)$ $\qquad \frac{\partial \tilde{y}}{\partial \tilde{t}} = \frac{\partial \tilde{y}}{\partial y}\frac{\partial t}{\partial \tilde{t}}\frac{\partial y}{\partial t}$

$\frac{\partial \tilde{x}}{\partial x} = \frac{d}{a} \quad \left[\frac{1}{ms}\right]\left[\frac{s}{1}\right]$ $\qquad \frac{\partial \tilde{y}}{\partial y} = \frac{b}{a} \quad \left[\frac{1}{gs}\right][s]$

$\frac{\partial t}{\partial \tilde{t}} = \frac{1}{a} \quad \left[\frac{s}{1}\right]$ $\qquad \frac{\partial t}{\partial \tilde{t}} = \frac{1}{a} \quad \left[\frac{s}{1}\right]$

$\frac{\partial \tilde{x}}{\partial \tilde{t}} = \left(\frac{d}{a}\right)\left(\frac{1}{a}\right)(ax - bxy) \quad \left[\frac{s}{m}\right]\left[\frac{m}{s} - \frac{m}{s}\right] = \frac{dx}{a} - \frac{dbxy}{a^2} = \underbrace{\frac{d}{a^2}}\overbrace{(ax - bxy)}^{\dot{x}} = \dot{\tilde{x}}$

$\frac{\partial \tilde{y}}{\partial \tilde{t}} = \left(\frac{b}{a}\right)\left(\frac{1}{a}\right)(-cy + dxy)\left[\frac{s}{g}\right]\left[\frac{g}{s} + \frac{g}{s}\right] = -\frac{by}{a} + \frac{bdxy}{a^2} = \frac{b}{a^2}\overbrace{(-cy + dxy)}^{\dot{y}} = \dot{\tilde{y}}$

$\frac{a^2}{d}\left(\dot{\tilde{x}}\right) = a\left(\tilde{x}\frac{a}{d}\right) - b\left(\tilde{x}\frac{a}{d}\right)\left(\tilde{y}\frac{a}{b}\right)$ $\qquad \dot{y} = \dot{\tilde{y}}\left(\frac{a^2}{b}\right) = -c\,\tilde{y}\left(\frac{a}{b}\right) + d\left(\tilde{x}\frac{a}{d}\right)\left(\tilde{y}\frac{a}{b}\right)$

$\Rightarrow \dot{\hat{x}} = \hat{x} - \hat{x}\hat{y}$ $\qquad \dot{\hat{y}} = -\hat{y}\left(\frac{c}{a}\right) + \hat{x}\hat{y}$

dimensionless param.

Jacobian = $\begin{pmatrix} \frac{\partial 1}{\partial x} & \frac{\partial 1}{\partial y} \\ \frac{\partial 2}{\partial x} & \frac{\partial 2}{\partial y} \end{pmatrix} = \begin{pmatrix} 1-\hat{y} & -\hat{x} \\ \hat{y} & -\frac{c}{a}+\hat{x} \end{pmatrix}$

fixed points of dimensionless

$\dot{\hat{x}} = 0 = \hat{x} - \hat{x}\hat{y}$

$\hat{y}_c = 1 \Rightarrow \dot{\hat{y}}(1) = 0 = (1)(-\frac{c}{a}) + \hat{x}(1)$

$\hat{x}_c = \frac{c}{a}$

$(\hat{x}_c, \hat{y}_c) = (\frac{c}{a}, 1)$

$Jac(\frac{c}{a}, 1) = \begin{pmatrix} 1-1 & -\frac{c}{a} \\ 1 & -\frac{c}{a}+\frac{c}{a} \end{pmatrix} = \begin{pmatrix} 0 & -\frac{c}{a} \\ 1 & 0 \end{pmatrix}$

