Risk Contribution is Exposure times Volatility times Correlation

Decomposing risk using the x-sigma-rho formula

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The aim of attribution analysis is to explain the impact of active management decisions on the risk and return of a portfolio. For such an analysis to be meaningful, the attribution model must reflect the investment process. Furthermore, as discussed by Menchero (2006/2007), alignment of risk and performance attribution along the same decision variables is essential to properly evaluate the tradeoff between risk and return.

The most general return attribution can be written as

$$R = \sum_m x_m g_m , \tag{1}$$

where x_m is the portfolio exposure to source *m*, and g_m is the return of the source. For *ex post* analysis (i.e., performance attribution), source exposures are known with certainty at the start of the period, but source returns are known only at the end of the period. By contrast, in the *ex ante* case (i.e., return forecasting), source exposures are known with certainty at the start of the period, but the return *distribution* must be *predicted* at the start of the period. Since this paper addresses the question of attributing risk forecasts, we focus our attention on *ex ante* returns.

The return contributions, denoted

$$Q_m = x_m g_m , \tag{2}$$

fully account for portfolio return. It is the job of the portfolio manager to forecast the source returns g_m and determine portfolio exposures x_m that optimally balance the risk/return tradeoff.

One approach to risk attribution is known as stand-alone volatility analysis, and centers on the volatility of each return contribution viewed in isolation,

$$\sigma(Q_m) = \sigma(x_m g_m). \tag{3}$$

A benefit of this approach is that the stand-alone volatility of each return contribution is an important and financially relevant quantity. The main drawback of this method is that it neglects the crucial role of correlation in determining portfolio risk. Clearly, a return contribution that is highly correlated with the portfolio is far riskier than one that is negatively uncorrelated. In the former case, the return contribution tends to co-move with the rest of the portfolio, thus offering little diversification. A negative correlation, by contrast, actually *reduces* risk, since it acts as a hedge for the rest of the portfolio. Another shortcoming of this approach is that since stand-alone volatilities do not sum to portfolio risk, part of the risk is left unattributed.[1] This lack of transparency limits the usefulness of stand-alone volatility analysis.

Another approach to risk attribution is based on the marginal contribution to risk (MCR), which is defined as a partial derivative,

$$\mathrm{MCR}_m = \frac{\partial \sigma(R)}{\partial x_m}. \tag{4}$$

[1] Neither do stand-alone *variances* add to portfolio variance. The difference is termed "unattributed covariance."

The financial interpretation of Equation (4) is that a small increment in source exposure Δx_m produces an approximate change in portfolio risk of $\mathrm{MCR}_m \Delta x_m$, holding all other exposures constant. In particular, portfolio risk can be reduced by increasing exposure to any source with a negative marginal contribution to risk.

As we show in the appendix, the source marginal contributions have two very useful properties. The first is the risk attribution formula,

$$\sigma(R) = \sum_m x_m \mathrm{MCR}_m \,, \tag{5}$$

which expresses the precise relation between the source marginal contributions and portfolio risk.

The second useful property is that for an unconstrained optimal portfolio (i.e., maximum information ratio), the expected source returns are directly proportional to the source marginal contributions,

$$E(g_m) = IR \cdot \mathrm{MCR}_m, \tag{6}$$

where *IR* is the portfolio information ratio. Equation (6) provides implied returns that serve as an important reality check on whether the actual portfolio is consistent with the manager's views.

The marginal contribution approach is extremely useful in that it fully accounts for portfolio risk, and provides the implied return of each source (under optimality assumptions). One drawback of the marginal contribution approach, however, is that it does not supply the stand-alone volatility of the return contribution — a useful and meaningful quantity. Another shortcoming of the marginal contribution approach is limited transparency. That is, the partial derivative in Equation (4) does little to provide further insight into the marginal contribution. Moreover, it is not immediately obvious why the risk of a portfolio (which has *fixed* exposures at any point in time) should depend on a partial derivative, which measures the rate of change in risk as the exposures are *varied.*[2]

In this article, we present a more insightful approach to risk attribution that combines the strengths of stand-alone volatility analysis with the virtues of the marginal contribution approach. Our approach identifies three intuitive drivers of risk: source exposures (x), source volatilities (σ), and source correlations with the portfolio (ρ). We therefore refer to this as the "*x-sigma-rho*" risk attribution methodology. Furthermore, by simply grouping x, σ, and ρ in different combinations, the other risk attribution methods are obtained as special cases. More specifically, we show that the stand-alone volatility of a return contribution is directly obtained from $(x\sigma)$, whereas the marginal contribution is given by $(\sigma\rho)$. The *x-sigma-rho* formulation may therefore be regarded as encompassing the other two approaches.

