A POISSON FORMULA FOR THE WAVE PROPAGATOR ON SCHWARZSCHILD-DE SITTER BACKGROUNDS

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Abstract. This paper proposes a Poisson formula for the wave propagator of the Schwarzschild–de Sitter (SdS) metric. That is done by proving a Poisson formula relating wave propagators and scattering resonances for a class of non-compactly supported potentials on the real line. That class includes the Regge–Wheeler potentials obtained from separation of variables for SdS. The novelty lies in allowing non-compact supports – all exact Poisson formulae of Lax–Phillips, Melrose, and other authors required compactness of the support of the perturbation.

1. Introduction

Quasinormal modes of black holes are supposed to measure the ringdown of gravitational waves – see [BCS09] for a survey, [Car+18; JMS21; YBF25] for more recent work and references. Their study has been of interest to mathematicians since the work of Bachelot–Motet-Bachelot [BM93] – see Hintz–Xie [HX21], Hitrik–Zworski [HZ25], and Jézéquel [Jéz24] for some recent advances.

The quasinormal modes[1] can be considered as scattering resonances for linear wave equations arising in general relativity – see [DZ19, Chapter 5] and references given there. One of the features of the theory of scattering resonances, going back to the work of Lax–Phillips, are Poisson formulas. They generalized the Poisson formula for the wave equation on compact Riemannian manifolds (M, g),

$$\operatorname{Tr}\cos t\sqrt{-\Delta_g} = \tfrac{1}{2} \sum_{\lambda_j^2 \in \operatorname{Spec}(-\Delta_g)} e^{i\lambda_j t}. \tag{1.1}$$

Here $-\Delta_g$ is the Laplace–Beltrami operator for the metric g and the traces are meant in the sense of distributions in t. The formula (1.1) is, of course, an immediate consequence of the spectral decomposition of $-\Delta_g$. In the non-compact setting, the left-hand side is not distributionally of trace class and it has to be renormalized. The right-hand side should then be replaced by a sum over scattering resonances with terms $e^{-i\lambda_j|t|}$ (since $\operatorname{Im}(\lambda_j) < 0$) and the formula is typically not valid at $t = 0$ – see [DZ19, §§2.6, 3.10, 7.4] for an introduction to this subject and references.

[1]We follow the informal convention, common in math and physics, of referring to both the frequencies and the corresponding states as modes; the former should be more correctly called quasinormal frequencies.

To investigate a possible Poisson formula for quasinormal modes of black holes, that is, a relation between (renormalized) traces of wave groups and sums over the modes, we consider the simplest case of Schwarzschild–de Sitter black holes. There, a standard Regge–Wheeler reduction gives quasi-normal modes as the union of scattering resonances of a family of potentials parametrized by angular momenta.

Motivated by this, we compute here the trace of the propagator for the wave equation $\partial_t^2 - \partial_x^2 + V(x)$ for V which include the Regge–Wheeler potentials and their perturbations – see §1.2.

1.1. **Trace formula for a class non-compactly supported potentials.** We consider potentials satisfying the following hypothesis.

There exist constants $A_\pm > 0$ and $R > 0$ such that

$$V(x)|_{\pm x > R} = F_\pm(\exp(\mp x A_\pm)), \tag{1.2}$$

where $z \mapsto F_\pm(z)$ are holomorphic in a neighborhood of $z = 0$ and $F_\pm(0) = 0$.

In particular this means there exist $\left\{V_j^\pm \in \mathbb{R} : j \in \mathbb{Z}_{\geq 1}\right\}$ and constants $A, C > 0$ so that

$$V(x)|_{\pm x > C} = \sum_{j=1}^{\infty} \left(e^{\mp x A_\pm}\right)^j V_j^\pm$$

with $\left|V_j^\pm\right| \leq A^j$.

The first result, which is implicit in [Dya11, §3] and earlier works, is the meromorphy of the resolvent of the corresponding Schrödinger operator (Green function):

Proposition 1.1. *The resolvent of* $P_V := D_x^2 + V(x)$*,* $D_x := \partial_x / i$*,*

$$R_V(\lambda) := (P_V - \lambda^2)^{-1} \colon L^2 \to L^2, \quad \operatorname{Im}(\lambda) \gg 1,$$

continues meromorphically to

$$R_V(\lambda) \colon L^2_{\mathrm{c}}(\mathbb{R}) \to L^2_{\mathrm{loc}}(\mathbb{R}), \quad \lambda \in \mathbb{C}.$$

In other words, for any $f, g \in L^2_{\mathrm{c}}$, $\lambda \mapsto \langle R_V(\lambda) f, g\rangle$ is a meromorphic function with poles independent of f and g. The poles are called *scattering resonances* and we denote their set (with elements included according to their multiplicity – see Proposition A.1) $\operatorname{Res}(V)$. For recent interesting developments in 1D scattering, see the work of Korotyaev [Kor04; Kor05; Kor17].

Define $\Box_V := \partial_t^2 + P_V$ and let $U_V(t) := \cos(t\sqrt{P_V})$ be the cosine wave propagator with respect to $\Box_V$. That is, if $u \in C_0^\infty(\mathbb{R})$:

$$\begin{cases} \Box_V(U(t)u(x))(t,x) = 0, \\ U(0)u(x) = u(x), \\ (\partial_t(U(t)u(x))\big|_{t=0} = 0. \end{cases} \tag{1.3}$$

Our main result relates the trace of the wave propagator minus the free wave propagator to a sum over resonances, analogous to (1.1).

Theorem (Main result)**.** *Suppose $V \in C^\infty(\mathbb{R};\mathbb{R})$ satisfies (1.2) and $U_V(t)$ is the wave propagator for $\Box_V$. Then for $t > 0$,*

$$\operatorname{Tr}(U_V(t) - U_0(t)) = \tfrac{1}{2} \sum_{\lambda_j \in \operatorname{Res}(V)} e^{-i\lambda_j t} + \mathcal{A}_+(t) + \mathcal{A}_-(t) - \tfrac{1}{2}, \tag{1.4}$$

in the sense of distributions. Here $\operatorname{Res}(V)$ is the set of resonances of $-\partial_x^2 + V$ and

$$\mathcal{A}_\pm(t) := -\frac{1}{2(e^{tA_\pm/2} - 1)}.$$

Example 1.1 (Pöschl-Teller potential)**.** *The scattering matrix, $S(\lambda)$, for the Pöschl-Teller potential*

$$V_{\mathrm{PT}}(x) := \frac{\ell^2 + 1/4}{\cosh^2(x)}$$

can be explicitly computed [Cev+16] and the resonances are given by

$$\left\{ \lambda_j^\pm = -i(j + 1/2) \pm \ell : j \in \mathbb{Z}_{\geq 0} \right\}.$$

The Birman–Kreĭn trace formula can be applied (exactly as will be done in §3, but with the poles of $\frac{d}{d\lambda} \log\det S(\lambda)$ explicit). It can then be shown that

$$\begin{aligned}
\operatorname{Tr}(U_{V_{\mathrm{PT}}}(t) - U_0(t)) &= \tfrac{1}{2} \sum_{\lambda_j \in \operatorname{Res}(V_{\mathrm{PT}})} e^{-i\lambda_j t} \underbrace{- \frac{1}{e^t - 1}}_{\mathcal{A}_+ + \mathcal{A}_-} - \tfrac{1}{2} \\
&= \frac{\cos(\ell t)}{2\sinh(t/2)} - \frac{e^{-t/2}}{2\sinh(t/2)} - \tfrac{1}{2} \\
&= \tfrac{1}{2} \left(\frac{\cos(\ell t) - e^{-t/2}}{\sinh(t/2)} - 1 \right), \quad t > 0.
\end{aligned} \tag{1.5}$$

We can compare this to a numerically computed approximation of the flat trace of $U_{V_{\mathrm{PT}}}(t)$, which is presented in Figure 1. Note that as $t \to 0^+$, the right-hand side of (1.5) goes to zero. This is a classical result for compactly supported potentials (see Smith [Smi23] and references given for a recent account).

When V is compactly supported, Proposition 1.1 is classical – see [DZ19, §2.2], and we have the Poisson formula

$$\operatorname{Tr}(U_V(t) - U_0(t)) = \tfrac{1}{2} \mathrm{p.v.} \sum_{\lambda_j \in \operatorname{Res}(V)} e^{-i\lambda_j |t|} - |\operatorname{chsupp} V| \delta_0(t) - \tfrac{1}{2}, \tag{1.6}$$

see [DZ19, Theorem 2.21]. Both sides are defined distributionally on $\mathbb{R}$ and p.v. denotes the distributional principal value obtained by summing over $|\lambda_j| \leq R$ and taking the $R \to \infty$ limit. The notation ch denotes the convex hull. The presence

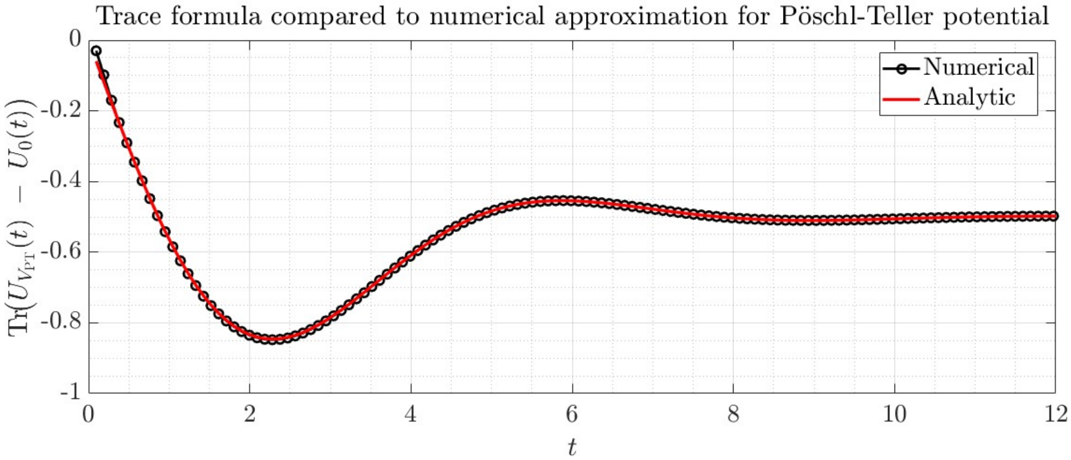

Figure 1. Comparison between a numerical approximation of the flat trace for the Pöschl–Teller potential (with $\ell = 1$) and the analytic expression in (1.5). The numerical trace is computed in MATLAB by truncating and discretizing the spatial domain, evolving a normalized Gaussian centered at each grid point, x_i, under the cosine propagator $\cos\left(t\sqrt{P_{V_{\mathrm{PT}}}}\right)$, subtracting the free wave evolution of the same Gaussian, and evaluating at x_i. This approximates $U_{V_{\mathrm{PT}}}(t, x_i, x_i) - U_0(t, x_i, x_i)$ (the Schwartz kernels). Computing the sum of these terms, multiplied by $x_i - x_{i-1}$ will approximate the flat trace.

of the $|\mathrm{chsupp} V|\delta_0(t)$ term explains the need for the renormalization terms $\mathcal{A}_\pm$ in the non-compact case. For $t > 0$, (1.6) was proved by Melrose [Mel82] and for $t > 2\mathrm{diam}(\mathrm{supp}\, V)$ by Lax–Phillips [LP78]. The proof of such trace identities often relies on the Eisenbud—Wigner delay operator, $\partial_\lambda S(\lambda) S(\lambda)^{-1}$. In the compactly supported setting, this operator is linked to the Breit-–Wigner series $\sum_{\lambda_j \in \mathrm{Res}(V)\setminus 0} \mathrm{Im}\,(\lambda_j)\,|\lambda_j|^{-2}$ via the Breit-–Wigner approximation (cf. [DZ19, Theorem 2.20]). For non-compactly supported potentials, however, the convergence or divergence of this approximation remains an open question; see Backus [Bac20] for a thorough discussion.

If we define $\iota\colon (t,x) \mapsto (t,x,x)$, $\pi\colon (t,x) \mapsto t$, and denote the kernel of $U_V(t)$ by $U_V(t,x,y)$, as a distribution on $(0,\infty) \times \mathbb{R}^2$, with the pairing using the Lebesgue measure, then the *flat trace* is defined as

$$\mathrm{Tr}^\flat\, U_V(t) \coloneqq \pi_* \iota^* U_V \tag{1.7}$$

(which is well-defined thanks to the usual wave front set [Hör09, §29.1]). Because $\mathrm{Tr}^\flat\, U_0(t) = 0$ and $\mathrm{Tr}^\flat(U_V - U_0) = \mathrm{Tr}(U_V - U_0)$, (1.4) can be written

$$\mathrm{Tr}^\flat\, U_V(t) = \tfrac{1}{2} \sum_{\lambda_j \in \mathrm{Res}(V)} e^{-i\lambda_j t} + \mathcal{A}_+(t) + \mathcal{A}_-(t) - \tfrac{1}{2}.$$

1.2. **Regge–Wheeler potentials.** As a consequence of our main theorem, we propose a trace formula for the wave propagator in the Schwarzschild-de Sitter metric. It necessarily involves a renormalization of the trace – not the flat trace – which can be used to pass from a trace formula for each Regge–Wheeler potential (parameterized by angular momenta), to a trace formula for the full potential. Because $\mathcal{A}_{\pm}(t)$ are independent of angular momentum, simply summing over angular momentum is too naive. On the other hand, thanks to recent work of Jézéquel [Jéz24] the sum of $e^{-i\lambda_j t}$ over all quasinormal modes (that is, the sum over $\mathrm{Res}(V_\ell)$) is a well-defined distribution.

Schwarzschild-de Sitter spacetime can be constructed in the following way (c.f. [Jéz24, §3.1]). Define $M := (r_-, r_+)_r \times \mathbb{S}^2$ where $0 < r_- < r_+$ are the positive roots of

$$G(r) := 1 - \frac{2m}{r} - \frac{\Lambda r^2}{3}$$

(for $0 < g/m^2 < 1$). Here $m > 0$ is the *black hole mass* and $\Lambda \in (0, (9m^2)^{-1})$ is the *cosmological constant*.

Define the Schwarzschild-de Sitter spacetime as $\hat{M} := \mathbb{R}_t \times M$ and let g be the Lorentzian metric:

$$g := -G\,\mathrm{d}t^2 + G^{-1}\,\mathrm{d}r^2 + r^2 g_{\mathbb{S}}$$

with $g_{\mathbb{S}}$ the standard metric on $\mathbb{S}^2$.

With this Lorentizan metric, we have the wave operator $\Box_g$, which satisfies, for $u \in C_0^\infty((r_-, r_+) \times \mathbb{S}^2)$

$$\Box_g u = -\frac{1}{G} D_t^2 u + r^{-2} D_r(r^2 G D_r u) + r^{-2}\Delta_{\mathbb{S}^2} u$$

(cf. [Cha98, Chapter 4]). Let $U_g(t)$ be the Dirichlet wave propagator with respect to $\Box_g$ (i.e., $U_g(t)$ satisfies (1.3) with $\mathbb{R}$ replaced by M and $\Box_V$ replaced by $\Box_g$).

Following [BZ97, §4], write:

$$\Box_g = -G^{-1} D_t^2 + G^{-1} P \tag{1.8}$$

where

$$P := G r^{-2} D_r (r^2 G) D_r + G r^{-2} \Delta_{\mathbb{S}^2}.$$

Let $\tilde{P} := r P r^{-1}$ and define $x = x(r)$ such that $x'(r) = G^{-1}$ so that

$$\tilde{P} = D_x^2 + G r^{-2}\Delta_{\mathbb{S}^2} + G r^{-1}(\partial_r G).$$

Decomposing a function $u(x,\omega)$ by spherical harmonics: $u(x,\omega) = \sum_{\ell,k} a_{\ell,k}(x) Y_\ell^k(\omega)$ (where $\Delta_\omega Y_\ell^k(\omega) = \ell(\ell+1) Y_\ell^k(\omega)$) we get that u is a generalized eigenfunction of $\tilde{P}$

with eigenvalue λ^2 if and only if:

$$(D_x^2 + \underbrace{Gr^{-2}(\ell(\ell+1) + r\partial_r G)}_{:=V_\ell(x)})a_{\ell,k} = \lambda^2 a_{\ell,k}. \tag{1.9}$$

Note for each ℓ, $V_\ell(x(r))$ is a smooth function on the interval $r \in (r_-, r_+)$ vanishing at $r = r_\pm$.

