Equation of angular motion

Let us consider a rigid body which rotates in an axis with angular displacement θ and linear displacement S, them from figure,

$$\mathbf{S = \theta r}$$

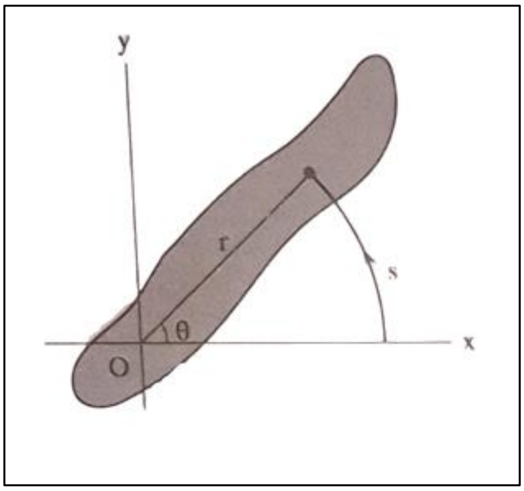


Fig: A rigid body rotating about an axis through

If the angular displacement is $\mathbf{d\theta}$ in time interval dt then angular velocity is given by,

$$\boldsymbol{\omega} = \frac{\mathbf{d\theta}}{\mathbf{dt}} - - - (\mathbf{i})$$

Also,

Angular acceleration $(\alpha) = \frac{\mathbf{d\omega}}{\mathbf{dt}} - - - (\mathbf{ii})$ (angular velocity not constant)

The relation between linear velocity and angular velocity is;

$\mathbf{V} = \frac{\mathbf{ds}}{\mathbf{dt}} = \frac{\mathbf{d(r\theta)}}{\mathbf{dt}} = \mathbf{r}\frac{\mathbf{d\theta}}{\mathbf{dt}} = \mathbf{r\omega} - - - (\mathbf{iii})$ [from (i)]

Again,

Angular acceleration and linear acceleration are related as

$\mathbf{a} = \frac{\mathbf{dv}}{\mathbf{dt}} = \mathbf{r}\frac{\mathbf{d\omega}}{\mathbf{dt}} = \mathbf{r\alpha} - - - (\mathbf{iv})$ [from (ii)]

Moment of inertia of a rotating body

The inability of a body to change its state of rest or state of uniform rotational motion by itself is called inertia. This inertia of a body in rotational motion is called rotational inertia or moment of inertia.

Let us consider a rigid body that consists of particles of masses m_1, m_2, m_3 --- m_n with distance r_1, r_2, r_3……… rn from the axis of rotational AB. The moment of inertia of the body about axis AB is defined by,

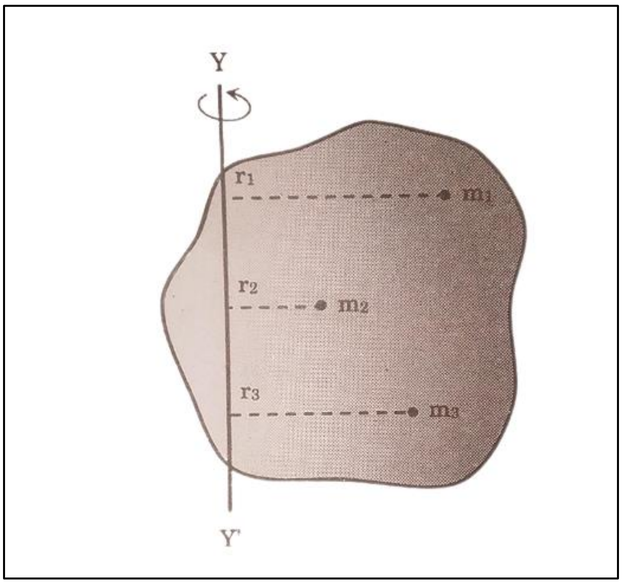


Fig: A rigid body rotation

$I = I_1 + I_2 + I_3 \ldots \ldots . + I_n$

$I = m_1{r_1}^2 + m_2{r_2}^2 + \frac{1}{2}m_3{r_3}^2 + \cdots + m_n{r_n}^2$

$\therefore\ I = \sum_{i=1}^{n} m_i{r_i}^2$

Thus, a moment of inertia of a rigid body is defined as the sum of the product of mass and the square of the distance from the axis of rotation of individual particles.

