FINAL PROJECT 2

CONTROL of ROBOTIC SYSTEMS

ENPM 667

Submitted by-

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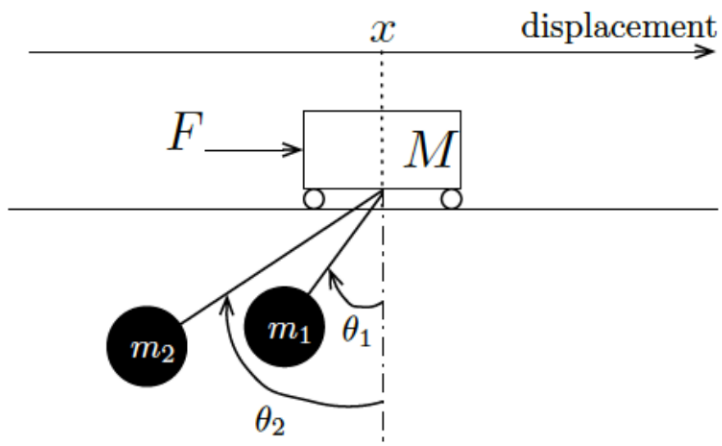
Contents

First Component (100 points): Consider a crane that moves along an one-dimensional track. It behaves as a frictionless cart with mass M actuated by an external force F that constitutes the input of the system. There are two loads suspended from cables attached to the crane. The loads have mass m_1 and m_2, and the lengths of the cables are l_1 and l_2, respectively. The following figure depicts the crane and associated variables used throughout this project.



Part A) Obtain the equations of motion for the system and the corresponding non-linear state space representation

Equations of motion are derived using the Euler Lagrange equations.

Euler-Lagrange equation in classical mechanics is equivalent to Newton's laws of motion and can be derived from Newton's second law of motion. Euler Lagrange equation is a second order partial differential equation whose solutions are the functions for which a given function is stationary.

Euler Lagrange for a single function of two variables with single derivative is given by –

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = F; \tag{1}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0; \tag{2}$$

Where x, θ are the two state variables (functions of time) and $\dot{x}$, $\dot{\theta}$ are their single derivates,

L is the single function here and is defined as the difference between Kinetic Energy (K) and Potential Energy (P).

Therefore , L can be written as $L = K - P$

Kinetic energy of a general system is the energy it possesses due to its motion and is defined as the work needed to accelerate a body of a given mass (M, m_1, m_2) from rest to its stated velocity $(\dot{x}, \dot{r}_1, \dot{r}_2)$. It can be written as

$$K = \frac{1}{2} m_i \dot{x}_i^{\,2}$$

For this system where 2 loads are suspended from a crane with the given parameters, the Kinetic energy can be written as

$$K = K(crane) + K(m_1) + K(m_2)$$

$$K = \frac{1}{2}[M\dot{x}^2 + m_1 \frac{dr_1}{dt} . \frac{dr_1}{dt} + m_2 \frac{dr_2}{dt} . \frac{dr_2}{dt}] \tag{3}$$

Where r_1, r_2 are the position vectors for mass m_1, m_2 respectively such that

$\frac{dr_1}{dt}.\frac{dr_1}{dt}$ represents a dot product of the position vector and gives a term representing the velocity of the mass (m_1) squared and,

$\frac{dr_2}{dt}.\frac{dr_2}{dt}$ represents a dot product of the position vector and gives a term representing the velocity of the mass (m_2) squared.

Also, position vector r_1, r_2 is given by

$$r_1 = (x - l_1 sin(\theta_1))i + l_1 cos(\theta_1)j$$

$$r_2 = (x - l_2 sin(\theta_2))i + l_1 cos(\theta_2)j$$

Substituting the values in (3), we get

$$K = \frac{1}{2}[M\dot{x}^2 + m_1(\dot{x} - l_1\dot{\theta}_1 cos(\theta_1))^2 + m_2(\dot{x} - l_2\dot{\theta}_2 cos(\theta_2))^2 + m_1(l_1\dot{\theta}_1 sin(\theta_1))^2 + m_2(l_2\dot{\theta}_2 sin(\theta_2))^2] \quad (4)$$

Potential energy of a general system is the energy held by an object because of its position relative to other objects and is independent of the trajectory of the body. For a body with given mass (m_1, m_2) and position vector (h_1, h_2) joining its initial and final position, the Potential energy can be written as

$$P = m_i g h_i$$

For this system where 2 loads are suspended from a crane with the given parameters, the Potential energy can be written as

$$P = P(m_1) + P(m_2)$$

$$P = m_1 g l_1 cos\theta_1 + m_2 g l_2 cos\theta_2 \quad (5)$$

Where g is the acceleration due to gravity, $g = 9.8m/s^2$

Therefore , function L can be written as

$$L = K - P$$

Substituting the values of K and P from (4) and (5) respectively and simplifying, we obtain

$$L = \frac{1}{2}[M\dot{x}^2 + m_1(\dot{x} - l_1\dot{\theta}_1 cos(\theta_1))^2 + m_2(\dot{x} - l_2\dot{\theta}_2 cos(\theta_2))^2 + m_1(l_1\dot{\theta}_1 sin(\theta_1))^2 + m_2(l_2\dot{\theta}_2 sin(\theta_2))^2] - [m_1 g l_1 cos\theta_1 + m_2 g l_2 cos\theta_2]$$

Taking the partial derivative of this function w.r.t $\dot{x}$, we get

$$\frac{\partial L}{\partial \dot{x}} = M\dot{x} + m_1(\dot{x} - l_1\dot{\theta}_1 cos(\theta_1)) + m_2(\dot{x} - l_2\dot{\theta}_2 cos(\theta_2)) \quad (6)$$

Taking Partial derivative of function L w.r.t x, we get

$$\frac{\partial L}{\partial x} = 0 \quad (7)$$

Taking time derivative of the partial derivative of the function L, we get

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = M\ddot{x} + m_1(\ddot{x} - l_1\ddot{\theta}_1 cos(\theta_1) + l_1\dot{\theta}_1{}^2 sin(\theta_1)) + m_2(\ddot{x} - l_2\ddot{\theta}_2 cos(\theta_2) + l_2\dot{\theta}_2{}^2 sin(\theta_2)) \quad (8)$$