Now, if $U_{\tilde{V}}(t)$ is the propagator for $-D_t^2 + \tilde{P}$ and $U_{V_\ell}(t)$ is the propagator for $-D_t^2 + D_x^2 + V_\ell(x)$, then for $u_0 \in C_0^\infty(\mathbb{R}_x \times \mathbb{S}^2_\omega)$

$$U_{\tilde{V}}(t)u_0 = \sum_{\ell=0}^{\infty} \sum_{k=-\ell}^{\ell} Y_\ell^k U_{V_\ell} \left\langle u_0, Y_\ell^k \right\rangle_{L^2_\omega(\mathbb{S}^2)}. \tag{1.10}$$

Here $\left\langle u_0, Y_\ell^k \right\rangle_{L^2_\omega(\mathbb{S}^2)}$ is integration of u_0 against Y_ℓ^k in the ω variable, so the resulting object is a function of x alone.

Using this, we can rewrite the propagator for $\Box_g$ (defined in (1.8)) as

$$[U_g(t)F](r,\omega) = r^{-1}[U_{\tilde{V}}(t)(rF)](r,\omega), \quad F = F(r,\omega). \tag{1.11}$$

It is straightforward to verify that $V_\ell(x)$ satisfies (1.2) with $A_\pm = |\partial_r G(r_\pm)|$. Indeed, by fixing $r_0 \in (r_-, r_+)$ we can define

$$x(r) = \int_{r_0}^{r} (G(s))^{-1} \, \mathrm{d}s$$

so that $x(r_\pm) = \pm\infty$. Near $r = r_+$, we have:

$$G(r) = (r_+ - r)f(r)$$

where $f(r)$ is nonzero and holomorphic near $r = r_+$. Taylor expanding f and integrating, we get for r near r_+

$$x(r) = -A_+^{-1}\log(r_+ - r) + b(r)$$

for b holomorphic near r_+. Therefore $w := \exp(-A_+ x) = (r - r_+)\exp(b(r))$. By the inverse function theorem, we get (for w near 0) $V_\ell(x(w)) = F_+(w)$ with F holomorphic near $w = 0$. We similarly get, for the change of coordinates $w = \exp(A_- x)$, $V_\ell(x(w)) = F_-(w)$ is holomorphic near $w = 0$.

Theorem (Trace formula for each spherical harmonic)**.** *For each $\ell \in \mathbb{Z}_{\geq 0}$, $t > 0$*

$$\operatorname{Tr}_{L^2(\mathbb{R})}(U_{V_\ell}(t) - U_0(t)) = \tfrac{1}{2} \sum_{\lambda_j \in \operatorname{Res}(V_\ell)} e^{-i\lambda_j t} - \frac{1}{2(e^{tA_-/2} - 1)} - \frac{1}{2(e^{tA_+/2} - 1)} - \tfrac{1}{2}$$

in the sense of distributions, where U_{V_ℓ} is the propagator for V_ℓ (defined in (1.9)), $\operatorname{Res}(V_\ell)$ is the set of resonances of V_ℓ, $A_\pm = |\partial_r(G(r_\pm))|$.

Remark 1.1. *The constants $A_\pm$ are related to the surface gravity of the event and cosmological horizon, κ_- and κ_+, respectively, by the simple relation $A_\pm = 2\kappa_\pm$ (cf. [Wal84, §12.5]). We can then write the terms in (1.2) as*

$$\frac{1}{e^{tA_\pm/2}-1} = -\sum_{m=0}^{\infty} e^{-mt\kappa_\pm} = -\sum_{m=0}^{\infty} e^{-i(-im\kappa_\pm)t}.$$

This suggests that these terms can be included in the sum over resonances. Interestingly, on the de Sitter space, there are resonances at $\{-im\kappa_\pm : m \in \mathbb{Z}_{\geq 0}\}$ (cf. [HX21]). Making sense of this connection could lead to a formulation of the Poisson formula for other metrics.

The trace on the left-hand side of (1.2) is written in the x variable. Changing variables x to r, the left-hand side of (1.2) is

$$\operatorname{Tr}_{L^2((r_-,r_+),G(r)^{-1}\,\mathrm{d}r)}(U_{V_\ell}(t) - U_0(t)). \tag{1.12}$$

Recalling the notion of flat trace (1.7), (1.12) can be written

$$\operatorname{Tr}^\flat_{L^2((r_-,r_+),G(r)^{-1}\,\mathrm{d}r)}(U_{V_\ell}(t)). \tag{1.13}$$

Conjugating U_{V_ℓ} by r allows us to write (1.13) as

$$\operatorname{Tr}^\flat_{L^2((r_-,r_+),r^2G(r)^{-1}\,\mathrm{d}r)}(r^{-1}U_{V_\ell}(t)r). \tag{1.14}$$

Let Π_ℓ be the orthogonal projection in the ω variable onto the space spanned by $\left\{Y_\ell^k(\omega) : k = -\ell, \ell+1, \dots, \ell\right\}$. Then by (1.10) and (1.11), we can rewrite (1.14) as

$$\operatorname{Tr}^\flat_{L^2((r_-,r_+)\times\mathbb{S}^2,r^2G^{-1}\,\mathrm{d}r\,\mathrm{d}\omega)}(\Pi_\ell U_g(t)\Pi_\ell).$$

So if we now define the *gravitationally renormalized trace* on

$$\mathrm{SdS} := L^2((r_-,r_+)\times\mathbb{S}^2, G(r)^{-1}r^2\,\mathrm{d}r\,\mathrm{d}\omega)$$

as

$$\widetilde{\operatorname{Tr}}^\flat_{\mathrm{SdS}}(A(t)) := \sum_{\ell=0}^{\infty}\left(\operatorname{Tr}^\flat_{\mathrm{SdS}}\Pi_\ell A(t)\Pi_\ell + \frac{1}{2(e^{tA_-/2}-1)} + \frac{1}{2(e^{tA_+/2}-1)} + \tfrac{1}{2}\right)$$

then (1.2) gives rise to the following global trace formula.

Theorem 1 (Global trace formula)**.** *The gravitationally renormalized trace of the wave propagator on the Schwarzschild-de Sitter metric is*

$$\widetilde{\operatorname{Tr}}^\flat_{\mathrm{SdS}}(U_g(t)) = \tfrac{1}{2}\sum_{\lambda\in\mathrm{Res}_g} e^{-it\lambda}, \quad t>0$$

where $\mathrm{Res}_g := \bigcup_\ell \mathrm{Res}(V_\ell)$ (and the term on the right-hand side is a well-defined distribution by [Jéz24, Theorem 3]).

1.3. **Outline of proof.** An outline of a proof of our main result is presented below. The order of steps below is *not* in the chronological order of the paper, but presented in a way that motivates the required analysis of §2.

(1) The Birman–Kreĭn trace formula (which we prove for our class of potentials in Section B) relates the trace of $U_V(t) - U_0(t)$ to a term depending on negative eigenvalues of P_V, a term involving the resonance of V at 0 (if it exists), and an integral involving the logarithmic derivative of the scattering matrix $S(\lambda)$ for V (see (3.3)). This is presented in §3.
(2) To apply the Birman–Kreĭn trace formula, we first need to prove the resolvent $(P_V - \lambda^2)^{-1}$ (defined for $\operatorname{Im}(\lambda) \gg 1$) can be meromorphically continued to $\mathbb{C}$. This is done by defining and estimating incoming and outgoing solutions $u_\pm(\lambda, x)$ to $(P_V - \lambda^2)u = 0$ (§2.1)
(3) The logarithmic derivative of the scattering matrix is $\frac{d}{d\lambda} \log(T(\lambda)/T(-\lambda))$ where $T(\lambda)$ is the transmission coefficient.
(4) The transmission coefficient is computed by taking the Wronskian of the incoming and outgoing solutions (see (2.15)).
(5) The term involving the logarithmic derivative of the scattering matrix will have singularities coming from $T(\lambda)$, which are decoupled into three sets (see (3.4)). Two of these end up giving the terms $\mathcal{A}_\pm(t)$, while the third term is related to the resonances of V (a Lemma relating zeros of the Wronskian of $u_\pm$ to resonances of V, with multiplicity, is required – which is proven in Appendix A).

Notation. We use the following notation throughout this paper. We write $D_x \coloneqq -i\partial_x$. We let L^2_c denote the space of L^2 functions with compact support and L^2_{loc} the space of locally L^2 functions We let $\|\cdot\|_1$ denote the trace norm. For functions $f(x)$ and $g(x)$, we write $f(x) \lesssim g(x)$ if there exists a constant $C > 0$ such that $f(x) \leq Cg(x)$ for all x. If C depends on a parameter α, we write $f(x) \lesssim_\alpha g(x)$. We write $f(x) \gg 1$ to mean $f(x) > C$ for some sufficiently large $C > 0$. If $g(x)$ is positive and $N \in \mathbb{Z}$, we write $f = \mathcal{O}(g^N)$ to mean there exists $C_N > 0$ such that $|f(x)| \leq C_N g(x)^N$. We write $f = \mathcal{O}(g^\infty)$ if $f = \mathcal{O}(g^N)$ for all $N > 0$ (and similarly $f = \mathcal{O}(g^{-\infty}) \iff f = \mathcal{O}(g^{-N})\ \forall\ N \in \mathbb{Z}_{>0}$).

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2. Analysis of incoming and outgoing solutions

In this section, we first construct and analyze the behavior of the incoming and outgoing solutions to

$$(D_x^2 + V(x))u(x) = \lambda^2 u(x)$$

where V has the asymptotics described by Equation (1.2) and $D_x \coloneqq -i\partial_x$. We will then use this to show that the corresponding resolvent operator $(P_V - \lambda^2)^{-1}$ can be meromorphically extended to the complex plane.

2.1. **Incoming and outgoing solutions.** We begin by recalling the definition for incoming and outgoing solutions for $P_V \coloneqq (D_x^2 + V(x) - \lambda^2)$. In the sequel, we assume V satisfies Equation (1.2) with $A_\pm$.

Definition 2.1 (Outgoing solution). *We say $u(x) = u(x, \lambda)$ is an outgoing solution for the operator*

$$P_\lambda \coloneqq (D_x^2 + V(x) - \lambda^2)$$

at $+\infty$ if $P_\lambda u = 0$ and there exists a globally defined, smooth profile $v_+(w) = v_+(w, \lambda)$ that is holomorphic in a neighborhood of zero, such that

$$u(x) = e^{i\lambda x} v_+(e^{-A_+ x}).$$

Analogously, $u(x) = u(x, \lambda)$ is an outgoing solution for P_λ at $-\infty$ if $P_\lambda u = 0$ and there exists a globally defined, smooth profile $v_-(w)$ that is holomorphic in a neighborhood of zero, such that

$$u(x) = e^{-i\lambda x} v_-(e^{A_- x}).$$

We also have a similar definition for incoming solutions.

Definition 2.2 (Incoming solution). *We say $u(x) = u(x, \lambda)$ is an incoming solution to P_λ at $+\infty$ if $P_\lambda u = 0$ and there exists a globally defined, smooth profile $w_+(w)$ that is holomorphic in a neighborhood of zero, such that*

$$u(x) = e^{-i\lambda x} w_+(e^{-A_+ x}).$$

Analogously, $u(x) = u(x, \lambda)$ is an incoming solution to P_λ at $-\infty$ if $P_\lambda u = 0$ and there exists a globally defined, smooth profile $w_-(w)$ that is holomorphic in a neighborhood of zero, such that

$$u(x) = e^{i\lambda x} w_-(e^{A_- x}).$$

Our first result in this section shows that for each λ, we can construct outgoing solutions with the asymptotics described above.

Proposition 2.3. *For each $\lambda \in \mathbb{C}$, there exist outgoing solutions for P_λ of the form*

$$u_\pm(x) = e^{\pm i\lambda x} v_\pm(e^{\mp A_\pm x})$$

where $v_\pm(w) = v_\pm(w,\lambda)$ are holomorphic near $w = 0$ in a neighborhood that is independent of λ and such that

$$v_\pm(0) = \frac{1}{\Gamma(1 - 2i\lambda A_\pm^{-1})}.$$

Moreover, there exists $C, M_\pm > 0$, independent of λ, such that for $x \in \mathbb{R}$

$$\left|\partial_x^j v_\pm(e^{\mp A_\pm x})\right| \le C \exp\Big(C\langle x - M_\pm\rangle \langle\lambda\rangle \log\langle\lambda\rangle\Big), \quad j = 0, 1. \tag{2.1}$$

where we write $\langle z\rangle := (1 + |z|^2)^{\frac{1}{2}}$.

Proof. We will provide the details for the case u_+ as the other case follows from analogous reasoning.

Step 1. We begin by making an ansatz

$$u_+(x) = e^{i\lambda x} v_+(e^{-A_+ x}).$$

It will be convenient in the forthcoming calculations to write as shorthand $w := e^{-A_+ x}$. A straightforward calculation shows that solving $P_\lambda u_+ = 0$ is equivalent to v_+ solving the equation

$$2\lambda D_x v_+(w) + D_x^2 v_+(w) + V(x) v_+(w) = 0. \tag{2.2}$$

We can write this equation in terms of of D_w by noting the simple identities $D_x = (-A_+ w) D_w$ and $D_x^2 = A_+^2 (w D_w)^2$. Hence, we find that $(D_x^2 + V(x) - \lambda^2) u_+(x) = 0$ if and only if

$$-2\lambda A_+ w D_w v_+ + A_+^2 (w D_w)^2 v_+ + V v_+ = 0. \tag{2.3}$$

To construct v_+, we will begin by formally expanding v_+ and V in a power series near $w = 0$:

$$v_+(w) = \sum_{j=0}^{\infty} v_j w^j, \quad V(w) = \sum_{j=1}^{\infty} V_j w^j, \tag{2.4}$$

where the coefficients V_j are defined as in (1.2). We note that the potential does not have a constant term in this expansion (i.e, V vanishes to at least first order at $w = 0$).

Substituting this into (2.3) and matching powers of w, we obtain the follow recursion relation for each $j \ge 1$:

$$j A_+^2 (j - 2i A_+^{-1} \lambda) v_j = \sum_{\ell=1}^{j} V_\ell v_{j-\ell}. \tag{2.5}$$

Step 2. We now prove the v_j's (the coefficients in the Taylor expansion of v_+) have sufficient decay to ensure that v_+ is analytic and has a nontrivial radius of convergence that is uniform in λ. We will begin by showing that the coefficients v_j defined by the recursion (2.5) are entire functions of λ.

To simplify notation, in the forthcoming calculations, we set

$$\alpha \coloneqq 2iA_+^{-1}\lambda.$$

We can write (2.5) as a $j \times j$ system as follows. For each j, we have

$$(\mathbf{\Lambda}_j - \mathbf{A}_j)\begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_j \end{pmatrix} = \begin{pmatrix} V_1 v_0 \\ V_2 v_0 \\ \vdots \\ V_j v_0 \end{pmatrix}$$

where $(\mathbf{\Lambda}_j) \coloneqq \operatorname{diag}((kA_+^2(k-\alpha))_{k=1}^j)$ and

$$\mathbf{A_j} \coloneqq \begin{pmatrix} 0 & \cdots & \cdots & 0 & 0 \\ V_1 & 0 & \cdots & \cdots & 0 \\ V_2 & V_1 & \cdots & \cdots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ V_{j-1} & \cdots & V_2 & V_1 & 0 \end{pmatrix}.$$

By Cramer's rule, as long as $\alpha \neq 1, 2, \ldots, j$, we can invert $\mathbf{\Lambda_j} - \mathbf{A}_j$ as:

$$(\mathbf{\Lambda}_j - \mathbf{A}_j)^{-1} = \frac{\mathbf{B}_j(\alpha)}{\det(\mathbf{\Lambda}_j - \mathbf{A}_j)} = \frac{\mathbf{B}_j(\alpha)}{A_+^{2j} j!(j-\alpha)\cdots(1-\alpha)}$$

where $\mathbf{B}_j(\alpha)$ is a $j \times j$ matrix with entries holomorphic in α.

