Chapter 4

Fundamental Groups of Algebraic Curves

In the previous chapter the Riemann Existence Theorem created a link between the category of compact connected Riemann surfaces and that of finite extensions of $\mathbf{C}(t)$. This hints at a possibility of developing a theory of the fundamental group in a purely algebraic way. We shall now present such a theory for curves over an arbitrary perfect base field, using a modest amount of algebraic geometry. Over the complex numbers the results will be equivalent to those of the previous chapter, but a new and extremely important feature over an arbitrary base field k will be the existence of a canonical quotient of the algebraic fundamental group isomorphic to the absolute Galois group of k. In fact, over a subfield of $\mathbf{C}$ we shall obtain an extension of the absolute Galois group of the base field by the profinite completion of the topological fundamental group of the corresponding Riemann surface over $\mathbf{C}$. This interplay between algebra and topology is a source for many powerful results in recent research. Among these we shall discuss applications to the inverse Galois problem, Belyi's theorem on covers of the projective line minus three points and some advanced results on 'anabelian geometry' of curves.

Reading this chapter requires no previous acquaintance with algebraic geometry. We shall, however, use some standard results from commutative algebra that we summarize in the first section. The next three sections contain foundational material, and the discussion of the fundamental group itself begins in Chapter 5.4.

4.1 Background in Commutative Algebra

We collect here some standard facts from algebra needed for subsequent developments. The reader is invited to use this section as a reference and consult it only in case of need. In what follows, the term 'ring' means a commutative ring with unit.

Recall that given an extension of rings $A \subset B$, an element $b \in B$ is said to be *integral* over A if it is a root of a *monic* polynomial $x^n + a_{n-1}x^{n-1} + \cdots + a_0 \in A[x]$. The *integral closure* of A in B consists of the elements of B integral over A; if this is the whole of B, then one says that the extension $A \subset B$ is *integral* or that B is *integral over A*. Finally, A is *integrally closed* in B if it equals its integral closure in B. In the special case when A is an integral domain and B is the fraction field of A one says that A is *integrally closed.* The basic properties of integral extensions may be summarized as follows.

Facts 4.1.1 *Let $A \subset B$ be an extension of rings.*

1. *An element $b \in B$ is integral over A if and only if the subring $A[b]$ of B is finitely generated as an A-module.*

2. *The integral closure of A in B is a subring of B, and moreover it is integrally closed in B.*

3. *Given a tower extensions $A \subset B \subset C$ with $A \subset B$ and $B \subset C$ integral, the extension $A \subset C$ is also integral.*

4. *If B is integral over A and $P \subset A$ is a prime ideal, there exists a prime ideal $Q \subset B$ with $Q \cap A = P$. Here P is a maximal ideal in A if and only if Q is a maximal ideal in B.*

All these facts are proven in [48], Chapter VII, §1, or [2], Chapter 5, 5.1–5.10.

Example 4.1.2 An unique factorisation domain A is integrally closed. Indeed, if an element a/b of the fraction field (with a, b coprime) satisfies a monic polynomial equation over A of degree n, then after multiplication by b^n we see that a^n should be divisible by b, which is only possible when b is a unit.

Assume now that $A \subset B$ is an integral extension of integrally closed domains, such that moreover the induced extension $K \subset L$ of fraction fields is Galois with finite Galois group G. Then B is stable by the action of G on L, being the integral closure of A in L. Given a maximal ideal $P \subset A$ denote by S_P the set of maximal ideals $Q \subset B$ with $Q \cap A = P$. The group G acts on the finite set S_P: each $\sigma \in G$ sends $Q \in S_P$ to the prime ideal $\sigma(Q) \in S_P$.

Fix $Q \in S_P$, and denote by D_Q its stabilizer in G for the above action. The action of each $\sigma \in D_Q$ on B induces an automorphism $\bar{\sigma}$ of $\kappa(Q)$ fixing $\kappa(P)$ elementwise. Moreover, the map $\sigma \mapsto \bar{\sigma}$ is a homomorphism $D_Q \to \mathrm{Aut}(\kappa(Q)|\kappa(P))$. Its kernel I_Q is a normal subgroup of D_Q called the *inertia subgroup* at Q.

Facts 4.1.3 *In the situation above the following statements hold.*

1. *The group G acts transitively on the set S_P; in particular, S_P is finite.*

2. *The subgroups D_Q and I_Q for $Q \in S_P$ are all conjugate in G.*

3. *It the extension $\kappa(Q)|\kappa(P)$ is separable, then it is a Galois extension and the homomorphism $D_Q/I_Q \to \mathrm{Gal}\,(\kappa(Q)|\kappa(P))$ defined above is an isomorphism.*

4. *If the order of I_Q is prime to the characteristic of $\kappa(P)$, then I_Q is cyclic.*

Statement (1) is [48], Chapter VII, Proposition 2.1, and (2) results from (1). Statement (3) is proven in [48], Chapter VII, Proposition 2.5. Finally, statement (4) results from [69], Theorem 9.12 (and the discussion preceding it).

We now state an important result concerning the finiteness of integral closure.

Facts 4.1.4 *Let A be an integral domain with fraction field K, and let $L|K$ be a finite extension. Assume that either*

a) A is integrally closed and $L|K$ is a separable extension, or

b) A is a finitely generated algebra over a field.

Then the integral closure of A in L is a finitely generated A-module.

For the proof of part *a*), see e.g. [2], Corollary 5.17; for *b*), see [18], Corollary 13.13.

An integral domain A is called a *Dedekind ring* if A is Noetherian (i.e. all of its ideals are finitely generated), integrally closed, and all nonzero prime ideals in A are maximal. Basic examples of Dedekind rings are polynomial rings in one variable over a field, the ring $\mathbf{Z}$ of integers and, more generally, the integral closure of $\mathbf{Z}$ in a finite extension of $\mathbf{Q}$.

Facts 4.1.5 *Let A be a Dedekind ring.*

1. *Every nonzero ideal $I \subset A$ decomposes uniquely as a product $I = P_1^{e_1} \cdots P_r^{e_r}$ of powers of prime ideals P_i.*

2. *For a prime ideal $P \subset A$ the localization A_P is a principal ideal domain.*

Recall that the localization A_P means the fraction ring of A with respect to the multiplicatively closed subset $A \setminus P$, which in our case is the subring of the fraction field of A constisting of fractions with denominator in $A \setminus P$. For a proof, see e.g. [2], Theorem 9.3 and Corollary 9.4.

Note that in view of Facts 4.1.1 (2), (4) and 4.1.4 *a*) the integral closure of a Dedekind ring in a finite separable extension of its fraction field is again a Dedekind ring. We then have the following consequence of the above facts.