[2] The underlying reason for this remarkable result rests on the mathematical property of *linear homogeneity.*

Other workers have also studied the question of risk attribution. Litterman (1996), for example, discussed the merits of marginal contribution analysis and also provided a very insightful geometric picture for combining sources of risk. He did not, however, discuss how the marginal contribution could be decomposed into a product of volatility and correlation. Menchero and Hu (2006) did show this, but did not take the next step of drilling into volatility and correlation, thus obtaining even greater insight into the sources of risk.

The main contribution of this article is to codify and extend the earlier work of Menchero and Hu (2006). This results in a flexible and general risk attribution framework capable of drilling into the finest level of granularity. We show how the method can be applied to a hierarchical investment process where the return sources are themselves composed of sub-sources. We also consider the opposite scenario in which a risk manager wants to understand the risk contribution from a collection of sources. Finally, we show how the correlation itself can be understood in terms of contributions from various sources. These concepts are illustrated with an extensive example that studies the financial market crisis of October 2008.

THE *X-SIGMA-RHO* FRAMEWORK

Given any attribution of portfolio return, the *x-sigma-rho* framework provides a corresponding attribution of portfolio risk. As shown in the appendix, the risk attribution corresponding to the general return attribution of Equation (1) is given by the *x-sigma-rho* formula,

$$\sigma(R) = \sum_m x_m \sigma(g_m) \rho(g_m, R), \tag{7}$$

where x_m is the portfolio exposure to source *m*, $\sigma(g_m)$ is the volatility of source *m*, and $\rho(g_m, R)$ is the correlation of source *m* with the portfolio. This result confirms the intuition that there are three distinct drivers of portfolio risk: The first driver is the exposure, which measures the size of the bet. This is directly under the control of the portfolio manager. The second driver is the stand-alone volatility of the return source. Sometimes this depends on the portfolio (as when the return driver is the active return in a sector), but usually it does not (e.g., when the return source represents a stock return or a factor return). The third driver of portfolio risk, of course, is the correlation.

In the appendix, we show that the marginal contribution to risk can be written as

$$\mathrm{MCR}_m = \sigma(g_m) \rho(g_m, R) \, , \tag{8}$$

consistent with Equation (5) and Equation (7). There are several key benefits to expressing the marginal contribution as the product of volatility and correlation. First, it is more intuitive than decomposing risk in terms of partial derivatives. Second, as we demonstrate in a moment, it is possible to drill into the source volatilities and correlations separately, yielding even greater insight into the sources of risk.

There is another, more subtle, benefit to expressing the marginal contribution as in Equation (8). According to Equation (5), the risk profile of a source is fully characterized by its exposure and marginal contribution. Decomposing the marginal contribution into a product of volatility and correlation, however, suggests that there is more to the story. Consider, for instance, two factors, both with the same exposure (say, 0.5), and the same marginal contribution (say 1%). Both factors thus contribute 50 bps to portfolio risk. Now suppose that the first factor has volatility 10 percent and correlation 0.10, while the second factor has volatility 2 percent and correlation 0.5. Although from a marginal contribution perspective the two factors are identical, the volatilities of their return contributions are quite different: For the first factor it is 5 percent, versus 1 percent for the second factor. The first factor, therefore, is far more likely to make a large contribution (either positive or negative) to portfolio return at the end of the period. Whether this makes the first factor "riskier" or not is debatable. What is clear, however, is that the *x-sigma-rho* methodology highlights the distinction between the two factors, whereas marginal contribution analysis does not.

The *x-sigma-rho* risk attribution scheme is driven from the corresponding return attribution. Since return attribution can be customized to fit any investment process, it follows that the *x-sigma-rho* methodology also applies to any investment process. Furthermore, it has the benefit of automatically attributing risk and return to the same decision variables. In this article, we consider distinct investment processes based on assets, sectors, and factors.

Asset-based Attribution

In a bottom-up investment process, the portfolio manager seeks to identify and overweight outperforming securities, while underweighting the underperformers. The active return R^A for such an investment process can be attributed as

$$R^A = \sum_n w_n^A (r_n - R^B), \tag{9}$$

where w_n^A is the active weight (source exposure) of security n, and $(r_n - R^B)$ is the relative return (source return) of the security.