If we set $v_0 \coloneqq \Gamma(1-\alpha)^{-1}$, then for all α (including integer values),

$$\begin{pmatrix} v_1 \\ \vdots \\ v_j \end{pmatrix} = \frac{\mathbf{B}_j(\alpha)}{j!(j-\alpha)\cdots(1-\alpha)\Gamma(1-\alpha)}\begin{pmatrix} V_1 \\ \vdots \\ V_j \end{pmatrix}$$
$$= \frac{\mathbf{B}_j(\alpha)}{j!\Gamma(j+1-\alpha)}\begin{pmatrix} V_1 \\ \vdots \\ V_j \end{pmatrix}$$

so that v_j are entire functions of α (and thus λ).

Step 3. For each j, we want to obtain a strong enough growth bound for $v_j(\alpha)$ for $\alpha \in \mathbb{C}$. We will do this in two steps. First, we will estimate $v_j(\alpha)$ for α away from $\mathbb{N}$, and then extend this to a global bound by using the maximum principle.

We now use the assumption that $|V_j| \leq A^j$ for some positive constant A. The following lemma will give us the required estimate away from $\mathbb{N}$.

Lemma 2.4. *Suppose $\varepsilon \in (0, A_+^{-2})$ and $\Omega_\varepsilon \coloneqq \{z \in \mathbb{C} : \operatorname{dist}(z, \mathbb{N}) > \varepsilon\}$, then for $\alpha \in \Omega_\varepsilon$,*

$$|v_j(\alpha)| \leq \frac{A^j\,|v_0|}{\varepsilon^j A_+^{2j}}.$$

Proof. We prove this by induction. For $j = 0$ this is clear. We then have

$$\begin{aligned} |v_j(\alpha)| &\leq \frac{1}{jA_+^2|j-\alpha|}\left|\sum_{k=0}^{j-1} v_k V_{j-k}\right| \leq \frac{|v_0|}{jA_+^2\varepsilon}\sum_{k=0}^{j-1} A^k\varepsilon^{-k}A_+^{-2k}A^{j-k} \\ &= \frac{|v_0|A^j}{jA_+^2\varepsilon}\sum_{k=0}^{j-1}\varepsilon^{-k}A_+^{-2k} \leq \frac{|v_0|A^j}{jA_+^2\varepsilon}\sum_{k=0}^{j-1}\varepsilon^{-(j-1)}A_+^{-2(j-1)} = \frac{|v_0|A^j}{\varepsilon^j A_+^{2j}}. \end{aligned}$$

The first inequality follows from (2.5), while the second inequality follows from the induction hypothesis, and the third inequality follows from the choice of ε. □

Next, we extend Proposition 2.4 to a global estimate. We fix $\varepsilon_0 \in (0, \min(A_+^{-2}, 1))$. By Lemma 2.4, and the choice of $v_0 = \Gamma(1-\alpha)^{-1}$ and the above steps, we have that $v_j(\alpha)$ are entire functions, and for $\alpha \in \Omega_{\varepsilon_0}$, we have

$$|v_j(\alpha)| \leq \frac{A^j}{\varepsilon_0^j A_+^{2j}\,|\Gamma(1-\alpha)|}. \tag{2.6}$$

To obtain a global bound for $v_j(\alpha)$, we use the following standard bound of the reciprocal of the Gamma function. It follows from the factorization of $\Gamma(z)$ and, for instance, [Hay64, Theorem 1.11].

Lemma 2.5. *There exist a constant $C > 0$ such that for $z \in \mathbb{C}$,*

$$\left|\frac{1}{\Gamma(z)}\right| \leq C\exp\left(C\,|z|\log(1+|z|)\right).$$

Now, for $k \in \mathbb{N}$, we apply the maximum principle and Lemma 2.5 to get that:

$$\begin{aligned} \max_{|\alpha-k|\leq\varepsilon_0}|v_j(\alpha)| &\leq \max_{|\alpha-k|=\varepsilon_0}\left|\frac{A^j}{\varepsilon_0^j A_+^{2j}\Gamma(1-\alpha)}\right| \leq \max_{|\alpha-k|=\varepsilon_0}\frac{CA^j}{A_+^{2j}\varepsilon_0^j}e^{C|1-\alpha|\log(|1-\alpha|+1)} \\ &\leq \frac{CA^j}{A_+^{2j}\varepsilon_0^j}e^{C(k+1)\log(k+2)} \leq \frac{CA^j}{A_+^{2j}\varepsilon_0^j}e^{Ck\log(k)} \end{aligned} \tag{2.7}$$

where the constant C may change in each inequality.

We can then use this to get a global bound on v_j. We have by (2.6) that for $\operatorname{dist}(\alpha, \mathbb{N}) > \varepsilon_0$:

$$|v_j(\alpha)| \leq \frac{A^j}{\varepsilon_0^j A_+^{2j}}e^{C|1-\alpha|\log(|1-\alpha|+1))}$$

and by (2.7), for $|\alpha - k| \leq \varepsilon_0$ (for some $k \in \mathbb{N}$):

$$|v_j(\alpha)| \leq \frac{CA^j}{A_+^{2j}\varepsilon_0^j} e^{Ck\log(k)}.$$

Therefore, there exists a $C > 0$ such that for all $\alpha \in \mathbb{C}$:

$$|v_j(\alpha)| \leq \frac{CA^j}{A_+^{2j}\varepsilon_0^j} e^{C(|\alpha|+1)\log(|1-\alpha|+1)}.$$

This shows that $v_+(w)$ as defined by (2.4) is well-defined in some neighborhood of zero with radius of convergence that is independent of λ.

Step 4. Apriori, the above construction only ensures that v_+ is defined in a neighborhood of $w = 0$ (uniformly in λ). We now carry out a standard energy-type estimate for P_λ to show that v_+ can be extended to a global (in w) smooth solution of (2.3). Along the way, we also show the bound (2.1). We begin by fixing $\delta > 0$ less than the radius of convergence of $\sum_0^\infty v_j w^j$ (which is independent of α), to ensure that there exists a $C = C(\delta)$ (independent of α) such that

$$\left|(\partial_w^j v_+)(\delta)\right| \leq C e^{C(|\alpha|+1)\log(|1-\alpha|+1)} \tag{2.8}$$

for $j = 0, 1$. We recall that in terms of the original variable x, we have $x(\delta) = A_+^{-1}\log(1/\delta)$. We will now use a simple energy estimate to estimate the growth of v_+ and its derivatives. By (2.2), v_+ satisfies the the ODE in the x variable:

$$2i\lambda\partial_x v_+(w) + \partial_x^2 v_+(w) - V(x)v_+(w) = 0.$$

Here, we can rewrite the above equation as a 2×2 system

$$\partial_x \begin{pmatrix} u_1(x) \\ u_2(x) \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 1 \\ V(x) & -2i\lambda \end{pmatrix}}_{:=\mathbf{A}(x)} \begin{pmatrix} u_1(x) \\ u_2(x) \end{pmatrix}. \tag{2.9}$$

where $u_1(x) := v_+(w(x))$ and $u_2(x) := \partial_x(v_+(w(x))$. Let $\mathbf{U}(x) := (u_1(x), u_2(x))^t$, so that $\partial_x\mathbf{U} = \mathbf{A}\mathbf{U}$. This gives the simple energy identity

$$\frac{d}{dx}|\mathbf{U}|^2 = 2\mathrm{Re}\left(\langle \mathbf{A}\mathbf{U}, \mathbf{U}\rangle\right).$$

By the Cauchy–Schwartz inequality

$$\frac{d}{dx}|\mathbf{U}(x)|^2 \geq -C\langle\lambda\rangle |\mathbf{U}(x)|^2$$

for some positive constant C (independent of λ). This gives us that:

$$\frac{d}{dx}\left(e^{C\langle\lambda\rangle x}|\mathbf{U}(x)|^2\right) \geq 0$$

so that integrating from x to some $x_0 > x$, we get that:

$$|\mathbf{U}(x)|^2 \leq e^{C\langle\lambda\rangle(x_0-x)}|\mathbf{U}(x_0)|^2.$$

Letting $x_0 = x(\delta)$, we get, by (2.8) that there exists a $C > 0$ such that for $j = 0, 1$

$$\left|\partial_x^j(v_+(w(x))\right| \leq Ce^{C\langle\lambda\rangle(x(\delta)-x)+C(|\alpha|+1)\log(|1-\alpha|+1)}.$$

This shows that v_+ can be continued to a global C^1 (in w) solution of $P_\lambda v_+ = 0$ and that (by elementary estimates) v_+ satisfies the bound (2.1). To show that v_+ is smooth, one can simply apply derivatives to (2.9) and inductively apply the above argument. □

2.2. **Meromorphic continuation of the resolvent.** We now prove a more precise version of Proposition 1.1 which meromorphically extends the resolvent $R_V(\lambda)$ to the entire complex plane. The poles of the meromorphically continued resolvent will be the zeros of the Wronskian of outgoing solutions to $P_V - \lambda^2$. This Wronskian is entire in λ. A technical Lemma is first required to prove that the Wronskian is not identically zero.

Proposition 2.6. *There exist purely imaginary λ in the upper half plane such that the outgoing solutions $u_+(x,\lambda)$ and $u_-(x,\lambda)$ from Proposition 2.3 are linearly independent. In particular, the Wronskian of $u_+(x,\lambda)$ and $u_-(x,\lambda)$*

$$W[u_+, u_-](\lambda) := (\partial_x u_+(x,\lambda))u_-(x,\lambda) - u_+(x,\lambda)(\partial_x u_-(x,\lambda)) \tag{2.10}$$

is not identically zero in λ.

Proof. Take $\lambda = i\mu$ for some sufficiently large real $\mu > 0$. It suffices to show that $u_+(x,\lambda)$ does not vanish as $x \to -\infty$ (since, by definition, we have $\lim_{x\to-\infty} u_-(x,\lambda) = 0$). Without loss of generality, we can assume u_+ is real-valued (as $\lambda^2 \in \mathbb{R}$, and thus $\mathrm{Re}(u_+)$ will also satisfy the equation $P_\lambda u = 0$). Our first aim will be to show that u_+ is non-negative and decreasing. We define

$$M := \sup\{x \in \mathbb{R} : u_+(x) \leq 0\}.$$

By definition of u_+ (as in Proposition 2.3), we know that $-\infty \leq M < \infty$. It will suffice to show that $M = -\infty$ and that u_+ is decreasing. We begin by selecting μ large enough so that

$$V(x) - \lambda^2 = V(x) + \mu^2 \geq 1, \quad \forall x \in \mathbb{R}.$$

Because $P_V u_+ = 0$, we have

$$\partial_x^2 u_+(x) = (V(x) + \mu^2)u_+(x) \geq u_+(x), \quad x \in (M, \infty).$$

Hence, on (M, ∞), we have

$$\partial_x^2 u_+ > 0.$$

Since $\partial_x u_+ \to 0$ as $x \to \infty$, we obtain by integrating from $x \in (M,\infty)$ to ∞,

$$\partial_x u_+ < 0, \quad x \in (M, \infty).$$

This implies u_+ is decreasing on (M, ∞). This further implies both that $M = -\infty$ and that (with strict inequality) $\liminf_{x \to -\infty} u_+(x) > 0$. □

As a consequence of the proposition, we have the following:

Proposition 2.7. *For* $\mathrm{Im}\,(\lambda) \gg 1$, *there exists an operator*

$$R_V(\lambda) \coloneqq (D_x^2 + V(x) - \lambda^2)^{-1} \colon L^2(\mathbb{R}) \to L^2(\mathbb{R})$$

which can be extended to a meromorphic family of operators

$$R_V(\lambda) \coloneqq (D_x^2 + V(x) - \lambda^2)^{-1} \colon L^2_{\mathrm{c}}(\mathbb{R}) \to L^2_{\mathrm{loc}}(\mathbb{R})$$

for all $\lambda \in \mathbb{C}$. *Moreover, if* $u_\pm(x, \lambda)$ *are outgoing solutions, as constructed in Proposition 2.3, then the poles of the resolvent are the* $\lambda \in \mathbb{C}$ *such that* $W[u_+, u_-](\lambda) = 0$ *(defined in (2.10)), which are notably independent of* x.

Proof. If $f \in L^2(\mathbb{R})$ and $\mathrm{Im}\,(\lambda) \gg 1$, then using the variation of parameters formula, a solution to

$$(D_x^2 + V(x) - \lambda^2)u = f$$

can be written as

$$u(x) = \int_{\mathbb{R}} R(x, y, \lambda) f(y)\,\mathrm{d}y$$

where:

$$R(x, y, \lambda) \coloneqq \frac{u_+(x, \lambda)u_-(y, \lambda)H(x - y)}{W[u_+, u_-](\lambda)} + \frac{u_-(x, \lambda)u_+(y, \lambda)H(y - x)}{W[u_+, u_-](\lambda)}. \tag{2.11}$$

Here, $u_\pm$ are solutions to $(D_x^2 + V(x) - \lambda^2)u_\pm = 0$ as constructed in Proposition 2.3, and $H(x)$ is the Heaviside function. By Proposition 2.6, the Wronskian for $u_\pm$ is not identically zero. Moreover, because the Wronskian is entire in λ, it follows that the resolvent can be extended to a meromorphic function of λ (with poles where the Wronskian is zero). □

2.3. **The Scattering Matrix.** Because V decays exponentially, we have the existence of the scattering matrix $S(\lambda)$ [Mel85, §5], whose components we write as

$$S(\lambda) \coloneqq \begin{pmatrix} T(\lambda) & R_+(\lambda) \\ R_-(\lambda) & T(\lambda) \end{pmatrix}$$

where $T(\lambda)$ is the transmission coefficient, $R_+(\lambda)$ is the right reflection coefficient, and $R_-(\lambda)$ is the left reflection coefficient. We recall that for compactly supported solutions to $(P_V - \lambda^2)u = 0$, $\lambda \in \mathbb{R} \setminus \{0\}$ have the form

$$u|_{x \ll -1} = A_- e^{i\lambda x} + B_- e^{-i\lambda x}, \quad u|_{x \gg 1} = A_+ e^{i\lambda x} + B_+ e^{-i\lambda x},$$

and $S(\lambda)$ takes the incoming terms to the outgoing terms, with the convention that for $V=0$ the scattering matrix is the identity:

$$S(\lambda)\begin{pmatrix}A_-\\B_+\end{pmatrix}=\begin{pmatrix}A_+\\B_-\end{pmatrix}.$$

In our situation the same definition applies but now in an asymptotic sense as $x\to\pm\infty$.

Let $u_\pm=u_\pm(x,\lambda)$ be outgoing solutions at $\pm\infty$ and let $w_\pm=w_\pm(x,\lambda)$ be incoming solutions at $\pm\infty$ (recalling Definitions 2.1 and 2.2) so that

$$u_\pm(x)|_{\pm x\gg 1}=e^{\pm i\lambda x}v_\pm(e^{\mp A_\pm x}),\quad w_\pm(x)|_{\pm x\gg 1}=e^{\mp i\lambda x}\tilde{v}_\pm(e^{\mp A_\pm x}).$$

For now, we will suppress the λ in the arguments of $u_\pm$ and $w_\pm$. Because u_+ and w_+ are linearly independent, there exist constants $a,b\in\mathbb{C}$ (depending on λ) such that

$$u_-(x)=au_+(x)+bw_+(x). \tag{2.12}$$

The scattering matrix will then satisfy

$$S(\lambda)\begin{pmatrix}0\\b\tilde{v}_+(0)\end{pmatrix}=\begin{pmatrix}av_+(0)\\v_-(0)\end{pmatrix}$$

so that

$$T(\lambda)b\tilde{v}_+(0)=v_-(0). \tag{2.13}$$

The transmission coefficient $T(\lambda)$ can be related to the Wronskian of u_+ and u_-. Indeed, using (2.12),

$$\begin{aligned}W[u_+,u_-](\lambda)&=bW[u_+,w_+](\lambda)=b\lim_{x\to\infty}W[e^{i\lambda x}v_+(e^{-A_+x}),e^{-i\lambda x}\tilde{v}_+(e^{-A_+x})]\\&=b2i\lambda v_+(0)\tilde{v}_+(0)=\frac{2i\lambda v_+(0)v_-(0)}{T(\lambda)}\end{aligned} \tag{2.14}$$

where in the last equality we used (2.13).