Proposition 4.1.6 *Let A be a Dedekind ring with fraction field K, and let B be the integral closure of A in a finite separable extension $L|K$. For a nonzero prime ideal $P \subset A$ consider the decomposition $PB = Q_1^{e_1} \cdots Q_r^{e_r}$ in B. Then*

$$\sum_{i=1}^{r} e_i[\kappa(Q_i) : \kappa(P)] = [L : K].$$

Proof: By the Chinese Remainder Theorem and Fact 4.1.5 (1) we have an isomorphism

$$B/PB \cong B/Q_1^{e_1} \oplus \cdots \oplus B/Q_r^{e_r}. \tag{4.1}$$

Since each Q_i generates a principal ideal (q_i) in the localization B_{Q_i} by Fact 4.1.5 (2), the map $b \mapsto q_i^j b$ induces isomorphisms $\kappa(Q_i) = B/Q_i \xrightarrow{\sim} Q_i^j/Q_i^{j+1}$ for all $0 \le j \le e_i - 1$. It follows that the left hand side of the formula of the proposition equals the dimension of the $\kappa(P)$-vector space B/PB. Choose elements $t_1 \dots, t_n \in B$ whose images B/PB form a basis of this vector space. By a form of Nakayama's lemma ([48], Chapter X, Lemma 4.3) the t_i generate the finitely generated A_P-module $B \otimes_A A_P$, hence they generate the K-vector space L. It remains to see that there is no nontrivial relation $\sum a_i t_i = 0$ with $a_i \in K$. Assume there is one. By multiplying with a suitable generator of the principal ideal PA_P we may assume the a_i lie in A_P and not all of them are in PA_P. Reducing modulo PA_P we then obtain a nontrivial relation with coefficients in $\kappa(P)$, which is impossible. □

The integers e_i in formula (4.1) are called the *ramification indices* at the prime ideals Q_i lying above P. As it will turn out later, it is by no means accidental that we are using the same terminology as for branched topological covers.

Corollary 4.1.7 *Let $A \subset B$ be an integral extension of Dedekind rings such that the induced extension $K \subset L$ of fraction fields is a finite Galois extension with group G, and let P be a maximal ideal of A. Assume that the extensions $\kappa(Q_i)|\kappa(P)$ are separable for all $Q_i \in S_P$. Then the ramification indices e_i in the above formula are the same for all i, and they equal the order of the inertia subgroups at the Q_i.*

Proof: It is enough to verify the second statement for the inertia subgroup at Q_1, the rest then follows from Proposition 4.1.3 (2). Let K_1 be the subfield of L fixed by D_{Q_1}, A_1 the integral closure of A in K_1 and $P_1 := Q_1 \cap A_1$. Since Q_1 is the only maximal ideal of B above P_1 by construction, the formula of the proposition reads $|D_{Q_1}| = e_1 \cdot [\kappa(Q_1) : \kappa(P_1)]$. On the other hand, Proposition 4.1.3 (3) implies $|D_{Q_1}| = |I_{Q_1}| \cdot [\kappa(Q_1) : \kappa(P_1)]$ (note that the extension $\kappa(Q_1)|\kappa(P_1)$ is separable, being a subextension of $\kappa(Q_1)|\kappa(P)$). The statement follows by comparing the two equalities. □

Before leaving this topic, we collect some facts about local Dedekind rings.

Fact 4.1.8 *The following are equivalent for an integral domain A.*

1. *A is a local Dedekind ring.*
2. *A is a local principal ideal domain that is not a field.*
3. *A is a Noetherian local domain with nonzero principal maximal ideal.*

For a proof, see e.g. [57], Theorem 11.2. Such rings are called *discrete valuation rings* in the literature.

Proposition 4.1.9 *Let A be a discrete valuation ring, and t a generator of its maximal ideal.*

1. *Every nonzero $a \in A$ can be written as $a = ut^n$ with some unit $u \in A$ and $n \geq 0$. Here n does not depend on the choice of t.*
2. *If x is an element of the fraction field K of A, then either x or x^{-1} is contained in A.*
3. *If $B \supset A$ is a discrete valuation ring with the same fraction field, then $A = B$.*

Proof: The intersection of the ideals (t^n) is 0 (this follows, for instance, from the fact that A is a principal ideal domain). Thus for $a \neq 0$ there is a unique $n \geq 0$ with $a \in (t^n) \setminus (t^{n+1})$, whence (1). By (1), if x is an element of the fraction field, we may write $x = ut^n$ with a unit u and $t \in \mathbf{Z}$, whence (2). For statement (3), assume $b \in B$ is not a unit. Then $b \in A$, for otherwise we would have $b^{-1} \in A \subset B$ by (2), which is impossible. It moreover follows that $b \in (t)$ (otherwise it would be a unit), so using (1) we see that t cannot be a unit in B. It follows that non-units in A are non-units in B, from which we conclude by (2) that the units of B lie in A. □

The first two statements imply:

Corollary 4.1.10 *Every nonzero $a \in K$ can be written as $a = ut^n$ with some unit $u \in A$ and $n \in \mathbf{Z}$. Here n does not depend on the choice of the generator t.*

In the notation of the corollary, the rule $a \mapsto n$ defines a well-defined map $v : K^\times \to \mathbf{Z}$ called the *discrete valuation associated with A*. It is actually a homomorphism of abelian groups and satisfies $v(x + y) \geq \min(v(x), v(y))$. One often extends v to a map $K \to \mathbf{Z} \cup \{\infty\}$ by setting $v(0) = \infty$.

Finally, let A be a finitely generated algebra over a field k. Recall that A is Noetherian by the Hilbert basis theorem. If A is an integral domain, we define its *transcendence degree* over k to be that of its fraction field K. Recall that this is the largest integer d for which there exist elements $a_1, \ldots, a_d$ in K that are algebraically independent over k, i.e. they satisfy no nontrivial polynomial relations with coefficients in k. As K is finitely generated over k, such a d exists.

Fact 4.1.11 (Noether's Normalization Lemma) *Let A be an integral domain finitely generated over a field k, of transcendence degree d. There exist algebraically independent elements $x_1, \ldots, x_d \in A$ such that A is finitely generated as a $k[x_1, \ldots, x_d]$-module.*

See [48], Chapter VIII, Theorem 2.1 for a proof. Notice that in the situation above $k[x_1, \ldots, x_d]$ is isomorphic to a polynomial ring.

Corollary 4.1.12 *If A is as above and $d = 1$, then every nonzero prime ideal in A is maximal.*

Hence if moreover A is integrally closed, it is a Dedekind ring.

