Bounds for Some Functions Concerning Dynamic Storage Allocation

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ABSTRACT The amount of store necessary to operate a dynamic storage allocation system, subject to certain constraints, with no risk of breakdown due to storage fragmentation, is considered. Upper and lower bounds are given for this amount of store, both of them stronger than those established earlier. The lower bound is the exact solution of a related problem concerning allocation of blocks whose size is always a power of 2.

KEY WORDS AND PHRASES: storage allocation, dynamic storage allocation, storage allocation strategy

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Introduction

A previous paper [1] discussed the function $N^*(M, n)$ defined as the size of store needed to implement a dynamic storage allocation strategy without possibility of breakdown where storage requests could be for arbitrary block sizes up to n words and the total number of words in use could not exceed M.

It is useful to consider also $\hat{N}(M, n)$ defined similarly except that requests for new blocks may still be made as long as the number of words in use before any request is less than M.

The proof in [1] that $N^*(M, 2^a) \leq M(a + 1)$ if $2^a \mid M$ in fact proves also that $\hat{N}(M, 2^a) \leq M(a + 1)$ if $2^a \mid M$, which is a slightly stronger result in view of the obvious inequality $\hat{N}(M, n) \geq N^*(M, n)$.

Another useful inequality is $N^*(M_1 + M_2, n) \leq \hat{N}(M_1, n) + N^*(M_2, n)$, which is shown by considering a store divided into two parts with sizes $\hat{N}(M_1, n)$ and $N^*(M_2, n)$ in each of which the relevant optimal strategy is followed, where a block is put into the first of these two parts whenever possible.

$\hat{N}(M_1 + M_2, n) \leq \hat{N}(M_1, n) + \hat{N}(M_2, n)$ is proved similarly.

A simplification to the problem which may not be too unrealistic is to assume that all requests are for blocks whose size is a power of 2.

Given this assumption, $N_2^*(M, n)$ and $\hat{N}_2(M, n)$, which are defined only for n a power of 2, are the equivalents of $N^*(M, n)$ and $\hat{N}(M, n)$. The proof in [1] that $N^*(M, 2^b) \geq 4Mb/13 - 16.2^b/13 + 9M/13$ is in fact a proof that $N_2^*(M, 2^b) \geq 4Mb/13 - 16.2^b/13 + 9M/13$.

$N_2^*(M, n) \leq N^*(M, n)$ is obvious.

$N_2^*(M_1 + M_2, n) \leq \hat{N}_2(M_1, n) + N_2^*(M_2, n)$ is easily shown in the same way as the similar inequality in N^* and $\hat{N}$.

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Results

1. $N_2^*(M, n) = M(1 + \frac{1}{2}\log_2 n) - n + 1$ if $n \mid M$. This implies that $N^*(M, 2^b) \geq M(1 + b/2) - 2^b + 1$ if $2^b \mid M$ and that no significantly stronger lower bound on $N^*(M, n)$ can be obtained by considering only allocations of blocks whose size is a power of 2.

In general

$$N^*(M, n) \geq [M/n]\cdot n(1 + [\log_2 n]/2) - 2^{[\log_2 n]} + 1$$
$$> (M - n)(1 + \log_2 n)/2 - n + 1.$$

2. Given $K > \{\frac{3}{2}\log 2 - \frac{1}{2} + \sum_{k=2}^{\infty}(1/2k)\log(1 + 1/k)\}^{-1}$, there is an X such that $\hat{N}(M, n) < M(X + K\log n) + 4n^2 + n$, which implies $N^*(M, n) < M(X + K\log n) + 4n^2 + n$.

The bound on K is a little less than 1.22 so for large enough M and n this result is considerably stronger than the bound $M(2 + \log n/\log 2)$ given in [1].

The Function $N_2^*(M, n)$ When $n \mid M$

(a) $N_2^*(M, n) \geq M(1 + \frac{1}{2}\log_2 n) - n + 1$ provided M is a multiple of n. The proof uses the following attacking strategy.

1. *Strategy.* There are $(1 + \log_2 n)$ moves; on the jth certain blocks are removed; and then as many blocks as possible of size 2^{j-1} are formed.

