Tables of Integral Transforms

CALIFORNIA INSTITUTE OF TECHNOLOGY
BATEMAN MANUSCRIPT PROJECT

A. Erdélyi, *Editor*

W. Magnus, F. Oberhettinger, F. G. Tricomi, *Research Associates*

Higher Transcendental Functions, 3 volumes.
Tables of Integral Transforms, 2 volumes.

TABLES OF INTEGRAL TRANSFORMS

Volume II

Based, in part, on notes left by

Harry Bateman

Late Professor of Mathematics, Theoretical Physics, and Aeronautics at the California Institute of Technology

and compiled by the

Staff of the Bateman Manuscript Project

Prepared at the California Institute of Technology under Contract No. N 6onr-244 Task Order XIV with the Office of Naval Research

Project Designation Number: NR 043-045

NEW YORK TORONTO LONDON
McGRAW-HILL BOOK COMPANY, INC.
1954

TABLES OF INTEGRAL TRANSFORMS, vol. II

PRINTED IN THE UNITED STATES OF AMERICA

Library of Congress Catalog Card Number: 54-6214

19550 9 10 11 12 13 14 15 16-MAMM-7 6 5 4

This work is dedicated to the
memory of

HARRY BATEMAN

as a tribute to the imagination which led him to undertake a project of this magnitude, and the scholarly dedication which inspired him to carry it so far toward completion.

STAFF OF THE BATEMAN MANUSCRIPT PROJECT

Director

Arthur Erdélyi

Research Associates

Wilhelm Magnus (1948-50)
Fritz Oberhettinger (1948-51)
Francesco G. Tricomi (1948-51)

Research Assistants

David Bertin (1951-52)
W.B. Fulks (1949-50)
A.R. Harvey (1948-49)
D.L. Thomsen, Jr. (1950-51)
Maria A. Weber (1949-51)
E.L. Whitney (1948-49)

Vari-typist

Rosemarie Stampfel

PREFACE

The aims, the history, and the organization of these *Tables of Integral Transforms* were described in the Introduction to vol. I. A little more than one half of the present second, and last, volume consists of tables of further integral transforms, the remaining part of this volume contains integrals of higher transcendental functions.

Under the generic name *Bessel transforms* we list not only the familiar Hankel transforms but also other transformations whose kernels are Bessel functions in the widest sense of the word. In addition to these we list fractional integrals, and also Stieltjes and Hilbert transforms. As far as we know, no extensive tables exist for any of the transformations included in this volume, in fact, for some of them there are comparatively few known transform pairs. A list of all transforms included in this work is given on p. xi ff.

The second part of the volume contains miscellaneous integrals involving higher transcendental functions. Some of these integrals cannot be written as transforms, others were not included in the transform tables and are given here. Generally speaking, an integral which can be written as a transform is more likely to be found in the transform tables than among integrals of higher transcendental functions. The latter are arranged according to their integrands. The "hierarchy" of functions given on p. xii of vol. I has been followed and, as in vol. I, composite functions are classified according to the "highest" function occurring in them. A list of definitions of higher transcendental functions is given in the Appendix.

Acknowledgments and thanks are due to the same persons and organizations as in connection with vol. I. Acknowledgments are also due to Mr. John B. Johnston who read the proofs and rendered other valuable technical assistance.

Corrections of errors, additions, and suggestions for improvement will be received gratefully by the Editor.

A. ERDÉLYI

STANDARD FORMS

Fourier cosine transform ($\mathfrak{F}_c$, Chapter I)

$$\int_0^\infty f(x) \cos(xy)\, dx$$

Fourier sine transform ($\mathfrak{F}_s$, Chapter II)

$$\int_0^\infty f(x) \sin(xy)\, dx$$

Exponential Fourier transform ($\mathfrak{F}_e$, Chapter III)

$$\int_{-\infty}^\infty f(x)\, e^{-ixy}\, dx$$

Laplace transform ($\mathfrak{L}$, Chapter IV)

$$\int_0^\infty f(t)\, e^{-pt}\, dt$$

Inverse Laplace transform (Chapter V)

$$\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} g(p)\, e^{pt}\, dp$$

Mellin transform ($\mathfrak{M}$, Chapter VI)

$$\int_0^\infty f(x)\, x^{s-1}\, dx$$

Inverse Mellin transform (Chapter VII)

$$\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} g(s)\, x^{-s}\, ds$$

Hankel transform ($\mathfrak{H}_\nu$, Chapter VIII)

$$\int_0^\infty f(x)\, J_\nu(xy)\, (xy)^{1/2}\, dx$$

Y-transform ($\mathfrak{Y}_\nu$, Chapter IX)

$$\int_0^\infty f(x)\, Y_\nu(xy)\, (xy)^{1/2}\, dx$$

K-transform ($\mathfrak{K}_\nu$, Chapter X)

$$\int_0^\infty f(x)\, K_\nu(xy)\,(xy)^{1/2}\, dx$$

H-transform (Chapter XI)

$$\int_0^\infty f(x)\, \mathbf{H}_\nu(xy)\,(xy)^{1/2}\, dx$$

Kontorovich-Lebedev transform (Chapter XII)

$$\int_0^\infty f(x)\, K_{ix}(y)\, dx$$

Riemann-Liouville fractional integral ($\mathfrak{R}_\mu$, Chapter XIII)

$$\frac{1}{\Gamma(\mu)} \int_0^y f(x)\,(y-x)^{\mu-1}\, dx$$

Weyl fractional integral ($\mathfrak{W}_\mu$, Chapter XIII)

$$\frac{1}{\Gamma(\mu)} \int_y^\infty f(x)\,(x-y)^{\mu-1}\, dx$$

Stieltjes transform ($\mathfrak{S}$, Chapter XIV)

$$\int_0^\infty \frac{f(x)}{x+y}\, dx$$

Generalized Stieltjes transform ($\mathfrak{S}_\rho$, Chapter XIV)

$$\int_0^\infty \frac{f(x)}{(x+y)^\rho}\, dx$$

Hilbert transform (Chapter XV)

$$\frac{1}{\pi} \int_{-\infty}^\infty \frac{f(x)}{x-y}\, dx$$

CONTENTS

BESSEL TRANSFORMS

CHAPTER VIII

HANKEL TRANSFORMS

HANKEL TRANSFORMS OF ORDER ν

CHAPTER IX

Y-TRANSFORMS

CHAPTER X

K-TRANSFORMS

CHAPTER XI

$\mathbf{H}$-TRANSFORMS

CHAPTER XII

KONTOROVICH-LEBEDEV TRANSFORMS

MISCELLANEOUS TRANSFORMS

CHAPTER XIII

FRACTIONAL INTEGRALS

CHAPTER XIV

STIELTJES TRANSFORMS

CHAPTER XV

HILBERT TRANSFORMS

INTEGRALS OF HIGHER TRANSCENDENTAL FUNCTIONS

CHAPTER XVI

ORTHOGONAL POLYNOMIALS

CHAPTER XVII

GAMMA FUNCTION, INCOMPLETE GAMMA FUNCTIONS AND RELATED FUNCTIONS

CHAPTER XVIII

LEGENDRE FUNCTIONS

CHAPTER XIX

BESSEL FUNCTIONS

CHAPTER XX

HYPERGEOMETRIC FUNCTIONS

APPENDIX

BESSEL TRANSFORMS

or

integral transforms whose kernels are Bessel functions or functions related to Bessel functions.

CHAPTER VIII

HANKEL TRANSFORMS

We call

$$g(y;\,\nu) = \mathfrak{H}_\nu\{f(x);\,y\} = \int_0^\infty f(x)\,J_\nu(xy)\,(xy)^{1/2}\,dx$$

the *Hankel transform of order* ν of $f(x)$ and take y to be a positive real variable. For the sake of brevity we often write $g(y)$ instead of $g(y;\,\nu)$. This form of the Hankel transform has the advantage of reducing to the Fourier sine or cosine transform when $\nu = \pm\,\frac{1}{2}$. Many authors regard

$$\int_0^\infty f(x)\,J_\nu(xy)\,x\,dx$$

or

$$\int_0^\infty f(x)\,J_\nu\,[2(xy)^{1/2}]\,dx$$

as the Hankel transform of order ν of $f(x)$. The Hankel transform is self-reciprocal [see 8.1(1)] and no table of inverse transforms is required.

Hankel's inversion theorem is proved in detail, and many Hankel transforms are evaluated in Watson's (1922) book on Bessel functions. The theory and application of Hankel transforms is described in several books on Fourier integrals, among which we mention Sneddon (1951) and Titchmarsh (1937).

From the transform pairs given in this chapter further transform pairs may be derived by means of the methods indicated in the introduction to volume I of this work, and also by means of the general formulas given in sec. 8.1. Tricomi (1935) discovered the relation

$$\mathfrak{L}\{t^{\frac{1}{2}\nu-\frac{1}{4}}\,g[(2t)^{1/2};\,\nu];\,s\} = s^{-\nu-1}\,\mathfrak{L}\{t^{\frac{1}{2}\nu-\frac{1}{4}}\,f[(2t)^{1/2}];\,s^{-1}\}$$

between Hankel transforms and Laplace transforms, and this relation may be used to evaluated Hankel transforms by means of the tables of Laplace transforms and inverse Laplace transforms given in chapters IV and V of volume I.

REFERENCES

Sneddon, I.N., 1951: *Fourier transforms*, McGraw-Hill, New York.

Titchmarsh, E.C., 1937: *Introduction to the theory of Fourier integrals*, Oxford.

Tricomi, Francesco, 1935: *Rend. dei Lincei* (6) 22, 564-571.

Watson, G. N., 1922: *A treatise on the theory of Bessel functions*, Cambridge.

HANKEL TRANSFORMS

8.1. General formulas

	$f(x)$	$\int_0^\infty f(x)\, J_\nu(xy)\,(xy)^{1/2}\,dx = g(y;\nu) \qquad y>0$
(1)	$\int_0^\infty g(y)\, J_\nu(xy)(xy)^{1/2}\,dy \qquad \operatorname{Re}\nu > -\tfrac{1}{2}$	$g(y)$
(2)	$f(ax) \qquad a>0$	$a^{-1}\, g(a^{-1}y;\nu)$
(3)	$x^m f(x) \qquad m=0,1,2,\dots$	$y^{1/2-\nu}\left(\dfrac{d}{y\,dy}\right)^m [y^{\nu-1/2+m}\, g(y;\nu+m)]$
(4)	$x^m f(x) \qquad m=0,1,2,\dots$	$(-1)^m\, y^{1/2+\nu}\left(\dfrac{d}{y\,dy}\right)^m \times [y^{-1/2+m-\nu}\, g(y;\nu-m)]$
(5)	$2\nu x^{-1} f(x)$	$y\,g(y;\nu-1)+y\,g(y;\nu+1)$
(6)	$x^{-1} f(x)$	$y^{1/2-\nu}\int_0^y \eta^{\nu-1/2}\, g(\eta;\nu-1)\,d\eta$
(7)	$x^{-1} f(x)$	$y^{1/2+\nu}\int_y^\infty \eta^{-\nu-1/2}\, g(\eta,\nu+1)\,d\eta$
(8)	$x^{-\mu} f(x) \qquad \operatorname{Re}\nu+1>\operatorname{Re}\mu>0$	$2^{1-\mu}[\Gamma(\mu)]^{-1}\, y^{1/2-\nu} \times \int_0^y \eta^{\nu-\mu+1/2}(y^2-\eta^2)^{\mu-1} g(\eta;\nu-\mu)\,d\eta$

General formulas (cont'd)

	$f(x)$	$\int_0^\infty f(x)\, J_\nu(xy)\,(xy)^{1/2}\,dx = g(y;\nu) \qquad y>0$
(9)	$x^{-\mu} f(x)$ $\operatorname{Re}\nu - 3/2 > \operatorname{Re}\mu > 0$	$2^{1-\mu}[\Gamma(\mu)]^{-1}\, y^{1/2+\nu} \times \int_y^\infty \eta^{1/2-\mu-\nu}(\eta^2-y^2)^{\mu-1}\, g(\eta;\nu+\mu)\,d\eta$
(10)	$2\nu f'(x)$	$(\nu-\frac{1}{2})\, y\, g(y;\nu+1) - (\nu+\frac{1}{2})\, y\, g(y;\nu-1)$
(11)	$x^{1/2-\nu}\left(\frac{d}{x\,dx}\right)^m [x^{\nu+m-1/2} f(x)]$ $m = 0, 1, 2, \ldots$	$y^m\, g(y;\nu+m)$
(12)	$x^{1/2+\nu}\left(\frac{d}{x\,dx}\right)^m [x^{m-\nu-1/2} f(x)]$ $m = 0, 1, 2, \ldots$	$(-y)^m\, g(y;\nu-m)$
(13)	$x^{1/2-\nu}\int_0^x \xi^{\nu-\mu+1/2}(x^2-\xi^2)^{\mu-1} \times f(\xi)\,d\xi$ $\operatorname{Re}\nu + \frac{1}{2} > \operatorname{Re}\mu > 0$	$2^{\mu-1}\,\Gamma(\mu)\, y^{-\mu}\, g(y;\nu-\mu)$
(14)	$x^{1/2+\nu}\int_x^\infty \xi^{1/2-\mu-\nu}(\xi^2-x^2)^{\mu-1} \times f(\xi)\,d\xi$ $\operatorname{Re}\nu + 1 > \operatorname{Re}\mu > 0$	$2^{\mu-1}\,\Gamma(\mu)\, y^{-\mu}\, g(y;\nu+\mu)$
(15)	$2^\lambda\,\Gamma(\lambda)\, x^{1/2-\nu} \times \int_0^x \xi^{1/2-\lambda-\mu+\nu}(x^2-\xi^2)^{\mu-1} \times f(\xi)\,d\xi$ $\operatorname{Re}\lambda > 0, \quad \operatorname{Re}\mu > 0$ $\operatorname{Re}\nu > \operatorname{Re}(\lambda+\mu) - \frac{1}{2}$	$2^\mu\,\Gamma(\mu)\, y^{1/2-\nu} \times \int_0^y \eta^{1/2-\lambda-\mu+\nu}(y^2-\eta^2)^{\lambda-1} \times g(\eta;\nu-\lambda-\mu)\,d\eta$

General formulas (cont'd)

	$f(x)$	$\int_0^\infty f(x)\, J_\nu(xy)\,(xy)^{1/2}\,dx = g(y;\nu) \qquad y>0$
(16)	$2^\lambda \Gamma(\lambda)\, x^{1/2+\nu} \times \int_x^\infty \xi^{1/2-\lambda-\mu-\nu}(\xi^2-x^2)^{\mu-1} \times f(\xi)\, d\xi \qquad \mathrm{Re}\,\lambda>0$ $\mathrm{Re}\,\mu>0, \quad \mathrm{Re}\,\nu>\lvert\mathrm{Re}(\lambda-\mu)\rvert-1$	$2^\mu \Gamma(\mu)\, y^{1/2+\nu} \times \int_y^\infty \eta^{1/2-\lambda-\mu-\nu}(\eta^2-y^2)^{\lambda-1} \times g(\eta;\nu+\lambda+\mu)\, d\eta$

8.2. Hankel transforms of order zero; Elementary functions

	$f(x)$	$\int_0^\infty f(x)\, J_0(xy)\,(xy)^{1/2}\,dx \qquad y>0$
(1)	$x^{-1/2}$	$y^{-1/2}$
(2)	$x^{-1/2} \qquad 0<x<1$ $0 \qquad 1<x<\infty$	$y^{1/2} J_0(y) + \tfrac{1}{2}\pi y^{1/2}[J_1(y)\,\mathbf{H}_0(y) - J_0(y)\,\mathbf{H}_1(y)]$
(3)	$0 \qquad 0<x<1$ $x^{-1/2} \qquad 1<x<\infty$	$y^{-1/2} - y^{1/2} J_0(y) + \tfrac{1}{2}\pi y^{1/2}[J_0(y)\,\mathbf{H}_1(y) - J_1(y)\,\mathbf{H}_0(y)]$
(4)	$x^{1/2}(a^2+x^2)^{-1/2} \qquad \mathrm{Re}\,a>0$	$y^{-1/2} e^{-ay}$
(5)	$x^{1/2}(a^2-x^2)^{-1/2} \qquad 0<x<a$ $0 \qquad a<x<\infty$	$y^{-1/2} \sin(ay)$
(6)	$0 \qquad 0<x<a$ $x^{1/2}(x^2-a^2)^{-1/2} \qquad a<x<\infty$	$y^{-1/2} \cos(ay)$
(7)	$x^{1/2}(x^2+a^2)^{-3/2} \qquad \mathrm{Re}\,a>0$	$a^{-1} y^{1/2} e^{-ay}$
(8)	$x^{1/2}(x^4+a^4)^{-1} \qquad \lvert\arg a\rvert<\tfrac{1}{4}\pi$	$-a^{-2} y^{1/2} \mathrm{kei}_0(ay)$

Elementary functions; Order zero (cont'd)

