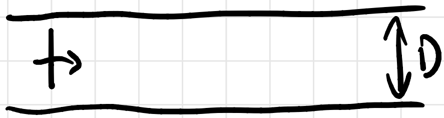
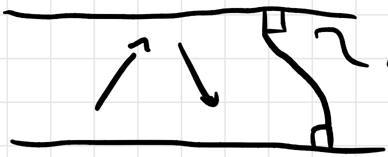
4.0 Higher Order Duct Acoustics

Previously: 1-D propagation, low freq approach



$\lambda < D/2$

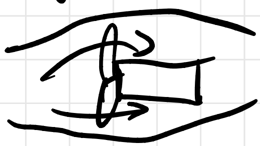
To higher freq's



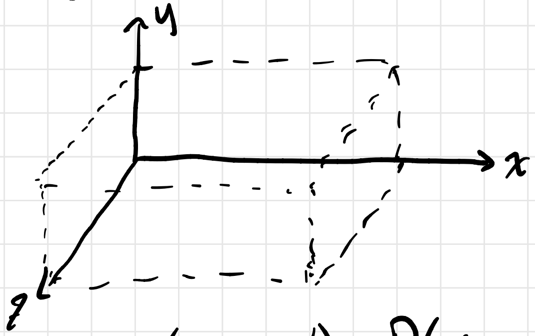
We will consider:

- Hard Walled, Constant cross-section ducts
- arbitrarily high freq

- HVAC Ducts
  - automotive
- Pipe Systems
- Jet Engines
- Refrigerating Compressors



Higher Order Modes in constant cross-section ducts



$$p(x,y,z,t) = P(x,y,z)\, e^{j\omega t}$$

$$P(x,y,z) = \underbrace{X_n(x)}_{\text{Axial}}\, \underbrace{\psi_n(y,z)}_{\text{Transverse}}$$

Convenient to make ψ_n's an orthonormal set

$$\frac{1}{A}\int \psi_n(y,z)\, \psi_{n'}(y,z)\, dS = \delta_{nn'}$$

integer's

Duct Area

$\delta_{nn'} = 1$ when $n' = n$

$\delta_{nn'} = 0$ otherwise

ψ_n's form a complete set

- a sum of these nodes can represent exactly any single valued function of y and z within the duct

$$\underbrace{f(y,z)}_{\text{pressure or velocity}} = \sum_n a_n \psi_n(y,z)$$

Single valued function: one value at any given point

ex. Square wave is not single valued

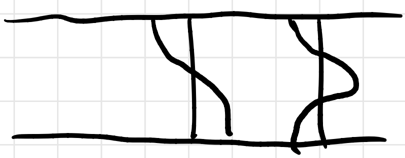
Physical Sound Pressure and particle velocity should be single valued

Express any transverse distribution within the duct as a sum of modes

$$f(y,z) = \sum_n a_n \psi_n(y,z)$$

$$a_n = \frac{1}{A} \int f(x,y)\, \psi_n(y,z)\, ds$$

↑ Transverse Distribution of the Sound field



$$\frac{\partial^2 X_n}{\partial x^2} + (k^2 - \alpha_n^2) X_n = 0 \qquad k_{x_n}^2 = k^2 - \alpha_n^2$$

axial wave number

$$\frac{d^2 X_n}{dx^2} + k_{x_n}^2 X_n = 0$$

$$X_n \propto e^{\pm j k_{x_n} x} \qquad X_n = \left[a_n e^{-j k_{x_n} x} + b_n e^{j k_{x_n} x}\right] e^{j\omega t}$$

mode n

$$P_n(x,y,z) = \psi_n(y,z)\left[a_n e^{-j k_{x_n} x} + b_n e^{j k_{x_n} x}\right]$$

* true regardless of shape of duct
so long as cross-section is constant
- also applies to locally reacting duct walls

Orthonormal $\int_S \psi_n \psi_{n'} \, dS = 0, \; n \neq n'$
$= A, \; n = n'$

forward going mode

$p_n(x,y,z) = \psi_n(y,z) a_n e^{-jk_{x_n} x}$

$\underbrace{k_{x_n}^2 = k^2 - \alpha_n^2}_{\text{positive or negative}}$ ~ discrete set of real positive numbers

k_{x_n} is either real or imaginary

cut-on if k_{x_n} is real : propagating mode

cut-off if k_{x_n} is imaginary : non-propagating mode

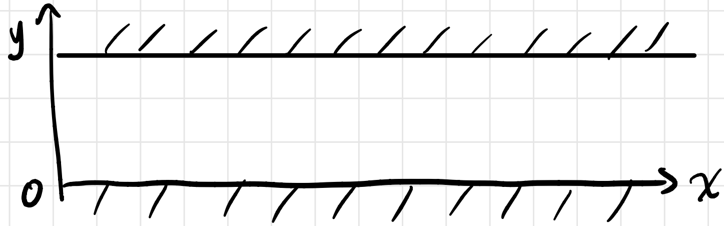
$\psi_n(y,z)$ solution of $\left(\frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\psi_n + \alpha_n^2 \psi_n = 0$

$\psi_n(y,z) = Y(y) Z(z)$

$$\left.\begin{aligned} \frac{\partial^2 Y}{\partial y^2} + k_y^2 Y = 0 \\ \frac{\partial^2 Z}{\partial z^2} + k_z Z = 0 \end{aligned}\right]$$

Subject to the condition

$k_{y_m}^2 + k_{z_n}^2 = \underbrace{\alpha_{mn}^2}_{\text{Two integers}}$



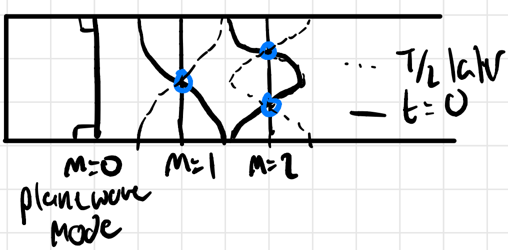
mode must satisfy hard-walled boundary condition

$y=0$ & $y=L_y$

$$\frac{d\Psi_n}{dy} = 0 \text{ at}$$

$Y(y) \propto \cos(k_y y)$ $\quad k_y = \frac{m\pi}{L_y}$ $\quad$ Satisfies b.c.'s

