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A NOVEL TWO-GRID METHOD FOR SEMILINEAR ELLIPTIC EQUATIONS*

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Abstract. A new finite element discretization technique based on two (coarse and fine) subspaces is presented for a semilinear elliptic boundary value problem. The solution of a nonlinear system on the fine space is reduced to the solution of two small (one linear and one nonlinear) systems on the coarse space and a linear system on the fine space. It is shown, both theoretically and numerically, that the coarse space can be extremely coarse and still achieve asymptotically optimal approximation. As a result, the numerical solution of such a nonlinear equation is not significantly more expensive than the solution of one single linearized equation.

Key words. elliptic boundary value problem, finite elements, two-grid

AMS subject classifications. 65M60, 65N15, 65N30

1. Introduction. In this paper, we shall present a new discretization technique for semilinear elliptic equations based on finite element spaces defined on two grids of different sizes. Methods along this direction have been studied in a recent paper [1] for linear (nonsymmetric or indefinite) and especially nonlinear elliptic partial equations. (See also references cited in [1] for other methods for nonlinear elliptic equations.) As for nonlinear equations, the idea in [1] is basically to use a coarse space to produce a rough approximation of the solution, and then use it as the initial guess for one Newton iteration on the fine grid. This procedure involves a nonlinear solve on the coarse space and a linear solve on the fine space. A remarkable fact about this simple approach is, as shown in [1], that the coarse mesh can be quite coarse and still maintain an optimal approximation. The purpose of this paper is to make a further refinement in the aforementioned process by solving one more linear equation on the *coarse* space. This additional correction step (which needs very little extra work) improves the accuracy of the algorithms in [1] up to one or two orders. The fact that a further coarse grid correction *after* the fine grid correction can actually improve the accuracy appears to be of great theoretical interest.

The rest of the paper is organized as follows. Section 2 is devoted to the standard finite element discretization for a model semilinear equation. Section 3 contains the new algorithm of the paper with error analysis. Section 4 gives a simple numerical example.

2. Preliminaries. Given a bounded domain $\Omega \subset \mathbb{R}^d$ ($d = 1, 2, 3$), we assume that $\partial\Omega$ is either smooth or convex and piecewise smooth. Let $\mathcal{W}_p^m(\Omega)$ be the standard Sobolev space with a norm $\|\cdot\|_{m,p}$ given by $\|v\|_{m,p}^p = \sum_{|\alpha|\le m} \|D^\alpha v\|_{L^p(\Omega)}^p$. For $p = 2$, we denote $\mathcal{H}^m(\Omega) = \mathcal{W}_2^m(\Omega)$ and $\mathcal{H}_0^1(\Omega)$ to be the subspace of $\mathcal{H}^1(\Omega)$ consisting of functions with vanishing trace on $\partial\Omega$. $\|\cdot\|_m = \|\cdot\|_{m,2}$ and $\|\cdot\| = \|\cdot\|_{0,2}$. The following Sobolev inequalities are well known:

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$$(2.1)\quad \|u\|_{0,p} \lesssim \|u\|_1 \quad (d=2 \quad \text{and} \quad 1 \le p < \infty) \quad \text{and} \quad \|u\|_{0,6} \lesssim \|u\|_1 \quad (d=3),$$

where the notation "$\lesssim$" is equivalent to "$\le C$" for some positive constant C.

We consider the following semilinear equation:

$$-\Delta u + f(x,u) = 0, \quad x \in \Omega, \quad u|_{\partial\Omega} = 0. \tag{2.2}$$

Here the function f is sufficiently smooth. For brevity, we shall drop the dependence of variable x in $f(x,u)$ in the following exposition.

We assume that the above equation has at least one solution $u \in \mathcal{H}_0^1(\Omega) \cap \mathcal{H}^2(\Omega)$ and the linearized operator $L_u \equiv -\Delta + f'(u)$ is nonsingular. As a result of this assumption, $L_u : \mathcal{H}^2(\Omega) \cap \mathcal{H}_0^1(\Omega) \mapsto \mathcal{L}_2(\Omega)$ is a bijection and satisfies

$$\|w\|_2 \le C_0 \|L_u w\| \quad \forall\, w \in \mathcal{H}^2(\Omega) \cap \mathcal{H}_0^1(\Omega)$$

for some constant C_0 depending on u.