Applying the *x-sigma-rho* formula to Equation (9), we obtain

$$\sigma(R^A) = \sum_n w_n^A \sigma(r_n - R^B)\rho(r_n - R^B, R^A)\,. \tag{10}$$

Here, $\sigma(r_n - R^B)$ represents the volatility of the relative return, and $\rho(r_n - R^B, R^A)$ is the correlation between the relative return and the active return.

Sector-based Attribution

In a sector-based strategy, the portfolio manager follows a two-step investment process. The first step is asset allocation across sectors. The second step is security selection within sectors.

The active return for a sector-based investment process can be decomposed according to Brinson and Fachler (1985),

$$R^A = \sum_i w_i^A (r_i^B - R^B) + \sum_i w_i^P (r_i^P - r_i^B)\,, \tag{11}$$

where the first term is the allocation effect, and the second term is the selection effect. For the allocation effect, the source exposure is the active weight (w_i^A), and the source return is the relative return of the sector ($r_i^B - R^B$). For the selection effect, the source exposure is the portfolio sector weight (w_i^P), and the source return is the active return of the sector ($r_i^P - r_i^B$).

The corresponding risk attribution can be immediately written using the *x-sigma-rho* formula:

$$\sigma(R^A) = \sum_i w_i^A \sigma(r_i^B - R^B)\, \rho(r_i^B - R^B, R^A) + \sum_i w_i^P \sigma(r_i^P - r_i^B) \rho(r_i^P - r_i^B, R^A)\,. \tag{12}$$

The first sum represents the total tracking error due to allocation decisions. The second sum gives the risk from security selection. An example of sector-based risk attribution is provided at the end of this article.

Factor-based Attribution

A factor model explains asset returns using a parsimonious set of explanatory variables. For equities, the factors might represent styles (Value, Growth, etc.), industries, and countries. For fixed income, the factors might correspond to characteristic movements in the yield curve (e.g., Shift or Twist), or to credit spreads in various quality and/or sector groupings. Factor models disentangle otherwise confounding effects (e.g., collinearity between styles and industries), thereby allowing the portfolio to be analyzed along multiple dimensions simultaneously.

In a factor model, asset returns are segmented into a systematic (factor) component and a diversifiable (specific) component:

$$r_n = \sum_k X_{nk} f_k + u_n\,. \tag{13}$$

Here, X_{nk} is the exposure of asset *n* to factor *k*, f_k is the factor return, and u_n is the specific return.

Factor returns are typically estimated by cross-sectional regression. As discussed by Grinold and Kahn (2000), they can be interpreted as the returns on long/short portfolios that have unit exposure to the factor under consideration, and zero exposure to other factors. The Value factor return, for example, is the return of a portfolio that has exposure of one to Value but is neutral to industries, countries, and other styles.

Factor and specific contributions can be rolled up to the portfolio level. That is, the portfolio active return is given by

$$R^A = \sum_k X_k^A f_k + \sum_n w_n^A u_n, \tag{14}$$

where w_n^A is the active weight in asset *n* and X_k^A is the active exposure to factor *k*,

$$X_k^A = \sum_n w_n^A X_{nk}. \tag{15}$$

Positive return contributions are obtained through positive exposure to factors with positive returns and by overweighting assets with positive specific returns.

The corresponding factor risk attribution can be immediately written using the *x-sigma-rho* formula:

$$\sigma(R^A) = \sum_k X_k^A \sigma(f_k)\rho(f_k, R^A) + \sum_n w_n^A \sigma(u_n)\rho(u_n, R^A)\,, \tag{16}$$

where $\sigma(f_k)$ is the factor volatility, $\rho(f_k, R^A)$ is the correlation between the factor return and the active return, $\sigma(u_n)$ is the specific volatility of stock *n*, and $\rho(u_n, R^A)$ is the correlation between the specific return and the active portfolio. The first sum is the total risk due to factors, while the second sum gives the total specific risk contribution. A detailed example is provided at the end of this paper.

VOLATILITY DRILLDOWN

It is sometimes useful to think of the return sources g_m in Equation (1) as being composed of contributions from sub-sources. That is, we write

$$g_m = \sum_l y_l h_l\,, \tag{17}$$

where y_l is the exposure to sub-source *l* and h_l is the return of the sub-source. For instance, g_m might represent the active return $(r_i^P - r_i^B)$ of a country in the Brinson model, which could be further decomposed using a factor model, as in Equation (14).

The volatility drilldown for source *m* is immediately obtained from the *x-sigma-rho* formula:

$$\sigma(g_m) = \sum_l y_l \sigma(h_l)\rho(h_l, g_m)\,, \tag{18}$$

where $\sigma(h_l)$ is the volatility of the sub-source, and $\rho(h_l, g_m)$ is the correlation between sub-source h_l and the source return g_m.