$Y(y) \propto \cos\left(m\pi y / L_y\right)$ $\quad m = 0, 1, 2, 3$



m = number of nodes in the mode

$Z(z) \propto \cos(k_{z_n} z)$ $\quad k_{z_n} = \frac{n\pi}{L_z}$ $\quad n = 0, 1, 2, 3$

same as $Y(y)$

$\alpha_{mn}^2 = k_{y_n}^2 + k_{z_n}^2$ $\quad$ specifies value of α_n^2

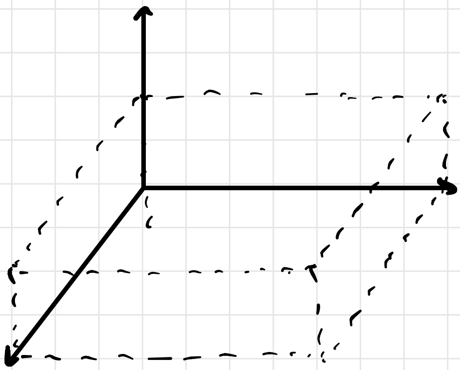
$k_{x_n} = \sqrt{k^2 - \alpha_n^2}$

$k_{x_{mn}} = \sqrt{k^2 - k_{y_m}^2 - k_{z_n}^2}$

$e^{-jk_{x_{mn}}x}$

as mode numbers increase, non-propagating solution of $k_{x_{mn}}$

Lecture Apr. 6.



$$P_n(x,y,z) = a_n \Psi_n(y,z) X_n e^{-jk_{x_n}x}$$

$$Y(y) \propto \frac{\cos(m\pi y)}{L_y}, \quad Z(z) \propto \frac{\cos(n\pi z)}{L_z}$$

$$\Psi_{mn} \propto Y_m(y) Z_n(z)$$

Rectangular Geometry: $\cos(k_m y)\cos(k_n z)$ $\quad k_{y_m} = m\pi/L_y$, $\quad k_{z_n} = n\pi/L_z$

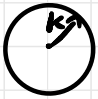
$m=0,\ n=0$

Plane wave mode $(0,0)$

For a solution to be sound

$$k^2 = k_{x_{mn}}^2 + k_{y_m}^2 + k_{z_n}^2 \qquad k = \frac{\omega}{c}$$

Allowed Solutions fall on the Surface of a sphere (Radiation Sphere)



$$k_{x_{mn}} = \sqrt{k^2 - k_{y_m}^2 - k_{z_n}^2}$$

$$P_{mn} \propto \Psi_{mn}(y,z) e^{-jk_{x_{mn}}x}$$

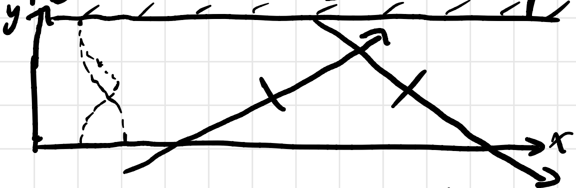
$$\cos(k_{y_m} y) = \frac{1}{2}\left(e^{jk_{y_m}y} + e^{-jk_{y_m}y}\right)$$

$$\cos(k_{z_n} z) = \cdots$$

$$P_{mn} \propto \left(e^{jk_{y_m}y} + e^{-jk_{y_m}y}\right)\left(e^{jk_{z_n}z} + e^{-jk_{z_n}z}\right) e^{-jk_{x_{mn}}x}$$

$$\underbrace{e^{\pm jk_{y_m}y}\, e^{\pm jk_{z_n}z}\, e^{-jk_{x_{mn}}x}}_{\text{4 plane waves travelling in directions fixed by component wave numbers}}$$

$$\underbrace{e^{-jk_{y_m}y}\, e^{-jk_{z_n}z}\, e^{-jk_{x_{mn}}x}}_{\text{Plane wave in 3d travelling in } +x,\ +y,\ +z \text{ directions}}$$



Same thing into the page

P_{mn} can be viewed as being the superposition of 4 propagating plane waves within the duct

The 'mode' is the interference pattern created by the interaction of plane waves

The 'mode' has no physical reality

Particular mode (m,n) is said to "propagate" when $k_{x_{mn}}$ is real

That is true when $\alpha_n^2 < k^2$ $\quad k_{y_m}^2 + k_{z_n}^2 = \alpha_n^2$

$$\left(\frac{m\pi}{l_y}\right)^2 + \left(\frac{n\pi}{l_z}\right)^2 < \left(\frac{\omega}{c}\right)^2$$ for propagation without attenuation

if $\alpha_n^2 > k^2$ $\quad k_{x_{mn}} = $ imaginary ... The mode is non-propagating

Frequency at which a mode begins to propagate

$$k^2 = \alpha_n^2 = (m\pi/L_y)^2 + (n\pi/L_z)^2 = \omega^2/c^2$$

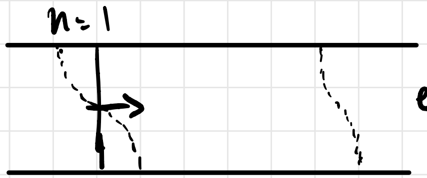
cut-on freq] The same thing
cut-off freq]

cut-off mode $\quad \alpha_n^2 > k^2$

$$k_{x_n}^2 = \sqrt{\underbrace{k^2 - \alpha_n^2}_{\text{negative}}} = \pm j\sqrt{\underbrace{\alpha_n^2 - k_n^2}_{\text{positive}}}$$

$$e^{-jk_{x_n}x} \rightarrow e^{\pm \underbrace{(\alpha_n^2 - k^2)^{1/2}}_{\text{pos + real}} x}$$

$n = 1$



exponential decay of amplitude

$$k_{x_{mn}} = \sqrt{k^2 - \left(\frac{m\pi}{y}\right)^2 - \left(\frac{n\pi}{z}\right)^2}$$

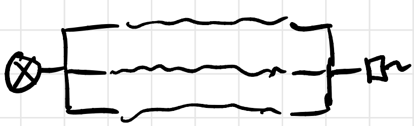
$m = 0, 1, 2, \ldots$
$n = 0, 1, 2, \ldots$

$k_{x_{00}} = k^2$ (0,0) plane wave mode ... Always propagates

→ All other modes do not propagate below a certain freq

→ As the frequency increases, more modes begin to propagate...

