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Reduction of Relativistic Two-Particle Wave Equations to Approximate Forms. III*

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The interaction between two fermions of charge $\epsilon_{\rm I}$ and $\epsilon_{\rm II}$ and intrinsic magnetic moment $\mu_{\rm I}$ and $\mu_{\rm II}$ is described by a sixteen-component wave equation of the Breit type. The method of reduction of two-particle wave equations, as given in two previous papers by Chraplyvy, is used to convert this equation to an approximate four-component Pauli equation. A simple perturbation calculation is used to determine the contribution of the intrinsic magnetic moments to the fine and hyperfine structure of hydrogen and positronium.

1. INTRODUCTION

IN two previous papers,[1] referred to hereafter as I and II, the Foldy-Wouthuysen canonical transformation[2] was generalized by Chraplyvy to the two-body problem both for the singular case of equal masses as well as for the case of unequal masses. We consider here several applications using, except where noted, the terminology and notation of I and II.

Four-component one-body relativistic equations of the Dirac type or sixteen-component two-body equations of the Breit type may now be reduced to two- or four-component approximate equations of the Pauli type by the Foldy-Wouthuysen method or by the procedure of expressing the small components of the spinor ψ in terms of its large components.[3] In Sec. 2, the differences between these methods are brought out in a discussion of a familiar one-body example which we shall have occasion to refer to later as the limiting case of a two-body problem.

In Sec. 3, we compare the Hermitian part of a three-dimensional Bethe-Salpeter equation with the corresponding Breit equation by applying the two-body transformation of I to both.

In Secs. 4 and 5, the Breit interaction between two fermions of charge $\epsilon_{\rm I}$ and $\epsilon_{\rm II}$ is amplified to include intrinsic magnetic moment and virtual annihilation terms. The resulting equations are converted to the Pauli representation by means of the two-body transformation. It is then a simple matter to calculate the contribution of the intrinsic magnetic moments to the fine and hyperfine structure of hydrogen and positronium.

2. THE HYDROGEN ATOM: ONE-BODY TREATMENT

Essentially the same differences exist between the results of the Foldy-Wouthuysen and the "large component" methods of reduction when applied to either the one or the two-body problem. Although these differences have been discussed qualitatively before,[1,2] a simple quantitative comparison may be of interest. We consider the hydrogen atom, approximated as a one-body problem.

The exact solution for the energy E of the Dirac equation[4] for hydrogen,

$$(\beta m+\boldsymbol{\alpha}\cdot\mathbf{p}-e^2/r)\psi=i\partial\psi/\partial t, \tag{1}$$

may be expanded in ascending powers of the fine structure constant α.

$$E=m-\frac{m\alpha^2}{2n^2}-\frac{m\alpha^4}{2n^4}\left[\frac{n}{j+\frac{1}{2}}-\frac{3}{4}\right]+\cdots. \tag{2}$$

If the Foldy-Wouthuysen transformation is applied to Eq. (1), one obtains

$$H'\psi'=\left\{\beta m+\frac{\beta p^2}{2m}-\frac{e^2}{r}-\frac{\beta p^4}{8m^3}+\frac{\pi e^2}{2m^2}\delta(\mathbf{r})+\frac{e^2}{8m^2}\boldsymbol{\sigma}\cdot\left(\frac{\mathbf{r}}{r^3}\times\mathbf{p}-\mathbf{p}\times\frac{\mathbf{r}}{r^3}\right)+\cdots\right\}\psi', \tag{3}$$

where H' is obviously Hermitian and ψ' is still a four-component spinor. This equation may be separated into two two-component equations referring to positive and

TABLE I. One-body relativistic energy level corrections in Eq. (3).

Operator	Expectation value	
$-\dfrac{p^4}{8m^3}$	$-\dfrac{m\alpha^4}{2n^4}\left(\dfrac{n}{l+\frac{1}{2}}-\dfrac{3}{4}\right)$	
$\dfrac{\pi e^2}{2m^2}\delta(\mathbf{r})$	$\dfrac{m\alpha^4}{2n^3}\delta_{0l}$	
$\dfrac{e^2}{4m^2}\boldsymbol{\sigma}\cdot\dfrac{\mathbf{r}}{r^3}\times\mathbf{p}$	$\dfrac{m\alpha^4}{2n^3}\dfrac{1}{(2l+1)(l+1)}$	$j=l+\frac{1}{2}$, $l\neq 0$
	$-\dfrac{m\alpha^4}{2n^3}\dfrac{1}{l(2l+1)}$	$j=l-\frac{1}{2}$

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[1] Z. V. Chraplyvy, Phys. Rev. 91, 388 and 92, 1310 (1953).

[2] L. L. Foldy and S. A. Wouthuysen, Phys. Rev. 78, 29 (1950).

[3] Other reduction methods exist. See R. A. Ferrell, thesis, Princeton University, Princeton, New Jersey, 1951 (unpublished).

[4] Units are chosen so that $\hbar=c=1$.