CORRELATION DRILLDOWN

Correlation is one of three drivers of portfolio risk. It is important to gain deeper insight into the sources of these correlations. As shown in the appendix, the correlation drilldown is given by

$$\rho(g_m, R) = \sum_n x_n \left[\frac{\sigma(g_n)}{\sigma(R)}\right] \rho(g_m, g_n), \tag{19}$$

where x_n is the portfolio exposure to source *n*, $[\sigma(g_n)/\sigma(R)]$ is the volatility ratio for source *n*, and $\rho(g_m, g_n)$ is the pair-wise correlation of sources *m* and *n*. This result states that for source n to make a significant contribution to the correlation between source m and the portfolio, three things are required: (a) the exposure to source n must be large, (b) the volatility of source n must be sizeable relative to portfolio risk, and (c) the pair-wise correlation must be significant.

RISK CONTRIBUTION DRILLDOWN

Often, we would like to view a collection of sources as a single group. More specifically, let

$$R = \sum_M Q_M, \tag{20}$$

where

$$Q_M = \sum_{m \in M} x_m g_m \,. \tag{21}$$

For example, Q_M might represent the return contribution from the group of all style factors. The *x-sigma-rho* formula immediately supplies the corresponding risk attribution:

$$\sigma(Q_M)\rho(Q_M, R) = \sum_{m \in M} x_m \sigma(g_m)\rho(g_m, R). \tag{22}$$

Equation (22) simply states that the risk contribution of a group *M* is the sum of the risk contributions of sources that comprise the group.

EXAMPLE

In this example, we use a factor approach to study the financial crisis of October 2008. We base our analysis on the MSCI *World Investable Market Index* (World IMI) which aims to capture the full breadth of investment opportunities available in developed markets. We take as our portfolio the GICS Financials sector carve-out of the World IMI, with the full World IMI serving as the benchmark, and the US dollar as the base currency.

We attribute the active risk to fundamental factors using the Barra Global Equity Model (GEM2), a multi-factor model described by Menchero, Morozov and Shepard (2009). The model is designed to dynamically adapt to changing market conditions. GEM2 decomposes active return into a local component and a currency component. The local component is further explained by a World equity factor, styles, industries, countries, and a stock-specific return.

In Table 1, we present a summary view of the *x-sigma-rho* risk attribution results. The active risk for October 2008 is 2.24%, stated on a monthly basis.[3] Virtually all of the risk is due to local-market effects, and none to currency effects. This can be understood by noting that although currencies have a stand-alone volatility of 14 bps, they are essentially uncorrelated with the active return.

Table 1 also drills down into the local market component of risk. In this case, the primary contributors to active risk are industries (155 bps) and styles (57 bps). Although industries have roughly twice the volatility of styles (185 bps versus 96 bps), industries make nearly three times the risk contribution. This is accounted for by the fact that industries are more strongly correlated (0.84) with the active return than styles (0.59). It is also interesting to note that countries contribute negatively to the overall active risk.

In Table 2, we drill into the styles risk contribution (57 bps). Notice that, by way of Equation (22), the styles risk contribution is attributed in two ways: first as the sum of the risk contributions of individual style factors, and second as the product of styles volatility (96 bps) and correlation (0.59) with active return. By far, the biggest contributors to style risk are Momentum (18 bps) and Volatility (28 bps). In the case of Momentum, the active portfolio has *negative* exposure (-0.32), implying that the Financial sector tended to underperform World IMI in the months leading up to October 2008. The Momentum factor has relatively high volatility of 113 bps per month. The Momentum factor returns are *negatively* correlated (-0.50) with active return, meaning that Momentum tends to perform well when Financials perform poorly. The product of exposure, volatility, and correlation gives a tracking error contribution of 18 bps from Momentum. Also reported in Table 2 is the marginal contribution to active risk (MCAR), which is simply the product of the factor volatility and the active correlation columns.