	$f(x)$	$\int_0^\infty f(x)\, J_0(xy)\,(xy)^{1/2}\,dx \qquad y>0$
(9)	$x^{5/2}(x^4+a^4)^{-1} \qquad \lvert\arg a\rvert < \tfrac{1}{4}\pi$	$y^{1/2}\,\mathrm{ker}_0(ay)$
(10)	$x^{5/2}(x^4-a^4)^{-1} \qquad a>0$	$\tfrac{1}{2}y^{1/2}[K_0(ay)-\tfrac{1}{2}\pi Y_0(ay)]$ The integral is a Cauchy Principal Value
(11)	$x^{-1/2}\dfrac{(x^2+a^2)^{1/2}-x}{(x^2+a^2)^{1/2}+x} \qquad \mathrm{Re}\, a>0$	$y^{-1/2}+2a^{-2}y^{-5/2}(aye^{-ay}+e^{-ay}-1)$
(12)	$x^{1/2}(x^4+2a^2x^2+b^4)^{-1/2}$ $a>b>0$	$y^{1/2}\, I_0[2^{-1/2}(a^2-b^2)^{1/2}y]$ $\times K_0[2^{-1/2}(a^2+b^2)^{1/2}y]$
(13)	$x^{1/2}(x^4+2a^2x^2+b^4)^{-1/2}$ $b>a>0$	$y^{1/2}\, J_0[2^{-1/2}(b^2-a^2)^{1/2}y]$ $\times K_0[2^{-1/2}(a^2+b^2)^{1/2}y]$
(14)	$x^{1/2}(b^2-x^2)(x^4\pm 2a^2x^2+b^4)^{-3/2}$ $0<a<b$	$2^{-1/2}(b^2\mp a^2)^{-1/2}y^{3/2}$ $\times J_1[(\tfrac{1}{2}b^2\mp\tfrac{1}{2}a^2)^{1/2}y]$ $\times K_0[(\tfrac{1}{2}b^2\pm\tfrac{1}{2}a^2)^{1/2}y]$
(15)	$x^{-1/2}(a^2-x^2)^{-1/2}\{[x+(x^2-a^2)^{1/2}]^{2n}$ $+[x-(x^2-a^2)^{1/2}]^{2n}\} \qquad 0<x<a$ $0 \qquad a<x<\infty$	$(-1)^n\pi a^{2n+1/2}y^{1/2}[J_n(\tfrac{1}{2}ay)]^2$
(16)	$x^{-1/2}(a^2+x^2)^{-1/2}[(a^2+x^2)^{1/2}+x]^{2\mu}$ $\mathrm{Re}\, a>0, \quad \mathrm{Re}\,\mu<\tfrac{3}{4}$	$y^{1/2}\, a^{2\mu}K_\mu(\tfrac{1}{2}ay)\, I_{-\mu}(\tfrac{1}{2}ay)$
(17)	$x^{1/2}(x^2+\alpha^2)^{-1/2}(x^2+\beta^2)^{-1/2}$ $\times[(x^2+\alpha^2)^{1/2}+(x^2+\beta^2)^{1/2}]$ $\mathrm{Re}\,\alpha>0, \quad \mathrm{Re}\,\beta>0$ $\mathrm{Re}\,\mu<\tfrac{3}{4}$	$y^{1/2}(\alpha^2-\beta^2)^{\mu}K_\mu[\tfrac{1}{2}y(\alpha+\beta)]$ $\times I_{-\mu}[\tfrac{1}{2}y(\alpha-\beta)]$

Elementary functions; Order zero (cont'd)

	$f(x)$	$\int_0^\infty f(x)\, J_0(xy)(xy)^{1/2}\,dx \qquad y>0$
(18)	$x^{-1/2}\, e^{-\alpha x} \qquad \mathrm{Re}\,\alpha>0$	$y^{1/2}(y^2+\alpha^2)^{-1/2}$
(19)	$x^{-3/2}(1-e^{-\alpha x}) \qquad \mathrm{Re}\,\alpha>0$	$y^{1/2}\sinh^{-1}(\alpha/y)$
(20)	$x^{n-1/2}\, e^{-\alpha x} \qquad \mathrm{Re}\,\alpha>0$	$\dfrac{n!\, y^{1/2}}{(\alpha^2+y^2)^{1/2 n+1/2}}\, \mathrm{P}_n\left[\dfrac{\alpha}{(\alpha^2+y^2)^{1/2}}\right]$
(21)	$x^{2\mu-3/2}\, e^{-x^2/2} \qquad \mathrm{Re}\,\mu>0$	$2^{\mu-1}\,\Gamma(\mu)\, y^{1/2}\, {}_1F_1(\mu;1;-\tfrac{1}{2}y^2)$
(22)	$x^{-1}\exp(-2\beta x^{1/2}) \qquad \mathrm{Re}\,\beta>0$	$\pi^{-1}\beta y^{-1/2} K_{1/4}(\tfrac{1}{2}e^{1/4 i\pi}\beta^2 y^{-1})$ $\times K_{1/4}(\tfrac{1}{2}e^{-1/4 i\pi}\beta^2 y^{-1})$
(23)	$x^{1/2}\exp[-\alpha(x^2+\beta^2)^{1/2}]$ $\mathrm{Re}\,\alpha>0, \quad \mathrm{Re}\,\beta>0$	$\alpha y^{1/2}(y^2+\alpha^2)^{-3/2}\exp[-\beta(y^2+\alpha^2)^{1/2}]$ $\times[1+\beta(y^2+\alpha^2)^{1/2}]$
(24)	$x^{1/2}(x^2+\beta^2)^{-1/2}$ $\times\exp[-\alpha(x^2+\beta^2)^{1/2}]$ $\mathrm{Re}\,\alpha>0, \quad \mathrm{Re}\,\beta>0$	$y^{1/2}(y^2+\alpha^2)^{-1/2}\exp[-\beta(y^2+\alpha^2)^{1/2}]$
(25)	$x^{1/2}(x^2+\beta^2)^{-1/2}$ $\times\exp[\pm i\alpha(x^2+\beta^2)^{1/2}]$ $\alpha>0, \quad \mathrm{Re}\,\beta>0$	$\pm\, iy^{1/2}(a^2-y^2)^{-1/2}$ $\times\exp[\pm i\beta(a^2-y^2)^{1/2}] \qquad 0<y<a$ $y^{1/2}(y^2-a^2)^{-1/2}$ $\times\exp[-\beta(y^2-a^2)^{1/2}] \qquad a<y<\infty$
(26)	$\pm\, ix^{1/2}(b^2-x^2)^{-1/2}$ $\times\exp[\pm i\,\alpha(b^2-x^2)^{1/2}]$ $0<y<b$ $x^{1/2}(x^2-b^2)^{-1/2}$ $\times\exp[-\alpha(x^2-b^2)^{1/2}]$ $b<y<\infty$ $\mathrm{Re}\,\alpha>0$	$y^{1/2}(y^2+\alpha^2)^{-1/2}\exp[\pm ib(y^2+\alpha^2)^{1/2}]$

Elementary functions; Order zero (cont'd)

	$f(x)$	$\int_0^\infty f(x)\, J_0(xy)(xy)^{1/2}\, dx \qquad y>0$
(27)	$x^{-1/2} \log x$	$-y^{-1/2} \log(2\gamma y)$
(28)	$x^{-1/2}(x^2+a^2)^{-1/2}$ $\times \log[x+(x^2+a^2)^{1/2}] \qquad a>0$	$y^{1/2}[\tfrac{1}{2}K_0^2(\tfrac{1}{2}ay)$ $+ \log a\, I_0(\tfrac{1}{2}ay)\, K_0(\tfrac{1}{2}ay)]$
(29)	$x^{-1/2}(x^2+a^2)^{-1/2}$ $\times \log \dfrac{(x^2+a^2)^{1/2}+x}{(x^2+a^2)^{1/2}-x} \qquad \mathrm{Re}\, a>0$	$y^{1/2} K_0^2(\tfrac{1}{2}ay)$
(30)	$x^{1/2} \log(1+a^2x^{-2}) \qquad \mathrm{Re}\, a>0$	$2y^{-1/2}[y^{-1}-a\, K_1(ay)]$
(31)	$x^{1/2} \log[ax^{-1}+(1+a^2x^{-2})^{1/2}]$ $\mathrm{Re}\, a>0$	$y^{-3/2}(1-e^{-ay})$
(32)	$x^{-1/2} \sin(ax) \qquad a>0$	$y^{1/2}(a^2-y^2)^{-1/2} \qquad 0<y<a$ $0 \qquad a<y<\infty$
(33)	$x^{-3/2} \sin(ax) \qquad a>0$	$\tfrac{1}{2}\pi y^{1/2} \qquad 0<y<a$ $y^{1/2} \sin^{-1}(a/y) \qquad a<y<\infty$
(34)	$x^{-1/2}(1+x)^{-1} \sin(1+x)$	$\tfrac{1}{2}\pi y^{1/2} J_0(y) \qquad 1\le y<\infty$
(35)	$x^{-1/2}(\beta^2+x^2)^{-1} \sin(ax)$ $a>0, \quad \mathrm{Re}\, \beta>0$	$y^{1/2}\beta^{-1} \sinh(\beta a)\, K_0(\beta y) \qquad a<y<\infty$
(36)	$x^{1/2}(\beta^2+x^2)^{-1} \sin(ax)$ $a>0, \quad \mathrm{Re}\, \beta>0$	$\tfrac{1}{2}\pi y^{1/2} e^{-a\beta} I_0(y\beta) \qquad 0<y<a$
(37)	$x^{-3/2} e^{-bx} \sin(ax)$	$y^{1/2} \sin^{-1}\left(\dfrac{2a}{r_1+r_2}\right)$ $r_1^2=b^2+(a+y)^2, \qquad r_1>0$ $r_2^2=b^2+(a-y)^2, \qquad r_2>0$

Elementary functions; Order zero (cont'd)

	$f(x)$	$\int_0^\infty f(x)\, J_0(xy)\,(xy)^{1/2}\, dx \qquad y>0$
(38)	$x^{1/2} \sin(\frac{1}{2}a^2x^2) \qquad a>0$	$a^{-2}y^{1/2}\cos(\frac{1}{2}a^{-2}y^2)$
(39)	$x^{-3/2}\sin(\frac{1}{2}a^2x^2)$	$\frac{1}{2}y^{1/2}\,\mathrm{si}(\frac{1}{2}a^{-2}y^2)$
(40)	$x^{-1}e^{-ax^{1/2}}\sin(ax^{1/2})$ $\lvert\arg a\rvert < \frac{1}{4}\pi$	$\frac{1}{2}y^{-1/2}\,a\,I_{1/4}(\frac{1}{4}a^2/y)\,K_{1/4}(\frac{1}{4}a^2/y)$
(41)	$x^{1/2}(\beta^2+x^2)^{-1/2}$ $\times\sin[a(\beta^2+x^2)^{1/2}]$ $a>0, \quad \mathrm{Re}\,\beta>0$	$y^{1/2}(a^2-y^2)^{-1/2}\cos[\beta(a^2-y^2)^{1/2}] \qquad 0<y<a$ $0 \qquad a<y<\infty$
(42)	$x^{-1/2}\cos(ax) \qquad a>0$	$0 \qquad 0<y<a$ $y^{1/2}(y^2-a^2)^{-1/2} \qquad a<y<\infty$
(43)	$x^{-3/2}[1-\cos(ax)] \qquad a>0$	$y^{1/2}\cosh^{-1}(a/y) \qquad 0<y<a$ $0 \qquad a<y<\infty$
(44)	$x^{-1/2}(x^2+\beta^2)^{-1}\cos(ax)$ $a>0, \quad \mathrm{Re}\,\beta>0$	$\frac{1}{2}\beta^{-1}\pi y^{1/2}e^{-a\beta}I_0(\beta y)$ $0<y<a$
(45)	$x^{1/2}(\beta^2+x^2)^{-1}\cos(ax)$ $a>0, \quad \mathrm{Re}\,\beta>0$	$y^{1/2}\cosh(\beta a)\,K_0(\beta y)$ $a<y<\infty$
(46)	$x^{-1/2}e^{-bx}\cos(ax)$	$y^{1/2}\,2^{-1/2}[(b^2+y^2-a^2)^2+4a^2b^2]^{-1/2}$ $\times\{[(b^2+y^2-a^2)^2+4a^2b^2]^{1/2}$ $+b^2+y^2-a^2\}^{1/2}$
(47)	$x^{1/2}\cos(\frac{1}{2}a^2x^2) \qquad a>0$	$a^{-2}y^{1/2}\sin(\frac{1}{2}a^{-2}y^2)$
(48)	$x^{-3/2}[1-\cos(\frac{1}{2}a^2x^2)]$	$-\frac{1}{2}y^{1/2}\,\mathrm{Ci}(\frac{1}{2}a^{-2}y^2)$

Elementary functions; Order zero (cont'd)

	$f(x)$	$\int_0^\infty f(x)\, J_0(xy)\,(xy)^{1/2}\,dx \qquad y>0$
(49)	$x^{-1}\, e^{-ax^{1/2}} \cos(ax^{1/2})$ $\lvert\arg a\rvert < \frac{1}{4}\pi$	$\frac{1}{2} a y^{-1/2}\, I_{-1/4}(\frac{1}{4}a^2 y^{-1})$ $\times K_{1/4}(\frac{1}{4}a^2 y^{-1})$
(50)	$x^{1/2}(\beta^2+x^2)^{-1/2}$ $\times \cos[a(\beta^2+x^2)^{1/2}]$ $a>0, \quad \operatorname{Re}\beta>0$	$-y^{1/2}(a^2-y^2)^{-1/2}\sin[\beta(a^2-y^2)^{1/2}]$ $0<y<a$ $y^{1/2}(y^2-a^2)^{-1/2}\, e^{-\beta(y^2-a^2)^{1/2}}$ $a<y<\infty$
(51)	$x^{-3/2}\, e^{-ax}\sinh(ax) \quad \operatorname{Re} a>0$	$\frac{1}{2} y^{1/2}\log[2ay^{-1}+(1+4a^2y^{-2})^{1/2}]$
(52)	$x^{-1/2}\, e^{-ax}\sinh(\beta x)$ $\operatorname{Re} a > \lvert\operatorname{Re}\beta\rvert$	$(a\beta y)^{1/2} r_1^{-1} r_2^{-1}(r_2-r_1)^{1/2}(r_2+r_1)^{-1/2}$ $r_1=[y^2+(\beta-a)^2]^{1/2}$ $r_2=[y^2+(\beta+a)^2]^{1/2}$
(53)	$x^{1/2}\, e^{-a/x}\sinh(ax^{-1}) \quad \operatorname{Re} a>0$	$2ay^{-1/2}\, J_1(2a^{1/2}y^{1/2})\, K_1(2a^{1/2}y^{1/2})$
(54)	$x^{-1/2}\, e^{-ax}\cosh(\beta x)$ $\operatorname{Re} a > \lvert\operatorname{Re}\beta\rvert$	$(a\beta y)^{1/2} r_1^{-1} r_2^{-1}(r_2+r_1)^{1/2}(r_2-r_1)^{-1/2}$ $r_1=[y^2+(\beta-a)^2]^{1/2}$ $r_2=[y^2+(\beta+a)^2]^{1/2}$
(55)	$x^{1/2}\sinh^{-1}(ax^{-1}) \qquad \operatorname{Re} a>0$	$y^{-3/2}(1-e^{-ay})$
(56)	$x^{-1/2}(1+x^2)^{-1/2}$ $\times\sinh(2\mu\sinh^{-1}x)$ $\lvert\operatorname{Re}\mu\rvert<\frac{1}{2}$	$\pi^{-1}\sin(\pi\mu)\, y^{1/2}[K_\mu(\frac{1}{2}y)]^2$
(57)	$x^{-1/2}(1+x^2)^{-1/2}$ $\times\cosh(2\mu\sinh^{-1}x)$ $\lvert\operatorname{Re}\mu\rvert<\frac{1}{2}$	$\frac{1}{2} y^{1/2} K_\mu(\frac{1}{2}y)$ $\times[I_\mu(\frac{1}{2}y)+I_{-\mu}(\frac{1}{2}y)]$