Kinetic energy of rotation of rigid body.

Let us consider a rigid body that consists of particles of masses m_1, m_2..., m_n with distances r_1, r_2…., r_n from the axis of rotation AB. Also, let **ω** be the angular velocity with which it is rotating and v_1, v_2…. v_n be the linear velocity of respective particles m_1, m_2…., m_n. Then

$v_1 = r_1\omega, v_2 = r_2\omega, v_3 = r_3\omega, \ldots \ldots . v_n = r_n\omega$

Rotational kinetic energy of particles of mass m_1,

$= \frac{1}{2}m_1{v_1}^2 = \frac{1}{2}m_1(\omega r_1)^2 = \frac{1}{2}m_1{r_1}^2\omega^2$

Similarly, the rotational kinetic energy of particles m_2, m_3…….m_n, are $\frac{1}{2}m_2{r_2}^2\omega^2$, $\frac{1}{2}m_3{r_3}^2\omega^2$………… $\frac{1}{2}m_n{r_n}^2\omega^2$ Respectively.

The rotational K.E of the rigid body is given by the sum of the kinetic energy of various constituent particles.

Rotational K.E of rigid body, K.E = $\frac{1}{2}m_1{r_1}^2\omega^2 + \frac{1}{2}m_2{r_2}^2\omega^2 + \frac{1}{2}m_3{r_3}^2\omega^2 + \ldots\ldots\ldots + \frac{1}{2}m_n{r_n}^2\omega^2$

$= \frac{1}{2}\left(m_1{r_1}^2 + m_2{r_2}^2 + m_3{r_3}^2 + \cdots \ldots \ldots . + m_n{r_n}^2\right)\omega^2$

$$= \frac{1}{2} \left(\sum_{i=1}^{n} m_i r_i^2 \right) \omega^2$$

$$= \frac{1}{2} I\omega^2 \quad [\therefore I = \sum_{i=1}^{n} m_i r_i^2]$$

$\therefore$ Rotational K.E of body = $\frac{1}{2} I\omega^2$

So, the rotational K.E of a body is equal to half of the product of the momentum of inertia of the body and the square of the angular velocity of the body about the given axis of rotation.

Radius of Gyration

The perpendicular distance between the center of mass and the axis of rotation of a rigid body is called the radius of gyration. If 'M' is the mass of the body and K is its radius of gyration, then the moment of inertia of the body is given by,

$$I = MK^2$$

Calculation of moment of inertia of rigid bodies

Moment of inertia of the uniform rod

a) About the axis passing through the center and perpendicular to its length

Let us consider a uniform rod of length 'l' and mass 'm'. Let YY' be the axis passing through the center and perpendicular to its length about which moment of inertia is to be determined.