Substituting (8) and (7) in (1), we obtain

$$M\ddot{x} + m_1(\ddot{x} - l_1\ddot{\theta}_1 cos(\theta_1) + l_1\dot{\theta}_1{}^2 sin(\theta_1)) + m_2(\ddot{x} - l_2\ddot{\theta}_2 cos(\theta_2) + l_2\dot{\theta}_2{}^2 sin(\theta_2)) = F$$

$$(M + m_1 + m_2)\ddot{x} + m_1(-l_1\ddot{\theta}_1 cos(\theta_1) + l_1\dot{\theta}_1{}^2 sin(\theta_1)) + m_2(-l_2\ddot{\theta}_2 cos(\theta_2) + l_2\dot{\theta}_2{}^2 sin(\theta_2)) = F$$

$$(M + m_1 + m_2)\ddot{x} = F - m_1(l_1\dot{\theta}_1{}^2 sin(\theta_1) - l_1\ddot{\theta}_1 cos(\theta_1)) - m_2(l_2\dot{\theta}_2{}^2 sin(\theta_2) - l_2\ddot{\theta}_2 cos(\theta_2))$$

$$\ddot{x} = \frac{F + m_1(l_1\ddot{\theta}_1 cos(\theta_1) - l_1\dot{\theta}_1{}^2 sin(\theta_1)) + m_2(l_2\ddot{\theta}_2 cos(\theta_2) - l_2\dot{\theta}_2{}^2 sin(\theta_2))}{(M + m_1 + m_2)}$$

Similarly, using (1) and (2), we can obtain expressions for other time varying state variable

$$\boxed{\ddot{\theta}_1 = \frac{\ddot{x}cos(\theta_1) - gsin(\theta_1)}{l_1}}$$

$$\boxed{\ddot{\theta}_2 = \frac{\ddot{x}cos(\theta_2) - gsin(\theta_2)}{l_2}}$$

For the Non-linear state space representation, it can be expressed as

$$\dot{X} = f(X, U) = \begin{bmatrix} \dot{x} \\ \ddot{x} \\ \dot{\theta}_1 \\ \ddot{\theta}_1 \\ \dot{\theta}_2 \\ \ddot{\theta}_2 \end{bmatrix}$$

Part B) Obtain the Linearized system around the equilibrium point specified by $x = 0$ and $\theta_1 = \theta_2 = 0$. Write the state space representation of the Linearized system

Linearized system is obtained by substituting the given equilibrium points and making a valid assumption that θ is very small and hence $sin(\theta) = \theta$ and $cos(\theta) = cos(\theta)$, therefore linearized system can be written as;

$$\ddot{x} = \frac{F + m_1(l_1\ddot{\theta}_1 cos(\theta_1) - l_1\dot{\theta}_1{}^2 \theta_1) + m_2(l_2\ddot{\theta}_2 cos(\theta_2) - l_2\dot{\theta}_2{}^2 \theta_2)}{(M + m_1 + m_2)}$$

Substituting the values of $\ddot{\theta}_1$ and $\ddot{\theta}_2$ we get,

$$\ddot{x} = \frac{F + m_1(l_1\left[\frac{\ddot{x}cos(\theta_1) - g(\theta_1)}{l_1}\right] cos(\theta_1) - l_1\dot{\theta}_1{}^2 \theta_1) + m_2(l_2\left[\frac{\ddot{x}cos(\theta_2) - g(\theta_2)}{l_2}\right] cos(\theta_2) - l_2\dot{\theta}_2{}^2 \theta_2)}{(M + m_1 + m_2)}$$

Since $\theta_1 = \theta_2 = 0$, therefore, $\dot{\theta}_1 = \dot{\theta}_2 = 0$, substituting this , we get

$$\ddot{x} \;=\frac{F+m_1(l_1\left[\frac{\ddot{x}-g(\theta_1)}{l_1}\right])+m_2(l_2\left[\frac{\ddot{x}-g(\theta_2)}{l_2}\right])}{(M+m_1+m_2)}$$

$$\ddot{x} \;=\frac{F+m_1([\ddot{x}-g(\theta_1)]\,)+m_2([\ddot{x}-g(\theta_2)]\,)}{(M+m_1+m_2)}$$

$$\ddot{x}(M+m_1+m_2)-\ddot{x}(m_1+m_2) \;=F+m_1[-g(\theta_1)]\;+m_2[-g(\theta_2)]$$

$$\boxed{\ddot{x}=\frac{F}{M}-\frac{m_1 g(\theta_1)}{M}-\frac{m_2 g(\theta_2)}{M}}$$

Similarly, $\ddot{\theta}_1$ and $\ddot{\theta}_2$ can be obtained for the linearized system as

$$\boxed{\ddot{\theta}_1=\frac{F}{l_1 M}-\frac{g(M+m_1)(\theta_1)}{l_1 M}-\frac{m_2 g(\theta_2)}{l_1 M}}$$

$$\boxed{\ddot{\theta}_1=\frac{F}{l_2 M}-\frac{g m_1(\theta_1)}{l_2 M}-\frac{g(M+m_2)(\theta_2)}{l_2 M}}$$

State-space representation of the Linearized Time Invariant system can be written as

$$\dot{X}(t)=AX(t)+BU(t)$$

$$Y(t)=CX(t)+DU(t)$$

$$\dot{X}(t)\ is\ a\ n\times 1\ vector, \qquad A\ is\ a\ n\times n\ matrix, \qquad B\ is\ a\ n\times m\ matrix, \qquad U(t)\ is\ a\ m\times 1\ vector$$

$$\begin{bmatrix}\dot{x}\\ \ddot{x}\\ \dot{\theta}_1\\ \ddot{\theta}_1\\ \dot{\theta}_2\\ \ddot{\theta}_2\end{bmatrix}=\begin{bmatrix}0 & 1 & 0 & 0 & 0 & 0\\ 0 & 0 & -\frac{m_1 g}{M} & 0 & -\frac{m_2 g}{M} & 0\\ 0 & 0 & 0 & 1 & 0 & 0\\ 0 & 0 & -\frac{g(M+m_1)}{l_1 M} & 0 & -\frac{m_2 g(\theta_2)}{l_1 M} & 0\\ 0 & 0 & 0 & 0 & 0 & 1\\ 0 & 0 & -\frac{g m_1}{l_2 M} & 0 & -\frac{g(M+m_2)}{l_2 M} & 0\end{bmatrix}\begin{bmatrix}x\\ \dot{x}\\ \theta_1\\ \dot{\theta}_1\\ \theta_2\\ \dot{\theta}_2\end{bmatrix}+\begin{bmatrix}0\\ \frac{1}{M}\\ 0\\ \frac{1}{l_1 M}\\ 0\\ \frac{1}{l_2 M}\end{bmatrix}F$$

$$Y(t) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \\ \theta_1 \\ \dot{\theta}_1 \\ \theta_2 \\ \dot{\theta}_2 \end{bmatrix}$$

<u>Part C)</u> Obtain conditions on $\mathbf{M}, m_1, m_2, l_1, l_2$ for which the linearized system is controllable

For any LTI system to be controllable, it must satisfy the rank condition

rank$\begin{bmatrix} B & AB & A^2B & \cdots & A^{n-1}B \end{bmatrix} = n$

* Using MATLAB , values of A, B were substituted in the above matrix to obtain a 6×6 matrix for checking the controllability. Multiple assumptions were made until finally $l_1 = l_2$ was assumed which resulted in the matrix shown below.

