Math Problem Of The Week: Problem 2
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Assume that x, y, and z are all positive, real numbers that satisfy the system of equations

$$\begin{aligned}
x + y + xy &= 8 \\
y + z + yz &= 15 \\
x + z + xz &= 35
\end{aligned}$$

Determine the value of $x + y + z + xyz$.

I begin by noting that the above expressions can be simplified notationally to

$$\begin{aligned}
x_i + x_j + x_i x_j &= \lambda_k \\
(x_i + 1)(x_j + 1) &= \lambda_k + 1
\end{aligned}$$

for $i \neq j \neq k$. This allows me to deduce each of the x_i:

$$x_i = \sqrt{\frac{(\lambda_k + 1)(\lambda_j + 1)}{\lambda_i + 1}} - 1$$

Thus

$$\begin{aligned}
x &= \frac{7}{2} \\
y &= 1 \\
z &= 7 \\
x + y + z + xyz &= 36
\end{aligned}$$

$\square$

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