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Axioma Research Paper
No. 060

August 7, 2015

Adjusted Factor-Based Performance Attribution

Factor-based performance attribution is ubiquitously employed in the asset management industry as a way to both understand and assess the management of a portfolio. Unfortunately, this attribution analysis can fail to tell the whole story. One reason is a strong correlation between the factor and specific return contributions that leads to potentially erroneous attributions. This correlation stems from a "misspecification" of the returns model and causes the factors to over- or under-explain the returns of a given portfolio. With the trend towards "smart beta", and factor-investing in general, this correlation is becoming more pervasive and accounting for it is critical when analyzing the return contributions of such factor-based strategies. We propose an *adjusted* factor-based performance attribution methodology that shifts the portion of the asset-specific contribution that is correlated with the factor contributions back into the factor portion. The resulting factor and specific contributions have near-zero correlation leading to factor contributions that do not over- or under-explain the returns of a portfolio. From a practical perspective, we find that the proposed methodology generally results in attributions that are more intuitive and provide stronger support of factor-based investment mandates.

Adjusted Factor-Based Performance Attribution

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August 13, 2015

1 Introduction

Factor-based performance attribution is commonly used to explain the sources of realized return of a portfolio. The methodology relies on a factor model of asset returns to decompose the return of a portfolio according to a set of factors. The portion of the portfolio return that can be explained by the model factors is called the *factor contribution* and the remainder is called the *asset-specific contribution*, or *specific contribution* for short. (For a description of factor-based performance attribution, see Fischer and Wermers, 2013, chap. 4)

Naturally, the choice of factors has a significant impact on the attribution of a given portfolio. If a portfolio manager, fundamental or quantitative, chooses which assets to overweight and underweight based on a criterion that is not captured by the factor model, then factor-based performance attribution will claim that the returns are asset-specific. The portfolio manager, knowing the criteria or factors used to construct the portfolio, finds these results to not only be unintuitive, but also lacking support for the manager's investment story. In order to better explain the returns coming from proprietary factors, the portfolio manager can construct a returns model that contains the factors used to construct the portfolio. The idea is that the portfolio return should be explained by those factors used to construct the portfolio to a greater extent than those present in any "standard" returns model (see Sivaramakrishnan and Stubbs, 2013).

In practice, we too often find unintuitive attribution results even after adding the portfolio-driving factors into the returns model that we use for attribution. Specifically, we frequently find the specific contribution to be significantly negative. To illustrate this, we constructed a portfolio rebalanced monthly from January 1995 to October 2013 according to the following strategy:

$$\begin{array}{ll} \text{maximize} & \text{Expected Return} \\ \text{subject to:} & \text{Long Only and Fully Invested} \\ & \text{Active Risk Constraint } 3\% \\ & \text{Active Sector Bounds of } \pm\ 4\% \\ & \text{Active Asset Bounds of } \pm\ 3\% \end{array} \tag{Strategy}$$

We used exposure to a particular Growth factor as the Expected Return and the Russell 1000 Index as the benchmark. We then considered two different returns models to use in attributing returns of this portfolio. The first model, RM1, uses 10 sector factors and 4 style factors: Market Sensitivity, Momentum, Size, and Value. The second model, RM2, adds the exact Growth factor that was used to construct the portfolio to those factors present in RM1. These factors are summarized in Table 1. Their definitions and a description of the returns models that use them are described in Appendix A.

RM1	**RM2**
	Growth
Market Sensitivity	Market Sensitivity
Momentum	Momentum
Size	Size
Value	Value
GICS Sectors	GICS Sectors

Table 1: Set of factors contained in the two returns models: RM1 and RM2.

The attribution results using these two models are summarized in Table 2. All contributions are annualized values computed using the linking methodology of Cariño (1999). The overall annualized factor contribution increases dramatically compared to RM1 when we use RM2, which contains the Growth factor. The main driver of this increase from -0.18% to 2.35% is the contribution of the Growth factor. Given that our portfolio maximized exposure to this factor, this seems to be exactly what we want. However, note that the asset-specific contribution decreased dramatically from 1.65% to -0.88%. While the factor contribution increased and the specific contribution decreased, we appear to have overshot. Not only is the specific contribution now negative, but the t-stat changed from being significantly positive (2.67) to almost significantly negative (-1.58).

Returns Model	RM1	RM2
Portfolio Return	10.72%	10.72%
Benchmark Return	9.25%	9.25%
Active Return	1.47%	1.47%
Specific Contribution (SC)	1.65%	-0.88%
Factor Contribution (FC)	-0.18%	2.35%
Industry	-0.14%	-0.15%
Styles	-0.04%	2.50%
Growth	–	2.65%
Market Sensitivity	-0.10%	-0.09%
Momentum	0.10%	0.09%
Size	0.13%	0.15%
Value	-0.17%	-0.29%
FC Volatility	1.13%	2.37%
SC Volatility	2.66%	2.41%
SC T-Stat	2.67	-1.58
FC T-Stat	-0.69	4.29
SC-FC Correlation	-0.09	-0.32

Table 2: Summary of performance attribution results. The values in the top section are the annualized active returns attributable to each factor or to a group of factors. In the bottom section, we give additional summary statistics for the Factor Contribution (FC) and Specific Contribution (SC).

This in itself is not a large problem. We could genuinely have such a large negative specific contribution. However, in Figure 1, we see that the cumulative factor and asset-specific contributions are moving in opposite directions throughout the backtest period suggesting that the contributions are negatively correlated. In fact, the correlation between the factor and specific contributions over the entire backtest is -0.32. So, for this particular portfolio, our "specific" contribution is actually related to our factors implying that some of our factors are over-explaining the portfolio returns.

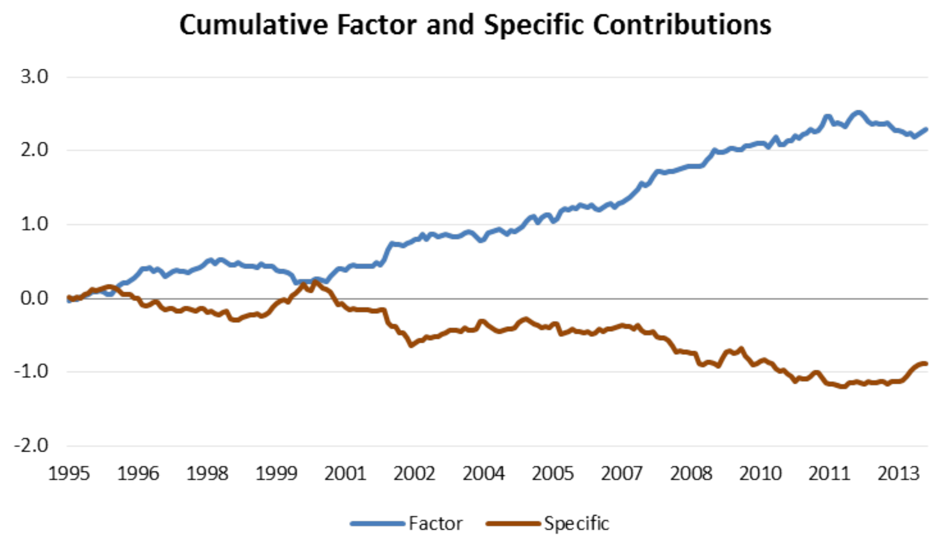


Figure 1: Cumulative factor and asset-specific contributions to return of the portfolio using the RM2 returns model for attribution.

