Math Problem Of The Week: Problem 1
Week of August 21, 2017

Benjamin Phillabaum
Allstate Insurance
Northbrook, IL

August 20, 2017

Rectangle I is inscribed in Rectangle II so each side of Rectangle II contains one and only one vertex of Rectangle I. If Rectangle I measures 1 unit by 2 units and if the area of Rectangle II is $\frac{22}{5}$ units squared, find the perimeter of Rectangle II.

Consider that Rectange I is composed of a, b, c, d and Rectangle II is composed of A, B, C, D where a is on the segment D and A, b is on the segment A and B, etc. Let θ be the angle $\angle Aab$, then we know that $\angle Aba = \frac{\pi}{2} - \theta$ and finally this ensures that $\theta = \angle Bbc = \angle Ccd = \angle Dda$. Finally let $\bar{ab} = \bar{cd} = \alpha$, $\bar{bc} = \bar{da} = \beta$, $\bar{AB} = \bar{CD} = \gamma$, and $\bar{BC} = \bar{DA} = \delta$. We are looking for $(\gamma, \delta) \sim f(\alpha, \beta, \gamma\delta)$.

$$\begin{aligned}
\gamma\delta &= \Lambda \\
\gamma &= \alpha\sin(\theta) + \beta\cos(\theta) \\
\delta &= \alpha\cos(\theta) + \beta\sin(\theta)
\end{aligned}$$

Doing some basic algebra:

$$\gamma\delta = \alpha\beta + \frac{\alpha^2 + \beta^2}{2}\sin(2\theta) = \Lambda$$

$$\theta = \sin^{-1}\left(\frac{\Lambda - \alpha\beta}{\frac{\alpha^2+\beta^2}{2}}\right)/2$$

Applying a simple half angle formula:

$$\begin{aligned}
\gamma &= \alpha\sqrt{\frac{1 - \sqrt{1 - \left(\frac{\Lambda-\alpha\beta}{\frac{\alpha^2+\beta^2}{2}}\right)^2}}{2}} + \beta\sqrt{\frac{1 + \sqrt{1 - \left(\frac{\Lambda-\alpha\beta}{\frac{\alpha^2+\beta^2}{2}}\right)^2}}{2}} \\
\delta &= \beta\sqrt{\frac{1 - \sqrt{1 - \left(\frac{\Lambda-\alpha\beta}{\frac{\alpha^2+\beta^2}{2}}\right)^2}}{2}} + \alpha\sqrt{\frac{1 + \sqrt{1 - \left(\frac{\Lambda-\alpha\beta}{\frac{\alpha^2+\beta^2}{2}}\right)^2}}{2}} \\
\texttt{Perimeter} &= 2(\gamma + \delta) = 2(\alpha + \beta)\left(\sqrt{\frac{1 - \sqrt{1 - \left(\frac{\Lambda-\alpha\beta}{\frac{\alpha^2+\beta^2}{2}}\right)^2}}{2}} + \sqrt{\frac{1 + \sqrt{1 - \left(\frac{\Lambda-\alpha\beta}{\frac{\alpha^2+\beta^2}{2}}\right)^2}}{2}}\right)
\end{aligned}$$

Plugging in the requisite numbers gives me $\frac{42}{5}$. $\square$