

Math Problem Of The Week: Problem 1
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Rectangle I is inscribed in Rectangle II so each side of Rectangle II contains one and only one vertex of Rectangle I. If Rectangle I measures 1 unit by 2 units and if the area of Rectangle II is $\frac{22}{5}$ units squared, find the perimeter of Rectangle II.

Consider that Rectangle I is composed of a, b, c, d and Rectangle II is composed of A, B, C, D where a is on the segment D and A , b is on the segment A and B , etc. Let θ be the angle $\angle Aab$, then we know that $\angle Aba = \frac{\pi}{2} - \theta$ and finally this ensures that $\theta = \angle Bbc = \angle Ccd = \angle Dda$. Finally let $\bar{ab} = \bar{cd} = \alpha$, $\bar{bc} = \bar{da} = \beta$, $\bar{AB} = \bar{CD} = \gamma$, and $\bar{BC} = \bar{DA} = \delta$. We are looking for $(\gamma, \delta) \sim f(\alpha, \beta, \gamma\delta)$.

$$\begin{aligned}\gamma\delta &= \Lambda \\ \gamma &= \alpha \sin(\theta) + \beta \cos(\theta) \\ \delta &= \alpha \cos(\theta) + \beta \sin(\theta)\end{aligned}$$

Doing some basic algebra:

$$\begin{aligned}\gamma\delta &= \alpha\beta + \frac{\alpha^2 + \beta^2}{2} \sin(2\theta) = \Lambda \\ \theta &= \sin^{-1}\left(\frac{\Lambda - \alpha\beta}{\frac{\alpha^2 + \beta^2}{2}}\right)/2\end{aligned}$$

Applying a simple half angle formula:

$$\begin{aligned}\gamma &= \alpha \sqrt{\frac{1 - \sqrt{1 - \left(\frac{\Lambda - \alpha\beta}{\frac{\alpha^2 + \beta^2}{2}}\right)^2}}{2}} + \beta \sqrt{\frac{1 + \sqrt{1 - \left(\frac{\Lambda - \alpha\beta}{\frac{\alpha^2 + \beta^2}{2}}\right)^2}}{2}} \\ \delta &= \beta \sqrt{\frac{1 - \sqrt{1 - \left(\frac{\Lambda - \alpha\beta}{\frac{\alpha^2 + \beta^2}{2}}\right)^2}}{2}} + \alpha \sqrt{\frac{1 + \sqrt{1 - \left(\frac{\Lambda - \alpha\beta}{\frac{\alpha^2 + \beta^2}{2}}\right)^2}}{2}} \\ \text{Perimeter} &= 2(\gamma + \delta) = 2(\alpha + \beta) \left(\sqrt{\frac{1 - \sqrt{1 - \left(\frac{\Lambda - \alpha\beta}{\frac{\alpha^2 + \beta^2}{2}}\right)^2}}{2}} + \sqrt{\frac{1 + \sqrt{1 - \left(\frac{\Lambda - \alpha\beta}{\frac{\alpha^2 + \beta^2}{2}}\right)^2}}{2}} \right)\end{aligned}$$

Plugging in the requisite numbers gives me $\frac{42}{5}$. \square