$$\begin{bmatrix} 0 & \frac{1}{M} & 0 & -\frac{g(m_1+m_2)}{l_1 M^2} & 0 & -\frac{g^2(m_1+m_2)(M+m_1+m_2)}{{l_1}^2 M^3} \\ \frac{1}{M} & 0 & -\frac{g(m_1+m_2)}{l_1 M^2} & 0 & -\frac{g^2(m_1+m_2)(M+m_1+m_2)}{{l_1}^2 M^3} & 0 \\ 0 & \frac{1}{l_1 M} & 0 & -\frac{g(M+m_1+m_2)}{(l_1 M)^2} & 0 & -\frac{(g(M+m_1+m_2))^2}{(l_1 M)^3} \\ \frac{1}{l_1 M} & 0 & -\frac{g(M+m_1+m_2)}{(l_1 M)^2} & 0 & -\frac{(g(M+m_1+m_2))^2}{(l_1 M)^3} & 0 \\ 0 & \frac{1}{l_1 M} & 0 & -\frac{g(M+m_1+m_2)}{(l_1 M)^2} & 0 & -\frac{(g(M+m_1+m_2))^2}{(l_1 M)^3} \\ \frac{1}{l_1 M} & 0 & -\frac{g(M+m_1+m_2)}{(l_1 M)^2} & 0 & -\frac{(g(M+m_1+m_2))^2}{(l_1 M)^3} & 0 \end{bmatrix}$$

It can be observed that row 4 and row 6 of this matrix are identical and hence this matrix loses rank (from 6 to 4), therefore the following conditions were obtained for a real Linear Time invariant system,

$M, m_1, m_2, l_1, l_2 > 0$, and

$l_1 \neq l_2$

Part D) Choose $\mathbf{M = 1000Kg}, \boldsymbol{m_1 = m_2 = 100Kg}, \boldsymbol{l_1 = 20m}, \boldsymbol{l_2 = 10m}$. Check that the system is controllable and obtain an LQR controller. Simulate the resulting response to initial conditions when the controller is applied to the linearized system and to the original nonlinear system. Adjust the parameters of the LQR cost until you obtain a suitable response. Use Lyapunov's indirect method to certify (locally or globally) of the closed loop system.

- The given parameters were substituted in the above matrix and the matrix obtained (using MATLAB) is shown below

$$1 * 10^{-3}\begin{bmatrix} 0 & 1 & 0 & -0.147 & 0 & 0.147 \\ 1 & 0 & -0.147 & 0 & 0.147 & 0 \\ 0 & 0.05 & 0 & -0.0319 & 0 & 0.0227 \\ 0.05 & 0 & -0.0319 & 0 & 0.0227 & 0 \\ 0 & 0.1 & 0 & -0.1127 & 0 & 0.1246 \\ 0.1 & 0 & -0.1127 & 0 & 0.1246 & 0 \end{bmatrix}$$

It is observed that none of the rows are identical or null, therefore the matrix is full rank and hence the system is controllable.

LQR Controller

When the system dynamics are described by a set of linear differential equations and the cost to be minimized is given by a quadratic function is called the LQ problem and the solution is provided by a feedback controller called Linear Quadratic regulator (LQR).

For a continuous -time linear system, defined on $t \in [t_0, t_1]$, described by

$$\dot{x} = Ax + Bu$$

with a quadratic cost function defined as

$$J = \int_{t_0}^{t_1} (x^T Q x + u^T R u) dt$$

where Q, R are positive definite matrices and,

the feedback control law that minimizes the value of the cost is

$$u = -Kx$$

where K is given by

$$K = R^{-1} B^T P$$

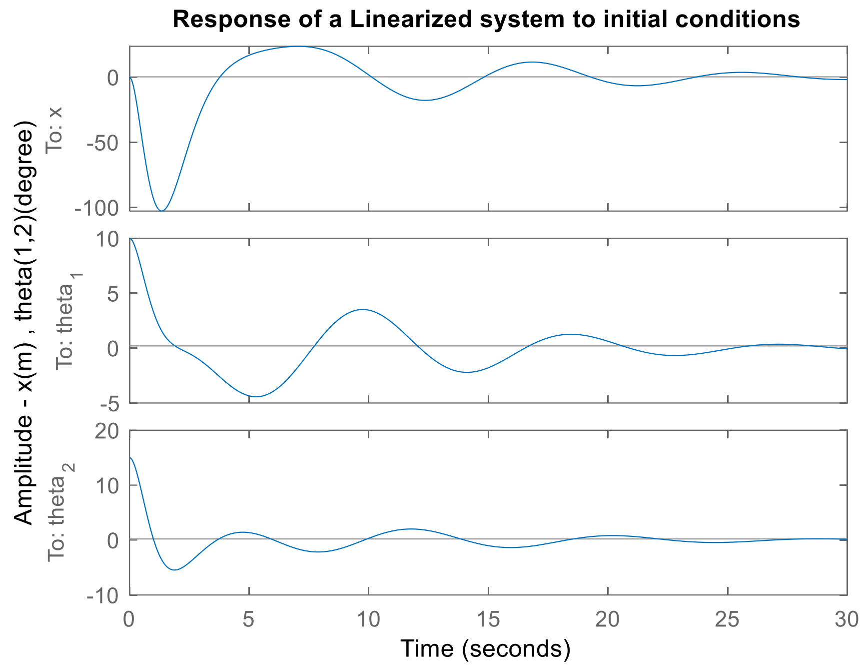
and P is the symmetric positive solution of the following stationary Riccati differential equation:

$$A^T P + PA - PBR^{-1} B^T P = -Q$$

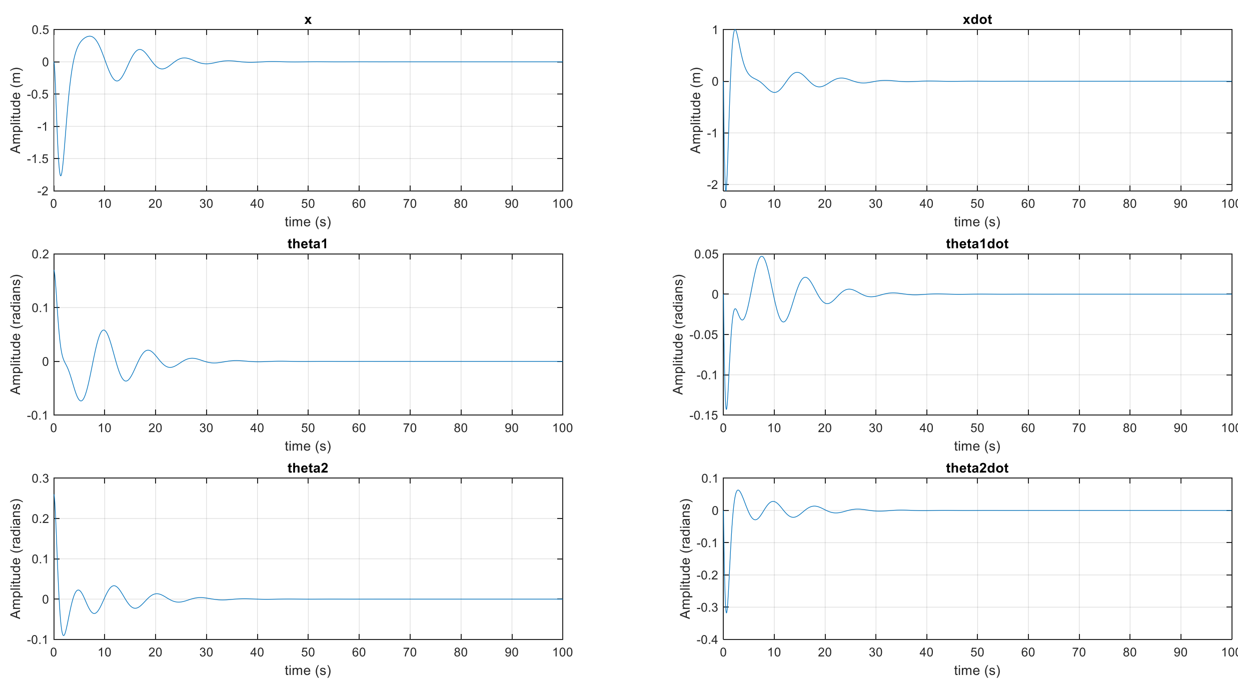
- Using MATLAB, an LQR controller was obtained with gain K given by the matrix