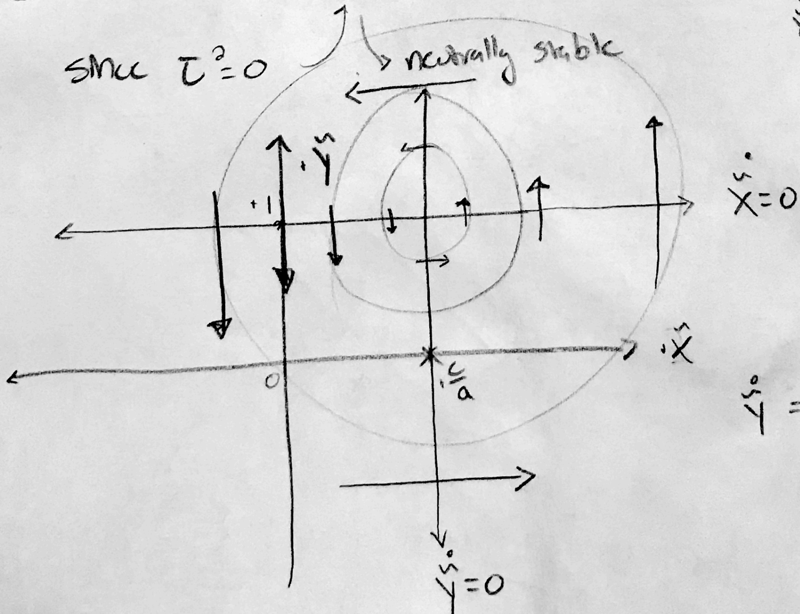
$\lambda = \frac{\tau \pm \sqrt{\tau^2 - 4\Delta}}{2} = \frac{\pm\sqrt{-4\frac{c}{a}}}{2} = \pm\sqrt{-\frac{c}{a}}$ imaginary

$\tau = 0 \quad \Delta = \frac{c}{a}$

$c, a > 0$

$\tau^2 - 4\Delta < 0$ = spirals/centers

since $\tau^2 = 0$ → neutrally stable



for $\hat{y} \to \infty$

$\dot{\hat{x}} = 0 = \hat{x} - \hat{x}\hat{y} \to -\infty$

$\hat{y} \to -\infty$

$\hat{x} - \hat{x}\hat{y} \to \infty$

for $\hat{x} = 0$

$\dot{\hat{y}} = 0 = -\frac{c}{a}\hat{y} + 0$

$\Rightarrow -\frac{c}{a}\hat{y} < 0$ for $\hat{y} > 0$

for $\hat{x} \to \infty$

$-\frac{c}{a}\hat{y} + \hat{x}\hat{y} \to +\infty$

#3. particular system involving

energy $E \rightarrow \left[kg \frac{m^2}{s^2}\right] \rightarrow$ mass $\frac{\text{length}^2}{\text{time}^2}$

density $D \rightarrow \left[\frac{kg}{m^3}\right] \rightarrow \frac{\text{mass}}{\text{length}^3}$

radius $R \rightarrow [m] \rightarrow$ length

time $t \rightarrow [s] \rightarrow$ time

Dimensions [mass, length, time]

Buckingham pi theorem

Parameters - Dimensions = Dimensionless parameters

$4 - 3 = 1$

$$\left(\cancel{kg}\frac{m^2}{s^2}\right)\left(\frac{m^3}{\cancel{kg}}\right)\left(s^2\right)\left(\frac{1}{m^5}\right)$$

$$(E)\left(\frac{1}{D}\right)\left(t^2\right)\left(\frac{1}{R^5}\right) = \text{dimensionless} = [1]$$

#4.

(p5)

$\dot{x} = a(1-x) - xy^2$ $\qquad \dot{x}(y_c=0) =$

$\dot{y} = xy^2 - (a+b)y \longrightarrow y_c = 0$

$\longrightarrow y_c = \frac{a+b}{x}$

$\dot{x}(y=0) = a(1-x) = 0 \ \to\ x_c = 1 \qquad fp\ (1,0)$

$\dot{x}\left(y=\frac{a+b}{x}\right) = a(1-x) - x\left(\frac{a+b}{x}\right)^2 = 0$

$= a - ax - \frac{(a+b)^2}{x} = 0$

$\overset{a}{-ax^2} + \overset{b}{ax} - \overset{c}{(a+b)^2} = 0$

$x_c = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-a \pm \sqrt{a^2 + 4(+a)(a+b)^2}}{-2a} = \frac{-a \pm \sqrt{a^2 - 4a(a+b)^2}}{2a}$

if $\sqrt{a^2 - 4a(a+b)^2} < 0$, x_c is imaginary

" ( )" > 0, x_c exists $\qquad$ thus when

$\underbrace{\sqrt{a^2 - 4a(a+b)^2}} = 0$ $\qquad$ the fixed points suddenly appear, characteristic of a bifurcation point, specifically a saddle-node bifurcation

$\Rightarrow a^2 - 4a(a+b)^2 = 0$

$\frac{a}{4} = (a+b)^2$

$\Rightarrow b_c = \frac{\pm\sqrt{a}}{2} - a$