ON THE ROLE OF EPISTEMIC SUBJECTIVITY IN THE EVOLUTION OF MATHEMATICAL LANGUAGE

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CERTIFICATION OF APPROVAL

I certify that I have read *ON THE ROLE OF EPISTEMIC SUBJECTIVITY IN THE EVOLUTION OF MATHEMATICAL LANGUAGE* by Brandon Patrick Hopkins, and that in my opinion this work meets the criteria for approving a thesis submitted in partial fulfillment of the requirements for the degree: Master of Arts in Philosophy at San Francisco State University.

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In Ladislav Kvasz's *Patterns of Change* (2008), he argues that one way in which mathematical language changes is that aspects of the *general form* of mathematical language are made increasingly explicit over time. His characterization of this pattern of change, which he calls *relativization*, is framed by a modified account of Wittgenstein's picture theory of meaning. In this essay I draw out a number of tensions between Wittgenstein's picture theory and the work Kvasz invokes it to do. I argue that various elements of Husserl's phenomenology can be brought together to better frame the characterization of relativization. More broadly, I show that Husserl's phenomenology is capable of spearheading a science of the relations between mathematical syntax and various elements of Kant's transcendental idealism.

I certify that the Abstract is a correct representation of the content of this thesis.

Chair, Thesis Committee Date

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1 Introduction

The content of mathematical language has always demanded special philosophical analysis. Whereas most, if not all, other kinds of linguistic content have been observed to depend at least partly on historical contingencies, the content of mathematical statements exhibits a stark degree of independence from historical contingencies. Even Kant, who put so much emphasis on subjectivity, maintained that mathematical statements obtain in virtue of cognitive-perceptual structure that is *transcendental* with respect to historical contingencies. Accordingly, the greater history of the philosophy of mathematics denied the relevance of historical contingencies in a correct account of mathematical knowledge. Even up through the early twentieth century, analytic orthodoxy separated questions pertaining to psychology, sociology, and history – "the context of discovery" – from questions pertaining to philosophy proper – "the context of justification."

In the middle of the twentieth century, however, philosophers began to rebel against the separation between these contexts. Most notably, Imre Lakatos argued in his *Proofs and Refutations* (1963–64, 1976) that the growth of mathematical knowledge is dependent on historically contingent *methodologies*, in which heuristics play a fundamental role both subjectively and intersubjectively. Moreover, in the wake of Thomas Kuhn's *The Structure of Scientific Revolutions* (1962), philosophers of mathematics began to pay increasing attention to methodologies of *change*, whereby new mathematical practices and paradigms are able to be justified in the context of old ones and succeed them. By the end of the twentieth century, this latter trend culminated in Donald Gillies' collection *Revolutions in Mathematics* (1992), in which a number of the authors pioneered various applications of aspects of the Kuhnian framework in the analysis of the history and structure of mathematical revolutions.

In reaction to these trends, Ladislav Kvasz argues in his book *Patterns of Change* (2008) that if the content of mathematics is taken to be fundamentally dependent on heuristics in addition to ordinary symbolic language, then the histories of different mathematical languages exhibit a number of robust patterns of change, dependent in determinable ways on features of this *linguistic* fundament. The novelty of Kvasz's approach, as opposed to the aforementioned approaches, is thus to examine historical changes in mathematics in terms of the *formal language* in which it was actually practiced. Since language is explicit, he argues, but methodology is not, studying patterns of linguistic change will allow the philosophy of the history of mathematics to attain a higher standard of analytic rigor than it otherwise would.

This comparative explicitness of language notwithstanding, one of the patterns of linguistic change that Kvasz identifies, which he calls "relativization," still – perhaps inevitably – involves reference to a number of *implicit* aspects of language. Relativization is characterized as a process in which *aspects of the general form* of mathematical language are made increasingly explicit within a particular mathematical language, where the explication of one of these aspects founds the implicit confrontation with another, which will also eventually be made explicit. Thus, as the aspects of the general form of mathematical language are made increasingly explicit within a particular mathematical language, that language is said to change according to the pattern of relativization, and possibly evolve.

The framework for Kvasz's analysis of relativization is Wittgenstein's picture theory of meaning—in particular, Wittgenstein's remarks on the distinction between the content of language, what it "depicts," and the *form* of language, which it can only “display.” In attempting to keep with this distinction, Kvasz argues that while aspects of the general form that mathematical language displays are indeed made explicit as a language evolves, these aspects do not have factual significance, but are only constrained by their interrelations among other aspects of this general form. Thus, Kvasz concludes, since the general form of mathematical language is only displayed, and not depicted, it provides the flexibility for relativization, and hence for linguistic evolution. Moreover, like Wittgenstein, Kvasz locates the *epistemic subject* – i.e., the historically situated subjectivity – in the *form* of language, in what is only displayed, rather than in its depicted content. For Kvasz, epistemic subjectivity is the aspect of the general form of mathematical language around whose explication mathematical language may evolve.

The goal of this essay is to expose and analyze an apparent tension between Kvasz's characterization of relativization and the Tractarian framework. In particular, if the evolution of mathematical languages can involve aspects of the general form of mathematical language being made increasingly explicit (in Kvasz's idiom), then this seems to entail that aspects of the general form of mathematical language can actually be depicted within language (in Wittgenstein's idiom). More precisely, this seems to contradict Wittgenstein's remarks on the place of epistemic subjectivity within language, and hence seems to undermine Kvasz's use of the Tractarian framework.

In order to bring out what is at stake in this tension, in §§2–4 I will give an overview Kvasz's project. This will be important because Kvasz's project is plausible enough in its own right that the apparent incompatibility with the Tractarian framework should not be seen as undermining the project at large. I will also attempt to fill in a number of gaps in Kvasz's narrative in order to better illustrate the role of epistemic subjectivity in his account.[1] This will prepare us to examine in §§5–9 Kvasz's remarks on the ontological status of aspects of the form of mathematical language in the context of Wittgenstein's picture theory of meaning, and to ask precisely what kind of ontological account of them should be sought. I will search for an account that can explain the significance of all of the aspects of the general form of mathematical language, but with particular emphasis on the role of epistemic subjectivity. In search of such an account, in §10 I turn away from Wittgenstein to Kant and Husserl. In §§11–19 I argue that, with the use of a variety of elements from within Husserl's far-reaching philosophy, which essentially absorbs Kant's, we can make much fairer sense of Kvasz's analysis than we can within the Tractarian framework. In particular, we can account for both the peculiar ontological status of the aspects of the general form of mathematical language as well as the sense in which it appears that Kvasz's account requires epistemic subjectivity to play two distinct roles.

Altogether, our analysis will take us in the direction of an answer to the question: By virtue of what is relativization possible? The broadest theme of this essay, in Kantian terms, is that this question cannot be answered except by appeal to transcendental conditions for the possibility of relativization, which conditions essentially transcend language itself. The theme of the later sections of the essay, however, is to give a qualified kind of *factual* significance to these transcendental conditions, in terms of Husserl's phenomenology. Thus, the major critique of Kvasz I put forward is that our linguistic turn, even when given such a highly historical bent, cannot dispense with the contents of transcendental arguments. Indeed, a truly scientific approach to the pattern of relativization will have to take these contents as factual.

2 Recoding, Linguistic Evolution, and Relativization in Brief

Like Lakatos, Kuhn, and others, if we take heuristics to be fundamental rather than incidental in mathematical practice, Kvasz argues, then we can actually observe a class of determinate patterns of linguistic change that show up *equally well* in the history of "iconic mathematical languages" as they do in the history of symbolic mathematical languages. The basis for this is that he regards mathematical languages as situated relatively on a symbolic–iconic *spectrum* – being different not in kind but only in relation to one another along this spectrum – where figures, pictures, and the like are all taken to be kinds of iconic linguistic expressions.

Hence, Kvasz argues that one pattern of linguistic change – what he calls "recodings" – can be regarded as syntheses of two languages relatively situated on opposite sides of this spectrum. His reconstruction of the history of recodings begins by taking elementary arithmetic and synthetic geometry to be historically primary, corresponding to the symbolic and iconic poles respectively. Accordingly: (1) algebra is regarded as a synthesis of linguistic resources from elementary arithmetic and synthetic geometry; (2) analytic geometry is regarded as a synthesis of linguistic resources from synthetic geometry and algebra; (3) differential and integral calculus is regarded as a synthesis of linguistic resources from algebra and analytic geometry; (4) iterative geometry is regarded as a synthesis of linguistic resources from analytic geometry and differential and integral calculus; (5) predicate calculus is regarded as a synthesis of linguistic resources from differential and integral calculus and iterative geometry; and (6) set theory is regarded as a synthesis of linguistic resources from iterative geometry and predicate calculus. Thus, contrary to the Fregean picture of arithmetic leading to algebra, and algebra leading to calculus, and calculus leading to predicate calculus, each of these (except arithmetic) is actually constituted – recoded – in interaction with mathematical language from the opposite side of the spectrum; the evolution of these languages is taken to be much more fluid than is usual.

With the notion of mathematical language so broadly construed – to include these fluctuations between the symbolic and iconic poles of language – Kvasz identifies six properties of mathematical languages that can be examined historically, i.e., which do not pertain to merely social or psychological dimensions of mathematical language, but to more analytically significant dimensions:

1. *Logical power* – how complex formulas can be proven in the language,
2. *Expressive power* – what new things can the language express, which were inexpressible in the previous stages,
3. *Explanatory power* – how the language can explain the failures which occurred in the previous stages,
4. *Integrative power* – what sort of unity and order the language enables us to conceive there, where we perceived just unrelated particular cases in the previous stages,
5. *Logical boundaries* – marked by occurrences of unexpected paradoxical expressions,
6. *Expressive boundaries* – marked by failures of the language to describe some complex situations.[2]

Thus, the *evolution* of mathematical languages can be seen as a series of increases along each of these dimensions, with different evolutionary leaps involving different sets of increases in different degrees. Hence, we may see how Kvasz's linguistic analysis allows for the context of discovery and change to be embedded within the context of justification. Kvasz is much more explicit about how these increases unfold in the history of recodings than in the history of relativizations, but they should be understood as implicit in relativizations, and equally explicable with sufficient care.

Furthermore, despite the alternation between the symbolic and iconic poles in the history of recodings, relativizations, on the other hand, *can* be observed as unfolding on each side of the symbolic–iconic spectrum more or less independently of the other. That is, whereas recodings alternate between symbolic and iconic language, relativizations need not, though they may. For the sake of simplicity, Kvasz's account of relativization separates out two historical threads, one in the history of synthetic geometry, the other in the history of algebra. It is important to emphasize, though, that this separation into two threads is not drawn upon the usual, post-Kantian division between geometry and algebra. As mentioned above, when iconic and symbolic linguistic resources are taken to be on a par, on a spectrum, there is a crucial sense in which interactions between them are constant and inevitable. Nevertheless, we can still examine the histories of geometry and algebra independently of one another, just so long as we keep in mind that these histories are not out of contact. Relativizations can be understood as *orthogonal* to recodings, but not unrelated. So even though these two threads of relativization are presented separately, there is nothing in principle barring a higher-order analysis of the interdependencies with relevant recodings. Indeed, such a higher-order analysis would presuppose the independent reconstructions.

As introduced in §1, relativization pertains to aspects of the *form* of mathematical language, which Kvasz characterizes in terms of Wittgenstein's picture theory of meaning. According to Wittgenstein, language comprises propositions,[3] and propositions are pictures of reality;[4] hence, language functions like a picture of reality. Moreover, Wittgenstein draws a distinction between what language *depicts* and what it can only *display*: its *pictorial form*.[5] What it depicts is explicit, but what it displays is only implicit; thus, for Wittgenstein, language has an implicit part – its form, by virtue of which its reference obtains – in addition to its explicit content. In keeping with these Wittgensteinian theses, Kvasz characterizes relativization as consisting in "two alternating processes – the explicit incorporation of the form of language which was at the previous stage only implicit, and the emergence of a new implicit form in the place of the previous one which was made explicit."[6]

He departs from Wittgenstein, however, in that he identifies eight progressively complex *forms* (i.e., plural) a mathematical language may pass through in its evolution: (1) the perspectivist, (2) the projective, (3) the coordinative, (4) the compositive, (5) the interpretive, (6) the integrative, (7) the constitutive, and (8) the conceptual.[7] Whereas Wittgenstein did not believe that different pictorial forms could be meaningfully specified (but only "displayed"), Kvasz relativizes Wittgenstein's notion of form to *particular* mathematical languages in order to observe how certain kinds of *local* formal change track with linguistic evolution. For Kvasz, there appears to be enough consistency in certain *aspects* of the form of mathematical language over time to meaningfully analyze them in a sort of historical meta-discourse. The odd numbers above correspond to forms in which some aspect of the general form of mathematical language is relatively implicit, which is then made explicit in the subsequent even-numbered form; the even-numbered forms are then the basis upon which another aspect of the general form of mathematical language can begin to be studied implicitly, before eventually also being made explicit.

The most tangible examples of relativization that Kvasz gives occur within the simpler forms of mathematical language, the perspectivist and the projective. We will now reconstruct Kvasz's account of this transition in the historical languages of synthetic geometry and algebra, respectively, so that we may get a better sense of how aspects of the general form of mathematical language are made explicit in the passage from one form to another on both sides of the symbolic–iconic spectrum.

3 Relativization in Context: Synthetic Geometry

The Perspectivist Form To examine the perspectivist form of synthetic geometry, Kvasz observes the transition from the pre-Renaissance style of painting into the Renaissance. Prior to the Renaissance, paintings lacked depth; it was not understood, e.g., how to give the impression of parallel lines running away from the perspective from which painting was constructed, i.e., toward the horizon. Florentine painter Giotto di Bondone is regarded in art history as one of the most advanced Medieval painters, having discovered how to *imply* parallel lines in certain linear features of objects in his paintings, e.g., in the angles between edges of the ceiling, etc. Yet Giotto did not understand how to synthesize these implications across multiple objects within the same painting; i.e., each object was constructed with the same angles as the others, giving the confused impression that the painting was constructed from multiple points of view at once. This was a common kind of feature of pre-Renaissance paintings.

It was not really until the 15th century, Kvasz argues, that the perspectivist form was made fully explicit by the Florentine artist Masaccio. In his fresco *The Holy Trinity*, for example, the three-dimensional perspective is uniformly presented in the relative sizes, colors, and outlines of all the objects.[8] Through the unified synthesis of these relativities, the *horizon* of the painting appears coherently in the background. Just as for Wittgenstein a picture does not depict but rather *displays* its form, the perspectivist painter does not paint the horizon but rather implies it; hence the horizon is an aspect of the form of mathematical language and not an object within it. Rather, in perspectivist paintings, things like buildings, rooms, houses, fountains, trees, and the like are the objects of the paintings, and the horizon appears by implication through the control of relativities among features of these objects.

The Projective Form Yet despite the successful implication of the horizon in the perspectivist painting, the *point of view*, while obviously implied, was, unlike the horizon, not directly attended to. Direct attention to the point of view from which the painting is constructed is the chief characteristic of the projective form of synthetic geometry. The implicit variant of this form that Kvasz cites is in fact a procedure for successfully generating a perspectivist painting, depicted by Albrecht Dürer circa 1500. According to this procedure, the painter would fix a transparent foil to a frame, situate the framed foil with respect to the painter so that the scene the painter wished to paint was directly on the opposite side of the foil from the painter's point of view, and then essentially *trace* the scene one dab of appropriately-colored paint at a time. Hence, this procedure directly concerned the point of view of the painter. But it was not until the work of Gérard Desargues, the founder of projective geometry, that the point of view found an explicit representation.