8.3. Hankel transforms of order zero; Higher transcendental functions

	$f(x)$	$\int_0^\infty f(x)\, J_0(xy)(xy)^{1/2}\,dx \qquad y>0$
(1)	$x^{1/2}\,P_n(1-2x^2) \qquad 0<x<1$ $0 \qquad 1<x<\infty$	$y^{-1/2}\,J_{2n+1}(y)$
(2)	$x^{5/2}\,P_n(1-2x^2) \qquad 0<x<1$ $0 \qquad 1<x<\infty$	$y^{-1/2}(2n+1)^{-1}$ $\times[(n+1)\,J'_{2n+2}(y)-n\,J'_{2n}(y)]$
(3)	$x^{1/2}\,e^{-\frac{1}{2}x^2}\,L_n(x^2)$	$(-1)^n\,e^{-\frac{1}{2}y^2}\,y^{1/2}\,L_n(y^2)$
(4)	$x^{1/2}\exp(-\tfrac{1}{2}\alpha x^2)\,L_n(\tfrac{1}{2}\beta x^2)$ $\operatorname{Re}\alpha>0$	$y^{1/2}\dfrac{(\alpha-\beta)^n}{\alpha^{n+1}}\exp(-\tfrac{1}{2}\alpha^{-1}x^2)$ $\times L_n\left[\dfrac{\beta y^2}{2\alpha(\beta-\alpha)}\right]$
(5)	$x^{1/2}\,e^{-x^2}\,L_n(x^2)$	$[n!]^{-1}\,2^{-2n-1}\,y^{2n+1/2}\,e^{-\frac{1}{4}y^2}$
(6)	$x^{-1/2}\operatorname{si}(ax) \qquad a>0$	$-y^{-1/2}\sin^{-1}(y/a) \qquad 0<y<a$ $0 \qquad a<y<\infty$
(7)	$x^{1/2}\operatorname{si}(a^2x^2) \qquad a>0$	$-2y^{-3/2}\sin(\tfrac{1}{4}x^2/a^2)$
(8)	$x^{1/2}\operatorname{Ci}(a^2x^2) \qquad a>0$	$2y^{-3/2}[1-\cos(\tfrac{1}{4}x^2/a^2)]$
(9)	$x^{-1/2}\operatorname{Ci}(a^2x^2) \qquad a>0$	$y^{-1/2}[\operatorname{Ci}(\tfrac{1}{4}x^2/a^2)+\log(\tfrac{1}{4}y^2x^2/a^2)]$
(10)	$x^{1/2}(1+x^2)^{-\nu-1}$ $\times P_\nu[(1-x^2)(1+x^2)^{-1}]$ $\operatorname{Re}\nu>0$	$[2^\nu\Gamma(\nu+1)]^{-2}\,y^{2\nu+1/2}\,K_0(y)$
(11)	$x^{1/2}\{P_{\lambda-1/2}[(1+a^2x^2)^{1/2}]\}^2$ $\operatorname{Re}a>0, \quad \lvert\operatorname{Re}\lambda\rvert<\tfrac{1}{4}$	$2\pi^{-2}\cos(\lambda\pi)\,a^{-1}y^{-1/2}[K_\lambda(\tfrac{1}{2}y/a)]^2$

Higher functions; Order zero (cont'd)

	$f(x)$	$\int_0^\infty f(x)\, J_0(xy)(xy)^{1/2}\,dx \qquad y>0$
(12)	$x^{1/2} P_{\sigma-1/2}[(1+a^2x^2)^{1/2}]$ $\times Q_{\sigma-1/2}[(1+a^2x^2)^{1/2}]$ $\operatorname{Re} a>0, \quad \operatorname{Re}\sigma>-1/4$	$a^{-1} y^{-1/2} I_\sigma(1/2\, y/a)\, K_\sigma(1/2\, y/a)$
(13)	$x^{1/2}\{P^\mu_{\sigma-1/2}[(1+a^2x^2)^{1/2}]\}^2$ $\operatorname{Re} a>0$ $\lvert\operatorname{Re}\sigma\rvert<1/4, \quad \operatorname{Re}\mu<1$	$-i\pi^{-1} y^{-3/2} W_{\mu,\sigma}(y/a)$ $\times [W_{\mu,\sigma}(e^{\pi i}y/a)-W_{\mu,\sigma}(e^{-\pi i}y/a)]$
(14)	$x^{1/2} P^\mu_{\sigma-1/2}[(1+a^2x^2)^{1/2}]$ $\times P^{-\mu}_{\sigma-1/2}[(1+a^2x^2)^{1/2}]$ $\operatorname{Re} a>0, \quad \lvert\operatorname{Re}\sigma\rvert<1/4$	$2\pi^{-1} y^{-3/2}\cos(\sigma\pi)$ $\times W_{\mu,\sigma}(y/a)\, W_{-\mu,\sigma}(y/a)$
(15)	$x^{1/2} P^\mu_{\sigma-1/2}[(1+a^2x^2)^{1/2}]$ $\times Q^\mu_{\sigma-1/2}[(1+a^2x^2)^{1/2}]$ $\operatorname{Re} a>0$ $\operatorname{Re}\sigma>-1/4, \quad \operatorname{Re}\mu<1$	$e^{\mu\pi i}\dfrac{\Gamma(1/2+\sigma-\mu)}{\Gamma(1+2\sigma)}\, y^{-3/2}$ $\times W_{\mu,\sigma}(y/a)\, M_{-\mu,\sigma}(y/a)$
(16)	$x^{-3/2}[1-J_0(ax)] \qquad a>0$	$0 \qquad y>a$ $y^{1/2}\log(a/y) \qquad y<a$
(17)	$x^{-1/2} J_0(ax)\, e^{-\beta x}$ $\operatorname{Re}\beta>\lvert\operatorname{Im} a\rvert$	$2\pi^{-1} y^{1/2} K(2a^{1/2}y^{1/2}/k)\, k^{-1/2}$ $k=[(a+y)^2+\beta^2]^{1/2}$
(18)	$x^{-1/2} J_1(ax) \qquad a>0$	$a^{-1} y^{1/2} \qquad 0<y<a$ $0 \qquad a<y<\infty$
(19)	$x^{-3/2} J_1(ax) \qquad a>0$	$2\pi^{-1} y^{1/2} E(y/a) \qquad 0<y<a$ $\dfrac{2y^{3/2}}{\pi a}\left[K\left(\dfrac{a}{y}\right)-\left(1-\dfrac{a^2}{y^2}\right)E\left(\dfrac{a}{y}\right)\right]$ $a<y<\infty$

Higher functions; Order zero (cont'd)

	$f(x)$	$\int_0^\infty f(x)\, J_0(xy)\, (xy)^{1/2}\, dx \qquad y>0$
(20)	$x^{1/2}\, J_\nu^2(\tfrac{1}{2}ax) \qquad \operatorname{Re}\nu>-\tfrac{1}{2}$	$2\pi^{-1}y^{-1/2}(a^2-y^2)^{-1/2} \times \cos[2\nu\sin^{-1}(y/a)] \qquad 0<y<a$ $0 \qquad a<y<\infty$
(21)	$x^{1/2}\, J_0(ax)\, Y_0(ax) \qquad a>0$	$0 \qquad 0<y<2a$ $-2\pi^{-1}y^{-1/2}(y^2-4a^2)^{-1/2} \qquad 2a<y<\infty$
(22)	$x^{-1/2}[\mathbf{H}_0(ax)-Y_0(ax)] \qquad a>0$	$4\pi^{-1}(a+y)^{-1}y^{1/2} \times K[\lvert a-y\rvert (a+y)^{-1}]$
(23)	$x^{-1/2}\cosh(\beta x)\, K_0(\alpha x) \qquad \operatorname{Re}\alpha>\lvert\operatorname{Re}\beta\rvert$	$y^{1/2}(u+v)^{-1/2}K(k)$ $u=\tfrac{1}{2}\{[(\alpha^2+\beta^2+y^2)^2-4\alpha^2\beta^2]^{1/2}+\alpha^2-\beta^2-y^2\}$ $v=\tfrac{1}{2}\{[(\alpha^2+\beta^2+y^2)^2-4\alpha^2\beta^2]^{1/2}-\alpha^2+\beta^2+y^2\}$ $k^2=v(u+v)^{-1}$
(24)	$x^{-1/2}\sinh(\beta x)\, K_1(\alpha x) \qquad \operatorname{Re}\alpha>\lvert\operatorname{Re}\beta\rvert$	$\alpha^{-1}y^{1/2}[uE(k)-K(k)E(u)+K(k)\operatorname{sn}u\operatorname{dn}u/(\operatorname{cn}u)]$ $\operatorname{cn}^2u=2y^2\{[(\alpha^2+\beta^2+y^2)^2-4\alpha^2\beta^2]^{1/2}-\alpha^2+\beta^2+y^2\}^{-1}$ $k^2=\tfrac{1}{2}\{1-(\alpha^2-\beta^2-y^2)\times[(\alpha^2+\beta^2+y^2)^2-4\alpha^2\beta^2]^{-1/2}\}$
(25)	$x^{1/2}\, J_0(\alpha x)\, K_0(\beta x) \qquad \operatorname{Re}\beta>\lvert\operatorname{Im}\alpha\rvert$	$y^{1/2}(\beta^4+\alpha^4+y^4-2\alpha^2y^2+2\alpha^2\beta^2+2\beta^2y^2)^{-1/2}$
(26)	$x^{3/2}\, J_1(\alpha x)\, K_0(\beta x) \qquad \operatorname{Re}\beta\ge\lvert\operatorname{Im}\alpha\rvert, \quad \operatorname{Re}\alpha>0$	$2\alpha y^{1/2}(\alpha^2+\beta^2-y^2)\times[(\alpha^2+\beta^2+y^2)^2-4y^2\alpha^2]^{-3/2}$

Higher functions; Order zero (cont'd)

	$f(x)$	$\int_0^\infty f(x)\, J_0(xy)\,(xy)^{1/2}\,dx \qquad y>0$
(27)	$x^{1/2}\, I_0(ax)\, K_0(\beta x)$ $\operatorname{Re}\beta > \operatorname{Re}a$	$y^{1/2}(a^4+\beta^4+y^4-2a^2\beta^2$ $+2a^2y^2+2\beta^2y^2)^{-1/2}$
(28)	$x^{3/2}\, I_0(ax)\, K_1(\beta x)$ $\operatorname{Re}\beta > \lvert\operatorname{Re}a\rvert$	$2y^{1/2}\beta(\beta^2+y^2-a^2)$ $\times[(y^2+a^2+\beta^2)^2-4a^2\beta^2]^{-3/2}$
(29)	$x^{1/2}\, I_1(ax)\, K_1(\beta x)$ $\operatorname{Re}\beta > \operatorname{Re}a > 0$	$2^{-1}y^{1/2}a^{-1}\beta^{-1}\{(a^2+\beta^2+y^2)$ $\times[(a^2+\beta^2+y^2)^2-4a^2\beta^2]^{-1/2}-1\}$
(30)	$x^{1/2}\, K_\mu(ax)\, I_\mu(\beta x)$ $\operatorname{Re}\mu > -1, \quad \operatorname{Re}a > \lvert\operatorname{Re}\beta\rvert$	$y^{1/2} r_1^{-1} r_2^{-1}(r_2-r_1)^\mu (r_2+r_1)^{-\mu}$ $r_1 = [y^2+(\beta-a)^2]^{1/2}$ $r_2 = [y^2+(\beta+a)^2]^{1/2}$
(31)	$x^{-1/2}\, I_\mu(\tfrac{1}{2}ax)\, K_\mu(\tfrac{1}{2}ax)$ $\operatorname{Re}a > 0, \quad \operatorname{Re}\mu > -\tfrac{1}{2}$	$a^{-1}y^{1/2}P_{\mu-1/2}[(1+y^2/a^2)^{1/2}]$ $\times Q_{\mu-1/2}[(1+y^2/a^2)^{1/2}]$
(32)	$x^{1/2}\, K_0^2(ax) \qquad \operatorname{Re}a > 0$	$y^{-1/2}(y^2+4a^2)^{-1/2}\log\dfrac{(y^2+4a^2)^{1/2}+y}{(y^2+4a^2)^{1/2}-y}$
(33)	$x^{1/2}\, K_\mu^2(ax)$ $\operatorname{Re}a > 0, \quad \lvert\operatorname{Re}\mu\rvert < 1$	$\pi 2^{-1-2\mu}a^{-2\mu}(\sin\mu\pi)^{-1}y^{-1/2}$ $\times(y^2+4a^2)^{-1/2}\{[(y^2+4a^2)^{1/2}+y]^{2\mu}$ $-[(y^2+4a^2)^{1/2}-y]^{2\mu}\}$
(34)	$x^{-1/2}\, J_\mu(a^2x^{-1})\, J_{-\mu}(a^2x^{-1})$ $a > 0, \quad \lvert\operatorname{Re}\mu\rvert < \tfrac{1}{4}$	$-i\csc(2\mu\pi)\,y^{-1/2}$ $\times[e^{2\pi i\mu}J_{2\mu}(2ay^{1/2}e^{-\frac{1}{4}\pi i})$ $\times J_{-2\mu}(2ay^{1/2}e^{\frac{1}{4}\pi i})$ $-e^{-2\pi i\mu}J_{2\mu}(2ay^{1/2}e^{\frac{1}{4}\pi i})$ $\times J_{-2\mu}(2ay^{1/2}e^{-\frac{1}{4}\pi i})]$

Higher functions; Order zero (cont'd)

	$f(x)$	$\int_0^\infty f(x)\, J_0(xy)\,(xy)^{1/2}\,dx \qquad y>0$
(35)	$x^{-1/2}[J_\mu^2(a^2x^{-1})-J_{-\mu}^2(a^2x^{-1})]$ $a>0, \quad \lvert\mathrm{Re}\,\mu\rvert<1/4$	$\sec(\mu\pi)\,y^{-1/2}$ $\times[J_{2\mu}(2ay^{1/2}e^{1/4\pi i})\,J_{2\mu}(2ay^{1/2}e^{-1/4\pi i})$ $-J_{-2\mu}(2ay^{1/2}e^{1/4\pi i})\,J_{-2\mu}(2ay^{1/2}e^{-1/4\pi i})]$
(36)	$x^{-1/2}H_\mu^{(1)}(a^2x^{-1})\,H_\mu^{(2)}(a^2x^{-1})$ $\lvert\arg a\rvert<1/4\pi, \quad \lvert\mathrm{Re}\,\mu\rvert<1/4$	$16\pi^{-2}\cos(\mu\pi)\,y^{-1/2}$ $\times K_{2\mu}(2ay^{1/2}e^{1/4\pi i})\,K_{2\mu}(2ay^{1/2}e^{-1/4\pi i})$
(37)	$x^{-1/2}I_\mu(a^2x^{-1})\,K_\mu(a^2x^{-1})$ $\lvert\arg a\rvert<1/4\pi, \quad \mathrm{Re}\,\mu>-1/4$	$2y^{-1/2}J_{2\mu}(2ay^{1/2})\,K_{2\mu}(2ay^{1/2})$
(38)	$x^{-1/2}J_\mu(ax^{1/2})\,K_\mu(ax^{1/2})$ $\lvert\arg a\rvert<1/4\pi, \quad \mathrm{Re}\,\mu>-1$	$1/2\,y^{-1/2}I_{1/2\mu}(1/4\,a^2y^{-1})\,K_{1/2\mu}(1/4\,a^2y^{-1})$
(39)	$x^{-1/2}Y_0(ax^{1/2})\,K_0(ax^{1/2})$ $\lvert\arg a\rvert<1/4\pi$	$-1/2\,\pi^{-1}y^{-1/2}[K_0(1/4\,a^2/y)]^2$
(40)	$x^{-1/2}Y_\mu(ax^{1/2})\,K_\mu(ax^{1/2})$ $\lvert\arg a\rvert<1/4\pi, \quad \lvert\mathrm{Re}\,\mu\rvert<1$	$-1/2\,y^{-1/2}\sec(1/2\,\mu\pi)\,K_{1/2\mu}(1/4\,a^2y^{-1})$ $\times[\pi^{-1}K_{1/2\mu}(1/4\,a^2y^{-1}$ $+\sin(1/2\,\mu\pi)\,I_{1/2\mu}(1/4\,a^2y^{-1})]$
(41)	$x^{-1/2}K_\mu(ae^{1/4\pi i}x^{1/2})$ $\times K_\mu(ae^{-1/4\pi i}x^{1/2})$ $\lvert\arg a\rvert<1/4\pi, \quad \lvert\mathrm{Re}\,\mu\rvert<1$	$2^{-4}\pi^2[\cos(1/2\,\mu\pi)]^{-1}$ $\times H_{1/2\mu}^{(1)}(1/4\,a^2/y)\,H_{1/2\mu}^{(2)}(1/4\,a^2/y)$
(42)	$x^{-1/2}D_n(ax)\,D_{n+1}(ax)$ $\lvert\arg a\rvert<1/4\pi$	$(-1)^n y^{-1/2}D_n(y/a)\,D_{n+1}(y/a)$
(43)	$x^{-1}D_\nu(a^{1/2}x^{1/2})\,D_{-\nu-1}(a^{1/2}x^{1/2})$ $\mathrm{Re}\,a>0$	$2^{-3/2}\pi a^{-1/2}y^{1/2}$ $\times P_{-1/4}^{\nu/2+1/4}[(1+4y^2/a^2)^{1/2}]$ $\times P_{-1/4}^{-\nu/2-1/4}[(1+4y^2/a^2)^{1/2}]$

Higher functions; Order zero (cont'd)