For comparison purposes, the stand-alone volatility analysis of styles risk is presented in Table 3. The stand-alone volatilities quoted in Table 3 are the volatilities of the return contribution of each style factor:

$$\sigma(X_k^A f_k) = |X_k^A| \sigma(f_k). \tag{23}$$

Up to a trivial sign, therefore, this just represents the "*x-sigma*" portion of the *x-sigma-rho* formula. For example, the Momentum stand-alone volatility (37 bps) is the product of the absolute Momentum active exposure (0.32) and the volatility of Momentum factor return (113 bps). Note that the stand-alone volatility is always positive, even for those return sources that actually *reduce* portfolio risk by virtue of negative correlation. In Table 1, for example, the stand-alone volatility of countries is 34 bps. Countries, however, are negatively correlated with active return, and actually decrease active risk by 5 bps. Stand-alone volatility analysis cannot capture this effect because it ignores correlation with the rest of the portfolio. Another consequence of ignoring correlation is that a large portion of the risk (19.63%) in Table 3 is left unattributed. In summary, the *x-sigma-rho* risk attribution formula includes the same information as in the stand-alone volatility analysis, but goes a step further by using correlation to fully account for the portfolio risk.

In Table 4, we present a drilldown into styles volatility (96 bps). The drilldown is similar to the styles risk contribution drilldown of Table 2, with one important difference: the correlations are with the return contribution of styles, $\rho(f_k, Q_{styles})$, rather than with the active return, $\rho(f_k, R^A)$.

[3] To annualize this risk, multiply by $\sqrt{12}$

This makes the factor risk contributions sum to the styles volatility (96 bps) rather than the styles active risk contribution (57 bps). It is notable that the Momentum factor accounts for a greater proportion of risk in Table 2 (18/57) than it does in Table 4 (23/96). This suggests that there are significant correlations between the Momentum factor and non-style factor return contributions. We verify this by drilling into the Momentum correlation.

We use Equation (19) to drill into the Momentum correlation with active return (-0.50). In Table 5, we see that two main drivers of this correlation are styles and industries. We first consider styles, which contribute -0.26 to the correlation. The styles group has a volatility ratio of 0.43, which is obtained as the volatility of the styles group (96 bps, from Table 1) divided by the total tracking error (224 bps). The correlation of Momentum with styles is -0.62. The strong negative correlation is to be expected, since the styles group has negative exposure to Momentum.

In Table 6, we drill into the correlation between Momentum and styles group. We see, unsurprisingly, that the largest contribution is coming from the Momentum factor itself (-0.38). The active exposure to Momentum is -0.32. The factor volatility ratio, 1.18, is given by the ratio of the Momentum volatility (113 bps, from Table 4) to the styles group volatility (96 bps). The Momentum factor, of course, is perfectly correlated with itself. Multiplying the three columns across gives the correlation contribution of -0.38.

As we saw in Table 5, industries contribute -0.30 out of a total correlation -0.50 between Momentum and the active portfolio. This is explained in large part by the strong negative correlation (-0.36) between Momentum and industries. In Table 7, we drill into this correlation. The dominant contribution (-0.19) comes from the Banks factor, but the other financial sectors also contribute significantly. Note that the large negative correlation (-0.39) between Momentum and Banks is a main driver of the negative correlation between Momentum and the industries group. Given that the Momentum factor portfolio has zero net exposure to Banks, it is interesting to consider how this negative correlation arises. Although the Momentum factor portfolio has zero net weight in the banking industry, it takes long positions in banking stocks that performed relatively well in the months leading up to October 2008, and short positions in those banks that performed relatively poorly. The beaten-down banks, which were extremely volatile, had negative weights in the Momentum factor portfolio but positive weights in the Banks factor. These stocks helped drive the returns of each factor, and thus contributed strongly to the negative correlation.

In Figure 1, we plot the time-series correlation of Momentum and Banks, as computed by the GEM2 model. It is interesting to note the strong positive correlation in 1998, followed by the sharp drop into negative correlation during the internet bubble. With the collapse of the bubble, the correlation reverted toward zero, remaining there for several years. With the onset of the financial crisis in August 2007, however, the correlation between Banks and Momentum becomes strongly negative.

Up to now, our example has focused on a factor-based investment process. For completeness, we now consider a sector-based investment process. In Table 8, we present a sector-based risk attribution segmented according to region. Virtually all of the tracking error comes from selection effect, and almost none from allocation effect. We may apply the techniques of this article to understand these sources of risk.

For the allocation effect, the source return is the relative return of the region, and the source exposure is the active weight in the region. For the selection effect, the source return is the active return of the region, and the source exposure is the portfolio weight in the region. From Table 8, we see that the source volatilities are roughly comparable in magnitude. However, the source exposures are much smaller for the allocation effect than for the selection effect. Likewise, the source correlations are much smaller for the allocation effect than for the selection effect. The small risk contribution from allocation effect, therefore, can be understood as the combined effect of small exposures and small correlations. Greater insight into the source correlations, of course, could be gained by application of the correlation drilldown in Equation (19), although we omit the details here.