The novelty of Desargues' approach was to reformulate the problem of representation as a relation between two two-dimensional surfaces, rather than as a relation between the three-dimensional world itself and the two-dimensional foil upon which it was traced. This simplified approach first afforded him insight into the fact that, if an object is projected from one surface onto the second, then the point of view from which the two impressions coincide constitutes the *center of the projection*; hence, the point of view is explicitly formalized as a *point*, i.e., within the language. Moreover, he observed that if one surface is placed orthogonally to the second – so that the first perhaps represents the ground of a scene to be painted and the second represents the canvas upon which that ground is to be painted – then the horizon, as it appears on the second surface, corresponds to the far edge of the first surface "at infinity." Thus, the notion of infinity is explicitly formalized as the *line* that the horizon of the first surface makes when it is projected onto the second; this gave an explicit linguistic foundation for referring to infinity. Finally, by substituting the three-dimensional world with a two-dimensional surface, Desargues was able to begin to implicitly study transformations between the surfaces *independently* of their objective contents—i.e., transformations directly concerning the *background* of a painting, its 'spatiality', rather than objects within it.

For our purposes, the points to keep in mind here are that between the perspectivist and projective forms of geometry, elements like the horizon, the epistemic subject, and, in a way, the background, are all beginning to appear within language.

4 Relativization in Context: Algebra

Now, while the titles for these forms of mathematical language – "perspectivist" and "projective" – have more obvious geometric sense than algebraic, Kvasz nonetheless draws systematic analogies for them in the history of algebra. We will here have to provide a first-pass of interpretation to make some of Kvasz's language clear, and in the next section we will expand on these interpretations.

The Perspectivist Form In the work of Muhammad ibn Musa al-Khwarizmi (ca. 9th century), the language in which algebraic equations were analyzed and solutions were sought was rather different from today's. The language consisted entirely in what we might today call (relatively) *natural* language; the set of numbers they regarded explicitly only included *positive rationals*; and while algebraists could imagine that irrational and negative solutions to algebraic equations existed (including simple roots), they had designed no direct means of referencing them beyond the expression of the equations themselves.[9] Moreover, the idea of the same variable being raised to different degrees in the same equation was not explicitly parsed in the language; the variables raised to different degrees were described by entirely separate words, without any independent symbol standing for the common variable itself: e.g., $2x^3 + 5x = 6$ in today's notation was expressed (but in Arabic) as "two cube plus five things equals six."

Like the perspectivist form of synthetic geometry, the perspectivist form of algebra was concerned only with "real" representation. For one, al-Khwarizmi did not represent the general form of algebraic equations as such, but only interacted with specific algebraic equations with positive rational coefficients and natural exponents. And while al-Khwarizmi recognized the formal possibility of irrational and negative solutions to quadratic equations, he designed no direct handles for irrational quantities and did not perceive negative quantities as "real." For another, as in the expression of the cubic equation above, the variable of degree three is described by the word 'cube' (but in Arabic), corresponding to the dimensionality of objects in the *natural* world. In fact, in general the degrees of the variables were described in terms of a kind of base-three number system, corresponding to the dimensionality of space; and while al-Khwarizmi recognized that degrees higher than three were possible in principle, since they were higher than the degree of natural space he regarded them just as a formal novelty and did not spend much time seeking their solutions.[10]

The analogy with the perspectivist form of synthetic geometry breaks down a bit, however, when we try to identify anything analogous to a horizon here. Anticipating this, Kvasz writes: "When we transfer the notion of the form of language from geometry to algebra we discover that objects like 0 or 1 play a role in algebra similar to that played in geometry by notions like center of projection or horizon."[11] But it is not made clear how 0 or 1, at this stage in the development of algebra, are supposed to represent anything like a center of projection or horizon. Perhaps 0 can be regarded as *like* a horizon insofar as it does not refer to anything in the natural world; or perhaps 0 or 1 can be regarded as *like* centers of projection insofar as they are used to describe the limiting cases of algebraic equations—but Kvasz does not say any more about this than the above quote.[12]

The Projective Form As for the projective form of algebra, Kvasz draws the analogy with projective geometry along the lines of the sense in which Desargues' approach involved substituting a representation for the real world. Just as Desargues' insights were afforded by his re-framing the problem of projection as a relation between two surfaces (rather than between the world and a canvas), Kvasz argues that Cardano's insight into the formula for solving cubic equations was afforded by his exploring the *method of substitution*.

In the two centuries prior to Cardano's insight, a gradual process had unfolded in which the linguistic expression for the procedure of root extraction became increasingly compact. Kvasz cites the advents of Regiomontanus (mid 15th century) and Michael Stifel (early-mid 16th century) as most notable. The former was the first to *symbolically* represent root extraction, denoting the operation with the capital 'R', from the Latin 'Radix': e.g., 'R cubica de 8'. The latter subsequently replaced the capital 'R' with a lowercase 'r' and extended the upper-bar of the 'r' over the first letter of the Latin word for the degree of the root: e.g., '$\sqrt{c}8$'.[13] This gradual condensation of root extraction occurred alongside the symbolic reification of mathematical notions like equality ('='), addition and subtraction ('+', '−'), unknown coefficients ('a, b, c'), and unknown quantities ('x, y, z'),[14] in what we might call a *spirit* of symbolic condensation. Thus, despite the fact that Cardano's solution to cubic equations was indeed already *expressible* in the language of the perspectivist form of algebra, it was not *practically discoverable* until, in the spirit of symbolic condensation, the relevant mathematical notions had been sufficiently reified that he could imagine to substitute one equation into another.

So rather than attempting to directly solve particular cubic equations for the unknown, Cardano explicitly considered the general cubic *form*, and he substituted the unknown quantity with *another* expression with two *different* unknown quantities. In modern notation – of which, again, Cardano did not enjoy the use – for the unknown x, he supposed that $x = \sqrt[3]{u} - \sqrt[3]{v}$. Then, solving for u and v and substituting those solutions back into $x = \sqrt[3]{u} - \sqrt[3]{v}$, he obtained the general solution to cubic equations:

$$x = \sqrt[3]{\frac{c}{2} + \sqrt{\left(\frac{c}{2}\right)^2 + \left(\frac{b}{3}\right)^3}} - \sqrt[3]{-\frac{c}{2} + \sqrt{\left(\frac{c}{2}\right)^2 + \left(\frac{b}{3}\right)^3}}.$$

Hence, as Desargues substituted a representation of an object for the object itself and studied projections from that representation into another representation, Cardano substituted a representation of a quantity for the quantity itself and studied substitutions of that representation in another representation. In this sense, Kvasz argues, Cardano's method of substitution can be analogized as a kind of *projection*. Moreover, whereas al-Khwarizmi's techniques were only given as implicit rules for manipulating particular algebraic equations, Cardano's solution explicitly represents the general *form* of cubic equations and formalizes the solution as a *formula*. In this way the projective form explicitly handles the point of view from which the objects of the language (equations and their solutions) are observed: the rules themselves, rather than just particular equations and positive rational quantities.

5 Aspects and Functions of Form

As remarked in §2, Kvasz identifies eight forms of language in the history of mathematics. In §§3–4 we observed the first two of these forms in the histories of synthetic geometry and algebra. This should be sufficient to begin to make some sense of Kvasz's conclusions regarding relativization and aspects of the form of mathematical language.

After reconstructing the histories of synthetic geometry and algebra to observe all eight forms, Kvasz concludes that the general form of mathematical language comprises six interrelated aspects:

epistemic subject of the language,	horizon of the language,
individua of the language,	fundamental categories of the language,[15]
ideal objects of the language,	background of the language.[16]

Each successive relativization within a particular mathematical language corresponds to one of these aspects of the general form of mathematical language being made explicit. Moreover, this organization represents three functions of linguistic form (corresponding to the rows) which are each jointly founded upon two of the six aspects (corresponding to the columns). The function of the first two is to incorporate the epistemic subject into the world, the function of the second two is to structure the world from the point of view of the epistemic subject, and the function of the third two is to help the epistemic subject find orientation in the world.[17] Hence, the form of language comprises three functions, each founded upon two aspects, which all collectively serve to integrate the epistemic subject into the language. We have seen the ways in which (mostly) the first pair of aspects appeared in the transition from the perspectivist forms to the projective.[18]

We observed the analogy between the perspectivist form of synthetic geometry and algebra mostly in terms of their explicit concern with "natural" objects. The perspectivist form of synthetic geometry explicitly concerned objects of the natural world, and through certain implicit rules for controlling relative features among these objects the horizon was able to be clearly displayed; the point of view, or epistemic subject, while implicit, was not directly attended to in the language. The perspectivist form of algebra explicitly concerned positive rational solutions to particular algebraic equations, and through certain implicit analytical techniques the possible solvability of cubic equations was clearly displayed; the rules algebraists used for solving the equations, while implicit in their techniques, were not directly attended to in the language.

Seeing the analogy this way,[19] it seems we can clarify the problem with the analogy of the horizon mentioned in the previous section. The difficulty seems to be rooted in the difference between the semantic contents of synthetic geometry and algebra. The former refers to the visible world, the latter to an "invisible world"; in a perspectivist painting worldly objects are transferred onto a canvas, while in perspectivist algebra positive rational numbers are associated with algebraic equations.

As I see it, therefore, the "horizon" in perspectivist algebra might actually the *fact* that cubic equations are possibly solvable; this fact was displayed by the techniques for analyzing particular cubic equations, but it was not formally describable, i.e., not able to be depicted within the language. This seems plausible to me, but I can't find Kvasz saying anything to similar effect; Kvasz's only direct comments on the horizon of perspectivist algebra identify 0 and 1 as possible candidates. To me it seems that if the horizon of perspectivist geometry was implied through relativities among objects of the language, then the horizon of perspectivist algebra should likewise be implied through the mathematical relationships among the elements of algebraic equations. What was so implied was the modal fact *that cubic equations are possibly solvable.* This, I think, is a useful clarification, which goes to show that even where Kvasz's analysis of a particular historical language is not exact, it can be made exact.

Furthermore, the analogy between the projective form of synthetic geometry and algebra was more obviously observable, I think, in terms of the first two aspects of the form of language above. By re-imagining the relation between the language and its semantic content as a relation between two representations, both languages wound up producing *new* explicit content. In projective geometry, the center of projection, the horizon, and the (ideal) notion of infinity are referenced in explicit terms, and reference to the background is clearly implicit even if no terms directly refer to it. In projective algebra, notions like root extraction, addition, equality, and the like, in the spirit of symbolic condensation, were sufficiently reified for Cardano to consider equations in terms of their *form* and suppose that those equations could have roots in addition to positive rational quantities. Hence, Cardano could construct a *formula* through which the entire set of solutions to cubic equations is explicitly referenced, even though the domain of possible solutions had not yet been sufficiently expanded to include names for all those referents.

Both the geometric and algebraic variants of the projective form of mathematical language were therefore able to explicitly handle their respective horizons, and indeed only by virtue of *simultaneously* devising an explicit handle for their epistemic subject. In the geometric variant, the horizon is represented by a line and the epistemic subject is represented by a point; and in the algebraic variant, the horizon is represented in the solution to cubic equations and the epistemic subject is represented in formulae. This way of drawing the analogy seems clear and plausible to me, even if Kvasz doesn't exactly spell it out this way.

We can now get a partial grip on the sense in which the progression of relativizations in a particular historical thread of mathematical language corresponds to growing explicitness and complexity of its epistemic subject. In the transition from the perspectivist to the projective forms, the horizon and the epistemic subject obtain explicit reference, and thereby the epistemic subject is incorporated into the world of the language. In Tractarian terms, the epistemic subject transitions from being merely displayed to being also at least partly depicted. Moreover, once these two aspects of the form of language are explicit within the language, it actually becomes possible to observe a *set* of perspectives on the content of the language. In projective geometry, while the epistemic subject is only symbolized by a point, we nonetheless gain access to a field of points of view: wherever a possible projection would be made to coincide with its original constitutes a possible point of view. In projective algebra, as the epistemic subject is symbolized in formulae, we gain access to a field of points of view in the newly-determinable set of formulae that preserve the meanings of the newly-reified mathematical notions. These newly-explorable sets are the basis upon which the next aspects of the form of the language can begin to be studied implicitly, before eventually also being made explicit.

Going a bit beyond our reconstructions, we can begin to imagine how, through the subsequent relativizations – i.e. through the subsequent forms of language – the world can begin to be structured from the point of view of the epistemic subject, and then ultimately the epistemic subject can become increasingly oriented in that world. That is, we can see how relativization is a process which is *oriented around the epistemic subject*. Moreover, we can imagine how, as the epistemic subject of the language achieves greater explicitness, the content of the language actually grows and changes: tokens are designed to stand for implicit aspects of the form of language, and the semantic content is thereby transformed.

6 Wittgenstein and Formal Ontology

With the foregoing before us, I now want to direct our attention to the peculiar ontological status of the aspects of the form of language. It was important to observe a bit of relativization in situ in order to gain a sense of how the aspects of the general form of mathematical language function in Kvasz's account. In the rest of this essay, the ontological status of the aspects of the general form of mathematical language is our main theme.

In a Tractarian vein, Kvasz claims that they are purely "formal, i.e., they have no factual meaning."[20] There is just one passage in which he says more about what he means by this:

> The aspects of the form of language, which constitute the identity, situatedness,

> individuality, similarity, orientation, and transparency, are not factual. The subject does not belong to the world (*Tractatus* 5.632: *The subject does not belong to the world: rather, it is a limit of the world.*), the horizon does not denote anything factual, there are no fixed individua, space does not really exist, not to speak about the ideal elements. But this is exactly the reason why the different aspects of the form of language are an ideal tool for the description of relativizations. Those elements of language that refer, in a direct or indirect way, are bound by the relation of reference and therefore they cannot change with sufficient flexibility. On the other hand the aspects of the form of language, precisely because they do not refer to anything in the world, are free. They are bound only by relations among each other. Therefore in cases when the development of the discipline requires a change, the aspects of the form of language offer sufficient space for innovation.[21]

The main point of this passage is indeed rather fascinating: since the six aspects of the form of language are so unconstrained by external reference, they are precisely what afford the changes necessary for this kind of evolution of mathematical languages—the only constraints within which they function are their interrelations among each other, but otherwise they are "free" to accommodate a wide variety of nominal changes. However, there is a sense in which this is puzzling. For Kvasz, as we have seen, relativization occurs precisely through explicitly incorporating the aspects of the general form of mathematical language into language. If there is nothing actually being incorporated into the language; or, if whatever is being incorporated does not have factual significance; or, if the aspects of the general form of mathematical language do not exist: then how could any of this business possibly contribute to the *evolution* of the language? Or, how could it even contribute to *change* in language? This obviously deserves further analysis.

The most obvious place to look next is deeper in Wittgenstein's picture theory of meaning. The Tractarian distinction between what a language can depict and what it can only display seems to get us somewhere: perhaps only what a language depicts has factual significance, whereas what it displays (form, the subject) does not. Indeed, for Wittgenstein, facts are the contents of linguistic pictures (i.e., what is depicted),[22] and facts are the existence of states of affairs[23]—so whatever a language depicts has factual significance and exists. And language cannot depict its pictorial form but only displays it, for a picture cannot place itself outside its pictorial form[24]—so the form of language does not have factual significance.