	$f(x)$	$\int_0^\infty f(x)\, J_0(xy)(xy)^{1/2}\, dx \qquad y>0$
(44)	$x^{-3/2}\, W_{\kappa,\mu}(ax)\, M_{-\kappa,\mu}(ax)$ $\operatorname{Re} a > 0$ $\operatorname{Re}\mu > -\tfrac{1}{2}, \quad \operatorname{Re}\kappa < \tfrac{3}{4}$	$e^{-i\kappa\pi}\dfrac{\Gamma(1+2\mu)}{\Gamma(\tfrac{1}{2}+\mu+\kappa)}\, y^{1/2}$ $\times P^{\kappa}_{\mu-1/2}[(1+y^2/a^2)^{1/2}]$ $\times Q^{\kappa}_{\mu-1/2}[(1+y^2/a^2)^{1/2}]$
(45)	$x^{-3/2}\, W_{k,\mu}(ax)\, W_{-k,\mu}(ax)$ $\operatorname{Re} a > 0, \quad -\tfrac{1}{2} < \operatorname{Re}\mu < \tfrac{1}{2}$	$\tfrac{1}{2}\pi\cos(\mu\pi)\, y^{1/2}\, P^{k}_{\mu-1/2}[(1+y^2/a^2)^{1/2}]$ $\times P^{-k}_{\mu-1/2}[(1+y^2/a^2)^{1/2}]$
(46)	$x^{1/2}\, {}_1F_1(\lambda; 1; -x^2) \qquad \operatorname{Re}\lambda > 0$	$[2^{2\lambda-1}\Gamma(\lambda)]^{-1} y^{2\lambda-3/2} \exp(-\tfrac{1}{4}y^2)$

8.4. Hankel transforms of order unity

	$f(x)$	$\int_0^\infty f(x)\, J_1(xy)(xy)^{1/2}\, dx \qquad y>0$
(1)	$x^{-1/2} \qquad 0<x<a$ $0 \qquad a<x<\infty$	$y^{-1/2}[1-J_0(ay)]$
(2)	$0 \qquad 0<x<a$ $x^{-1/2} \qquad a<x<\infty$	$y^{-1/2}\, J_0(ay)$
(3)	$x^{-1/2}(x^2+a^2)^{-1/2} \qquad \operatorname{Re} a>0$	$a^{-1} y^{-1/2}(1-e^{-ay})$
(4)	$x^{-1/2}(a^2-x^2)^{-1/2} \qquad 0<x<a$ $0 \qquad a<x<\infty$	$a^{-1} y^{-1/2}[1-\cos(ay)]$
(5)	$0 \qquad 0<x<a$ $x^{-1/2}(x^2-a^2)^{-1/2} \qquad a<x<\infty$	$a^{-1} y^{-1/2}\sin(ay)$

Order unity (cont'd)

	$f(x)$	$\int_0^\infty f(x)\, J_1(xy)\,(xy)^{1/2}\,dx \qquad y>0$
(6)	$x^{-1/2}\, e^{-\alpha x} \qquad \mathrm{Re}\,\alpha>0$	$y^{-1/2}[1-\alpha(\alpha^2+y^2)^{-1/2}]$
(7)	$x^{-1/2}\, e^{-1/4 \alpha x^2} \qquad \mathrm{Re}\,\alpha>0$	$y^{-1/2}(1-e^{-y^2/\alpha})$
(8)	$x^{3/2}\, e^{-\alpha x^2/4} \qquad \mathrm{Re}\,\alpha>0$	$4\alpha^{-2}\, y^{3/2}\, e^{-y^2/\alpha}$
(9)	$x^{-1/2}\exp[-\alpha(x^2+\beta^2)^{1/2}]$ $\mathrm{Re}\,\alpha>0, \quad \mathrm{Re}\,\beta>0$	$y^{-1/2}[e^{-\beta\alpha}-\alpha(\alpha^2+y^2)^{-1/2}$ $\times e^{-\beta(\alpha^2+y^2)^{1/2}}]$
(10)	$x^{-1/2}(\beta^2+x^2)^{-1/2}$ $\times \exp[-\alpha(\beta^2+x^2)^{1/2}]$ $\mathrm{Re}\,\alpha>0, \quad \mathrm{Re}\,\beta>0$	$y^{-1/2}\,\beta^{-1}[e^{-\beta\alpha}-e^{-\beta(\alpha^2+y^2)^{1/2}}]$
(11)	$x^{-1/2}\log x$	$-y^{-1/2}\log(\tfrac{1}{2}\gamma y)$
(12)	$x^{-1/2}\log(a^2+x^2)$	$2y^{-1/2}[K_0(ay)+\log a]$
(13)	$x^{-1/2}\log(1+x^4)^{1/2}$	$2y^{-1/2}\,\mathrm{ker}_0\, y$
(14)	$x^{-1/2}\sin(ax) \qquad a>0$	$0 \qquad 0<y<a$ $ay^{-1/2}(y^2-a^2)^{-1/2} \qquad a<y<\infty$
(15)	$x^{-3/2}\, e^{-ax}\sin(bx)$	$by^{-1/2}(1-r)$ $b^2=\dfrac{y^2}{1-r^2}-\dfrac{a^2}{r^2}$
(16)	$x^{-1/2}\sin(\tfrac{1}{4}ax^2) \qquad a>0$	$y^{-1/2}\sin(y^2/a)$
(17)	$x^{-1/2}\sin^2(\tfrac{1}{4}ax^2) \qquad a>0$	$\tfrac{1}{2}y^{-1/2}\cos(\tfrac{1}{2}y^2/a)$

Order unity (cont'd)

	$f(x)$	$\int_0^\infty f(x)\, J_1(xy)\,(xy)^{1/2}\,dx \qquad y>0$
(18)	$x^{-1/2}(x^2+a^2)^{-1/2}$ $\times \sin[b(x^2+a^2)^{1/2}]$ $\operatorname{Re} a>0, \quad b>0$	$a^{-1}y^{-1/2}\{\sin(ab)-\sin[a(b^2-y^2)^{1/2}]\}$ $\quad 0<y<b$ $a^{-1}y^{-1/2}\sin(ab) \quad b<y<\infty$
(19)	$x^{-1/2}\cos(ax) \quad a>0$	$y^{-1/2}[1-a(a^2-y^2)^{-1/2}] \quad 0<y<a$ $y^{-1/2} \quad a<y<\infty$
(20)	$x^{-1/2}\cos(\tfrac{1}{4}ax^2) \quad a>0$	$2y^{-1/2}\sin^2(\tfrac{1}{2}y^2/a)$
(21)	$x^{-1/2}(x^2+a^2)^{-1/2}$ $\times\cos[b(x^2+a^2)^{1/2}]$ $\operatorname{Re} a>0, \quad b>0$	$a^{-1}y^{-1/2}\{-\cos[a(b^2-y^2)^{1/2}]$ $+\cos(ab)\} \quad 0<y<b$ $a^{-1}y^{-1/2}\{\cos(ab)$ $-\exp[-a(y^2-b^2)^{1/2}]\} \quad b<y<\infty$
(22)	$x^{-1/2}\tan^{-1}(x^2)$	$-2y^{1/2}\operatorname{kei}_0 y$
(23)	$x^{3/2}P_n(1-2x^2) \quad 0<x<1$ $0 \quad 1<x<\infty$	$(2n+1)^{-1}y^{-1/2}[(n+1)J_{2n+2}(y)$ $-n\,J_{2n}(y)]$
(24)	$x^{-1/2}[D_n(ax)]^2 \quad \lvert\arg a\rvert<\tfrac{1}{4}\pi$	$(-1)^{n-1}y^{-1/2}[D_n(y/a)]^2$
(25)	$x^{-1/2}\operatorname{si}(a^2x^2) \quad a>0$	$y^{-1/2}[-\operatorname{si}(\tfrac{1}{4}x^2/a^2)-\tfrac{1}{2}\pi]$
(26)	$x^{-1/2}J_0(ax) \quad a>0$	$0 \quad 0<y<a$ $y^{-1/2} \quad a<y<\infty$
(27)	$x^{-3/2}J_0(ax)$	$2a\pi^{-1}y^{-1/2}[E(y/a)-(1-y^2/a^2)K(y/a)]$ $\quad 0<y<a$ $2\pi^{-1}y^{1/2}E(a/y) \quad a<y<\infty$

Order unity (cont'd)

	$f(x)$	$\int_0^\infty f(x)\, J_1(xy)(xy)^{1/2}\,dx \qquad y>0$
(28)	$x^{-5/2}[J_0(ax)-1] \qquad a>0$	$-\frac{1}{4}y^{3/2}[1+2\log(a/y)] \qquad 0<y<a$ $-\frac{1}{4}y^{1/2}a^2 \qquad a<y<\infty$
(29)	$x^{-1/2}J_0(ax)\,J_0(bx) \qquad a,\,b>0$	$0 \qquad 0<y<\lvert a-b\rvert$ $\pi^{-1}y^{-1/2}\cos^{-1}[(a^2+b^2-y^2)/(2ab)] \qquad \lvert a-b\rvert<y<a+b$ $y^{-1/2} \qquad a+b<y<\infty$
(30)	$x^{-5/2}J_1(ax) \qquad a>0$	$\dfrac{y^{1/2}(y+a)}{\pi}\left[E\left(\dfrac{2iy^{1/2}a^{1/2}}{\lvert y-a\rvert}\right)-K\left(\dfrac{2iy^{1/2}a^{1/2}}{\lvert y-a\rvert}\right)\right]$
(31)	$x^{-1/2}Y_0(ax) \qquad a>0$	$-\pi^{-1}y^{-1/2}\log(1-y^2/a^2) \qquad 0<y<a$
(32)	$x^{1/2}\,\mathrm{kei}_0\,x$	$-\frac{1}{2}y^{-1/2}\tan^{-1}(y^2)$
(33)	$x^{-1/2}\,\mathrm{ker}_0\,x$	$\frac{1}{2}y^{-1/2}\log(1+y^4)^{1/2}$

HANKEL TRANSFORMS OF ORDER ν

8.5. Algebraic functions and powers with arbitrary index

	$f(x)$	$\int_0^\infty f(x)\, J_\nu(xy)(xy)^{1/2}\,dx \qquad y>0$
(1)	$1 \qquad 0<x<1$ $0 \qquad 1<x<\infty$ $\mathrm{Re}\,\nu>-3/2$	$2^{1/2}y^{-1}\dfrac{\Gamma(\frac{3}{4}+\frac{1}{2}\nu)}{\Gamma(\frac{1}{4}+\frac{1}{2}\nu)}+(\nu-\frac{1}{2})\,J_\nu(y)$ $\times S_{-1/2,\,\nu-1}(y)-J_{\nu-1}(y)\,S_{1/2,\,\nu}(y)$

Algebraic functions (cont'd)

	$f(x)$	$\int_0^\infty f(x)\, J_\nu(xy)(xy)^{1/2}\, dx \qquad y>0$
(2)	$0 \qquad 0<x<1$ $1 \qquad 1<x<\infty$	$J_{\nu-1}(y)\, S_{1/2,\nu}(y)$ $+(1/2-\nu)\, J_\nu(y)\, S_{-1/2,\nu-1}(y)$
(3)	$x^{-1/2} \qquad \mathrm{Re}\,\nu>-1$	$y^{-1/2}$
(4)	$x^{1/2-\nu} \qquad 0<x<1$ $0 \qquad 1<x<\infty$	$\dfrac{2^{1-\nu} y^{\nu-3/2}}{\Gamma(\nu)} - y^{-1/2}\, J_{\nu-1}(y)$
(5)	$x^{\nu-1/2} \qquad 0<x<1$ $0 \qquad 1<x<\infty$ $\mathrm{Re}\,\nu>-1/2$	$2^{\nu-1}\, y^{1/2-\nu}\, \pi^{1/2}\, \Gamma(\nu+1/2)$ $\times[J_\nu(y)\, \mathbf{H}_{\nu-1}(y) - \mathbf{H}_\nu(y)\, J_{\nu-1}(y)]$
(6)	$x^{\nu+1/2} \qquad 0<x<1$ $0 \qquad 1<x<\infty$ $\mathrm{Re}\,\nu>-1$	$y^{-1/2}\, J_{\nu+1}(y)$
(7)	$x^\mu \qquad -\mathrm{Re}\,\nu-3/2<\mathrm{Re}\,\mu<-1/2$	$2^{\mu+1/2}\, y^{-\mu-1}\, \dfrac{\Gamma(1/2\mu+1/2\nu+3/4)}{\Gamma(1/2\nu-1/2\mu+1/4)}$
(8)	$x^\mu \qquad 0<x<1$ $0 \qquad 1<x<\infty$ $\mathrm{Re}(\mu+\nu)>-3/2$	$y^{-\mu-1}\Big[(\nu+\mu-1/2)y\, J_\nu(y)$ $\times S_{\mu-1/2,\nu-1}(y) - y\, J_{\nu-1}(y)\, S_{\mu+1/2,\nu}(y)$ $+ 2^{\mu+1/2}\, \dfrac{\Gamma(1/2\mu+1/2\nu+3/4)}{\Gamma(1/2\nu-1/2\mu+1/4)}\Big]$
(9)	$x^{\nu-1/2}(x+a)^{-1} \qquad \lvert\arg a\rvert<\pi$ $-1/2<\mathrm{Re}\,\nu<3/2, \quad \nu\neq 1/2$	$1/2\,\pi a^\nu \sec(\nu\pi)\, y^{1/2}\, [\mathbf{H}_{-\nu}(ay)$ $- Y_{-\nu}(ay)]$

Algebraic functions (cont'd)

	$f(x)$	$\int_0^\infty f(x)\, J_\nu(xy)(xy)^{1/2}\, dx \qquad y>0$
(10)	$x^{\rho-3/2}(x+\alpha)^{-\mu-1}$ $\quad \lvert\arg\alpha\rvert<\pi$, $\mathrm{Re}(\rho+\nu)>0$, $\mathrm{Re}(\rho-\mu)<5/2$	$\dfrac{y^{1/2}\pi\alpha^{\rho-\mu-1}}{\sin(\rho+\nu-\mu)\pi\,\Gamma(\mu+1)} \times\left\{\sum_{m=0}^{\infty}\dfrac{(-1)^m(\tfrac{1}{2}\alpha y)^{\nu+2m}\Gamma(\rho+\nu+2m)}{m!\,\Gamma(\nu+m+1)\Gamma(\rho+\nu-\mu+2m)} - \sum_{m=0}^{\infty}\dfrac{(\tfrac{1}{2}\alpha y)^{\mu+1-\rho+m}\Gamma(\mu+m+1)}{m!\,\Gamma[\tfrac{1}{2}(\mu+\nu-\rho+m+3)]} \times\dfrac{\sin\tfrac{1}{2}(\rho+\nu-\mu-m)\pi}{\Gamma[\tfrac{1}{2}(\mu-\nu-\rho+m+3)]}\right\}$
(11)	$x^{-1/2}(x^2+\alpha^2)^{-1/2}$ $\quad \mathrm{Re}\,\alpha>0, \quad \mathrm{Re}\,\nu>-1$	$y^{1/2} I_{\frac{1}{2}\nu}(\tfrac{1}{2}\alpha y)\, K_{\frac{1}{2}\nu}(\tfrac{1}{2}\alpha y)$
(12)	$x^{\nu+1/2}(x^2+\alpha^2)^{-1}$ $\quad \mathrm{Re}\,\alpha>0, \quad -1<\mathrm{Re}\,\nu<3/2$	$\alpha^\nu y^{1/2} K_\nu(\alpha y)$
(13)	$x^{\nu-1/2}(x^2+\alpha^2)^{-1}$ $\quad \mathrm{Re}\,\alpha>0, \quad -1/2<\mathrm{Re}\,\nu<5/2$	$\tfrac{1}{2}\pi\alpha^{\nu-1}\sec(\nu\pi)\, y^{1/2}[I_\nu(\alpha y)-\mathbf{L}_{-\nu}(\alpha y)]$
(14)	$x^{-\nu-1/2}(x^2+\alpha^2)^{-1}$ $\quad \mathrm{Re}\,\alpha>0, \quad \mathrm{Re}\,\nu>-\tfrac{1}{2}$	$\tfrac{1}{2}\pi\alpha^{-\nu-1} y^{1/2}[I_\nu(\alpha y)-\mathbf{L}_\nu(\alpha y)]$
(15)	$x^{\nu+1/2}(x^2+\alpha^2)^{-1/2}$ $\quad \mathrm{Re}\,\alpha>0, \quad -1<\mathrm{Re}\,\nu<\tfrac{1}{2}$	$2^{1/2}\pi^{-1/2}\alpha^{\nu+1/2} K_{\nu+1/2}(\alpha y)$
(16)	$x^{1/2-\nu}(x^2+\alpha^2)^{-1/2}$ $\quad \mathrm{Re}\,\alpha>0, \quad \mathrm{Re}\,\nu>-\tfrac{1}{2}$	$\pi^{1/2}2^{-1/2}\alpha^{1/2-\nu}[I_{\nu-1/2}(\alpha y)-\mathbf{L}_{\nu-1/2}(\alpha y)]$
(17)	$x^{-\nu-1/2}(x^2+\alpha^2)^{-\nu-1/2}$ $\quad \mathrm{Re}\,\alpha>0, \quad \mathrm{Re}\,\nu>-\tfrac{1}{2}$	$2^\nu\alpha^{-2\nu}y^{1/2+\nu}\dfrac{\Gamma(\nu+1)}{\Gamma(2\nu+1)} \times I_\nu(\tfrac{1}{2}\alpha y)\, K_\nu(\tfrac{1}{2}\alpha y)$

Algebraic functions (cont'd)

