Midterm Project: Numerical Solutions of Shallow Water Equations

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1 Introduction

Shallow Water Equations model the propagation of disturbances in water and other incompressible fluids and are used to describe the dynamics of important phenomenon like tsunami.The underlying assumption is that the depth of the fluid is small compared to the wave length of the disturbance. The conservative form of the shallow water equations is,

$$\begin{aligned} \frac{\partial h}{\partial t} + \frac{\partial (hu)}{\partial x} + \frac{\partial (hv)}{\partial y} &= 0 \\ \frac{\partial (hu)}{\partial t} + \frac{\partial (hu^2 + \frac{1}{2}gh^2)}{\partial x} + \frac{\partial (huv)}{\partial y} &= fhv \qquad (1) \\ \frac{\partial (hv)}{\partial t} + \frac{\partial (huv)}{\partial x} + \frac{\partial (hv^2 + \frac{1}{2}gh^2)}{\partial y} &= -fhu \end{aligned}$$

Here $h \geq 0$ is the fluid height, u and v are the horizontal and vertical velocities, g is the acceleration due to gravity (9.8m/s2 on Earth) and f is the Coriolis force. We let

$$U = \begin{bmatrix} h \\ hu \\ hv \end{bmatrix},\ F(U) = \begin{bmatrix} hu \\ hu^2 + \frac{1}{2}gh^2 \\ huv \end{bmatrix},\ G(U) = \begin{bmatrix} hv \\ huv \\ hv2 + \frac{1}{2}gh^2 \end{bmatrix},\ S(U) = \begin{bmatrix} 0 \\ fhv \\ -fhu \end{bmatrix},$$

which can now be rewritten in a more compact form,

$$\frac{\partial U}{\partial t} + \frac{\partial F(U)}{\partial x} + \frac{\partial G(U)}{\partial y} = S(U) \qquad (2)$$

In the absence of the Coriolis force, we get the standard form of the conservation law,

$$\frac{\partial U}{\partial t} + \frac{\partial F(U)}{\partial x} + \frac{\partial G(U)}{\partial y} = 0 \qquad (3)$$

We specify either periodic boundary condition, or "free boundary" conditions for h and "reflective" boundary conditions for uh and uv. Free boundary conditions mean the boundary exerts no stress, while reflective boundary conditions mean the boundary behaves like a mirror.

2 1D Solver

Consider the shallow water equations in one dimension,

$$\frac{\partial U}{\partial t} + \frac{\partial F(U)}{\partial x} = 0, \qquad (4)$$

where $U = [h, hu]^T$ and $F(U) = [hu,\ hu^2 + \frac{1}{2} + gh^2]^T$. Define $q(x,t) = u(x,t)h(x,t)$ so we have

$$U = \begin{bmatrix} h, \\ q \end{bmatrix}, \; F(U) = \begin{bmatrix} q, \\ \frac{q^2}{h} + \frac{1}{2}gh^2 \end{bmatrix}.$$

Let A be a Jacobian of $F(U)$. Thus

$$A = \begin{bmatrix} 0 & 1 \\ -\frac{q^2}{h^2} + gh & \frac{2q}{h} \end{bmatrix} \tag{5}$$

with eigenvalues $\lambda = \frac{q}{h} \pm \sqrt{gh}$. Since $h(x,t) > 0$ and g is a positive constant eigenvalues of the Jacobian are real and distinct (i.e A has full set of eigenvectors). Hence, the system is hyperbolic.

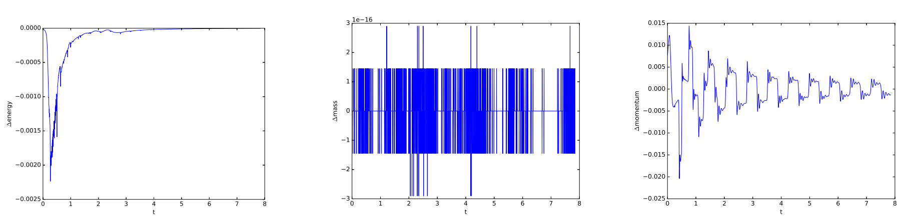
The Lax-Wendroff method for equation (4) is,

$$U_i^{n+1} = U_i^n - \frac{\Delta t}{2}((I - \frac{\Delta t}{\Delta x} A_{i+1/2}^n)(D_+ F_i^n) + ((I + \frac{\Delta t}{\Delta x} A_{i-1/2}^n)(D_- F_i^n). \tag{6}$$

where $F_i^n = F(U_i^n), A_{i+1/2}$ is the Jacobian matrix of F evaluated at $U_{i+1/2}$ and D_+ and D_- are the standard forward and backward difference operators defined as,

$$D_{\pm} w(x) = \frac{\pm w(x \pm \Delta x) - \pm w(x))}{\Delta x}. \tag{7}$$

Assuming we have N_x points in the x-direction with $\Omega = [0\ 1]$, and $1 \leq i \leq n$, periodic boundary conditions can be imposed by $U_1 = U_{N_x}$, reflective boundary conditions at$x = 0$ by $U_1 = -U_2$, and free boundary conditions at $x = 0$ by $U_1 = U_2$. These equations describe how conditions are imposed on the entire vector U, here we implement reflective boundary conditions for hu and free boundary conditions for h. The intial conditions are chosen to be interesting, such that $h = 4 + sin(2\text{pi}x)$ for $u = 0$, and $h = e^{-(\frac{x-\mu}{\sigma})^2} for\ u = 0$.



(a) Energy (b) Mass (c) Momentum

Figure 1: 1D conservation plots.