This much is clear enough, but it exposes a puzzle. If the form of language cannot be depicted, then how does Kvasz so successfully depict it in his analysis? And how do relativizing mathematical languages come to depict the aspects of their general form? That is, if the form of language, by virtue of not being able to be depicted, has no factual significance, then (a) how can Kvasz's analysis appear cogent, much less plausible, and (b) how can mathematical languages come to depict aspects of their general form in subsequent, more evolved languages?

There is a possible Tractarian answer in the vicinity of the above-mentioned propositions: in that a picture represents its content from a position outside it, and in that a picture cannot place itself outside its pictorial form.[25] Implicit in these remarks is the possibility that, while it is indeed impossible for a picture to depict its own pictorial form, a picture *can* depict *another* picture's pictorial form. These remarks do not appear to bar the possibility of aspects of form functioning as objects in separate, higher-order pictures—such as in a *meta-discourse*, either mathematical (as in a successive form of mathematical language) or philosophical (as in Kvasz's analysis). As far as I can tell this is a consistent interpretation of Wittgenstein's remarks, which bodes well for Kvasz's use of him.

Plausible though this is, it presents us with yet another puzzle. If the aspects of the form of language function as objects in higher-order pictures, and if pictures are facts, then this would seem to entail that the aspects of the form of language have some kind of factual significance after all. That is, on simply Tractarian grounds, Kvasz's claim that the aspects of the general form of mathematical language have no factual significance is simply false.

This could be fine for Wittgenstein, I think, *but for the fact that* Kvasz regards relativization as reification of the aspects of form, which center around the explication of epistemic subjectivity in particular. For Wittgenstein, on the other hand, the subject, as with general form, is the "limit of the world," and therefore cannot actually be depicted within language. For Wittgenstein, while we can indeed write sentences of the form, "A says p," and so on – in which something like an epistemic subject appears to be related to an object as another (kind of) object; e.g., "A is integrated into L" – these sentences really just express propositions of the form "'p' says that p,"[26], or "L is integrated into L." For Wittgenstein, there is no such *thing* as the subject; the subject is not an object to be related to other objects in facts. Thus, for Wittgenstein, the evolution of mathematical languages cannot be characterized in terms of a subject being made explicit. For Wittgenstein all this would really cash out to is that mathematical languages evolve as "what is made explicit is made explicit"—and this is far short of what Kvasz is actually saying.

In order to address this puzzle, it appears that Kvasz would have to make a distinction between two kinds of epistemic subjectivity: there is the Wittgensteinian subject, which, like the form of language, has no factual significance within itself, and resides only at the "limit of the world"; and then there is another kind of subject, which, like the form of a particular language, which is accessible from outside – from the vantage of another language – is actually accessible to factual discourse. Hence, just as there must be another language *outside of* a particular language in order to describe the form of that language, there must be another epistemic subject *outside of* a particular epistemic subject to describe that subject.

In the rest of this essay, then, two bugs in Kvasz's use of Wittgenstein's picture theory of meaning will be analyzed, in terms of the following two questions: (1) How can we understand the ontological status of the aspects of the general form of mathematical language; and (2) How can we understand the unique role of epistemic subjectivity in Kvasz's account? In (2) we focus on epistemic subjectivity not simply as an exemplar from within the set of aspects of the general form of mathematical language, but because it appears to have a unique role, as the center point of all the functions the aspects enable. The rest of this essay may be observed as pointing in the direction of answers to each of these questions.

7 Epistemic Subjectivities and Beyond

The puzzle, as we now see it, regards how to resolve the tension between the following two notions:

1. On the one hand, according to Kvasz's account of relativization, the evolution of mathematical languages is partly a function of the epistemic subjectivity implicit in mathematical languages being made increasingly explicit in subsequent "meta-languages." There is, at one moment, an implicit subject, at the limit of the language, which then, at the next moment, becomes explicit.
2. On the other hand, according to the *Tractatus*, while we have indeed drawn out a sense in which Wittgenstein might assent to the possibility of factual

> meta-discourse regarding "forms" of language, Wittgenstein is clearly opposed to the possibility of the subject of language being depicted within language. The subject is strictly a formal feature of language, which cannot be factually described within language.

Can this tension be resolved? I think it can, partly on the same ground that we saw the possibility of factual meta-discourse, and partly on entirely new ground.

The first part of the solution is to posit two distinctions, as briefly indicated above. First, we have to observe a distinction between (a) an epistemic subjectivity associated with (the form of) a particular mathematical language, and (b) a "meta-subjectivity" associated with (the form of) a particular meta-discourse which is used to factually describe (a). Second, we have to observe an analogous but separate distinction between, again, (a) an epistemic subjectivity associated with (the form of) a particular mathematical language, and (b*) a "meta-subjectivity" associated with (the form of) a more evolved language that supersedes and factually refers to (a). Thus, schematically, we separate two kinds of "meta-subjectivity", (b) and (b*): philosophical meta-subjectivity, as implicit in analytical meta-discourse such as Kvasz's overall project, and mathematical meta-subjectivity, as implicit in subsequent mathematical languages which take prior (features of) epistemic subjectivities as mathematical objects. At any rate, in either case, we must distinguish between two languages, one of which refers to the other, and both of which involve distinct epistemic subjectivities associated with their own respective semantic contents, of a higher order than the contents of the languages they take as objects. And from the vantage of the meta-language – whether philosophical or mathematical – the epistemic subjectivity of the language it describes is considered objectively, as an object which exists in states of affairs, i.e., factually.[27]

This much appears clear and plausible, and I think goes a long way toward clarifying the framework within which Kvasz's analysis is playing out. We see, then, that there is actually no way to make sense of certain features of epistemic subjectivity *except* as objective existents, *in* the world that the discourses are taken to describe. In this sense, we do actually have grounds to determine the truth-values of expressions in the meta-discourses, because they just describe the world, which includes an epistemic subject *qua* object; the world of the meta-language includes subjectivity as constituent. Thus, in one sense at least, epistemic subjectivity – contrary to Kvasz's statements quoted above – *does* indeed exist. He just has to be wrong about this.

Alas! however plausible this clarification is, we can now begin to observe the same limit that, I think, compelled the early Wittgenstein to prohibit reference to the epistemic subject of language in the first place. Let's grant that the foregoing story makes of analyses of the histories and evolution of *particular* mathematical languages, with reference to *particular* respective philosophical and mathematical meta-discourses. More precisely, let's grant that this story makes sense when reflecting on *past* languages. Nevertheless, things get tricky when we try to understand the explication of epistemic subjectivity as something that is happening in real-time.

We can think of it this way. As we have been saying, we can observe an explicit epistemic subjectivity (a) of a language L from the vantage of a meta-language ML, with which we *ideally associate* a corresponding (b) or (b*), according to Kvasz's account of relativization. Schematically, the analogy is:

$$\text{(a)} : L \quad :: \quad \text{(b)/(b*)} : ML \qquad \text{(and so on).}$$

What I want to get at is the nature of this "ideal association" between the meta-language and its corresponding meta-subjectivity. Notice that this schema involves a peculiar presupposition: Whatever language we are currently using, it succeeds *ML* enough that we can treat *ML*'s subjectivity explicitly. In other words, when the meta-language itself is also something upon which we are reflecting – i.e., when it is *also* a language in the past, beyond which we have evolved – it is easy enough to observe the corresponding (b) or (b*) as implicit. The reason for this is that *we are already profiting from the subsequent meta-meta-languages which took the preceding epistemic subjectivities of the preceding meta-languages as objects.*

This is really an abstract point, but it's critical to understand, so let's dwell on it for a moment. When we look far enough into the past, we are often already perched from within an advanced enough language that we can achieve reference to *a bunch* of different things, in an order:

1. some initial object-language *L*, and
2. its epistemic subjectivity (a);
3. some meta-language *ML*, and
4. its epistemic subjectivity (b) or (b*); *plus*
5. some meta-meta-language *MML*, and *possibly*
6. its epistemic subjectivity (c) or (c*);
7. and possibly so on, depending on how far back we are looking and how much our language has evolved.

When looking this far back, so that we see not just an initial object language but a *number* of successive meta-languages, we tend to confuse an important aspect of this process of succession: *a language emerges before its epistemic subjectivity can be referred to.* Hence, in the transition from step 5 to step 6, I write, "and *possibly*. . . ."

We can note here the similarity between this point and Kant's edict that "The *I think* must *be able* to accompany all my representations."[28] According to Kant, it is not that the *I think* in fact always does accompany my representations, but that it can be made to accompany my representations in various peculiar sorts of synthetic a priori judgments involving them. That is, for Kant, it is not exactly that the *I think* is *implicit* in my representations, but that it can in principle always be invoked in synthetic a priori judgments involving those representations.

What does this mean for Kvasz's account of relativization? It means, basically, that we have to take a step back, or forward, or at any rate *beyond* this talk of epistemic subjectivity. We have to ask: What is happening when, as Kvasz would say, an epistemic subjectivity is made explicit? As we are seeing, in real-time we cannot appeal to an epistemic subjectivity as in any way *functioning* in the explication of epistemic subjectivities. There is something going on prior to judgments pertaining to the epistemic subject which makes them possible in the first place. Hence, our question has become a bit more Kantian: What are the conditions for the possibility of explicating epistemic subjectivity?

Another way to understand our problem is that, in the present moment, there is no such *thing* as a "purely implicit" epistemic subjectivity to be made explicit. As we saw above, what exists in the present moment is only an object-language of its own order (a highly advanced meta-language) which in principle cannot appeal to such a thing as its own "purely implicit subjectivity" without presupposing a *pretend future point of view.* Therefore, in real-time, we cannot factually describe our current state of affairs, and so the notion that "our current language can evolve as we make ourselves explicit" remains incoherent. It is this notion, in quotes, that we must now begin to make sense of.

8 Epistemic Subjectivity: To Be or Not To Be?

Indeed, for Kvasz, the non-existence of the epistemic subject is a central *feature* of epistemic subjectivity, and not a bug, as he writes in the focal passage from §6:

> the aspects of the form of language, precisely because they do not refer to anything in the world, are free. They are bound only by relations among each other. Therefore in cases when the development of the discipline requires a change, the aspects of the form of language offer sufficient space for innovation.[29]

But it appears that Kvasz's remarks here jump the gun on explaining what we might loosely call the *mechanics* of this part of the process of relativization. It appears that he is so quick to get to the (indeed intriguing) point that the non-existence of the epistemic subject explains how mathematical language can change and evolve that *he fails to explain how such a non-existent 'thing' could play such a role in the first place*! We are left without an ontology capable of making sense of this possibility. Indeed, on our interpretation of the *Tractatus* we can get something about the qualified existence of epistemic subjectivities of past languages back; but in real-time, insofar as epistemic subjectivity, *qua* aspect of general form, is supposed to somehow be affording the freedom required for innovation, it does not appear that we can refer to it as an object in a meaningful relation. That is, we cannot attribute the property "affords (part of) the freedom required for change" to "the real-time epistemic subject of a language" in a meaningful sense, for the latter does not refer to anything.

The explanatory gap remaining here is reminiscent of an old joke:

> A: "Did you see the fish the one-armed fisherman caught?"
> B: "No."

A: "It was *this* long." (holding out one arm)

How could something that does not exist in the first place function so centrally in an account of how mathematical languages can evolve? How can something that does not exist have such a central role in relativization? This same question applies to the rest of the six aspects of form: if they don't presently exist, then how can they possibly function, how can they presently afford the possibility of innovation? Our problem is, without some positive sense of the *arguments* of a function, we are left with as good as pure fiction. This is not to say that the function is pure fiction, but only that we need a positive sense of its arguments for it to be significant.

On our interpretation of the *Tractatus*, we can get so far with Kvasz: from the vantage of an evolved language, we can look back at the lesser languages which led up to our present and say that they were pregnant with features of our current language all along. We can also say, for example, that the solution to cubic equations was somehow already implicit in al-Khwarizmi's methodology for solving algebraic equations, even if it had not yet been seen in al-Khwarizmi's time just the way that Cardano eventually came to see it. We can even, indeed plausibly, say that there is an epistemic subjectivity implicit in al-Khwarizmi's methodology that became explicit in Cardano's, whose explication *afforded* the solution to cubic equations; and we can even say, moreover, that there was an epistemic subjectivity implicit in Cardano's methodology which also eventually became increasingly explicit throughout the subsequent forms of algebra, as relativization unfolded, whose explication afforded solutions to many other types of algebraic problems. We can *even* say, with Kvasz, that these so-explicated epistemic subjectivities did not properly exist prior to their explication in the meta-languages which operated over them, which took them as arguments of their functions.—And all this makes a fair amount of intuitive sense, up to a point.

But when we try to think it through the point of view of the the working mathematician, in real-time – or, rather, when we try to read this pattern into our present moment, with respect to *me*, the working mathematician, in real-time – it does not appear quite right to say that I do not yet exist—*especially* if I, *qua* epistemic subject, am somehow supposed to be (part of) the seat of the affordance of the possibility of linguistic change and innovation. Semantics aside, I obviously exist enough to be part of this. Our problem now is, how should we understand this part? What is the relationship between epistemic subjectivity and me, the *I*?

The situation we are in, then, is this: We can follow Kvasz insofar as we restrict ourselves to a particular mathematical language or meta-discourse about particular mathematical language, i.e., insofar as we observe previous languages as being pregnant with features (such as the epistemic subjectivity) of subsequent languages. In this sort of case, we are always speaking from within a meta-language in which the epistemic subjectivities of previous languages are considered entirely as objects, i.e. as argument in the functions of relativization—and this is fine, except for the (relatively minor) fact that it entails that epistemic subjectivity, and the other aspects of form, do have some qualified sort of existence. *Even so*, we cannot assert the pattern of relativization as a general rule, which can be construed to apply to the present; alternatively, and equivalently, we cannot understand relativization as it unfolded in real-time from the point of view of working mathematicians of the past. Somehow, here, the ones whom we would typically call innovators are hardly called upon in order to explain the process of relativization.

9 A Preliminary Consultation with Kant

Preliminarily, we can make a Kantian diagnosis of Kvasz's error. Although Kvasz's only remarks on Kant's critical philosophy pertain to Kant's philosophy of mathematics, in a major sense, Kvasz's analysis of relativization, insofar as it involves appeal to an epistemic subjectivity, appears to try to keep with Kant's conclusions regarding the paralogisms of pure reason.

Kant's problem in the paralogisms is that there is a tendency to misjudge the nature and structure of pure reason. According to Kant, some of the most common misjudgments regarding pure reason posit the existence of a *transcendental being* – or, we might rather say, a transcendental subject – which is ontologically simple and persistent through time. The first three paralogisms are the most relevant for our purposes, and the general concept is the same in each. Kant traces these arguments all explicitly back to Descartes' metaphysics, but they are also implicit in the metaphysics of Spinoza and Leibniz and others.

The first paralogism diagnoses the error in the argument leading to the judgment that the pure subject of experience exists *as substance*. Along the same lines, the second paralogism diagnoses the error in the argument leading to the judgment that the pure subject of experience is ontologically simple. The third diagnoses the error in the argument that the pure subject of experience is persistent through time. The arguments go:

> What cannot be thought otherwise than as [subject / unitary / identical across time] does not exist otherwise than as [subject / unitary / identical across time], and is therefore [substance / simple / persistent].
>
> Now a thinking being, considered merely as such, cannot be thought otherwise than as [subject / unitary / identical across time].