S_0 is the set of all words of the store; S_1 consists of alternate words of S_0; in general S_{j+1} consists of alternate words of S_j $(0 \leq j < \log_2 n - 1)$.

On the jth move the blocks removed are all those which do not lie partly on a word of S_{j-1} $(0 < j < \log_2 n + 1)$. The last move is given later. Thus after the jth move, every block lies on one and only one word of S_{j-1}.

At the start of the $(j + 1)$-th move, S_j is chosen from the two halves of S_{j-1} so as to minimize the sum over the blocks B chosen to be removed of (2^j-size of B) $(0 < j < \log_2 n)$; i.e.

$$\sum_{\substack{\text{removed}\\ \text{blocks}}} (2^j\text{-size of block}) \leq \sum_{\substack{\text{retained}\\ \text{blocks}}} (2^j\text{-size of block}). \tag{1}$$

2. *Definitions*

(a) The number of blocks of size 2^i existing after the move creating 2^j blocks (the $(j + 1)$-th move) is $K_{i,j}$. The number of blocks of size 2^i removed in the first phase of that move is $R_{i,j}$. Hence $K_{i,j-1} = K_{i,j} + R_{i,j}$ $(i < j)$.

The inequality (1) can be written as $\sum_{i=0}^{j-1} R_{i,j}(2^j - 2^i) \leq \sum_{i=0}^{j-1} K_{i,j}(2^j - 2^i)$.

(b) Odd $(j) = \sum_{i=0}^{j} K_{i,j}(2^j - 2^i)$.

3. *Proof.*

$$\begin{aligned}
\text{Odd }(j) - \text{odd }(j-1) &= \sum_{i=0}^{j} K_{i,j}(2^j - 2^i) - \sum_{i=0}^{j-1} K_{i,j-1}(2^{j-1} - 2^i)\\
&= \sum_{i=0}^{j-1} K_{i,j}(2^j - 2^i) - \sum_{i=0}^{j-1} K_{i,j-1}(2^{j-1} - 2^i)\\
&= \sum_{i=0}^{j-1} \{(K_{i,j-1} - R_{i,j})(2^j - 2^i) - K_{i,j-1}(2^{j-1} - 2^i)\}\\
&= \sum_{i=0}^{j-1} \{K_{i,j-1}(2^{j-1}) - R_{i,j}(2^j - 2^i)\},
\end{aligned}$$

but

$$\sum_{i=0}^{j-1} R_{i,j}(2^j - 2^i) \leq \sum_{i=0}^{j-1} K_{i,j}(2^j - 2^i),$$

which implies

$$\sum_{i=0}^{j-1} R_{i,j}(2^j - 2^i) \le \tfrac{1}{2}\sum_{i=0}^{j-1} (K_{i,j} + R_{i,j})(2^j - 2^i) = \tfrac{1}{2}\sum_{i=0}^{j-1} K_{i,j-1}(2^j - 2^i).$$

Thus

$$\text{odd } (j) - \text{odd } (j-1) \ge \sum_{i=0}^{j-1} K_{i,j-1}\{2^{j-1} - \tfrac{1}{2}(2^j - 2^i)\} = \tfrac{1}{2}\sum_{i=0}^{j-1} K_{i,j-1}\cdot 2^i.$$

This last sum is precisely the number of words in use after the jth move.

Let $y_j = M - \sum_{i=0}^{j-1} K_{i,j-1}\cdot 2^i$, i.e. the number of unused words after the jth move $y_j < 2^{j-1}$.

Therefore y_j = residue of $(-\sum_{i=0}^{j-1} K_{i,j-1}\cdot 2^i)$ mod 2^{j-1}, since $2^{j-1} \mid n \mid M$.

Hence y_j = residue of $(\sum_{i=0}^{j-1} K_{i,j-1}\cdot(2^{j-1} - 2^i))$ mod 2^{j-1} = residue of odd $(j-1)$ mod 2^{j-1}.

Now odd $(j) \ge Mj/2$ can be proved inductively. The case odd $(0) = 0$ is trivial.

Suppose odd $(j-1) \ge M(j-1)/2$. Then odd $(j-1) \ge M(j-1)/2 + y_j$ (for $j \le \log_2 n$), so odd $(j) \ge M(j-1)/2 + y_j + \tfrac{1}{2}(M - y_j) \ge Mj/2$.