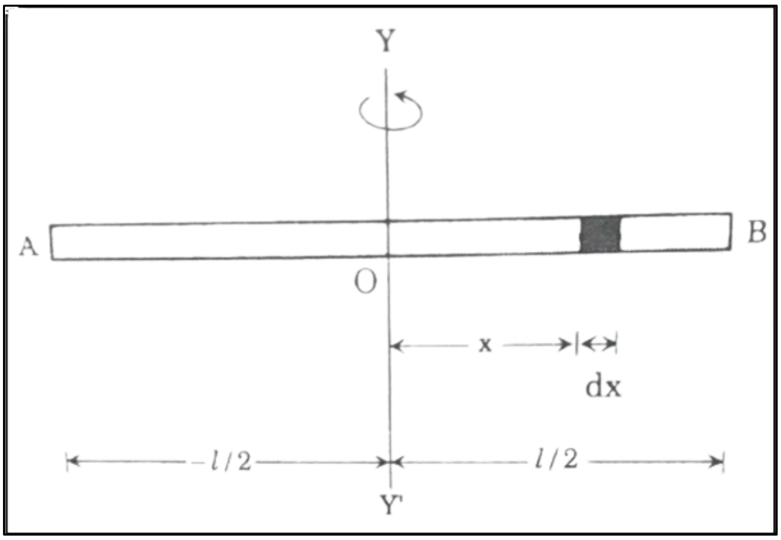


Fig: Moment of inertia of the uniform rod

Consider a small segment of length 'dx' at a distance 'x' from axis YY' shown in figure.

Now mass per unit length of the rod = $\frac{m}{l}$

So, mass of the small segment (dm) = $\left(\frac{m}{l}\right) . dx$

Now, a moment of inertia of a small segment about axis YY',

$$dI = dmx^2$$
$$dI = \frac{m}{l}x^2dx - - - (i)$$

Thus, a moment of inertia of the whole rod about axis AB is obtained by integrating equation (i) as,

$$I = \int_{-\frac{l}{2}}^{\frac{l}{2}} dI$$

$$= \int_{-\frac{l}{2}}^{\frac{l}{2}} \frac{m}{l}x^2dx$$

$$= \frac{m}{l}\left[\frac{x^3}{3}\right]_{-\frac{l}{2}}^{\frac{l}{2}}$$

$$= \frac{m}{l}\left[\frac{\left(\frac{l}{2}\right)^3}{3} - \frac{\left(-\frac{l}{2}\right)^3}{3}\right]$$

$$= \frac{m}{l}\left[\frac{(l)^3}{24} + \frac{(l)^3}{24}\right]$$

$$= \frac{m}{l}\left[\frac{l^3+l^3}{24}\right]$$

$$= \frac{m}{l}\left[\frac{2l^3}{24}\right]$$

$$I = \frac{ml^2}{12}$$

This is a required expression for M.I of a uniform rod when the axis passes through the center and is perpendicular to its length.

b. About the axis passing through one end and perpendicular to the length.

Let us consider a uniform rod of length 'l' and mass 'm'. Let AB be the axis passing through one end and perpendicular to its length about which moment of inertia is to be determined.

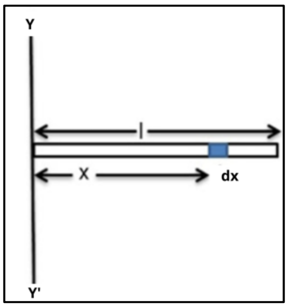


Fig: moment of inertia of the rod

Also, consider a small segment of length 'dx' at distance 'x' from axis YY' as shown in the figure;

Now, mass per unit length of the rod = $\frac{m}{l}$

So, mass of segment (dm) = $\left(\frac{m}{l}\right).dx$

Now, the moment of inertia of a small segment about axis AB,

$$dI = dmx^2$$
$$dI = \frac{m}{l}x^2dx --- (i)$$

Thus, a moment of inertia of the whose rod about the axis of AB is obtained by integrating equation (i) as,

$$I = \int_{-\frac{l}{2}}^{\frac{l}{2}} dI$$

$$= \int_0^l \frac{m}{l}x^2dx$$

$$= \frac{m}{l}\left[\frac{x^3}{3}\right]_0^l$$

$$= \frac{m}{l} * \frac{l^3}{3}$$

$$I = \frac{ml^2}{3}$$

This is a required expression for 'I' of a uniform rod when the axis passes through one end and perpendicular to its length.

Moment of inertia of circular ring.