> Therefore it also exists only as [subject / unitary / identical across time], i.e., as [substance / simple / persistent].[30]

The error, according to Kant, is that whereas in the major premise, the term "thought," in the clause "what cannot be *thought* otherwise than as [subject / unitary / identical across time]," is taken to operate over things that can be *both* conceived *and* intuited, in the minor premise it is taken to operate over things which can *only* be conceived and cannot be intuited. The conclusion, therefore, in each case, mistakenly judges something that is only conceivable as something which also intuitable.

On the one hand, we can see Kvasz trying to keep with Kant in some sense by denying epistemic subjectivity a place in the intuitable world: Kvasz wants the function of the epistemic subject not to entail its factual existence; he is not bound to the ontological simplicity of the epistemic subject, and is comfortable with it being understood (relationally) in terms of other aspects of the form of *particular* languages in which it dwells; and he is even comfortable with the epistemic subjectivities of subsequent mathematical languages being different from the epistemic subjectivities of prior ones, perhaps just in virtue of the differences in the particular languages in which they dwell. But on the other hand, the way in which we have exposed Kvasz's implicit account of "real-time relativization" bears some concerning similarities with the transcendental illusions that Kant's critique is in the business of clarifying. That is, if we try to read epistemic subjectivity into the present moment of the working, relativizing, mathematician, then we appear to commit ourselves to a transcendental *I* that exists, is simple, and persists through time.

In some places we can see that Kvasz's arguments are empirical, suggesting that epistemic subjectivity is somehow to be located purely within the form of mathematical language, and purely determined by that language bottom-up; and yet insofar as relativization is understood as unfolding in real-time, he wants to be transcendental, suggesting that some part of epistemic subjectivity – and, indeed, the other aspects of form – over and above the particular mathematical languages is supposed to afford the possibility formal change and innovation within those languages. But these two impulses, taken together, present us immediately with three problems, in order of increasing difficulty:

(1) They are simply incompatible *prima facie*: an epistemic subjectivity which is *reducible* to the form(s) of the particular mathematical languages in which it dwells cannot carry any *additional* properties which do not also so dwell; i.e., reducibility does not permit anything to hang "over and above": If X reduces to Y, then nothing about X cannot be explained in terms of Y. This would mean that whatever Kvasz means by the aspects of the general form of mathematical language, they cannot be anything different from a particular mathematical language itself.

(2) If the epistemic subjectivity that is reducible to the form(s) of the particular mathematical languages upon which it supervenes is the same epistemic subjectivity that is supposed to actually afford the possibility of linguistic change and innovation, then we are right back where we started, having gained nothing: as Wittgenstein would have put it, mathematical languages would just then be said to evolve as "what is made explicit is made explicit"—and this is far short of what Kvasz is trying to say regarding epistemic subjectivity.[31]

(3) Finally, and perhaps most devastatingly from the Kantian point of view, by calling one thing that's supposed to be empirically grounded (intuitable) and

> another thing that's supposed to be a transcendental ground (pure concept) by the same name, "epistemic subjectivity," Kvasz is accidentally exposing his account to Kant's critique of the paralogisms: he is taking something that in one sense is not intuitable but only conceivable, the transcendental ground, to be also intuitable, empirical ground. Without some explicit safeguard against this confusion, we are liable to infer that purely conceptual phenomena entail corresponding physical phenomena, or to make otherwise paralogical inferences.

Without immediately addressing these problems – for they will become largely thematic in the rest of this essay – we can for the moment conclude in their lights that, at a minimum, Kvasz's account of the real-time role of epistemic subjectivity in relativization ought to be sensitive to a peculiar kind of distinction, i.e., between empirically accessible subjectivities that dwell in the different forms of mathematical language, and the transcendental possibility of the novel explication of epistemic subjectivities. But how can we salvage the content of Kvasz's remarks in light of this distinction?

The key to the way forward, I think, is in the work that Kvasz is invoking this "purely implicit" epistemic subjectivity to do, or, in the *role* it is supposed to play, in real-time relativization. As he remarks in the focal passage, what is really important about this presently non-existent, purely implicit epistemic subjectivity is that *something about it* – a kind of *freedom* or *flexibility* – *affords the possibility of formal change and innovation.* That is the real role of the purely posited epistemic subjectivity: it is supposed to be (part of) that by virtue of which formal linguistic change is possible; it is a necessary precondition for the possibility of formal linguistic innovation. But exactly how should this role be understood, especially in connection with the merely supervenient subjectivities?

10 Interlude: Husserl's Place in Kvasz's Project

Unfortunately, there is no explicit path from Kvasz's account of relativization back to Kant's critical philosophy – except for Kvasz's passing comments on Kant's philosophy of mathematics – so while we can make this preliminary diagnosis in Kantian terms, it would not be easy to follow it through for treatment; and in any case doing so would probably end up either transforming Kvasz's linguistic theory into a thoroughly transcendental one, or else Kant's transcendental theory into a linguistic one. So while a Kantian path is indeed possibly available to us – and could prove immensely fruitful – I see another, more direct path available that would not necessarily entail any radical transformation in the *linguistic* spirit of Kvasz's project. It will take some time to set up, but later we will return to parts of the foregoing Kantian diagnosis, albeit in a rather different set of idioms.

In an important sense, Kvasz's account of relativization can be explicitly traced to his understanding of Edmund Husserl's method of historical analysis, particularly as delineated in the latter's *The Crisis of European Sciences and Transcendental Phenomenology.* Although Kvasz's book – of which we are presently in an analysis – hardly mentions Husserl at all, much less to significant effect, the project can nonetheless rightly be interpreted partly as an application of Husserl's methodology to the history of mathematical languages. His debt to Husserl is clarified in his paper, "Galileo, Descartes, and Newton – Founders of the Language of Physics," an earlier draft of which was originally supposed to be included as a chapter of his book, but was excluded due to its pertaining less obviously to mathematics than to physics.[32]

While it is beyond our purview to reconstruct the specifics of that paper in detail here, it will be useful to outline some of its content so that we can begin to observe part of Husserl's place in Kvasz's project.

In the aforementioned paper, Kvasz examines the scientific revolution of the 17th century as it unfolded through the work of Galileo, Descartes, and Newton. He begins from Husserl's interpretation of Galileo's significance in the history of physics—one of the main themes of *Crisis*. According to Kvasz, "Husserl's achievement was the insight that the subject matter of physics is formed by *intentional objects* that are constituted in the process of *idealization*, in which some aspect of the lifeworld is *replaced by a mathematical ideality*."[33] We will explain more precisely what this means in due course, but for now we can just note that for Kvasz, by way of Husserl, idealization is the process that effects the replacement of our ordinary sense of the world with exact concepts.

According to Husserl, it was Galileo who originally effected this replacement with respect to nature as a whole; for as he puts it in *Crisis*, "through Galileo's *mathematization of nature*, *nature itself* is idealized under the guidance of the new mathematics; nature itself becomes ... a mathematical manifold."[34] In other words, Husserl attributes the original metaphysical position that nature *just is* mathematical to Galileo. Yet despite the import of Husserl's insight into the hidden hypothetical nature of Galileo's position, Kvasz argues that "a fuller understanding of idealization requires complementing the analysis of Galileo's works by a similar analysis of the works of Descartes and Newton. Only these three layers of language—the Galilean, the Cartesian, and the Newtonian—taken together constitute the intentional objects of physics."[35] Thus in his analyses of the linguistic innovations of each of these figures – both in the course of their own life's work as well as in connection with their predecessors – Kvasz supplements Husserl's account of idealization by identifying two additional phases of linguistic development that are *preconditions* for what Husserl calls idealization. These phases he calls "objectivization" and "representation." Most importantly for our analysis, in *Patterns of Change*, Kvasz renames these preconditions for idealization, respectively, *relativization* and *recoding*.[36] In other words, Kvasz views relativization and recoding as preliminary phases of what Husserl calls idealization, and the two former should be understood as filling in details of the latter. That is, for Kvasz, Husserl's account of idealization should be understood as a broad-stroke depiction of a process which in fact presupposes relativization and recoding.

Hence, insofar as Kvasz's account of relativization (and recoding) will have to be consistent with Husserl's account of idealization, we can look to Husserl's account of idealization for cues as to how to understand Kvasz's account of relativization, in particular epistemic subjectivity.

11 Abstraction via (Co-)Idealization

Yet even as Kvasz's accounts of relativization and recoding are intended to fill in missing details of Husserl's account of idealization, features of Husserl's account of idealization were nevertheless sufficiently exact to be appropriated by Kvasz without much essential modification. We will now introduce one of the larger of these features that Kvasz treats explicitly in his paper on Galileo, Descartes, and Newton: the relationship between *abstraction* and *idealization*. This will set us up to observe the role language plays in idealization in the next section.

According to Kvasz, abstraction and idealization are often confused because of the prevalence of the Platonist assumption that the cognition of perfect geometric objects is fundamentally independent of the perception of imperfect material objects. When we thus presuppose a stark enough difference *in kind* between ideal mathematical objects and their worldly correlates, it is easy to say that the ideal correlates are just what remain when we abstract away all worldly accidents; those ideal correlates are the "metaphysical remainder" of this act of abstraction. However, Kvasz argues, this obfuscates the sense in which we actually have to *constitute* our relation to *new* idealities *prior* to our having any real knowledge of the fact that they might "remain" after certain worldly accidents are abstracted away. When we think about the 'cutting edge' of our sense of the realm of perfect mathematical idealities, we have to ask ourselves: How do we actually *get at* the stuff over the edge? To answer "by abstraction" is to gloss over a lot of constitutive activity on our part, which essentially *founds* our sense of the new idealities. Hence, for Kvasz, "idealization" is the name for this process by which the two kinds of contents separated in abstraction are constituted in the first place, prior to abstraction. As Kvasz writes, "abstraction presupposes idealization, therefore it cannot explain it."[37] Thus, abstraction also presupposes relativization and recoding, just insofar as idealization presupposes them.[38]

I will here submit that this characterization of the difference between idealization and abstraction can be traced directly back to at least one specific passage in Husserl's *Crisis*, with which we know Kvasz is familiar.[39] In the context of examining Galileo's assumption that the "intuited plena" of sense-qualities are in principle mathematizable (and hence mathematical *a priori*), Husserl writes:

> In every application to intuitively given nature, pure mathematics must give up its abstraction from the intuited plenum, whereas it leaves intact what is idealized in the shapes (spatial shapes, duration, motion, deformation). But

> in one respect this involves the performance of coidealization of the sensible plena belonging to the shapes. The extensive and intensive infinity which was substructed through the idealization of the sensible appearances, going beyond all possibilities of actual intuition—separability and divisibility *in infinitum*, and thus everything belonging to the mathematical continuum—implies a substruction of infinities for the *plenum*-qualities which themselves are *eo ipso* co-substructed. The whole concrete world of bodies is thus charged with infinities not only of shape but also of plena.[40]

The relevant part of this passage for our purposes is Husserl's account of the relationship between "what is idealized in the shapes" of pure geometry and the "*coidealization* of the sensible plena belonging to the shapes" in the intuitively given world. According to Husserl, these idealizations occur *concurrently.* As Kvasz argued above, it is not as if we have an abstract realm of idealities available to us prior to some constitutive activity on our part. We do not, as it were, discover what is abstracted from, the sensible plenum, by reference to a pre-existing "abstracted to," the ideal objects; neither should be understood as more primary than the other. Rather, the constitution of the ideal objects to and from which we may eventually abstract is part of a process in which the sensible plena to and from which we may eventually abstract are simultaneously (co-)idealized. The two idealizations – that of the abstracta and that of the sensible plena – are mutually determined in an act of co-idealization.

In this we can see how Husserl would agree with Kvasz that "abstraction presupposes idealization": abstraction does not just presuppose the idealization of the mathematical shapes to which we abstract, but also presupposes the idealization of the sensible plena which are abstracted away. These are concurrent parts of the same process of idealization that founds the possibility of abstraction. And not only is idealization the idealization of the mathematical objects, for this would be equally "Realist" with respect to the material world. Indeed, the radicality of Husserl's approach is that it does not take the Ideal *or* the Material realm to be prior to the other, but takes them to be co-constituted in the same act of co-idealization.

12 Idealization via Linguistic Sedimentation

Moreover, as for Kvasz, for Husserl this process of co-idealization is fundamentally a *linguistic* process. As we remarked above, insofar as Kvasz argues that idealization is founded on acts of relativization and recoding, abstraction is equally founded on these acts—*linguistic* acts. As Kvasz writes:

> what has to be preserved in the process of abstraction is not arbitrary. The abstract object must fit into the linguistic framework, in our example into the linguistic framework of geometry or of physics. ***Abstraction is a linguistic reduction***; it is the replacing of reality by its linguistic representation. The syntax of the language leads us in the process of abstraction—it determines which properties can be neglected and which not.[41]

Hence, for Kvasz, abstraction is always relative to linguistic foundations, which determine precisely which aspects of an object are distinguished ("coidealized") and which are then subtracted ("abstracted"). Formulaically, we might say the abstract object equals the real object minus the sensible object; or inversely, the sensible object equals the real object minus the abstract object—where the "real object" is taken to be the cognitive-perceptual unity prior to the coidealization of its cognitive and perceptual parts.[42]

Within *Crisis*, two passages in particular support the claim that Husserl would have accepted Kvasz's account of the fundamental function of overt language – i.e., of both symbolic and iconic tokens – in the process of idealization. One comes from Husserl's chapter on "Galileo's mathematization of nature,"[43] the other from the appendix on "The Origin of Geometry."[44] Both of these chapters trace the process whereby the abstract realm of pure (synthetic) geometric shapes was idealized and divorced from its sensible real-objective counterpart. His concern is with just how it was possible for "perfect" geometric "limit-shapes" to obtain a *sui generis* status independent of their imperfect foundations.

In the former chapter, regarding these perfect limit-shapes, Husserl writes:

> Like all cultural acquisitions which arise out of human accomplishment, they remain objectively knowable and available without requiring that the formulation of their meaning be repeatedly and explicitly renewed. On the basis of sensible embodiment, e.g., in speech and writing, they are simply apperceptively grasped and dealt with in our operations. Sensible "models" function in a similar way, including especially the drawings on paper which are constantly used during work, printed drawings in textbooks for those who learn by reading, and the like. It is similar to the way in which certain cultural objects (tongs, drills, etc.) are understood, simply "seen," with their specifically cultural properties, without any renewed process of making intuitive what gave such properties their true meaning. Serving in the methodical praxis of mathematicians, in this form of long-understood acquisitions, are significations which are, so to speak, sedimented in their embodiments. And thus they make mental manipulation possible in the geometrical world of ideal objects.[45]

Hence, Husserl is here in agreement with Kvasz that when the imperfect sensible objective embodiments of the perfect geometric shapes are abstracted away, it is only by virtue of these objective embodiments being *replaced* with another kind of sensible embodiment, "e.g., in speech and writing." The need for the sensible embodiment does not go away; rather, the need for the real-objective embodiment is transcended by a need for a *linguistic* embodiment, and the linguistic embodiment supplants the real-objective one to found a new kind of ideal activity.

Husserl's account here also goes a bit beyond Kvasz's in its reference to the process of "sedimentation," which is based on Husserl's theory of memory and retention.[46]

This theory is basically (but not entirely) Humean in that a sufficient number of experienced associations of a sufficient number of discrete linguistic tokens with certain relevant real objects enables the mind to engage with the tokens themselves in place of the aspects of real-objects they are take to represent in practice. The semantic content of the real objects becomes co-sedimented in the associated linguistic tokens—or, as we will soon say, (re-)founded upon them. The language therefore founds our reasoning about the objects whose sense it has absorbed, and all kinds of linguistic variations afford subtler idealizations and hence higher abstractions.