This problem is not unique to this particular case. While we illustrated the problem by looking at a returns model that contained the same Growth factor used to construct the portfolio, varying forms of the problem can manifest themselves in nearly all combinations of portfolios and returns models. When the exact factors used to construct a portfolio are present in the attribution model, we typically see that the time-series of factor- and specific-return contributions are negatively correlated. This ultimately produces a factor contribution that is too large (and positive) and an asset-specific contribution that is too large (and negative). When factors related to, but not identical to, those used to construct the portfolio are used in the attribution model, we often find that the factor and specific contributions are positively correlated. Thus the attribution gives a factor contribution that is not large enough and a specific contribution that is positive and too large. Saxena and Stubbs (2013) analyze the effects of factor misalignment on risk estimation and Saxena and Stubbs (2015) consider the effects of misalignment in portfolio construction. In this paper, we provide computational results that illustrate the effects of varying degrees of alignment between the portfolio's alpha factors and the attribution's underlying factors. We also provide explanations as to why the correlation between factor and specific contributions is sometimes positive and sometimes negative.

In the remainder of the paper, we will discuss the cause of correlated factor and specific contributions, explain why correlated factor and specific contributions are a problem, and propose a performance-attribution methodology that corrects the contributions such that the resulting factor and specific contributions are uncorrelated. With the trend towards "smart beta", and factor-investing in general, this correlation is becoming more pervasive and accounting for it is critical in the generation of meaningful attributions of such factor-based strategies. The proposed adjusted attribution methodology can produce attributions that better illustrate that the return of the fund is indeed being driven primarily by factors that are known to exhibit the significant risk premia associated with smart-beta factors over long horizons. In our motivating example, the methodology we introduce in this paper reduces the correlation between period-level factor and specific contributions to near zero, and drives the annualized specific contribution to near zero while the contribution from the Growth factor remains large and highly significant.

2 Background in Linear Factor Models

Linear factor models are based on the assumption that asset returns can be modeled with a set of linear equations satisfying a set of properties as follows:

$$\begin{aligned} &r_t = X_{t-1} f_t + \varepsilon_t \\ &\text{corr}(f_{tk}, \varepsilon_{ti}) = 0 \text{ for all } i, k \\ &\text{corr}(\varepsilon_{ti}, \varepsilon_{tj}) = 0 \text{ for all } i \neq j. \end{aligned} \tag{1}$$

That is, the assumptions are that each asset's specific return, ε_{ti}, is uncorrelated to each factor return and that each asset's specific returns are uncorrelated to each other. As we saw in our motivating example, a portfolio's specific contribution and its factor contribution may very well be significantly correlated.

Assume that X_{t-1} is given and let the factor returns in (1) be determined by a cross-sectional weighted least-squares regression with weighting matrix W_t. Then the factor returns, f_t, are given as

$$f_t = (X_t^T W_t X_t)^{-1} X_t^T W_t r_t. \tag{2}$$

These are the returns of a set of factor mimicking portfolios (FMPs). The jth FMP can be written as

$$W_t X_t (X_t^T W_t X_t)^{-1} e_j \tag{3}$$

where e_j is the jth column of the identity matrix.

Now, we consider the attribution of a portfolio h. Dropping the time subscript from our returns model and taking the inner product with h, we can decompose the return of the portfolio in a given period as follows:

$$\begin{aligned} h^T r &= h^T(Xf + \varepsilon) \\ &= h^T Xf + h^T(r - Xf) \\ &= \underbrace{(h^T X(X^T WX)^{-1} X^T Wr)}_{\text{factor contribution}} + \underbrace{(h^T(I - X(X^T WX)^{-1} X^T W)r)}_{\text{specific contribution}} \end{aligned} \tag{4}$$

We can write the factor and specific contributions as the sum of the individual factor contributions as

$$\text{factor contribution}(h^T r) = \sum_i \underbrace{(h^T X_i)}_{\text{exposure to factor } i} \cdot \underbrace{(e_i^T (X^T W X)^{-1} X^T W r)}_{\text{return of factor } i} \tag{5}$$

$$\text{specific contribution}(h^T r) = [h^T r - \sum_i (h^T X_i) \cdot (e_i^T (X^T W X)^{-1} X^T W r)], \tag{6}$$

where X_i is the i-th column of X. The total factor contribution is the sum of each factor's individual contribution. The contribution of an individual factor is the product of the factor exposure and the return of the corresponding FMP. The asset-specific contribution corresponds to the return that cannot be explained by the factors, i.e., the specific contribution of return in a given period is the total portfolio return of the portfolio during the period less the factor contribution.

Let us consider how the portfolio returns decompose for a couple of very specific portfolio scenarios. These will help illustrate what is happening in our original example and motivate our solution.

Scenario 1

Let portfolio h be a pure FMP for factor j that is constructed with the same set of factors X and weights W that was used to construct the returns model, i.e., h is given by (3). Using our decomposition of portfolio returns defined in (4), we can write the returns of this particular h as

$$\begin{aligned} h^T r &= \underbrace{(e_j^T (X^T W X)^{-1} X^T W) X f}_{\text{factor contribution}} + \underbrace{[e_j^T (X^T W X)^{-1} X^T W r - (e_j^T (X^T W X)^{-1} X^T W) X f]}_{\text{specific contribution}} \\ &= e_j^T f + [0] \\ &= f_j \end{aligned} \tag{7}$$

So, the returns of the FMP for factor j are 100% explained by the factor portion of the model. This is exactly what portfolio managers would expect.

Scenario 2

Now, let portfolio h be a pure FMP for factor j that is constructed with the same set of factors, X, but with a different set of weights, $\tilde{W}$, than was used to construct the returns model. Then the decomposition of returns of this portfolio can be written as

$$\begin{aligned} h^T r &= (e_j^T (X^T \tilde{W} X)^{-1} X^T \tilde{W}) X f + [e_j^T (X^T \tilde{W} X)^{-1} X^T \tilde{W} r - (e_j^T (X^T \tilde{W} X)^{-1} X^T \tilde{W}) X f] \\ &= f_j + [\tilde{f}_j - f_j]. \end{aligned} \tag{8}$$

Though h is a pure FMP, its return attributable to the factors is equal to the factor return of the FMP constructed with the original weighting matrix W, not $\tilde{W}$. The "asset-specific" return is then equal to the difference between the two factor returns computed using different weighting matrices.

Now, consider what happens when $(\tilde{f}_j - f_j)$ is correlated with f_j. This is not an unlikely occurrence. Using different universes and/or weighting matrices to compute factor returns for the same signal is likely to produce two sets of factor returns that are correlated. Suppose that the two factor returns are multiples of each other, e.g., $f_j = 2 \cdot \tilde{f}_j$. In this extreme case, the correlation between the factor and specific return contributions for this portfolio is -1.0. The factor contribution to return is a factor of 2 too large, thus creating a significant negative specific contribution to balance the return decomposition.