Finally using Proposition 2.3, we can rewrite (2.14) as

$$T(\lambda)=\frac{2i\lambda}{\Gamma(1-2i\lambda A_+^{-1})\Gamma(1-2i\lambda A_-^{-1})W[u_+,u_-](\lambda)} \tag{2.15}$$

which is a meromorphic function and has poles only at the zeros (in λ) of $W[u_+,u_-](\lambda)$.

Remark 2.1. *Observe that, besides $\lambda=0$, the only zeros of $T(\lambda)$ are at the poles of $\Gamma(1-2i\lambda A_+^{-1})$ and $\Gamma(1-2i\lambda A_-^{-1})$ where $W[u_-,u_+](\lambda)\neq 0$, which are along the negative imaginary axis at the points*

$$\left\{-\frac{1}{2}A_\pm i(k+1):k\in\mathbb{Z}_{\geq 0}\right\}.$$

By unitarity of the scattering matrix, we see that these points provide "false" poles of the scattering matrix along the positive imaginary axis at the points:

$$\left\{\frac{1}{2}A_{\pm}i(k+1) : k \in \mathbb{Z}_{\geq 0}\right\} \setminus \{\lambda \in \mathbb{C} : W[u_-, u_+](-\lambda) = 0\}.$$

The existence of such poles contradicts Heisenberg's original hypothesis that all poles of the scattering matrix correspond to resonances.[2] *These "false" poles will give rise to the $\mathcal{A}_{\pm}(t)$ appearing in our trace formula (1.4).*

3. Applying the Birman–Kreĭn trace formula

We are now prepared to apply the Birman–Kreĭn trace formula to prove our main result. Recall we are computing $\operatorname{Tr}(U_V(t) - U_0(t))$, where $U_V(t) := \cos(t\sqrt{P_V})$ is the wave propagator for a potential $V(x)$. We fix $\varphi \in C_0^\infty((0,\infty))$[3], so that

$$\begin{aligned}\operatorname{Tr}(U_V(t) - U_0(t))(\varphi) &= \operatorname{Tr}(\cos(t\sqrt{P_V}) - \cos(t\sqrt{P_0}))(\varphi) && (3.1)\\ &= \operatorname{Tr}(f(P_V) - f(P_0)) && (3.2)\end{aligned}$$

where $f(\lambda^2) = \frac{1}{2}(\hat{\varphi}(\lambda) + \hat{\varphi}(-\lambda))$. Equation (3.2) follows by writing

$$\begin{aligned}\cos\left(t\sqrt{P_V}\right)(\varphi) &= \tfrac{1}{2}\int_{-\infty}^{\infty}\left(e^{it\sqrt{P_V}} + e^{-it\sqrt{P_V}}\right)\varphi(t)\,\mathrm{d}t\\ &= \tfrac{1}{2}(\hat{\varphi}(\sqrt{P_V}) + \hat{\varphi}(-\sqrt{P_V})) = f(P_V).\end{aligned}$$

By the Birman–Kreĭn trace formula (Theorem 2), we obtain

$$\begin{aligned}\operatorname{Tr}(f(P_V) - f(P_0)) = \frac{1}{2\pi i}&\int_0^\infty f(\lambda^2)\operatorname{Tr}(S(\lambda)^{-1}\partial_\lambda S(\lambda))\,\mathrm{d}\lambda\\ &+ \sum_{j=1}^{k} f(E_j) + \tfrac{1}{2}(m_R(0) - 1)f(0)\end{aligned} \tag{3.3}$$

where E_j are the negative eigenvalues of $D_x^2 + V(x)$. Note that if V is positive (which we do not necessarily assume here), there are no negative eigenvalues.

We now focus on the first term on the right-hand side of (3.3). The term we are integrating against $f(\lambda)$ is

$$\mathcal{G}(\lambda) := \operatorname{Tr}\left(S(\lambda)^{-1}\partial_\lambda S(\lambda)\right) = \frac{d}{d\lambda}\log\det(S(\lambda)) = \frac{d}{d\lambda}\log\left(T(\lambda)/T(-\lambda)\right)$$

[2]See for instance [New13, §12.1.2] for a discussion of false poles in the context of Jost functions. See also [Pai95] for a bibliographical account of Jost which includes a brief discussion, with references, of "false" poles. As explained by Graham–Zworski [GZ03], these "false" poles also play an important role in scattering on asymptotically hyperbolic spaces and its relation to the conformal structure of the boundary at infinity.

[3]There is a singularity at $t = 0$ which requires us to choose φ supported in positive time.

where the first equality follows from Jacobi's formula and the second is a standard fact about the scattering matrix (see, for instance, [DZ19, §2.10 Exercise 3]). Moreover, we have by (2.15)

$$T(\lambda) = \frac{2i\lambda}{\Gamma(1-\alpha_-)\Gamma(1-\alpha_+)F(\lambda)}$$

where

$$F(\lambda) \coloneqq W[u_+(\cdot,\lambda), u_-(\cdot,\lambda)]$$

is entire in λ, with u_+ and u_- constructed in Proposition 2.3 and $\alpha_\pm = 2i\lambda A_\pm^{-1}$. Therefore, we can expand

$$\begin{aligned}\log(T(\lambda)/T(-\lambda)) = \log(\Gamma(1+\alpha_-)) - \log(\Gamma(1-\alpha_-)) + \log(\Gamma(1+\alpha_+))\\ - \log(\Gamma(1-\alpha_+)) + \log(F(-\lambda)) - \log(F(\lambda)).\end{aligned}$$

Computing the derivative, we obtain

$$\mathcal{G}(\lambda) = \frac{d}{d\lambda}\log(T(\lambda)/T(-\lambda))(\lambda) = \mathcal{G}_\Gamma^+(\lambda) + \mathcal{G}_\Gamma^-(\lambda) + \mathcal{G}_F(\lambda) \tag{3.4}$$

where we define

$$\begin{aligned}\mathcal{G}_\Gamma^\pm(\lambda) &\coloneqq \frac{d}{d\lambda}\left(\log(\Gamma(1+\alpha_\pm)) - \log(\Gamma(1-\alpha_\pm))\right)\\ &= \frac{2iA_\pm^{-1}\Gamma'(1+\alpha_\pm)}{\Gamma(1+\alpha_\pm)} + \frac{2iA_\pm^{-1}\Gamma'(1-\alpha_\pm))}{\Gamma(1-\alpha_\pm)}\end{aligned} \tag{3.5}$$

and

$$\mathcal{G}_F(\lambda) \coloneqq \frac{d}{d\lambda}(\log(F(-\lambda)) - \log(F(\lambda))). \tag{3.6}$$

Inserting this into the first term on the right-hand side of (3.3) gives

$$\begin{aligned}&\frac{1}{2\pi i}\int_0^\infty f(\lambda^2)\,\mathrm{Tr}(S(\lambda)^{-1}\partial_\lambda S(\lambda))\,\mathrm{d}\lambda\\ =&\frac{1}{2\pi i}\int_0^\infty f(\lambda^2)\left(\mathcal{G}_F(\lambda) + \mathcal{G}_\Gamma^+(\lambda) + \mathcal{G}_\Gamma^-(\lambda)\right)\mathrm{d}\lambda\end{aligned} \tag{3.7}$$

It will turn out to be relatively straightforward to analyze the contribution of $\mathcal{G}_\Gamma^\pm$ in (3.7) using the well-known series expansion of the digamma function. On the other hand, to analyze $\mathcal{G}_F$, we will need some precise quantitative information about the distribution of the zeros of F. Our starting point is to observe that, thanks to (2.1) and the fact that the Wronskian is constant in x, we have the upper bound

$$|F(\lambda)| \lesssim e^{C|\lambda|\log(1+|\lambda|)}.$$

Since F is entire in λ, this yields the Hadamard factorization (cf. [Hay64, Theorem 1.9])

$$F(\lambda) = e^{a\lambda+b}\lambda^m \prod_{j=1}^{\infty} \left(1 - \frac{\lambda}{\lambda_j}\right) e^{\lambda/\lambda_j}.$$

for some $a, b \in \mathbb{C}$. Here m is the multiplicity of the zero at 0 and the λ_j are the nonzero roots of F. We then observe the following (formal at this point) expansion of $\log(F(\lambda))$,

$$\log(F(\lambda)) = (a\lambda + b) + m\log(\lambda) + \sum_{j=1}^{\infty} \left(\log\left(1 - \frac{\lambda}{\lambda_j}\right) + \frac{\lambda}{\lambda_j}\right)$$

so that

$$\frac{d}{d\lambda}\log(F(\lambda)) = a + \frac{m}{\lambda} + \sum_{j=1}^{\infty} \left(\frac{1}{\lambda - \lambda_j} + \frac{1}{\lambda_j}\right).$$

Similarly, we have

$$\frac{d}{d\lambda}\log(F(-\lambda)) = -a + \frac{m}{\lambda} + \sum_{j=1}^{\infty} \left(\frac{1}{\lambda_j + \lambda} - \frac{1}{\lambda_j}\right).$$

Therefore, we have the following formal expansion of $\mathcal{G}_F$ (defined in (3.6)),

$$\mathcal{G}_F(\lambda) = -2a + \sum_{j=1}^{\infty} \left(\frac{1}{\lambda_j + \lambda} + \frac{1}{\lambda_j - \lambda} - \frac{2}{\lambda_j}\right). \tag{3.8}$$

We observe in particular that $\mathcal{G}_F$ is an even function in λ. In order to compute the contribution of this term in (3.7) and rigorously justify the forthcoming manipulations, we will need the following proposition, which gives a precise description of the location and density of the zeros of F.

Proposition 3.1. *The roots of $F(\lambda) = W[u_+, u_-](\lambda)$ satisfy the following properties*

(1) *There exists $C > 0$ such that:*

$$\left\{F^{-1}(0)\right\} \subset \{iy : y \in [0, C]\} \cup \{z \in \mathbb{C} : \mathrm{Im}\,(z) < 0\}.$$

(2) *If $F(\lambda) = 0$, then $F(-\bar{\lambda}) = 0$.*[4]

(3) *There exists $\delta > 0$ such that F does not vanish anywhere on the strip $S_\delta :=$ $\{\lambda \in \mathbb{C} : -\delta < \mathrm{Im}(\lambda) < 0\}$.*[5]

(4) *For each $\varepsilon > 0$, there exists a $C > 0$ such that for all $r > 0$*

$$\#\{z \in \mathbb{C} : F(z) = 0, |z| \le r\} \le C(1 + r^{1+\varepsilon}).$$

[4] In fact for all $z \in \mathbb{C}$, $F(z) = \overline{F(-\bar{z})}$.

[5] We remark that [Mar02] can be used to show that F does not vanish on a logarithmic region, however this is not needed in this article.

For ease of exposition, we postpone the proof of this proposition until the end of the section.

We now compute the contribution of $\mathcal{G}_F$ in (3.7). Using that $\mathcal{G}_F$ is even, we observe the expansion

$$\begin{aligned}\frac{1}{2\pi i}\int_0^\infty f(\lambda^2)\mathcal{G}_F(\lambda)\,\mathrm{d}\lambda &= \frac{1}{4\pi i}\int_0^\infty (\hat{\varphi}(\lambda)+\hat{\varphi}(-\lambda))\mathcal{G}_F(\lambda)\,\mathrm{d}\lambda \\ &= \frac{1}{4\pi i}\int_{-\infty}^\infty \hat{\varphi}(\lambda)\mathcal{G}_F(\lambda)\,\mathrm{d}\lambda \qquad (3.9)\end{aligned}$$

To compute the integral in the second line, our first objective will be to establish the uniform summability bound

Lemma 3.2. *Let $\mathcal{G}_F^j$ denote the jth summand in (3.8). Then, for every $\varepsilon > 0$ sufficiently small, we have the uniform bound*

$$\sum_{j=1}^\infty |\mathcal{G}_F^j(\lambda)| \leq C_\varepsilon \langle\lambda\rangle^{1+\varepsilon}, \qquad \lambda \in \mathbb{R}$$

with $C_\varepsilon > 0$.

The above lemma asserts, in particular that for each λ, $\mathcal{G}_F^j$ is absolutely summable with a sub-polynomial upper bound in $\lambda \in \mathbb{R}$. We note importantly that this bound holds for real λ.

Proof. Without loss of generality, we assume λ_j are indexed in ascending order of magnitude. That is,

$$|\lambda_1| \leq |\lambda_2| \leq ... \leq |\lambda_k| \leq \cdots .$$

Since F is entire and $\lambda_j \neq 0$, we have the lower bound

$$\inf_{j\in\mathbb{N}} |\lambda_j| > c_0$$

for some $c_0 > 0$ depending on the profile of F. Moreover by Property 4 in Proposition 3.1, we have

$$|\lambda_j| \geq Cj^{1-\varepsilon} \qquad (3.10)$$

where $C > 0$ depends on ε.[6] As above, in the sequel, we will adopt the convention that $C > 0$ denotes a generic constant (which can change from line to line) depending on ε and on c_0, but is otherwise universal. Now we consider separately the regions

[6]Indeed, if (3.10) were not true, then for every $\tilde{C} > 0$, there would exist a subsequence $j_k \to \infty$ such that $|\lambda_{j_k}| < \tilde{C}(j_k)^{1-\varepsilon}$. But because λ_j are ordered by absolute value, we have that $j_k \leq \#\left\{|\lambda_j| \leq (j_k)^{1-\varepsilon}\tilde{C}\right\} \leq C(1+((j_k)^{1-\varepsilon}\tilde{C})^{1+\varepsilon})$. This implies that $j_k \leq C(1+(j_k)^{1-\varepsilon^2}\tilde{C}^{1+\varepsilon})$ which cannot hold as $k \to \infty$.