3 modes



4/8 Duct Modes

$\psi(y,z)\, e^{-jk_{x_{mn}} x}$

$k_{x_{mn}} = \sqrt{k^2 - k_{y_m}^2 - k_{z_n}^2}$ in rectangular duct: $k_{y_m} = \frac{m\pi}{L_y}$

$k_{z_n} = \frac{n\pi}{L_z}$

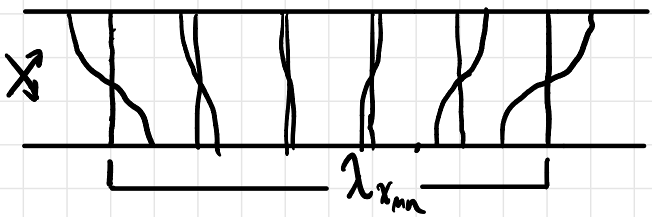
(m,n)

$k_{x_{0,0}} = k \quad (0,0)$

$e^{-jk_{x_{mn}} x}$ ~ propagation factor

Define a modal wavelength in the x-direction

$k_{x_{mn}} \lambda_{x_{mn}} = 2\pi \qquad \lambda_{x_{mn}} = \frac{2\pi}{k_{x_{mn}}}$



Pattern moves one-modal wavelength down the duct in one temporal period $T = \frac{1}{f} = 2\pi/\omega$

Pattern is moving with an apparent phase speed

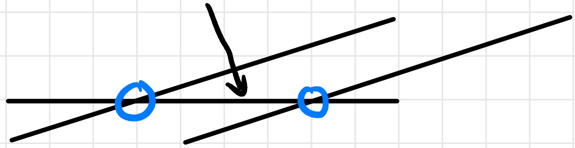
$c_{p_{mn}}$] modal phase velocity

$\lambda_{x_{mn}} = c_{p_{mn}} \cdot T = c_{p_{mn}} \frac{2\pi}{\omega} = 2\pi / k_{x_{mn}}$

$c_{p_{mn}} = \frac{\omega}{k_{x_{mn}}} \qquad k_{x_{mn}} = \sqrt{k^2 - k_{y_m}^2 - k_{z_n}^2}$

When the mode begins to propagate (cuts-on)

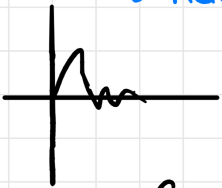
$k_{x_{mn}} = 0$ so $c_{p_{mn}} = \infty$



$C_{p_{mn}}$: function of frequency

Modal Propagation is dispersive

- different freq components travel at different speeds



$$C_{p_{mn}} = \frac{\omega}{k_{x_{mn}}} = \frac{\omega}{\sqrt{k^2 - k_{y_m}^2 - k_{z_n}^2}} = \frac{c}{\sqrt{1 - \frac{(k_{y_m}^2 - k_{z_n}^2)}{k^2}}}$$

$$\frac{C_{p_{mn}}}{c} = \frac{1}{\sqrt{1 - \underbrace{\frac{(k_{y_m}^2 - k_{z_n}^2)}{k^2}}_{<1 \text{ for propagating mode}}}}$$

$\frac{C_{p_{mn}}}{c} > 1$ for propagating mode

$$k^2 > k_{y_m}^2 + k_{z_n}^2$$

$$\omega^2 > c^2 (k_{y_m}^2 + k_{z_n}^2)$$ Cut-on freq for $(m,n)^{th}$ mode

$$\omega_{mn}^2 = c^2 (k_{y_m}^2 + k_{z_n}^2)$$

$$k_{y_m} = \frac{m\pi}{L_y} \qquad k_{z_n} = \frac{n\pi}{L_z}$$

$$\omega_{mn}^2 = c^2 \left[\left(\frac{m\pi}{L_y}\right)^2 + \left(\frac{n\pi}{L_z}\right)^2 \right]$$

$\omega = 0$ ... plane wave mode

if $L_y > L_z$ $m=1, n=0$ is next mode

if $L_z > L_y$ $m=0, n=1$ is next mode

Order in which modes cut-on depends on the geometry

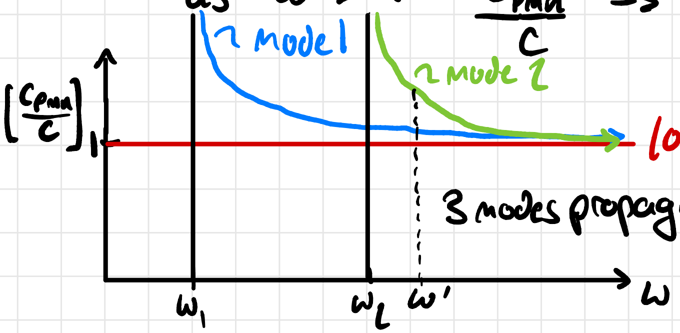
ω_{mn}'s increase with mode number

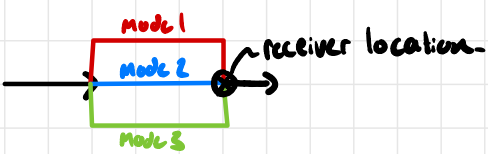
At the cut-on frequency

$\frac{c_{pmn}}{c} = \infty$ phase speed is infinite...

c_{pmn} decreases and ultimately tends toward c

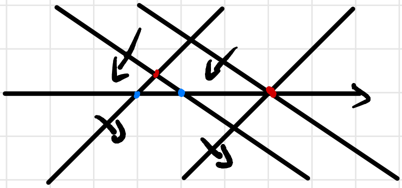
as $\omega \to \infty$ $\frac{c_{pmn}}{c} \to 1$





Phase Speed

- Speed at which points of constant phase move down the duct
- interference pattern moves at a speed $c_{pmn} > c$
- but the pattern is created by waves that are themselves propagating at ambient sound speed



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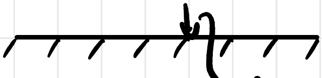


$u_n = 0$

$\psi(y,z)e^{-jk_{x_{mn}}x}$

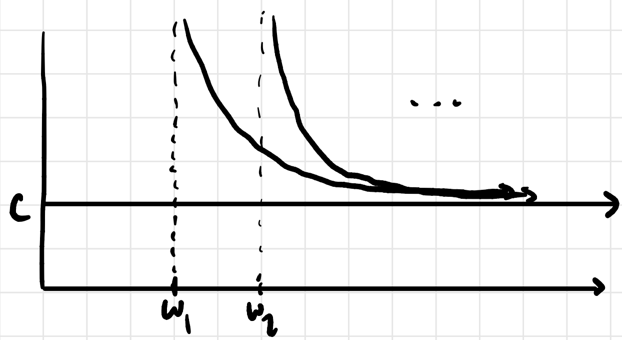
$k_{x_{mn}} = \sqrt{k^2 - \underbrace{k_{y_m}^2 - k_{z_n}^2}}$