Proof: Let $P \subset A$ be a nonzero prime ideal, and use the normalization lemma to write A as an integral extension of the polynomial ring $k[x]$. The prime ideal $P \cap k[x]$ of $k[x]$ is nonzero, because if t is a nonzero element of P, it satisfies a monic polynomial equation $t^n + a_{n-1}t^{n-1} + \cdots + a_0 = 0$ with $a_i \in k[x]$. Here we may assume a_0 is nonzero, but it is an element of $P \cap k[x]$ by the equation. Now all nonzero prime ideals in $k[x]$ are generated by irreducible polynomials and hence they are maximal. Thus P is maximal by Fact 4.1.1 (4). □

Corollary 4.1.13 *Let A be an integral domain finitely generated over a field k, and let $M \subset A$ be a maximal ideal. Then A/M is a finite algebraic extension of k.*

Proof: Apply Noether's Normalization Lemma to A/M. If A/M had positive transcendence degree d, it would be integral over the polynomial ring $k[x_1, \ldots, x_d]$ This contradicts Fact 4.1.1 (4) (with $P = Q = 0$), because A/M is a field but the polynomial ring isn't. □

Corollary 4.1.14 *Let k be algebraically closed. Every maximal ideal M in the polynomial ring $k[x_1, \ldots, x_n]$ is of the form $(x_1 - a_1, \ldots, x_n - a_n)$ with appropriate $a_i \in k$.*

Proof: As k is algebraically closed, we have an isomorphism $k[x_1, \ldots, x_n]/M \cong k$ by the previous corollary. Let a_i be the image of x_i in k via this isomorphism. Then M contains the maximal ideal $(x_1 - a_1, \ldots, x_n - a_n)$, hence they must be equal. □

Corollary 4.1.15 *Let k be a field, and $\phi : A \to B$ a k-homomorphism of finitely generated k-algebras. If M is a maximal ideal in B, then $\phi^{-1}(M)$ is a maximal ideal in A.*

Proof: By replacing A with $\phi(A)$ we may assume A is a subring of B and $\phi^{-1}(M) = M \cap A$. In the tower of ring extensions $k \subset A/(M \cap A) \subset B/M$ the field B/M is a finite extension of k by Corollary 4.1.13, so $A/(M \cap A)$ is an integral domain algebraic over k. By Fact 4.1.1 (4) it must be a field, i.e. $M \cap A$ is maximal in A. □

Corollaries 4.1.13 and 4.1.14 are weak forms of Hilbert's Nullstellensatz. Here is a statement that may be considered as a strong form. Recall that the *radical* $\sqrt{I}$ of an ideal I in a ring A consists of the elements $f \in A$ satisfying $f^m \in I$ with an appropriate $m > 0$.

Fact 4.1.16 *Let A be an integral domain finitely generated over a field k, and let $I \subset A$ be an ideal. The radical $\sqrt{I}$ is the intersection of all maximal ideals containing I.*

This follows from [48], Chapter IX, Theorem 1.5 in view of the previous corollary. See also [18], Corollary 13.12 combined with Corollary 2.12.

4.2 Curves over an Algebraically Closed Field

We now introduce the main objects of study in this chapter in the simplest context, that of affine curves over an algebraically closed field. When the base field is $\mathbf{C}$, we shall establish a connection with the theory of Riemann surfaces.

We begin by defining affine varieties over an *algebraically closed field* k. To this end, let us identify points of *affine n-space* $\mathbf{A}^n$ over k with

$$\mathbf{A}^n(k) := \{(a_1, \ldots, a_n) : a_i \in k\}.$$

Given an ideal $I \subset k[x_1, \ldots, x_n]$, we define

$$V(I) := \{P \in \mathbf{A}^n(k) : f(P) = 0 \quad \text{for all}\, f \in I\}.$$

Definition 4.2.1 The subset $X := V(I) \subset \mathbf{A}^n(k)$ is called the *affine closed set* defined by I.

Remark 4.2.2 According to the Hilbert Basis Theorem there exist finitely many polynomials $f_1, \ldots, f_m \in k[x_1, \ldots, x_n]$ with $I = (f_1, \ldots, f_m)$. Therefore

$$V(I) = \{P \in \mathbf{A}^n(k) : f_i(P) = 0 \quad i = 1, \ldots, m\}.$$

Example 4.2.3 Let us look at the simplest examples. For $n = 1$ each ideal in $k[x]$ is generated by a single polynomial f; since k algebraically closed, f factors in linear factors $x - a_i$ with some $a_i \in k$. The affine closed set we obtain is a finite set of points corresponding to the a_i.

For $n = 2, m = 1$ we obtain the locus of zeros of a single two-variable polynomial f in $\mathbf{A}^2$: it is a *plane curve*. In general it may be shown that an affine closed set in $\mathbf{A}^2$ is always the union of a plane curve and a finite set of points.

The following lemma is an easy consequence of the definition; its proof is left to the readers.

Lemma 4.2.4 *Let $I_1, I_2, I_\lambda\,(\lambda \in \Lambda)$ be ideals in $k[x_1, \ldots, x_n]$. Then*

a) $I_1 \subseteq I_2 \Rightarrow V(I_1) \supseteq V(I_2)$;

b) $V(I_1) \cup V(I_2) = V(I_1 \cap I_2) = V(I_1 I_2)$;

c) $V(\langle I_\lambda : \lambda \in \Lambda \rangle) = \bigcap_{\lambda \in \Lambda} V(I_\lambda)$.

The last two properties imply that the affine closed sets may be used to define the closed subsets in a topology on $\mathbf{A}^n$ (note that $\mathbf{A}^n = V(0)$, $\emptyset = V(1)$). This topology is called the *Zariski topology* on $\mathbf{A}^n$, and affine closed sets are equipped with the induced topology. A basis for the Zariski topology is given by the open subsets of the shape $D(f) := \{P \in \mathbf{A}^n : f(P) \neq 0\}$, where $f \in k[x_1, \ldots, x_n]$ is a fixed polynomial. Indeed, each closed subset $V(I)$ is the intersection of subsets of the form $V(f)$.

If $I = \sqrt{I}$, then by Fact 4.1.16 it is the intersection of the maximal ideals containing it. These are of the form $(x_1 - a_1, \ldots, x_n - a_n)$ with some $a_i \in k$ according to Corollary 4.1.14, therefore I consists precisely of those $f \in k[x_1, \ldots, x_n]$ that vanish at all $P \in V(I)$. Thus in this case the ideal I and the set $X = V(I)$ determine each other, and we call X an *affine variety.*

Definition 4.2.5 If $X = V(I)$ is an affine variety, we define the *coordinate ring* of X to be the quotient $\mathcal{O}(X) := k[x_1, \ldots, x_n]/I$. Its elements are called *regular functions on* X; the images $\bar{x}_i$ of the x_i are called the *coordinate functions* on X.

We may evaluate a regular function $f \in \mathcal{O}(X)$ at a point $P = (a_1, \ldots, a_n) \in X$ by putting $f(P) := \tilde{f}(a_1, \ldots, a_n)$ with a preimage $\tilde{f}$ of f in $k[x_1, \ldots, x_n]$; the value does not depend on the choice of $\tilde{f}$.