	$f(x)$	$\int_0^\infty f(x)\, J_\nu(xy)(xy)^{1/2}\, dx \qquad y>0$
(18)	$x^{\nu+1/2}(x^2+a^2)^{-\nu-1/2}$ $\operatorname{Re} a>0, \quad \operatorname{Re}\nu>-1/2$	$\dfrac{\pi^{1/2}\, y^{\nu-1/2}}{2^\nu e^{ay}\Gamma(\nu+1/2)}$
(19)	$x^{\nu+1/2}(x^2+a^2)^{-\nu-3/2}$ $\operatorname{Re} a>0, \quad \operatorname{Re}\nu>-1$	$\dfrac{y^{\nu+1/2}\,\pi^{1/2}}{2^{\nu+1}\, a e^{ay}\,\Gamma(\nu+3/2)}$
(20)	$x^{\nu+1/2}(x^2+a^2)^{-\mu-1} \qquad \operatorname{Re} a>0$ $-1<\operatorname{Re}\nu<2\operatorname{Re}\mu+3/2$	$\dfrac{a^{\nu-\mu}y^{\mu+1/2}K_{\nu-\mu}(ay)}{2^\mu\Gamma(\mu+1)}$
(21)	$x^{\lambda-3/2}(x^2+a^2)^{-\mu-1} \qquad \operatorname{Re} a>0$ $-\operatorname{Re}\nu<\operatorname{Re}\lambda<2\operatorname{Re}\mu+7/2$	$\dfrac{y^{\nu+1/2}\Gamma(\frac12\lambda+\frac12\nu)\Gamma(\mu-\frac12\lambda-\frac12\nu+1)}{2^{\nu+1}a^{2\mu-\lambda-\nu+2}\Gamma(\mu+1)\Gamma(\nu+1)}$ $\times {}_1F_2(\frac12\lambda+\frac12\nu;\frac12\lambda+\frac12\nu-\mu,\nu+1;\frac14y^2a^2)$ $+\dfrac{y^{2\mu-\lambda+5/2}\Gamma(\frac12\lambda+\frac12\nu-\mu-1)}{2^{2\mu-\lambda+3}\Gamma(\frac12\nu-\frac12\lambda+\mu+2)}$ $\times {}_1F_2(\mu+1;\mu+2+\frac12\nu-\frac12\lambda,$ $\mu+2-\frac12\lambda-\frac12\nu;\frac14y^2a^2)$
(22)	$x^{-1/2}(a^2-x^2)^{-1/2} \qquad 0<x<a$ $0 \qquad a<x<\infty$ $\operatorname{Re}\nu>-1$	$\frac12\pi y^{1/2}[J_{\frac12\nu}(\frac12ay)]^2$
(23)	$0 \qquad 0<x<a$ $x^{-1/2}(x^2-a^2)^{-1/2} \qquad a<x<\infty$	$-\frac12\pi y^{1/2}J_{\frac12\nu}(\frac12ay)\,Y_{\frac12\nu}(\frac12ay)$
(24)	$x^{1/2-\nu}(a^2-x^2)^{-1/2} \qquad 0<x<a$ $0 \qquad a<x<\infty$	$2^{-1/2}\pi^{1/2}a^{1/2-\nu}\mathbf{H}_{\nu-1/2}(ay)$
(25)	$x^{\nu-1/2}(a^2-x^2)^{\nu-1/2} \qquad 0<x<a$ $0 \qquad a<x<\infty$ $\operatorname{Re}\nu>-1/2$	$2^{\nu-1}\pi^{1/2}\Gamma(\nu+\frac12)a^{2\nu}y^{1/2-\nu}$ $\times[J_\nu(\frac12ay)]^2$

Algebraic functions (cont'd)

	$f(x)$	$\int_0^\infty f(x)\, J_\nu(xy)(xy)^{1/2}\, dx \qquad y>0$
(26)	$0 \qquad 0<x<a$ $x^{-\nu-1/2}(x^2-a^2)^{-\nu-1/2} \qquad a<x<\infty$ $\lvert \mathrm{Re}\,\nu \rvert < 1/2$	$-2^{-\nu-1}a^{-2\nu}\,\Gamma(1/2-\nu)\,y^{\nu+1/2}\,\pi^{1/2}$ $\times J_\nu(1/2\, ay)\, Y_\nu(1/2\, ay)$
(27)	$x^{\nu+1/2}(a^2-x^2)^{-\nu-1/2} \qquad 0<x<a$ $0 \qquad a<x<\infty$ $\lvert \mathrm{Re}\,\nu \rvert < 1/2$	$\pi^{-1/2}\,2^{-\nu}\,\Gamma(1/2-\nu)\,y^{\nu-1/2}\sin(ay)$
(28)	$0 \qquad 0<x<a$ $x^{-\nu+1/2}(x^2-a^2)^{\nu-1/2} \qquad a<x<\infty$ $\lvert \mathrm{Re}\,\nu \rvert < 1/2$	$\pi^{-1/2}\,2^{-\nu}\,\Gamma(1/2+\nu)\,y^{-\nu-1/2}\cos(ay)$
(29)	$x^{\nu+1/2}(a^2-x^2)^{-\nu-3/2} \qquad 0<x<a$ $0 \qquad a<x<\infty$ $-1<\mathrm{Re}\,\nu<-1/2$	$2^{-1-\nu}\,\pi^{-1/2}\,\Gamma(-1/2-\nu)\,a^{-1}\cos(ay)$ $\times y^{\nu+1/2}$
(30)	$0 \qquad 0<x<a$ $x^{1/2-\nu}(x^2-a^2)^{\nu-3/2} \qquad a<x<\infty$ $1/2<\mathrm{Re}\,\nu<5/2$	$2^{-\nu-1}\,\pi^{-1/2}\,a^{-1}\,\Gamma(\nu-1/2)\,y^{1/2-\nu}\sin(ay)$
(31)	$x^{1/2-\nu}(a^2-x^2)^{\mu} \qquad 0<x<a$ $0 \qquad a<x<\infty$ $\mathrm{Re}\,\mu>-1$	$\dfrac{2^{1-\nu}\,a^{\mu-\nu+1}\,s_{\nu+\mu,\,\mu-\nu+1}(ay)}{y^{\mu+1/2}\,\Gamma(\nu)}$
(32)	$0 \qquad 0<x<a$ $x^{1/2-\nu}(x^2-a^2)^{\mu} \qquad a<x<\infty$ $\mathrm{Re}\,\mu>-1, \quad \mathrm{Re}(\nu-2\mu)>1/2$	$2^{\mu}\,\Gamma(\mu+1)\,a^{1+\mu-\nu}\,y^{-\mu-1/2}\,J_{\nu-\mu-1}(ay)$

Algebraic functions (cont'd)

	$f(x)$	$\int_0^\infty f(x)\, J_\nu(xy)(xy)^{1/2}\, dx \qquad y>0$
(33)	$x^{\nu+1/2}(a^2-x^2)^\mu \qquad 0<x<a$ $0 \qquad a<x<\infty$ $\operatorname{Re}\nu>-1, \quad \operatorname{Re}\mu>-1$	$2^\mu\Gamma(\mu+1)\, y^{-\mu-1/2}\, a^{\nu+\mu+1}$ $\times J_{\nu+\mu+1}(ay)$
(34)	$x^{\mu-1/2}(a^2-x^2)^\lambda \qquad 0<x<a$ $0 \qquad a<x<\infty$ $\operatorname{Re}\lambda>-1, \quad \operatorname{Re}(\mu+\nu)>-1$	$\dfrac{a^{2\lambda+\mu+\nu+1}y^{\nu+1/2}\, \mathrm{B}(\lambda+1, \frac{1}{2}\mu+\frac{1}{2}\nu+\frac{1}{2})}{2^{\nu+1}\,\Gamma(\nu+1)}$ $\times {}_1F_2\left(\dfrac{1+\mu+\nu}{2}; \nu+1, \dfrac{3+\mu+\nu}{2}+\lambda; -\dfrac{a^2y^2}{4}\right)$
(35)	$x^{-\nu-1/2}(a^2+2x)^{-1/2}$ $\times[(a^2+2x)^{1/2}-a]^{2\nu}$ $\operatorname{Re}\nu>-\frac{1}{2}$	$2^\nu\Gamma(\nu+\frac{1}{2})\,\pi^{-1/2} D_{-\nu-1/2}(ae^{\frac{1}{4}\pi i}y^{1/2})$ $\times D_{-\nu-1/2}(ae^{-\frac{1}{4}\pi i}y^{1/2})$
(36)	$x^{-1/2}(x^2+a^2)^{-1/2}$ $\times[(x^2+a^2)^{1/2}+x]^{\nu-1}$ $\operatorname{Re}a>0, \quad -1<\operatorname{Re}\nu<5/2$	$2\pi^{-1/2}a^{\nu-3/2}\sinh(\frac{1}{2}ay)K_{\nu-1/2}(\frac{1}{2}ay)$
(37)	$x^{-1/2}(x^2+a^2)^{-1/2}$ $\times[(x^2+a^2)^{1/2}+x]^{1-\nu}$ $\operatorname{Re}a>0, \quad \operatorname{Re}\nu>-\frac{1}{2}$	$\pi^{1/2}\, a^{1/2-\nu}\, e^{-\frac{1}{2}ay}\, I_{\nu-1/2}(\frac{1}{2}ay)$
(38)	$x^{-1/2}(x^2+a^2)^{-1/2}$ $\times[(x^2+a^2)^{1/2}\pm x]^\mu \quad \operatorname{Re}a>0$ $\operatorname{Re}\nu>-1, \quad \operatorname{Re}\mu<3/2$	$y^{1/2}\, a^\mu\, I_{\frac{1}{2}(\nu\mp\mu)}(\frac{1}{2}ay)\, K_{\frac{1}{2}(\nu\pm\mu)}(\frac{1}{2}ay)$
(39)	$x^{-\nu+1/2}(x^2+a^2)^{-1/2}$ $\times[(x^2+a^2)^{1/2}-a]^\nu$ $\operatorname{Re}a>0, \quad \operatorname{Re}\nu>-1$	$y^{-1/2}\, e^{-ay}$
(40)	$x^{-\mu-1/2}(x^2+a^2)^{-1/2}$ $\times[(x^2+a^2)^{1/2}+a]^\mu$ $\operatorname{Re}a>0, \quad \operatorname{Re}(\nu-\mu)>-1$	$\dfrac{\Gamma(\frac{1}{2}+\frac{1}{2}\nu-\frac{1}{2}\mu)}{ay^{1/2}\,\Gamma(\nu+1)} W_{\frac{1}{2}\mu, \frac{1}{2}\nu}(ay)\, M_{-\frac{1}{2}\mu, \frac{1}{2}\nu}(ay)$

Algebraic functions (cont'd)

	$f(x)$	$\int_0^\infty f(x)\, J_\nu(xy)\,(xy)^{1/2}\,dx \qquad y>0$
(41)	$0 \qquad 0<x<a$ $x^{-1/2}(x^2-a^2)^{-1/2}\{[x+(x^2-a^2)^{1/2}]^{\nu+1}$ $+[x-(x^2-a^2)^{1/2}]^{\nu+1}\}$ $a<x<\infty$ $\mathrm{Re}\,\nu<1/2$	$-\pi^{1/2}a^{\nu-3/2}[\sin(\tfrac{1}{2}ay)\,J_{\nu+1/2}(\tfrac{1}{2}ay)$ $+\cos(\tfrac{1}{2}ay)\,Y_{\nu+1/2}(\tfrac{1}{2}ay)]$
(42)	$0 \qquad 0<x<a$ $x^{-1/2}(x^2-a^2)^{-1/2}\{[x+(x^2-a^2)^{1/2}]^{\nu-1}$ $+[x-(x^2-a^2)^{1/2}]^{\nu-1}\}$ $a<x<\infty$ $\mathrm{Re}\,\nu<5/2$	$\pi^{1/2}a^{\nu-3/2}[\cos(\tfrac{1}{2}ay)\,J_{\nu-1/2}(\tfrac{1}{2}ay)$ $-\sin(\tfrac{1}{2}ay)\,Y_{\nu-1/2}(\tfrac{1}{2}ay)]$
(43)	$x^{-1/2}(a^2-x^2)^{-1/2}\{[x+i(a^2-x^2)^{1/2}]^{\mu}$ $+[x-i(a^2-x^2)^{1/2}]^{\mu}\} \qquad 0<x<a$ $0 \qquad a<x<\infty$ $\mathrm{Re}(\mu+\nu)>-1$	$\pi a^{\mu} y^{1/2}\, J_{\frac{1}{2}(\nu+\mu)}(\tfrac{1}{2}ay)\, J_{\frac{1}{2}(\nu-\mu)}(\tfrac{1}{2}ay)$
(44)	$0 \qquad 0<x<a$ $x^{-1/2}(x^2-a^2)^{-1/2}\{[x+(x^2-a^2)^{1/2}]^{\mu}$ $+[x-(x^2-a^2)^{1/2}]^{\mu}\} \qquad a<x<\infty$ $\mathrm{Re}\,\mu<3/2$	$-\tfrac{1}{2}\pi y^{1/2}a^{\mu}[J_{\frac{1}{2}(\mu+\nu)}(\tfrac{1}{2}ay)\,Y_{\frac{1}{2}(\nu-\mu)}(\tfrac{1}{2}ay)$ $+J_{\frac{1}{2}(\nu-\mu)}(\tfrac{1}{2}ay)\,Y_{\frac{1}{2}(\nu+\mu)}(\tfrac{1}{2}ay)]$
(45)	$x^{-2\mu-1/2}(a^2-x^2)^{-1/2}$ $\times\{[a+(a^2-x^2)^{1/2}]^{2\mu}$ $+[a-(a^2-x^2)^{1/2}]^{2\mu}\} \qquad 0<x<a$ $0 \qquad a<x<\infty$ $\mathrm{Re}(2\mu)<\mathrm{Re}\,\nu+1$	$\dfrac{a^{\nu}\,\mathrm{B}(\tfrac{1}{2}+\tfrac{1}{2}\nu+\mu,\ \tfrac{1}{2}+\tfrac{1}{2}\nu-\mu)\, y^{\nu+1/2}}{\Gamma(1+\nu)}$ $\times\,{}_1F_1(\tfrac{1}{2}+\tfrac{1}{2}\nu-\mu;\nu+1;-iay)$ $\times\,{}_1F_1(\tfrac{1}{2}+\tfrac{1}{2}\nu-\mu;\nu+1;iay)$
(46)	$x^{\mu-1/2}(1-2ax+a^2)^{-1/2} \qquad 0<x<1$ $0 \qquad 1<x<\infty$ $\mathrm{Re}(\nu+\mu+\tfrac{1}{2})>0$	see Bose, S. K., 1946: *Bull. Calcuta Math. Soc.*, 38, 177-180.