$k = \omega/c$

- frequency Geometry -

$c_{p_{mn}} = \dfrac{\omega}{k_{x_{mn}}} \geq c$

$\omega_{mn}^2 = c^2\left(k_{y_m}^2 + k_{z_n}^2\right)$



Plane Wave mode is always propagating and non-dispersive

$k^2 = k_{x_{mn}}^2 + k_{z_n}^2 + k_{y_m}^2$

$\bar{k} = k_{x_{mn}}\hat{i} + k_{y_m}\hat{j} + k_{z_n}\hat{k}$ wave vector

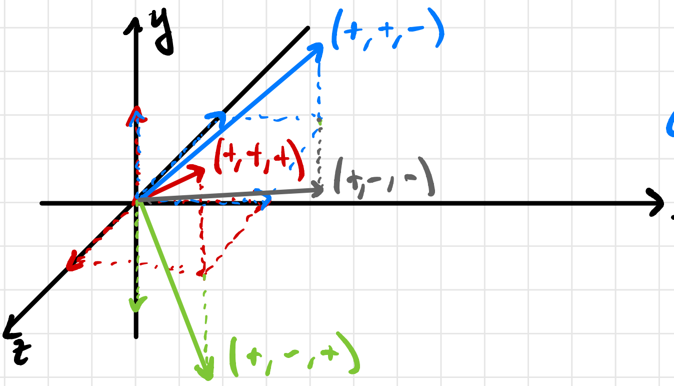
$\omega, m, n]$ - determining direction of wave propagation

Rectangular Duct Case -

$$\Psi_{mn}(y,z)e^{-jk_{x_{mn}}x} \propto \cos(k_{y_m}y)\cos(k_{z_n}z)e^{-jk_{x_{mn}}x}$$

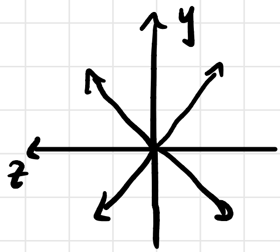
$$\frac{1}{2}\left(e^{jk_{y_m}y}+e^{-jk_{y_m}y}\right)\cdot\frac{1}{2}\left(e^{jk_{z_n}z}+e^{-jk_{z_n}z}\right)e^{-jk_{x_{mn}}x}$$

$$\underbrace{e^{\pm jk_{y_m}y}e^{\pm jk_{z_n}z}}_{\text{4 possible Combos}}e^{-jk_{x_{mn}}x}$$



$$e^{-jk_{y_m}y}e^{-jk_{z_n}z}e^{-jk_{x_{mn}}x}$$

$$e^{-jk_{y_m}y}e^{jk_{z_n}z}e^{-jk_{x_{mn}}x}$$



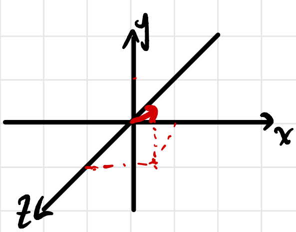
At cut-on frequency: "folded" back on y-z plane

As ω increases

"folds" towards x-axis

$$k\cos\theta_{mn} = k_{x_{mn}} \quad \therefore \cos\theta_{mn} = \frac{k_{x_{mn}}}{k} = \frac{k_{x_{mn}}\cdot c}{\omega}$$

Each $\Psi_{mn}(y,z)$ can be viewed as a superposition of 4 plane waves each propagating at an angle θ_{mn} with respect to the x-axis

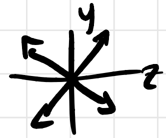


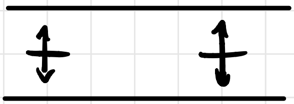
$$k\cos\theta_{mn} = k_{x_{mn}}$$

In general: $\cos\theta_{mn} = k_{x_{mn}} \cdot \frac{c}{\omega} = \sqrt{1-(\alpha_n^2/k^2)}$

at the cut-on frequency

$k^2=\alpha_n^2$ $\quad \cos\theta = \sqrt{1-1} \Rightarrow \theta = \pi/2$

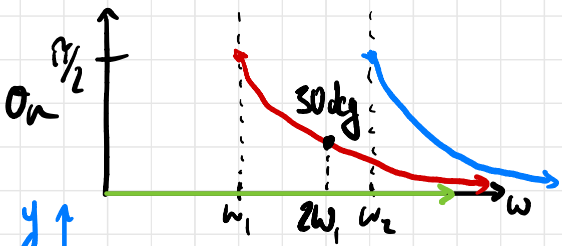




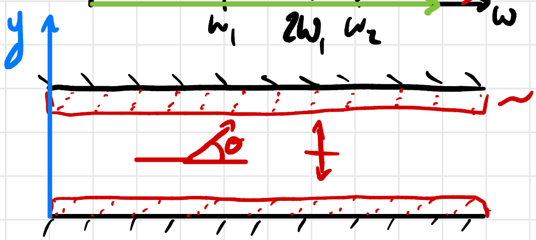
as $\omega \to \infty$ $\quad \frac{\alpha_n^2}{k^2} \to 0$

$\cos\theta_n \to 1 \quad \theta_n \to 0$

wave vectors collapse onto the x-axis as $\theta_n \to 0$

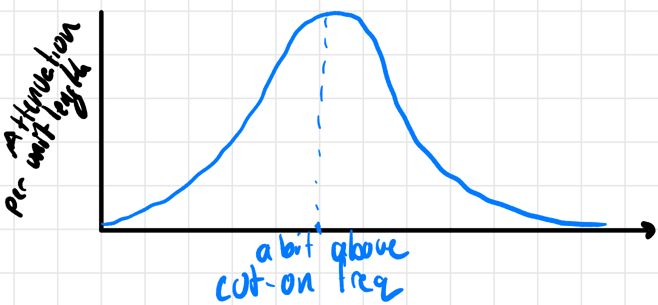


• Plane Wave Mode



~ Absorbing layers $\quad Z_{cn} = \rho_0 c/\cos\theta$

Match the impedance of the plane wave to the impedance of the lining material



Absorbing matl's used as duct liners:
- Most effective just above the cut-on freq for the equivalent hard walled duct mode - ensure plane waves "see" the absorptive lining
- optimal performance occurs when component wave impedance