We have just shown what can happen when using a factor-mimicking portfolio that was constructed with a different set of weights than those used to construct the returns model to perform the attribution. The point of these scenarios is to show that we can have two portfolios that both have identical exposures to the factors in a particular returns model and yet have considerably different returns and attributions of return. To illustrate this specific point in less contrived scenarios, we created a time-series of six FMP-like portfolios that have unit exposure to the Value factor and zero exposure to all other factors. Each of these portfolios is optimal with respect to the following strategy:

$$\begin{array}{llll} \text{minimize} & h^T Q h & & \\ \text{subject to:} & X_j^T h = 1 & & \text{Unit exposure to factor } j \\ & X_i^T h = 0 & \forall i \neq j & \text{Zero exposure to all other factors} \\ & h \in \mathcal{C} & & h \text{ must satisfy user constraints} \end{array} \tag{FMP Strategy}$$

where $\mathcal{C}$ corresponds to a set of additional constraints. The six portfolios are labeled Pure FMP, Pure FMP (MCAP), Pure FMP (sqrt(MCAP)), Asset Bounds, Long-Only, and TO Limit. The first three optimize (FMP Strategy) with no user constraints and Q set to a diagonal matrix of asset-specific variances, reciprocal of market-caps, and reciprocal of square-root of market-caps, respectively. The last three portfolios all use a factor risk model as Q. The Asset Bounds strategy limited the holdings to be within $\pm 3\%$. The TO Limit strategy restricted the turnover to be no more than 1% each day. The Long-Only strategy is a little different. There we maximized exposure to the factor subject to the constraint, $h \geq -b$ on asset holdings, and a 3% risk constraint. The purpose of this strategy is to simulate the active holdings of a long-only strategy.

We then looked at the correlation between the factor and specific contributions of these portfolios when analyzed with three different returns models. The first returns model is the aforementioned RM2; the second and third use the same factors, but use square-root of market-cap (Pure FMP sqrt(MCAP)) and inverse of specific variance (Pure FMP) as the weights as opposed to the market-cap weights used in RM2. The results are summarized in Table 3. Notice that most of the correlations are large in magnitude. They were computed

	RM2 Factor	Pure FMP (sqrt(MCAP))	Pure FMP
Pure FMP	-0.52	-0.23	–
Pure FMP (MCAP)	–	0.35	-0.72
Pure FMP (sqrt(MCAP))	-0.81	–	-0.87
Asset Bounds	-0.25	-0.07	-0.52
Long-Only	-0.61	-0.22	-0.75
TO Limit	-0.64	-0.32	-0.66

Table 3: Correlations between specific contributions and factor contributions for various combinations of portfolios and asset-return models.

over a sample of more than 4700 days, so even the smallest value of -0.07 is statistically significant.

The root cause of the unacceptably large correlations between factor and specific contributions is that asset exposures may not accurately represent sensitivities to the factor returns. As illustrated in Table 3, portfolios with the exact same factor exposures do not necessarily behave the same. Since the factor returns are returns of a set of factor mimicking portfolios, we do know that the exposure of at least a set of portfolios, the FMPs used to generate the returns, have exposures that equal their sensitivities. Therefore, there will be no specific return for an FMP. Even if we try to create a portfolio whose active holdings equal those of a combination of FMPs, constraints generally force our active holdings to be quite different. Therefore, we need for any portfolio that has unit exposure to a factor and no exposure to other factors to behave similarly to the FMP that drives the factor return. If this happen for all factors, then the correlation between factor and specific return contributions will be near zero. When a portfolio with unit exposure to a factor and no exposure to any other factor fails to behave almost identically to the FMP, a correlation between factor and specific contributions is induced and we must explicitly account for the correlation in the attribution.

We can now see when we are likely to have negative or positive correlations between the factor and specific contributions. In the hypothetical Scenario 2, we saw a negative correlation when we assumed that the return of the factor used to perform the attribution was twice the return of the factor used to construct the portfolio. A similar case exists when we use the exact same factor to explain returns that we used to construct the portfolio in a realistic scenario. Suppose that we construct a long-only portfolio that tries to maximize exposure to a value factor. The active holdings of this portfolio are likely be to be similar to the "Long-Only" FMP constructed above. The construction of this "Long-Only" FMP is identical to the construction of the pure FMP for the same factor except that it has the additional constraint that the holdings in the FMP cannot short more than negative of the benchmark weights. With this restriction, we expect the "Long-Only" FMP to be highly correlated to the pure FMP, but whose returns are generally lower because of the additional constraint. This leads to an attribution that tries to explain our active holdings, which look like the "Long-Only" FMP, with the pure FMP of the same factor. Therefore, the large correlation between the returns of these two FMPs and the likelihood that the returns of the constrained FMP are a fraction of those of the pure FMP leads to a negative correlation between the factor and specific contributions of our active portfolio. The reverse situation occurs when we tilt on a factor that is not in the returns model used for attribution. Suppose that the factor our portfolio tilts on is slightly related to a factor in the attribution model. The return of our portfolio will be partially explained by the related factor in the attribution model, but the majority of the return will be a specific contribution. Because the factor in the attribution model is related to the tilt factor, its return is likely positively correlated with the return of the active portfolio and thus the specific contribution. This scenario leads to a positive correlation between the factor and specific contributions.

3 Solution: Adjusted Attribution

We have demonstrated that a violation of the uncorrelated factor and specific returns assumption of our returns model (at least when aggregated at the portfolio level) is leads to unintuitive, and even erroneous, attribution results in many cases. We will now propose a methodology to correct for this correlation between factor and specific contributions, by forcing it to be zero. Let ε^t be the specific contributions in period t for Scenario 2 described previously and consider the time-series model:

$$\varepsilon^t = \tilde{f}_j^t - f_j^t = \beta \cdot f_j^t + u^t. \tag{9}$$

That is, we model the specific contributions of (8) with the factor contributions. The OLS solution to this time-series regression under the assumption that $f_j = 2 \cdot \tilde{f}_j$ is $\beta = -0.5$. So, using our model of specific contributions given in (9), we can modify the attribution equation of our Scenario 2 portfolio to be

$$h^T r = f_j + \varepsilon = (1+\beta) f_j + u. \tag{10}$$

Our asset-specific returns are now given by u. And, under our assumption that $f_j = 2 \cdot \tilde{f}_j$, it must be the case that $u = 0$ because the factors can explain the entire portfolio. Additionally, the factor contribution is now half of its original value due to a reduction in the exposure to the factor from 1.0 to 0.5.

In a more general scenario, the time-series model of our specific contributions is given by:[1]

$$\varepsilon^t = \sum_i \beta_i \cdot ((h^t)^T X_i^t f_i^t) + u^t. \tag{11}$$

An OLS estimate of β is then used to adjust the attribution of returns according to

$$\begin{aligned} h^T r &= \sum_i h^T X_i f_i + \varepsilon \\ &= \sum_i (1+\beta_i) h^T X_i f_i + u. \end{aligned} \tag{12}$$

This is the basis of our proposed adjusted attribution. We explain the time-series of specific contributions with factor contributions and reallocate this explainable portion back into the factors. The resulting adjusted factor and specific contributions are then uncorrelated by construction.