Note that by definition the finitely generated k-algebra $\mathcal{O}(X)$ is reduced, i.e. it has no nilpotent elements. It may have zero-divisors, however; it is an integral domain if and only if I is a prime ideal. In that case we say that X is an *integral* affine variety over k.

Example 4.2.6 If we look back at the examples in 4.2.3, we see that for $n = 1$ the affine closed set we obtain is a variety if and only if the a_i are distinct, and it is integral if and only if it is a single point, i.e. $f = x - a_i$.

The affine plane curve $X = V(f) \subset \mathbf{A}^2$ is an integral variety if and only if f is irreducible.

We may use the notion of regular functions to define morphisms of affine varieties, and hence obtain a category.

Definition 4.2.7 Given an affine variety $Y = V(J)$, by a *morphism* or *regular map* $Y \to \mathbf{A}^m$ we mean an m-tuple $\phi = (f_1, \ldots, f_m) \in \mathcal{O}(Y)^m$. Given an affine variety $X \subset \mathbf{A}^m$, by a morphism $\phi : Y \to X$ we mean a morphism $\phi : Y \to \mathbf{A}^m$ such that $\phi(P) := (f_1(P), \ldots, f_m(P)) \in \mathbf{A}^m$ lies in X for all points $P \in Y$.

Example 4.2.8 If $X = V(f)$ is a plane curve, a morphism $X \to \mathbf{A}^1$ is defined by a polynomial $f_1 \in k[x_1, x_2]$. The polynomial $f_1 + fg$ defines the same morphism $X \to \mathbf{A}^1$ for all $g \in k[x_1, x_2]$. If $Y = V(h)$ is another plane curve, a morphism $Y \to X$ is defined by a pair of polynomials (h_1, h_2) such that $f \circ (h_1, h_2)$ is a multiple of h. Again, the h_i are determined up to adding multiples of h.

If $\phi : Y \to X$ is a morphism of affine varieties, there is an induced k-algebra homomorphism $\phi^* : \mathcal{O}(X) \to \mathcal{O}(Y)$ given by $\phi^*(f) = f \circ \phi$. Note that ϕ^* takes functions vanishing at a point $P \in X$ to functions vanishing at the points $\phi^{-1}(P)$, so the preimage in $\mathcal{O}(X)$ of the ideal of $\mathcal{O}(Y)$ corresponding to a point $Q \in \phi^{-1}(P)$ is precisely the ideal defined by P.

Remark 4.2.9 A morphism $\phi : Y \to X$ is continuous with respect to the Zariski topology. Indeed, it is enough to see that the preimage of each basic open set $D(f) \subset X$ is open in Y, which holds because $\phi^{-1}(D(f)) = D(\phi^*(f))$.

Proposition 4.2.10 *The maps $X \to \mathcal{O}(X)$, $\phi \to \phi^*$ induce an anti-equivalence between the category of affine varieties over k and that of finitely generated reduced k-algebras.*

Proof: For fully faithfulness, let $\bar{x}_1, \ldots, \bar{x}_m$ be the coordinate functions on X. Then $\phi^* \mapsto (\phi^*(\bar{x}_1), \ldots, \phi^*(\bar{x}_m))$ defines an inverse for the map $\phi \mapsto \phi^*$. For essential surjectivity, simply write a finitely generated reduced k-algebra as a quotient $A \cong k[x_1, \ldots, x_n]/I$. Then $X = V(I)$ is a good choice. □

To proceed further, we define regular functions and morphisms on open subsets of an integral affine variety X. First, the *function field* $K(X)$ of X is the fraction field of the integral domain $\mathcal{O}(X)$. By definition, an element of $K(X)$ may be represented by a quotient of polynomials f/g with $g \notin I$, with two quotients f_1/g_1 and f_2/g_2 identified if $f_1 g_2 - f_2 g_1 \notin I$.

Next let P be a point of X. We define the *local ring* $\mathcal{O}_{X,P}$ at P as the subring of $K(X)$ consisting of functions that have a representative with $g(P) \neq 0$. It is the same as the localization of $\mathcal{O}(X)$ by the maximal ideal corresponding to P. One thinks of it as the ring of functions 'regular at P'. For an open subset $U \subset X$ we define the *ring of regular functions on U* by

$$\mathcal{O}_X(U) := \bigcap_{P \in U} \mathcal{O}_{X,P},$$

the intersection being taken inside $K(X)$. The following lemma shows that for $U = X$ this definition agrees with the previous one.

Lemma 4.2.11 *For an integral affine variety X one has $\mathcal{O}(X) = \bigcap_{P \in X} \mathcal{O}_{X,P}$.*

Proof: To show the nontrivial inclusion, pick $f \in \bigcap_P \mathcal{O}_{X,P}$, and choose for each P a representation $f = f_P/g_P$ with $g_P \notin P$. By our assumption on the g_P none of the maximal ideals of $\mathcal{O}(X)$ contains the ideal $I := \langle g_P : P \in X \rangle \subset \mathcal{O}(X)$, so Fact 4.1.16 implies $I = \mathcal{O}(X)$. In particular, there exist $P_1, \ldots, P_r \in X$ with $1 = g_{P_1} h_{P_1} + \ldots g_{P_r} h_{P_r}$ with some $h_{P_i} \in \mathcal{O}(X)$. Thus

$$f = \sum_{i=1}^{r} f g_{P_i} h_{P_i} = \sum_{i=1}^{r} (f_{P_i}/g_{P_i}) g_{P_i} h_{P_i} = \sum_{i=1}^{r} f_{P_i} h_{P_i} \in \mathcal{O}(X),$$

as required. □

Now given integral affine varieties X and Y and open subsets $U \subset X$, $V \subset Y$, we define a morphism $\phi : V \to U$ similarly as above: we consider X together with its embedding in $\mathbf{A}^m$, and we define ϕ as an m-tuple $\phi = (f_1, \ldots, f_m) \in \mathcal{O}_Y(V)^m$ such that $\phi(P) := (f_1(P), \ldots, f_m(P))$ lies in U for all points $P \in Y$. We say that ϕ is an isomorphism if it has a two-sided inverse.

We now restrict the category under consideration. First a definition:

Definition 4.2.12 The *dimension* of an integral affine k-variety X is the transcendence degree of its function field $K(X)$ over k.

Remark 4.2.13 We can give a geometric meaning to this algebraic notion as follows. First note that since $\mathcal{O}(\mathbf{A}^n_k) = k[x_1, \ldots, x_n]$, affine n-space has dimension n, as expected. Next, let X be an integral affine variety of dimension n. The Noether Normalization Lemma (Fact 4.1.11) together with Proposition 4.2.10 shows that there is a surjective morphism $\phi : X \to \mathbf{A}^n$ so that moreover $\mathcal{O}(X)$ is a finitely generated $k[x_1, \ldots, x_n]$-module. The latter property implies that ϕ has finite fibres. Indeed, if $P = (a_1, \ldots, a_n)$ is a point in $\mathbf{A}^n$ and $M_P = (x_1 - a_1, \ldots, x_n - a_n)$ the corresponding maximal ideal in $k[x_1, \ldots, x_n]$, then $\mathcal{O}(X)/M_P\mathcal{O}(X)$ is a finite dimensional k-algebra, and as such has finitely many maximal ideals. Their preimages in $\mathcal{O}(X)$ correspond to the finitely many points in $\phi^{-1}(P)$. Thus n-dimensional affine varieties are 'finite over $\mathbf{A}^n$'.