$\frac{\rho_0 c}{\cos\theta_n} \simeq$ Surface impedance of the liner

Acoustic Intensity - Quantifies amount of energy going down the duct

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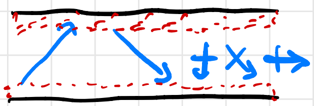
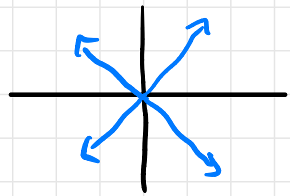
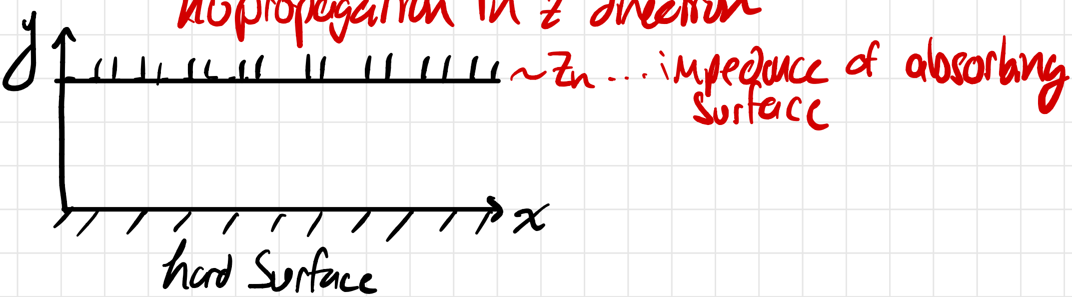




4.4 Rectangular Ducts with an impedance lining

No propagation in z direction



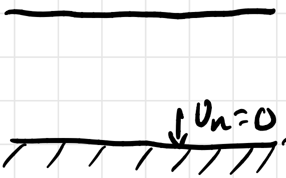
Absorbing Surface is locally reactive
(impedance not a function of angle)

Zero Flow M<0.1 low speed flows

Scalar Helmholtz Equation Applies in the duct space

$$\nabla^2 p + k^2 p = 0$$

$$p(x,y) = \left(A e^{-jk_y y} + B e^{jk_y y}\right) e^{-jk_x x}$$



~ Hard Surface ... $v_y = 0$ at $y = 0$

$$v_y = \frac{-1}{j\omega\rho_0}\frac{\partial p}{\partial y} \Rightarrow B = A$$

$$p(x,y) = 2A\cos k_y y \cdot e^{-jk_x x}$$

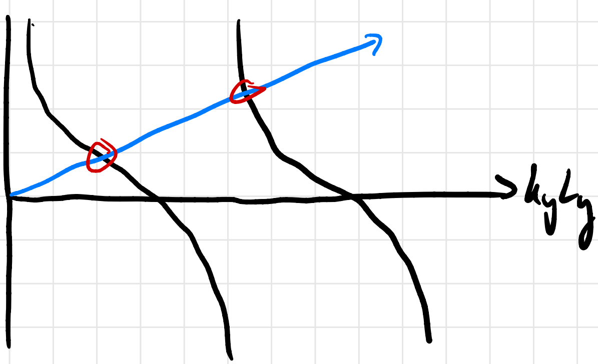
Impedance b.c. at $y = L_y$

$$Z_n = \frac{p(x,L_y)}{v_y(x,L_y)} \qquad v_y(x,L_y) = -\frac{1}{j\omega\rho_0}\frac{\partial p}{\partial y}\Big|_{y=L_y}$$

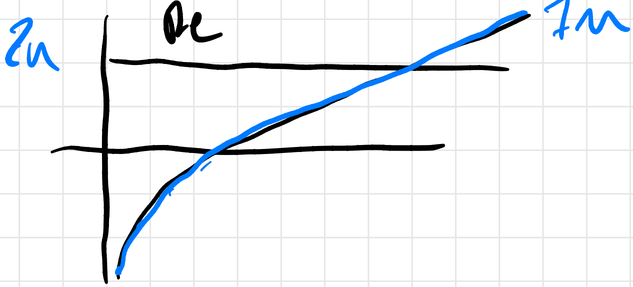
$$u_y = \frac{k_y}{j\omega\rho_0}\cdot 2A\sin(k_yL_y)\,e^{-jk_xx}$$

$$Z_n = \frac{p(x,L_y)}{v_y(x,L_y)} = \frac{j\omega\rho_0}{k_y}\cdot\frac{\cancel{2}\cancel{A}\cos(k_yL_y)\,\cancel{e^{-jk_xx}}}{\cancel{2}\cancel{A}\sin(k_yL_y)\,\cancel{e^{-jk_xx}}}$$

$$Z_n = \frac{j\omega\rho_0}{k_y}\cot(k_yL_y) \Rightarrow \frac{Z_n}{j\rho_0 c(kL_y)}(k_yL_y) = \cot(k_yL_y)$$



Slightly More Complicated
b.c. Z_n is generally
Complex



$\therefore$ $k_y l_y$ will be complex if Z_n is complex

Since cot is periodic ∴ infinite number of solutions

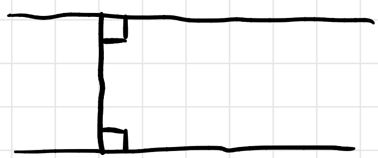
k_x is also complex - always finite real + imag parts

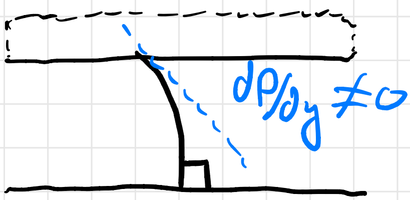
Most interested in imaginary part:

$Im\{k_x\}$ gives us the rate of attenuation

$$e^{-jk_x x} \to e^{-j(\beta - j\alpha)x} \to \underbrace{e^{-j\beta x}}_{\text{Propagating}} \underbrace{e^{-\alpha x}}_{\text{Amplitude decay}}$$

hard walled duct





Since v_y at $l_y \neq 0$

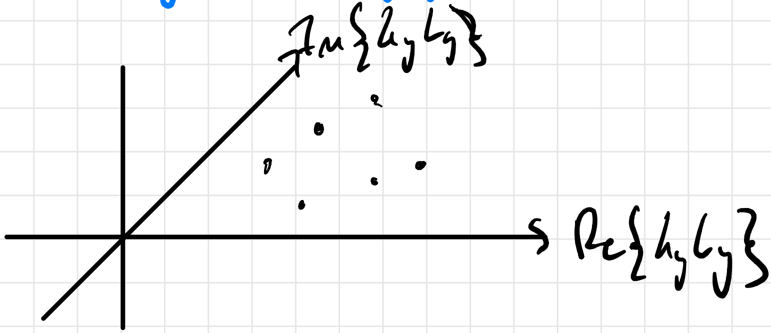
Some curvature of the wavefront is necessary to satisfy b.c. at $y = L_y$

We no longer have the perfect plane wave

$$\frac{Z_n}{j\rho_0 c (k L_y)} (k_y L_y) = \cot(k_y L_y)$$

D = constant at a particular frequency

$$D (k_y L_y) = \cot(k_y L_y)$$



2D search for $k_y L_y$

Result: k_{y_n} ... $n = 1, 2, 3 \ldots$

$$k_{x_n} = \sqrt{k^2 - k_{y_n}^2}$$

Mode: $p_n = 2A_n \cos(k_{y_n} y) e^{-jk_{x_n} x}$ $\quad$ k_{x_n} is complex

$k_{x_n} = \beta_{x_n} - j\alpha_{x_n}$ $\quad$ $e^{-jk_{x_n}x} \to \underbrace{e^{-j\beta_{x_n}x}}_{\text{propagating}} \underbrace{e^{-\alpha_{x_n}x}}_{\text{decay}}$

Now - All modes propagate + All modes decay

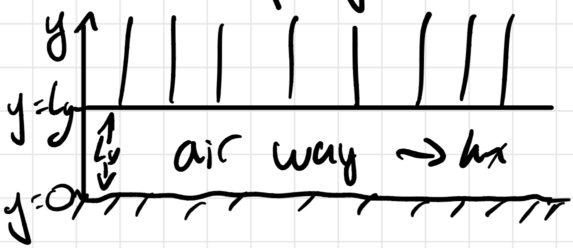
We need to be concerned with the so-called "Least Attenuated Mode" (LAM)

- smallest α_{x_n} from the various solutions

That mode is usually but not always the first solution...

Special Case:

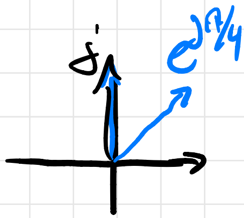
- narrow duct $|k_y L_y| \ll 1$
- low freq
- Z_n is purely real



~ deep layer of absorbing Material

$$\frac{Z_n}{\omega \rho_0} = -j\frac{\cot(k_y L_y)}{k_y} \quad \text{when} |k_y L_y| \ll 1$$

$$\frac{Z_n}{\omega \rho_0} \simeq j\frac{1}{k_y^2 L_y} \Rightarrow k_y^2 = j\frac{\omega \rho_0}{Z_n L_y}$$



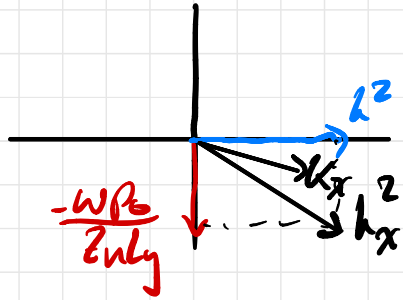
$$k_y = \sqrt{j\frac{\omega \rho_0}{Z_n L_y}} = \sqrt{j}\cdot\sqrt{\frac{\omega \rho_0}{Z_n L_y}}$$

$e^{j\pi/4} = \sqrt{j}$

$k_y = e^{j\pi/4}\sqrt{\frac{\omega\rho_0}{Z_n l_y}}$ k_y is complex with equal real + imag parts

$k_x^2 = k^2 - k_y^2$

$k_x = \sqrt{k^2 - j\frac{\omega\rho_0}{Z_n l_y}}$



k_x has a real part + imag part is negative

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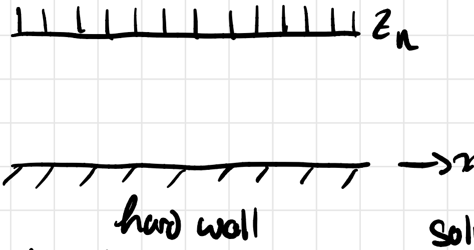
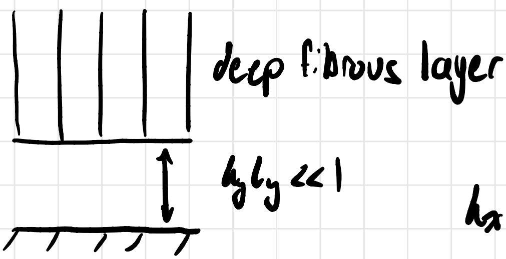


$\frac{Z_n}{j\rho_0 c(k l_y)}(k_y l_y) = \cot(k_y l_y)$

solve $k_{y_n} \rightarrow k_{x_n} = \sqrt{k^2 - k_{y_n}^2}$



$k_x = \sqrt{k^2 - j\frac{\omega\rho_0}{Z_n l_y}} = \beta - j\alpha$

$e^{-j\beta_x x} e^{-\alpha_x x}$

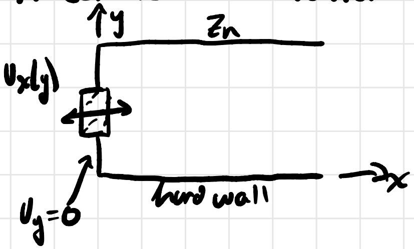
In general: k_{y_n} $n = 1, 2, 3 \ldots$ (no longer have plane wave mode)

$p_n = A_n \cos(k_{y_n} y)\, e^{-j k_{x_n} x}$

If Z_n is locally reacting $(Z_n \neq f(\theta))$

duct modes are orthogonal

Assemble a solution as a sum of modes



$$p(x,y) = \sum_{n=1}^{\infty} \boxed{A_n} \cos(k_{y_n} y)\, e^{-jk_{x_n} x}$$ unknown

$$U_x(x,y) = \frac{-1}{j\omega\rho_0}\frac{\partial p}{\partial x}$$ => evaluate U_x at $x=0$