Therefore odd $(j) \ge Mj/2$ for all $j < \log_2 n$. Therefore odd $(j) \ge Mj/2 + y_{j+1}$ (for $j < \log_2 n$).

Before the last move, there are $\sum_{i=0}^{b-1} K_{i,b-1}$ words of S_{b-1} occupied (writing b for $\log_2 n$), but

$$\begin{aligned}\sum_{i=0}^{b-1} K_{i,b-1} &= \frac{1}{2^{b-1}}\left\{\sum_{i=0}^{b-1} K_{i,b-1}(2^{b-1} - 2^i) + K_{i,b-1}\cdot 2^i\right\}\\ &= \frac{1}{2^{b-1}}\left\{\text{odd } (b-1) + \sum_{i=0}^{b-1} K_{i,b-1}\cdot 2^i\right\}\\ &\ge \frac{1}{2^{b-1}}\{M(b-1)/2 + y_b + M - y_b\} = M(b+1)/2^b.\end{aligned}$$

Now for the last move choose a set of alternate blocks which contains at least half the words in use and remove all such blocks which do not lie next to a gap containing an odd number of words of S_{b-1}. Then form as many n blocks as possible.

The number of words which would have been available for n blocks without the "odd gap rule" is at least $M/2$. Thus if there were x "odd gaps" the number actually available is at least $M/2 - x\cdot 2^{b-1}$. If there were U unused words of S_{b-1} before the removals, at most $(U - x)/2$ n blocks could have been placed since each occupies two adjacent words of S_{b-1} and the "odd gap rule" ensures that even after removals have been made, it is still the case that at most $(U - x)/2$ n blocks can be placed.

Therefore $(U - x)/2 \ge [(M/2 - x\cdot 2^{b-1})/2^b] \ge M/2^{b+1} - x/2 - (2^b - 1)/2^b$; i.e. $U/2 \ge M/2^{b+1} - (2^b - 1)/2^b$ and $U \ge M/2^b - (2^b - 1)/2^{b-1}$. Thus, since U is an integer, $U \ge M/2^b - 1$.

Therefore the total number of words of S_{b-1} is greater than or equal to $M(b+1)/2^b + M/2^b - 1$, which equals $M(b+2)/2^b - 1$.

Hence the store size is greater than or equal to $2^{b-1}\{M(b+2)/2^b - 1\} - (2^{b-1} - 1)$, which equals $M(1 + b/2) - 2^b + 1 = M(1 + \tfrac{1}{2}\log_2 n) - n + 1$.

Thus $N_2^*(M, n) \ge M(1 + \tfrac{1}{2}\log_2 n) - n + 1$ whenever $n \mid M$.

(b) $N_2^*(M, n) \le M(1 + \tfrac{1}{2}\log_2 n) - n + 1$ whenever $n \mid M$.

LEMMA 1. $N_2^*(2^{x+1}, 2^x) \le 2^x + x\cdot 2^x + 1$.

A defensive strategy which uses this area of store is as follows.

Divide the store into overlapping zones: the first zone has size 2^{x+1}; the 2^j zone has size 2^x and overlaps the last 2^{j-1} words of the 2^{j-1} zone $0 < j \le x$.

Define a 2^i area as a sequence of 2^i consecutive words, all in one zone, distant $k\cdot 2^i$ *from the start of the zone for some k*.

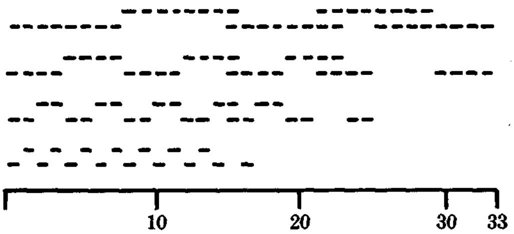

FIG. 1. Acceptable positions for blocks in Lemma 1

For $i < x$, place a 2^i block in any 2^i area of any 2^j zone ($j \le i$) except the last 2^i area of the 2^i zone, or place it in the second 2^i area of the 2^{i+1} zone. If $i = x$, place the 2^i block in any 2^i area. Figure 1 illustrates this in the case $x = 3$.