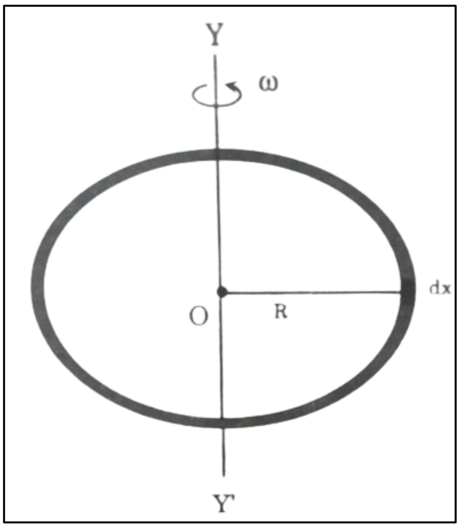


Fig: Moment of inertia of a circular ring

Let us consider a circular ring having radius R and mass 'M' then, Circumference of the ring = $2\pi R$

Mass per unit length = $\frac{M}{2\pi R}$

Let us consider an axis YY' passing through a center of ring and perpendicular to its plane and take a small element of length 'dx' in the ring then its mass is given by

$$dm = \left(\frac{M}{2\pi R}\right) dx$$

Moment of inertia of element about the axis YY'

$$dI = dm.\, R^2$$

$$= \left(\frac{M}{2\pi R}\right) dx\, R^2$$

$$= \frac{MR}{2\pi}.\, dx --- \text{(i)}$$

The total moment of inertia of circular ring axis passing through center of ring is obtained by integrating equation(i) from 0 to $2\pi R$.

$$I = \int_0^{2\pi R} \frac{MR}{2\pi}.\, dx$$

$$= \frac{MR}{2\pi}\int_0^{2\pi R} dx$$

$$= \frac{MR}{2\pi}[x]_0^{2\pi R}$$

$$= \frac{MR}{2\pi}[\, 2\pi R - 0]$$

$$\therefore I = MR^2$$

This equation gives the moment of inertia of circular ring.

1.4.3 Moment of inertia of thin circular disc.

Let us consider a circular disc having radius 'R' and mass 'M' rotating about an axis YY' passing through its center 'o' and perpendicular to its plane as shown in figure above. Then,

Area = πR^2

Mass per unit area $= \frac{M}{\pi R^2}$

Also, consider a thin element of disc of thickness 'dx' at distance 'x' from axis then,

Area of elementary position (da) = dx x $2\pi x$

And

Mass of elementary position (dm)= $\frac{M}{\pi R^2}$ x dx x $2\pi x$

$$= \frac{2Mx}{R^2} dx$$

Now, Moment of inertia element about axis YY'

$= dmx^2$

$= \left(\frac{2Mx}{R^2} dx\right) x^2$

$= \frac{2Mx^3}{R^2} dx - - - \text{(i)}$

The total inertia of circular disc axis passing through center and perpendicular to the plane is obtained by integrating equation (i) from 0 to R

$$I = \int_0^R \frac{2Mx^3}{R^2} . dx$$

$$= \frac{M}{R^2} \int_0^R x^3 dx$$

$$= \frac{MR}{R^2} \left[\frac{x^4}{4}\right]_0^R$$

$$= \frac{M}{R^2} \left[\frac{R^4}{4}\right]$$

$$\therefore I = \frac{MR^2}{2}$$

This is the equation gives the moment of inertia of a thin circular disc about an axis passing through its center and perpendicular to its plane.

Torque and angular acceleration for a rigid body.

Torque

The turning effect of a force in a body is called torque. It is denoted by τ and given by