Algebraic functions (cont'd)

	$f(x)$	$\int_0^\infty f(x)\, J_\nu(xy)(xy)^{1/2}\, dx \qquad y>0$
(47)	$x^{\nu+5/2}(x^4+4a^4)^{-\nu-1/2}$ $\lvert\arg a\rvert<\pi/4, \quad \mathrm{Re}\,\nu>1/6$	$\dfrac{\pi^{1/2} y^{1/2+\nu} J_{\nu-1}(ay)\, K_{\nu-1}(ay)}{2^{3\nu-1} a^{2\nu-2}\Gamma(\nu+\frac12)}$
(48)	$x^{\nu+1/2}(x^4+4a^4)^{-\nu-1/2}$ $\lvert\arg a\rvert<\frac14\pi, \quad \mathrm{Re}\,\nu>-\frac12$	$\dfrac{y^{\nu+1/2}\pi^{1/2} J_\nu(ay)\, K_\nu(ay)}{a^{2\nu}2^{3\nu}\Gamma(\nu+\frac12)}$
(49)	$x^{\nu+1/2}(x^4\pm 2a^2x^2+b^4)^{-1/2}$ $\times[b^2+x^2+(x^4\pm 2a^2x^2+b^4)^{1/2}]^{-2\nu}$ $0<a<b$ $\mathrm{Re}\,\nu>-\frac12$	$(b^2\mp a^2)^{-\nu}\, 2^\nu y^{1/2} K_\nu[(\frac12 b^2\pm\frac12 a^2)^{1/2} y]$ $\times J_\nu[(\frac12 b^2\mp\frac12 a^2)^{1/2} y]$
(50)	$0 \qquad 0<x<a$ $x^{1/2-\nu}(x^2-a^2)^{-1/2}$ $\times\{[a+(a^2-x^2)^{1/2}]^\nu$ $+[a-(a^2-x^2)^{1/2}]^\nu\} \qquad a<x<\infty$ $\mathrm{Re}\,\nu>-1$	$2y^{-1/2}\cos(ay-\frac12\nu\pi)$

8.6. Exponential and logarithmic functions

(1)	$x^{-1/2}e^{-ax}$ $\mathrm{Re}\,a>0, \quad \mathrm{Re}\,\nu>-1$	$y^{1/2-\nu}(a^2+y^2)^{-1/2}[(a^2+y^2)^{1/2}-a]^\nu$
(2)	$x^{-3/2}e^{-ax}$ $\mathrm{Re}\,a>0, \quad \mathrm{Re}\,\nu>0$	$\nu^{-1}y^{1/2-\nu}[(a^2+y^2)^{1/2}-a]^\nu$
(3)	$x^{m+1/2}e^{-ax} \qquad \mathrm{Re}\,\nu>-m-2$	$(-1)^{m+1}y^{1/2-\nu}\dfrac{d^{m+1}}{da^{m+1}}\{(a^2+y^2)^{-1/2}$ $\times[(a^2+y^2)^{1/2}-a]^\nu\}$

Exponential and logarithmic functions (cont'd)

	$f(x)$	$\int_0^\infty f(x)\, J_\nu(xy)(xy)^{1/2}\, dx \qquad y>0$
(4)	$x^{\nu+1/2}\, e^{-ax}$ $\mathrm{Re}\, a>0, \quad \mathrm{Re}\,\nu>-1$	$\pi^{-1/2} 2^{\nu+1}\Gamma(\nu+3/2)\, a y^{\nu+1/2}$ $\times(a^2+y^2)^{-\nu-3/2}$
(5)	$x^{\nu-1/2}\, e^{-ax}$ $\mathrm{Re}\, a>0, \quad \mathrm{Re}\,\nu>-1/2$	$2^\nu \pi^{-1/2}\,\Gamma(\nu+1/2) y^{\nu+1/2}(a^2+y^2)^{-\nu-1/2}$
(6)	$x^{\mu-3/2}\, e^{-ax}$ $a>0, \quad \mathrm{Re}(\mu+\nu)>0$	$y^{1/2}(a^2+y^2)^{-1/2\mu}\Gamma(\mu+\nu)$ $\times P_{\mu-1}^{-\nu}[a(a^2+y^2)^{-1/2}]$
(7)	$x^{\mu-3/2}\, e^{-ax}$ $\mathrm{Re}\, a>0, \quad \mathrm{Re}(\mu+\nu)>0$	$\dfrac{y^{\nu+1/2}\,\Gamma(\mu+\nu)}{2^\nu a^{\mu+\nu}\Gamma(\nu+1)}$ $\times {}_2F_1\left(\dfrac{\mu+\nu}{2}, \dfrac{\mu+\nu+1}{2}; \nu+1; -\dfrac{y^2}{a^2}\right)$ $=\dfrac{y^{\nu+1/2}\,\Gamma(\mu+\nu)}{2^\nu(a^2+y^2)^{1/2(\mu+\nu)}\Gamma(\nu+1)}$ $\times {}_2F_1\left[\dfrac{\mu+\nu}{2}, \dfrac{1-\mu+\nu}{2}; \nu+1; \dfrac{y^2}{(a^2+y^2)}\right]$
(8)	$x^{-1/2}\, e^{-ax^2}$ $\mathrm{Re}\, a>0, \quad \mathrm{Re}\,\nu>-1$	$\dfrac{\pi^{1/2} y^{1/2}}{2a^{1/2}} \exp\left(-\dfrac{y^2}{8a}\right) I_{1/2\nu}\left(\dfrac{y^2}{8a}\right)$
(9)	$x^{1/2}\, e^{-ax^2}$ $\mathrm{Re}\, a>0, \quad \mathrm{Re}\,\nu>-2$	$\dfrac{\pi^{1/2} y^{3/2}}{8a^{3/2}} \exp\left(-\dfrac{y^2}{8a}\right)$ $\times\left[I_{1/2\nu-1/2}\left(\dfrac{y^2}{8a}\right) - I_{1/2\nu+1/2}\left(\dfrac{y^2}{8a}\right)\right]$
(10)	$x^{\nu+1/2}\, e^{-ax^2}$ $\mathrm{Re}\, a>0, \quad \mathrm{Re}\,\nu>-1$	$\dfrac{y^{\nu+1/2}}{(2a)^{\nu+1}} \exp\left(-\dfrac{y^2}{4a}\right)$

Exponential and logarithmic functions (cont'd)

	$f(x)$	$\int_0^\infty f(x)\, J_\nu(xy)(xy)^{1/2}\, dx \qquad y>0$
(11)	$x^{\nu-3/2}\, e^{-\alpha x^2}$ $\operatorname{Re}\alpha>0, \quad \operatorname{Re}\nu>0$	$2^{\nu-1}\, y^{1/2-\nu}\, \gamma\left(\nu, \dfrac{y^2}{4\alpha}\right)$
(12)	$x^{\nu+1/2}\, e^{\pm i a x^2}$ $a>0, \quad -1<\operatorname{Re}\nu<1/2$	$\dfrac{y^{\nu+1/2}}{(2a)^{\nu+1}} \exp\left[\pm i\left(\dfrac{\nu+1}{2}\pi - \dfrac{y^2}{4a}\right)\right]$
(13)	$x^{2n+\nu+1/2}\, e^{-1/4 x^2}$ $\operatorname{Re}\nu>-1-2n$	$2^{2n+\nu+1}\, n!\, y^{\nu+1/2}\, e^{-y^2}\, L_n^{\nu}(y^2)$
(14)	$x^{\mu-1/2}\, e^{-\alpha x^2}$ $\operatorname{Re}\alpha>0, \quad \operatorname{Re}(\mu+\nu)>-1$	$\dfrac{y^{\nu+1/2}\Gamma(1/2\nu+1/2\mu+1/2)}{2^{\nu+1}\,\alpha^{1/2(\mu+\nu+1)}\,\Gamma(\nu+1)}$ $\times {}_1F_1\left(\dfrac{\nu+\mu+1}{2}; \nu+1; -\dfrac{y^2}{4\alpha}\right)$ $=\dfrac{\Gamma(1/2\nu+1/2\mu+1/2)}{y^{1/2}\,\alpha^{1/2\mu}\,\Gamma(\nu+1)} \exp\left(-\dfrac{y^2}{8\alpha}\right)$ $\times M_{1/2\mu,\, 1/2\nu}\left(\dfrac{y^2}{4\alpha}\right)$
(15)	$x^{-3/2}\, e^{-\alpha/x} \qquad \operatorname{Re}\alpha>0$	$2y^{1/2}\, J_\nu[(2\alpha y)^{1/2}]\, K_\nu[(2\alpha y)^{1/2}]$
(16)	$x^{-3/2}\, e^{-\alpha/x-\beta x}$ $\operatorname{Re}\alpha>0, \quad \operatorname{Re}\beta>0$	$2y^{1/2}\, J_\nu\{(2\alpha)^{1/2}[(\beta^2+y^2)^{1/2}-\beta]^{1/2}\}$ $\times K_\nu\{(2\alpha)^{1/2}[(\beta^2+y^2)^{1/2}+\beta]^{1/2}\}$
(17)	$x^{-1}\, e^{-\alpha x^{1/2}}$ $\operatorname{Re}\alpha>0, \quad \operatorname{Re}\nu>-1/2$	$\pi^{-1/2}\, 2^{1/2}\, \Gamma(\nu+1/2)\, D_{-\nu-1/2}(2^{-1/2}\alpha e^{1/4\pi i} y^{-1/2})$ $\times D_{-\nu-1/2}(2^{-1/2}\alpha e^{-1/4\pi i} y^{-1/2})$

Exponential and logarithmic functions (cont'd)

	$f(x)$	$\int_0^\infty f(x)\, J_\nu(xy)(xy)^{1/2}\, dx \qquad y>0$
(18)	$x^{\nu+1/2}\, e^{a(1-x^2)} \qquad 0<x<1$ $0 \qquad 1<x<\infty$ $\operatorname{Re}\nu>-1/2$	$(2ia)^{-\nu-1}\, y^{1/2+\nu}[U_{\nu+1}(2ia,y)$ $-i\,U_{\nu+2}(2ia,y)]$
(19)	$x^{\nu+1/2}\exp[-a(x^2+\beta^2)^{1/2}]$ $\operatorname{Re}a>0,\ \operatorname{Re}\beta>0,\ \operatorname{Re}\nu>-1$	$(\tfrac{1}{2}\pi)^{-1/2}\, a\beta^{\nu+3/2} y^{\nu+1/2}(y^2+a^2)^{-\nu-3/4}$ $\times K_{\nu+3/2}[\beta(y^2+a^2)^{1/2}]$
(20)	$x^{-1/2}(\beta^2+x^2)^{-1/2}$ $\times\exp[-a(\beta^2+x^2)^{1/2}]$ $\operatorname{Re}a>0, \quad \operatorname{Re}\beta>0$ $\operatorname{Re}\nu>-1$	$y^{1/2}\, I_{\frac{1}{2}\nu}\{\tfrac{1}{2}\beta[(a^2+y^2)^{1/2}-a]\}$ $\times K_{\frac{1}{2}\nu}\{\tfrac{1}{2}\beta[(a^2+y^2)^{1/2}+a]\}$
(21)	$x^{\nu+1/2}(\beta^2+x^2)^{-1/2}$ $\times\exp[ia(\beta^2+x^2)^{1/2}] \qquad a>0$ $\operatorname{Re}\beta>0, \quad -1<\operatorname{Re}\nu<1/2$	$i\,2^{-1/2}\pi^{1/2}\beta^{1/2+\nu}(a^2-y^2)^{-1/4-1/2\nu}\ y^{1/2+\nu}$ $\times H^{(1)}_{-\nu-1/2}[\beta(a^2-y^2)^{1/2}]$ $0<y<a$ $2^{1/2}\pi^{-1/2}\beta^{1/2+\nu}y^{1/2+\nu}(y^2-a^2)^{-1/4-1/2\nu}$ $\times K_{\nu+1/2}[\beta(y^2-a^2)^{1/2}]$ $a<y<\infty$
(22)	$x^{\nu+1/2}(\beta^2+x^2)^{-1/2}$ $\times\exp[-a(\beta^2+x^2)^{1/2}]$ $\operatorname{Re}a>0,\ \operatorname{Re}\beta>0,\ \operatorname{Re}\nu>-1$	$(\tfrac{1}{2}\pi)^{-1/2}\beta^{\nu+1/2}y^{\nu+1/2}(a^2+y^2)^{-1/2\nu-1/4}$ $\times K_{\nu+1/2}[\beta(a^2+y^2)^{1/2}]$
(23)	$x^{-\nu+1/2}(x^2+\beta^2)^{-1/2}$ $\times[(x^2+\beta^2)^{1/2}-\beta]^{\nu}$ $\times\exp[-a(x^2+\beta^2)^{1/2}]$ $\operatorname{Re}a>0,\ \operatorname{Re}\beta>0,\ \operatorname{Re}\nu>-1$	$y^{\nu+1/2}[a+(y^2+a^2)^{1/2}]^{-\nu}(y^2+a^2)^{-1/2}$ $\times\exp[-\beta(y^2+a^2)^{1/2}]$
(24)	$x^{\sigma-1/2}(x^2+\beta^2)^{-1/2}[(x^2+\beta^2)^{1/2}+\beta]^{-\sigma}$ $\times\exp[-a(x^2+\beta^2)^{1/2}]$ $\operatorname{Re}a>0, \quad \operatorname{Re}\beta>0$ $\operatorname{Re}(\nu+\sigma)>-1$	$\dfrac{\Gamma(\tfrac{1}{2}\nu+\tfrac{1}{2}\sigma+\tfrac{1}{2})}{\beta\Gamma(\nu+1)\,y^{1/2}}\, M_{\frac{1}{2}\sigma,\frac{1}{2}\nu}\{\beta[(y^2+a^2)^{1/2}-a]\}$ $\times W_{-\frac{1}{2}\sigma,\frac{1}{2}\nu}\{\beta[(y^2+a^2)^{1/2}+a]\}$

Exponential and logarithmic functions (cont'd)

	$f(x)$	$\int_0^\infty f(x)\, J_\nu(xy)(xy)^{1/2}\, dx \qquad y > 0$
	For other Hankel transforms containing exponential functions see Laplace transforms.	
(25)	$x^\mu \log x$ $-\mathrm{Re}\,\nu - 3/2 < \mathrm{Re}\,\mu < 0$	$\dfrac{2^{\mu-1/2}\,\Gamma(\frac{1}{2}\mu+\frac{1}{2}\nu+\frac{3}{4})}{\Gamma(\frac{1}{2}\nu-\frac{1}{2}\mu+\frac{1}{4})\,y^{\mu+1}}$ $\times[\psi(\frac{1}{2}\mu+\frac{1}{2}\nu+\frac{3}{4})$ $+\psi(\frac{1}{2}\nu-\frac{1}{2}\mu+\frac{1}{4})-\log(\frac{1}{4}y^2)]$

8.7. Trigonometric and inverse trigonometric functions

(1)	$x^{-1/2}\sin(ax)$ $a > 0, \quad \mathrm{Re}\,\nu > -2$	$\cos(\frac{1}{2}\pi\nu)\, y^{\nu+1/2}(a^2-y^2)^{-1/2}$ $\times[a+(a^2-y^2)^{1/2}]^{-\nu} \qquad 0 < y < a$ $y^{1/2}(y^2-a^2)^{-1/2}\sin[\nu\sin^{-1}(a/y)]$ $a < y < \infty$
(2)	$x^{-3/2}\sin(ax)$ $a > 0, \quad \mathrm{Re}\,\nu > -1$	$\nu^{-1}\sin(\frac{1}{2}\nu\pi)\, y^{\nu+1/2}$ $\times[a+(a^2-y^2)^{1/2}]^{-\nu} \qquad 0 < y \le a$ $\nu^{-1} y^{1/2}\sin[\nu\sin^{-1}(a/y)]$ $a < y < \infty$
(3)	$x^{\nu+1/2}\sin(ax)$ $a > 0, \quad -3/2 < \mathrm{Re}\,\nu < -1/2$	$-2^{1+\nu}\pi^{-1/2}\sin(\nu\pi)\,\Gamma(\nu+3/2)\,a$ $\times y^{\nu+1/2}(a^2-y^2)^{-\nu-3/2}$ $0 < y < a$ $-2^{1+\nu}\pi^{-1/2}\,\Gamma(\nu+3/2)$ $\times a y^{\nu+1/2}(y^2-a^2)^{-\nu-3/2}$ $a < y < \infty$
(4)	$x^{\nu-1/2}\sin(ax)$ $a > 0, \quad -1 < \mathrm{Re}\,\nu < \frac{1}{2}$	$[\Gamma(\frac{1}{2}-\nu)]^{-1}\pi^{1/2}\, 2^\nu y^{\nu+1/2}$ $\times(a^2-y^2)^{-\nu-1/2} \qquad 0 < y < a$ $0 \qquad a < y < \infty$

Trigonometric functions (cont'd)

	$f(x)$	$\int_0^\infty f(x)\, J_\nu(xy)(xy)^{1/2}\, dx \qquad y>0$
(5)	$x^{1/2-\nu}\sin(ax)$ $a>0, \quad \operatorname{Re}\nu>1/2$	$0 \qquad 0<y<a$ $2^{1-\nu}\pi^{1/2}a[\Gamma(\nu-1/2)]^{-1}y^{1/2+\nu}\times(y^2-a^2)^{\nu-3/2} \qquad a<y<\infty$
(6)	$x^{-\nu+2n+1/2}\sin(ax)$ $a>0, \quad \operatorname{Re}\nu>2n+1/2$	$0 \qquad 0<y<a$ $(-1)^n 2^{\nu-2n-2}y^{2n-\nu+3/2}(2n+1)!\times\Gamma(\nu-2n-1)[\Gamma(2\nu-2n-1)]^{-1}\times(y^2-a^2)^{\nu-2n-3/2}C_{2n+1}^{\nu-2n-1}(ay^{-1})$ $a<y<\infty$
(7)	$x^{\mu-3/2}\sin(ax)$ $a>0, \quad -\operatorname{Re}\nu<\operatorname{Re}\mu<3/2$	$\dfrac{y^{\nu+1/2}\Gamma(\nu+\mu)\sin[\frac{1}{2}\pi(\nu+\mu)]}{2^\nu a^{\nu+\mu}\Gamma(\nu+1)}\times{}_2F_1\left(\dfrac{1+\nu+\mu}{2},\dfrac{\nu+\mu}{2};\nu+1;\dfrac{y^2}{a^2}\right)$ $0<y<a$ $\dfrac{2^\mu a\,\Gamma(1/2+1/2\nu+1/2\mu)}{y^{\mu+1/2}\Gamma(1/2+1/2\nu-1/2\mu)}\times{}_2F_1\left(\dfrac{1+\nu+\mu}{2},\dfrac{1+\mu-\nu}{2};\dfrac{3}{2};\dfrac{a^2}{y^2}\right)$ $a<y<\infty$
(8)	$x^{\nu-1/2}(x^2+\beta^2)^{-1}\sin(ax) \quad a>0$ $\operatorname{Re}\beta>0, \quad -1<\operatorname{Re}\nu<3/2$	$\beta^{\nu-1}\sinh(a\beta)y^{1/2}K_\nu(\beta y) \qquad y\geq a$
(9)	$x^{1/2-\nu}(x^2+\beta^2)^{-1}\sin(ax) \quad a>0$ $\operatorname{Re}\beta>0, \quad \operatorname{Re}\nu>-1/2$	$\frac{1}{2}\pi\beta^{-\nu}e^{-a\beta}y^{1/2}I_\nu(\beta y)$ $0<y\leq a$
(10)	$x^{-3/2}e^{-xa\cos\phi\cos\psi}\sin(xa\sin\psi)$ $\operatorname{Re}\nu>-1, \quad a>0, \quad 0<\phi,\psi<\frac{1}{2}\pi$	$a^{1/2}\nu^{-1}(\sin\phi)^{1/2}(\tan\frac{1}{2}\phi)^\nu\sin(\nu\psi)$ $y=a\sin\phi$