⇓

Apply B.C. at $x=0$
$U_x|_{x=0} = U_x(y)$

- multiply $U_x(y)$ by one of the modes
- integrate both sides of the equation from $y=0 \to y=L_y$

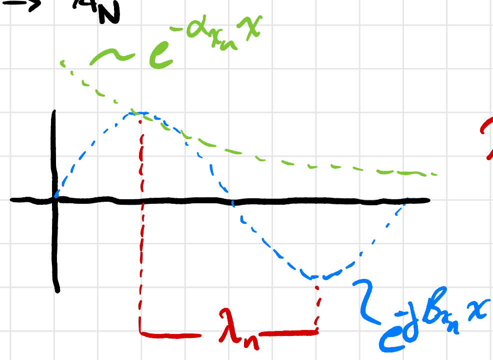
→ Algebraic expression
solve for A_n as a function of $U_x(y)$

→ One mode at a time

from $A_1 \to A_N$

Since k_{x_n} is complex

$$k_{x_n} = \beta_{x_n} - j\alpha_{x_n}$$



$$\lambda_{x_n} = \frac{2\pi}{\beta_{x_n}}$$

$$\beta_{x_n} = \frac{2\pi}{\lambda_{x_n}}$$

k_{x_n} is different for each mode
usually (not always) higher mode
has faster decay

Beyond a certain distance only the Least Attenuated Mode
is significant - Maybe only need to worry about this one

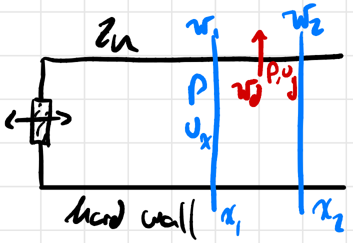
In that case: $p(x,y) = A_1 \cos(k_{y,1} y)\, e^{-jk_{x,1}x}$ if $n=1$ is the LAM

Hard Walled Duct -

- propagating modes: k_{x_n} is real → oscillatory solutions
- non-propagating modes: k_{x_n} is imag → exponential decay

Finite Impedance Lined duct -

- k_{x_n} is always complex
- no clear distinction between prop & non-prop modes
- ★ - every mode propagates & every mode attenuates



$$I_y = \frac{1}{2} Re\{p u_y^*\}$$

$$W_d = \int_{x_1}^{x_2} I_y \, dx$$

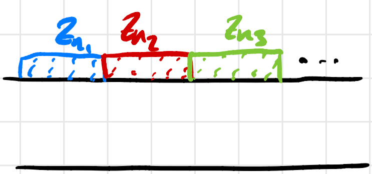
$$I_x = \frac{1}{2} Re\{p u_x^*\}$$

$$\int_S I_x \, ds = W_1$$

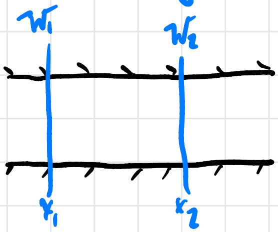
$W_2 - W_1 = W_d$ ← want this to be big

From design POV - choose Z_n to maximize W_d

- but everything is frequency dependent -
- to increase bandwidth of absorption →



Acoustic Intensity



hard walled duct - $W_1 = W_2$

$$I_x = \frac{1}{2} Re\{ p u_x^* \}$$

particle velocity

$$u_x = \frac{-1}{j\omega\rho_0}\frac{\partial p}{\partial x}$$ harmonic

$$p = \sum_n p_n$$

$$\frac{\partial p}{\partial x} = \frac{\partial}{\partial x}\sum_n p_n = \sum_n \frac{\partial p_n}{\partial x}$$

u_{x_n} = particle velocity associated with mode n

$$= \frac{-1}{j\omega\rho_0}\frac{\partial p_n}{\partial x} \qquad p_n = \psi_n(y,z)e^{\pm jk_{x_n}x}$$

$$u_{x_n} = \pm\frac{\cancel{j}k_{x_n}}{\cancel{j}\omega\rho_0}\psi_n(y,z)e^{\pm jk_{x_n}x} = \pm\frac{k_{x_n}}{\omega\rho_0}\psi_n(y,z)e^{\pm jk_{x_n}x}$$

$$= \frac{k_{x_n}}{\omega\rho_0}\psi_n(y,z)e^{-jk_{x_n}x}$$

$$\text{or } \frac{-k_{x_n}}{\omega\rho_0}\psi_n(y,z)e^{jk_{x_n}x}$$

$\frac{p}{u} = \rho_0 c$ for +ve going

$\frac{p}{u} = -\rho_0 c$ for -ve going

For a propagating mode : k_{x_n} is real

particle velocity and pressure are in phase

u_{x_n} + p_n are in phase with eachother

∴ the mode carries energy down the duct

for non-propagating modes: u_{x_n} + p_n are out of phase (90°)

∴ no energy is carried by that mode

$I_{x_n} = 0$

Sound Power is carried by a finite number of modes

+ the number of propagating modes increases as the frequency goes up

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Acoustic Intensity:

$$U_{x_n} = \frac{-1}{j\omega\rho_0}\frac{dp}{dx} \qquad p_n = \Psi_n(y,z)\, e^{-jk_{x_n}x}$$

$$U_x = \sum_{\langle n \rangle} U_{x_n} \qquad U_{x_{n+}} = \frac{k_{x_n}}{\omega\rho_0}\Psi_n(y,z)\, e^{-jk_{x_n}x}$$

$$U = \pm \frac{p}{\rho_0 c}$$

Modal Impedance

$$Z_n = \frac{p_n}{U_n} = \frac{\omega\rho_0}{k_{x_n}} \quad \text{for positive going wave}$$

$$= \frac{-p_n}{U_n} = \frac{-\omega\rho_0}{k_{x_n}} \quad \text{for negative going wave}$$

$$k_{x_n} = \frac{\omega}{c_{p_n}} \qquad c_{p_n} \text{ modal phase speed}$$

$$Z_n = \rho_0 \cdot c_{p_n}$$

$$= \frac{\rho_0 c}{k_{x_n}/k} = \frac{\rho_0 c}{\cos\theta_n}$$

$$Z_n = \frac{\omega\rho_0}{k_{x_n}} = \frac{\omega\rho_0}{\sqrt{k^2 - \alpha_n^2}}$$

in 2d : $\alpha_{mn}^2 = (m\pi/L_y)^2 + (n\pi/L_z)^2$

for rectangular hard-walled duct

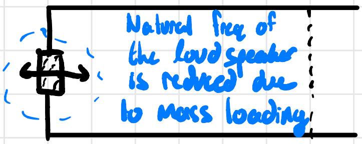
when $k^2 < \alpha^2$ $\quad k_{xn}$ is imaginary (non-prop mode)

$$\sqrt{k^2 - \alpha_n^2} = \pm j \underbrace{\sqrt{\alpha_n^2 - k^2}}_{\text{real + positive}} \quad \text{when } k^2 < \alpha_n^2$$

