simplicial complexes : 

23/10/18

Homology group of s.c : $C_n(K) :=$ free abelian grp generated by oriented n-simplexes $/\sim$

$\sigma + \sigma' = 0$ iff σ, σ' are simplexes with opposite orientation.

- $\partial_n : C_n(K) \longrightarrow C_{n-1}(K)$

$$\partial_n([v_0 \cdots v_{n+1}]) = \sum_i (-1)^i [v_0 .. \hat{v}_i .. v_{n+1}]$$

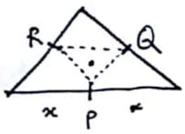
$$H_n(K) := \frac{\ker \partial_n}{\operatorname{Im} \partial_{n+1}}$$

- X be top. space. $X = |K|$ ↳ homotopy on sc K

$$H_n(X) := H_n(K)$$

Q. $X = |K| = |K'|$, K, K' are diff. simplicial complexes. We need $H_n(K) = H_n(K')$. — We will prove some intermediate results —

- Bary centric subdivision

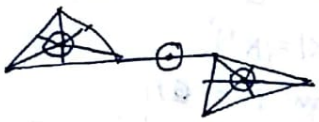


① Take P, Q, R as midpts
② $\Delta PQR \to$ Take its bary center

Let A be a simplex with vertices $v_0, v_1, .., v_n$

Barycenter of A, $\hat{A} := \frac{1}{n+1}(v_0 + v_1 + .. + v_n)$

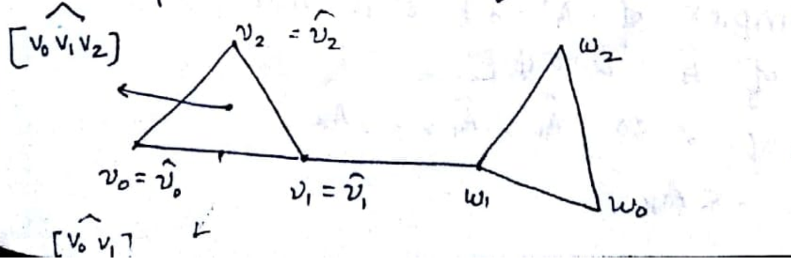
- Let K be a simplicial complex

E.g:  union of all three pts ⇒ First barycentric subdiv

K' (first barycentric subdivision of K) : vertices of K' are all barycenters of simplexes of K

- A collection $\hat{A}_0, \hat{A}_1, .., \hat{A}_k$ bary centers forms a k-simplex of K' iff

$$A_{\sigma(0)} < A_{\sigma(1)} < A_{\sigma(2)} < \cdots < A_{\sigma(k)}$$ for some permutation σ of $\{0, 1, .., k\}$



Why σ perm ? $0 \leftrightarrow 1$ may not be faces of each other, some other vertex may be.

E.g :

First B.C subdiv

second B.C. subdivision

K : finite simplicial cx.

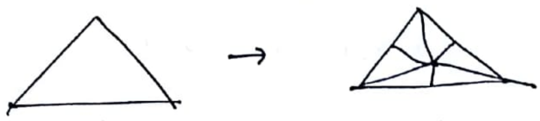
- Mesh : $\mu(K)$: maximum of the diameter of its simplexes.

- Lemma : (i) K' is a simplicial cx
  (ii) Each simplex of K' lies in simplex of K
  (iii) $|K'| = |K|$
  (iv) If $\dim(K) = n$, $\mu(K') \leq \frac{n}{n+1} \mu(K)$

Proof : Let Δ be a simplex of K' whose vertices are $[\hat{A}_0, \hat{A}_1, \ldots, \hat{A}_K]$ s.t $A_0 < A_1 < \ldots < A_K$ where A_i are simplexes of K.

We prove K' is a simplicial complex & $|K'| = |K|$ by induction on the number of simplexes in K.

<u>Base case</u> : # simplexes in $K = 1 \Rightarrow K$ is just a simplex.



K

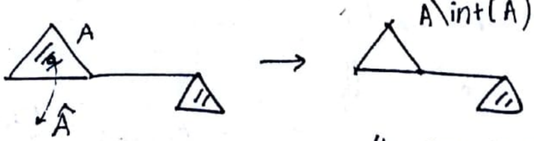
K' is a simplicial cx (easy to observe)

Any simplex in $K' \subseteq K \Rightarrow |K| = |K'|$

<u>Induction case</u> : # simplexes in $K = n$

let A be a simplex in K of max dimension

Remove the interior(A) from A to form a new simplicial complex L.



simplexes $= n-1$

By ind hyp : L' is s.c, $|L'| = |L|$

Let σ be a simplex of A' s.t σ is not a baricenter of $\hat{A}$ & $\sigma \notin L'$

label vertices of σ as $\hat{A}_0, \hat{A}_1, \ldots, \hat{A}_K$
s.t. $A_0 < A_K < \ldots < A_{..}$

The vertices $\hat{A}_0, \dots, \hat{A}_{k-1}$ determine a face τ of σ where τ lies in L & $\tau = \sigma \cap L'$

σ, σ' are 2 simplexes of K'

$\tau' = \sigma' \cap L'$

case(i) $\tau \cap \tau' \neq \emptyset \Rightarrow$ they intersect in L' ($\because$ both in L')

$\Downarrow$

& L' is simp. complex

$\Downarrow$

$\tau \cap \tau'$ is a simplex

Now consider the vertices of $\tau \cap \tau'$ & $\hat{A}$ form a face of $\sigma \cap \sigma'$.

case(ii) $\tau \cap \tau' = \emptyset$

$\sigma \cap \sigma'$

$\left.\begin{matrix} \sigma \setminus \hat{A} \in L', \ \sigma' \setminus \hat{A} \subset L' \end{matrix}\right\} \Rightarrow \sigma \cap \sigma' = \hat{A}$

$\Rightarrow K_4'$ is a simplicial complex.

- $\mu(\Delta') \leq \frac{n}{n+1} \mu(\Delta)$

  Δ : n-simplex

- simplicial map :- K, L simplicial complexes. A map $s : |K| \to |L|$ is called simplicial if s takes simplexes of K linearly onto simplexes of L. i.e.

- $x \in A$, A is a simplex with vertices $v_0 \dots v_k$, $x \in A$

  $x = \sum \lambda_i v_i$ then $s(x) = \sum \lambda_i s(v_i)$

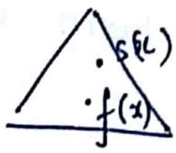
- If x is a vertex, s takes $s(x)$ is also a vertex in L

  But $s(v_i) \neq s_i$ i.e. mapping is not ordered

- Def^n : ~~X~~ consider a cont. map $f : |K| \to |L|$. Given $x \in |K|$. the pt $f(x)$ is either a vertex or contained in the interior of a unique simplex of $|L|$.

- Given $x \in |K|$, the pt x is either a vertex of contained in the interior of a unique simplex of $|L|$.
  This simplex is said to be carrier of x.

- simplicial appxmation: $f : |K| \to |L|$ cont. map
  A simplicial map $s : |K| \to |L|$ is said to be a simplicial approximation of $f : |K| \to |L|$ if $s(x)$ lies in the carrier of $f(x)$ $\forall x \in |K|$



CL (Both lie in the same simplex)

Q. Is s homotopic to f?
$\because$ $s(x)$, $f(x)$ lie in same simplex, $[s(x), f(x)]$ lie in the same simplex.

- Next class: $f : |K| \to |L|$
  $\exists$ m s.t $\ni$ mth barycentric subdiv of K s.t
  $f : |K^m| \to |L|$ st $\exists$ simplicial appx s: of f.