In (12), the exposures are adjusted by an amount that is relative to the standard factor exposure calculation. Instead of using the factor contributions as the independent variables as we did in (11), we could use factor returns as the independent variables. Such a model of the specific contributions leads to an absolute adjustment of the factor exposures:

$$\begin{aligned} h^T r &= \sum_i h^T X_i f_i + \varepsilon \\ &= \sum_i (h^T X_i + \beta_i) f_i + u. \end{aligned} \tag{13}$$

In our experience, exposures are typically off by a relative amount rather than an absolute amount. In Appendix B, we illustrate how a relative adjustment can also be more appropriate for cases where factor exposures are changing through time. For these reasons, we prefer the relative adjustment to the absolute adjustment. Nevertheless, an absolute adjustment may work better in some situations and the remainder of this paper is relevant to the absolute form of adjusted attribution as well.

Up to this point, we have primarily looked at motivational examples and solutions. Before looking at results on realistic portfolios, we consider the issue of potentially over-fitting the adjustment regression. Over-fitting would allow factors to account for some of the noise in the portfolio returns rather than accounting for only factor contributions that exist in the portfolio residual returns. In order to avoid this, we turn our attention to the procedure used to estimate the time-series model in (11). Using contributions as opposed to factor returns (relative adjustment versus absolute adjustment) has advantages with regard to this issue in addition to previously mentioned advantages. Since we are making relative adjustments to the exposures, the adjustment procedure will not suddenly allow a factor to explain a large portion of returns when the unadjusted factor exposure was near zero. If the exposure was near zero prior to adjustment, it will remain near zero after the adjustment. In this sense, using adjusted attribution with contributions behaves like a Bayesian method with the standard exposures as the prior.

To avoid having numerical issues with time-series regressions that contain a large number of correlated factor contributions, we use a variable selection scheme to select a reduced set of factors. Rather than use an OLS estimate of β, we use only a set of statistically significant betas computed over a subset of factors to adjust the factor and specific contributions of the portfolio. Betas for all other factors will be set to zero.

Specifically, we use a heuristic variable selection scheme to select the independent variables (factor contributions) of (11) based on their statistical significance as measured by their p-value. We use an iterative regression scheme that starts with all variables present. After each iteration, we remove the one variable having the greatest p-value if that p-value is greater than the specified tolerance 0.02. If none of the p-values exceed the tolerance then the iterative procedure of removing factors is stopped. Thereafter, we employ a reentry procedure where we consider rejected variables' reentry in the regression one at a time. A variable can reenter the regression only if its entry does not increase the p-value of any variable (including itself) above the tolerance. After the reentry trials, a final regression is run with the selected variables to compute the final estimate of β. The resulting β is then used to adjust factor contributions as prescribed by (12).

Though not essential, we can generally further improve the intuition behind the solution by pre-selecting a subset of factors to be candidates for adjustment. In order for a factor to have any real impact on the adjusted attribution, the exposure to the factor should be relatively large. The factors that are likely to have large exposures are those factors that are being intentionally bet on such as alpha factors. So, for a typical portfolio where we are performing attribution on the active holdings, we generally select the style factors (or a subset of) to be the initial candidate set of factors.

4 Examples

Now, we will look at results for many different portfolios exhibiting varying degrees of correlated factor and specific contributions when using standard factor-based attribution. In all of the computational results shown here, we restrict our candidate set of factor exposures to be adjusted to the style factors. For the strategies used in our tests, these are the only factors likely to have a large contribution to returns due to the constraints applied to the portfolio construction strategy.

First, we will return to the motivating example that we considered in the introduction and look at the adjusted attribution results when RM2 is used to explain returns. The regression statistics for the adjustment regression are summarized in Table 4 and Table 5. Two factors ended up in the final list of statistically significant factors: Value and Growth. The β for Value was -0.65 and for Growth was -0.39. This means that our exposures to Value and Growth were too large and should be only 35% and 61% of their original values, respectively. The average active exposures before and after attribution are in Table 6. In all tables below, the shaded cells indicate those that notably changed from the standard attribution.

Regression Statistics	
Multiple R Square	0.41
R Square	0.17
Adjusted R Square	0.16
Standard Error	0.0064
Observations	224

Table 4: Regression statistics for calculation of betas

	Coefficient	Standard Error	T-Stat	P-Value
Value	-0.65	0.21	-3.07	0.00
Growth	-0.39	0.06	-6.39	0.00

Table 5: Beta coefficients and their significance determined by the time-series regression.

	Active Exposures	
Attribution	Standard	Adjusted
Growth	1.010	0.616
Market Sensitivity	0.012	0.012
Momentum	0.017	0.017
Size	-0.114	-0.114
Value	-0.205	-0.072

Table 6: Active portfolio exposures to style factors using standard and adjusted attribution.

In Table 7, we compare the attribution results using adjusted performance attribution (PA) to those results originally shown in Table 2 that used standard PA methodology. Comparing the last two columns of Table 7, we see that the correlation between the adjusted factor and specific contributions moved from -0.32 to 0.09. Furthermore, the annualized factor contribution decreased from 2.35% to 1.50% and the annualized specific contribution increased from -0.88% to -0.03%. It is important to note that in the proposed methodology, we consider the relationship between period factor and asset-specific contributions through time. It is possible to have cumulative specific contributions even if the period returns are Gaussian noise. So, while the adjusted attribution significantly reduced the annualized specific contribution in our example, it is not necessary for this to be the case in general. Our goal is to eliminate the relationship between factor and specific contributions.

Returns Model	RM1	RM2	RM2 (Adjusted PA)
Portfolio Return	10.72%	10.72%	10.72%
Benchmark Return	9.25%	9.25%	9.25%
Active Return	1.47%	1.47%	1.47%
Specific Contribution (SC)	1.65%	-0.88%	-0.03%
Factor Contribution (FC)	-0.18%	2.35%	1.50%
Industry	-0.14%	-0.15%	-0.15%
Styles	-0.04%	2.50%	1.65%
Growth	–	2.65%	1.60%
Market Sensitivity	-0.10%	-0.09%	-0.09%
Momentum	0.10%	0.09%	0.09%
Size	0.13%	0.15%	0.15%
Value	-0.17%	-0.29%	-0.10%
FC Volatility	1.13%	2.37%	1.53%
SC Volatility	2.66%	2.41%	2.21%
SC T-Stat	2.67	-1.58	-0.06
FC T-Stat	-0.69	4.29	4.22
SC-FC Correlation	-0.09	-0.32	0.09

Table 7: Summary of performance attribution results. The values are the annualized returns attributable to each factor or to a group of factors.

To further validate the proposed adjusted PA methodology, we ran a variety of backtests and used different returns models with and without the adjustment in the attributions of the portfolios. We start with twelve backtests (combination of three benchmarks and four alphas) using US market data. Our backtests used the Russell 1000, Russell 2000, and Russell 3000 indexes as benchmarks as well as the investable universes. We used Growth, Value, Momentum, and an equal-weighted combination of Value and Momentum as four different expected returns in our backtests. The optimization strategy was as described in (Strategy). All backtests started with cash in January 1995 and were rebalanced monthly through October 2013. All attributions used the RM2 returns model whose factors are listed in Table 1. In our computational results, we show the results for two factor-based attributions: the standard factor-based attribution methodology labeled as *Prior* and adjusted factor-based attribution using the same model labeled as *Adjusted*.