Torque = force x perpendicular distance from the axis of rotation

or, $\tau = rF$

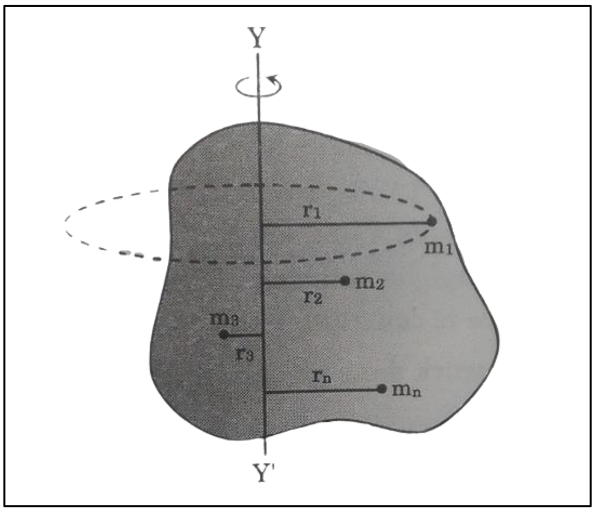
Angular acceleration

The rate of change of angular velocity at any instant of time is called angular acceleration. It is denoted by α and given by

$$\alpha = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2}$$

Relation between torque and moment of inertia

Let us consider a rigid body that consists of particles of masses m_1, m_2, m_3, --- m_n, with distance r_1, r_2, r_3 --- r_n from the axis of rotation AB. Suppose torque (T) is applied to the body which produces angular acceleration (α) on the body. Let F_1, F_2, F_3... F_n are the forces acting on individual particles producing acceleration a_1, a_2, a_3 ... a_n such that $a_1 = r_1\alpha$, $a_2 = r_2\,\alpha$ --- $a_n = r_n\alpha$



Now, the force acting on the first particles,

$F_1 = m_1\, a_1$

$\quad = m_1 r_1\, \alpha$

Again,

Torque acting on this particle about axis of rotation,

$\tau_1 = r_1\, F_1$

$= r_1 (m_1 r_1 \alpha)$

$= m_1 r_1^2 \alpha$

Similarly, Torque acting on other particles,

$\tau_2 = m_2 r_2^2 \alpha$, $\tau_3 = m_3 r_2^2 \alpha$, ,... $\tau_n = m_n r_n^2 \alpha$

Thus,

Now torque on the whole body = Sum of individual torque

$\tau = \tau_1 + \tau_2 + \tau_3 + \ldots\ldots. + \tau_n$
$= m_1 r_1^2 \alpha + m_2 r_2^2 \alpha + m_3 r_2^2 \alpha + \ldots\ldots\ldots.. + m_n r_n^2 \alpha$
$= (m_1 r_1^2 + m_2 r_2^2 + m_3 r_2^2 + \ldots\ldots\ldots.. + m_n r_n^2)\alpha$
$\therefore \tau = \alpha I$ $[\therefore I = \sum mr^2]$

This is the relation between the moment of inertia of a body and torque.

Work done by a couple and power in rotational motion

Two equal and unlike parallel forces acting at two different points of rigid body forms couple.

Let us consider, that a wheel is to be acted upon by a couple of forces (F, F) at points A and B. Let the wheel turns through angle 'θ' in a time 'dt' such that points A and B are displaced to points A' and B' and let 'S' be the linear displacement.

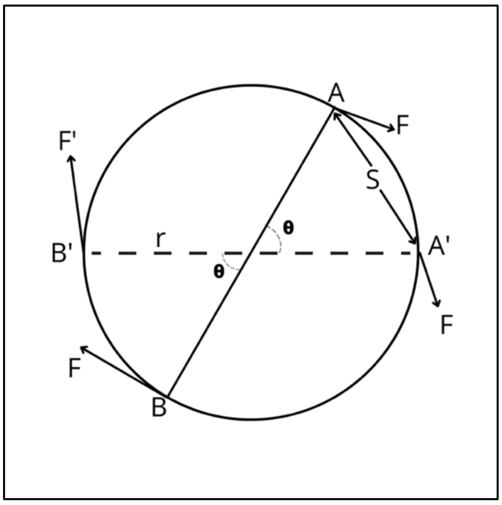


Fig: Work done by the torque

Now, work done by force at point A,

$W_A = F.S$

Also, the work done by force at point B,

$W_B = F.S$

$\because$ Total work done (W) = $W_A + W_B$

= F.S +F.S

$= 2FS ---(i)$

Also, from the figure,

$\boldsymbol{\theta = \frac{s}{r}}$

$\mathbf{i.e\ S = \theta r}$

$\because$ Work done (w) = 2f θr

= (f.2r) θ

$= \tau\theta$ [$\because$ f.2r is the torque due to couple]

So, work done by a couple (w) = τθ

Thus, work done by a couple is the product of torque and the angle of rotation of the rigid body.