$$\boxed{K = 10^4 [0.3162 \quad 0.6325 \quad 3.0486 \quad -0.4719 \quad 1.9580 \quad -2.4293]}$$

- Response to the initial conditions when the controller is applied to the Linearized system-



- Response to the initial conditions when the controller is applied to the original nonlinear system-



It is observed that both Linearized and original non-linear system converge at around 40 seconds when an LQR controller is applied. Gain K is obtained (MATLAB) using the following function

$$K = lqr(A, B, Q, R)$$

Values for parameters Q, R were changed to obtain a suitable response shown above.

- Lyapunov's indirect method is used to certify stability of the closed-loop system and the eigen values for the closed loop matrix A_c is found using MATLAB and the following results are obtained

$$eigen\ values = \begin{bmatrix} -1.4432 + 1.6258i \\ -1.4432 - 1.6258i \\ -0.2261 + 0.7296i \\ -0.2261 - 0.7296i \\ -0.1703 + 0.7296i \\ -0.1703 - 0.7296i \end{bmatrix}$$

It is observed that the real part of all the eigen values is negative i.e. $Re\{\lambda_i\} < 0 \; \forall \; i = 1, \ldots 6$
Also, it is known that for a Linear time varying closed loop system , local stability = global stability. Therefore, it can be said that the system is both locally and globally stable or it is at least Locally stable around the equilibrium points.

<u>Part E)</u> Suppose that you select the following output vectors: $x(t), (\theta_1(t), \theta_2(t)), (x(t), \theta_2(t))$ *or* $(x(t), \theta_1(t), \theta_2(t))$. Determine for which output vectors the linearized system is observable.

- <u>Observability</u> – It is a measure of how well internal states of a system can be inferred from the knowledge of its external outputs. If a system is observable, it means that the internal states of the system can be inferred.
  Observability can be checked by calculating the rank of the matrix given by-

$$rank[C \quad CA \quad CA^2 \quad CA^3 \quad CA^4 \quad CA^5]^T = n$$

If the rank of the above matrix which is given by $nm \times n$, where matrix C is of dimensions $m \times 1$ and matrix A is of the dimension $n \times n$ is same as the rank of the matrix $A(6 \times 6)$, then the closed loop system is said to be observable.

a. <u>Output vector - $x(t)$</u>

$$rank\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -0.9800 & 0 & -0.9800 & 0 \\ 0 & 0 & 0 & -0.9800 & 0 & -0.9800 \\ 0 & 0 & 0.6243 & 0 & 1.1045 & 0 \\ 0 & 0 & 0 & 0.6243 & 0 & 1.1045 \end{bmatrix}$$

No two rows or columns are identical and hence the matrix does not lose its rank, and the rank of the matrix is 6, which is same as the rank of the matrix A, and hence system with output vector $x(t)$ is <u>observable</u>.

b. <u>Output vector - $\theta 1(t), \theta 2$</u>

$$rank\begin{bmatrix} 0 & 0 & 1.0000 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1.0000 & 0 \\ 0 & 0 & 0 & 1.0000 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1.0000 \\ 0 & 0 & -0.5390 & -0.9800 & -0.0490 & 0 \\ 0 & 0 & -0.0980 & 0 & -1.0780 & 0 \\ 0 & 0 & 0 & -0.5390 & 0 & -0.0490 \\ 0 & 0 & 0 & -0.0980 & 0 & -1.0780 \\ 0 & 0 & 0.2953 & 0 & 0.0792 & 0 \\ 0 & 0 & 0.1585 & 0 & 1.1669 & 0 \\ 0 & 0 & 0 & 0.2953 & 0 & 0.0792 \\ 0 & 0 & 0 & 0.1585 & 0 & 1.1669 \end{bmatrix}$$

Columns 1 and columns 2 are identical and hence the matrix loses its rank, and the rank of the matrix is 4, which is less than the rank of the matrix A, and hence the system with output vector $(\theta_1(t), \theta_2(t))$ is not observable.

c. Output vector - $x(t), \theta_2(t))$

$$rank\begin{bmatrix} 1.0000 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1.0000 & 0 \\ 0 & 1.0000 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1.0000 \\ 0 & 0 & -0.9800 & 0 & -0.9800 & 0 \\ 0 & 0 & -0.0980 & 0 & -1.0780 & 0 \\ 0 & 0 & 0 & -0.9800 & 0 & -0.9800 \\ 0 & 0 & 0 & -0.0980 & 0 & -1.0780 \\ 0 & 0 & 0.6243 & 0 & 1.1045 & 0 \\ 0 & 0 & 0.1585 & 0 & 1.1669 & 0 \\ 0 & 0 & 0 & 0.6243 & 0 & 1.1045 \\ 0 & 0 & 0 & 0.1585 & 0 & 1.1669 \end{bmatrix}$$

No two rows or columns are identical and hence the matrix does not lose its rank, and the rank of the matrix is 6, which is same as the rank of the matrix A, and hence system with output vector $(x(t), \theta_2(t))$ is observable.

d. Output vector - $x(t), \theta_1(t), \theta_2(t)$

$$rank\begin{bmatrix} 1.0000 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1.0000 & 0 & 0 & 0 \\ 0 & 1.0000 & 0 & 0 & 1.0000 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1.0000 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1.0000 \\ 0 & 0 & -0.9800 & 0 & -0.9800 & \\ 0 & 0 & -0.5390 & 0 & -0.0490 & 00 \\ 0 & 0 & -0.0980 & 0 & -1.0780 & 0 \\ 0 & 0 & 0 & -0.9800 & 0 & -0.9800 \\ 0 & 0 & 0 & -0.5390 & 0 & -0.0490 \\ 0 & 0 & 0 & -0.0980 & 0 & -1.0780 \\ 0 & 0 & 0.6243 & 0 & 1.1045 & 0 \\ 0 & 0 & 0.2953 & 0 & 0.0792 & 0 \\ 0 & 0 & 0.1585 & 0 & 1.1669 & 0 \\ 0 & 0 & 0 & 0.6243 & 0 & 1.1045 \\ 0 & 0 & 0 & 0.2953 & 0 & 0.0792 \\ 0 & 0 & 0 & 0.1585 & 0 & 1.1669 \end{bmatrix}$$

No two rows or columns are identical and hence the matrix does not lose its rank, and the rank of the matrix is 6, which is same as the rank of the matrix A, and hence system with output vector $x(t), \theta_1(t), \theta_2(t)$ is observable.