Given $h \in (0,1)$, we assume that $V_h \subset \mathcal{H}_0^1(\Omega)$ is a piecewise linear finite element space defined on a quasi-uniform triangulation (with a meshsize h) of Ω satisfying

$$\inf_{\chi \in V_h} \{\|v-\chi\|_{0,p} + h\|v-\chi\|_{1,p}\} \lesssim h\|v\|_{2,p} \quad \forall\, v \in \mathcal{W}_p^2(\Omega) \cap \mathcal{H}_0^1(\Omega),\ 1 \le p \le \infty.$$

The standard finite element discretization of (2.2) is to find $u_h \in V_h$ so that

$$(\nabla u_h, \nabla\chi) + (f(u_h), \chi) = 0 \quad \forall\, \chi \in V_h. \tag{2.3}$$

It can be proven that (cf. [1] and the references cited therein) if h is sufficiently small, the above equation has a (locally unique) solution u_h satisfying

$$\|u-u_h\|_{0,p} + h\|u-u_h\|_{1,p} \le C(\|u\|_{2,p})h^2 \quad \forall\, 2 \le p < \infty \tag{2.4}$$

and

$$\|u-u_h\|_{0,\infty} \le C(\|u\|_{2,\infty})h^2|\log h|, \qquad \|u-u_h\|_{1,\infty} \le C(\|u\|_{2,\infty})h.$$

LEMMA 2.1. *There exists a constant $\delta > 0$ such that for any $v \in \mathcal{H}_0^1(\Omega) \cap \mathcal{L}_\infty(\Omega)$ with $\|u-v\|_{0,\infty} \le \delta$, the following statements are valid.*

1. *The operator $L_v \equiv -\Delta + f'(v) : \mathcal{H}_0^1(\Omega) \cap \mathcal{H}^2(\Omega) \mapsto \mathcal{L}_2(\Omega)$ is bijective and there exists a constant $C = C(\delta)$, so that*

$$\|w\|_2 \le C(\delta)\|L_v w\| \quad \forall\, w \in \mathcal{H}_0^1(\Omega) \cap \mathcal{H}^2(\Omega).$$

2. *If h is sufficiently small, there exists a constant $c = c(\delta)$ such that*

$$\sup_{\chi \in V_h} \frac{a_v(w_h, \chi)}{\|\chi\|_1} \ge c(\delta)\|w_h\|_1 \quad \forall\, w_h \in V_h,$$

where

$$a_v(w_h, \chi) = (\nabla w_h, \nabla\chi) + (f'(v)w_h, \chi). \tag{2.5}$$

Proof. Since f is smooth, we have

$$\|f'(u) - f'(v)\|_{0,\infty} \le C_1\|u-v\|_{0,\infty} \le C_1\delta$$

for some constant C_1 depending on f and u. Thus

$$\|(L_u - L_v)w\| = \|(f'(u) - f'(v))w\| \leq \|f'(u) - f'(v)\|_{0,\infty}\|w\| \leq C_1\delta\|w\|_2,$$

and, if $\delta \leq (2C_0C_1)^{-1}$,

$$\|w\|_2 \leq C_0\|L_u w\| \leq C_0(\|L_v w\| + \|(L_u - L_v)w\|) \leq C_0\|L_v w\| + \tfrac{1}{2}\|w\|_2.$$

The first statement is then justified with $C = 2C_0$.

The assumption that L_u is nonsingular implies that

$$\sup_{\chi\in V_h} \frac{a_u(w_h, \chi)}{\|\chi\|_1} \geq c_0\|w_h\|_1 \quad \forall\, w_h \in V_h$$

for some constant c_0 and sufficiently small h. But

$$a_v(w_h, \chi) \geq a_u(w_h, \chi) - C_1\delta\|w_h\|_1\|\chi\|_1.$$

The second statement then follows with $c = c_0/2$ for $\delta \leq c_0/(2C_1)$. □

3. Two-grid algorithms based on the Newton method. In this section, we shall present the main algorithm of this paper. The basic ingredient in our approach is another finite element space V_H ($\subset V_h \subset \mathcal{H}_0^1(\Omega)$) defined on a coarser quasi-uniform triangulation (with meshsize $H > h$) of Ω. Note that all the results for V_h in the previous section are valid for V_H if H is sufficiently small.

Setting $a_H(v, \phi) = a_{u_H}(v, \phi)$ (see (2.5)), the main algorithm of the paper is as follows.