$|\lambda_j| > 2|\lambda|$ and $|\lambda_j| \le 2|\lambda|$. When $|\lambda_j| \ge 2|\lambda|$, we have by straightforward algebraic manipulation and (3.10),

$$\begin{aligned}\left|\frac{1}{\lambda_j+\lambda}+\frac{1}{\lambda_j-\lambda}-\frac{2}{\lambda_j}\right| &= \left|\frac{\lambda}{\lambda_j(\lambda_j-\lambda)}-\frac{\lambda}{\lambda_j(\lambda_j+\lambda)}\right| \\ &\le \left|\frac{\lambda}{\lambda_j(\lambda_j-\lambda)}\right|+\left|\frac{\lambda}{\lambda_j(\lambda_j+\lambda)}\right| \\ &\le C\frac{|\lambda|}{j^{2(1-\varepsilon)}}\end{aligned}$$

Above, in the last estimate, we used that $|\lambda_j+\lambda| \approx |\lambda_j-\lambda| \approx |\lambda_j|$. Hence,

$$\sum_{|\lambda_j|\ge 2|\lambda|} |\mathcal{G}_F^j(\lambda)| \le C|\lambda|.$$

In the region $|\lambda_j| \le 2|\lambda|$, we may assume $|\lambda| \ge c_0/2$, otherwise this region is empty and we are done. We begin by showing the bound

$$\frac{1}{|\lambda_j \pm \lambda|}+\frac{1}{|\lambda_j|} \le C. \tag{3.11}$$

We already have the required bound for $|\lambda_j|^{-1}$ in view of (3). To estimate $|\lambda_j \pm \lambda|$, we have for some $\delta > 0$ (in view of the fact that λ is real),

$$|\lambda_j \pm \lambda| \ge |\operatorname{Im}(\lambda_j)| \ge \delta.$$

In the second inequality, we used Property 3 of Proposition 3.1. Therefore, we have (3.11), and thus (in light of Property 4 of Proposition 3.1), we have

$$\sum_{|\lambda_j|\le 2|\lambda|} |\mathcal{G}_F^j(\lambda)| \le C\#\{j : |\lambda_j| \le 2|\lambda|\} \le C(1+|\lambda|^{1+\varepsilon}) \le C\langle\lambda\rangle^{1+\varepsilon}.$$

Combining everything yields (3.2), as desired. □

Now, we return to our analysis of (3.9). Using Proposition 3.2, that $\hat{\varphi}$ is Schwartz, $\varphi(0) = 0$, and Fubini's Theorem, we compute

$$\frac{1}{4\pi i}\int_{-\infty}^{\infty} \hat{\varphi}(\lambda)\mathcal{G}_F(\lambda)\,\mathrm{d}\lambda = \frac{1}{4\pi i}\sum_{j=1}^{\infty}\int_{-\infty}^{\infty} \hat{\varphi}(\lambda)\mathcal{G}_F^j(\lambda)\,\mathrm{d}\lambda.$$

We note that the term $-2a$ from (3.8) does not contribute to the sum as $\int_{-\infty}^{\infty} \hat{\varphi}(\lambda)\,\mathrm{d}\lambda = 0$. It remains to compute each of the above summands. For this, we will carry out a simple contour deformation. We let $R_n \to \infty$ be a sequence to be chosen. Clearly we have for each j

$$\int_{-\infty}^{\infty} \hat{\varphi}(\lambda)\mathcal{G}_F^j(\lambda)\,\mathrm{d}\lambda = \int_{-R_n}^{R_n} \hat{\varphi}(\lambda)\mathcal{G}_F^j(\lambda)\,\mathrm{d}\lambda + O(R_n^{-\infty}).$$

Let γ_1^n be the line from $-R_n$ to $-R_n - iR_n$, γ_2^n be the line from $-R_n - iR_n$ to $R_n - iR_n$, and γ_3^n be the line from $R_n - iR_n$ to R_n. We denote by γ^n, the union of these three lines oriented clockwise and let D_n denote the rectangle enclosed by γ^n and the real axis. Choosing R_n such that the contour does not intersect any of the poles of $\mathcal{G}_F$, we have by the residue theorem,

$$\int_{-R_n}^{R_n} \hat{\varphi}(\lambda)\mathcal{G}_F^j(\lambda)\,\mathrm{d}\lambda = -\int_{\gamma^n} \hat{\varphi}(\lambda)\mathcal{G}_F^j(\lambda)\,\mathrm{d}\lambda - 2\pi i \mathbb{1}_{D_n}(\lambda_j)\mathrm{Res}(\hat{\varphi}(\lambda)\mathcal{G}_F^j(\lambda), \lambda_j). \quad (3.12)$$

Here $\mathbb{1}_{D_n}(\lambda_j)$ is 1 if $\lambda_j \in D_n$ and 0 otherwise. By Property 4 of Proposition 3.1 and pigeonholing, we can further arrange for the sequence R_n to satisfying the following properties:

(1) (Approximate unit spacing). The sequence R_n satisfies

$$|R_n - n| < 1.$$

(2) (Quantitative avoidance of poles) For $\lambda \in \gamma^n$, we have

$$\inf_j |\lambda_j \pm \lambda| \geq Cn^{-2}. \quad (3.13)$$

Remark 3.1. *n^{-2} is not optimal but will suffice for our purposes.*

It remains to estimate the contributions of γ_i^n for each $i = 1, 2, 3$. First, for λ in γ_1^n or γ_3^n, we compute from (3.13) and the fact that the length of γ^n is on the order of n,

$$\int_{\gamma^n} |\hat{\varphi}(\lambda)||\mathcal{G}_F^j(\lambda)|\,\mathrm{d}\lambda \leq Cn^3 \sup_{\lambda\in\gamma^n} |\hat{\varphi}(\lambda)|.$$

In view of the support properties of φ and that λ is in the lower half plane, we have for $\lambda \in \gamma_1^n \cup \gamma_3^n$

$$n^4|\hat{\varphi}(\lambda)| \leq C|\lambda|^4|\hat{\varphi}(\lambda)| \leq C\int_0^\infty |\varphi^{(4)}(t)|e^{t\,\mathrm{Im}(\lambda)}\,\mathrm{d}t \leq C\|\varphi^{(4)}\|_{L^1(\mathbb{R})}.$$

For λ on the contour γ_2^n, we use the fact that $\mathrm{Im}\,(\lambda) = -R_n$ to obtain

$$|\hat{\varphi}(\lambda)| \leq Ce^{-R_n}\|\varphi\|_{L^1(\mathbb{R})}.$$

Combining the above, we obtain

$$\int_{\gamma^n} \hat{\varphi}(\lambda)\mathcal{G}_F^j(\lambda)\,\mathrm{d}\lambda \to 0, \quad \text{as } n \to \infty.$$

It remains to compute the sum of the second term on the right-hand side of (3.12) over j. Indeed, we have

$$\begin{aligned}\sum_{j=1}^{\infty}\int_{-\infty}^{\infty}\hat{\varphi}(\lambda)\mathcal{G}_F^j(\lambda)\,\mathrm{d}\lambda &= \sum_{j=1}^{\infty}\lim_{n\to\infty}\int_{-R_n}^{R_n}\hat{\varphi}(\lambda)\mathcal{G}_F^j(\lambda)\,\mathrm{d}\lambda \\ &= -2\pi i\sum_{j=1}^{\infty}\lim_{n\to\infty}\mathbb{1}_{D_n}(\lambda_j)\mathrm{Res}(\hat{\varphi}(\lambda)\mathcal{G}_F^j(\lambda),\lambda_j) \\ &= -2\pi i\left(\sum_{\mathrm{Im}(\lambda_j)>0}\hat{\varphi}(-\lambda_j)-\sum_{\mathrm{Im}(\lambda_j)<0}\hat{\varphi}(\lambda_j)\right) \end{aligned} \tag{3.14}$$

From the definition of f and the fact that the λ_j with positive imaginary part satisfy $\mathrm{Re}(\lambda_j)=0$ (by Proposition 3.1 Property (1)), we can further write

$$\sum_{\mathrm{Im}(\lambda_j)>0}\hat{\varphi}(-\lambda_j) = 2\sum_{\mathrm{Im}(\lambda_j)>0}f(\lambda_j^2)-\sum_{\mathrm{Im}(\lambda_j)>0}\hat{\varphi}(\lambda_j) \tag{3.15}$$

Let $\tilde{E}_j=\lambda_j^2$ for $\mathrm{Im}\,(\lambda_j)>0$. By Proposition 3.1, there are finitely many $\tilde{E}_j$ (whose cardinality we denote by k') which are negative real numbers (we will later show they are the negative eigenvalues of P_V). We can now rewrite (3.14) using (3.15) as

$$\sum_{j=1}^{\infty}\int_{-\infty}^{\infty}\hat{\varphi}(\lambda)\mathcal{G}_F^j(\lambda)\,\mathrm{d}\lambda = -2\pi i\left(2\sum_{j=1}^{k'}f(\tilde{E}_j)-\sum_{j=1}^{\infty}\hat{\varphi}(\lambda_j)\right)$$

Thanks to (3.12), we can insert this into (3.3) to obtain

$$\begin{aligned}\mathrm{Tr}(f(P_V)-f(P_0)) = &\frac{1}{2}\sum_{j=1}^{\infty}\hat{\varphi}(\lambda_j)+\frac{1}{4\pi i}\int_{-\infty}^{\infty}\hat{\varphi}(\lambda)(\mathcal{G}_\Gamma^+(\lambda)+\mathcal{G}_\Gamma^-(\lambda))\,\mathrm{d}\lambda \\ &+\frac{1}{2}(m_R(0)-1)f(0)+\sum_{j=1}^{k}f(E_j)-\sum_{j=1}^{k'}f(\tilde{E}_j).\end{aligned} \tag{3.16}$$

It remains to compute the contributions of $\mathcal{G}_\Gamma^+$ and $\mathcal{G}_\Gamma^-$. We recall from (3.5) the formula,

$$\mathcal{G}_\Gamma^\pm(\lambda) = \frac{2iA_\pm^{-1}\Gamma'(1+\alpha_\pm)}{\Gamma(1+\alpha_\pm)}+\frac{2iA_\pm^{-1}\Gamma'(1-\alpha_\pm))}{\Gamma(1-\alpha_\pm)}$$

where $\alpha_\pm=2i\lambda A_\pm^{-1}$. The main ingredient here is the following well-known formula for the digamma function,

$$\Psi(z) := \frac{\Gamma'(z)}{\Gamma(z)} = -\gamma+\sum_{j=0}^{\infty}\left(\frac{1}{j+1}-\frac{1}{j+z}\right),$$

where γ is the Euler–Mascheroni constant. Using this expansion, one can carry out a contour deformation argument as in the analysis of the term $\mathcal{G}_F(\lambda)$ to write

$$\begin{aligned}
&\frac{1}{2\pi i}\int_0^\infty f(\lambda^2)(\mathcal{G}_\Gamma^+(\lambda)+G_\Gamma^-(\lambda))\,\mathrm{d}\lambda\\
&=\tfrac{1}{4\pi i}\int_{-\infty}^\infty \hat{\varphi}(\lambda)(\mathcal{G}_\Gamma^+(\lambda)+\mathcal{G}_\Gamma^-(\lambda))\,\mathrm{d}\lambda\\
&=-\tfrac{1}{2}\sum_{j=0}^\infty\left(\hat{\varphi}\left(-iA_+\frac{j+1}{2}\right)+\hat{\varphi}\left(-iA_-\frac{j+1}{2}\right)\right).
\end{aligned}$$

The sum can be simplified as

$$\begin{aligned}
&\sum_{n=0}^\infty\left(\hat{\varphi}\left(\frac{-A_-(n+1)i}{2}\right)+\hat{\varphi}\left(\frac{-A_+(n+1)i}{2}\right)\right)\\
&=\sum_{n=0}^\infty\int_{-\infty}^\infty\left(e^{-tA_-(n+1)/2}+e^{-tA_+(n+1)/2}\right)\varphi(t)\,\mathrm{d}t\\
&=\int_{-\infty}^\infty\varphi(t)\sum_{n=0}^\infty\left(e^{-tA_-/2}e^{-tA_-n/2}+e^{-tA_+/2}e^{-tA_+n/2}\right)\mathrm{d}t\\
&=\int_{-\infty}^\infty\varphi(t)\left(\frac{e^{-tA_-/2}}{1-e^{-tA_-/2}}+\frac{e^{-tA_+/2}}{1-e^{-tA_+/2}}\right)\mathrm{d}t\\
&=\int_0^\infty\varphi(t)\left(\frac{1}{e^{tA_-/2}-1}+\frac{1}{e^{tA_+/2}-1}\right)\mathrm{d}t
\end{aligned}$$

Combining this with (3.16) and (3.1) and using that φ is supported on $(0,\infty)$, we obtain

$$\begin{aligned}
\operatorname{Tr}(U(t)-U(0))&=\tfrac{1}{2}\sum_{j=1}^\infty e^{-i\lambda_j t}-\tfrac{1}{2}\left(\frac{1}{e^{tA_-/2}-1}+\frac{1}{e^{tA_+/2}-1}\right)\\
&\quad+\tfrac{1}{2}(m_R(0)-1)+\sum_{j=1}^k f(E_j)-\sum_{j=1}^{k'} f(\tilde{E}_j)
\end{aligned}\tag{3.17}$$

in the sense of distributions. To conclude the proof of (1.4), we need to show that the λ_j appearing in the first term on the right-hand side correspond to non-zero resonances of V and that the $\tilde{E}_j$ correspond to the negative eigenvalues for V. This is the content of the following lemma, which is proved in Section A.

Lemma 3.3. *λ_0 is a pole of $R_V(\lambda)$ with multiplicity m if and only if λ_0 is an order m zero of $F(\lambda)=W[u_+(\cdot,\lambda),u_-(\cdot,\lambda)]$. Moreover, if 0 is a resonance (see Proposition A.1) of $-\partial_x^2+V(x)$, then it is simple.*

With this, we see that $\tilde{E}_j$ are the (negative) eigenvalues (corresponding to the poles of $R_V(\lambda)$ in the upper half plane) of P_V so the sum of the last two sums in (3.17) are zero, and

$$\tfrac{1}{2}\sum_{j=0}^{\infty} e^{-i\lambda_j t} + \tfrac{1}{2}(m_R(0)-1) = \tfrac{1}{2}\sum_{\lambda_j\in\mathrm{Res}(V)} e^{-i\lambda_j t} - \tfrac{1}{2}$$

which establishes the trace formula (1.4). It remains to finally prove Proposition 3.1, which we originally postponed.

Proof of Proposition 3.1. Properties 1 and 2 are standard results in scattering theory for real-valued potentials (see, for instance, [DZ19, §2.2]). Indeed, the roots of F correspond to the resonances of P_V by Proposition 3.3. Because the spectrum of P_V is contained in the positive real axis with possibly finitely many negative eigenvalues, the resolvent $(P_V-\lambda^2)$ is a meromorphic operator for $\mathrm{Im}\,(\lambda)>0$ on L^2 with finitely many poles along the imaginary axis corresponding to negative eigenvalues of P_V (this proves 1). Property 2 follows by taking the complex conjugate of outgoing solutions to $P_V-\lambda^2$.

We next prove Property 3. In view of Property 1, it suffices to show that there is some $\delta>0$ such that F does not vanish anywhere on $S_{M,\delta} := S_\delta \cap \{\lambda : |\lambda|\geq M\}$ where $\delta>0$ and $M>0$ are some positive constants to be chosen. Moreover, thanks to Proposition 3.3, it is sufficient to show that $S_{M,\delta}$ is resonance-free. In view of this, we need to show that for every $\rho\in C_c^\infty(\mathbb{R})$ and $\lambda\in S_{M,\delta}$, there holds

$$\|\rho R_V \rho\|_{L^2\to L^2} < \infty. \tag{3.18}$$

To set the stage, we use the standard formula for the resolvent, see for instance the discussion before [Fro97, Lemma 3.1]. For that we define

$$V^{\frac{1}{2}} := \mathrm{sgn}(V)|V|^{\frac{1}{2}}, \qquad \mathbf{R}_V(\lambda) := V^{\frac{1}{2}} R_0(\lambda)|V|^{\frac{1}{2}}$$

(note that $V^{\frac{1}{2}}|V|^{\frac{1}{2}} = V$). Multiplying the identity $R_0(\lambda) = R_V(\lambda) + R_0(\lambda) V R_V(\lambda)$ from the left and the right by $V^{1/2}$ and $|V|^{1/2}$ respectively, we then have (omitting λ for improved readability)

$$(\mathrm{Id}+\mathbf{R}_V)V^{\frac{1}{2}} R_V |V|^{\frac{1}{2}} = \mathbf{R}_V \implies V^{\frac{1}{2}} R_V |V|^{\frac{1}{2}} = (\mathrm{Id}+\mathbf{R}_V)^{-1}\mathbf{R}_V,$$

provided $\mathrm{Id}+\mathbf{R}_V$ is invertible. Inserting $R_V = R_0 - R_V V R_0$ into $R_V = R_0 - R_0 V R_V$ we obtain

$$R_V = R_0 - R_0 V R_0 + R_0 |V|^{\frac{1}{2}} V^{\frac{1}{2}} R_V |V|^{\frac{1}{2}} V^{\frac{1}{2}} R_0,$$

and hence, if $\mathrm{Id}+\mathbf{R}_V$ is invertible, we obtain the formal identity.

$$R_V = R_0 - R_0 V R_0 + R_0|V|^{\frac{1}{2}}(\mathrm{Id}+\mathbf{R}_V)^{-1}\mathbf{R}_V V^{\frac{1}{2}} R_0. \tag{3.19}$$

To establish (3.18), we need the following simple lemma, which will allow us to invert $\mathrm{Id}+\mathbf{R}_V$ on $S_{M,\delta}$ (and thus make use of (3.19)) using the Neumann series.