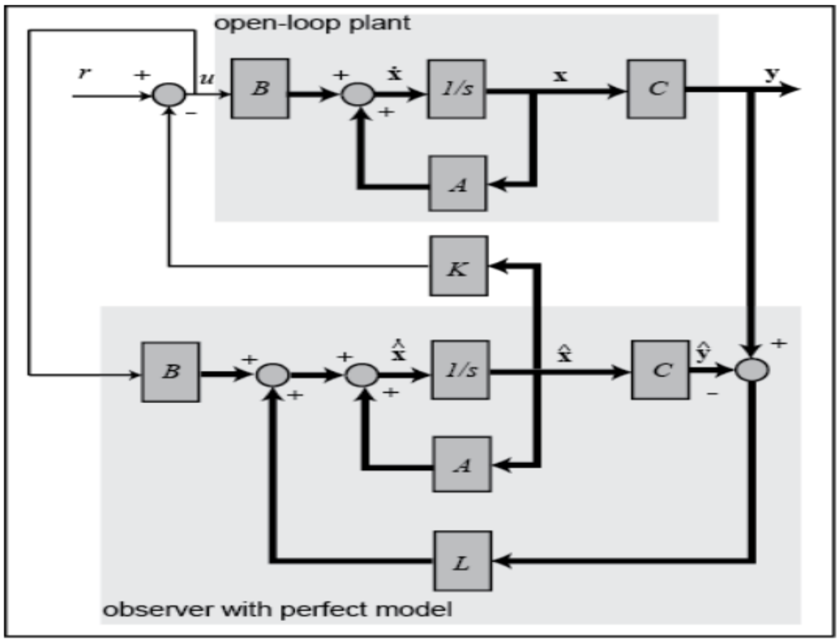
Conclusion – It can be concluded that the closed loop system is not observable if the state variable $x(t)$ is not present in the output vector, i.e. the feedback loop does not have this state vector and hence the internal states of the system cannot be inferred without the presence of the state vector $x(t)$ in the output vector.

<u>Part F)</u> Obtain your 'best' Luenberger observer for each one of the output vectors for which the system is observable and simulate its response to initial conditions and unit step input. The simulation should be done for the observer applied to both the linearized system and the original nonlinear system.

According to Luenberger , *any system driven by the output of the given system can server as an observer for that system.* Luenberger observer is obtained by subtracting the output of the observer from the output of the plant and multiplied by matrix L, which is added to the equation for the state of the observer. For an LTI system, it is written as

$$\dot{\hat{X}}(t) = A\hat{X}(t) + L[y(t) - \hat{y}(t)] + BU(t)$$

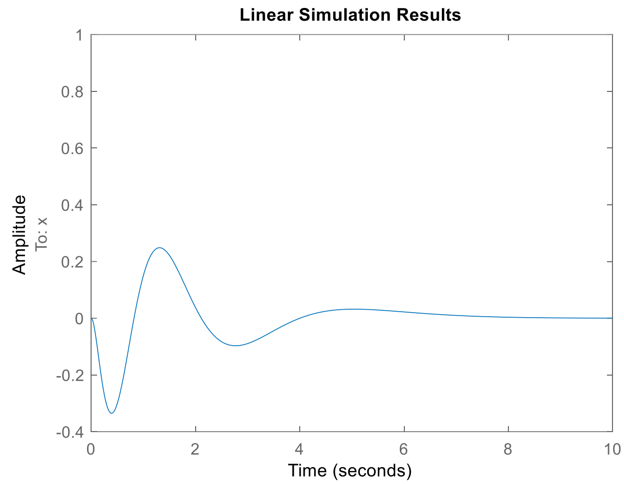
$$\hat{Y}(t) = C\hat{X}(t) + DU(t)$$



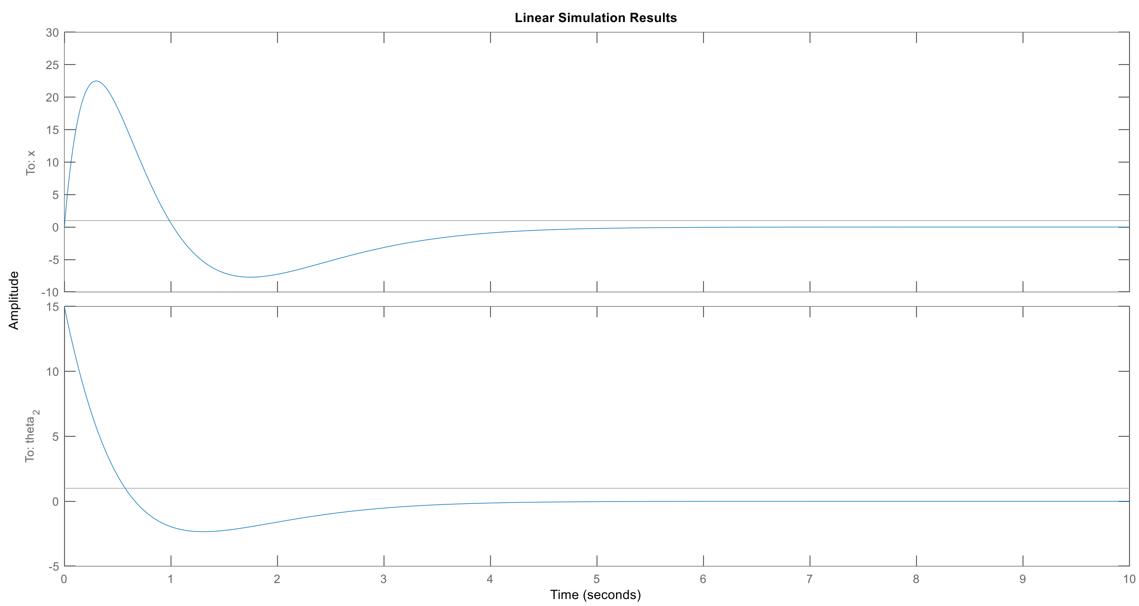
For the initial conditions $x(0) = [0; 0; 10; 0; 15; 0]$ and unit step input, we assume poles from the eigen value as