In the latter chapter of *Crisis*, Husserl adds to this by distinguishing a unique feature of the sedimentation of significances in linguistic tokens from the sedimentation of significances in real objects. Regarding the "ideal objectivity" of the geometric limit-shapes, he writes:

> Works of this class do not, like tools (hammers, pliers) or like architectural and other such products, have a repeatability in many like exemplars. The Pythagorean theorem, [indeed] all of geometry, exists only once, no matter how often or even in what language it may be expressed. It is identically the same in the "original language" of Euclid and in all "translations"; and within each language it is again the same, no matter how many times it has been sensibly uttered, from the original expression and writing-down to the innumerable oral utterances or written and other documentations. The sensible utterances have spatiotemporal individuation in the world like all corporeal occurrences, like everything embodied in bodies as such; but this is not true of the spiritual form itself, which is called an "ideal object". In a certain way ideal objects do exist objectively in the world, but it is only in virtue of these two-leveled repetitions and ultimately in virtue of sensibly embodying repetitions. For language itself, in all its particularizations (words, sentences, speeches), is, as can easily be seen from the grammatical point of view, thoroughly made up of ideal objects.[47]

Hence, what distinguishes the ideal objects that are founded on language from the significances of tools is that they are essentially *independent* of the particular sensible embodiments upon which they are founded. Whereas the significance of, e.g., a particular hammer is essentially dependent on the hammer, the ideal objects of language, even though they are indeed dependent on linguistic foundations *in general*, are independent of any *particular* linguistic foundation. In the process of the semantic contents of individual objects being sedimented in linguistic foundations, these contents thereby become *free* of their original, particular foundation in the object.

So then, simply, the broad structure of the process of idealization is this: In the first instance, there is a phenomenon; for example, a mathematical phenomenon, or rather, a phenomenon among phenomena, which might show up in the world or in our perception of the world, but always among other attending phenomena.[48] Next, there have to be more experiences of that phenomenon and its attending phenomena, i.e., enough experiences that it becomes sedimented *qua* phenomenon in our consciousness. Once sedimented *qua* phenomenon, we may begin to idealize the phenomenon by establishing a linguistic token, and indeed a *set* of linguistic tokens, with which to associate and differentiate it from its attending phenomena. And then, and only then, once the cluster of idealizations has been founded upon its respective linguistic foundations, i.e., in words, we may *subtractively* abstract the one phenomenon from its attending co-idealized phenomena. That is, we may then, and only then, refer to, e.g., perfect triangles as the *residuum* after subtracting out all the attending sensory plena. Accordingly, we *constitute* the idea, e.g., of perfect triangles.

13 Two Kinds of Foundation Relations

So far we have seen how, for Kvasz, relativization is supposed to be a precondition for the possibility of what Husserl calls idealization. We have also seen how both

Kvasz and Husserl regard the essential difference between idealization and abstraction: idealization is a precondition for the possibility of abstraction (and hence so is relativization, though we have not seen precisely how yet). We have also seen how both Kvasz and Husserl regard linguistic tokens as foundational in constituting the domain of mathematics. We will now provide a brief account of Husserl's notion of foundation in order to flesh out the sense in which Husserl's theory of formal ontology might actually clarify the conflicting sorts of epistemic subjectivity, the different forms of language, and the aspects of the form of language more generally.

In Husserl's Third Logical Investigation he presents the beginnings of a formal theory of part–whole relations, or formal ontology.[49] This theory is intended to make precise the possible dependence relations among various sorts of phenomena, regarded as either wholes with parts or as parts of wholes. According to Husserl there are two kinds of parts of a given whole: its *pieces* – which are independent with respect to the whole, and which serve to found it – and its *moments* – which are essentially dependent on the whole, and which are founded by it. This theory is assumed by much of Husserl's subsequent work, and wherever he refers to "foundation" or "grounding" relations he intends for his remarks to be able to be squared with this theory.[50]

The presentation of foundation relations in the Third Investigation is highly abstract by design, as it is intended to be sufficiently general to apply to all possible foundation relations. Insofar as anything at all can be regarded as a whole, it can be regarded as possibly being founded by certain pieces and founding certain moments; and insofar as anything at all can be regarded as a part of another whole, it can be regarded as either founding that whole (being a piece of it) or being founded by it (being a moment of it). Each of these kinds of foundation relation comes in two possible forms: object foundation, or dependence relations among *particular* objects, and species foundation, or dependence relations among *species* of objects. Thus, there are four broad types of foundation relation:

- Object Foundation:
  - A common example of a **piece** of a whole is a particular leg of a particular table: the particular leg is independent of the particular table because it is still *this* leg whether or not is is mounted to *this* table; the identity of the leg is not essentially dependent on the particular table to which it is mounted. However, this particular table is not independent of this particular leg, because to be *this* table means also to have *these* legs as pieces.
  - An example of a **moment** of the table, then, might be, e.g., its *particular* color: *this* color cannot be separated from *this* table, for it is just *this* table's color; it is a part of it that is essentially dependent on it. If it is the same color, but not founded on *this* table, then it is simply not *this* table's color.
- Species Foundation:
  - As above, the **piece-species** 'furniture leg' is independent of the whole-species 'table' because legs can be legs of other whole-species, e.g., chairs or beds; legs do not derive their essence from their being pieces of tables exclusively. However, the species 'table' is not independent of the species 'leg', because to be a table means to have legs; hence, the whole-species 'table' is indeed founded on the piece-species 'leg'.

- As above, the **moment-species** 'table color' cannot be separated from its foundation in the whole-species 'table'; table colors are founded on tables; this is just a logical point; table colors are parts of tables that are essentially dependent on them, whatever similarities the table colors might have with other kinds of colors.

So, to repeat, wholes are founded upon their pieces, and moments are founded upon their wholes; both pieces and moments are *parts* of their wholes, but the former is independent of it whereas the latter is dependent on it.

These examples are a bit simplistic, but we can spell out the import for our purposes as follows: Whereas object foundation is maximally particularized, species foundation opens up the possibility of degrees of independence among various essentially dependent phenomena. This, of course, is to be expected as we relax the supposition of particularity, and move into a world of kinds. For example, what founds the moment-species *color* in general? Upon what is it founded? Indeed, anything in the species *extended body* will suffice;[51] we might rather say, then, that what we mean by the moment-species *color* is *extended body color.* We can thus see that, given this framing, in terms of species, we do not need any *particular* extended body to found any *particular* color. Any body, in principle, is capable of founding the species 'color'. Thus, particular colors can be viewed as *highly independent* of particular bodies, despite the fact that they are indeed essentially dependent on bodies. More to the point: colors are highly independent of particular bodies *by virtue of* their foundation relation being a species relation rather than an object relation. Colors do not need particular bodies, nor do bodies need particular colors—and yet the ideas of "bodies" and "colors" do not thereby become ambiguous or non-individuated.

14 Foundation Relations and Relativization

Let's now apply this rather abstract theory of foundation relations to Kvasz's account of relativization. To begin, let's take note of two ways in which Kvasz's account might fail to be appropriately sensitive to Husserl's theory.

1. In one of the passages from *Crisis* cited above, Husserl writes, "The Pythagorean Theorem, indeed all of geometry, exists only once... It is identically the same in the 'original language' of Euclid and in all 'translations'."[52] As we saw earlier, Kvasz's arguments are to the effect that patterns of linguistic change in mathematics are always predicated on highly historically contingent modifications in features of particular mathematical languages. Hence, Kvasz may wish to maintain that, e.g., the Pythagorean Theorem is not so unique. Indeed, there are many different articulations of the theorem, some of which have afforded alternative methods of its proof (and different systemic consequences), which Kvasz might be inclined to view as discrete phenomena, each involving different semantic contents corresponding to the differences in the respective linguistic foundations. Especially regarding Husserl's claim that "all of geometry exists only once," Kvasz would seem to be inclined to distinguish, at minimum, between synthetic, projective, analytic, and fractal, as all being differentiated according to differences in their respective *particular* foundations.
2. In the same passage, Husserl writes, "in a certain way ideal objects do exist objectively in the world."[53] Of course, this appears to contradict Kvasz's claim that "the horizon does not denote anything factual, there are no fixed individua, space does not really exist, not to speak about the ideal elements."

> Indeed, according to Kvasz, it is precisely by virtue of the non-existence of these phenomena that they can be said to afford the possibility of linguistic change.

Simply, then, we have two discrepancies to clarify between Kvasz and Husserl: First, we have to make sense of how geometry can be, at once, both highly dependent on particular linguistic foundations *as well as* highly independent of particular linguistic foundations. Second, we have to make sense of the precise way in which the aspects of the form of mathematical language can be said to exist, and moreover, how they can be said to afford the possibility of linguistic change.

With an eye toward clarifying these discrepancies, I will offer an initial parsing of the various foundation relations implicit in each of these phenomena, as Kvasz presents them. We will see that this goes a long way toward clarifying Kvasz's two apparently conflicting impulses – the empiricist and the transcendentalist. In turn, these clarifications will set us up for further clarifications regarding how to understand the role of epistemic subjectivity in Kvasz's account of relativization.

Foundations of Forms of Language Most obviously, it appears that Husserl could well regard, e.g., geometry as a particular 'object' founded upon a large set of particular linguistic (hence, ideal) pieces—a set comprising both symbolic and iconic tokens as well as various explicit and implicit features of these. In this sense, Husserl would regard geometry as essentially dependent on its particular linguistic (ideal) foundations, much in the sense of Kvasz's analysis; for in this case the sense of 'geometry' is completely determined by the set of particular linguistic (ideal) elements it comprises. This is in keeping with Kvasz's emphasis on the fine-grained historical details as the basis for understanding mathematical disciplines at large: they are determined bottom-up by their particular, historical, linguistic foundations.

More abstractly, however, if we regard 'geometry' as a species rather than an object – which appears to be consistent with Kvasz's recognition of multiple types of geometry, i.e., synthetic, projective, analytic, and fractal – then Husserl could similarly, and plausibly, regard it as founded on the pieces-species 'mathematical languages' or 'linguistic (ideal) tokens'. Of course, the different species of geometry would have to be observed as being founded on different species (or sets of species) of linguistic foundations – the language for describing which we currently don't have before us – but in general this framing is valid: geometry as a whole is founded upon a set of mathematical languages or linguistic tokens, depending on the granularity of an intended analysis.

In both these senses, then – of object and species foundation – we can see that for both Kvasz and Husserl, geometry, e.g., can be regarded as a whole founded by – essentially dependent on – its linguistic (ideal) pieces. Moreover, the difference between these two ways of regarding the foundations of geometry clarifies the first apparent contradiction between Kvasz and Husserl described above. On the one hand, when geometry is regarded as a particular object with particular objectual foundations, it is essentially determined by those foundations; geometry *just is* those foundations, or rather, is in a way reducible to those foundations. This accords with Kvasz's possible inclination to disregard Husserl's propositions that the Pythagorean Theorem is unique or that geometry is independent of particular linguistic foundations; in this sense Kvasz is regarding geometry in the sense of object foundation. On the other hand, Husserl's theory also entails that geometry can be viewed as independent of particular linguistic foundations, in terms of the relation of species foundation. The species ‘geometry’ is not just to be understood as whatever particular language it happens to be expressed in, though it does require language – indeed, with certain structural features – for its expression. The species ‘geometry’ is thus not reducible to any particular linguistic (ideal) substrate, to any particular geometric language; it is a species that is founded upon the a particular species of mathematical languages, but is not essentially dependent on any one of those languages.[54]

Foundations of the General Form and its Six Aspects Now, what foundation relations characterize what Kvasz calls the *general form of mathematical language*?

There is rather an interesting mistake one might be inclined to make at first. In light of the foregoing, and given that his analysis of form unfolds with such intimate reference to particular events in the histories of geometry and algebra, one might first suppose that the general form of language will also be dependent on particular linguistic tokens. Of course, to an extent this is right: without particular linguistic foundations, there could be no general form of language; the general form has to somehow be founded upon particular linguistic tokens. However, this appears to put the cart before the horse; for the general form of language is supposed to be implicit in *all possible* mathematical languages, not just all actual ones. This, in conjunction with the fact that the general form of mathematical language has historically entailed eight successive forms of language, across two historical threads, indicates that we should try to understand the general form as a species, to which the eight successive forms belong as members.

Now, at another glance it might seem that the form of language is a species-whole which is founded upon, e.g., certain kinds of languages, corresponding to the eight successive forms. This sounds a bit better, but it still obfuscates the sense in which we will want to regard the form of language as a *part* of language, as in, "form and content are parts of language"; for, if the general form of language *and* the eight forms of language are *both* regarded as parts of language, then one cannot properly be regarded as the foundation of the other, because foundation relations obtain only between parts and wholes. Therefore, I take it, the general form of language must be a moment-species founded by a species-whole of some kind, e.g., of mathematical languages in general. On this classification we will preserve the sense in which the form of mathematical language is both independent of particular mathematical languages – though, of course, not independent of some mathematical language, nor of mathematical languages in general – as well as the sense in which the form of language should be regarded as a part of mathematical language.[55]

Furthermore, what foundation relations characterize the six aspects of the form of language: the epistemic subject, the horizon, the individua, the categories, the ideal objects[56], and the background? For one, with respect to the general form of language, they appear just to be founding pieces; i.e., the general form of language is nothing over and above the six aspects it comprises, it is constitutively dependent on them—alternatively, the general form of language appears to be founded upon the three functions that are each founded by two aspects of the form of language: to incorporate the epistemic subject into the world, to structure the world from the point of view of the epistemic subject, and to help the subject find its orientation in the world. Hence, we could say that the general form is an object-whole founded by its pieces; and the species of the general form are then founded upon species of these aspects of form, variously apparent as they are in different mathematical languages.

Hence, we have here distinguished two senses of the general form of mathematical languages: the sense in which it is a species-whole, which is founded upon the eight successive forms of language Kvasz identifies, and the sense in which it is an object-whole, which is founded upon its six aspects.

But especially when regarded in this latter sense, we seem to be posed with the deeper question: what is the foundation of the six aspects? In one sense, the aspects of the general form of mathematical language have to be founded upon linguistic tokens—*of course*. Moreover, insofar as they are supposed to be independent of particular mathematical languages, but not independent of mathematical languages in general – i.e., insofar as they appear across many different languages, including novel ones – the six aspects must each be a species rather than an object. In this sense we would be able to say, e.g., that the epistemic subject of perspectivist geometry is a member of the species of epistemic subjectivity more generally, and hence projective geometry can also have an epistemic subject—and this makes sense for us. Thus, we know that the six aspects of form are each both founded (i.e., dependent) and a species—but are they wholes or moments?

I am inclined to think that they are moments, i.e., dependent parts of mathematical language. This makes sense of the way in which we can say that they show up in *any* mathematical language, past, present, or future. The reason is, they are moments of the species-whole of mathematical languages, and can be founded upon any particular mathematical language—just like the relationship between color and extended bodies. That is, as in the relationship between color and extended bodies, the aspects of form and mathematical languages appear to be *co-dependent*: just as you cannot have color or extension without the other, you cannot have mathematical language or the aspects of form without the other.