All twelve cases in this first set of tests fall in the perfect alignment case where the alpha factors used as expected returns in the construction of the strategy are also present in the returns model used to perform the attribution. All twelve cases have very large negative correlations between the factor and specific contributions as shown in Figure 2. For each of these cases, the adjusted attribution reduced this correlation to near zero. If we had used an OLS solution of (11) with all factors, the adjusted correlation would be zero. But, using only the set of significant factors is more intuitive and nearly eliminated the issue. The annualized contributions for the prior and adjusted cases that used the Russell 1000 Index as the benchmark are shown in Figure 3. Notice that most of the factor contribution adjustments took place in the alpha factors and the specific contributions were adjusted accordingly. That is, if Value was used as the expected return, then the factor contribution of Value was adjusted downwards and the specific contribution was adjusted by the opposite amount. In Figure 3d, where the expected return is a combination of the Value and Momentum exposures, we can see that the Value contribution was significantly adjusted downwards, but the Momentum contribution was unchanged. In this case, the t-stat for the Value contribution was -10.54 while the t-stat for the Momentum contribution was only -1.52. Having a large p-value of

13%, the Momentum contribution did not meet our significance threshold of 2%, thus the Momentum contribution was not adjusted. This is perhaps not surprising given the relatively small adjustment made in the Momentum-only alpha case (Figure 3b) compared to the large adjustment made in the Value-only alpha case (Figure 3c).

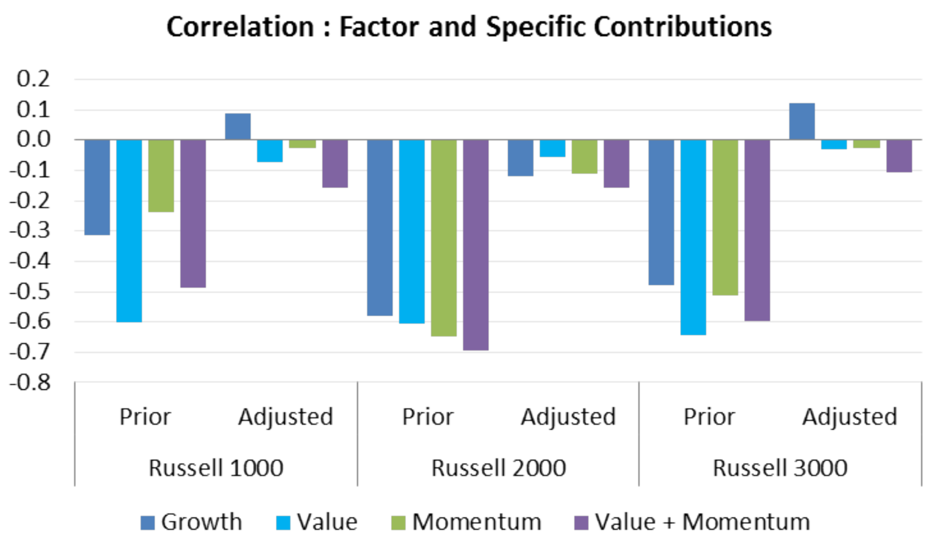
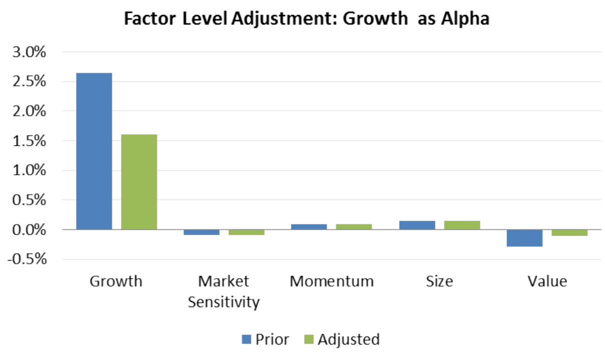


Figure 2: Correlation of Factor and Specific Contributions: Aligned case and 12 combinations of expected returns and benchmarks.

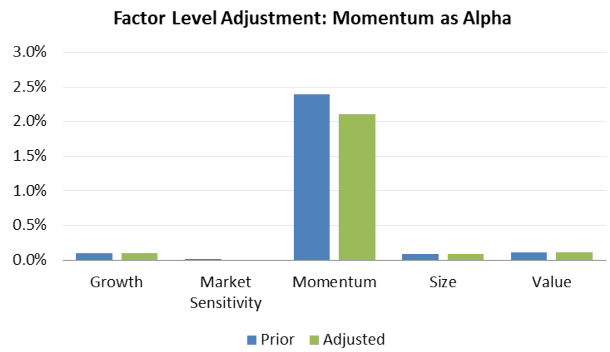
Figure 4 shows the split of total active returns between factor and specific contributions. In all cases, the annualized specific contribution increased when using adjusted attribution. When using standard attribution, the exposures to the alpha factors were overestimated thus causing a downward-biased specific contribution.

We have only discussed attribution thus far, but the decomposition of realized risk will also change when using adjusted attribution. In general, a greater portion of the realized risk should be attributable to factors and less to asset-specific bets. However, the absolute volatility of the factor contributions may be reduced when using adjusted attribution due to a reduction in exposures. As shown in Figure 5, the volatilities of both the factor and specific contributions were reduced in all twelve cases.

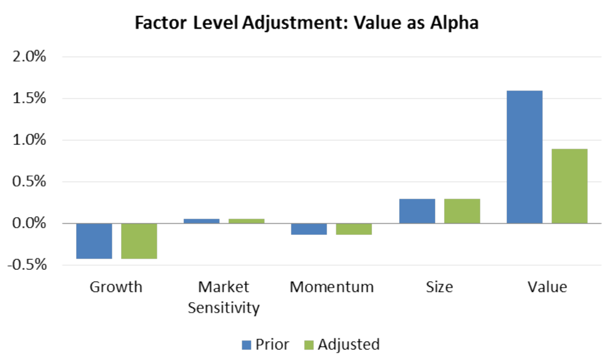
Next, we consider the partially misaligned case where the exact definition of alpha used to construct the portfolio is not in the returns model, but a related factor is. We ran three backtests under this scenario with Estimated E/P as the alpha factor and the same three Russell indexes used in prior experiments as the benchmarks. The same strategy and risk model were used to construct the portfolios. In Figure 6, we can see that the correlation between factor and specific contributions prior to adjustment is positive in each of the three cases. This is the opposite of what we saw in the aligned case. While the adjustment regression is able to explain a portion of specific contributions, it is not able to explain the portion that could potentially be explained by Estimated E/P as opposed to the B/P as is used in the returns model. In Figure 7a, we can see that the specific contribution was positive in each case prior to adjustment. Again, this is the opposite of what we saw in the perfectly aligned case. The volatility of the specific contributions is reduced when using adjusted PA