Trigonometric functions (cont'd)

	$f(x)$	$\int_0^\infty f(x)\, J_\nu(xy)(xy)^{\frac{1}{2}}\, dx \qquad y>0$
(11)	$x^{\nu+\frac{1}{2}} e^{-ax\cos\phi\,\cos\psi}\sin(ax\sin\psi)$ $a>0, \quad 0<\phi,\ \psi<\frac{1}{2}\pi$ $\operatorname{Re}\nu>-3/2$	$2^{\nu+1}\pi^{-1/2}\Gamma(\nu+3/2)\,a^{-\nu-3/2}$ $\times(\sin\phi)^{\nu+1/2}(\cos^2\psi$ $+\sin^2\psi\cos^2\phi)^{-\nu-3/2}$ $\times\sin[(\nu+3/2)\alpha] \qquad y=a\sin\phi$ $\tan(\frac{1}{2}\alpha)=\tan\psi\cos\phi$
(12)	$x^{\nu-\frac{1}{2}} e^{-xa\cos\phi\,\cos\psi}\sin(xa\sin\psi)$ $a>0, \quad 0<\phi,\ \psi<\frac{1}{2}\pi$ $\operatorname{Re}\nu>-1$	$2^{\nu}\pi^{-\frac{1}{2}}a^{-\nu-\frac{1}{2}}\Gamma(\nu+\frac{1}{2})(\sin\phi)^{\nu+\frac{1}{2}}$ $\times(\cos^2\psi+\sin^2\psi\cos^2\phi)^{-\nu-\frac{1}{2}}$ $\times\sin[(\nu+3/2)\alpha] \qquad y=a\sin\phi$ $\tan(\frac{1}{2}\alpha)=\tan\psi\cos\phi$
(13)	$x^{-\frac{1}{2}}\sin(ax^2)$ $a>0, \quad \operatorname{Re}\nu>-3$	$-\dfrac{\pi^{\frac{1}{2}}y^{\frac{1}{2}}}{2a^{\frac{1}{2}}}\sin\left(\dfrac{y^2}{8a}-\dfrac{\nu+1}{4}\pi\right)$ $\times J_{\frac{1}{2}\nu}\left(\dfrac{y^2}{8a}\right)$
(14)	$x^{\frac{1}{2}}\sin(ax^2)$ $a>0, \quad \operatorname{Re}\nu>-4$	$\dfrac{\pi^{1/2}y^{3/2}}{8a^{3/2}}\left[\cos\left(\dfrac{y^2}{8a}-\dfrac{\nu\pi}{4}\right)J_{\frac{1}{2}\nu-\frac{1}{2}}\left(\dfrac{y^2}{8a}\right)\right.$ $\left.-\sin\left(\dfrac{y^2}{8a}-\dfrac{\nu\pi}{4}\right)J_{\frac{1}{2}\nu+\frac{1}{2}}\left(\dfrac{y^2}{8a}\right)\right]$
(15)	$x^{\nu+\frac{1}{2}}\sin(ax^2)$ $a>0, \quad -2<\operatorname{Re}\nu<\frac{1}{2}$	$\dfrac{y^{\nu+\frac{1}{2}}}{2^{\nu+1}a^{\nu+1}}\cos\left(\dfrac{y^2}{4a}-\dfrac{\nu\pi}{2}\right)$
(16)	$x^{\nu+\frac{1}{2}}\sin(ax^2) \qquad 0<x<b$ $0 \qquad b<x<\infty$ $\operatorname{Re}\nu>-2$	$(2a)^{-\nu-1}y^{\nu+\frac{1}{2}}[\sin(ab^2)U_{\nu+1}(2ab^2, by)$ $-\cos(ab^2)U_{\nu+2}(2ab^2, by)]$
(17)	$x^{-1}e^{-ax^{\frac{1}{2}}}\sin(ax^{\frac{1}{2}})$ $a>0, \quad \operatorname{Re}\nu>-1$	$i\,2^{-\frac{1}{2}}\pi^{-\frac{1}{2}}\Gamma(\nu+\frac{1}{2})\,D_{-\nu-\frac{1}{2}}(ay^{-\frac{1}{2}})$ $\times[D_{-\nu-\frac{1}{2}}(iay^{-\frac{1}{2}})-D_{-\nu-\frac{1}{2}}(-iay^{-\frac{1}{2}})]$

Trigonometric functions (cont'd)

	$f(x)$	$\int_0^\infty f(x)\, J_\nu(xy)(xy)^{1/2}\, dx \qquad y>0$
(18)	$x^{\nu+1/2} \sin[a(x^2+\beta^2)^{1/2}] \qquad a>0$ $\operatorname{Re}\beta>0, \quad -1<\operatorname{Re}\nu<-1/2$	$(1/2\,\pi)^{1/2}\, a\beta^{\nu+3/2} y^{\nu+1/2} (a^2-y^2)^{-\nu/2-3/4}$ $\times\{\sin(\nu\pi) J_{\nu+3/2}[\beta(a^2-y^2)^{1/2}]$ $+\cos(\nu\pi) Y_{\nu+3/2}[\beta(a^2-y^2)^{1/2}]\}$ $0<y<a$ $-(1/2\,\pi)^{-1/2}\, a\beta^{\nu+3/2} y^{\nu+1/2}(y^2-a^2)^{-\nu/2-3/4}$ $\times K_{\nu+3/2}[\beta(y^2-a^2)^{1/2}] \quad a<y<\infty$
(19)	$x^{-1/2}(x^2+\beta^2)^{-1/2}$ $\times\sin[a(x^2+\beta^2)^{1/2}] \qquad a>0$ $\operatorname{Re}\beta>0, \quad \operatorname{Re}\nu>-1$	$1/2\,\pi y^{1/2} J_{1/2\nu}\{1/2\,\beta[a-(a^2-y^2)^{1/2}]\}$ $\times J_{-1/2\nu}\{1/2\,\beta[a+(a^2-y^2)^{1/2}]\}$ $0<y<a$
(20)	$x^{\nu+1/2}(\beta^2+x^2)^{-1/2}$ $\times\sin[a(\beta^2+x^2)^{1/2}] \qquad a>0$ $\operatorname{Re}\beta>0, \quad -1<\operatorname{Re}\nu<1/2$	$2^{-1/2}\pi^{1/2}\beta^{1/2+\nu} y^{1/2+\nu}(a^2-y^2)^{-1/4-1/2\nu}$ $\times J_{-\nu-1/2}[\beta(a^2-y^2)^{1/2}] \quad 0<y<a$ $0 \qquad a<y<\infty$
(21)	$x^{\nu+1/2}(b^2+x^2)^{-2}$ $\times\sin[a(x^2+b^2)^{1/2}] \qquad a>0$ $b>0, \quad -1<\operatorname{Re}\nu<7/2$	$ay^{1/2}\, b^\nu K_\nu(yb) \qquad y>a$
(22)	$x^{-1/2}(a^2-x^2)^{-1/2}$ $\times\sin[b(a^2-x^2)^{1/2}] \quad 0<x<a$ $-x^{-1/2}(x^2-a^2)^{-1/2}$ $\times\exp[-b(x^2-a^2)^{1/2}]$ $a<x<\infty$ $b>0, \quad \operatorname{Re}\nu>-1$	$1/2\,\pi y^{1/2} J_{1/2\nu}\{1/2\,a[(b^2+y^2)^{1/2}-b]\}$ $\times Y_{1/2\nu}\{1/2\,a[(b^2+y^2)^{1/2}+b]\}$
(23)	$x^{\nu+1/2}(a^2-x^2)^{-1/2}$ $\times\sin[b(a^2-x^2)^{1/2}] \quad 0<x<a$ $-x^{\nu+1/2}(x^2-a^2)^{-1/2}$ $\times\exp[-b(x^2-a^2)^{1/2}]$ $a<x<\infty$ $b>0, \quad \operatorname{Re}\nu>-1$	$2^{-1/2}\pi^{1/2} a^{\nu+1/2}(b^2+y^2)^{-1/2\nu-1/4}\, y^{\nu+1/2}$ $\times Y_{\nu+1/2}[a(b^2+y^2)^{1/2}]$

Trigonometric functions (cont'd)

	$f(x)$	$\int_0^\infty f(x)\, J_\nu(xy)(xy)^{1/2}\, dx \qquad y>0$
(24)	$0 \qquad 0<x<a$ $x^{1/2-\nu}\sin[b(x^2-a^2)^{1/2}]$ $a<x<\infty$ $b>0, \quad \mathrm{Re}\,\nu>1/2$	$0 \qquad 0<y<b$ $2^{-1/2}\pi^{1/2}a^{3/2-\nu}by^{1/2-\nu}$ $\times(y^2-b^2)^{\nu/2-3/4}$ $\times J_{\nu-3/2}[a(y^2-b^2)^{1/2}]$ $b<y<\infty$
(25)	$x^{1/2-\nu}(x^2+\beta^2)^{-1/2}[(x^2+\beta^2)^{1/2}-\beta]^\nu$ $\times\sin[a(x^2+\beta^2)^{1/2}] \qquad a>0$ $\mathrm{Re}\,\beta>0, \quad \mathrm{Re}\,\nu>-1$	$y^{\nu+1/2}[a+(a^2-y^2)^{1/2}]^{-\nu}(a^2-y^2)^{-1/2}$ $\times\cos[\beta(a^2-y^2)^{1/2}+\tfrac{1}{2}\pi\nu]$ $0<y<a$ $y^{-1/2}(y^2-a^2)^{-1/2}\exp[-\beta(y^2-a^2)^{1/2}]$ $\times\sin[\nu\sin^{-1}(a/y)] \qquad a<y<\infty$
(26)	$0 \qquad 0<x<c$ $x^{1/2-\nu}(x^2+b^2)^{-1}$ $\times\sin[a(x^2-c^2)^{1/2}] \qquad c<x<\infty$ $\mathrm{Re}\,\nu>-3/2$	$\tfrac{1}{2}\pi y^{1/2}b^{-\nu}e^{-a(c^2+b^2)^{1/2}}\, I_\nu(by)$ $0<y<a$
	For other Hankel transforms containing sines see the table of Fourier sine transforms.	
(27)	$x^{-3/2}\cos(ax) \qquad a>0$ $\mathrm{Re}\,\nu>0$	$\nu^{-1}\cos(\tfrac{1}{2}\nu\pi)y^{\nu+1/2}[a+(a^2-y^2)^{1/2}]^{-\nu}$ $0<y\le a$ $\nu^{-1}y^{1/2}\cos[\nu\sin^{-1}(a/y)]$ $a<y<\infty$
(28)	$x^{\nu+1/2}\cos(ax)$ $a>0, \quad -1<\mathrm{Re}\,\nu<-1/2$	$2^{1+\nu}\pi^{1/2}a[\Gamma(-\tfrac{1}{2}-\nu)]^{-1}y^{\nu+1/2}$ $\times(a^2-y^2)^{-\nu-3/2} \qquad 0<y<a$ $0 \qquad a<y<\infty$
(29)	$x^{\nu-1/2}\cos(ax)$ $a>0, \quad \lvert\mathrm{Re}\,\nu\rvert<1/2$	$-2^\nu\pi^{-1/2}\sin(\nu\pi)\Gamma(\tfrac{1}{2}+\nu)$ $\times y^{\nu+1/2}(a^2-y^2)^{-\nu-1/2} \qquad 0<y<a$ $2^\nu\pi^{-1/2}\Gamma(\tfrac{1}{2}+\nu)y^{\nu+1/2}(y^2-a^2)^{-\nu-1/2}$ $a<y<\infty$

Trigonometric functions (cont'd)

	$f(x)$	$\int_0^\infty f(x)\, J_\nu(xy)(xy)^{1/2}\, dx \qquad y>0$
(30)	$x^{-\nu-1/2}\cos(ax)$ $a>0, \quad \mathrm{Re}\,\nu>-1/2$	$0 \qquad 0<y<a$ $\dfrac{\pi^{1/2}(y^2-a^2)^{\nu-1/2}}{2^\nu y^{\nu-1/2}\Gamma(\nu+1/2)} \qquad a<y<\infty$
(31)	$x^{-\nu+2n-1/2}\cos(ax)$ $a>0, \quad \mathrm{Re}\,\nu>2n-1/2$	$0 \qquad 0<y<a$ $(-1)^n y^{-\nu+2n+1/2}\, 2^{\nu-2n-1}\,\Gamma(\nu-2n)$ $\times[\Gamma(2\nu-2n)]^{-1}(2n)!$ $\times(y^2-a^2)^{\nu-2n-1/2}\, C_{2n}^{\nu-2n}(ay^{-1})$ $a<y<\infty$
(32)	$x^{\mu-3/2}\cos(ax)$ $a>0, \quad -\mathrm{Re}\,\nu<\mathrm{Re}\,\mu<3/2$	$\dfrac{y^{\nu+1/2}\,\Gamma(\nu+\mu)\cos[\tfrac{1}{2}\pi(\nu+\mu)]}{2^\nu a^{\nu+\mu}\,\Gamma(\nu+1)}$ $\times\, {}_2F_1\left(\dfrac{\nu+\mu}{2}, \dfrac{\nu+\mu+1}{2}; \nu+1; \dfrac{y^2}{a^2}\right)$ $0<y<a$ $\dfrac{2^{\mu-1}\, y^{1/2-\mu}\,\Gamma(\tfrac{1}{2}\nu+\tfrac{1}{2}\mu)}{\Gamma(1+\tfrac{1}{2}\nu-\tfrac{1}{2}\mu)}$ $\times\, {}_2F_1\left(\dfrac{\nu+\mu}{2}, \dfrac{\mu-\nu}{2}; \dfrac{1}{2}; \dfrac{a^2}{y^2}\right)$ $a<y<\infty$
(33)	$x^{\nu+1/2}(x^2+\beta^2)^{-1}\cos(ax)$ $a>0, \ \mathrm{Re}\,\beta>0, \ -1<\mathrm{Re}\,\nu<1/2$	$\beta^\nu \cosh(a\beta)\, y^{1/2}\, K_\nu(\beta y) \qquad y\geq a$
(34)	$x^{-\nu-1/2}(x^2+\beta^2)^{-1}\cos(ax)$ $a>0, \ \mathrm{Re}\,\beta>0, \ \mathrm{Re}\,\nu>-3/2$	$\tfrac{1}{2}\pi\beta^{-\nu-1}e^{-a\beta}\, y^{1/2}\, I_\nu(\beta y)$ $0<y\leq a$

Trigonometric functions (cont'd)

	$f(x)$	$\int_0^\infty f(x)\, J_\nu(xy)(xy)^{1/2}\, dx \qquad y>0$
(35)	$x^{-3/2}\, e^{-xa\cos\phi\cos\psi}$ $\times\cos(xa\sin\psi) \qquad a>0$ $0<\phi,\ \psi<\tfrac{1}{2}\pi, \quad \operatorname{Re}\nu>0$	$a^{1/2}\,\nu^{-1}(\sin\phi)^{1/2}(\tan\tfrac{1}{2}\phi)^\nu\cos(\nu\psi)$ $y=a\sin\phi$
(36)	$x^{\nu+1/2}\, e^{-ax\cos\phi\cos\psi}$ $\times\cos(ax\sin\psi) \qquad a>0$ $0<\phi,\ \psi<\tfrac{1}{2}\pi, \quad \operatorname{Re}\nu>-1$	$2^{\nu+1}\pi^{-1/2}\Gamma(\nu+3/2)\,a^{-\nu-3/2}$ $\times(\sin\phi)^{\nu+1/2}(\cos^2\psi$ $+\sin^2\psi\cos^2\phi)^{-\nu-3/2}$ $\times\cos[(\nu+3/2)\alpha] \qquad y=a\sin\phi$ $\tan(\tfrac{1}{2}\alpha)=\tan\psi\cos\phi$
(37)	$x^{\nu-1/2}\, e^{-xa\cos\phi\cos\psi}$ $\times\cos(xa\sin\psi) \qquad a>0$ $0<\phi,\ \psi<\tfrac{1}{2}\pi, \quad \operatorname{Re}\nu>-\tfrac{1}{2}$	$2^\nu\pi^{-1/2}a^{-\nu-1/2}\Gamma(\nu+\tfrac{1}{2})(\sin\phi)^{\nu+1/2}$ $\times(\cos^2\psi+\sin^2\psi\cos^2\phi)^{-\nu-1/2}$ $\times\cos[(\nu+\tfrac{1}{2})\alpha] \qquad y=a\sin\phi$ $\tan(\tfrac{1}{2}\alpha)=\tan\psi\cos\phi$
(38)	$x^{-1/2}\cos(ax^2)$ $a>0, \quad \operatorname{Re}\nu>-1$	$\dfrac{\pi^{1/2}y^{1/2}}{2a^{1/2}}\cos\left(\dfrac{y^2}{8a}-\dfrac{\nu+1}{4}\pi\right)$ $\times J_{\frac{1}{2}\nu}\left(\dfrac{y^2}{8a}\right)$
(39)	$x^{1/2}\cos(ax^2)$ $a>0, \quad \operatorname{Re}\nu>-2$	$\dfrac{\pi^{1/2}y^{3/2}}{8a^{3/2}}\left[\cos\left(\dfrac{y^2}{8a}-\dfrac{\nu\pi}{4}\right)\right.$ $\times J_{\frac{1}{2}\nu+\frac{1}{2}}\left(\dfrac{y^2}{8a}\right)$ $\left.+\sin\left(\dfrac{y^2}{8a}-\dfrac{\nu\pi}{4}\right)J_{\frac{1}{2}\nu-\frac{1}{2}}\left(\dfrac{y^2}{8a}\right)\right]$
(40)	$x^{\nu+1/2}\cos(ax^2) \quad -1<\operatorname{Re}\nu<\tfrac{1}{2}$	$\dfrac{y^{\nu+1/2}}{2^{\nu+1}a^{\nu+1}}\sin\left(\dfrac{y^2}{4a}-\dfrac{\nu\pi}{2}\right)$