Integral affine varieties of dimension 1 are called integral affine *curves*. The following lemma shows that their Zariski topology is particularly simple.

Lemma 4.2.14 *All proper Zariski closed subsets of an integral affine curve are finite.*

Proof: Quite generally, in a Noetherian ring every proper ideal I satisfying $I = \sqrt{I}$ is an intersection of finitely many prime ideals (a consequence of primary decomposition; see e.g. [2], Theorem 7.13). Therefore the lemma follows from Corollary 4.1.12. □

We now impose a further restriction, this time of local nature.

Definition 4.2.15 A point P of an integral affine variety X is *normal* if the local ring $\mathcal{O}_{X,P}$ is integrally closed. We say that X is normal if all of its points are normal.

Remark 4.2.16 In fact, X is normal if and only if $\mathcal{O}(X)$ is integrally closed. Indeed, if $\mathcal{O}(X)$ is integrally closed, then so is each localization $\mathcal{O}_{X,P}$; the converse follows from Lemma 4.2.11.

Normality is again an algebraic condition, but in dimension 1 the geometric meaning is easy to describe. In this case normality means by definition that the $\mathcal{O}_{X,P}$ are discrete valuation rings. We first look at the key example of plane curves.

Example 4.2.17 Let $X = V(f) \subset \mathbf{A}^2$ be an integral affine plane curve. Write x and y for the coordinate functions on $\mathbf{A}^2$ and assume that P is a point such that one of the partial derivatives $\partial_x f(P), \partial_y f(P)$ is nonzero; such a point is called a *smooth* point. Then $\mathcal{O}_{X,P}$ is a discrete valuation ring, i.e. P is a normal point.

To see this, we may assume after a coordinate transformation that $P = (0,0)$ and $\partial_y f(P) \neq 0$. The maximal ideal M_P of $\mathcal{O}_{X,P}$ is generated by x and y. Regrouping terms in the equation f we may write $f = \phi(x)x + \psi(x,y)y$, where $\phi \in k[x]$ and $\psi \in k[x,y]$. The constant term of ψ is $\partial_y(P)$, which is nonzero by assumption. Thus in $\mathcal{O}_{X,P}$ we may write $y = gx$, where g is the image of $-\phi\psi^{-1}$ in $\mathcal{O}_{X,P}$, and hence $M_P = (x)$. We conclude by Fact 4.1.8.

We now show that in characteristic 0 every normal affine curve is locally isomorphic to one as in the above example.

Proposition 4.2.18 *Assume k is of characteristic 0, and let X be an integral affine curve. Every normal point P of X has a Zariski open neighbourhood isomorphic to an open neighbourhood of a smooth point on an affine plane curve.*

Proof: The local ring $\mathcal{O}_{X,P}$ is a discrete valuation ring, so its maximal ideal is principal, generated by an element t. Since we are in characteristic 0, by the theorem of the primitive element we find $s \in K(X)$ such that $K(X) = k(t,s)$. Replacing s by st^m for m sufficiently large if necessary, we may assume $s \in \mathcal{O}_{X,P}$. Taking the minimal polynomial of s over $k(t)$ and multiplying by a common denominator of the coefficients we find an irreducible polynomial $f \in k[x,y]$ such that $f(t,s) = 0$ and moreover the fraction field of the ring $k[x,y]/(f)$ is isomorphic to $K(X)$. It follows that the map $(t,s) \mapsto (x,y)$ defines an isomorphism of $K(X)$ onto the function field of the plane curve $V(f) \subset \mathbf{A}^2_k$. If we choose $U \subset X$ so that $t, s \in \mathcal{O}_X(U)$, then the above map defines a morphism $\rho : U \to V(f)$. Conversely, the map $x \mapsto t, y \mapsto s$ defines a morphism $V(f) \to X$ that is an inverse to ρ on $\rho(U)$; in particular, $\rho(U)$ is open in $V(f)$. We conclude that X and $V(f)$ contain the isomorphic open subsets U and $\rho(U)$, with U containing P.

We finally show that $(\partial_y f)(\rho(P)) \neq 0$. The image of P by ρ is a point of the form $(0, \alpha)$; by composing ρ with the map $(x, y) \mapsto (x, y - \alpha)$ we may assume $\rho(P) = (0, 0)$. Since t generates the maximal ideal of $\mathcal{O}_{X,P} \cong \mathcal{O}_{V(f),(0,0)}$, we find $\bar{a}, \bar{b} \in \mathcal{O}(V(f))$ with $\bar{b}((0, 0)) \neq 0$ and $s = (\bar{a}/\bar{b})t$. Lifting them to polynomials $a, b \in k[x, y]$, we get the equality $by = ax + cf$ in $k[x, y]$. Taking partial derivative with respect to y gives $(\partial_y b)y + b = (\partial_y a)x_1 + (\partial_y c)f + c\partial_y f$. Evaluating at $(0, 0)$ we obtain $b(0, 0) = c(0, 0) \cdot \partial_y f(0, 0)$. Here the left hand side is nonzero since $\bar{b}((0, 0)) \neq 0$, hence so is $\partial_y f(0, 0)$. □

Remarks 4.2.19

1. The only place in the above proof where we used the characteristic 0 assumption is where we applied the theorem of the primitive element. But if t is a generator of the maximal ideal of a normal point as in the above proof, the extension $K(X)|k(t)$ is always separable (see e.g. [100], Proposition II.1.4), and hence the theorem applies. Thus the proposition extends to arbitrary characteristic.

2. Readers should be warned that in dimension greater than 1 the normality condition is weaker than smoothness (which is in general a condition on the rank of the Jacobian matrix of the equations of the variety; see Definition 5.1.30 and the subsequent discussion).

In the case $k = \mathbf{C}$ the above considerations enable us to equip a normal affine curve X with the structure of a Riemann surface.

Construction 4.2.20 Let X be an integral normal affine curve over $\mathbf{C}$, P a point of X. Choose a generator t of the maximal ideal of $\mathcal{O}_{X,P}$. By the discussion above we find an open neighbourhood U of P and a function $u \in \mathcal{O}_X(U)$ such that the map $(t, u) \mapsto (x, y)$ yields an isomorphism ρ of U onto a Zariski open subset of some $V(f) \subset \mathbf{A}^2_{\mathbf{C}}$ satisfying $(\partial_y f)(\rho(P)) \neq 0$. Equip $V(f)$ with the restriction of the complex topology of $\mathbf{C}^2$. As in Example 3.1.3 (4) we find a complex open neighbourhood V of $\rho(P)$ (which we may choose so small that it is contained in $\rho(U)$) where x defines a complex chart on $V(f)$. Now define a 'complex' topology on $\rho^{-1}(V)$ by pulling back the complex topology of V and declare $x \circ \rho$ to be a complex chart in the neighbourhood $\rho^{-1}(V)$ of P.