To see that this method cannot fail, suppose to the contrary that a 2^i block has been requested and cannot be placed.

First observe that if there is a set of permitted blocks which obstruct the 2^i block, then there is such a set which does not occupy any of the overlap areas, for the obstructive effect of a 2^j block on the overlap of the 2^{j-1} and 2^j zones is no more than that of two 2^{j-1} blocks on the penultimate 2^{j-1} area of the 2^{j-1} zone and the second 2^{j-1} area of the 2^j zone, and similarly for larger blocks.

Next, therefore assume the overlap areas are empty and consider the number of words in use in each zone.

Case i. $i = 0$. The first zone must contain $2^{x+1} - 1$ words in use. The second zone must contain 1 word in use. Thus 2^{x+1} words are in use, and the request for a 1 block is illegitimate.

Case ii. $0 < i < x$. Each 2^i area of the first zone must have at least one word in use. Thus the first zone contains $2^{x+1}/2^i$ words in use.

Each 2^i area of the 2^j zone $(0 < j < i)$ must have at least 2^j words in use except that the first 2^i area may have only 2^{j-1} words. Thus the 2^j zone contains $(2^x/2^i)2^j - 2^{j-1}$ words in use.

Each 2^i area of the 2^i zone contains a block except possibly the last one. Thus the 2^i zone has $2^x - 2^i - 2^{i-1}$ words in use.

The second 2^i area of the 2^{i+1} zone must be full. Thus the total number of words in use is at least

$$2^i + 2^{x+1-i} + 2^x \sum_{j=1}^{i} 2^{j-i} - \sum_{j=1}^{i} 2^{j-1} - 2^i = 2 \cdot 2^x - 2^i + 1,$$

so that the request for a 2^i block is illegitimate.

Case iii. $i = x$. As in case ii the first zone contains $2^{x+1}/2^x$ words in use and the 2^j zone contains $2^j - 2^{j-1}$ $(0 < j < x)$ words.

The 2^x area in the 2^x zone must contain at least 2^{x-1} words. Thus the total number of busy words is greater than or equal to $2 + \sum_{j=1}^{x} 2^{j-1} = 2^x + 1$, so that the request for a 2^x block is unacceptable.

This concludes the proof of Lemma 1.

LEMMA 2. *$\hat{N}_2(2kn, n) \le 2kn(1 + \frac{1}{2} \log_2 n)$ for integral k.*

Again, the defensive strategy divides the store into zones (this time nonoverlapping), and divides the zones into areas.

The first zone has size $2kn$. Every other 2^j zone has size kn.

A 2^i block can be placed in any 2^i area in any 2^j zone ($j \le i$). If a 2^i block cannot be placed, every 2^i area of every 2^j zone ($j \le i$) contains at least 2^j busy words.

Thus the number of busy words is greater than or equal to $2kn/2^i + kn \sum_{j=1}^{i} 2^{j-i} = 2kn$.

This concludes the proof of Lemma 2.

It has not been specified which of the 2^i areas is to be chosen if there are several. If it is specified that the first one is to be chosen, then the strategy is a "first fit buddy." That is, it differs from the strategy of [2] only in that the first adequate free block, rather than a minimal one, is chosen to be used or halved.

LEMMA 3. $N_2^*(n, n) \le N_2^*(n, n/2)$ $(n \ge 2)$.

If an n block is requested the store must be empty and the n block will be removed before any other allocations are made, so an n block can be placed anywhere and other blocks can be dealt with in the usual way for $N_2^*(n, n/2)$.

Now the result follows from the inequality:

$$\begin{aligned}
N_2^*(M_1 + M_2, n) &\le N_2^*(M_1, n) + \hat{N}_2(M_2, n)\\
N_2^*(2kn, n) &\le N_2^*(2n, n) + \hat{N}_2((2k-2)n, n)\\
&\le n + n\log_2 n + 1 + (2k-2)n(1 + \tfrac{1}{2}\log_2 n)\\
&= 2kn(1 + \tfrac{1}{2}\log_2 n) - n + 1,\\
N_2^*((2k+1)n, n) &\le N_2^*(n, n) + \hat{N}_2(2kn, n) \le N_2^*(n, n/2) + \hat{N}_2(2kn, n)\\
&= n/2 + n/2(\log_2 n - 1) + 1 + 2kn(1 + \tfrac{1}{2}\log_2 n)\\
&= (2k+1)n(1 + \tfrac{1}{2}\log_2 n) - n + 1.
\end{aligned}$$

(c) Combining (a) and (b), $N_2^*(M, n) = M(1 + \frac{1}{2}\log_2 n) - n + 1$ if $n \mid M$.