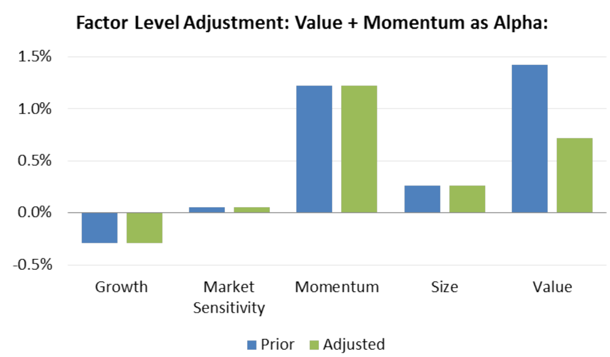
(a) Style Contributions: Growth



(b) Style Contributions: Momentum

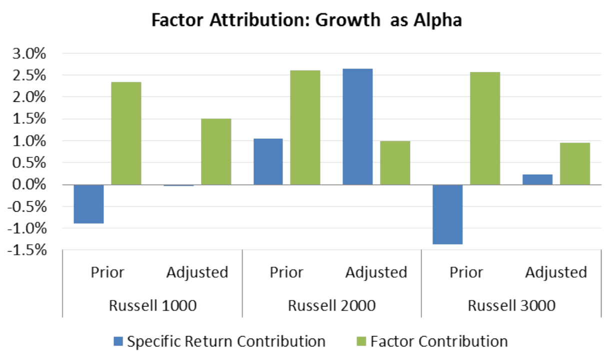


(c) Style Contributions: Value

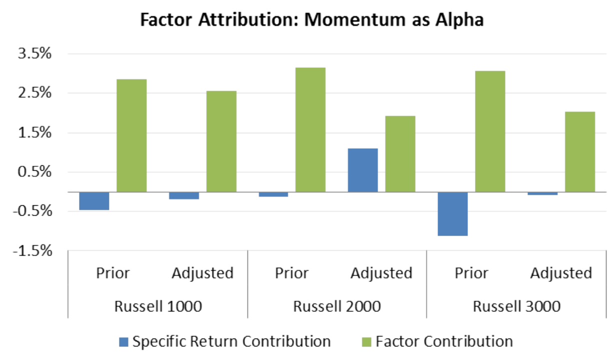


(d) Style Contributions: Value + Momentum

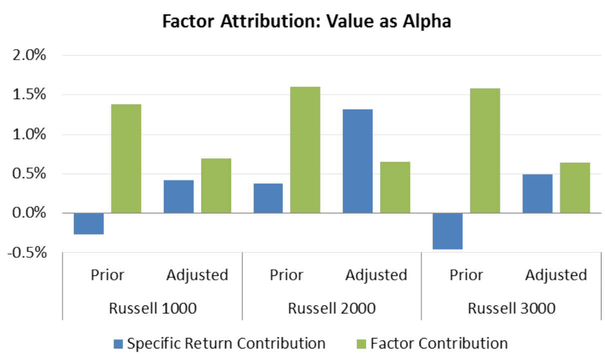
Figure 3: Style Contributions With and Without Adjustment – Aligned Returns Model. These results are for the case where we used the Russell 1000 Index as the benchmark and investable universe.



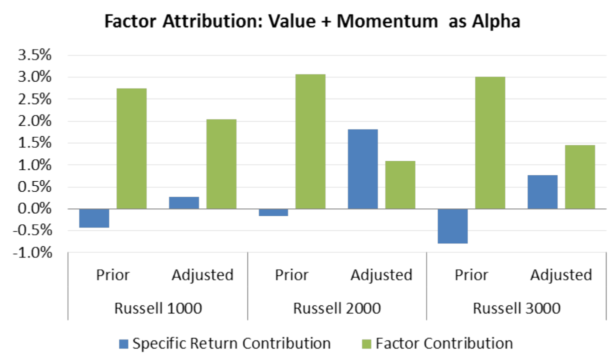
(a) Factor and Specific Contributions: Growth



(b) Factor and Specific Contributions: Momentum

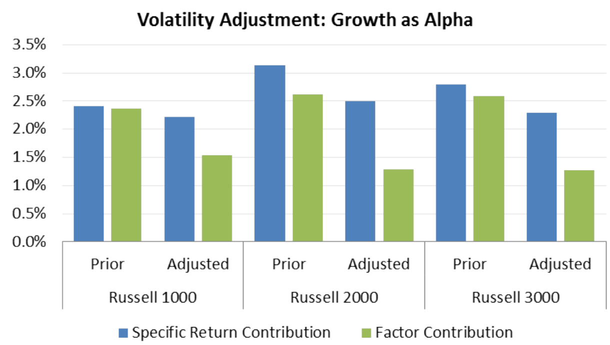


(c) Factor and Specific Contributions: Value

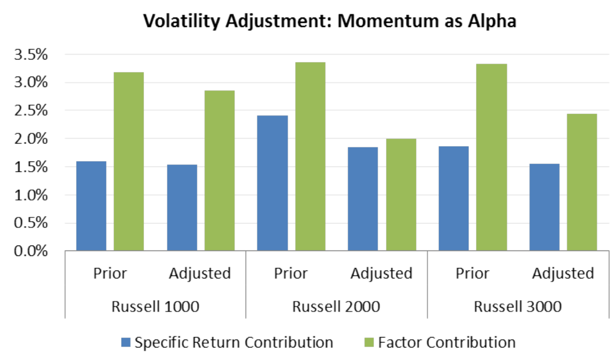


(d) Factor and Specific Contributions: Value + Momentum

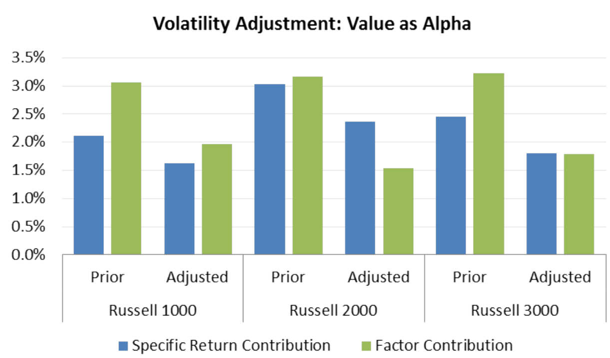
Figure 4: Performance Attribution – Aligned Returns Model



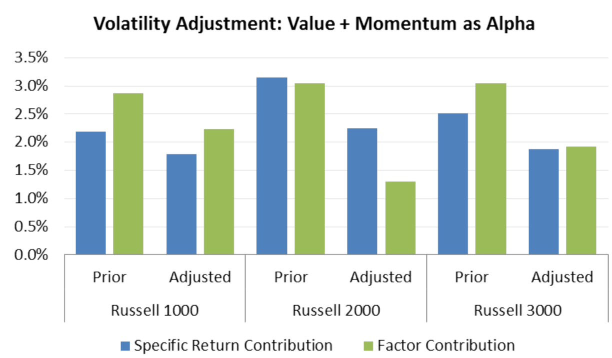
(a) Factor and Specific Contribution Volatility: Growth



(b) Factor and Specific Contribution Volatility: Momentum



(c) Factor and Specific Contribution Volatility: Value



(d) Factor and Specific Contribution Volatility: Value + Momentum

Figure 5: Volatility Adjustment – Aligned Returns Model

as seen in Figure 7b, but the reduction is not nearly as large as it was in the completely aligned case.