We contend that performing this construction for all $P \in X$ yields a well-defined topology and a complex atlas on X. Indeed, if $P' \in \rho^{-1}(V)$ is a point for which the complex chart is constructed via a morphism $\rho' : (t', s') \mapsto (x, y)$, the map $\tau : (t, s) \mapsto (t', s')$ defines an algebraic isomorphism between some Zariski open neighbourhoods of P and P'. The composite $\rho' \circ \tau \circ \rho^{-1}$ induces a holomorphic isomorphism between suitable small complex neighbourhoods of $\rho(P)$ and $\rho'(P)$ (an algebraic function regular at a point is always holomorphic in some neighbourhood). It follows that the topologies and the complex charts around P and P' are compatible.

Remarks 4.2.21

1. In fact, one sees that the complex chart $x \circ \rho$ in the neighbourhood of P viewed as a **C**-valued function is nothing but t. For this reason generators of the maximal ideal of $\mathcal{O}_{X,P}$ are called *local parameters at* P.

2. Given a morphism $\phi : Y \to X$ of normal affine curves over **C**, an examination of the above construction shows that ϕ is holomorphic with respect to the complex structures on Y and X.

4.3 Affine Curves over a General Base Field

We now extend the theory of the previous section to an arbitrary base field. The main difficulty over a non-closed field is that there is no reasonable way to identify a variety with a point set. For instance, though the polynomial $f = x^2+y^2+1$ defines a curve $V(f)_{\mathbf{C}}$ in $\mathbf{A}^2_{\mathbf{C}}$, it has no points with coordinates in **R**. Still, it would make no sense to define the real curve defined by f to be the empty set. Furthermore, the 'coordinate ring' $\mathbf{R}[x,y]/(x^2+y^2+1)$ still makes sense. If we tensor it with **C**, we obtain the ring $\mathcal{O}(V(f)_{\mathbf{C}})$ whose maximal ideals are in bijection with the points of $V(f)_{\mathbf{C}}$ as defined in the previous section. These points come in conjugate pairs: each pair corresponds to a maximal ideal in $\mathbf{R}[x,y]/(x^2+y^2+1)$.

If we examine the situation of the last section further, we see from Proposition 4.2.10 that the coordinate ring $\mathcal{O}(X)$ completely determines an affine variety X over the algebraically closed field. In particular, we may recover the Zariski topology: the open sets correspond to sets of maximal ideals not containing some ideal $I \subset \mathcal{O}(X)$. When X is integral, the function field, the local rings and the ring of regular functions on an open subset $U \subset X$ are all constructed out of $\mathcal{O}(X)$. Moreover, we see that for each pair $V \subset U$ of open subsets there are natural restriction homomorphisms $\mathcal{O}_X(U) \to \mathcal{O}_X(V)$, and thus the rule $U \mapsto \mathcal{O}_X(U)$ defines a presheaf of rings on X. It is immediate to check that the sheaf axioms are satisfied, so we obtain a sheaf of rings $\mathcal{O}_X$ on X, the *sheaf of regular functions*. To proceed further it is convenient to formalize the situation.

Definition 4.3.1 A *ringed space* is a pair $(X, \mathcal{F})$ consisting of a topological space X and a sheaf of rings $\mathcal{F}$ on X.

We now give the general definition of integral affine curves. This will be a special case of the definition of affine schemes to be discussed in the next chapter, but there are some simplifying features.

Construction 4.3.2 We define an integral affine curve over an arbitrary field k as follows. Start with an integral domain $A \supset k$ finitely generated and of transcendence degree 1 over k. By Corollary 4.1.12 every nonzero prime ideal in A is maximal. We associate a topological space X with A whose underlying set is the set of prime ideals of A, and we equip it with the topology in which the open subsets are X and those that do not contain a given ideal $I \subset A$. Note that all nonempty open subsets contain the point (0); it is called the *generic point* of X. The other points come from maximal ideals and hence are closed as one-point subsets; we call them *closed points*. By the same argument as in Corollary 4.2.14 the open subsets in X are exactly the subsets whose complement is a finite (possibly empty) set of closed points.

Given a point P in X, we define the local ring $\mathcal{O}_{X,P}$ as the localization A_P; note that for $P = (0)$ we obtain $\mathcal{O}_{X,(0)} = K(X)$, the fraction field of A. Finally, we put

$$\mathcal{O}_X(U) := \bigcap_{P \in U} \mathcal{O}_{X,P},$$

for an open subset $U \subset X$. As above, it defines a sheaf of rings on X. We define an integral affine curve over k to be a ringed space $(X, \mathcal{O}_X)$ constructed in the above way. We usually drop the sheaf $\mathcal{O}_X$ from the notation. When we would like to emphasize the relationship with A, we shall use the scheme-theoretic notation $X = \operatorname{Spec}(A)$.

Next we introduce morphisms for the curves just defined. They are to be morphisms of ringed spaces, whose general definition is as follows.

Definition 4.3.3 A morphism $(Y, \mathcal{G}) \to (X, \mathcal{F})$ of ringed spaces is a pair $(\phi, \phi^\sharp)$, where $\phi : Y \to X$ is a continuous map, and $\phi^\sharp : \mathcal{F} \to \phi_*\mathcal{G}$ a morphism of sheaves on X. Here $\phi_*\mathcal{G}$ denotes the sheaf on X defined by $\phi_*\mathcal{G}(U) = \mathcal{G}(\phi^{-1}(U))$ for all $U \subset X$; it is called the *pushforward* of $\mathcal{G}$ by ϕ.

In more down-to-earth terms, a morphism $Y \to X$ of integral affine curves is a continuous map $\phi : Y \to X$ of underlying spaces and a rule that to each regular function $f \in \mathcal{O}_X(U)$ defined over an open subset $U \subset X$ associates a function $\phi_U^\sharp(f)$ in $\mathcal{O}_Y(\phi^{-1}(U))$. One should think of $\phi_U^\sharp(f)$ as the composite $f \circ \phi$.

Remark 4.3.4 In the case when k is algebraically closed, this definition is in accordance with that of the previous section. Indeed, the morphisms defined there are continuous maps (Remark 4.2.9) and induce maps $\phi^\sharp$ of sheaves via the rule $f \mapsto f \circ \phi$. Conversely, to see that a morphism $\phi : X \to \mathbf{A}_k^n$ in the new sense induces a morphism as in the previous section it is enough to consider the n-tuple $(\phi^\sharp(x_1), \dots, \phi^\sharp(x_n))$.

