Hyperkähler metrics and application to the twistor space

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The quest for hyperkähler metrics

Hyperkähler structures can be defined purely in terms of Riemannian geometry. A hyperkähler manifold is then a Riemannian manifold having the symplectic group as holonomy group.

One of the many consequences of Yau's theorem is the existence of a unique (Ricci-flat) hyperkähler metric in any kähler class of a compact irreducible holomorphic symplectic manifold. However there is no explicit expression of such metric.

Actually, very few compact hyperkähler manifolds are known, all stemming from K3 surfaces. Expressions of hyperkähler metrics are even scarcer.

Quiver construction

Nakajima (see [3, 4]) developed a very general construction of (non-compact) holomorphic symplectic varieties using the *hyperkähler quotient* techniques presented in [2]. Such quotient variety can be endowed with any tensor-structure coming from above, assuming the tensor satisfy some invariance properties. Moreover the quotient metric can be explicitly described and would give examples of hyperkähler metrics.

For example Hitchin has recovered several known hyperkähler metrics, such as Calabi-Eguchi-Hanson's metric on $T^*\mathbb{P}^n$, using this construction.

Twistor space

Any hyperkähler manifold comes with a $\mathbb{P}^1$-family of complex structures. All those data can be gathered into a single manifold : *the twistor space* Z. It is fibered over $\mathbb{P}^1$, and any fiber correspond to the given complex structure on M_{diff}. It happens that the twistor space is a complex manifold and the forenamed fibration is holomorphic.

$$\begin{tikzcd} Z \arrow[r] \arrow[d] & M_{\mathsf{diff}} \\ \mathbb{P}^1 & \end{tikzcd}$$

In the moduli space of hyperkähler manifolds the rational curves arising from the twistor construction carry a lot of information [5].

My research

Thus my research focusses on finding exact expressions of metrics on hyperkähler manifolds. In case the metric has an exact expression, it would certainly be analytic, and therefore one could effectively compute local coordinates on the twistor space using integrability theorems and the expression of the almost complex structure given in [2].

Such coordinates would allow to unravel many mysteries of this twistor space and would have many applications from algebraic geometry to theoretical physics.

A first example would be to recover the result of A. Fujiki [1] identifying the twistor space of $T^*\mathbb{P}^1$ with an open subset of Nagata's threefold (the first known example of non-projective proper smooth algebraic variety).

Other

Among my other (but related) research interests, one can cite quaternionic and octonionic geometry and also most of Penrose's work : including tiling, his notation for monoïdal categories, twistor theory and also his "mathematical planonism".

References

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[3] Hiraku Nakajima, *Quiver varieties and kac-moody algebras*, Duke Mathematical Journal **91** (1998), no. 3, 515–560.

[4] Olivier Schiffmann, *Variétés carquois de nakajima*, Astérisque **317** (2008), no. 976, 295–344.

[5] M. Verbitsky, *A global torelli theorem for hyperkahler manifolds*, ArXiv e-prints (2009).