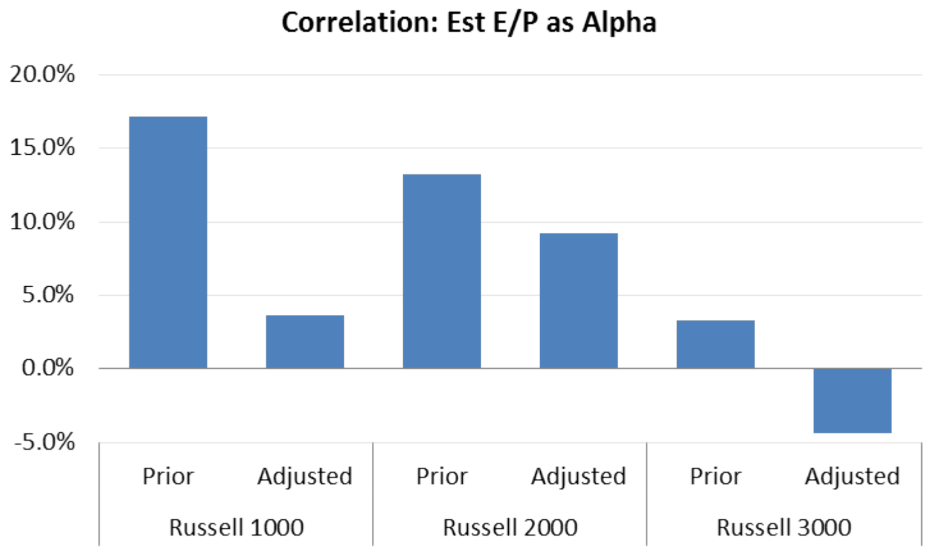
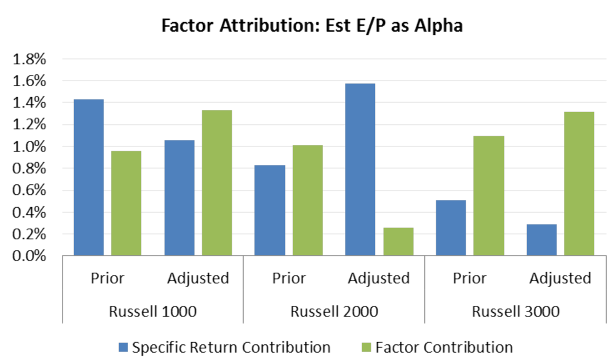
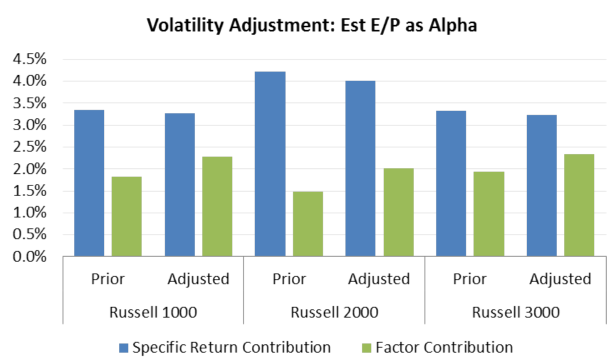


Figure 6: Correlation Adjustment: Partially Misaligned Case



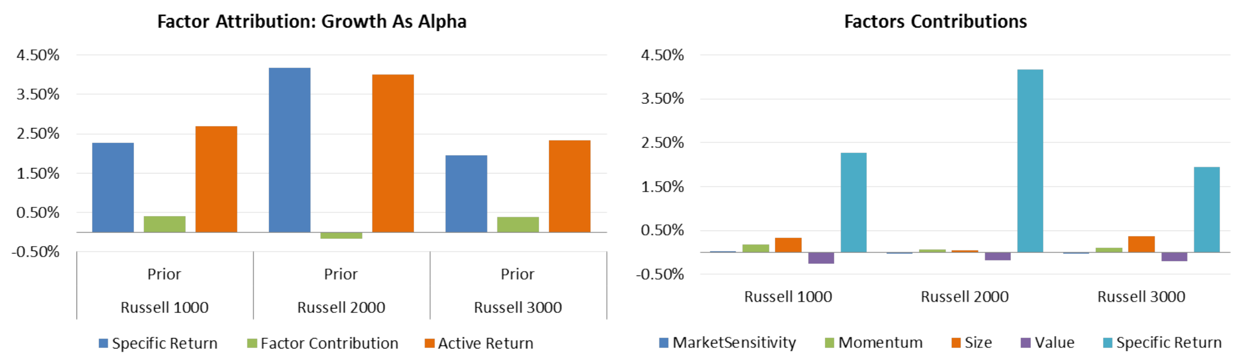
(a) Factor and Specific Contribution: Estimated E/P



(b) Volatility Adjustment: Estimated E/P

Figure 7: Performance Attribution – Partially Misaligned Returns Model

Thus far, we have shown that when we use factor-based attribution with a returns model that contains either similar factors or the exact factors used to construct the portfolio, adjusted attribution is needed to correct for the correlation between factor and specific contributions. One might wonder whether omitting all factors that are similar to the alpha factor(s) would eliminate this problem and produce better attribution results. In this last set of tests, we consider such a completely misaligned case where the alpha factor and similar factors are missing from the returns model. We ran three more backtests here where we used the Growth factor as our expected returns and used a risk model based on the RM1 returns model (that did not contain the Growth factor). This is the same scenario that we used as our motivating example in the introduction. The results are shown in Figure 8a. Because no factor related to the alpha factor is present in the returns model, most of the active return is attributed to asset-specific bets. In Figure 8b, contributions from the individual styles are plotted alongside the specific contribution to further show that factor contributions to each factor were small. So, while the issues related to correlated factor and specific contributions may not be present, the attribution does not show the true source of skill. And as we saw in the introduction, the specific contribution in such a scenario may not be statistically significant while a factor contribution would be.



(a) Factor and Specific Contribution: Growth

(b) Style Contributions: Growth

Figure 8: Performance Attribution – Completely Misaligned Returns Model

To illustrate this point, we look at one case in which the alpha factor is completely missing from the returns model. The results are summarized in Table 8. We used Estimated E/P as the alpha factor and the returns model had only four styles: Growth, Market Sensitivity, Momentum, and Size. In the "Prior" column, one can see the contributions of the alpha factor are attributed to specific contributions. No adjustment regression is run in this case. Even though alpha factor contributions are mixed with genuine specific contributions there is no detectable skill in the specific contributions as the t-stat of specific contribution is insignificant. In the "Adjusted" column, the attribution was run with a returns model that has the alpha factor and adjusted PA methodology. Now, having identified the alpha factor's contributions, the specific contributions are reduced appropriately and their t-stats stay insignificant. However, the t-stat for the factor contributions improved significantly.

	Prior	Adjusted	
Factor Contributions (FC)	0.53%	1.32%	Up
Specific Contributions (SC)	1.09%	0.29%	Down
Total Active Return	1.61%	1.61%	Same
T-stat(SC)	1.30	0.38	Stays Insignificant
T-stat(FC)	1.41	2.43	Becomes Significant

Table 8: Adjustment Regression with Missing Factor in the Returns Model

5 Conclusions

Explaining portfolio returns with factors is advantageous for several reasons. First, it generally supports the story of any portfolio based on factor investing. Perhaps more importantly, it improves the statistical significance of outperformance. Factor mimicking portfolios associated with alpha factors often have very significant and persistent returns, so explaining more of the portfolio with these factors increases the portion of the return explainable by a persistent source of return. Most portfolios contain "noise" due to constraints and other frictions that will show up as a specific contribution (see Clarke et al., 2005). If a would-be contribution from a factor missing from the returns model is mixed with the specific contribution, the statistical significance of the mixture contribution will generally be less than that of the would-be factor contribution. In other words, attributing the return to a factor that is truly driving the returns of the portfolio allows us to separate the signal from the noise, thus reducing the contribution from noise and increasing the probability that the portfolio return will be persistent. But, introducing the portfolio-driving factor into the attribution can introduce correlated factor and specific contributions.