We now establish an analogue of Proposition 4.2.10 for affine curves. To begin with, a finitely generated integral domain A of transcendence degree 1 over a field determines an integral affine curve $X = \operatorname{Spec}(A)$; conversely, an integral affine curve X gives rise to an A as above by setting $A = \mathcal{O}_X(X)$. By construction, these two maps are inverse to each other. We shall also use the notation $\mathcal{O}(X)$

instead of $\mathcal{O}_X(X)$. This is in accordance with the notation of the previous chapter, and we may also call $\mathcal{O}(X)$ the coordinate ring if X.

For affine curves $X = \mathrm{Spec}\,(A)$ and $Y = \mathrm{Spec}\,(B)$ a morphism $\phi : X \to Y$ induces a ring homomorphism $\phi_X^\sharp : A \to B$ given by $\mathcal{O}(X) \to (\phi_* \mathcal{O}_Y)(X) = \mathcal{O}(Y)$. Associating a morphism of curves with a ring homomorphism is a bit more complicated.

Lemma 4.3.5 *Given a homomorphism $\rho : A \to B$ with A and B as above, there is a unique morphism* $\mathrm{Spec}\,(\rho) : Y \to X$ *such that* $(\mathrm{Spec}\,(\rho))_X^\sharp : \mathcal{O}(X) \to \mathcal{O}(Y)$ *equals ρ.*

Proof: For a prime ideal $P \subset B$ the ideal $\rho^{-1}(P) \subset A$ is a prime ideal (indeed, the map $A/(\rho^{-1}(P)) \to B/P$ is injective, hence $A/(\rho^{-1}(P))$ is an integral domain). This defines a map of sets $\mathrm{Spec}\,(\rho) : Y \to X$ that is easily seen to be a continuous map of topological spaces. But in our situation we can say more. There are two cases.

Case 1: ρ is injective. In this case A is a subring of B via ρ, and moreover by Corollary 4.1.15 if P is a maximal ideal in B, then $\rho^{-1}(P) = P \cap A$ is a maximal ideal in A. Of course, we have $\rho^{-1}((0)) = (0)$.

Case 2: ρ is not injective. As we are dealing with curves, the ideal $M := \ker(\rho)$ is maximal in A, and so $\rho^{-1}(P) = M$ for all prime ideals $P \subset B$. This corresponds to a 'constant morphism' $Y \to \{M\}$.

We define morphisms of sheaves $\mathrm{Spec}\,(\rho)^\sharp : \mathcal{O}_X \to \mathrm{Spec}\,(\rho)_* \mathcal{O}_Y$ in each case. In Case 1 we have an inclusion of function fields $K(X) \subset K(Y)$ and also of localizations $A_{(P \cap A)} \subset B_P$ for each maximal ideal $P \subset B$. By taking intersections this defines maps $\mathcal{O}_X(U) \to \mathcal{O}_Y(\mathrm{Spec}\,(\rho)^{-1}(U))$ for each open set $U \subset X$; for $U = X$ we get $\rho : A \to B$ by the same argument as in Lemma 4.2.11. In Case 2 we define $\mathcal{O}_X(U) \to \mathcal{O}_Y(\mathrm{Spec}\,(\rho)^{-1}(U))$ to be the composite

$$\mathcal{O}_X(U) \to A_M \to A_M/MA_M \xrightarrow{\sim} A/M \to B$$

if $M \in U$, and to be 0 otherwise. The reader will check that this indeed yields a morphism of sheaves. □

The lemma and the arguments preceding it now imply:

Proposition 4.3.6 *The assignments $A \mapsto \mathrm{Spec}\,(A)$, $\rho \mapsto \mathrm{Spec}\,(\rho)$ and $X \mapsto \mathcal{O}(X)$, $\phi \mapsto \phi_X^\sharp$ yield mutually inverse contravariant functors between the category of integral domains finitely generated and of transcendence degree 1 over a field, and that of integral affine curves.*

Note that the conclusion here is stronger than in Proposition 4.2.10, because here we say that the two categories are actually *anti-isomorphic*: there is an arrow-reversing bijection between objects and morphisms. In Proposition 4.2.10 this was only true up to isomorphism, because an affine variety as defined there may have several embeddings in affine spaces.

We now discuss an important construction related to extensions of the base field.

Construction 4.3.7 Let $X = \mathrm{Spec}\,(A)$ be an integral affine curve over a field k, and $L|k$ a field extension for which the tensor product $A \otimes_k L$ is an integral domain. Then the integral affine curve $X_L = \mathrm{Spec}\,(A \otimes_k L)$ is defined. We call the resulting curve over L the *base change* of X to L. There is a natural morphism $X_L \to X$ corresponding by the previous proposition to the map $A \to A \otimes_k L$ sending a to $a \otimes 1$.

Assume that $A \otimes_k \bar{k}$ is an integral domain for an algebraic closure $\bar{k}|k$; in this case X is called *geometrically integral*. Then $A \otimes_k L$ is an integral domain for all algebraic extensions $L|k$, so that the above assumption on L is satisfied. Thus for a fixed algebraic extension $L|k$ the rule $X \mapsto X_L$ defines a functor from the category of geometrically integral affine k-curves to that of integral affine L-curves.

Example 4.3.8 We can now discuss the $\mathbf{R}$-curve with equation $x^2 + y^2 + 1 = 0$ rigorously. It is defined as $X := \mathrm{Spec}\,(\mathbf{R}[x, y]/(x^2 + y^2 + 1))$. The closed points of X correspond to the maximal ideals in $\mathbf{R}[x, y]$ containing $(x^2 + y^2 + 1)$; for each such ideal M we must have $\mathbf{R}[x, y]/M \cong \mathbf{C}$, as $\mathbf{C}$ is the only nontrivial finite extension of $\mathbf{R}$ and X has no points over $\mathbf{R}$. Under the base change morphism $X_{\mathbf{C}} \to X$ there are two closed points lying above each closed point of X, because $\mathbf{C} \otimes_{\mathbf{R}} \mathbf{C} \cong \mathbf{C} \oplus \mathbf{C}$. If we make $\mathrm{Gal}\,(\mathbf{C}|\mathbf{R})$ act on the tensor product via its action on the second term, then on the right hand side the resulting action interchanges the components.

To give a concrete example, the ideal $M = (x^2, y^2 + 1) \subset \mathbf{R}[x, y]$ contains $(x^2 + y^2 + 1)$, hence defines a point of X. The maximal ideals of $\mathbf{C}[x, y]$ lying above M are $(x, y+i)$ and $(x, y-i)$, corresponding to the points $(0, -i)$ and $(0, i)$. They are indeed conjugate under the Galois action.

We say that an integral affine curve is *normal* if its local rings are integrally closed. As in the previous section, this is equivalent to requiring that the coordinate ring $\mathcal{O}(X)$ is integrally closed.

