#7.1.2 $\dot{r} = r(1-r^2)(9-r^2)$ $r_c = \pm 1, \pm 3$ (0)

$\dot{\theta} = 1$

Jacobian $\begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix} = \begin{pmatrix} 9-30r^2+5r^4 & 0 \\ 0 & 0 \end{pmatrix}$

$J(\pm 1)$:

$\begin{pmatrix} -16 & 0 \\ 0 & 0 \end{pmatrix}$

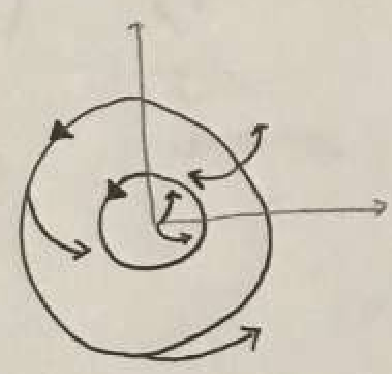
$\tau < 0$ stable

$J(\pm 3)$:

$\begin{pmatrix} 144 & 0 \\ 0 & 0 \end{pmatrix}$

$\tau > 0$ unstable

for all r, $\dot{\theta} = 1$



7.1.3

p2

$\dot{r} = r(1-r^2)(4-r^2)$ $\quad r_c = \pm 1, \pm 2 \quad (0)$

$\dot{\theta} = 2 - r^2$

$\dot{\theta}(\pm 1) > 0, \ \dot{\theta}(\pm 2) < 0$

Jacobian:

$$\begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix} = \begin{pmatrix} 4 - 15\overset{-60}{r^2} + 5\overset{80}{r^4} & 0 \\ -2r & 0 \end{pmatrix}$$

$J(\pm 1)$

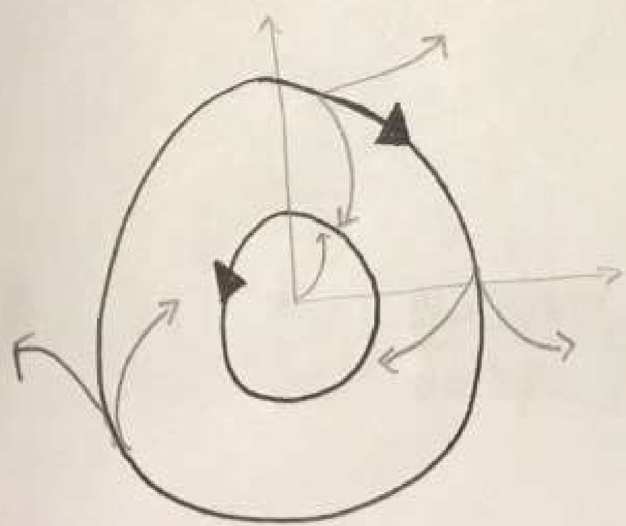
$$\begin{pmatrix} -6 & 0 \\ \pm 2 & 0 \end{pmatrix}$$

$\tau < 0$ stable

$J(\pm 2)$

$$\begin{pmatrix} 24 & 0 \\ \pm 4 & 0 \end{pmatrix}$$

$\tau > 0$ unstable



#7.1.6

a)

~~voltage~~ current across a capacitor is known to be

$$i = C_{ap}\frac{\partial V}{\partial t}$$

$$\Rightarrow \frac{\partial V}{\partial t} = \dot{V} = \frac{i}{C_{ap}}$$

using the governing equations of Kirchhoffs Law we know $V_s = V_c + V_L$

and when i is low $V_s = -V_s$

(from question "negative resistor" at low current)

thus

$$V_c = V_s + V_L$$

$$V_c = f(i) + L\frac{\partial i}{\partial t}$$

b) $\frac{\partial w}{\partial \tau} = \frac{\partial w}{\partial t}\left(\frac{\partial t}{\partial \tau}\right)$

$$\omega = C^{1/2} V$$

$$\frac{\partial w}{\partial t} = C^{1/2}\frac{\partial V_c}{\partial t} = C^{1/2}\left(\frac{-i}{C_{ap}}\right)$$

$$\tau = t(LC)^{1/2}$$

$$\frac{\partial \tau}{\partial t} = (LC)^{-1/2}$$

$$\frac{\partial w}{\partial t}\left(\frac{\partial t}{\partial \tau}\right) = \frac{C^{1/2}(-i)(LC)^{1/2}}{C}$$

$$\Rightarrow \frac{\partial w}{\partial \tau} = -i\sqrt{L}$$

where $x = i\sqrt{L}$

$$\frac{\partial w}{\partial \tau} = -x$$

7.3.1

$\dot{x} = x - y - x(x^2 + 5y^2) = f$

$\dot{y} = x + y - y(x^2 + y^2) = g$

a) fixed point @ $(0,0)$

Jacobian

$$\begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix} = \begin{pmatrix} 1-3x^2-5y^2 & -1-10yx \\ 1-2yx & 1-x^2-3y^2 \end{pmatrix} \Bigg|_{(0,0)} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