The way in which (c) depends on (b) and (b) depends on Lemma 2 implies that equality holds in Lemma 2, i.e. $\hat{N}_2(2kn, n) = 2kn(1 + \frac{1}{2}\log_2 n)$, and this optimal value is given by the first fit buddy strategy.

The Upper Bound

Given $K > \{\frac{3}{2}\log 2 - \frac{1}{2} + \sum_{k=2}^{\infty}(1/2k)\log(1 + 1/k)\}^{-1}$, there exists an X such that $\hat{N}(M, n) < M(X + K\log n) + 4n^2 + n$.

Proof. Let

$$\Sigma = \tfrac{3}{2}\log 2 - \tfrac{1}{2} + \sum_{k=2}^{\infty}(1/2k)\log(1 + 1/k) \text{ and}$$

$$\begin{aligned}
\Sigma(j) = \tfrac{1}{2}\log 2 + \sum_{k=1}^{j-1}\{&(1/2k)\log((2k^2 + k - 1)/2k^2)\\
&+ (1/2(k+1))\log((2k+2)/(2k+1)) + \log((2k^2+k)/(2k^2+k-1))\\
&- 1/2k(k+1)\} - (1/2j)\log(j).
\end{aligned}$$

Since

$$\begin{aligned}
&\sum_{k=1}^{j-1}\{(1/2k)\log((2k^2 + k - 1)/2k^2) + (1/2(k+1))\log((2k+2)/(2k+1))\}\\
&\qquad = \sum_{k=2}^{j-1}(1/2k)\log((2k^2 + k - 1)/2k^2) + \sum_{k=2}^{j}(1/2k)\log(2k/(2k-1))\\
&\qquad = \sum_{k=2}^{j-1}(1/2k)\log((k+1)/k) + (1/2j)\log(2j/(2j-1))
\end{aligned}$$

and

$$\begin{aligned}
\sum_{k=1}^{j-1}\log((2k^2+k)/(2k^2+k-1)) &= \sum_{k=1}^{j-1}\{\log(2k+1) - \log(2k-1) + \log(k)\\
&\qquad - \log(k+1)\}\\
&= \log(2j-1) - \log(1) + \log(1) - \log(j)\\
&= \log((2j-1)/j)
\end{aligned}$$

and

$$\sum_{k=1}^{j-1}(1/2k(k+1)) = \sum_{k=1}^{j-1}(1/2k - 1/2(k+1)) = \tfrac{1}{2} - 1/2j,$$

$$\lim_{j\to\infty}\Sigma(j) = \sum_{k=2}^{\infty}(1/2k)\log((k+1)/k) + \log(2) - \tfrac{1}{2} + \tfrac{1}{2}\log(2) = \Sigma.$$

Thus there exists J such that $\Sigma(j) > 1/K$ for all $j \geq J$.

Now an allocation strategy can be described which demonstrates that

$$\hat{N}(M, n) \leq KM \sum_{i=1}^{2n} 1/i + 2JM + 4n^2 + n;$$

i.e. taking $X = K(1 + \log 2) + 2J$,

$$\hat{N}(M, n) \leq M(X + K \log n) + 4n^2 + n.$$

The strategy is as follows.

Divide the store into nonoverlapping i zones ($i = 1, \cdots, 2n$) such that the area of the i zone is

$$\begin{cases} KM + 2JM + n \text{ for } i = 1, \text{ or} \\ KM/i + n \text{ rounded up to a multiple of } i \text{ for } i > 1. \end{cases}$$

The i zone is divided into "i areas" of size i.

A block of size j may be placed either in any i zone ($i \leq j$) provided the number of i areas wholly or partly covered is less than or equal to $2j/i$, or on the last j squares of an i area ($i \leq 2j$).