ALGORITHM A1. Find $u_h^* = u_H + e_h + e_H$ such that

1. $u_H \in V_H, \quad (\nabla u_H, \nabla\phi) + (f(u_H), \phi) = 0 \quad \forall\, \phi \in V_H;$
2. $e_h \in V_h, \quad a_H(e_h, \chi) = -(f(u_H), \chi) - (\nabla u_H, \nabla\chi) \quad \forall\, \chi \in V_h;$
3. $e_H \in V_H, \quad a_H(e_H, \phi) = -\tfrac{1}{2}(f''(u_H)e_h^2, \phi) \quad \forall\, \phi \in V_H.$

The new feature of the above algorithm mainly lies in step 3 where a further coarse grid correction is performed. We notice that the linearized operator used in this step is based on the first coarse grid approximation u_H (instead of the more accurate $u_H + e_h$). As we shall see later, such a correction indeed improves the accuracy of the approximation.

Corresponding to the form $a_H(\cdot, \cdot)$, we define a projection $P_H : \mathcal{H}_0^1(\Omega) \mapsto V_H$ by

$$a_H(\phi, P_H v) = a_H(\phi, v) \quad \forall\, \phi \in V_H, v \in \mathcal{H}_0^1(\Omega).$$

By Lemma 2.1, it can be easily shown that there exists $H_0 > 0$, if $H \leq H_0$, P_H is well defined and satisfies for $k = 1, 2$

$$\|w - P_H w\| + H\|w - P_H w\|_1 \leq C(H_0)H^k\|w\|_k \quad \forall\, w \in \mathcal{H}_0^1(\Omega) \cap \mathcal{H}^k(\Omega). \tag{3.1}$$

We begin our analysis for Algorithm A1 with a simple lemma.

LEMMA 3.1. *For any* $\chi \in V_h$

$$a_H(u_H + e_h, \chi) = (-f(u_H) + f'(u_H)u_H, \chi), \tag{3.2}$$

$$\begin{aligned} a_H(u_h^*, \chi) = {} & (-f(u_H) + f'(u_H)u_H - \tfrac{1}{2}f''(u_H)e_h^2, \chi) \\ & + \tfrac{1}{2}(f''(u_H)e_h^2, \chi - P_H\chi). \end{aligned} \tag{3.3}$$

Proof. The first equality follows directly for the definition of u_H and e_h. Again, by the definition of P_H and e_H,

$$a_H(e_H, \chi) = a_H(e_H, P_H\chi) = -\tfrac{1}{2}(f''(u_H)e_h^2, P_H\chi).$$

Adding this to the first equality then leads to the second one. □

LEMMA 3.2. *If* $u \in \mathcal{W}_p^2(\Omega)$ *with* $p > 2$ *for* $d = 2$ *and* $p = \frac{12}{5}$ *for* $d = 3$, *then*

$$\|u_h - (u_H + e_h)\|_1 \lesssim H^4.$$

Proof. It follows from (2.3) that

$$a_H(u_h, \chi) = -(f(u_h), \chi) + (f'(u_H)u_h, \chi),$$

which, together with (3.2), gives

$$a_H(u_h - (u_H + e_h), \chi) = (f(u_H) - f(u_h) + f'(u_H)(u_h - u), \chi) = (b(u_h - u_H)^2, \chi).$$

Here we have

$$b = -\int_0^1 (1-t)f''(u_H + t(u_h - u_H))dt.$$

By assumption and (2.4), it is easy to see that b is a uniformly (with respect to both H and h) bounded function on $\bar{\Omega}$. Using the Hölder inequality and (2.1), we then deduce that

$$((u_h - u_H)^2, \chi) \le \|(u_h - u_H)^2\|_{0,\frac{p}{2}} \|\chi\|_{0,\frac{p}{p-2}} \lesssim \|u_h - u_H\|_{0,p}^2 \|\chi\|_1.$$

It follows from Lemma 2.1 and (2.4) that

$$\|u_h - (u_H + e_h)\|_1 \lesssim \|u_h - u_H\|_{0,p}^2 \lesssim H^4. \qquad \square$$

The estimate in Lemma 3.2, essentially contained in [1], is already quite remarkable because of the high power on the coarse meshsize H. But more remarkable estimates will be seen in the next theorem.