In the next section we will turn to more exactly what is involved in these claims, but we can here first clarify the second apparent contradiction between Kvasz and Husserl, described above. When Kvasz writes that the six aspects of the general form of mathematical language are purely "formal, i.e., they have no factual meaning," he actually means that they denote contents internal to mathematical language rather than in the transcendent world. This appears to be just what Husserl means when he writes, "In a certain way ideal objects do exist objectively in the world, but it is only in virtue of these two-leveled repetitions and ultimately in virtue of sensibly embodying repetitions." As Husserl continues in that passage, "For language itself, in all its particularizations (words, sentences, speeches), is, as can easily be seen from the grammatical point of view, thoroughly made up of ideal objects." That is, for Husserl, insofar as the aspects of the form of language are themselves denoted by language and hence ideal objects, they can also be regarded as existing "objectively in the world"—but, that is, only inasmuch as their linguistic foundation is included in the package.

15 On the Idealization of Form

But let us linger here a while longer. As we have just said, on Husserl's theory of formal ontology, we can regard the aspects of the general form of mathematical language as existing, but *only insofar as we take the linguistic foundations to be part of the package.* The logic here is actually enormously instructive: linguistic foundations are a necessary (pre-)condition for idealization of the aspects of form—but are they sufficient? Is there anything else that is needed to found the aspects of form, beyond language alone?

Kvasz really tries to maintain a negative answer here, that mathematical language is sufficient to found its general form. He accordingly adopts the Tractarian position that language comprises what is implicit in addition to what is explicit. Thus, as we saw it just above, Kvasz is trying to maintain that the semantic content of the six aspects of form, their "meaning," is determined *purely intensionally*, i.e., without any reference to an exterior domain of phenomena. Language refers to language, and thus the significance of the form of language is to be found within language. But is this not too contrived? Must the linguistic turn be 180 degrees?

Well, as we may recall from our discussion of the way in which meanings are sedimented in linguistic tokens, for Husserl, this whole story about the significance of the aspects of general form presupposes the presence of particular idealized phenomena. That is, an idealization – such as of the six aspects of form – *presupposes* the existence of some phenomenon to be idealized. Without the original phenomenon, there is no significance to be re-founded upon a linguistic token in the first place. The sedimentation of an ideal meaning presupposes a relationship with a particular phenomenon, which over time, by being ideally conjoined with the linguistic token, becomes sedimented in it as its foundation. Hence, there must be phenomena that correlate with the six aspects of form. Moreover, implicit in this account of sedimentation is a primitive *distinction* between the phenomenon and the linguistic foundation of its idealization. This is important, for it indicates that the origin of the phenomenon has to be distinct from language, though, of course, expressed in it.

Before we continue, let's briefly address an objection Kvasz could make here. Perhaps he disagrees with us here and says, "No, the phenomenon does not have to be distinct from language; it just has to be distinct from the language in which it is expressed. Thus, just as we construed the Tractarian position regarding depicting the (aspects of the) forms of languages from inside *other* languages, *outside* of the former, we depict the general form of language reflectively, from a linguistic position outside the languages whose form we are describing. This is consistent. Hence, we do not need an extra-linguistic phenomenon to found our depiction of the general form of mathematical languages, but only *other*, *prior*, or *exterior* mathematical languages, but mathematical languages nevertheless."

This is indeed an interesting point, which I here, in a Husserlian vein, interpret to the effect that past language is *always part* of the foundation of what enables our depiction of general form—but again, it is simply *insufficient*. Yes, it is true that relativization is a fundamentally linguistic phenomenon—but not entirely; language is not the entire fundament to observe here. It appears here that Kvasz's restricted focus on language is precisely what entails *two* contradictions with Husserl's account of idealization: (1) as we have already begun to see, it contradicts the sense in which Husserl would regard the aspects of form as *existing*, albeit *dependently*, and now (2), it contradicts the sense in which Husserl's account of idealization presupposes the presence of a phenomenon *prior* to its significance being re-founded (sedimented) in a linguistic token. Regarding (2), we can imagine it this way: by virtue of *what* is mathematical language, with its peculiar form, possible? For fear of circularity, mathematical language cannot be the entire ground of the form of mathematical language.

So, if we are to maintain Kvasz's claim that relativization is a precondition for idealization, then we should have to square relativization with these other preconditions Husserl describes for idealization. So, again, we ask: What else, other than language, is needed to found the six aspects of the form of mathematical language? Presumably, each will correspond with its own phenomenon upon which its linguistic expression depends, by virtue of which mathematical language is possible:

epistemic subject of the language	horizon of the language
individua of the language	fundamental categories of the language
ideal objects of the language	background of the language

$$\downarrow$$

[epistemic subject of the language]	[horizon of the language]
[individua of the language]	[fundamental categories of the language]
[ideal objects of the language]	[background of the language]

Once again, we will examine this question and answer it in Husserlian terms.

16 Transcendental Idealism and Phenomenology

It will be instructive to characterize Husserl's phenomenology against the background of Kant's transcendental idealism. The clearest way to observe the difference is to review Husserl's three famous *reductions*: the eidetic reduction, the transcendental reduction, and the phenomenological reduction. These will, in turn, expose the sense in which Husserl was sensitive to Kant's philosophy as well as the sense in which Husserl's philosophy goes beyond Kant's. In the course of this section we will see how they could agree to explain the significance of the aspects of the form of language as well as how Husserl would begin to think Kant's explanation inadequate.[57]

Much of Husserl's work was an attempt at elucidating the sphere of phenomenological analysis proper. The way in which Husserl leads us to this sphere is by means of his famous epochē, or "bracketing" of various domains of experience. The epochē entails a *refraining* from making ontological judgments regarding particular classes of phenomena. Thus, one does not *negate* the existence of those phenomena, but simply refrains from entering into thought regarding their existence. Accordingly, the phenomena remain within the field of consciousness, albeit *qua* mere phenomena. With each successive application of the epochē, the phenomenologist *reduces* the domain of experience over which their existential judgments operate – so as to observe them for what they *really* are for consciousness – until eventually bottoming-out on the phenomenological domain proper.

The Eidetic Reduction. The first of Husserl's reductions is basically an exposition of the domain of ideas, as distinct from the domain of ordinary worldly objects. By bracketing the natural world – suspending the "natural standpoint" in which objects are taken to exist in space and time – Husserl exposes the *eidetic* domain, or the domain of the essences of the phenomena we perceive from the natural standpoint. In fact, this reduction just gets us up to the point of view of Aristotle (not much beyond Plato): we can recognize the difference between objects in the world, what the ancients called *matter*, and the ideas which those objects instantiate, what the ancients called *forms*. This is the first reduction: the bracketing of the material world in order to expose the domain of essences. From the point of view of the eidetic reduction, then, we can observe the distinction between the science of the world, "natural science," and the science of the ideas that are instantiated in objects in the world, "eidetic science." The latter is taken to include mathematics as well as systems of logic applicable to the natural world.[58] Thus, in the first reduction, Husserl's epochē exposes one of the first dualistic themes in the history of philosophy: the distinction between form and matter.

Moreover, this helps to elucidate a minor part of Kvasz's analysis: the ideas expressed in mathematical language as opposed to its reference in the world. For Husserl, forms are instantiated in material contents, and yet they are not reducible to them; we co-idealize both and then abstract away the content to focus on the form. The same holds true in how Kvasz's analysis directs our attention to the form of mathematical language: we co-idealize the natural content of mathematical language and its formal structure together, and then subtract away the content to focus on the form. Accordingly, there is obviously going to be a sense in which the content constrains the form that can be instantiated in it. It is not as if form is totally unconstrained by content; the only forms that are possible are those that can be instantiated in actual contents. Hence, whatever Kvasz's "form of language" amounts to, it has to be constrained in this minimal sense by the natural content that instantiates it.[59]

The Transcendental Reduction. The second reduction proceeds by bracketing the phenomena we are directed toward in either the natural standpoint or the eidetic standpoint. That is, after the eidetic reduction, while we have indeed suspended our ontological judgments about natural objects, we have not suspended our ontological judgments about the ideas that are instantiated in natural objects. The transcendental reduction thus involves bracketing all these phenomena which the mind can be directed toward after the eidetic reduction. That means, we suspend our judgments about the existence of the ideas we elucidated with the eidetic reduction, and we concentrate our mind on the *transcendental acts* that make the perceptions of the phenomena in the natural and eidetic standpoints possible in the first place. Thus, in the second reduction, Husserl's epochē exposes the dualistic theme of modern philosophy: the distinction between all of our actual ideas and the transcendental conditions that make them possible.

At this level Husserl finds a certain tripartite structure implicit in both the natural and the eidetic attitudes: a noema, a noesis, and hyle. The noemata are whatever phenomena take the place of the object-pole of consciousness, in this case natural and eidetic phenomena; the noeses are whatever *transcendental acts* occupy the place of the subject-pole of consciousness, conditioning the possibility of consciousness of the noemata; and the hyle are the sensory matter that constrain the development of our noemata and noeses. This, in effect, describes the domain exposed by Kant's transcendental deductions, except for one major difference. Whereas Kant's analysis distinguished between the phenomena of possible experience and the transcendental conditions of their possibility, Husserl's analysis distinguishes between noemata and noeses, respectively. The difference is in the nature of the relationship between the elements of these distinctions. For Kant, it is a problem of validity: his concern is with the question, supposing our experiences are as they are, what transcendental conditions must obtain? For Husserl, it is a factual problem: his concern is not with what transcendental conditions must obtain, but with what transcendental acts *actually* obtain in the consciousnesses of certain phenomena. Hence, noeses are *factually* related to noemata rather than deduced. Schematically, we can say that *whereas Kant's transcendental philosophy proceeded by the transcendental deduction, Husserl's proceeds by the transcendental reduction.*

The Phenomenological Reduction. This brings us to the phenomenological reduction. According to Husserl, this last reduction is relatively straightforward after the first two, although even more abstract, and therefore never until him seized upon. So far, we have exposed three domains of inquiry: the natural domain, the eidetic domain (by suspending ontological judgments about objects the natural domain), and the transcendental domain (by suspending ontological judgments about objects from both the natural domain and the eidetic domain). The transcendental domain is the domain of relationships between different transcendental acts, or noeses, and their corresponding actual and possible objects, or noemata. The most noteworthy example of a transcendental philosopher is Kant, whose transcendental deductions were an admirable attempt to prove the possibility of noeses–noemata correlations; Husserl, then, in a move beyond Kant takes the noeses themselves up *qua* objects rather than as mere necessary conditions for the possibility of objective experience.

Moving even further beyond Kant, Husserl makes a further exposition with the use of a further epochē. Having before us these two dimensions – the noetic-noematic(-hyletic) and eidetic–natural – Husserl notices that yet another domain of inquiry can be distinguished. By bracketing the the transcendental acts *themselves*, which we have hitherto exposed *as* acts through the transcendental reduction, yet another domain of inquiry emerges. There is, of course, the domain of various noeses correlated with various natural and eidetic noemata; but there now appears to be a "meta-noetic" domain, which cannot be captured in terms of noeses. Hence, when noetic acts themselves are subjected to the epochē, then we direct our consciousness *directly* upon consciousness itself. Hence, at the far limit of refraining from ontological judgment, Husserl exposes the domain of phenomenology proper: the domain of *meta-noetic consciousness*.

In other words, whereas Kant's transcendental philosophy took the domains of preconditions for the possibility of experience and preconditions for the possibility of synthetic a priori judgments to comprise just one domain – a domain which conditioned the possibility of empirical science – Husserl indeed sees these as two distinct domains of inquiry. According to Husserl, Kant's transcendental idealism amounts to a kind of palatable *metaphysics* of the particular noeses that condition the possibility of synthetic science, both a posteriori and a priori, but it is not adequate as a *phenomenological* science, or a science of consciousness more generally.[60] Kant's transcendental deductions determine the metaphysics of particular kinds of noetic–noematic events, but they are incapable of reaching into the nature of meta-noetic consciousness itself, much less of *describing* this consciousness. Hence, according to Husserl, the task of phenomenological inquiry, at this highest level, is to come to descriptive terms with meta-noetic consciousness in general. We will return to this theme shortly.

17 On a Kantian Take on Aspects of Form

As we have said, the first major difference between Husserl and Kant is that whereas Kant's metaphysics are trading in validity, Husserl's phenomenology is trading in fact. This is relevant to what we are after. Our concern is with exactly what affords the possibility of formal change in relativization; i.e., with what exactly affords the possibility of novel expressions of the form of mathematical language. Therefore, we cannot simply settle for necessary preconditions for this possibility, but should rather be looking for the *actual* preconditions for this possibility. Nevertheless, Kant's transcendental deductions are not totally irrelevant for us; as we have seen, Husserl's phenomenology is designed to take us from Kant's arguments that certain preconditions *must* be in effect to the factual question of which preconditions are *actually* in effect. Therefore, it is instructive to begin from Kant, and to then fill out how

Husserl would "factualize" our concern.

Indeed, we can guess how Kant might go about re-telling Kvasz's story rather easily. Let's return again to the six aspects of form:

epistemic subject of the language	horizon of the language
individua of the language	fundamental categories of the language
ideal objects of the language	background of the language

How can we trace these back to elements of Kant's transcendental idealism? For example, it appears that ideal objects could correlate with what Kant would call objects of perception, and the background of language with the transcendental aesthetic. Hence, we might say, the function of the aspects of form to help the epistemic subject find orientation in the world is, in part, founded upon the transcendental aesthetic, i.e., the a priori forms of inner and outer sense which spontaneously differentiate an intuited object from its spatial and temporal background. Moreover, it appears that the individua could correlate with forms of intuitions of particulars and the fundamental categories might might be variously derivative of (synthetic a priori) judgments which employ pure concepts of the understanding. Hence, we might say, the function of the aspects of form to structure the world from the point of view of the epistemic subject is, in part, founded upon the meta-conceptual categories of the understanding and the synthetic judgments they make possible. Finally, it appears that the epistemic subject could correlate with the unity of apperception, and the horizon of language might correlate with noumena. Hence, we might say, the function of the aspects of form to incorporate the epistemic subject into the world is founded upon the fundamental interdependence of synthetic a priori and synthetic a posteriori judgments.

At any rate – regardless of whether these initial proposals mark exactly the best ways to correlate the aspects of form with Kant's transcendental idealism (I even doubt this, not being an expert on Kant)[61] – I think it is plausible that, as far as idealization presupposes extra-linguistic phenomena to be idealized, the lion's share of the phenomena presupposed by Kvasz's account of the aspects of the general form of mathematical language are, in some way or other, correlates of Kant's conditions for the possibility of experience. But since Kant's conditions for the possibility of experience are traded in validity rather than factuality, insofar as we are searching for an *actual* ground for what affords relativization, we should rather trade in terms of noeses and noema. Hence, I take it, the phenomena presupposed by Kvasz's account of relativization are going to be certain kinds of noeses, which more or less map to Kant's transcendental deductions.

But let us linger here for a moment with Kant's deductions and try to observe how Kant could construe Kvasz's claim that relativization unfolds as these functions of *epistemic subjectivity* are explicated in language. As we said before, it appears that the closest analogue to the epistemic subject in Kant's transcendental deductions is the unity of apperception. What what does Kant mean by this? Well, according to Kant, experience is unified into *one* experience, of *one* consciousness, by virtue of a "threefold synthesis" in which three distinguishable aspects of experience are synthesized into a single coherent experience. On one level sensations are apprehended in a proto-intelligible format; on another level they are reproduced as unfolding spatiotemporally; and on another level they are comprehended in terms of concepts—this all happens simultaneously. By virtue of the activity of this latter synthesis, otherwise disparate experiences are conceivable as connected to other experiences; and at the "boundary" of all conceptual activity, experiences are conceived as experiences of one and the same subject.[62] Hence, for Kant, the unity of this subject of experience is not simply given, but is actively constituted in acts of understanding, i.e., in acts of synthetic judgment (either a priori or a posteriori). This synthetic activity of the understanding is the precondition which allows the 'I think' to be brought to bear on representations.