Lemma 3.4. *There exist parameters $\delta, M > 0$ such that for every $\lambda \in S_{M,\delta}$, there holds*

$$\|\mathbf{R}_V\|_{L^2\to L^2} \le \tfrac{1}{2}.$$

Proof. From the representation formula for the free resolvent, it suffices to establish the estimate

$$|\lambda|^{-1}\sup_{x\in\mathbb{R}}\int_{\mathbb{R}} |V(x)|^{\frac{1}{2}}|V(y)|^{\frac{1}{2}}e^{|\mathrm{Im}(\lambda)||x-y|}\,\mathrm{d}y \le \tfrac{1}{2}, \qquad \lambda \in S_{M,\delta}.$$

Since V satisfies the sub-exponential bound $|V(x)| \le Ce^{-\alpha|x|}$ for some $C, \alpha > 0$, it follows that by taking δ small enough, there holds

$$|\lambda|^{-1}\sup_{x\in\mathbb{R}}\int_{\mathbb{R}} |V(x)|^{\frac{1}{2}}|V(y)|^{\frac{1}{2}}e^{|\mathrm{Im}(\lambda)||x-y|}\,\mathrm{d}y \le \frac{C_\alpha}{M}, \qquad \lambda \in S_{M,\delta}.$$

The desired bound then follows by taking M sufficiently large. □

By slightly modifying the above argument, we have for every $\rho \in C_c^\infty(\mathbb{R})$

$$\|\rho R_0|V|^{\frac{1}{2}}\|_{L^2\to L^2} + \|V^{\frac{1}{2}}R_0\rho\|_{L^2\to L^2} \le C_\rho \tag{3.20}$$

for $\lambda \in S_{M,\delta}$ (where δ and M are exactly as above, and do not depend on the choice of ρ). At this point, (3.18) follows immediately from the formula (3.19), Proposition 3.4, and (3.20). This completes the proof of (3).

We finally prove Property 4. The bound is clear for $r \le 1$ thanks to the fact that F is non-constant and entire. We now assume $r \ge 1$. Thanks to (2.1) and the fact that the Wronskian is constant in x, we have the upper bound

$$|F(\lambda)| \lesssim e^{C|\lambda|\log(1+|\lambda|)}.$$

Moreover, since F is non-constant, we can find λ_0 in the unit ball such that $F(\lambda_0) \neq 0$. Jensen's formula then yields

$$\#\{z \in \mathbb{C} : F(z) = 0, |z| \le r\} \le C|r|\log(1+|r|) \lesssim_\varepsilon r^{1+\varepsilon}, \qquad r \ge 1$$

for some constant C depending on $F(\lambda_0)$, but not on r. □

Appendix A. Proof of Proposition 3.3

In proving Proposition 3.1, we used Proposition 3.3, which states that the poles of $R_V(\lambda)$ coincide (with multiplicity) with the zeros of $F(\lambda) = W[u_+(\cdot,\lambda), u_-(\cdot,\lambda)]$. We define the multiplicity of a resonance in the usual way.

Definition A.1. *For a potential $V(x)$ satisfying (1.2), a resonance is a pole of the meromorphic continuation of the resolvent $R_V(\lambda)$ (recall Proposition 2.7). If $R_V(\lambda)$ has a pole at $\lambda_0 \in \mathbb{C}$, then we can write*

$$R_V(\lambda) = \sum_{j=1}^{J} (\lambda - \lambda_0)^{-j} B_j + B_0(\lambda)$$

for finite rank operators B_j, $B_0(\lambda)$ holomorphic near $\lambda = \lambda_0$, and $J \in \mathbb{Z}_{\geq 1}$ (the order of the pole). The ***multiplicity of the pole*** *λ_0 is the dimension of the space spanned by $B_j(L^2_c(\mathbb{R}))$ for $j = 1, \ldots, J$.*

By (2.11), the poles of $R_V(\lambda)$ are exactly the poles of the resolvent function

$$R(x, y, \lambda) = \frac{u_+(x, \lambda)u_-(y, \lambda)H(x - y) + u_-(x, \lambda)u_+(y, \lambda)H(y - x)}{F(\lambda)} \tag{A.1}$$

where $H(\cdot)$ is the Heaviside function. We now prove Proposition 3.3. We begin with the first part, where we show that λ_0 is a pole of $R_V(\lambda)$ with multiplicity m if and only if λ_0 is a zero of order m for $F(\lambda)$. Clearly, from the formula for $R(x, y, \lambda)$, if $F(\lambda_0) \neq 0$, then λ_0 is not a pole of the resolvent. Hence, it suffices to show that if $\lambda_0 \in \mathbb{C}$ is a zero of order $m \geq 1$ for F, if and only if λ_0 is a pole of $R_V(\lambda)$ with multiplicity m. We will begin with the forward implication, which is more difficult. In our proof, we will need the following technical lemma relating the spaces spanned by the λ derivatives of u_+ and u_- at $\lambda = \lambda_0$.

Lemma A.2. *If $F(\lambda)$ vanishes to order $m \geq 1$ at $\lambda_0 \in \mathbb{C}$, then there exists a polynomial $p_{m-1}(\lambda)$ of degree $m - 1$ (depending only on λ) such that the function*

$$r_{m-1}(x, \lambda) \coloneqq u_+(x, \lambda) - p_{m-1}(\lambda)u_-(x, \lambda) \tag{A.2}$$

vanishes to order m at $\lambda = \lambda_0$. That is, r_{m-1} satisfies

$$(\partial_l^\lambda r_{m-1})_{|\lambda=\lambda_0} = 0, \quad 0 \leq l \leq m - 1. \tag{A.3}$$

In particular, we have

$$\operatorname{Span}\{(\partial_\lambda^j u_+)_{|\lambda=\lambda_0}\}_{j=0}^{m-1} \subset \operatorname{Span}\{(\partial_\lambda^j u_-)_{|\lambda=\lambda_0}\}_{j=0}^{m-1}.$$

Proof. First, the case $m = 1$ is obvious as the $W(u_+, u_-)(\lambda_0) = 0$ implies that u_+ and u_- are linearly dependent at λ_0. For $m > 1$, we construct r_{m-1} by induction (for fixed m). Assume that for $0 \leq j < m - 1$, we have constructed $r_0, ..., r_j$ satisfying (A.2) and (A.3) (the case $j = 0$ follows easily). We now aim to construct r_{j+1}. First, we observe that there holds

$$\partial_\lambda^{j+1} W(u_+, u_-)_{|\lambda=\lambda_0} = \partial_\lambda^{j+1} W(r_j, u_-)_{|\lambda=\lambda_0} = W(\partial_\lambda^{j+1} r_j, u_-)_{|\lambda=\lambda_0} = 0 \tag{A.4}$$

where in the first equality, we used the formula (A.2) and in the second, we used the Leibniz rule and (A.3). Moreover, we claim that $(\partial_\lambda^{j+1} r_j)(x,\lambda_0)$ satisfies $(P_V - \lambda_0^2)(\partial_\lambda^{j+j} r(x,\lambda_0))$. To see this, we compute

$$0 = \partial_\lambda^{j+1}((P_V - \lambda^2)r_j) = (P_V - \lambda^2)\partial_\lambda^{j+1} r_j - 2(j+1)\lambda\partial_\lambda^j r_j - j(j+1)\partial_\lambda^{j-1} r_j$$

and see by (A.3) that the second two terms on the right-hand side vanish. This, with (A.4), shows that there exists a constant c_j such that

$$(\partial_\lambda^{j+1} r_j)(x,\lambda_0) = c_j u_-(x,\lambda_0).$$

We then define r_{j+1} by correcting r_j by a suitable degree $j+1$ monomial. We can simply take

$$r_{j+1}(x,\lambda) \coloneqq r_j(x,\lambda) - \frac{c_j}{(j+1)!}(\lambda-\lambda_0)^{j+1} u_-(x,\lambda).$$

or equivalently

$$p_{j+1}(\lambda) \coloneqq p_j(\lambda) + \frac{c_j}{(j+1)!}(\lambda-\lambda_0)^{j+1}.$$

It is straightforward to verify that r_{j+1} satisfies (A.2) and (A.3). This completes the proof. □

Now, returning to the proof of Proposition 3.3, let us suppose F has a zero of order $m \geq 1$ at $\lambda_0 \in \mathbb{C}$. Then $F = (\lambda-\lambda_0)^m G(\lambda)$ for G an entire function nonzero near λ_0. The Schwartz kernel of the resolvent can be written

$$R(x,y,\lambda) = \frac{u_+(x,\lambda)u_-(y,\lambda)H(x-y) + u_-(x,\lambda)u_+(y,\lambda)H(y-x)}{(\lambda-\lambda_0)^m G(\lambda)}. \tag{A.5}$$

By applying Proposition A.2 and multiplying (A.2) by u_-, we find that there exists a polynomial $p(\lambda)$

$$\partial_\lambda^j\big|_{\lambda_0}(u_+(y,\lambda)u_-(x,\lambda)) = \partial_\lambda^j\big|_{\lambda_0}(u_+(x,\lambda)u_-(y,\lambda)) = \partial_\lambda^j\big|_{\lambda_0}(p(\lambda)u_-(x,\lambda)u_-(y,\lambda)),$$

for $0 \leq j \leq m-1$. Therefore, by Taylor expanding $u_+(x,\lambda)u_-(y,\lambda)$ and $u_-(x,\lambda)u_+(y,\lambda)$ about $\lambda = \lambda_0$ and using the identity $H(x-y)+H(y-x) = 1$, we can write the Schwartz kernel of the resolvent in (A.5) as

$$R(x,y,\lambda) = \sum_{j=0}^{m-1} \frac{\partial_\lambda^j\big|_{\lambda_0}\big(p(\lambda)u_-(x,\lambda)u_-(y,\lambda)\big)}{j!G(\lambda)(\lambda-\lambda_0)^{m-j}} + \tilde{R}(x,y,\lambda)$$

with $\tilde{R}(x,y,\lambda)$ holomorphic in λ near λ_0. For notational convenience, we define $T_j(\lambda)$ as the operators with Schwartz kernel

$$T_j(x,y,\lambda) \coloneqq \partial_\lambda^j(p(\lambda)u_-(x,\lambda)u_-(y,\lambda))$$

so that

$$R(x,y,\lambda) = \sum_{j=0}^{m-1} \frac{T_j(x,y,\lambda_0)}{j!G(\lambda)(\lambda-\lambda_0)^{m-j}} + \tilde{R}(x,y,\lambda).$$

Let B_j be the operator with kernel $T_j(x, y, \lambda_0)$. It suffices to show that

$$\dim(\mathrm{Span}(B_0(L_c^2), B_1(L_c^2), \ldots, B_{m-1}(L_c^2)) = m. \tag{A.6}$$

To this end, we first observe that

$$T_0(x, y, \lambda) = p(\lambda)u_-(x, \lambda)u_-(y, \lambda) \tag{A.7}$$

so that B_0 is a rank one operator and $\mathrm{Span}(B_0(L_c^2)) = \mathrm{Span}(u_-(x, \lambda_0))$. For $j \geq 1$, by the Leibniz rule, $T_j(x, y, \lambda)$ is of the form

$$T_j(x, y, \lambda) = p(\lambda)\partial_\lambda^j u_-(x, \lambda)u_-(y, \lambda) + \sum_{|\alpha|<j} c_\alpha \partial_\lambda^{\alpha_1} p(\lambda)\partial_\lambda^{\alpha_2} u_-(x, \lambda)\partial_\lambda^{\alpha_3} u_-(y, \lambda) \tag{A.8}$$

for some coefficients $c_\alpha \in \mathbb{Z}_{\geq 0}$. To determine the rank of B_j, we need the following lemma.

Lemma A.3. *For each $j = 1, \ldots, m$, $\partial_\lambda^j\big|_{\lambda_0} u_-(x, \lambda)$ is not in the span of*

$$\left\{\partial_\lambda^0\big|_{\lambda_0} u_-(x, \lambda), \partial_\lambda\big|_{\lambda_0} u_-(x, \lambda), \ldots, \partial_\lambda^{j-1}\big|_{\lambda_0} u_-(x, \lambda)\right\}.$$

Proof. Our first observation is to notice that by differentiating the equation $(P - \lambda^2)u_- = 0$ a total of $1 \leq \ell \leq j$ times in λ, there holds

$$(P - \lambda_0^2)(\partial_\lambda^l\big|_{\lambda_0} u_-(x, \lambda)) = 2l\lambda_0 \partial_\lambda^{l-1}\big|_{\lambda_0} u_- + (l-1)l\partial_\lambda^{l-2}\big|_{\lambda_0} u_- \tag{A.9}$$

with the convention that $(l-1)l\partial_\lambda^{l-2}\big|_{\lambda_0} u_- = 0$ when $l = 1$. Note that the right-hand side of (A.9) involves derivatives of order at most $\ell - 1$. We can therefore inductively apply (A.9) to get that for each $\ell \in \mathbb{Z}_{\geq 0}$

$$(P - \lambda_0^2)^{\ell+1}(\partial_\lambda^\ell\big|_{\lambda_0} u_-(x, \lambda)) = 0. \tag{A.10}$$

Let $j > 1$ and suppose by contradiction that $\partial_\lambda^j\big|_{\lambda_0} u_-(x, \lambda)$ is in the span of the vectors $\left\{\partial_\lambda^\ell|_{\lambda_0} u_-(x, \lambda)\right\}_{\ell=0}^{j-1}$. By (A.10), $(P - \lambda_0^2)^j(\partial_\lambda^j\big|_{\lambda_0} u_-(x, \lambda)) = 0$. While on the other hand, by repeatedly using (A.9),

$$0 = (P - \lambda_0^2)^j \left(\partial_\lambda^j\big|_{\lambda_0} u_-(x, \lambda)\right) = cu_-(x, \lambda_0)$$

for some nonzero $c \in \mathbb{C}$. Because $u_-(x, \lambda_0)$ is not identically zero, this gives a contradiction. □

We will also need the following lemma, which along with Proposition A.3 proves (A.6)

Lemma A.4. *For $j = 0, \ldots, m$:*

$$\begin{aligned}&\mathrm{Span}(B_0(L_c^2), B_1(L_c^2), \ldots, B_{j-1}(L_c^2))\\ &\qquad = \mathrm{Span}(\partial_\lambda^0\big|_{\lambda_0} u_-(x, \lambda), \partial_\lambda^1\big|_{\lambda_0} u_-(x, \lambda), \ldots, \partial_\lambda^{j-1}\big|_{\lambda_0} u_-(x, \lambda)).\end{aligned} \tag{A.11}$$

Proof. The case where $j = 0$ was established by (A.7). We now assume (A.11) is true for a $j \leq m-1$. Let $u \in L^2_c$ be such that $\int u(y)u_-(y,\lambda)\,\mathrm{d}y \neq 0$, so by (A.8)

$$B_j u = \sum_{\ell=0}^{j} c_\ell \partial_\lambda^\ell|_{\lambda_0} u_-(x,\lambda)$$

for $c_\ell \in \mathbb{C}$ and $c_j \neq 0$. Therefore, by the inductive hypothesis, we have $\partial_\lambda^j|_{\lambda_0} u_-(x,\lambda) \in \mathrm{Span}(B_0(L^2_c), B_1(L^2_c), \ldots, B_j(L^2_c))$, and thus

$$\begin{aligned}\mathrm{Span}(\partial_\lambda^0|_{\lambda_0} u_-(x,\lambda), \partial_\lambda^1|_{\lambda_0} u_-(x,\lambda), \ldots, \partial_\lambda^j|_{\lambda_0} u_-(x,\lambda))& \\ \subset \mathrm{Span}(B_0(L^2_c), B_1(L^2_c), \ldots, B_j(L^2_c)).&\end{aligned}$$

But we also immediately have the converse inclusion as for $u \in L^2_c$, $B_j u$ is a linear combination of $\partial_\lambda^\ell|_{\lambda_0} u_-(x,\lambda)$ for $\ell = 0, \ldots, j$. □

We have now shown that if $F(\lambda)$ is a zero of order m at λ_0, then the resolvent $R(\lambda)$ has a pole of multiplicity m.

For the converse direction, we require the following Lemma.