THEOREM 3.1. *If* $u \in \mathcal{W}_4^2(\Omega)$ *is the solution of* (2.2) *and* $u_h \in V_h$ *is the solution of* (2.3) (*satisfying* (2.4)), *then*

$$\|u_h - u_h^*\|_1 \lesssim H^5, \qquad \|u_h - u_h^*\| \lesssim H^6.$$

Consequently,

$$\|u - u_h^*\|_1 \lesssim h + H^5, \quad \|u - u_h^*\| \lesssim h^2 + H^6.$$

Proof. By the definition of u_h and the Taylor expansion, we have

$$\begin{aligned} a_H(u_h, \chi) = (&-f(u_H) + f'(u_H)u_H - \tfrac{1}{2}f''(u_H)(u_h - u_H)^2, \chi) \\ &+(O(u_h - u_H)^3, \chi), \end{aligned}$$

which, together with (3.3), gives that for any $\chi \in V_h$,

$$\begin{aligned} a_H(u_h - u_h^*, \chi) = &-\tfrac{1}{2}(f''(u_H)(e_h^2 - (u_h - u_H)^2), \chi) + \tfrac{1}{2}(f''(u_H)e_h^2, \chi - P_H\chi) \\ &+(O(u_h - u_H)^3, \chi). \end{aligned}$$

By the Hölder inequality and (2.1)

$$\begin{aligned}(e_h^2-(u_h-u_H)^2,\chi) &\le \|(u_h-(u_H+e_h))(e_h+u_h-u_H)\|_{0,\frac{6}{5}}\|\chi\|_{0,6}\\ &\le \|u_h-(u_H+e_h)\|_{0,3}\|e_h+u_h-u_H\|_{0,2}\|\chi\|_{0,6}\\ &\le \|u_h-(u_H+e_h)\|_1(\|e_h\|+\|u_h-u_H\|)\|\chi\|_1.\end{aligned}$$

It follows from (2.4) and Lemma 3.2 that

$$(e_h^2-(u_h-u_H)^2,\chi)\le H^6\|\chi\|_1.$$

By the Schwarz inequality and Lemma 3.2,

$$(f''(u_H)e_h^2,(I-P_H)\chi)\lesssim \|e_h\|_{0,4}^2\|(I-P_H)\chi\|\lesssim H^4\|(I-P_H)\chi\|.$$

By the Hölder inequality, (2.4), and (2.1),

$$(O(u_h-u_H)^3,\chi)\lesssim \|(u_h-u_H)^3\|_{0,\frac{6}{5}}\|\chi\|_{0,6}\lesssim \|u_h-u_H\|_{0,4}^3\|\chi\|_1\lesssim H^6\|\chi\|_1.$$

Consequently,

$$a_H(u_h-u_h^*,\chi)\lesssim H^6\|\chi\|_1+H^4\|(I-P_H)\chi\|. \tag{3.4}$$

This, together with Lemma 2.1 and (3.1), immediately implies the first estimate of the theorem. To derive the estimate in $\mathcal{L}_2(\Omega)$ norm, we use a duality argument by considering the auxiliary problem: Find $w\in\mathcal{H}_0^1(\Omega)\cap\mathcal{H}^2(\Omega)$ so that

$$-\Delta w+f'(u_H)w=u_h-u_h^*.$$

By Lemma 2.1, there exists $H_0>0$ and $C(H_0)>0$, so that if $H\le H_0$,

$$\|w\|_2\le C(H_0)\|u_h-u_h^*\|.$$

Then, if $w_h\in V_h$ is the nodal value interpolation of w, we have

$$\|u_h-u_h^*\|^2=a_H(u_h-u_h^*,w)=a_H(u_h-u_h^*,w-w_h)+a_H(u_h-u_h^*,w_h).$$

Note that

$$a_H(u_h-u_h^*,w-w_h)\lesssim \|u_h-u_h^*\|_1\|w-w_h\|_1\lesssim H^5h\|w\|_2\lesssim H^6\|u_h-u_h^*\|.$$

It follows from (3.4) that

$$\begin{aligned}a_H(u_h-u_h^*,w_h) &\lesssim H^6\|w_h\|_1+H^4\|(I-P_H)w_h\|\\ &\lesssim H^6\|w\|_2+H^4(\|(I-P_H)w\|+H\|w-w_h\|_1)\\ &\lesssim H^6\|w\|_2\lesssim H^6\|u_h-u_h^*\|.\end{aligned}$$

The second estimate of the theorem then follows. □

Comparing Theorem 3.1 with Lemma 3.2, we find that one coarse grid correction leads to a one-order improvement in $\mathcal{H}^1(\Omega)$ norm and possibly a two-order improvement in $\mathcal{L}_2(\Omega)$ norm.