CONCLUSION

The *x-sigma-rho* framework is a flexible methodology for attributing portfolio risk to the same decision variables used to attribute portfolio return. It can be applied to any investment process, and provides a basis for evaluating the tradeoffs between risk and return. For each return source, the *x-sigma-rho* formula decomposes the risk contribution into the product of source exposure, source volatility and source correlation with the portfolio. Moreover, the *x-sigma-rho* framework provides drilldowns into risk contributions, volatilities, and correlations, thus providing insight and transparency at any desired level of granularity.

APPENDIX

To obtain the *x-sigma-rho* risk attribution of Equation (7), let portfolio return be given by

$$R = \sum_m x_m g_m. \tag{A1}$$

Computing portfolio variance in terms of Equation (A1) and expanding the covariance yields the portfolio variance attribution:

$$\mathrm{var}(R) = \sum_m x_m \mathrm{cov}(g_m, R). \tag{A2}$$

Dividing Equation (A2) through by portfolio volatility and applying the usual relation between covariance and correlation yields the *x-sigma-rho* attribution

$$\sigma(R) = \sum_m x_m \sigma(g_m)\rho(g_m, R). \tag{A3}$$

Similarly, to obtain the risk contribution drilldown, consider a portfolio return contribution given by

$$Q_M = \sum_{m \in M} x_m g_m \,. \tag{A4}$$

Computing the covariance with portfolio return of each side of Equation (A4) and expanding the covariance yields

$$\mathrm{cov}(Q_M, R) = \sum_{m \in M} x_m \mathrm{cov}(g_m, R)\,. \tag{A5}$$

Dividing Equation (A5) by portfolio volatility and applying the usual relation between covariance and correlation yields the risk contribution drilldown

$$\sigma(Q_M)\rho(Q_M, R) = \sum_{m \in M} x_m \sigma(g_m)\rho(g_m, R)\,, \tag{A6}$$

which is Equation (22) of the main text. To obtain the correlation drilldown formula, begin with the usual relation:

$$\rho(g_m, R) = \frac{\mathrm{cov}(g_m, R)}{\sigma(g_m)\sigma(R)}\,. \tag{A7}$$

Substituting the return attribution of Equation (A1) for the second argument of the covariance and expanding yields

$$\rho(g_m, R) = \sum_n x_n \frac{\sigma(g_n)}{\sigma(R)} \frac{\mathrm{cov}(g_m, g_n)}{\sigma(g_m)\sigma(g_n)}\,. \tag{A8}$$

Applying again the usual relation between covariance and correlation yields the correlation drilldown:

$$\rho(g_m, R) = \sum_n x_n \left[\frac{\sigma(g_n)}{\sigma(R)}\right] \rho(g_m, g_n) \ , \tag{A9}$$

which is Equation (19) of the main text. To express marginal contribution to risk as a product of volatility and correlation, expand the second argument of covariance in Equation (A2) to obtain

$$\mathrm{var}(R) = \sum_{n,m} x_n x_m \mathrm{cov}(g_m, g_n) \,. \tag{A10}$$

Differentiating Equation (A10) with respect to x_m yields

$$2\sigma(R)\frac{\partial\sigma(R)}{\partial x_m} = 2\sum_n x_n \mathrm{cov}(g_m, g_n) \,. \tag{A11}$$

Solving Equation (A11) for the desired partial derivative yields

$$\frac{\partial\sigma(R)}{\partial x_m} = \frac{\mathrm{cov}(g_m, R)}{\sigma(R)}. \tag{A12}$$

The left-side of Equation (A12) is the marginal contribution to risk of source *m*, and the right side is the product of volatility and correlation. Therefore,

$$\mathrm{MCR}_m = \sigma(g_m)\rho(g_m, R) \,, \tag{A13}$$

which is Equation (8) of the main text. The marginal contribution-based risk attribution in Equation (5) is proved by combining Equation (A13) with the *x-sigma-rho* risk attribution in Equation (A3).