How does this bear on Kvasz's account of the role of epistemic subjectivity in relativization? We see that the transcendental condition of the 'I think' is the synthetic activity of the understanding, which amounts to the making of synthetic judgments. So, how can acts of synthetic judgment be the ground of the explication of epistemic subjectivity? For Kant, we might say that the epistemic subjectivity of a mathematical language is made explicit as synthetic judgments are made to the effect that disparate representations fall under one concept. For example, when al-Khwarizmi interacted with the elements of his algebraic activity, his activity was in attempt to unify the elements conceptually; hence, Cardano's formalization of al-Khwarizmi's methodology is the completion of this conceptual unification, in language. Likewise, when Masaccio painted *The Holy Trinity*, it was in attempt to unify the perspective of the painter through the relativities among the elements of the painting; hence, Desargues' formalization of Masaccio's methodology is the completion of this conceptual unification, in language. Thus, synthetic judgments (largely a priori ones), as acts of the understanding, condition the possibility of the explication of epistemic subjectivity.

This gets us a lot; but it also appears to leave something out. On the one hand, there is, e.g., al-Khwarizmi's methodology which is symbolized by Cardano, in virtue of synthetic acts of understanding; but on the other hand, there is *al-Khwarizmi him-*

self, who is lost in the symbolization, and who was an integral part of the constitution of its phenomenal fundament in the first place. The same extends to Cardano, and to whoever comes in subsequently to symbolize *his* methodology, etc. Are not, e.g., al-Khwarizmi and Cardano themselves *actual* parts of what afforded relativization? We can agree with Kant that the synthetic acts of the understanding – which condition the synthetic unity of apperception – are necessary parts of the story here; but they appear not to be sufficient. Our account requires reference not just to acts of the understanding which unify experience, but to an *actor* whose experience is unified in part by acts of understanding. More succinctly, the actor idealizes the previously enacted methodology in novel mathematical language, and thus a noesis is explicated *qua* noema. *The actor is distinct from the noesis that is explicated.* Thus, an actor's actions effect higher-order noeses via the idealization of lower-order noeses, an idealization which is founded upon a bit of language. To be cheeky, but indeed precise, we can say: an actor effects the *noematization of noeses*, and thereby founds a higher-order noesis.

We may observe here that there are actually *two* distinct sorts of foundation relations simultaneously in effect, and running in "opposite" directions. On the one hand, the noematization of noeses – via idealization – itself founds a higher-order noesis, and, indeed founds the possibility of many more higher-order noeses. On the other hand, we are explaining the possibility of the noematization of a noesis in terms of a higher-order noesis *undertaken by an actor*. The former kind of foundation relation is a bottom-up foundation – from language and lower-order noeses to a higher-order noesis – whereas the latter is top-down – from a source of the possibility of a novel noematization of a pre-existing but un-noematized noesis. We are thus attributing a founding not just to a noematized noesis, but to a noematizing actor.

The precise nature of this "source" will become thematic in the following section of this essay, but before transitioning let us observe one final feature of the foregoing propositions. As we have seen, e.g., when Cardano takes up al-Khwarizmi's methodology as mathematical object, and then as subsequent algebraists take up Cardano's methodology as mathematical object, these various methodologies lose their original connection to the particular actors by whom they were originally enacted. Hence, it is not simply that relativization unfolds by virtue of the fact that the epistemic subject, *qua* aspect of form, does not refer; it is more precisely by virtue of the fact that the epistemic subject does not necessarily refer to any *particular* epistemic subject. The same is true of the other aspects of form: they *do* refer, albeit to *species* rather than to particulars. It is by virtue of the essence of the species that novel explications of the aspects of form can be construed as novel explications of those very aspects of form; otherwise they would simply be construed as explications of entirely novel transcendental acts, via entirely new aspects of linguistic form.[63]

18 Intentionality and the Transcendental Ego

Considerations like these, I take it, are what lead to Husserl's proposition that there is an *essence of mind*, i.e., that what we have called "meta-noetic" consciousness can become a theme for phenomenological science.[64] This is in part what I mean above by *species* of subjects, rather than particular subjects. According to Husserl, the essence of mind, or *transcendental consciousness*, is the ultimate ground of all possible experience—the ground not only of the natural attitude, but of the eidetic attitude, the transcendental attitude, *and* the phenomenological attitude; of all noeses and therefore all noemata. Transcendental consciousness in general is the ground of the possibility of all differentiations of consciousness into different sub-species of subjectivit*ies*. And here we have a fundamental if simple kind of answer to our question: What founds the possibility of relativization? At the phenomenological limit, it is transcendental consciousness. But what is transcendental consciousness? And how does it exceed Kant's answer to the question of what the conditions for the possibility of experience are? How does it enable us to understand the role of the working-mathematician in relativization?

The true starting point here is with the notion of *intentionality*. Husserl's theory of intentionality is derived primarily from Brentano's characterization of the term in the following oft-cited paragraphs:

> Every mental phenomenon is characterized by what the Scholastics of the Middle Ages called the intentional (or mental) inexistence of an object, and what we might call, though not wholly unambiguously, reference to a content, direction toward an object (which is not to be understood here as meaning a thing), or immanent objectivity. Every mental phenomenon includes something as object within itself, although they do not do so in the same way. In presentation, something is presented, in judgment something is affirmed or denied, in love loved, in hate hated, in desire desired and so on.
>
> This intentional inexistence is characteristic exclusively of mental phenomena. No physical phenomenon exhibits anything like it. We can, therefore, define mental phenomena by saying that they are those phenomena which contain an object intentionally within themselves.[65]

Hence, for Husserl, it is, at least in part, *intentionality* which characterizes the essence of mind. The mind is always directed *toward* something, consciousness is always *of* something, namely, its intentional correlate. Schematically, we might say, intentionality comprises the following "structure," variously articulated for the sake of effect:

intending	intends	intended
mind	to	world
absolute subject	toward	absolute object
transcendental consciousness	of	transcendent something
I think	about	things
A	$\rightarrow$	X

What differentiates Husserl's account of intentionality from Kant's transcendental idealism is that, after the phenomenological epochē has been carried out – after the final suspension of ontological judgment – we are no longer able to understand the phenomenon in the A column as an intentional correlate, as an existing X. That is, whereas Kant took the transcendental conditions for the possibility of experience up *as* intentional correlates themselves – albeit as necessary preconditions for the possibility of empirical consciousness – Husserl is directing our attention upon the intentional structure of consciousness which makes this type of undertaking possible in the first place.

This is a bit puzzling, for how can we attend to consciousness if we do not take consciousness as an intentional correlate? Indeed, this is perhaps the most difficult barrier to overcome to understand phenomenology. Simply put, when considering the nature of transcendental consciousness, while indeed we have to *idealize* it in such a way that an "absolute subject" A stands in an intentional relation to an intended object X – i.e., *linguistically* – what we are actually referring to by "A" is "something" that in principle resists being intended, and yet which we can nonetheless in some sense intend. That is, transcendental consciousness, as what stands universally opposed to the intended, is literally *unintendable*—and yet phenomenology can somehow successfully idealize and describe it. In Kantian terms, it is truly "what cannot be thought otherwise than as subject," but in the sense of what cannot be intended otherwise than in the sense of unintendable.

This in fact puts Husserl at odds with Kant's arguments in the paralogisms. First, recall that, for Husserl, idealization presupposes an extra-linguistic phenomenon as well as a linguistic token upon which to re-found that phenomenon. So construed, the idealization of transcendental consciousness presupposes the *phenomenon* of transcendental consciousness, and not merely a concept of it; for Husserl, the concept is to be understood as idealized in language and founded upon the original extra-linguistic phenomenon. That is, whereas Kant's arguments in the paralogisms proceed on the ground that while we can have a concept of a transcendental being – "transcendental consciousness" – *we can have no intuition* of such a thing, Husserl's analysis rejects this. According to Husserl, we *do* have something equivalent to an "intuition" of transcendental consciousness, or, at any rate, we do experience transcendental consciousness as a phenomenon: we experience it as the subject-pole of intentionality which perpetually resists direct intention, which is always consciousness of something *else*. It is an *immanent* intuition rather than a transcendent one. The transcendent intuition founds the possibility of the immanent intuition, but the immanent intuition is still regarded as real. We may thus idealize "being directed toward a transcendent X" – this "being" is transcendental consciousness.

Conjoining this conclusion with our expositions of the different kinds of foundation relations above, we can notice just how peculiar transcendental consciousness is. On the one hand, it is grounded in a transcendent X; but on the other hand, it is the immanent ground of our being directed upon X. Thus, while we say that noeses are grounded in their noemata – and that higher-order noeses can be grounded in lower-order noeses (*qua* noemata) – in a highly unusual way, all of these are simultaneously grounded within consciousness itself. We cannot simply explain, e.g., the *noematization of noeses* from the bottom up, in terms of the noeses that become noematized. This leaves the noematization itself unexplained, as an absolute miracle; all we thereby explain is the transcendent ground of yet another noesis, and not the transcendental ground of the *noematization*. We must also appeal to transcendental consciousness as the ground of the possibility of this noematization. While an original noesis is itself grounded in the noemata toward which it is directed, the noematization of that noesis must be grounded from "above." Schematically, we can say: the noematization of noesis is fundamentally grounded in the *noesisization of consciousness*; consciousness itself affords the possibility of novel noesisization.

Having thus exposed the intentional structure of transcendental consciousness at the "limit" of the epochē, Husserl asks, What else remains in this phenomenological residuum? What is the "extension" of the phenomenological domain, the immanent domain of transcendental consciousness? Of course, the reductions have already excluded all *transcendent* reality; we have bracketed all judgments pertaining to the existence of the intentional correlates of all noeses and noemata. All that remains before the phenomenological eye is what is transcendentally immanent, and nothing of what is transcendent. At some point here, the question emerges quite naturally: Do *I* remain in this phenomenological residuum, or is my existence also bracketed in the epochē? Do I transcend consciousness, or am I immanent in it? What of me, if anything, remains after the phenomenological epochē? Husserl addresses this in a passage which is worth quoting at length:

> So much is clear from the outset, that after carrying this reduction through, we shall never stumble across the pure Ego as an experience among others within the flux of manifold experiences which survives as transcendental

> residuum; nor shall we meet it as a constitutive bit of experience appearing with the experience of which it is an integral part and again disappearing. The Ego appears to be permanently, even necessarily, there, and this permanence is obviously not that of a stolid unshifting experience, of a "fixed idea." On the contrary, it belongs to every experience that comes and streams past, its "glance" goes "through" every actual *cogito*, and towards the object. This visual ray changes with every *cogito*, shooting forth afresh with each new one as it comes, and disappearing with it. But the Ego remains self-identical. In principle, at any rate, every *cogitatio can* change, can come and go, even though it may be open to doubt whether each is *necessarily* perishable, and not merely, as we find it, perishable *in point of fact*. But in contrast the pure Ego appears to be *necessary* in principle, and as that which remains absolutely self-identical in all real and possible changes of experience, it can *in no sense* be reckoned *as a real part or phase* of the experiences themselves.
>
> In every actual *cogito* it lives out its life in a special sense, but all experiences also within the mental background belong to it and it to them, and all of them, as belonging to *one* single stream of experience, that, namely, which is mine, *must* permit of being transformed into actual *cogitationes* or of being inwardly absorbed into such; in the words of Kant, "The *'I think' must be able to accompany all my presentations.*"
>
> If as residuum of the phenomenological suspension of the world and the empirical subjectivity that belongs to it there remains a pure Ego (a fundamentally different one, then, for each separate stream of experiences), a *quite peculiar* transcendence simultaneously presents itself—a non-constituted transcendence—*a transcendence in immanence*. Given the immediately essential part which this transcendence plays in every *cogito*, we should not be free to suspend it. . . .[66]

This is rather a lengthy passage which contains much vocabulary with which we are not here acquainted, but we can observe its import thusly. Husserl is proposing that transcendental consciousness – as pure consciousness of consciousness rather than judgments of consciousness – remains *my* consciousness from the phenomenological standpoint, despite being indefinitely differentiated according to the intentional correlates on lower levels of consciousness. Just as consciousness is always consciousness of an X, for Husserl, it also always remains A's consciousness of an X. For example, I can perceive X in one moment, Y in another, and Z in yet another – whatever these may be. In one sense, I am different in each of these moments just according to the differences between X, Y, and Z; insofar as my experience is regarded as founded upon each, my being is dependent upon them. And yet, in another sense, I remain completely unchanged and completely unchangeable; insofar as they are regarded as objects of my consciousness, their being is dependent upon me.

Accordingly, therefore, Husserl interprets Kant's edict that "the 'I think' must be able to accompany all my representations" as not just grounded in transcendental consciousness, but as grounded in a *transcendental ego* which necessarily remains self-identical throughout all modifications of consciousness. More precisely, that is, for Husserl, the 'I think' is able to accompany all my representations *by virtue of the phenomenological fact that* a transcendental ego stands perpetually at the subject-pole of all actual and possible consciousnesses. The synthetic 'I think' of Kant is always able to be invoked by virtue of the transcendental ego being always actual on the subject-pole of intentional consciousness. Indeed, it is always possible in part by virtue of synthetic acts of the understanding, but for Husserl these synthetic acts are to be understood as transcendentally grounded in the transcendental ego. Take away the transcendental ego and judgments to the effect that "I think X, Y, and Z" lose their actual truth-conditions. It is not just a judgment to the effect that "the I that thinks X is the I that thinks Y is the I that thinks Z" that justifies the inference that "I think X, Y, and Z," although this is certainly part of the story; the inference is ultimately justified by the *fact* that I think X, Y, and Z.

The import for our purposes is that relativization can now be observed as not just a peculiar pattern of change that shows up in the history of mathematical languages, but as a pattern of change that is fundamentally related to the nature of consciousness itself. The immanent transcendental ego is *implicit* all noematization of noesis, as the absolute-subject pole of intentional consciousness; therefore, relativization may unfold as the egos that are implicit in noematizations of noeses are founded upon language, and indeed, as those noeses themselves are founded upon language. Furthermore, we are not struck with any pre-given limit for this explicatory process; mathematical languages will continue to be relativized as more and more aspects of mathematical noeses – aspects of the mathematical activity of the transcendental ego – are expressed in language. This does not entail an infinite life-span for relativization, but only an indefinite one. There is no reason a priori why there should not be a "limit" to the transcendental structure of mathematical noesis. Indeed, we typically take mathematics to be inexhaustible, but this is not necessarily an attack on finitism with respect to mathematical syntax. For all we know, relativization may terminate in a finite, transcendentally grounded mathematical (or logico-mathematical) syntax.

19 Phenomenological Reflections on Relativization

Thus, with a bit of help from Husserl, Kvasz's account of relativization can overcome the two difficulties we observed when regarding his notion of epistemic subjectivity from the Tractarian and the Kantian points of view.