Therefore:

$$Z_n = \pm \frac{\omega \rho_0}{j(\alpha_n^2 - k^2)^{1/2}}$$ ... purely imaginary

Forward going mode: $Z_n = \dfrac{j\,\omega \rho_0}{(\alpha_n^2 - k^2)^{1/2}}$] mass-like impedance



- operating below cut-on frequency for first higher order mode
- only plane waves can propagate

Need many higher order modes to represent the particle velocity in the near-field

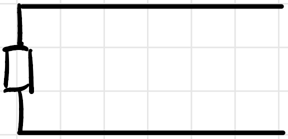
non-prop modes have mass like impedances

- evanescent near-field (non-propagating modes)
  - mass like load on the source

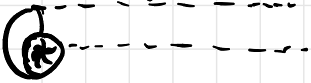
$$Z_n = +j \frac{\omega \rho_0}{(\alpha_n^2 - k^2)^{1/2}}$$

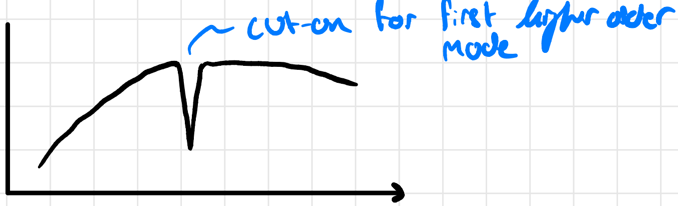
At modal cut-on $Z_n \rightarrow \infty$

The mode presents an infinite load to the source at cut-on



Turbocharger

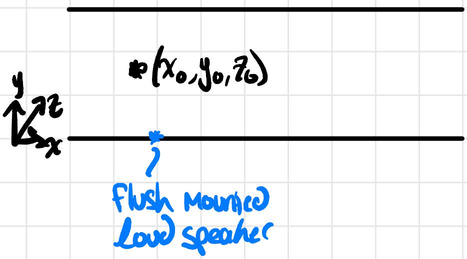




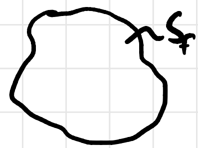
Finite impedance source can be stopped by this effect at the modal cut-on frequencies

Axial Energy Flow

- Sources in Ducts
  - Point Source



hard walled constant cross-section



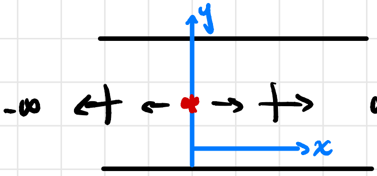
Inhomogenous Helmholtz Equation

$$\nabla^2 p + k^2 p = -4\pi A\, \delta(\bar{x} - \bar{x}_0)$$

~ source location

Source Strength

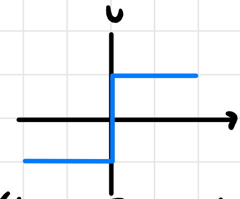
field point



$$x_n = a_n e^{-jk_{x_n} x}$$

Pressure in infinite duct is symmetric

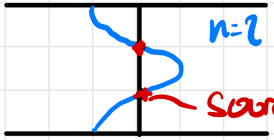
Particle velocity changes sign at $x = x_0$



$$P_t(x,y,z) = \sum_n a_n e^{-jk_{x_n} x} \psi(y,z)$$

(Modal Participation factor)

$$a_n = \frac{2\pi A}{S_t} \psi(y,z)$$ ~ strength of each mode (Modal Amplitude)



In General:

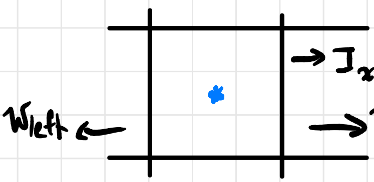
If a source is placed at a node where $\psi(y,z)=0$, the mode n does not contribute to the solution

$$P_n(r,y,z) = \sum_n a_n e^{-jk_{x_n}x}\psi_n(y,z) = \frac{2\pi A}{S_t}\sum_n \underbrace{\psi_n(y_0,z_0)}_{\text{Source Location}}\underbrace{\psi_n(y,z)e^{-jk_{x_n}x}}_{\text{Receiver Location}}$$

Cannot drive a mode if the source is placed at a node

Cannot sense a mode if the sensor is placed at a node

Important implications for active control



Infinite Series

$$W = \int_{S_t} I_x\, dS \qquad I_x = \mathrm{Re}\{p u^*\}$$

Modes are orthogonal

$$W = S_t \sum I_{x_n} \Rightarrow W_{right} = \frac{2\pi^2 A^2}{\omega\rho S_t}\mathrm{Re}\left\{\sum_n \psi_n(y_0,z_0)/k_{x_n}\right\}$$

ψ_n are real numbers

axial wave numbers are real or imag

Only propagating modes contribute to the sound power

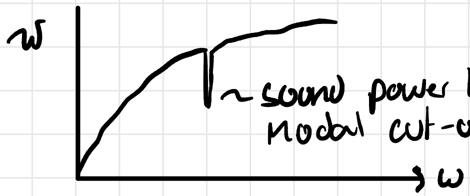
- number of modes contributing increases progressively as more modes cut-on

- but still, at any finite frequency, the sum is finite

$$W_{right} = \frac{2\pi^2 |A|^2}{\omega\rho_0 S_t} \sum_{\tilde{n}} \underbrace{\frac{\psi_n^2(y_0, z_0)}{\sqrt{k^2-\alpha^2}}}_{=0 \text{ when mode } n \text{ cuts on}}$$

$\tilde{n}$ ... sum over propagating modes only

Power output of the source goes to ∞ as each mode cuts-on



sound power tends to be reduced at modal cut-on, due to finite source internal impedance