Lemma A.5. *If the resolvent $R(\lambda)$ has a pole of order $m \in \mathbb{Z}_{>0}$ at λ_0, then $F(\lambda)$ has a zero of order m at λ_0.*

Proof. Given the hypothesis, we may write the Schwartz kernel of $R(\lambda)$ as

$$R(\lambda) = \sum_{j=0}^{m} \frac{R_j(x,y,\lambda)}{(\lambda-\lambda_0)^{m-j}}$$

with R_j holomorphic near λ_0 and $R_0 \neq 0$. This must agree with the expression of $R(\lambda)$ given in (A.1), which immediately implies that F has a zero of order m at λ_0. □

We now suppose that $R(\lambda)$ has a resonance of multiplicity m at λ_0. By Proposition A.5, this implies the order of the resonance is m. Indeed, if the order of the resonance at λ_0 was $m' \neq m$, then by Proposition A.5, F would have a zero of order m' at λ_0. We could then apply the forward direction to see that this implies the multiplicity of the resonance is m', which is a contradiction.

To conclude the proof of Proposition 3.3, it remains to show that if 0 is a resonance, then it is simple. For this, we recall from (2.15) that the transmission coefficient $T(\lambda)$ can be expressed in terms of $F(\lambda)$ as

$$T(\lambda) = \frac{2i\lambda}{\Gamma(1-\alpha_+)\Gamma(1-\alpha_-)F(\lambda)}. \tag{A.12}$$

By [Kla88, Theorem 2.1], for a large class of potentials (which include V), $T(0) \neq 0$. If 0 is a resonance of $-\partial_x^2 + V(x)$, then by (A.12) and that $T(0) \neq 0$, $F(\lambda)$ must have a zero of order 1. So by the above argument, 0 is a resonance of multiplicity 1. This finally concludes the proof of Proposition 3.3.

Appendix B. Birman-Kreĭn trace formula for exponentially decaying potentials

In this section we prove the Birman-Kreĭn trace formula for exponentially decaying potentials (i.e., where $|V(x)| \leq e^{-c|x|}$). The assumption on the decay is stronger than necessary but allows an easy definition of the multiplicity of the resonance at zero as then the resolvent continues meromorphically to a strip.

Theorem 2 (**Birman–Kreĭn trace formula in one dimension**). *Let $V \in C^\infty(\mathbb{R};\mathbb{R})$ be such that $|V(x)| \leq e^{-c|x|}$ for some $c > 0$, and set $P_V = D_x^2 + V(x)$. Then for $f \in \mathscr{S}(\mathbb{R})$ the operator $f(P_V) - f(P_0)$ is of trace class and*

$$\operatorname{Tr}\left(f(P_V) - f(P_0)\right) = \frac{1}{2\pi i}\int_0^\infty f(\lambda^2)\operatorname{Tr}\left(S(\lambda)^{-1}\partial_\lambda S(\lambda)\right) \mathrm{d}\lambda + \sum_{j=1}^{K} k f(E_j) + \tfrac{1}{2}(m_R(0) - 1) f(0)\,,$$

where $S(\lambda)$ is the scattering matrix, $m_R(0)$ is the multiplicity of the resonance at 0, and E_j are the (negative) eigenvalues of P_V.

The proof of Theorem 2 is a mild modification of the proof of the one-dimensional Birman-Kreĭn trace theorem for compactly supported potentials found in [DZ19, §2.6].

Our proof critically uses a determinant identity to analyze the scattering matrix. A version of this identity was proven in [Fro97], but for super-exponentially decaying potentials. A minor adjustment to the proof will enable us to prove the identity for $\{\operatorname{Im}(\lambda) > -\delta\} \setminus \{0\}$. This is a weaker result, but will suffice for our purposes. The main technical input is the following mild adaptation of [Fro97, Lemma 7.6] to our setting.

Lemma B.1. *Let $V \in L^\infty(\mathbb{R})$ be a function satisfying the exponential decay bound*

$$|V(x)| \leq Ce^{-2c_0|x|}$$

for constants $C, c_0 > 0$. Then for $\lambda \in \mathbb{C}$ such that $0 < 4|\operatorname{Im}(\lambda)| < c_0$, $\mathbf{R}_V(\lambda)$ is trace class and

$$\det S(\lambda) = \frac{\det(\mathrm{Id} + \mathbf{R}_{\mathrm{V}}(-\lambda))}{\det(\mathrm{Id} + \mathbf{R}_{\mathrm{V}}(\lambda))}, \tag{B.1}$$

where $\mathbf{R}_V(\lambda) := V^{\frac{1}{2}} R_0(\lambda)|V|^{1/2}$ (which was introduced in the proof of Proposition 3.3).

The proof follows by making minor adjustments to the argument in [Fro97] (specifically [Fro97, Lemma 3.1 and 7.6]). We include the short argument for the convenience to the reader.

Proof. **Step 1.** The first step is to show that

$$\lim_{L\to\infty} V^{\frac{1}{2}}\mathbf{R}_{\chi_L}(\lambda)|V|^{\frac{1}{2}} = \mathbf{R}_V(\lambda) \tag{B.2}$$

where the limit is taken in the trace norm and χ_L denotes the indicator function on the interval $[-L, L]$.

The hypothesis on V ensures that we have the bound $|V|^{\frac{1}{2}} \le Ce^{-c_0|x|}$. Our first goal will be to show that it suffices to establish (B.2) with V replaced by a simpler step function approximation w of $Ce^{-c_0|x|}$. To define w, we define the dyadic-like telescoping sequence μ_k by

$$\mu_1 \coloneqq Ce^{-c_0}, \quad \mu_k \coloneqq C(e^{-c_0k} - e^{-c_0(k+1)})$$

where $k > 1$. We then define w by

$$w(x) = \sum_{k=1}^{\infty} \mu_k\chi_k(x)$$

where χ_k is the indicator function on $[-k, k]$. As a consequence of the telescoping identity

$$\sum_{k\ge j} \mu_k = Ce^{-c_0j},$$

there holds $|V|^{\frac{1}{2}}w^{-1} \le 1$. Consequently, it suffices to establish (B.2) with $V^{\frac{1}{2}}$ and $|V|^{\frac{1}{2}}$ replaced by w. To this end, we estimate

$$\begin{aligned}
&\|w\chi_L R_0\chi_L w - wR_0w\|_1 \\
&\qquad = \|w\chi_L R_0(\lambda)\chi_L w - wR_0(\lambda)\chi_L w + wR_0(\lambda)\chi_L w - wR_0(\lambda)w\|_1 \\
&\qquad \le 2\|wR_0(\lambda)(1-\chi_L)w\|_1 \\
&\qquad \le 2\|\sum_{i\ge1}\mu_i\chi_i R_0(\lambda)\sum_{j>L}\mu_j\chi_j\|_1 \\
&\qquad \le 2\sum_{i}\sum_{j>L}\mu_i\mu_j\|\chi_i R_0(\lambda)\chi_j\|_1 \\
&\qquad \le 2\sum_{i\le j}\sum_{j>L}\mu_i\mu_j\|\chi_j R_0(\lambda)\chi_j\|_1 + 2\sum_{i>j}\sum_{j>L}\mu_i\mu_j\|\chi_i R_0(\lambda)\chi_i\|_1 \\
&\qquad \le \frac{C}{|\operatorname{Im}(\lambda)|}\left(\sum_{i\ge1}\mu_i\sum_{j>L} je^{j(4|\operatorname{Im}(\lambda)|-c_0)} + \sum_{i\ge1} ie^{i(4|\operatorname{Im}(\lambda)|-c_0)}\sum_{j>L}\mu_j\right)
\end{aligned} \tag{B.3}$$

where in the last line, we used the elementary bound (see for instance, [Fro97, Corollary 7.5])

$$\|\chi_k R_0(\lambda)\chi_k\|_1 \leq \frac{C}{|\operatorname{Im}(\lambda)|} k e^{4k|\operatorname{Im}(\lambda)|}.$$

Thanks to the hypothesis $|\operatorname{Im}(\lambda)| < \frac{c_0}{4}$ and the summability of μ_i, it follows that the last line of (B.3) goes to zero as $L \to \infty$.

Step 2. [Fro97, Lemma 3.1] follows without modification. Indeed, by [Fro97, Proposition 7.2],

$$\mathbf{R}_{\chi_1}(\lambda) = \mathbf{R}_{\chi_1}(-\lambda) + F_2(\lambda) \tag{B.4}$$

where $F_2(\lambda)$ is a rank two operator. Conjugating (B.4) by unitary dilation $\psi(x) \mapsto L^{-1/2}\psi(x/L)$, multiplying the left by $V^{1/2}$ and right by $|V|^{1/2}$, and using (B.2), we get

$$\mathbf{R}_V(\lambda) = \mathbf{R}_V(-\lambda) + F_V(\lambda) \tag{B.5}$$

where $F_V(\lambda)$ is a rank two operator. Rearranging (B.5), and taking the Fredholm determinant gives (B.1) (see [Fro97, eq. 3.4], and subsequent discussion).

□

Proof of Theorem 2. For simplicity, we assume that there are *no* negative eigenvalues as their contribution is easy to analyze.

Step 1. We first show that $f(P_V) - f(P_0)$ is trace class. Because P_V is self-adjoint, we have by Stone's formula

$$\begin{aligned} f(P_V) &= \frac{1}{2\pi i}\int_0^\infty f(\lambda^2)(R_V(\lambda) - R_V(-\lambda))2\lambda\,\mathrm{d}\lambda \\ &= \frac{1}{4\pi i}\int_{\mathbb{R}} f(\lambda^2)(R_V(\lambda) - R_V(-\lambda))2\lambda\,\mathrm{d}\lambda, \end{aligned}$$

using that the integrand is even in λ. The integral on the right-hand side should be interpreted as an operator $L^2_{\mathrm{c}} \to L^2_{\mathrm{loc}}$. We therefore have

$$\begin{aligned} f(P_V) - f(P_0) &= \frac{1}{4\pi i}\int_{\mathbb{R}} f(\lambda^2)(R_V(\lambda) - R_0(\lambda) - (R_V(-\lambda) - R_0(\lambda))2\lambda\,\mathrm{d}\lambda \\ &= \frac{1}{4\pi i}\int_{\mathbb{R}} f(\lambda^2)(R_V(-\lambda)VR_0(-\lambda) - R_V(\lambda)VR_0(\lambda))2\lambda\,\mathrm{d}\lambda, \end{aligned} \tag{B.6}$$

where we use that

$$R_V(\lambda) - R_0(\lambda) = -R_V(\lambda)VR_0(\lambda). \tag{B.7}$$

We now define the meromorhic family of operators

$$B(\lambda) := 2\lambda R_V(\lambda)VR_0(\lambda)\colon L^2_{\mathrm{c}} \to L^2_{\mathrm{loc}}. \tag{B.8}$$

We observe that this family is holomorphic in the closed upper half plane $\{\mathrm{Im}\,(\lambda) \geq 0\}$. This is because $R_V(\lambda)VR_0(\lambda)$ has a simple pole at $\lambda = 0$ (canceling the λ is the definition of $B(\lambda)$). This follows by using (B.7) and Proposition 3.3.

We can then rewrite (B.6) as

$$f(P_V) - f(P_0) = \frac{1}{4\pi i}\int_{\mathbb{R}} f(\lambda^2)(B(-\lambda) - B(\lambda))\,\mathrm{d}\lambda. \tag{B.9}$$

Fixing $\varepsilon > 0$ sufficiently small, let $g \in \mathcal{S}(\mathbb{C})$, $\operatorname{supp} g \subset \{\lambda \in \mathbb{C} : \mathrm{Im}\,(\lambda) < \varepsilon\}$, be an *almost analytic extension* of $f(\lambda)$ (see for instance [DZ19, §B.2]):

$$g(\lambda) = f(\lambda^2),\ \lambda \in \mathbb{R},\ \overline{\partial}_\lambda g(\lambda) = \mathcal{O}(|\mathrm{Im}\,(\lambda)|^\infty). \tag{B.10}$$

The support of g can be made arbitrarily close to the real axis. This is critical, as we will eventually use Proposition B.1 – which is only valid in a neighborhood of the real axis (which depends on the decay of V i.e., $A_\pm$). The Cauchy–Green formula [DZ19, (D.1.1)] applied to the right-hand side of (B.9) shows that

$$f(P_V) - f(P_0) = \sum_{\pm} \pm\frac{1}{2\pi}\int_{\mathrm{Im}(\lambda)>0} \overline{\partial}_\lambda g(\lambda) B(\pm\lambda)\,\mathrm{d}m(\lambda).$$

Since there are no negative eigenvalues, the spectral theorem gives the bound

$$\|R_V(\lambda)\|_{L^2\to L^2} = \frac{1}{d(\lambda^2, \mathbb{R}_+)} \leq \frac{1}{|\lambda|\mathrm{Im}\,(\lambda)},\quad \mathrm{Im}\,(\lambda) > 0\,. \tag{B.11}$$

In particular, when $V = 0$ we have the following $L^2 \to H^2$ bound when $\mathrm{Im}\,(\lambda) > 0$,

$$\begin{aligned}\|\lambda R_0(\lambda)\|_{L^2\to H^2} &\lesssim |\lambda|(\|D_x^2 R_0(\lambda)\|_{L^2\to L^2} + \|R_0(\lambda)\|_{L^2\to L^2})\\ &\lesssim |\lambda|(1 + |\lambda|^2)\|R_0(\lambda)\|_{L^2\to L^2}\\ &\lesssim \frac{(1 + |\lambda|^2)}{\mathrm{Im}\,(\lambda)}.\end{aligned}$$

Combining this estimate with the sub-exponential bound $|V| \leq e^{-c|x|}$, ensures $VR_0(\lambda)$ is a trace class operator for $\mathrm{Im}\,(\lambda) > 0$ and that

$$\|\lambda VR_0(\lambda)\|_{\mathcal{L}_1} \lesssim \frac{1 + |\lambda|^2}{\mathrm{Im}\,(\lambda)},\quad \mathrm{Im}\,(\lambda) > 0.$$

From this bound, as well as (B.8) and (B.11), we find that for $\mathrm{Im}\,(\lambda) > 0$, there holds

$$\|B(\lambda)\|_{\mathcal{L}_1} \lesssim \frac{1 + |\lambda|^2}{|\mathrm{Im}\,(\lambda)\,|^2|\lambda|} \lesssim \frac{1 + |\lambda|^2}{|\mathrm{Im}\,(\lambda)\,|^3}. \tag{B.12}$$

Using (B.12) and (B.10) we conclude that for any $N > 0$, and in particular for $N \geq 4$,

$$\begin{aligned}\Big\| \int_{\pm\mathrm{Im}(\lambda)>0} \overline{\partial}_\lambda g(\lambda) B(\pm\lambda)\, \mathrm{d}m(\lambda) \Big\|_{\mathcal{L}_1} \\ \leq C_N \int_{0<\pm\mathrm{Im}(\lambda)<1} |\mathrm{Im}\,(\lambda)\,|^N (1+|\lambda|)^{-N+2} |\mathrm{Im}\,(\lambda)\,|^{-3}\, \mathrm{d}m(\lambda) < \infty\,.\end{aligned}$$

It follows that $f(P_V) - f(P_0)$ is trace class, as desired.

Step 2. We now relate $B(\lambda)$ to the scattering matrix.