Again,

Power (p)= $\frac{dW}{dt}$

$= \frac{d(\tau\theta)}{dt}$

$= \tau\frac{d\theta}{dt}$

$\therefore p = \tau\omega$

In rotational motion the product of torque and angular velocity is power.

Angular momentum

The moment of linear momentum of an object is called angular momentum. It is denoted by 'L' and given by

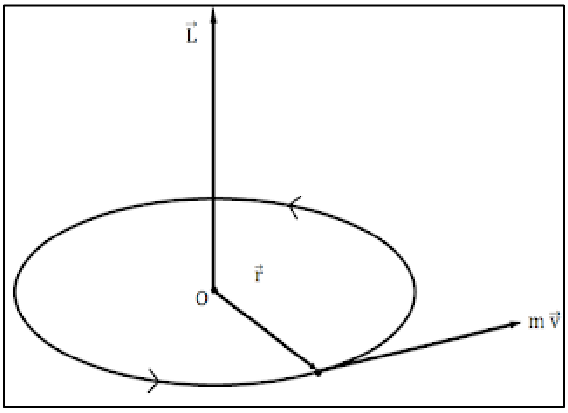


Fig: Angular momentum of a particle

L = linear momentum X perpendicular distance from the axis of rotation

$L = mvr \;---\; (i)$

Since $v = \omega r$

$L = m(\omega r)r$

$L = m\omega r^2 \;---\; (ii)$

Equations (i) and (ii) is an expression for angular momentum. It is a vector quantity and its unit is kgm^2/s

Relation between angular momentum and moment of inertia

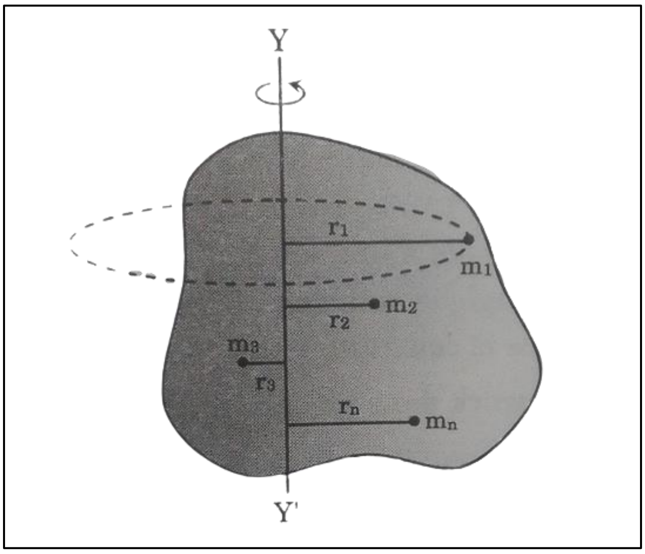


Fig: A rigid body rotating about an axis YY' has angular momentum Iω.

Let us consider a rigid body that consists of particles of masses m_1, m_2, m_3, --- m_n with distance r_1, r_2, r_3 --- r_n from the axis of rotation AB. Also let the body be rotating with angular velocity 'ω' and v_1, v_2, v_3... v_n be the linear velocities of respective particles m_1, m_2, m_3, --- m_n then,

$v_1 = r_1\omega$, $v_2 = r_2\omega$, $v_3 = r_3\omega$, ………… $v_n = r_n\omega$

Now linear momentum of 1st particles

$p = m_1v_1$

$\;\; = m_1 r_1\omega$

So, angular momentum of 1st particles

$L_1 = r_1p_1$

$\;\; = r_1(m_1 r_1\omega)$

$= m_1r_1^2\ \omega$

Similarly, angular momentums of the particle are

$L_2 = m_1r_1^2\ \omega$, $L_3 = m_3r_2^2\ \omega$, ,... $L_n = m_nr_n^2\ \omega$