Having correlated factor and asset-specific portfolio contributions violates one of the basic assumptions of a factor model of asset returns and thus introduces error into factor-based performance attribution results. This error can be found to some extent when performing factor-based attribution with nearly every real portfolio and any returns models. We proposed a solution of adjusting the factor-based performance attribution methodology to account for the correlation between the factor and asset-specific contributions of return that are computed with standard factor-based PA. In essence, the proposed attribution is a combination of factor-based attribution and style analysis (see Sharpe, 1988). Here, the "styles" are the most significant factor contributions. And, because the same factor contributions are already present in the attribution, we distribute the asset-specific contributions that can be explained by the factors back into contributions to the factors rather than separately accounting for the explanation of specific contributions.

The adjusted attribution introduced in this paper can have a significant and positive impact on attribution results. And perhaps the proposal is natural. When discussing the assumptions of a linear factor model, William Sharpe (see Sharpe, 2015) makes the following statements (with changes in notation to match that of this paper):

> *First, the residual return ε_i is assumed to be uncorrelated with each of the factors: corr$(\varepsilon_i, f_j) = 0$ for every i, j. This is not as restrictive as it may seem. Consider, for example, a case in which the residual return is correlated with factor 1. By adjusting the factor exposure X_{i1} appropriately, the correlation of the residual with the factor can be made to equal zero. Moreover, this can be done for every factor. In fact, in simple settings using historic data, multiple regression procedures can be used to find a set of factor exposures X_{ij} that will give residual returns that are uncorrelated with each of the factors. Why? Because standard linear multiple regression methods select slope coefficients (here, the $X'_{ij}s$) that minimize the variance of the residual (here ε_i). But this will insure that the residual is uncorrelated with each of the independent variables (here, the f_j's), since the removal of any such correlation by changing one or more X_{ij}'s will reduce the variance of the residual.*

So, following Sharpe's statements, we can indeed drive the correlation between asset-specific and factor contributions to zero. The proposed approach that uses only statistically significant adjustments to the original exposures does so without introducing noise and keeps the analysis parsimonious.

The methodology introduced here largely corrects for the problem we set out to address – that of erroneous factor-based attribution results caused by correlated factor and specific return contributions. The methodology can be further enhanced to account for portfolios whose exposures change significantly through time. We have tested dynamic betas using a moving window on many portfolios using daily returns and found the β values to be sufficiently stable through time to warrant the use of static values of β. That is, even if the exposures of the portfolio are changing through time, we find that the relative adjustments to the exposures given by $(1+\beta)$ are stable. Nevertheless, we do not see the need to restrict the methodology in this way. There will be portfolios for which dynamic values of β are more appropriate and portfolio managers may need to consider a more dynamic approach such as a state space model or dynamic conditional beta (see Engle, 2012) if the benefit warrants the cost of the additional complexity. We have found a relatively simple moving window regression to compute β to be adequate for the factor-based portfolios that we have investigated.

Notes

[1]We do not add a constant term to the adjusted attribution regression. If we did, then the constant term would still go into the adjusted specific contribution. If a factor contribution contains a constant return, we would rather the factor explain the contribution than have it be a specific contribution. By not including a constant term, we insure that the factor explains the constant if possible. However, we ensure that the correlation between the adjusted factor and specific contributions is zero only if we include a constant term in the regression or if the mean of the adjusted specific contributions is zero.

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Kartik Sivaramakrishnan and Robert A. Stubbs. Improving the investment process with a custom risk model: A case study with the GLER model. *The Journal of Investing*, 22(4): 129–147, Winter 2013.

A Factor Definitions and Factor Returns

The five styles were Growth, Momentum, Market Sensitivity, Size, and Value. Our style factor exposures are defined as follows: Growth is the plowback ratio times return-on-equity, Momentum is the cumulative return over past year excluding most recent month, Market Sensitivity is the beta of time-series regression of an asset's return against the market return using six month daily data, Size is the natural logarithm of market capitalization, and finally Value is Book-to-price (B/P). The style factors were normalized over the estimation universe by subtracting the market-cap weighted mean from the raw descriptor and dividing by the equal-weighted deviation from zero. The ten sectors are defined by GICS. The factor returns were estimated using daily cross-sectional regressions weighted using stocks' market capitalization.

B Absolute Adjustment of Factor Exposures

Instead of using the factor contributions as our independent variables as we did in (11), we could use factor returns as the independent variables:

$$\varepsilon^t = \sum_i \beta_i f_i^t + u^t. \tag{14}$$

The adjusted attribution of a portfolio h is then written as:

$$\begin{aligned} h^T r &= \sum_i h^T X_i f_i + \varepsilon \\ &= \sum_i (h^T X_i + \beta_i) f_i + u. \end{aligned} \tag{15}$$

In this form of adjusted regression, the factor exposures are adjusted by the absolute amount β_i as opposed to the relative amount present in (12).

In our experience, exposures are generally off by a relative amount as opposed to an absolute amount. Additionally, the relative adjustment of exposures may be more capable of handling cases where factor exposures are changing significantly through time. Suppose that our portfolio is a multiple of an FMP that was determined with weights $\widetilde{W}$ and that this multiple oscillates through time in a cyclical nature. We use the sine function to proxy cyclical factor exposures. Specifically, let the exposure at time t be $sin(t)$ and assume that $f_j = 2 \cdot \tilde{f}_j$ as in Scenario 2. Then the return of the portfolio in period t is decomposed as

$$(h^t)^T r^t = \underbrace{sin(t) \cdot f_j^t}_{\text{factor contribution}} + \underbrace{-{}^1\!/\!_2 \cdot sin(t) \cdot f_j^t}_{\text{specific contribution}}. \tag{16}$$

The correlation between the specific and factor contributions is still -1.0. Now, we try to explain the specific contributions with the returns model via the time-series regression

$$-{}^1\!/\!_2 \cdot sin(t) \cdot f_j^t = f_j^t \cdot \beta + u. \tag{17}$$

The solution to this regression is $\beta = 0$ which means that no adjustment would occur. The adjusted attribution procedure would fail to correct for the correlation between factor and specific contributions. The reason for this is that the factor exposure is both over- and under-estimated through time by equal amounts such that the average adjustment is zero.

Rather than try to explain specific contributions with factor returns, we propose to explain the specific contributions with factor *contributions*. In this simple example, if we use factor contributions rather than factor returns as the independent variables, our adjusting time-series regression becomes

$$-{}^1\!/\!_2 \cdot sin(t) \cdot f_j^t = sin(t) f_j^t \cdot \beta + u. \tag{18}$$

The OLS solution is $\beta = -{}^1\!/\!_2$, forcing $u = 0$, and our adjusted exposure to be ${}^1\!/\!_2$. So, by using factor contributions rather than factor returns, we are able to correct for correlated factor and specific contributions even in this example of dynamically changing exposures.

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