Define the operator

$$\mathcal{R}(\lambda) \coloneqq -\partial_\lambda \mathbf{R}_V(\lambda)(\mathrm{Id} + \mathbf{R}_V(\lambda))^{-1}.$$

This operator is related to the scattering matrix. Indeed, by taking the logarithmic derivative of both sides of (B.1), we get

$$\mathrm{Tr}(\partial_\lambda S(\lambda) S(\lambda)^{-1}) = \mathrm{Tr}\,\mathcal{R}(\lambda) + \mathrm{Tr}\,\mathcal{R}(-\lambda). \tag{B.13}$$

Lemma B.2. *For* $\mathrm{Im}\,(\lambda) > 0$*, the operator* $\mathcal{R}(\lambda)$ *is trace class and* $\mathrm{Tr}\,\mathcal{R}(\lambda) = \mathrm{Tr}\,B(\lambda)$*.*

Proof. We observe that for $\lambda \in \mathbb{C}$, $\mathcal{R}(\lambda)$ is a meromorphic family of operators in $\mathcal{L}_1(L^2)$, and has no poles in the region $\mathrm{Im}\,(\lambda) > 0$ (recalling our assumption that there are no negative eigenvalues). We then have a direct analog of [DZ19, (2.2.33)]:

$$\mathrm{Id} + \mathbf{R}_V(\lambda) = U_1(\lambda)(Q_0 + \lambda^{-1} Q_{-1} + \lambda Q_1) U_2(\lambda),$$

where $U_j(\lambda)$ are invertible, holomorphic, and Q_j are finite rank operators such that

$$\mathrm{rank}\, Q_{-1} = 1, \quad \mathrm{rank}\, Q_1 = m_R(0), \quad Q_j Q_k = \delta_{jk} Q_j,$$

see [DZ19, Theorem C.10]. Hence,

$$\mathrm{Tr}\,\mathcal{R}(\lambda) = -\,\mathrm{Tr}(\lambda^{-1} Q_1 - \lambda^{-1} Q_{-1}) + \varphi(\lambda) = \frac{1 - m_R(0)}{\lambda} + \varphi(\lambda), \tag{B.14}$$

where $\varphi(\lambda)$ is holomorphic in $\mathrm{Im}\,(\lambda) \geq 0$. Equation (B.14) follows by writing

$$\begin{aligned}\mathrm{Tr}\,\mathcal{R}(\lambda) = &-\partial_\lambda \log\det(U_1(\lambda)) - \partial_\lambda \log\det(U_2(\lambda)) \\ &- \log(\det(Q_0 + \lambda^{-1} Q_{-1} + \lambda Q_1)),\end{aligned}$$

using that $Q_j Q_k = \delta_{jk} Q_j$ (and Jacobi's formula) to rewrite the third term as

$$-\,\mathrm{Tr}(\lambda^{-1} Q_1 - \lambda^{-1} Q_{-1}).$$

In order to control $\varphi(\lambda)$, we first observe the $L^2 \to H^2$ bound for $\mathbf{R}_V$

$$\|\mathbf{R}_V(\lambda)\|_{L^2 \to H^2} \leq C|\lambda| e^{C(\mathrm{Im}(\lambda))_-}, \quad |\lambda| \geq 1.$$

From the Cauchy formula, we also have that for $\mathrm{Im}\,(\lambda) \geq 0$, $|\lambda| \gtrsim 1$,

$$\|\partial_\lambda \mathbf{R}_V(\lambda)\|_{\mathcal{L}_1} \leq C\|\partial_\lambda \mathbf{R}_V(\lambda)\|_{L^2\to H^2} \leq C'|\lambda|.$$

Combining this bound with the definition of $\mathcal{R}(\lambda)$, and the invertibility of $I + \mathbf{R}_V(\lambda)$ in the region $|\lambda| \gg 1$, $\mathrm{Im}\,(\lambda) \geq 0$, we obtain for some $R_0 > 0$

$$|\operatorname{Tr}\mathcal{R}(\lambda)| \leq C|\lambda|, \quad |\lambda| \geq R_0, \quad \mathrm{Im}\,(\lambda) \geq 0.$$

Then, since $\varphi(\lambda)$ is holomorphic in the region $\mathrm{Im}\,(\lambda) \geq 0$, we obtain the bound

$$|\varphi(\lambda)| \leq C(1 + |\lambda|), \quad \mathrm{Im}\,(\lambda) \geq 0.$$

We next show that

$$\operatorname{Tr}\mathcal{R}(\lambda) = \operatorname{Tr} B(\lambda)\,, \tag{B.15}$$

where $B(\lambda)$ is given by (B.8).

Observe by cyclicity of the trace (cf. [DZ19, Theorem B.4.9])

$$\begin{aligned}
\operatorname{Tr}(B(\lambda)) &= -2\lambda \operatorname{Tr}(R_V(\lambda) V R_0(\lambda)) \\
&= -2\lambda \operatorname{Tr}(R_0(\lambda) V R_0(\lambda)) - 2\lambda \operatorname{Tr}((R_V(\lambda) - R_0(\lambda) V R_0(\lambda)) \\
&= -2\lambda \operatorname{Tr}(V^{1/2} R_0^2(\lambda)|V|^{1/2}) - 2\lambda \operatorname{Tr}((R_V(\lambda) - R_0(\lambda) V R_0(\lambda)) \\
&= -2\lambda \operatorname{Tr}(V^{1/2} R_0^2(\lambda)|V|^{1/2}) + 2\lambda \operatorname{Tr}(R_0(\lambda) V R_V(\lambda) V R_0(\lambda))
\end{aligned} \tag{B.16}$$

where in the last equality, we use the resolvent identity $(R_0 - R_V)VR_0 = R_0 V R_V$.

To compute $\operatorname{Tr}\mathcal{R}(\lambda)$, we first use the fact that $R_0(\lambda)$ is bounded on L^2 for $\mathrm{Im}\,(\lambda) > 0$ and hence

$$\partial_\lambda(\mathbf{R}_V(\lambda)) = 2\lambda V^{\frac{1}{2}} R_0(\lambda)^2 |V|^{\frac{1}{2}}\,.$$

Therefore $\operatorname{Tr}\mathcal{R}(\lambda)$ is equal to

$$\begin{aligned}
&- 2\lambda \operatorname{Tr}(V^{1/2} R_0(\lambda)^2 |V|^{1/2}(1 + \mathbf{R}_V(\lambda)^{-1}) \\
&= -2\lambda \operatorname{Tr}(V^{1/2} R_0(\lambda)^2 |V|^{1/2}) - 2\lambda \operatorname{Tr}(V^{1/2} R_0^2(\lambda)|V|^{1/2}((1 + \mathbf{R}_V(\lambda))^{-1} - 1))
\end{aligned} \tag{B.17}$$

The second term on the right-hand side can be written

$$\begin{aligned}
&- 2\lambda \operatorname{Tr}(V^{1/2} R_0^2(\lambda)|V|^{1/2}(1 + \mathbf{R}_V(\lambda))^{-1}(1 - (1 + \mathbf{R}_V(\lambda)))) \\
&\quad = 2\lambda \operatorname{Tr}(V^{1/2} R_0^2(\lambda)|V|^{1/2}(1 + \mathbf{R}_V(\lambda))^{-1}(\mathbf{R}_V(\lambda))) \\
&\quad = 2\lambda \operatorname{Tr}(V^{1/2} R_0^2(\lambda) V R_V(\lambda)|V|^{1/2}) = 2\lambda \operatorname{Tr}(R_0(\lambda) V R_V(\lambda) V R_0(\lambda))
\end{aligned} \tag{B.18}$$

where in the penultimate equality we use that

$$(1 + \mathbf{R}_V(\lambda))^{-1}\mathbf{R}_V(\lambda) = V^{1/2} R_V(\lambda)|V|^{1/2}$$

and in the final equality we use cyclicity of the trace.

Comparing (B.16) with (B.17) and (B.18) gives (B.15). $\square$

Step 3. We lastly compute the trace of $f(P_V) - f(P_0)$.

From Proposition B.2 we get for $\varepsilon > 0$

$$\operatorname{Tr}\left(f(P_V) - f(P_0)\right) = \frac{1}{2\pi}\sum_{\pm} \pm \int_{\{\pm\operatorname{Im}(\lambda)>0\}\cap\{|\lambda|>\varepsilon\}} \overline{\partial}_\lambda g(\lambda) \operatorname{Tr}\mathcal{R}(\pm\lambda)\,\mathrm{d}m(\lambda)$$
$$+ \frac{1}{2\pi}\sum_{\pm} \pm \int_{\{\pm\operatorname{Im}(\lambda)>0\}\cap\{|\lambda|\le\varepsilon\}} \overline{\partial}_\lambda g(\lambda) \operatorname{Tr}\mathcal{R}(\pm\lambda)\,\mathrm{d}m(\lambda).$$

The second term is $\mathcal{O}(\varepsilon^\infty)$ using that $\operatorname{Tr}\mathcal{R}(\lambda) = \operatorname{Tr} B(\lambda)$ (B.15), the trace norm bound on B (B.12), and $\overline{\partial}$ bounds on g (B.10). We next apply the Cauchy–Green formula (cf. [DZ19, (D.1.1)]) on the first term to get

$$\frac{1}{2\pi}\sum_{\pm} \pm \int_{\{\pm\operatorname{Im}(\lambda)>0\}\cap\{|\lambda|>\varepsilon\}} \overline{\partial}_\lambda g(\lambda) \operatorname{Tr}\mathcal{R}(\pm\lambda)\,\mathrm{d}m(\lambda)$$
$$= \frac{1}{4\pi i}\sum_{\pm} \int_{\gamma_\pm(\varepsilon)} g(\lambda)\operatorname{Tr}\mathcal{R}(\pm\lambda)\,\mathrm{d}\lambda$$

where $\gamma_\pm(\varepsilon)$ is the contour going from $-\infty$ to $-\varepsilon$, then along the semicircle $|z| = \varepsilon$ to $+\varepsilon$ such that $\pm\operatorname{Im}(\gamma_\pm(\varepsilon)) \ge 0$, then from $+\varepsilon$ to ∞. Note that

$$\int_{\gamma_\pm(\varepsilon)} g(\lambda)\operatorname{Tr}\mathcal{R}(\pm\lambda)\,\mathrm{d}\lambda = \int_{\gamma_\pm(\varepsilon)\cap\mathbb{R}} g(\lambda)\operatorname{Tr}\mathcal{R}(\pm\lambda)\,\mathrm{d}\lambda$$
$$+ \int_{\gamma_\pm(\varepsilon)\setminus\mathbb{R}} g(\lambda)\operatorname{Tr}\mathcal{R}(\pm\lambda)\,\mathrm{d}\lambda. \tag{B.19}$$

The sum over $\pm$ of the second term can be estimated using (B.14)

$$\sum_{\pm}\int_{\gamma_\pm(\varepsilon)\setminus\mathbb{R}} g(\lambda)\operatorname{Tr}\mathcal{R}(\pm\lambda)\,\mathrm{d}\lambda = (m_R(0)-1)f(0)\oint_{|\lambda|=1}\frac{\mathrm{d}\lambda}{\lambda} + \mathcal{O}(\varepsilon)$$
$$= 2\pi i(m_R(0)-1)f(0) + \mathcal{O}(\varepsilon).$$

The sum over $\pm$ of the first term of (B.19) can be written, using (B.13),

$$\int_{-\infty}^{\infty} f(\lambda^2)\operatorname{Tr}\partial_\lambda S(\lambda)S(\lambda)^{-1}\,\mathrm{d}\lambda - \int_{-\varepsilon}^{\varepsilon} f(\lambda^2)\operatorname{Tr}\partial_\lambda S(\lambda)S(\lambda)^{-1}\,\mathrm{d}\lambda.$$

The second term is $\mathcal{O}(\varepsilon)$. Sending $\varepsilon \to 0$, and using that $\operatorname{Tr}\partial_\lambda S(\lambda)S(\lambda)^{-1}$ is even, gives the theorem.

□

References

[Bac20] A. Backus. "The Breit-Wigner series for noncompactly supported potentials on the line". In: *arXiv preprint arXiv:2005.13765* (2020).

[BCS09] E. Berti, V. Cardoso, and A. O. Starinets. "Quasinormal modes of black holes and black branes". In: *Classical and Quantum Gravity* 26.16 (2009), p. 163001.

[BM93] A. Bachelot and A. Motet-Bachelot. "Les résonances d'un trou noir de Schwarzschild". In: *Annales de l'IHP Physique théorique* 59.1 (1993), pp. 3–68.

[BZ97] A. S. Barreto and M. Zworski. "Distribution of resonances for spherical black holes". In: *Mathematical Research Letters* 4.1 (1997), pp. 103–121.

[Car+18] V. Cardoso, J. L. Costa, K. Destounis, P. Hintz, and A. Jansen. "Quasinormal modes and strong cosmic censorship". In: *Physical review letters* 120.3 (2018), p. 031103.

[Cev+16] D. Cevik, M. Gadella, Ş. Kuru, and J. Negro. "Resonances and antibound states for the Pöschl–Teller potential: Ladder operators and SUSY partners". In: *Physics Letters A* 380.18-19 (2016), pp. 1600–1609.

[Cha98] S. Chandrasekhar. *The mathematical theory of black holes*. Vol. 69. Oxford university press, 1998.

[Dya11] S. Dyatlov. "Quasi-normal modes and exponential energy decay for the Kerr-de Sitter black hole". In: *Communications in mathematical physics* 306.1 (2011), pp. 119–163.

[DZ19] S. Dyatlov and M. Zworski. *Mathematical theory of scattering resonances*. Vol. 200. American Mathematical Soc., 2019.

[Fro97] R. Froese. "Asymptotic distribution of resonances in one dimension". In: *Journal of differential equations* 137.2 (1997), pp. 251–272.

[GZ03] C. R. Graham and M. Zworski. "Scattering matrix in conformal geometry". In: *Inventiones mathematicae* 152 (2003), pp. 89–118.

[Hay64] W. K. Hayman. *Meromorphic Functions*. Oxford Mathematical Monographs. Oxford: Clarendon Press, 1964.

[Hör09] L. Hörmander. *The analysis of linear partial differential operators IV: Fourier integral operators*. Springer, 2009.

[HX21] P. Hintz and Y. Xie. "Quasinormal modes and dual resonant states on de Sitter space". In: *Physical Review D* 104.6 (2021), p. 064037.

[HZ25] M. Hitrik and M. Zworski. "Overdamped QNM for Schwarzschild black holes". In: *Annals of PDE* 11.2 (2025), p. 28.

[Jéz24] M. Jézéquel. "Upper bound on the number of resonances for even asymptotically hyperbolic manifolds with real-analytic ends". In: *Analysis & PDE* 17.10 (2024), pp. 3623–3670.

[JMS21] J. L. Jaramillo, R. P. Macedo, and L. A. Sheikh. "Pseudospectrum and black hole quasinormal mode instability". In: *Physical Review X* 11.3 (2021), p. 031003.

[Kla88] M. Klaus. “Low-energy behaviour of the scattering matrix for the Schrodinger equation on the line”. In: *Inverse Problems* 4.2 (1988), p. 505.

[Kor04] E. Korotyaev. “Stability for inverse resonance problem”. In: *International Mathematics Research Notices* 73 (2004), pp. 3927–3936.

[Kor05] E. Korotyaev. “Inverse resonance scattering on the real line”. In: *Inverse Problems* 21 (2005), pp. 325–341.

[Kor17] E. Korotyaev. “Lieb-Thirring type inequality for resonances”. In: *Bulletin of Mathematical Sciences* 7 (2017), pp. 211–217.

[LP78] P. D. Lax and R. S. Phillips. “The Time Delay Operator and a Related Trace Formula”. In: *Topics in Functional Analysis*. Advances in Mathematics Supplementary Studies 3 (1978), pp. 197–215.

[Mar02] A. Martinez. “Resonance free domains for non globally analytic potentials”. In: *Annales Henri Poincaré* 3.4 (2002), pp. 739–756.

[Mel82] R. Melrose. “Scattering theory and the trace of the wave group”. In: *Journal of Functional Analysis* 45.1 (1982), pp. 29–40.

[Mel85] A. Melin. “Operator methods for inverse scattering on the real line”. In: *Communications in partial differential equations* 10.7 (1985), pp. 677–766.

[New13] R. G. Newton. *Scattering theory of waves and particles*. Springer Science & Business Media, 2013.

[Pai95] A. Pais. “Res Jost January 10, 1918 – October 3, 1990”. In: *Das Märchen vom Elfenbeinernen Turm: Reden und Aufsätze*. Springer Berlin Heidelberg, 1995, pp. 1–9.

[Smi23] H. F. Smith. “On the trace of the wave group and regularity of potentials”. In: *Annales de l’Institut Fourier* 73.4 (2023), pp. 1453–1488.

[Wal84] R. M. Wald. *General relativity*. The University of Chicago Press, 1984.

[YBF25] Y. Yang, E. Berti, and N. Franchini. “Black hole quasinormal mode resonances”. In: *arXiv preprint arXiv:2504.06072* (2025).

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