Putting this another way, a block of size j may be placed in the i zone subject to the following rules:

if $i/2 \leq j < i$, the block may only be at the end of an i area.
if $i \leq j < 3i/2$, the block may straddle only one i area boundary.
if $j \geq 3i/2$, the block may be placed anywhere in the i zone.

Figure 2 shows part of a 10 zone containing blocks of these three types.

The proof depends on showing that if a block of size j cannot be placed by this strategy, then in each i zone ($1 \leq i \leq 2j$) there is at least a certain density $\delta(i, j)$ of busy words, except perhaps in the last n words of the zone. Thus the total number of busy words is at least $\sum_{i=1}^{2j} [(KM/i)\delta(i, j) + 2JM\delta(1, j)]$, which will be shown to be at least M. The details of the proof are mainly concerned with the determination of the density function.

In the i zone a block whose size is less than i is called a "small" block and any other block a "large" block.

(a) If $j \leq i \leq 2j$ and a block of size j cannot be placed in the i zone, every i area is at least partly covered. In the i areas spanned by any block, the density is greater than or equal to $\frac{1}{2}$. Thus

$$\delta(i, j) \geq \tfrac{1}{2}. \tag{i}$$

(b) If $\frac{2}{3}j \leq i \leq j$ and a j block cannot be placed in the i zone, divide the i zone into segments as follows:

First divide the zone at the end of every small block; then subdivide each division into segments so that every segment has size $2i$ except the last in each division, which has size i or $2i$. Every segment consists of one or two i areas. Since the j block cannot go into a segment of size $2i$, part at least of a block is already there. If there is a whole block not at either end of the segment, it is a large block, and the density in the segment is greater than or equal to $i/2i$. If the obstruction is caused by one or more blocks at or overlapping the ends, these blocks leave a gap of less than j, so the density is greater than $(2i - j)/2i$. In a segment of length i, except possibly at the end of the zone, there is a small block giving density greater than or equal to $\frac{1}{2}$. Thus the density is greater

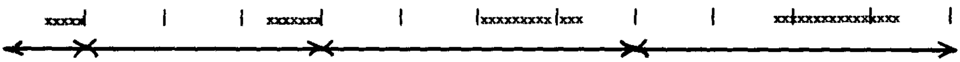

FIG. 2. Division of part of a 10 zone into twelve 10 areas and four sections

than or equal to $(2i - j)/2i$ over all the zone except possibly the last i words; i.e.

$$\delta(i, j) \geq (2i - j)/2i \quad (i \leq j \leq \tfrac{3}{2}i). \tag{ii}$$

(c) In the general case it is convenient to consider the integer k such that $ki \leq j < (k + 1)i$ (including $k = 1$ when $i < \frac{2}{3}j$).

Divide the i zone into sectors, the divisions occurring at the end of any i area in which a block ends (Figure 2 shows this division), and suppose a j block cannot be placed in the i zone.

If there is an empty sector at the end of the zone, its size is less than j and so less than n. Thus at least KM/i words are contained in nonempty sectors. There are three types of nonempty sectors to consider:

(1) The sector ends with a small block and has length less than or equal to ki; density $\geq \frac{1}{2}i/ki = 1/2k$.

(2) The sector ends with a small block and has length $= (k + 1)i$ and there is no other block in the sector; density $\geq \frac{1}{2}i/(k + 1)i = 1/2(k + 1)$, and density $>$ $[(k + 1)i - j]/(k + 1)i$ since otherwise there would be a gap of j before the small block.

(3) The sector ends with an i area containing all or part of a large block. This large block starts in or before the $(k + 1)$-th i area of the sector. If it ends in or before the $(k + 2)$-th i area, density $\geq i/(k + 2)i = 1/(k + 2)$. Otherwise it extends at least from the jth word of the sector to the first word of the last i area of the sector giving density $\geq [(k + 2 + x)i + 2 - j]/(k + 3 + x)i$ where $(k + 3 + x)i$ is the size of the sector and $x \geq 0$, density $\geq [(k + 2)i + 2 - j]/(k + 3)i$.