Thus, the total angular momentum of whole body be,

$L= L_1+ L_2+ L_3 + \ldots\ldots + L_n$
$= m_1r_1^2\ \omega + m_2r_2^2\ \omega + m_3r_2^2\ \omega + \ldots\ldots\ldots. +m_nr_n^2\ \omega$
$= (m_1r_1^2 + m_2r_2^2 + m_3r_2^2 + \ldots\ldots\ldots. +m_nr_n^2)\ \omega$
$\therefore \tau = \omega I$ $[\therefore I=\sum mr^2]$

This is the required relation between angular momentum and the moment of inertia of a body which shows that the magnitude of angular momentum of a body about a given axis is equal to the product of the moment of inertia 'I' of the body and its angular velocity 'w 'about that axis.

Relation between angular momentum and torque

We have, the angular momentum (L) of a rigid body rotating about an axis with angular velocity 'ω' is

$$L = \omega I \quad --- \ (i)$$

Where, I = Moment of inertial of a body

Differentiating equation (i) with respect to time, we get

$$\frac{dL}{dt} = \frac{d(\omega I)}{dt}$$

$$\frac{dL}{dt} = I\frac{d\omega}{dt}$$

$$\frac{dL}{dt} = I\alpha --- (ii)$$

Where $\alpha = dw / dt$ = angular acceleration

Also,

torque on the body is,

$\tau = I\alpha$ --- (iii)

Comparing equations (ii) and (iii) we get

$$\tau = \frac{dL}{dt}$$

This is the required relation between torque and angular momentum and this relation shows that torque acting on a body is equal to the rate of change of angular momentum of the body.

Principle of Conservation of Angular Momentum

It states that " If no external torques act on the system, then total angular momentum remains constant."

i.e. If $\tau = 0$ then,

L = Constant

IW = constant

In general, $I_1W_1 = I_2W_2$

proof

Since torque acting on a body is equal to the rate of change of angular momentum.

i. e $\tau = \frac{dL}{dt}$

If $\tau = 0$ then,

$\frac{dL}{dt} = 0$

$dL = 0$

On integrating, we get

L= constant
IW = constant
In general $I_1W_1 = I_2W_2$

This proves the principle of conservation of angular momentum.

Translation Motion	**Rotational Motion**
1. Linear displacement, s	1. Angular displacement, θ
2. Linear velocity, $v = \frac{ds}{dt}$	2. Angular velocity, $\omega = \frac{d\theta}{dt}$
3. Linear acceleration, $a = \frac{dv}{dt} = \frac{d^2s}{dt^2}$	3. Angular acceleration, $\alpha = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2}$
4. Mass, m	4. Moment of inertia, I
5. Linear momentum, $P = mv$	5. Angular momentum, $L = I\,\omega$
6. Force, $F = \frac{dp}{dt} = m\,a$	6. Torque, $\tau = \frac{dL}{dt} = I\,\frac{d\omega}{dt} = I\alpha$
7. Work done by force, $W = Fs$	7. Work done by torque, $W = \tau\,\theta$
8. Translational $K.E. = \frac{1}{2}mv^2$	8. Rotational $K.E. = \frac{1}{2}I\omega^2$
9. Equations of translational motion. i) $s = ut$ ii) $v = u + at$ iii) $s = ut + \frac{1}{2}at^2$ iv) $v^2 = u^2 + 2as$	9. Equations of rotational motion. i) $\theta = \omega t$ ii) $\omega = \omega_0 + \alpha\,t$ iii) $\theta = \omega_0\,t + \frac{1}{2}\,\alpha\,t^2$ iv) $\omega_2 = \omega_0^2 + 2\,\alpha\,\theta$