We now prove an analogue of Theorem 3.3.7 for integral affine curves. For this we have to restrict the morphisms under the consideration. We say that a morphism $\phi : Y \to X$ of integral affine curves is *finite* if $\mathcal{O}(Y)$ becomes a finitely generated $\mathcal{O}(X)$-module via the map $\phi_X^\sharp : \mathcal{O}(X) \to \mathcal{O}(Y)$. A finite morphism has finite fibres, by the same argument as in Remark 4.2.13. This property is shared by proper holomorphic maps of Riemann surfaces.

Remark 4.3.9 A finite morphism of integral affine curves is always surjective. Indeed, Case 2 of the proof of Lemma 4.3.5 cannot occur for a finite morphism, and in Case 1 we may apply Fact 4.1.1 (4). An example of a non-finite morphism is given by the inclusion $\mathbf{A}^1_k \setminus \{0\} \to \mathbf{A}^1_k$, corresponding to the natural ring homomorphism $k[x] \to k[x, x^{-1}]$ (over any field k).

Now assume given a finite morphism $Y \to X$ integral affine curves. We have just remarked that the corresponding homomorphism $\mathcal{O}(Y) \to \mathcal{O}(X)$ of coordinate rings is injective, whence an inclusion of function fields $\phi^* : K(X) \subset K(Y)$. As the morphism is finite, this must be a finite extension.

Theorem 4.3.10 *Let X be an integral normal affine curve. The rule $Y \mapsto K(Y)$, $\phi \mapsto \phi^*$ induces an anti-equivalence between the category of normal integral affine curves equipped with a finite morphism $\phi : Y \to X$ and that of finite field extensions of the function field $K(X)$.*

Proof: For essential surjectivity take a finite extension $L|K(X)$, and apply Fact 4.1.4 *b*) with $A = \mathcal{O}(X)$. It implies that the integral closure B of $\mathcal{O}(X)$ in L is a finitely generated k-algebra, which is also integrally closed by Fact 4.1.1 (2). As L is finite over $K(X)$, it is still of transcendence degree 1. Applying Proposition 4.3.6 to the ring extension $\mathcal{O}(X) \subset B$ we obtain an integral affine curve $Y = \mathrm{Spec}\,(B)$ and a morphism $\phi : Y \to X$ inducing the ring inclusion $\mathcal{O}(X) \subset B$ above. Again by Fact 4.1.4 *b*) the morphism ϕ is finite. Fully faithfulness is proven by a similar argument as in Theorem 3.3.7. □

The affine curve Y constructed in the first part of the proof is called the *normalization* of X in L.

Examples 4.3.11

1. The theorem is already interesting over an algebraically closed field k. For instance if we take $X = \mathbf{A}^1_k$ and $L = k(x)[y]/(y^2 - f)$, where $f \in k[x]$ is of degree at least 3 having no multiple roots, then the normalization of $\mathbf{A}^1_k$ in L is the normal affine plane curve $V(y^2 - f) \subset \mathbf{A}^2_k$.

2. Over a non-closed field we get other kinds of examples as well. If we assume that X is geometrically integral, then for every finite extension $L|k$ we may look at the normalization of X in $L \otimes_k K(X)$. It will be none other than the base change X_L, because tensorizing with L does not affect integral closedness.

Remark 4.3.12 The concept of normalization is also interesting for a non-normal integral affine curve X. Taking the integral closure B of $\mathcal{O}(X)$ in $K(X)$ yields via Proposition 4.3.6 a normal integral affine curve $\widetilde{X}$ with function field $K(X)$ that comes equipped with a finite surjective morphism $\widetilde{X} \to X$. This implies a characterization of normality: an integral affine curve X is normal if and only if every finite morphism $\phi : Y \to X$ inducing an isomorphism $\phi^* : K(X) \xrightarrow{\sim} K(Y)$ is an isomorphism. As in the proof of Proposition 4.2.18 one sees that the condition $\phi^* : K(X) \xrightarrow{\sim} K(Y)$ can be rephrased by saying that ϕ is an isomorphism over an open subset. So the criterion becomes: X is normal if and only if every finite surjective morphism $Y \to X$ inducing an isomorphism over an open subset is in fact an isomorphism.

4.4 Proper Normal Curves

When one compares the theory developed so far with the theory of finite covers of Riemann surfaces, it is manifest that our presentation is incomplete at one point: the preceding discussion does not include the case of *compact* Riemann surfaces, only those with some points deleted. For instance, we have an algebraic definition of the affine line, but not that of the projective line. We now fill in this gap by considering proper normal curves.

We shall give the scheme-theoretic definition, which is in fact quite close to what Zariski and his followers called an 'abstract Riemann surface'. Its starting point is the study of the local rings $\mathcal{O}_{X,P}$ of an integral normal affine curve X over a field k. They are all discrete valuation rings having the same fraction field, namely the function field $K(X)$ of X, and they all contain the ground field k.

Lemma 4.4.1 *The local rings of an integral normal affine curve X are exactly the discrete valuation rings R with fraction field $K(X)$ containing $\mathcal{O}(X)$.*

Proof: If R is such a ring, the intersection of its maximal ideal M with $\mathcal{O}(X)$ is nonzero, for otherwise the restriction of the projection $R \to R/M$ to $\mathcal{O}(X)$ would be injective, and the field R/M would contain $K(X)$, which is absurd. Thus $M \cap \mathcal{O}(X)$ is a maximal ideal in $\mathcal{O}(X)$, and R contains the local ring $\mathcal{O}_{X,P}$. But then by Proposition 4.1.9 (3) we have $R = \mathcal{O}_{X,P}$. □

We now consider the simplest example.

Example 4.4.2 The rational function field $k(x)$ is the function field of the affine line $\mathbf{A}^1_k$ over k; we have $\mathcal{O}(\mathbf{A}^1_k) = k[x]$. But $k(x)$ is also the fraction field of the ring $k[x^{-1}]$, which we may view as the coordinate ring of another copy of $\mathbf{A}^1_k$ with coordinate function x^{-1}. By Proposition 4.1.9 (3) every discrete valuation ring $R \supset k$ with fraction field $k(x)$ contains either x or x^{-1}, and hence by the preceding discussion R is a local ring of one of the two copies of $\mathbf{A}^1_k$. In fact, there is only one localization of $k[x^{-1}]$ that does not contain x: the localization at the ideal (x^{-1}). Thus there is only one discrete valuation ring R as above that is not a local ring on the first copy of $\mathbf{A}^1_k$; it corresponds to the 'point at infinity'. The whole discussion is parallel to the construction of the complex structure on the Riemann surface $\mathbf{P}^1(\mathbf{C})$ in Example 3.1.3 (2): there we took a copy of $\mathbf{C}$ around 0, another copy around ∞, and outside these two points we identified the two charts