Defining

$$\left.\begin{aligned} D_1(i, j) &= 1/2k, \\ D_2(i, j) &= \max(1/2(k + 1), [(k + 1)i - j]/(k + 1)i), \\ D_3(i, j) &= [(k + 2)i + 2 - j]/(k + 3)i, \\ D_4(i, j) &= 1/(k + 2), \end{aligned}\right\} \text{ where } k = [j/i],$$

and

$$D(i, j) = \begin{cases} \frac{1}{2} & (j \leq i \leq 2j), \quad \text{(i)} \\ (2i - j)/2i & (i < j \leq \frac{3}{2}i), \quad \text{(ii)} \\ \min(D_1(i, j), D_2(i, j), D_3(i, j), D_4(i, j)), & \text{otherwise,} \end{cases}$$

then $\delta(i, j) \geq D(i, j)$ wherever $\delta(i, j)$ is defined.

Suppose for the time being $j > 3i/2$ to determine which of D_1, D_2, D_3, D_4 is minimum.

Then if $k = 1$, $[(k + 1)i - j]/(k + 1)i < (2i - 3i/2)/2i = \frac{1}{4} = 1/2(k + 1)$, so $D_2(i, j) = \frac{1}{4} < D_4(i, j)$, while if $k > 1$, $D_1(i, j) \leq D_4(i, j)$.

Moreover if $k = 1$, $D_3(i, j) > [(k + 2)i - 2i]/(k + 3)i = \frac{1}{4} = D_2(i, j)$, while if $k > 1$, $D_3(i, j) > [(k + 2)i - j]/(k + 3)i$; but $[(k + 2)i - j]/(k + 3)i > [(k + 1)i - j]/(k + 1)i$ and $[(k + 2)i - j]/(k + 3)i > i/(k + 3)i = 1/(k + 3) \geq 1/2(k + 1)$; i.e. $D_3(i, j) > D_2(i, j)$.

Hence $\min(D_1(i, j), D_2(i, j), D_3(i, j), D_4(i, j)) = \min(D_1(i, j), D_2(i, j))$.

There are three cases.

Case i. $j > (k + \frac{1}{2})i$. $(k + 1)i - j < \frac{1}{2}i$, so $D_2(i, j) = 1/2(k + 1) < 1/2k = D_1(i, j)$. Hence

$$D(i, j) = D_2(i, j) = 1/2(k + 1). \tag{iii}$$

Case ii. $j < [(2k^2 + k - 1)/2k]i$. $j < (k + \frac{1}{2})i$. Hence $(k + 1)i - j > \frac{1}{2}i$, so $D_2(i, j) = [(k + 1)i - j]/(k + 1)i$. Thus $D_2(i, j) > [2k(k + 1)i - (2k^2 + k - 1)i]/2k(k + 1)i = 1/2k = D_1(i, j)$. Thus

$$D(i, j) = D_1(i, j) = 1/2k. \tag{iv}$$

Case iii. $[(2k^2 + k - 1)/2k]i \le j \le (k + \frac{1}{2})i$. For $k = 1$, this is the case $i \le j \le \frac{3}{2}i$, which is irrelevant to the minimization, so assume $k > 1$.

Again $j \le (k + \frac{1}{2})i$, so that $D_2(i, j) = [(k + 1)i - j]/(k + 1)i$. Hence $D_2(i, j) \le [2k(k + 1)i - (2k^2 + k - 1)i]/2k(k + 1)i = 1/2k = D_1(i, j)$. Thus

$$D(i, j) = D_2(i, j) = [(k + 1)i - j]/(k + 1)i, \tag{v}$$

which can be seen from (ii) to be valid also for $k = 1$.

Comparing (i), (iii), (iv), and (v), $D(i, j)$ is seen to be continuous as i varies and to be constant except in the ranges $[(2k^2 + k - 1)/2k]i \le j \le (k + \frac{1}{2})i$ where it rises from $1/2(k + 1)$ to $1/2k$.

If a j block cannot be placed in any i zone $(1 \le i \le 2j)$, the total number of busy words is greater than or equal to $\sum_{i=1}^{2j} (KM/i)\delta(i, j) + 2JM\delta(1, j)$, which is greater than or equal to $KM \sum_{i=1}^{2j} (D(i, j)/i + 2JM/2j)$.