Table 1
Summary *x-sigma-rho* risk attribution

Source	Volatility	Active Correlation	Active Risk Contribution
Local Excess	2.25%	0.99	2.24%
World	0.00%	0.00	0.00%
Styles	0.96%	0.59	0.57%
Industries	1.85%	0.84	1.55%
Countries	0.34%	-0.15	-0.05%
Specific	0.61%	0.27	0.17%
Currency	0.14%	0.00	0.00%
Total	2.24%	1.00	2.24%

Table 2
Styles risk drilldown

Factor	Portfolio Exposure	Benchmark Exposure	Active Exposure	Factor Volatility	Active Correlation	MCAR	Active Risk Contrib
Momentum	-0.30	0.03	-0.32	1.13%	-0.50	-0.57%	0.18%
Volatility	0.53	-0.06	0.59	1.24%	0.38	0.48%	0.28%
Value	0.24	0.02	0.21	0.52%	0.03	0.02%	0.00%
Size	0.06	0.04	0.02	0.45%	0.17	0.08%	0.00%
Nonlinear Size	-0.03	-0.02	-0.01	0.32%	0.12	0.04%	0.00%
Growth	-0.31	-0.01	-0.30	0.38%	-0.14	-0.06%	0.02%
Liquidity	0.31	0.05	0.26	0.40%	0.36	0.14%	0.04%
Leverage	0.58	0.01	0.57	0.28%	0.28	0.08%	0.04%
Total				0.96%	0.59	0.57%	0.57%

Table 3
Styles stand-alone volatility analysis

Factor	Volatility of Contribution	Active Variance Contribution	Active Variance % Contribution
Momentum	0.37%	0.13	14.61%
Volatility	0.73%	0.54	59.04%
Value	0.11%	0.01	1.34%
Size	0.01%	0.00	0.01%
Nonlinear Size	0.00%	0.00	0.00%
Growth	0.12%	0.01	1.47%
Liquidity	0.10%	0.01	1.18%
Leverage	0.16%	0.02	2.72%
Styles Covariance	n/a	0.18	19.63%
Total	0.96%	0.91	100.00%

Table 4
Styles volatility drilldown

Factor	Portfolio Exposure	Benchmark Exposure	Active Exposure	Factor Volatility	Styles Correlation	Styles MCR	Styles Risk Contrib
Momentum	-0.30	0.03	-0.32	1.13%	-0.62	-0.70%	0.23%
Volatility	0.53	-0.06	0.59	1.24%	0.83	1.04%	0.61%
Value	0.24	0.02	0.21	0.52%	0.17	0.09%	0.02%
Size	0.06	0.04	0.02	0.45%	-0.04	-0.02%	0.00%
Nonlinear Size	-0.03	-0.02	-0.01	0.32%	0.21	0.07%	0.00%
Growth	-0.31	-0.01	-0.30	0.38%	0.13	0.05%	-0.02%
Liquidity	0.31	0.05	0.26	0.40%	0.42	0.17%	0.04%
Leverage	0.58	0.01	0.57	0.28%	0.45	0.13%	0.07%
Total				0.96%	1.00	0.96%	0.96%

Table 5
Momentum factor correlation with active return

Group	Group Volatility Ratio (Active)	Momentum Correlation with Group	Momentum Correlation Contribution (Active)
World	0.00	0.00	0.00
Styles	0.43	-0.62	-0.26
Industries	0.83	-0.36	-0.30
Countries	0.15	0.24	0.04
Specific	0.27	0.00	0.00
Currencies	0.11	0.23	0.02
Total			-0.50

Table 6
Momentum factor correlation with styles

Factor	Active Exposure	Factor Volatility Ratio (Styles)	Momentum Correlation with Factor	Momentum Correlation Contribution (Styles)
Momentum	-0.32	1.18	1.00	-0.38
Volatility	0.59	1.30	-0.18	-0.14
Value	0.21	0.54	-0.10	-0.01
Size	0.02	0.48	-0.03	0.00
Nonlinear Size	-0.01	0.34	-0.18	0.00
Growth	-0.30	0.40	0.24	-0.03
Liquidity	0.26	0.42	-0.12	-0.01
Leverage	0.57	0.29	-0.27	-0.04
Total				-0.62