$\tau = 2 > 0$ unstable $\quad \Delta = 2$

$\tau^2 - 4\Delta < 0$ spiral

b) rewrite in polar: $r\dot{r} = x\dot{x} + y\dot{y}$, $\dot{\theta} = \frac{(x\dot{y} - y\dot{x})}{r^2}$

$$r\dot{r}(f,g) = x\left(x - y - x(x^2 + 5y^2)\right) + y\left(x + y - y(x^2 + y^2)\right)$$

$$= x^2 - xy - x^2(x^2 + 5y^2) + yx + y^2 - y^2(x^2 + y^2)$$

$$= \underbrace{x^2 + y^2}_{r^2} \underbrace{- x^4 - y^4}_{-(x^4 + y^4)} - 6x^2y^2$$

$(x^2 + y^2)(x^2 + y^2) = x^4 + y^4 + 2x^2y^2$

$$= -\left(\underbrace{(x^2 + y^2)^2}_{r^2} - 2x^2y^2\right)$$

$$= r^2 - r^4 + 2x^2y^2 - 6x^2y^2 = r^2 - r^4 - 4x^2y^2$$

$$= r^2 - r^4 - 4(r\cos\theta)^2(r\sin\theta)^2$$

$$r\dot{r} = r^2(1 - r^2 - 4r^2\cos^2\theta\sin^2\theta)$$

$$\dot{r} = r(1 - r^2 - 4r^2\cos^2\theta\sin^2\theta)$$

$$r^2\dot{\theta} = (x\dot{y} - y\dot{x}) = x\left(x+y-y(x^2+y^2)\right) - y\left(x-y-x(x^2+5y^2)\right)$$

$$= x^2 + \cancel{xy} - \cancel{yx(x^2+y^2)} - \cancel{yx} + y^2 + \cancel{yx}(x^2+5y^2)$$

$$= x^2+y^2+4y^3x = r^2 + 4(r\sin\theta)^3(r\cos\theta)$$

$$\dot{\theta}r^2 = r^2(1+4r^2\sin^3\theta\cos\theta)$$

$$\begin{bmatrix} \dot{\theta} = 1 + 4r^2\sin^3\theta\cos\theta \\ \dot{r} = r - r^3(1+4\cos^2\theta\sin^2\theta) \end{bmatrix}$$

c) Determine r_{max} such that all trajectories have radially outward for any θ $\quad \dot{r} > 0$

$(\sin\theta\cos\theta)$ maximized @ $\theta = \frac{\pi}{4}$

$$\left(\sin\tfrac{\pi}{4}\right)\left(\sin\tfrac{\pi}{4}\right)\left(\cos\tfrac{\pi}{4}\right)\left(\cos\tfrac{\pi}{4}\right) = \left(\tfrac{\sqrt{2}}{2}\right)\left(\tfrac{\sqrt{2}}{2}\right)\left(\tfrac{\sqrt{2}}{2}\right)\left(\tfrac{\sqrt{2}}{2}\right) = \frac{4}{16} = \frac{1}{4}$$

$$\dot{r} = r - r^3\left(1 + 4\left(\tfrac{1}{4}\right)\right) = r - 2r^3 > 0$$

$$r(1-2r^2) > 0 \quad \text{true for}$$

$$0 < r < \sqrt{1/2}$$

max radius outward

if $r=1$

$\dot{r} = -1 < 0$

(P.6)

d) Determine minimum radius such that all any θ trajectories have radially inward ~~outward~~ comp. $\dot{r} < 0$

$\dot{r} = r - r^3(1 + 4\cos^2\theta \sin^2\theta) < 0$

← minimize

$(\sin\theta\cos\theta)$ minimized @ $\theta = 0, \frac{\pi}{2}$

$\sin(0)\sin(0)\cos(0)\cos(0) = 0$

min radius inward

$\dot{r} = r - r^3 < 0 \qquad r_c = 1$

if $r < 1$ $\quad$ true for

$\dot{r} > 0$ $\quad$ ~~all~~ $r > 1$

e) $r = \sqrt{1/2}$ is the maximum radius for outward radially $\qquad$ $r = 1$ is minimum radius for inward radially

these radii form the boundaries of a confined trajectory excluding the point $(0,0)$ this satisfies the Poincaré-Bendixson theorem conditions and lead to the conclusion that a limit cycle exists in the bounded region of r $\quad \sqrt{1/2} < r < 1$