$$P = [-1.7\ -1.8\ -1.9\ -2\ -2.1\ -2.2]$$

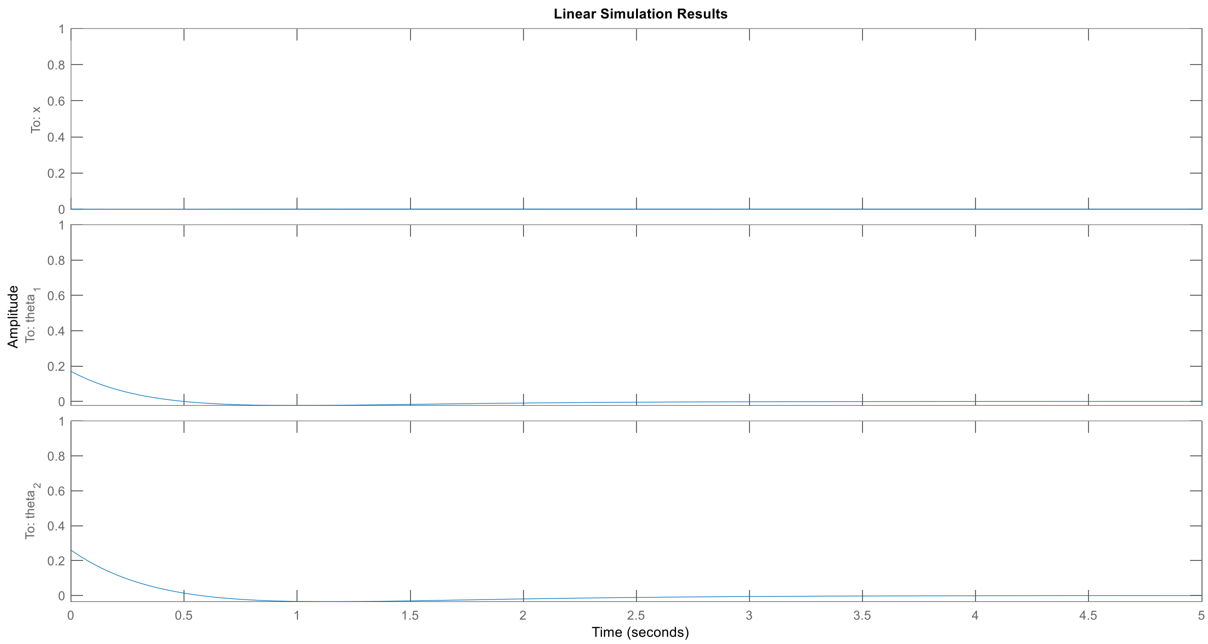
a. <u>Luenberger Observer for Output vector - $x(t)$</u>



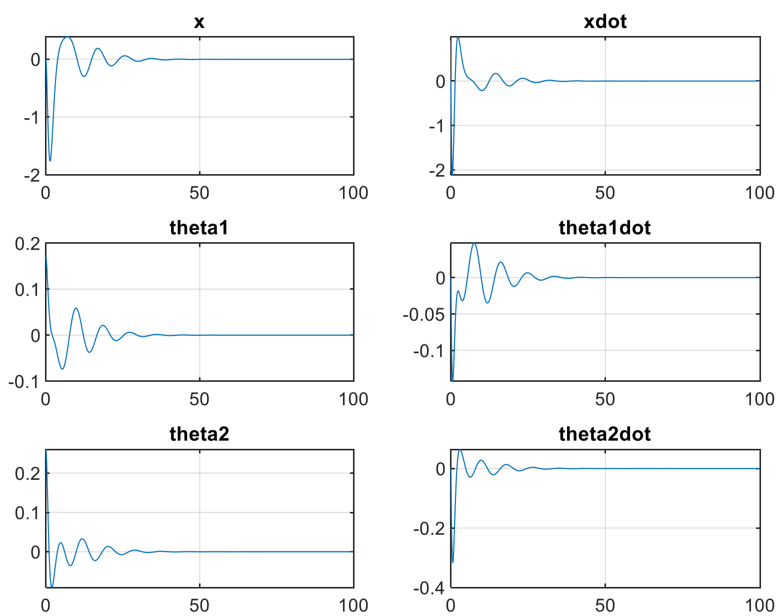
b. <u>Luenberger Observer for Output vector - $x(t), \theta_2(t)$</u>



c. <u>Luenberger Observer for Output vector - $x(t), (\theta_1(t), \theta_2(t))$</u>



G) Design an output feedback controller for your choice of the "smallest" output vector. Use the LQG method and apply the resulting output feedback controller to the original nonlinear system. Obtain your best design and illustrate its performance in simulation. How would you reconfigure your controller to asymptotically track a constant reference on x ? Will your design reject constant force disturbances applied on the cart ?



APPENDIX

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%Ethan Quist and Himanshu Singhal
%%
%B) Linearized system around equilibrium points x=0, theta1=0, and theta2=0
syms g M m1 m2 L1 L2
A1 = [0 1 0 0 0 0; 0 0 (-g*m1)/M 0 (-g*m2)/M 0; 0 0 0 1 0 0; 0 0 (-g*(M+m1))/(L1*M)
0 (-g*m2)/(L1*M) 0; 0 0 0 0 0 1; 0 0 (-g*m1)/(L2*M) 0 (-g*(M+m2))/(L2*M) 0];
B1 = [0; 1/M; 0; 1/(L1*M); 0; 1/(L2*M)];
Ranker1 = [B1 A1*B1 (A1^2)*B1 (A1^3)*B1 (A1^4)*B1 (A1^5)*B1];
rank(Ranker1);

%%
%C)Conditions for controllability
syms g M m1 m2 L1

L2=L1;
A1 = [0 1 0 0 0 0; 0 0 (-g*m1)/M 0 (-g*m2)/M 0; 0 0 0 1 0 0; 0 0 (-g*(M+m1))/(L1*M)
0 (-g*m2)/(L1*M) 0; 0 0 0 0 0 1; 0 0 (-g*m1)/(L2*M) 0 (-g*(M+m2))/(L2*M) 0];
B1 = [0; 1/M; 0; 1/(L1*M); 0; 1/(L2*M)];
Ranker1 = [B1 A1*B1 (A1^2)*B1 (A1^3)*B1 (A1^4)*B1 (A1^5)*B1];
rank(Ranker1);

%%
%D)Simulate the response
M = 1000;
m1 = 100;
m2 = 100;
L1 = 20;
L2 = 10;
g = 9.8;

