1. **Context.** Let J be a complex structure on X satisfying $IJ = -JI$ and such that (X, g, J) is Kähler. Therefore

$$\nabla J = 0 \tag{1.1}$$

and moreover one can take coordinates x^a on X in a neighboorhood of O such that

$$\forall a, b \quad \nabla_{\partial_a} \partial_b = 0 \tag{1.2}$$

at the point O. Such coordinate need not be normal at O ($g_{ij} \neq \delta_{ij}$).

2. **Vanishing of the first derivative.** In such coordinate, (1.1) readily gives

$$\partial_a J_i^j|_O = 0 \tag{2.1}$$

Indeed, (1.2) implies the vanishing of the Christoffel symbols Γ_{ab}^c at O and parallelism given by (1.1) reads

$$\partial_a J_i^j = -\Gamma_{ak}^j J_i^k + \Gamma_{ai}^k J_k^j \tag{2.2}$$

which therefore goes to 0 at O.

3. **Second derivative of the tensor J.** One wants to compute higher derivative of the tensor J at O

$$\partial_b \partial_a J_i^j|_O \tag{3.1}$$

Take $A = \partial_a$, $B = \partial_b$, $X = \partial_i$, $Y = \partial_j$, then

$$\partial_b \partial_a \left(J_i^k g_{kj} \right) = B\left(A\left(g\left(JX, Y\right)\right)\right) \tag{3.2}$$

The right-hand side can be expanded as follows

$$\begin{aligned}
\partial_b \partial_a \left(J_i^k g_{kj} \right) &= B\left(A\left(g\left(JX, Y\right)\right)\right) \\
&= B\left(g\left(\nabla_A (JX), Y\right) + g\left(JX, \nabla_A Y\right)\right) \\
&= g\left(\nabla_B \nabla_A (JX), Y\right) + g\left(\nabla_A (JX), \nabla_B Y\right) + g\left(\nabla_B (JX), \nabla_A Y\right) + g\left(JX, \nabla_B \nabla_A Y\right)
\end{aligned}$$

by using the parallelism of the metric g under the Levi-Civita connection ∇.

Moreover (1.1) gives for all vector C and vector field Z,

$$\nabla_C (JZ) = J(\nabla_C Z) \tag{3.3}$$

Combining the equations above with (1.2) yields

$$\partial_b \partial_a \left(J_i^k g_{kj} \right)|_O = g\left(J \nabla_B \nabla_A X, Y\right) + g\left(JX, \nabla_B \nabla_A Y\right) \tag{3.4}$$

On the other hand,

$$\partial_b \partial_a \left(J_i^k g_{kj} \right) = g_{kj} \left(\partial_b \partial_a J_i^k \right) + \left(\partial_a J_i^k \right) \left(\partial_b g_{kj} \right) + \left(\partial_b J_i^k \right) \left(\partial_a g_{kj} \right) + J_i^k \left(\partial_b \partial_a g_{kj} \right) \tag{3.5}$$

As for x goes to O, (2.1) yields

$$\partial_b\partial_a\left(J_i^k g_{kj}\right)|_O = g_{kj}\left(\partial_b\partial_a J_i^k\right)|_O + J_i^k\left(\partial_b\partial_a g_{kj}\right)|_O \tag{3.6}$$

4. **The computation of $\nabla_B\nabla_A X$.** In all generality, one has

$$\nabla_C Z = C^a\left(\nabla_a X\right)^k = C^a\left(\partial_a Z^k + \Gamma_{al}^k Z^l\right)\partial_k \tag{4.1}$$

Hence, with $C = B = \partial_b$

$$\nabla_B Z = \left(\partial_b Z^k + \Gamma_{bl}^k Z^l\right)\partial_k \tag{4.2}$$

Applying this to $Z = \nabla_A X = \left(\partial_a \delta_i^k + \Gamma_{al}^k \delta_i^l\right)\partial_k = \Gamma_{ai}^k \partial_k$, one finally obtains

$$\nabla_B\nabla_A X = \left(\partial_b \Gamma_{ai}^k + \Gamma_{bl}^k \Gamma_{ai}^l\right)\partial_k \tag{4.3}$$

And moreover evaluating in O kills the Christoffel symbols and gives

$$\nabla_B\nabla_A X|_O = \partial_b \Gamma_{ai}^k \partial_k \tag{4.4}$$

And similarly

$$\nabla_B\nabla_A Y|_O = \partial_b \Gamma_{aj}^k \partial_k \tag{4.5}$$

5. **Conclusion and links with curvature.** In this paragraph, all derivative will be implicitly evaluated at O.

First by pluggin (4.4) and (4.5) in (3.4), one gets

$$J_k^l g_{lj}\partial_b\Gamma_{ai}^k + J_i^l g_{kl}\partial_b\Gamma_{aj}^k \tag{5.1}$$

And therefore combininig (5.1) and (3.6) one gets

$$g_{kj}\left(\partial_b\partial_a J_i^k\right) + J_i^k\left(\partial_b\partial_a g_{kj}\right) = J_k^l g_{lj}\partial_b\Gamma_{ai}^k + J_i^l g_{kl}\partial_b\Gamma_{aj}^k \tag{5.2}$$

which yields

$$\partial_b\partial_a J_i^j = J_k^j\partial_b\Gamma_{ai}^k + g^{jm}g_{kl}J_i^l\partial_b\Gamma_{am}^k - g^{jm}J_i^k\left(\partial_b\partial_a g_{km}\right) \tag{5.3}$$

The Riemann curvature tensor express in O in the coordinates x^a

$$R_{mab}^k = \partial_a\Gamma_{bm}^k - \partial_b\Gamma_{am}^k \tag{5.4}$$

keeping in mind the symmetry of the Christoffel symbols, namely

$$\Gamma_{ij}^k = \Gamma_{ji}^k \tag{5.5}$$

And

$$\partial_b\partial_a g_{km} = -\frac{1}{3}\left(R_{kbma} - R_{kamb}\right) \tag{5.6}$$

Hence

$$g^{jm}\left(\partial_b\partial_a g_{km}\right) = -\frac{1}{3}\left(g^{jm}R_{makb} - g^{jm}R_{mbka}\right) = -\frac{1}{3}\left(R_{akb}^j - R_{bka}^j\right) \tag{5.7}$$

After some manipulation one gets

$$\partial_b\partial_a J_i^j = J_k^j \partial_b \Gamma_{ai}^k + J_i^l g_{kl} g^{jm} \partial_b \Gamma_{am}^k + \frac{1}{3}\left(R_{akb}^j - R_{bka}^j\right) J_i^k \tag{5.8}$$

which only depends on $J(O)$ and on the curvature and the christoffel symbols of g. For instance assuming ...

$$\partial_b\partial_a J_i^j = J_k^l \left(\delta_l^j \partial_b \Gamma_{ai}^k + \delta_i^k \left(g_{pl} g^{jm} \partial_b \Gamma_{am}^p + \frac{1}{3}\left(R_{alb}^j - R_{bla}^j\right)\right)\right) \tag{5.9}$$

Annexe A. other relations

Orthogonality of J

$$g(JZ, Z') = -g(Z, JZ') \tag{0.10}$$

in coordinates

$$g_{kl} J_i^l = -J_k^l g_{li} \tag{0.11}$$