3 2D Solver

3.1 1D Conservation Laws

We expect the following quantities to be conserved: mass h, momentum or mass velocity, hu and hv, and energy $0.5(hv^2+gh^2)$. Potential vorticity is only looked at in the 2D solver. Invesigation of conservation of momentum in solver. Is the momentum conserved, what explanation do we have?
TO DO 1D FIGURES!!!
The spatial domain for the 2D solver is $\Omega = [1] \times [1]$. To go about solving the system in 2D there are many options, we compare two schemes. Lax-Wendfoff, and a two step version of Richtmeyer proposed.

3.2 Dimensional Splitting (Lax Wendroff)

Using the 1D equations of the Lax-Wendroff scheme for system (4) a dimenional split can be implemented by doing a step in x, followed by a step in y:

$$U_{i,j}^{*} = U_{i,j}^{n} - \frac{\Delta t}{2}((I - \frac{\Delta t}{\Delta x} A_{i+1/2,j}^{n}) D_{+}^{x} F_{i,j}^{n} + (I + \frac{\Delta t}{\Delta x} A_{i-1/2,j}^{n}) D_{-}^{x} F_{i,j}^{n}) \quad (8)$$

$$U_{i,j}^{n+1} = U_{i,j}^{*} - \frac{\Delta t}{2} B(I - \frac{\Delta t}{\Delta y} B_{i,j+1/2}^{*}) D_{+}^{y} G_{i,j}^{*}) + (I + \frac{\Delta t}{\Delta x} B_{i,j-1/2}^{*}) D_{-}^{y} F_{i,j}^{*}) \quad (9)$$

3.3 Two-Step Lax-Wendroff Method (Richtmeyer)

Richtmeyer proposed a two-step version of the Lax-Wendroff method, which is much simpler than the original, especially in multi-dimensional problems. The first step uses Lax's method and the second step is a midpoint leapfrog calculation.

$$U_i^{n+1} = \frac{1}{2}(U_{i+1}^n + U_{i-1}^n) - \Delta t \Big(\frac{F_{i+1}^n - F_{i-1}^n}{2\Delta x} \Big) \quad (10)$$

$$U_i^{n+2} = U_i^n + (2\Delta t) \Big(\frac{F_{i+1}^{n+1} - F_{i-1}^{n+1}}{2\Delta x} \Big) \quad (11)$$

The values $F_{i\pm1}^{n+1}$ in the second step are based on the $U_{i\pm1}^{n+1}$ results on the first step. The first step is considered a provisonal step, with signifigance attached only to the results of the second step in each sequence. This method does not look anything like the original Lax-Wendroff method of (8), but substitution

of (12) and (13) shows that the methods are equivalent.
The extension to multiple dimensions is obvious and neat.

$$\frac{\partial U}{\partial t} = -\left(\frac{\partial F}{\partial x} + \frac{\partial G}{\partial y}\right) \tag{12}$$

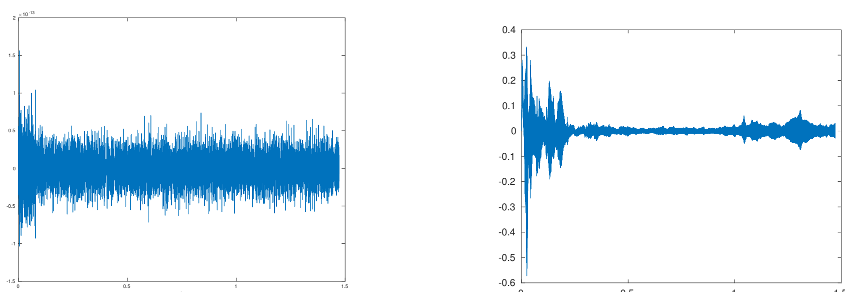
$$\begin{aligned} U_i^{n+1} = \frac{1}{4}\Big(&U_{i+1,j}^n + U_{i-1,j}^n + U_{i,j+1}^n + U_{i,j-1}^n\Big) \\ &- (\Delta t)\Big(\frac{F_{i+1,j}^n - F_{i-1,j}^n}{2\Delta x} + \frac{G_{i,j+1}^n - G_{i,j-1}^n}{2\Delta x}\Big) \end{aligned} \tag{13}$$

This 2D-scheme requires about a fourth of the computational time as the original Lax-Wendroff method, and produces less shock overshoot than the orignal.

The boundary conditions are prescribed for two cases. The first, periodic for both h, hu, and hv, and the second, and more complicated conditions, free for h, reflective in the horizontal direction and free in the vertical direction for uh, and reflective in the vertical direction and free in the horizontal direction for vh. The initial conditions are then chosen to be piecewise,

$$u_0(x,y) = \begin{cases} 8, & \text{if}(x-0.3)^2 + (y-0.3)^2 \\ 1, & \text{otherwise} \end{cases} \tag{14}$$

interestingly forming a cylindrical column.



(a) Energy

(b) Mass

Figure 2: 2d conservation plots for mass, and energy.

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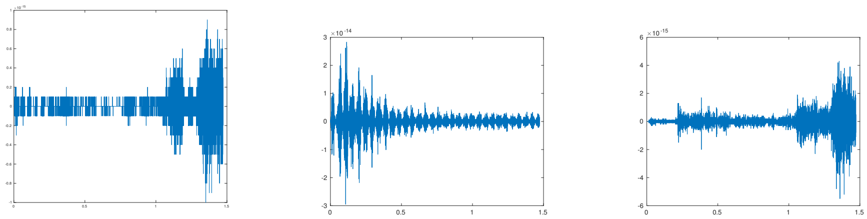

(a) u-momentum (b) v-momentum (c) vorticity

Figure 3: 2D Conservation plots.

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