A = [0 1 0 0 0 0; 0 0 (-g*m1)/M 0 (-g*m2)/M 0; 0 0 0 1 0 0; 0 0 (-g*(M+m1))/(L1*M)
0 (-g*m2)/(L1*M) 0; 0 0 0 0 0 1; 0 0 (-g*m1)/(L2*M) 0 (-g*(M+m2))/(L2*M) 0];
B = [0; 1/M; 0; 1/(L1*M); 0; 1/(L2*M)];
C = [1 0 0 0 0 0; 0 0 1 0 0 0; 0 0 0 0 1 0];
D = [0;0;0];

states = {'x' 'x_dot' 'theta_1' 'theta_1_dot' 'theta_2' 'theta_2_dot'};
inputs = {'F'};
outputs = {'x'; 'theta_1'; 'theta_2'};

sys_ss = ss(A,B,C,D,'statename',states,'inputname',inputs,'outputname',outputs);

co = ctrb(sys_ss);
controllability = rank(co);

%%
%D) With given parameters we applied the following LQR controller
%Linear System
%Define LQR parameters
Q = diag([100 1 10000 1 10000 1]);
R = 0.00001;
K = lqr(A,B,Q,R);
```

```
%Closed Loop
Ac = [(A-B*K)];
Bc = [B];
Cc = [C];
Dc = [D];

states = {'x' 'x_dot' 'theta_1' 'theta_1_dot' 'theta_2' 'theta_2_dot'};
inputs = {'r'};
outputs = {'x'; 'theta_1'; 'theta_2'};

%Defining the closed loop system
sys_cl                                                                  =
ss(Ac,Bc,Cc,Dc,'statename',states,'inputname',inputs,'outputname',outputs);

%This code was used for testing the system to find 30second convergence
%initial(sys_cl,x0)

%Defining the time interval
t = 0:0.01:30;
r =0.2*ones(size(t));

%initial conditions theta1 = 10, theta2 = 15
x0 = [0; 0; 10; 0; 15; 0]; %radians or degrees? - need to find out

%Linearized system System Response with Initial condition x0 for 30s
%lsim(sys_cl,r,t,x0)
%title('Response of a Linearized system to initial conditions')
%ylabel('Amplitude - x(m) , theta(1,2)(degree)')

%Nonlinear system
% M = 1000;
% m1 = 100;
% m2 = 100;
% L1 = 20;
% L2 = 10;
% g = 9.8;
% %U = -K*x;
% Xp = zeros(6,1);
% Xp(1)=x(2);
%   Xp(2)=(   U   +   (x(4)^2)*m1*L1*sin(x(3))   -   (x(6)^2)*m2*L2*sin(x(5))   -
g*m1*sin(x(3))*cos(x(3)) - g*m2*sin(x(5))*cos(x(5)) )/(M+m1+m2);
% Xp(3)=x(4);
%     Xp(4)=(     U*cos(x(3))      -      (x(4)^2)*m1*L1*sin(x(3))*cos(x(3))      -
(x(6)^2)*m2*L2*sin(x(5))*cos(x(3))      -      g*m1*sin(x(3))*(cos(x(3))^2)      -
g*m2*sin(x(5))*cos(x(5))*cos(x(3))    -    g*sin(x(3))*(M    +    m1*sin(x(3))^2    +
m2*sin(x(5))^2) )/(L1*(M + m1*sin(x(3))^2 + m2*sin(x(5))^2 ));
% Xp(5)=x(6);
%      Xp(6)=(U*cos(x(5))       -       (x(4)^2)*m1*L1*sin(x(3))*cos(x(5))       -
(x(6)^2)*m2*L2*sin(x(5))*cos(x(5))     -     g*m1*sin(x(3))*cos(x(3))*cos(x(5))     -
g*m2*sin(x(5))*(cos(x(5))^2)     -     g*sin(x(5))*(M     +     m1*sin(x(3))^2     +
m2*sin(x(5))^2))/(L2*(M + m1*sin(x(3))^2 + m2*sin(x(5))^2 ));

%%%above code is in function nonlinearpendulum.m
```

```
Response of a nonlinear system
f1=@nonlinearpendulum;

Tf = 100;
T = [0:0.01:Tf];

%Intial conditions as theta1=0.17 radians, theta2=0.26 radians
x0 = [0; 0; 0.17; 0; 0.26; 0];

%no gains for initial state - dont need these plots
%K = [ 0 0 0 0 0 0];

%Part D gains
%gains from lqr and linearized state
K = [3162.3 6324.5 30486 -4718.8 19580 -24293];


[t,x] = ode45(f1, T, x0, [], K);

figure(1)
title('response of a Nonlinear system to initial conditions')
subplot(3,2,1)
plot(t, x(:,1))
title('x')
xlabel('time (s)')
ylabel('Amplitude (m)')
grid

subplot(3,2,2)
plot(t, x(:,2))
title('xdot')
xlabel('time (s)')
ylabel('Amplitude (m)')
grid

subplot(3,2,3)
plot(t, x(:,3))
title('theta1')
xlabel('time (s)')
ylabel('Amplitude (radians)')
grid

subplot(3,2,4)
plot(t, x(:,4))
title('theta1dot')
xlabel('time (s)')
ylabel('Amplitude (radians)')
grid

subplot(3,2,5)
plot(t, x(:,5))
title('theta2')
xlabel('time (s)')
ylabel('Amplitude (radians)')
grid
```

```
subplot(3,2,6)
plot(t, x(:,6))
title('theta2dot')
xlabel('time (s)')
ylabel('Amplitude (radians)')
grid

%Checking for stability
eig(Ac);

%%
% E)observable
ob = obsv(sys_ss);
observability = rank(ob);

%output vector x(t)
obsv(A,C(1,:));
ObCheck1 = rank(obsv(A,C(1,:)));

%output vector theta1(t), theta2(t)
obsv(A,C([2 3],:));
ObCheck2 = rank(obsv(A,C([2 3],:)));

%output vector x(t), theta2(t)
obsv(A,C([1 3],:));
ObCheck3 = rank(obsv(A,C([1 3],:)));

%output vector x(t), theta1(t), theta2(t)
obsv(A,C);
ObCheck4 = rank(obsv(A,C));

%All of the other checks were observable
% x(t) yes
%(theta1,theta2) no
%(x(t),theta2) yes
%(x(t),theta1,theta2) yes

%%
%F)Luenberger Observer
%Make 10 times faster than slowest controller pole
P = [-1.7 -1.8 -1.9 -2.0 -2.1 -2.2];

%place the poles for first Luenberger with just x(t)
C1 = C(1,:);
L1 = place(A',C1',P)';

Ae1 = A - L1*C1;
Be1 = B;
Ce1 = C1;
De1 = 0;

states = {'x' 'x_dot' 'theta_1' 'theta_1_dot' 'theta_2' 'theta_2_dot'};
inputs = {'F'};
outputs = {'x'};
```

```
sys_est1                                                                  =
ss(Ae1,Be1,Ce1,De1,'statename',states,'inputname',inputs,'outputname',outputs);
t = 0:0.01:10;
r = ones(size(t));
%Theta1 theta2 in degrees
x0 = [0; 0; 10; 0; 15; 0];
%sim for F 1
%lsim(sys_est1,r,t,x0);