Table 7
Momentum factor correlation with industries

Factor Name	Active Exposure	Factor Volatility Ratio (Industries)	Momentum Correlation with Factor	Momentum Correlation Contribution (Industries)
Energy Equipment & Services	-0.02	2.68	0.25	-0.01
Oil, Gas & Consumable Fuels	-0.07	1.71	0.32	-0.04
Oil & Gas Exploration & Production	-0.02	3.15	0.32	-0.02
Chemicals	-0.03	0.83	-0.01	0.00
Construction, Containers, Paper	-0.01	0.84	-0.33	0.00
Aluminum, Diversified Metals	-0.02	2.83	0.28	-0.01
Gold, Precious Metals	-0.01	4.49	0.28	-0.01
Steel	-0.01	1.82	0.15	0.00
Capital Goods	-0.08	0.67	-0.13	0.01
Commercial & Professional Services	-0.01	0.70	-0.12	0.00
Transportation Non-Airline	-0.02	0.92	-0.27	0.01
Airlines	0.00	2.75	-0.36	0.00
Automobiles & Components	-0.02	1.14	-0.29	0.01
Consumer Durables & Apparel	-0.02	0.86	-0.48	0.01
Consumer Services	-0.01	1.02	-0.34	0.00
Media	-0.02	0.77	-0.13	0.00
Retailing	-0.02	1.25	-0.45	0.01
Food & Staples Retailing	-0.02	0.72	-0.24	0.00
Food, Beverage & Tobacco	-0.05	0.63	-0.07	0.00
Household & Personal Products	-0.02	1.20	-0.22	0.00
Health Care Equipment & Services	-0.03	1.16	0.04	0.00
Biotechnology	-0.01	2.10	0.15	0.00
Pharmaceuticals, Life Sciences	-0.06	1.27	0.06	-0.01
Banks	0.32	1.55	-0.39	-0.19
Diversified Financials	0.20	1.07	-0.29	-0.06
Insurance	0.16	1.07	-0.17	-0.03
Real Estate	0.10	1.14	-0.24	-0.03
Internet Software & Services	-0.01	1.39	0.08	0.00
IT Services, Software	-0.03	0.88	0.04	0.00
Communications Equipment	-0.02	1.53	-0.08	0.00
Computers, Electronics	-0.03	1.10	-0.08	0.00
Semiconductors	-0.01	1.81	-0.09	0.00
Telecommunication Services	-0.04	0.83	0.00	0.00
Utilities	-0.05	0.95	0.23	-0.01
Total				-0.36

Figure 1
Time series correlation of Momentum factor and Banks factor

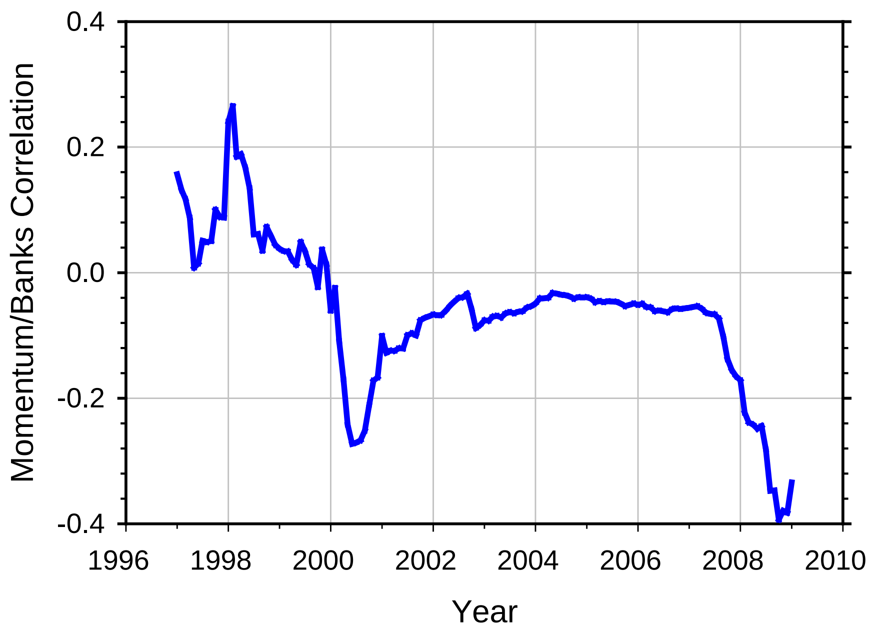

Table 8
Sector-based risk attribution, segmented by region

Region	Port Weight	Bench Weight	Active Weight	Relative Volatility	Relative Correl.	Alloc. Risk Contrib	Active Volatility	Active Correl.	Select. Risk Contrib	Region Risk Contrib
UK	0.10	0.10	0.01	2.24%	-0.03	0.00%	3.33%	0.79	0.28%	0.28%
Europe (ex UK)	0.24	0.20	0.04	1.76%	0.15	0.01%	2.28%	0.89	0.50%	0.51%
North America	0.47	0.56	-0.09	1.22%	-0.09	0.01%	2.51%	0.95	1.12%	1.13%
Japan	0.10	0.10	-0.01	3.74%	0.04	0.00%	2.73%	0.76	0.20%	0.20%
Pacific (ex Japan)	0.09	0.04	0.04	3.00%	0.00	0.00%	2.03%	0.71	0.13%	0.13%
Total	1.00	1.00	0.00	0.25%	0.08	0.02%	2.25%	0.99	2.22%	2.24%

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