Hence if $j \le J$ the number of busy words is greater than or equal to M, while if $j > J$ the number of busy words is greater than or equal to $KM \sum_{i=1}^{2j} (D(i, j)/i)$.

If $D(i, j)/i$ was a monotonic decreasing function of i, this would imply busy words greater than or equal to $KM \int_{i=1}^{2j+1} (D(i, j)/i)di$, but in fact $D(i, j)/i$ is decreasing everywhere except in the ranges $[(2k^2 + k - 1)/2k]i \le j \le (k + \frac{1}{2})i$, where it is equal to $[(k + 1)i - j]/(k + 1)i^2 = 1/i - j/(k + 1)i^2$. In these ranges

$$(d/di)\{D(i, j)/i\} = -1/i^2 + 2j/(k + 1)i^3 = [2j - (k + 1)i]/(k + 1)i^3 \ge 0.$$

So $D(i, j)/i$ is increasing in the range and increases there by

$$\begin{aligned}
&(2k^2 + k - 1)/2kj - (2k^2 + k - 1)^2/4k^2(k + 1)j - (k + \tfrac{1}{2})/j + (k + \tfrac{1}{2})^2/(k + 1)j \\
&\quad = [2k^2 + k - 1 - 2k(k + \tfrac{1}{2})]/2kj \\
&\qquad + [4k^2(k^2 + k + \tfrac{1}{4}) - (4k^4 + k^2 + 1 + 4k^3 - 4k^2 - 2k)]/4k^2(k + 1)j \\
&\quad = -1/2kj + (4k^2 + 2k - 1)/4k^2(k + 1)j = (2k^2 - 1)/4k^2(k + 1)j < 1/2j(k + 1).
\end{aligned}$$

Hence

$$\begin{aligned}
\sum_{i=1}^{2j} (D(i,j)/i) &> \int_1^{2j+1} (D(i,j)/i)\, di - \sum_{k=1}^{j-1} (1/2j(k + 1)) \\
&> \int_j^{2j+1} (1/2i)\, di - (1/2j) \log (j) + \sum_{k=1}^{j-1} \int_{j/(k+1)}^{j/k} (D(i, j)/i)\, di \\
&> \tfrac{1}{2} \log 2 - (1/2j) \log (j) + \sum_{k=1}^{j-1} \left\{ \int_{j/(k+1)}^{j/(k+\frac{1}{2})} (1/2(k + 1)i)\, di \right. \\
&\qquad + \left. \int_{j/(k+\frac{1}{2})}^{2kj/(2k^2+k-1)} ((k + 1)i - j)/(k + 1)i^2\, di + \int_{2kj/(2k^2+k-1)}^{j/k} (1/2ki)\, di \right\} \\
&= \tfrac{1}{2} \log 2 - (1/2j) \log j + \sum_{k=1}^{j-1} \left\{ (1/2(k + 1)) \log ((2k + 2)/(2k + 1)) \right. \\
&\qquad + \log ((2k^2 + k)/(2k^2 + k - 1)) \\
&\qquad \left. - (1/2k(k + 1)) + (1/2k) \log ((2k^2 + k - 1)/2k^2) \right\} \\
&= \Sigma(j) > 1/K \qquad \text{since } j > J.
\end{aligned}$$

Hence the number of busy words is greater than or equal to $KM \sum_{i=1}^{2j} D(i, j)/i > KM/K = M$.

This implies $\hat{N}(M, n) \le 2JM + \sum_{i=1}^{2n} (KM/i + i + n) \le 2JM + \sum_{i=1}^{2n} (KM/i) + 4n^2 + n < 2JM + KM(1 + \log 2n) + 4n^2 + n = M(X + K \log n) + 4n^2 + n$.

The value given for X and the term $4n^2 + n$ could no doubt be reduced considerably by variations on this argument. The limit on K is the best possible for this strategy if all legal allocations are possible; e.g. if when there is a choice it is made at random. However, if this choice is made on some sensible basis such as a first fit rule, it is possible that the value of K can be lowered by a more detailed analysis. I would conjecture that for this strategy or some other, K can be reduced to 1.

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