%place the poles for first Luenberger with x(t) and theta 2
C2 = C([1 3],:);
L2 = place(A',C2',P)';

Ae2 = A - L2*C2;
Be2 = B;
Ce2 = C2;
De2 = 0;

states = {'x' 'x_dot' 'theta_1' 'theta_1_dot' 'theta_2' 'theta_2_dot'};
inputs = {'F'};
outputs = {'x'; 'theta_2'};

sys_est2                                                                  =
ss(Ae2,Be2,Ce2,De2,'statename',states,'inputname',inputs,'outputname',outputs);
t = 0:0.01:10;
r = ones(size(t));
x0 = [0; 0; 10; 0; 15; 0];
%sim for F 2
%lsim(sys_est2,r,t,x0);

%place the poles for first Luenberger with x(t) theta 1 and theta 2
C3 = C;
L3 = place(A',C3',P)';

Ae3 = A - L3*C3;
Be3 = B;
Ce3 = C3;
De3 = 0;

states = {'x' 'x_dot' 'theta_1' 'theta_1_dot' 'theta_2' 'theta_2_dot'};
inputs = {'F'};
outputs = {'x'; 'theta_1'; 'theta_2'};

sys_est3                                                                  =
ss(Ae3,Be3,Ce3,De3,'statename',states,'inputname',inputs,'outputname',outputs);
t = 0:0.01:10;
r = ones(size(t));
x0 = [0; 0; 10; 0; 15; 0];
%sim for F 3
lsim(sys_est3,r,t,x0);

%%
%G) L3 seems to be the "best" L
%Linear System
```

```
Ace = [(A-B*K) (B*K);
       zeros(size(A)) (A-L3*C)];
Bce = [B;
       zeros(size(B))];
Cce = [Cc zeros(size(Cc))];
Dce = [0;0;0];

states = {'x' 'x_dot' 'theta_1' 'theta_1_dot' 'theta_2' 'thet_2_dot' 'e1' 'e2' 'e3'
'e4' 'e5' 'e6'};
inputs = {'F'};
outputs = {'x'; 'theta_1'; 'theta_2'};


sys_est_cl =
ss(Ace,Bce,Cce,Dce,'statename',states,'inputname',inputs,'outputname',outputs);
xe0 = [0; 0; 0.17; 0; 0.26; 0; 0;0;0;0;0;0];

t = 0:0.01:30;
r =zeros(size(t));
%sim for part G
%lsim(sys_est_cl,r,t,xe0)

 %Nonlinear system
function Xp = nonlinearpendulumG(t, x, K)

M = 1000;
m1 = 100;
m2 = 100;
L1 = 20;
L2 = 10;
g = 9.8;


%For part G, K now has different gains
U = -K*x;

Xp = zeros(12,1);

Xp(1)=x(2);
Xp(2)=( U - (x(4)^2)*m1*L1*sin(x(3)) - (x(6)^2)*m2*L2*sin(x(5)) -
g*m1*sin(x(3))*cos(x(3)) - g*m2*sin(x(5))*cos(x(5)) )/(M + m1*sin(x(3))^2 +
m2*sin(x(5))^2 );
Xp(3)=x(4);
Xp(4)=( U*cos(x(3)) - (x(4)^2)*m1*L1*sin(x(3))*cos(x(3)) -
(x(6)^2)*m2*L2*sin(x(5))*cos(x(3)) - g*m1*sin(x(3))*(cos(x(3))^2) -
g*m2*sin(x(5))*cos(x(5))*cos(x(3)) - g*sin(x(3))*(M + m1*sin(x(3))^2 +
m2*sin(x(5))^2) )/(L1*(M + m1*sin(x(3))^2 + m2*sin(x(5))^2 ));
Xp(5)=x(6);
Xp(6)=(U*cos(x(5)) - (x(4)^2)*m1*L1*sin(x(3))*cos(x(5)) -
(x(6)^2)*m2*L2*sin(x(5))*cos(x(5)) - g*m1*sin(x(3))*cos(x(3))*cos(x(5)) -
g*m2*sin(x(5))*(cos(x(5))^2) - g*sin(x(5))*(M + m1*sin(x(3))^2 +
m2*sin(x(5))^2))/(L2*(M + m1*sin(x(3))^2 + m2*sin(x(5))^2 ));
Xp(7)=x(8);
```

```
Xp(8)=(    U    -    (x(4)^2)*m1*L1*sin(x(3))    -    (x(6)^2)*m2*L2*sin(x(5))    -
g*m1*sin(x(3))*cos(x(3))  -  g*m2*sin(x(5))*cos(x(5))  )/(M  +  m1*sin(x(3))^2  +
m2*sin(x(5))^2 );
Xp(9)=x(10);
Xp(10)=(       U*cos(x(3))       -       (x(4)^2)*m1*L1*sin(x(3))*cos(x(3))       -
(x(6)^2)*m2*L2*sin(x(5))*cos(x(3))      -      g*m1*sin(x(3))*(cos(x(3))^2)      -
g*m2*sin(x(5))*cos(x(5))*cos(x(3))    -    g*sin(x(3))*(M    +    m1*sin(x(3))^2    +
m2*sin(x(5))^2) )/(L1*(M + m1*sin(x(3))^2 + m2*sin(x(5))^2 ));
Xp(11)=x(12);
Xp(12)=(U*cos(x(5))         -         (x(4)^2)*m1*L1*sin(x(3))*cos(x(5))         -
(x(6)^2)*m2*L2*sin(x(5))*cos(x(5))    -    g*m1*sin(x(3))*cos(x(3))*cos(x(5))    -
g*m2*sin(x(5))*(cos(x(5))^2)     -     g*sin(x(5))*(M     +     m1*sin(x(3))^2     +
m2*sin(x(5))^2))/(L2*(M + m1*sin(x(3))^2 + m2*sin(x(5))^2 ));

%different code that calls the function
f1=@nonlinearpendulumG;

Tf = 100;
T = [0:0.01:Tf];

x0 = [0; 0; 0.17; 0; 0.26; 0; 0; 0; 0; 0; 0; 0];

%
%Gains from lqr and linearized state and L gains for estimator.
K = [3162.3 6324.5 30486 -4718.8 19580 -24293 11.7 55.333 -201.54 81.915 70.215 -
209.3];

[t,x] = ode45(f1, T, x0, [], K);

figure(1)
subplot(3,2,1)
plot(t, x(:,1))
title('x')
grid

subplot(3,2,2)
plot(t, x(:,2))
title('xdot')
grid

subplot(3,2,3)
plot(t, x(:,3))
title('theta1')
grid

subplot(3,2,4)
plot(t, x(:,4))
title('theta1dot')
grid

subplot(3,2,5)
plot(t, x(:,5))
title('theta2')
grid
```

```
subplot(3,2,6)
plot(t, x(:,6))
title('theta2dot')
grid



%%%%%%%%%%%%%%%%%%%%%
```