The strict Tractarian interpretation of relativization entailed that we could not meaningfully refer to an epistemic subject as in any way involved in the mechanics of relativization, for the subject was a limit of language rather than an object which could be depicted by language. Husserl saves the concern implicit in the Tractarian interpretation by distinguishing between consciousness as differentiated according to particular intentional correlates and consciousness as undifferentiated according to particular intentional correlates; consciousness as founded as consciousness as founding. The former correspond to the different subjectivities that dwell in different mathematical languages, whereas the latter corresponds to the subjectivity that is, *in point of phenomenological fact*, *immanent* in all possible consciousnesses, and therefore able to be explicated in real-time.[67]

Moreover, the strict Kantian interpretation of relativization entailed that the epistemic subject of mathematical languages had to be a product of the synthetic activity of the understanding – which actually got us far, but was insufficient insofar as it did not provide us with an *actual* ground for relativization and it did not allow for primary reference to the *actors* implicated in the affair. Husserl's account of transcendental consciousness allows us to fill in, in point of fact, precisely what the ground of the explicated epistemic subject is: it is the transcendental ego, which resides at the limit of the subject-pole of transcendental consciousness, even after the phenomenological epochē has been carried through. Thus, according to Husserl, relativization may unfold as noematizations of noeses are idealized within the language of a working mathematician. The activity of the working mathematician – their mathematical noeses – can then be further idealized in higher-order noetic acts undertaken by subsequent mathematicians.

This much gets us a great deal in the way of saving Kvasz's original account of relativization. We can now completely follow Kvasz in saying that relativization unfolds as the aspects of, and hence the general form of, mathematical language are made increasingly explicit. The aspects of the form of language that are made increasingly explicit are founded upon noetic acts, which are eventually intended *qua* noemata in higher-order noeses. These higher-order noeses are then the phenomenal basis for further idealizations (via linguistic sedimentation), and hence may become explicated in higher-order meta-languages, both philosophical and mathematical. In other words, we can ground relativization in phenomenology: as aspects of the general form of language are made explicit within a mathematical language, noematized noeses are sedimented in language; the linguistic sedimentation of these noeses then founds the possibility of higher-order mathematical noeses, which may be similarly sedimented in language.

We can also completely follow Kvasz in maintaining the special dual-role of epistemic subjectivity in his account. On the one hand, the explications of implicit noeses are specifically designed to (a) integrate the transcendental ego into the world, (b) structure the world from the point of view of the transcendental ego, and (c) help the transcendental ego become oriented in the world. More succinctly, the mathematical explication of noeses is intended by the transcendental ego of the working mathematician, which intentions themselves may eventually become thematized and idealized. On the other hand, it remains consistent for us to speak of particular egos as associated with particular mathematical languages; these particular egos are just to be regarded as moments of the transcendental ego in general, but equivalently, as members of the species 'transcendental ego' which resides at the subject-pole of mathematical noeses—moments, which are simultaneously, in a different sense, taken as object-wholes founded upon their linguistic pieces.[68]

All we need to reject, therefore, is Kvasz's ambiguous remarks to the effect that the aspects of form do not refer to anything; they do refer to things, but to *kinds* of things rather than particulars; that is, to species rather than objects. They refer to *transcendental* species – kinds of noeses and egos – which condition the possibility not just of actual mathematical languages but of novel expressions of mathematical language, i.e., the possibility of novel particularizations of the aspects of form. Of course, this possibility of novel particularizations is partly founded on the fact that linguistic syntax, regarded as *mere* syntax, has no necessary connection with semantic content; but it is also partly founded on the foregoing facts of phenomenology, which we might say entail a necessary connection between the general form of mathematical language and mathematical noesis. Thus, most broadly construed, the point of this essay has been to draw a line pointing from the form of mathematical language back into the structure of mathematical consciousness.[69]

Thus this essay may be concluded. We began by observing a slice of Kvasz's account of how aspects of the form of mathematical language are made explicit in what he calls relativization. The problem with Kvasz's account we zeroed in on revolved around some ambiguous remarks to the effect that the aspects of the form of mathematical language can be rendered explicit in new forms of language because they do not refer to anything, they do not exist, etc. We have shown by way of several detours through Husserl's phenomenology how this can be made sense of.

NOTES

1. It should be noted here that, if I'd had more time, I would have tightened up these sections to not deviate so much from the thread of my main argument, and to be more sensitive to some disagreements Prof. Islami and I had regarding how to interpret Kvasz where his expositions were not so unambiguous. There are, however, two reasons in particular why I couldn't do this: (1) This thesis was submitted during the Spring 2020 semester, which, as you might recall, was the onset of the COVID-19 pandemic. As it happened, I came down with the virus about as soon as I finished my thesis defense. I think I was exposed a week or two before my defense but suppressed it until after I'd finished. For this and other reasons, I'm sure, I had a particularly bad case of the disease for someone my age, etc., which lasted nearly four weeks, leaving me not much time to catch up on everything that was due before the official end of the semester. Moreover, (2) the content of these sections, while not completely germane to the main argument of the essay, and possibly not perfectly reflective of how Kvasz would like his expositions to be understood, nonetheless accomplishes things further in the general direction in which I am pointing. That said, the reader may gloss these sections without offending the author; but be aware that there is, indeed, content within them that is fully germane to the main argument of the essay. This disclosure may also extend to other parts of the essay, albeit less noticeably. I include several endnotes throughout this essay to indicate places where revisions or extrapolations might have been made had I not gotten sick.
2. Kvasz, *Patterns of Change*, p. 16.
3. Wittgenstein, *Tractatus*, p. 22, prop. 4.001.
4. Ibid., pp. 23–24 passim, esp. props. 4.01 and 4.021.
5. Ibid., p. 11, prop. 2.172. Cf. p. 32, prop. 4.121.
6. Kvasz, *Patterns*, p. 110.
7. I say "may pass through" here because, in fact, Kvasz does not locate the coordinative or the compositive forms within the development of non-Euclidean geometry. Cf. ibid., p. 202.
8. Ibid., pp. 114–118 passim.
9. Cf., e.g., Burton, *The History of Mathematics*, pp. 238–242.
10. Perhaps, therefore, the crowning achievement of the perspectivist form of algebraic language was that it nevertheless contained linguistic resources sufficient for Cardano to express his formula for solving cubic equations seven centuries later, however

insufficient for Cardano to have discovered his formula.

11. Kvasz, *Patterns*, p. 163.
12. It should be noted here that Prof. Islami does not agree with my problematization of Kvasz's finding the analogue of the horizon in 0 or 1. According to her, 0 and 1 are supposed to act like a horizon in that most algebraic activity in al-Khwarizmi's time was standardized by setting algebraic expressions equal to 0 or 1. That is, it was in terms of 0 and 1 that patterns were sought within the algebraic expressions. So, as the horizon in the physical world was a reference in terms of which relativities among physical objects were oriented, 0 and 1 were the reference(s) in terms of which relativities among the elements of algebraic expressions were oriented. While I agree with her that this is likely how Kvasz intends for his remarks to be interpreted, as a keen reader may observe in what immediately follows, it appears to entail a breakdown with respect to his division between the perspectivist and projective forms. More precisely, whereas the horizon in perspectivist geometry is *displayed* according to the relativities depicted within the objects, and then depicted within projective geometry, in perspectivist algebra 0 and 1 are already explicit, i.e., not displayed by rather already depicted. In any case, none of what follows in the main argument of this essay hinges on this breakdown; it is rather just the beginning of an interesting discussion about how exactly to draw the boundaries between the perspectivist and projective forms and how exactly to draw the analogies between symbolic and iconic languages.
13. Ibid., p. 167. It is interesting to note that this tendency to condense the natural language expressions, carried on for another hundred or so years, eventually led to Descartes putting the number whose root was to be extracted under the '$\sqrt{\ }$', and putting the degree of the root on the upper-left, i.e., '$\sqrt[n]{x}$'.
14. Cf., e.g., Burton, *The History of Mathematics*, pp. 345–348.
15. According to Kvasz these middle two are present only in the evolution of algebra. I think a case might be made against this, however, if we are careful in how we draw the analogies. They may not pertain to explicit parts of geometric languages but may nonetheless be implicit. But this will require consideration beyond the present context.
16. Kvasz, *Patterns*, p. 205.
17. Ibid., pp. 206–207 passim.
18. This should be sufficient for a careful thinker about what follows, but the reader may wish to consult more thoroughly ibid., pp. 107–223.
19. Here the keen reader may recall n. 12, from p. 13 above. My suspicion is that a middle-ground between Prof. Islami's take and my own can be found, although I do

not currently have the time to search for it here. My premature suspicion, however, is that my own take, while requiring some reworking of Kvasz's story about perspectivist algebra, is closer to the truth; but it's complicated because there is nonetheless a sense in which 0 and 1 are precisely horizon-like.

20. Ibid., p. 205.
21. Ibid., p. 208.
22. Wittgenstein, *Tractatus*, p. 9, prop. 2.1; p. 10, prop. 2.141.
23. Ibid., p. 5, prop. 2.
24. Ibid., p. 11, props. 2.17–2.174.
25. Ibid.
26. Ibid., pp. 64–65, props. 5.54–5.5423.
27. An alternate analysis on similar grounds might proceed by suggesting a difference between (b) and (b*) that entails some difference in what follows. I am skeptical that such an alternate analysis would entail anything that contradicts what follows.
28. Kant, *Critique*, B131.
29. Kvasz, *Patterns*, p. 208.
30. Kant, *Critique*, B410-411, A351–366; language adopted from Rosenberg, *Accessing Kant*, pp. 258–263.
31. (1) and (2) here can be understood as two sides of one and the same problem: on the one hand, if E is reducible to L, then E cannot involve additional properties that cannot be explained in terms of L; on the other hand, if E is reducible to L, then E must be explained in terms of L.
32. This fact was related to me by Prof. Islami, whom Kvasz told directly.
33. Kvasz, "Galileo, Descartes, and Newton," p. 523.
34. Husserl, *Crisis*, p. 23.
35. Kvasz., "Galileo, Descartes, and Newton," p. 524.
36. Ibid., p. 530, fn. 4.
37. Ibid., p. 532.
38. The sensitive reader will here begin to notice a parallel between "getting over the edge" of our sense of the realm of perfect mathematical idealities and "getting over the edge" of our sense of epistemic subjectivity. This will become thematic in what follows.
39. I start with the *Crisis* because this is the only work of Husserl's that Kvasz references in his paper. In the following sections, we will trace part of this back to much earlier in Husserl's oeuvre.

40. Husserl, *Crisis*, p. 38.
41. Kvasz, "Galileo, Descartes, and Newton," p. 532.
42. We may also observe in passing that this entails a certain amount of flexibility in the meaning of abstraction: since it is bounded primarily by language rather than any *a priori* sense of mathematical *or* sensible idealities, abstraction is a highly flexible, relative to the linguistic foundations within a particular cognitive-perceptual region in which an abstraction is to occur; it is absolutely context-dependent, but nonetheless unique.
43. Husserl, *Crisis*, §9, pp. 23–59.
44. Ibid., Appendix VI, pp. 353–378.
45. Ibid., §9a, pp. 26–27.
46. Cf, e.g., Husserl, *Ideas*, §§81–82; also, *Cartesian Meditations*, §§35–39; also, in much more detail, *On the Phenomenology of the Consciousness of Internal Time*.
47. Husserl, *Crisis*, Appendix VI, p. 357.
48. Cf. Husserl, *Ideas*, §§35–38.
49. Cf. E. Husserl, *Logical Investigations*, "Third Investigation." For attempts at formalizing the theory, see also: P. Simons, "The Formalization of Husserl's Theory of Wholes and Parts"; K. Fine, "Part–Whole"; and F. Correia, "Husserl on Foundation".
50. Throughout the rest of this paper these words – "founds" and "grounds" and their cognates – will be used basically interchangeably, however there are often contextual factors that determine the choice. I do not here concern myself with those contextual factors, however they are becoming increasingly thematic for me.
51. This is not strictly true, in terms of physics, but we can grant the point for the sake of argument. In principle, we should be able to render this language consistent with physics.
52. Husserl, *Crisis*, Appendix VI, p. 357.
53. Ibid.
54. But again, further clarification within the species of mathematical languages is warranted for a precise analysis.
55. This leaves space for the more specific dependence relations that will obtain between each of the eight successive forms of language and the types of linguistic tokens that can found them. But again, as far as I can see, Kvasz's account does not include the distinctions that would seem to be necessary to characterize these linguistic types. It is interesting here to note that it might be very useful to try to find apt language for describing the types involved here—but this is a highly linguistically technical project which I currently do not have the space, much less the expertise, to undertake.

56. It should be noted here that our investigation of Husserl's account of idealization has rendered Kvasz's use of this particular locution fuzzy. For Husserl ideal objects are the intentional contents of language, not only the mathematical constructions Kvasz uses this locution to denote.
57. These expositions of the reductions are largely inspired by Føllesdal in his "Husserl's Reductions." The comparisons with Kant are original.
58. An interesting contemporary example of the kind of logic meant here is described in Kit Fine, "Essence and Modality." What he there calls "essences" are as close to what Husserl means by this term as I can find in contemporary analytic literature. Fine differs from Husserl, however, insofar as he proposes to reduce modal necessity to laws of essence; it is not clear to me that Husserl would accept this, although I have some ideas as to how he might.
59. In fact, the precise way in which sense-contents constrain ideas is very opaque. For example, we know that the general form of language has to answer to answer to the structure of the sense data. One answer to this problem immediately follows, which is the Kantian answer, the transcendental answer, that the order of constraint goes the other direction. Another answer is more realist with respect to sense-contents; I leave open the possibility of a line of research which grounds form in structured-content in this way, although many claim to rule this possibility out a priori. In any case, none of what follows hinges on these remarks.
60. Cf. Husserl, *Crisis*, §§25–35.
61. In particular, I remain puzzled as to the possible Kantian correlate of Kvasz's notion of the horizon.
62. Kant, *Critique*, A95–130, esp. A119; B130–141. Cf. Rosenberg, *Accessing Kant*, pp. 108–139.
63. I do not preclude this possibility, of new aspects of form – indeed, I think we can't – but am only making the point that there must be an essence of the species 'epistemic subject' that affords the possibility of relativization by affording the possibility of novel instantiation.
64. Cf. Husserl, *Ideas*, §§33–34.
65. Brentano, *Psychology from an Empirical Standpoint*, p. 68.
66. Husserl, *Ideas*, §57, pp. 156–157.
67. For a great variety of comparisons between the early Husserl's and the early Wittgenstein's positions on many of the themes of this essay, see Smith, "Intentionality and Picturing."
68. Once again, due to my untimely experience with COVID-19, I was unable to really open up another problem I began to notice in the later stages of writing this essay,

regarding precisely how the transcendental ego should be understood. This problem became pronounced after reading Sartre's *Transcendence of the Ego*, which was recommended to me by Prof. van Fraassen. The latter has written at least two essays on how he thinks the transcendental ego ought to be understood, from a relatively logical point of view: "The Transcendence of the Ego (The Non-Existent Knight)" and "How Can We Understand Transcendence of the Ego?" If the reader is interested in where these concerns go, they may contact me directly, assuming I am able to be located. At the time I write this, I cannot imagine that the implications of Sartre's essay, if correct, interact nicely with the conclusions of the present essay; they might, but I can't currently see how.

69. At this point I wonder if there is any real difference between the following two locutions: "the structure of mathematical consciousness" and "the mathematical structure of consciousness." It is not clear to me that these are necessarily different, or even that Husserl would think so.

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