According to Theorem 3.1, to obtain the asymptotically optimal accuracy, it suffices to take $H=O(h^{1/3})$ for both $\mathcal{L}_2(\Omega)$ and $\mathcal{H}^1(\Omega)$ norms (for $\mathcal{H}^1(\Omega)$ norm, it even suffices to take $H=O(h^{1/5})$). As a result, the dimension of V_H can be much smaller than the dimension of V_h, and thus the dominated part of the work in Algorithm A1 is the solution of the linear system in step 2.

With a standard treatment near the boundary where it is curved, the estimates corresponding to Theorem 3.1 for Algorithm A1 for elements of degree r are as follows:

$$\|u_h - u_h^*\|_1 \lesssim H^{2r+3}, \qquad \|u_h - u_h^*\| \lesssim H^{2r+4}.$$

Thus

$$\|u - u_h^*\|_1 \lesssim h^r + H^{2r+3}, \qquad \|u - u_h^*\| \lesssim h^{r+1} + H^{2r+4}.$$

A proper choice of meshsizes would be $h = O(H^{2+2/(r+1)})$ (or even $h = O(H^{2+3/r})$).

For most practical purposes, Algorithm A1, which involves only one Newton iteration on the fine grid, is sufficient for applications. Nevertheless, a more dramatic result can be derived if one more Newton iteration is performed on the fine grid.

ALGORITHM A2. Find $\tilde{u}_h = u_h^* + e^h$ such that
1. $u_h^* \in V_h$ is obtained by Algorithm A1.
2. $e^h \in V_h, \quad a_{u_h^*}(e^h, \chi) = -(f(u_h^*), \chi) - (\nabla u_h^*, \nabla \chi) \quad \forall\, \chi \in V_h.$

For this algorithm, we can prove that, if $u \in W_4^2(\Omega)$, then

$$\|u_h - \tilde{u}_h\|_1 \lesssim H^{12} |\log h|^{\frac{1}{2}}.$$

In fact, by Taylor expansion, we have

$$a_{u_h^*}((u_h - \tilde{u}_h), \chi) = (O(u_h - u_h^*)^2, \chi).$$

For $d = 2$, using the well-known inequality that $\|\chi\|_{0,\infty} \lesssim |\log h|^{1/2} \|\chi\|_1$, we deduce that

$$(O(u_h - u_h^*)^2, \chi) = \|u_h - u_h^*\|^2 \|\chi\|_{0,\infty} \lesssim \|u_h - u_h^*\|^2 \|\chi\|_1 |\log h|^{\frac{1}{2}} \lesssim H^{12} |\log h|^{\frac{1}{2}} \|\chi\|_1.$$

The desired estimate for $d = 2$ then follows. The proof for $d = 3$ depends on the $L_p(\Omega)$ estimates of $u_h - u_h^*$, and the technical details will be omitted here.

It is also possible to make an additional coarse grid correction in Algorithm A2 to further improve the estimate, but we suspect such an improvement will not be that important since the order of H is already so high.

One final remark is that the technique presented in this paper extends naturally to other discretization methods such as finite difference and spectral methods.

4. A numerical example. In this section we present a simple numerical result to demonstrate the efficiency of our proposed schemes. Our model problem is

$$-\Delta u + u^3 = f(x) \quad \text{in } \Omega \quad \text{and} \quad u = 0 \quad \text{on } \partial\Omega.$$

Here Ω is the unit square $(0,1) \times (0,1)$ and f is so chosen that $u = \sin \pi x \sin \pi y$ is the exact solution. The domain Ω is divided into families T_H and T_h of quadrilaterals, and $V_H, V_h \subset H_0^1(\Omega)$ are linear spaces of piecewise continuous bilinear functions defined on T_H and T_h, respectively.

On the coarse grid level, we solve the nonlinear problem by the Newton iteration. (Because of its small size, this nonlinear system takes very little time to solve compared to the larger linear systems.) The fine grid linearized equations (which are symmetric positive definite for this example) are solved by the conjugate gradient method or multigrid method. We take $h = H^4$ with $H = \frac{1}{4}$. Notice that $\dim V_h = 65,025$ while $\dim V_H = 49$. The numerical results are shown in Table 1.

TABLE 1
Errors in $\mathcal{H}^1$ and $\mathcal{L}_2$ norms.

	$\mathcal{H}^1$-norm	$\mathcal{L}_2$-norm
$u - u_H$	5.01E−1	3.19E−02
$u - (u_H + e_h)$	7.90E−3	1.45E−04
$u - (u_H + e_h + e_H)$	7.87E−3	3.19E−05
$u - u_h$	7.87E−3	7.85E−06

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