Trigonometric functions (cont'd)

	$f(x)$	$\int_0^\infty f(x)\, J_\nu(xy)(xy)^{1/2}\, dx \qquad y>0$
(41)	$x^{\nu+1/2}\cos(ax^2) \qquad 0<x<b$ $0 \qquad b<x<\infty$ $\operatorname{Re}\nu>-1$	$(2a)^{-\nu-1}y^{\nu+1/2}$ $\times[\sin(ab^2)\, U_{\nu+2}(2ab^2, by)$ $+\cos(ab^2)\, U_{\nu+1}(2ab^2, by)]$
(42)	$x^{-1}e^{-ax^{1/2}}\cos(ax^{1/2})$ $a>0, \quad \operatorname{Re}\nu>-1/2$	$2^{-1/2}\pi^{-1/2}\Gamma(\nu+1/2)\, D_{-\nu-1/2}(ay^{-1/2})$ $\times[D_{-\nu-1/2}(iay^{-1/2})+D_{-\nu-1/2}(-iay^{-1/2})]$
(43)	$x^{\nu+1/2}\cos[a(x^2+\beta^2)^{1/2}] \qquad a>0$ $\operatorname{Re}\beta>0, \quad -1<\operatorname{Re}\nu<-1/2$	$(1/2\,\pi)^{1/2}a\,\beta^{\nu+3/2}y^{\nu+1/2}$ $\times(a^2-y^2)^{-\nu/2-3/4}$ $\times\{\cos(\pi\nu)\, J_{\nu+3/2}[\beta(a^2-y^2)^{1/2}]$ $-\sin(\pi\nu)\, Y_{\nu+3/2}[\beta(a^2-y^2)^{1/2}]\}$ $0<y<a$ $0 \qquad a<y<\infty$
(44)	$x^{-1/2}(x^2+\beta^2)^{-1/2}$ $\times\cos[a(x^2+\beta^2)^{1/2}] \qquad a>0$ $\operatorname{Re}\beta>0, \quad \operatorname{Re}\nu>-1$	$-1/2\,\pi y^{1/2}J_{1/2\nu}\{1/2\,\beta[a-(a^2-y^2)^{1/2}]\}$ $\times Y_{-1/2\nu}\{1/2\,\beta[a+(a^2-y^2)^{1/2}]\}$ $0<y<a$
(45)	$x^{\nu+1/2}(x^2+\beta^2)^{-1/2}$ $\times\cos[a(x^2+\beta^2)^{1/2}] \qquad a>0$ $\operatorname{Re}\beta>0, \quad -1<\operatorname{Re}\nu<1/2$	$-2^{-1/2}\pi^{1/2}\beta^{1/2+\nu}y^{1/2+\nu}(a^2-y^2)^{-1/4-1/2\nu}$ $\times Y_{-\nu-1/2}[\beta(a^2-y^2)^{1/2}] \quad 0<y<a$ $2^{1/2}\pi^{-1/2}\beta^{1/2+\nu}y^{1/2+\nu}(y^2-a^2)^{-1/4-1/2\nu}$ $\times K_{\nu+1/2}[\beta(y^2-a^2)^{1/2}] \quad a<y<\infty$
(46)	$x^{\nu+1/2}(x^2+b^2)^{-3/2}$ $\times\cos[a(x^2+b^2)^{1/2}] \qquad a>0$ $b>0, \quad -1<\operatorname{Re}\nu<5/2$	$y^{1/2}b^\nu K_\nu(by) \qquad y>a$
(47)	$x^{-1/2}(a^2-x^2)^{-1/2}\cos[b(a^2-x^2)^{1/2}]$ $0<x<a$ $0 \qquad a<x<\infty$ $\operatorname{Re}\nu>-1$	$1/2\,\pi y^{1/2}J_{1/2\nu}\{1/2\,a[(b^2+y^2)^{1/2}-b]\}$ $\times J_{1/2\nu}\{1/2\,a[(b^2+y^2)^{1/2}+b]\}$

Trigonometric functions (cont'd)

	$f(x)$	$\int_0^\infty f(x)\, J_\nu(xy)(xy)^{1/2}\, dx \qquad y>0$
(48)	$x^{\nu+1/2}(a^2-x^2)^{-1/2}$ $\times\cos[b(a^2-x^2)^{1/2}] \quad 0<x<a$ $0 \quad a<x<\infty$ $\mathrm{Re}\,\nu>-1$	$2^{-1/2}\pi^{1/2}a^{\nu+1/2}y^{\nu+1/2}(b^2+y^2)^{-1/2\nu-1/4}$ $\times J_{\nu+1/2}[a(b^2+y^2)^{1/2}]$
(49)	$0 \quad 0<x<a$ $x^{1/2-\nu}(x^2-a^2)^{-1/2}$ $\times\cos[b(x^2-a^2)^{1/2}] \quad a<x<\infty$ $b>0, \quad \mathrm{Re}\,\nu>-1/2$	$0 \quad 0<y<b$ $2^{-1/2}\pi^{1/2}a^{1/2-\nu}y^{1/2-\nu}(y^2-b^2)^{1/2\nu-1/4}$ $\times J_{\nu-1/2}[a(y^2-b^2)^{1/2}] \quad b<y<\infty$
(50)	$x^{1/2-\nu}(x^2+\beta^2)^{-1/2}$ $\times[(x^2+\beta^2)^{1/2}-\beta]^\nu$ $\times\cos[a(x^2+\beta^2)^{1/2}] \quad a>0$ $\mathrm{Re}\,\beta>0, \quad \mathrm{Re}\,\nu>-1$	$-y^{\nu+1/2}[a+(a^2-y^2)^{1/2}]^{-\nu}(a^2-y^2)^{-1/2}$ $\times\sin[\beta(a^2-y^2)^{1/2}+1/2\pi\nu]$ $0<y<a$ $y^{-1/2}(y^2-a^2)^{-1/2}\exp[-\beta(y^2-a^2)^{1/2}]$ $\times\cos[\nu\sin^{-1}(a/y)] \quad a<y<\infty$
(51)	$0 \quad 0<x<c$ $x^{1/2-\nu}(x^2+b^2)^{-1}(x^2-c^2)^{-1/2}$ $\times\cos[a(x^2-c^2)^{1/2}] \quad c<x<\infty$ $\mathrm{Re}\,\nu>-5/2$	$1/2\pi y^{1/2}b^{-\nu}(c^2+b^2)^{-1/2}$ $\times e^{-a(c^2+b^2)^{1/2}}\, I_\nu(by) \quad 0<y<a$
	For other Hankel transforms containing cosines see the table of Fourier cosine transforms.	
(52)	$x^{-1/2}(x^2-1)^{-1/2}$ $\times\cos[(\nu-1)\cos^{-1}x]$ $0<x<1$ $0 \quad 1<x<\infty$ $\mathrm{Re}\,\nu>0$	$\pi^{1/2}\sin(1/2y)\, J_{\nu-1/2}(1/2y)$
(53)	$x^{-1/2}(1-x^2)^{-1/2} \quad 0<x<1$ $\times\cos[(\nu+1)\cos^{-1}x]$ $0 \quad 1<x<\infty$ $\mathrm{Re}\,\nu>-1$	$\pi^{1/2}\cos(1/2y)\, J_{\nu+1/2}(1/2y)$

Trigonometric functions (cont'd)

	$f(x)$	$\int_0^\infty f(x)\, J_\nu(xy)(xy)^{1/2}\,dx \qquad y>0$
(54)	$x^{-1/2}(1-x^2)^{-1/2}\cos(\mu\cos^{-1}x)$ $\quad 0<x<1$ $0 \quad 1<x<\infty$ $\mathrm{Re}(\mu+\nu)>-1$	$\frac{1}{2}\pi y^{1/2}\, J_{\frac{1}{2}(\mu+\nu)}(\frac{1}{2}y)\, J_{\frac{1}{2}(\nu-\mu)}(\frac{1}{2}y)$
(55)	$0 \quad 0<x<1$ $x^{1/2}(x^2-1)^{-1/2}\cos(\nu\cos^{-1}x^{-1})$ $\quad 1<x<\infty$ $\mathrm{Re}\,\nu>-1$	$y^{-1/2}\cos(y-\frac{1}{2}\nu\pi)$

8.8. Hyperbolic and inverse hyperbolic functions

(1)	$x^{\nu-1/2}\, e^{-\frac{1}{2}\pi x}\operatorname{csch}(\frac{1}{2}\pi x)$ $\mathrm{Re}\,\nu>-\frac{1}{2}$	$\pi^{-1/2}\, 2^{\nu+1}\,\Gamma(\nu+\frac{1}{2})\, y^{\nu+1/2}$ $\times \sum_{n=1}^{\infty}(n^2\pi^2+y^2)^{-\nu-1/2}$
(2)	$x^{1/2}\dfrac{x\cosh x+\sinh x}{\sinh(2x)+2x}$	For this and similar integrals see Boit, M. A., 1935: *J. Appl. Phys.*, 6, 367-375.
(3)	$x^{\nu+1/2}\sinh(ax)\operatorname{csch}(\pi x)$ $\lvert\mathrm{Re}\,a\rvert<\pi, \quad \mathrm{Re}\,\nu>-1$	$2\pi^{-1}y^{1/2}\sum_{n=1}^{\infty}(-1)^{n-1}n^{\nu+1}\sin(na)$ $\times K_\nu(ny)$
	For other similar integrals see Weber, H., 1873 : *J. of Math.* 75, 75-105.	
(4)	$x^{-1/2}(1+x^2)^{-1/2}\sinh(2\mu\sinh^{-1}x)$ $\mathrm{Re}\,\nu>-1, \quad \lvert\mathrm{Re}\,\mu\rvert<\frac{3}{4}$	$\frac{1}{2}y^{1/2}[\, I_{\frac{1}{2}\nu-\mu}(\frac{1}{2}y)\, K_{\frac{1}{2}\nu+\mu}(\frac{1}{2}y)$ $-I_{\frac{1}{2}\nu+\mu}(\frac{1}{2}y)\, K_{\frac{1}{2}\nu-\mu}(\frac{1}{2}y)]$

Hyperbolic functions (cont'd)

	$f(x)$	$\int_0^\infty f(x)\, J_\nu(xy)(xy)^{1/2}\, dx \qquad y>0$
(5)	$x^{-1/2}(1+x^2)^{-1/2}\cosh(2\mu\sinh^{-1}x)$ $\mathrm{Re}\,\nu>-1, \quad \lvert\mathrm{Re}\,\mu\rvert<3/4$	$\tfrac{1}{2}y^{1/2}[I_{\frac{1}{2}\nu-\mu}(\tfrac{1}{2}y)K_{\frac{1}{2}\nu+\mu}(\tfrac{1}{2}y)$ $+I_{\frac{1}{2}\nu+\mu}(\tfrac{1}{2}y)K_{\frac{1}{2}\nu-\mu}(\tfrac{1}{2}y)]$
(6)	$0 \qquad 0<x<1$ $x^{-1/2}(x^2-1)^{-1/2}$ $\times\cosh[(\nu-1)\cosh^{-1}x]$ $1<x<\infty$ $-1/2<\mathrm{Re}\,\nu<5/2$	$\tfrac{1}{2}\pi^{1/2}[\cos(\tfrac{1}{2}y)J_{\nu-\frac{1}{2}}(\tfrac{1}{2}y)$ $-\sin(\tfrac{1}{2}y)Y_{\nu-\frac{1}{2}}(\tfrac{1}{2}y)]$
(7)	$0 \qquad 0<x<1$ $x^{-1/2}(x^2-1)^{-1/2}$ $\times\cosh[(\nu+1)\cosh^{-1}x]$ $1<x<\infty$ $-5/2<\mathrm{Re}\,\nu<1/2$	$-\tfrac{1}{2}\pi^{1/2}[\sin(\tfrac{1}{2}y)J_{\nu+\frac{1}{2}}(\tfrac{1}{2}y)$ $+\cos(\tfrac{1}{2}y)Y_{\nu+\frac{1}{2}}(\tfrac{1}{2}y)]$
(8)	$0 \qquad 0<x<1$ $x^{-1/2}(x^2-1)^{-1/2}\cosh(\mu\cosh^{-1}x)$ $1<x<\infty$ $\lvert\mathrm{Re}\,\mu\rvert<3/2$	$-\tfrac{1}{4}\pi y^{1/2}[J_{\frac{1}{2}(\mu+\nu)}(\tfrac{1}{2}y)Y_{\frac{1}{2}(\nu-\mu)}(\tfrac{1}{2}y)$ $+J_{\frac{1}{2}(\nu-\mu)}(\tfrac{1}{2}y)Y_{\frac{1}{2}(\nu+\mu)}(\tfrac{1}{2}y)]$

8.9. Orthogonal polynomials

(1)	$x^{-1/2}(1-x^2)^{-1/2}T_n(x) \qquad 0<x<1$ $0 \qquad 1<x<\infty$ $\mathrm{Re}\,\nu>-n-1$	$\tfrac{1}{2}\pi y^{1/2}J_{\frac{1}{2}(\nu+n)}(\tfrac{1}{2}y)J_{\frac{1}{2}(\nu-n)}(\tfrac{1}{2}y)$
(2)	$x^{\nu+1/2}e^{-x^2}L_n^\nu(x^2) \qquad \mathrm{Re}\,\nu>-1$	$2^{-2n-\nu-1}(n!)^{-1}y^{2n+\nu+1/2}\exp(-\tfrac{1}{4}y^2)$
(3)	$x^{\nu+1/2}e^{-\frac{1}{2}x^2}L_n^\nu(x^2) \qquad \mathrm{Re}\,\nu>-1$	$(-1)^n e^{-\frac{1}{2}y^2}y^{\nu+1/2}L_n^\nu(y^2)$

Orthogonal polynomials (cont'd)

	$f(x)$	$\int_0^\infty f(x)\, J_\nu(xy)(xy)^{1/2}\, dx \qquad y>0$
(4)	$x^{2n+\nu+1/2}\, e^{-1/2 x^2}\, L_n^{\nu+n}(\tfrac{1}{2}x^2)$ $\operatorname{Re}\nu>-1$	$y^{2n+\nu+1/2}\, e^{-1/2 y^2}\, L_n^{\nu+n}(\tfrac{1}{2}y^2)$
(5)	$x^{\nu+1/2}\, e^{-\beta x^2}\, L_n^{\nu}(ax^2)$ $\operatorname{Re}\beta>0, \quad \operatorname{Re}\nu>0$	$2^{-\nu-1}\beta^{-\nu-n-1}(\beta-a)^n y^{\nu+1/2}$ $\times\exp\left(-\dfrac{y^2}{4\beta}\right) L_n^{\nu}\left[\dfrac{ay^2}{4\beta(a-\beta)}\right]$
(6)	$x^{\nu+1/2}\, e^{-ax^2}\,[L_n^{1/2\nu}(ax^2)]^2$ $\operatorname{Re}a>0, \quad \operatorname{Re}\nu>-1$	$(2a)^{-\nu-1}\, y^{\nu+1/2}$ $\times\exp\left(-\dfrac{y^2}{4a}\right)\left[L_n^{1/2\nu}\left(\dfrac{y^2}{4a}\right)\right]^2$
(7)	$x^{\nu+1/2}\, e^{-\beta x^2}\,[L_n^{1/2\nu}(ax^2)]^2$ $\operatorname{Re}\beta>0, \quad \operatorname{Re}\nu>-1$	$\dfrac{y^{1/2+\nu}}{\pi\, n!}\Gamma(n+1+\tfrac{1}{2}\nu)(2\beta)^{-\nu-1}$ $\times\exp\left(-\dfrac{y^2}{4\beta}\right)$ $\times\sum_{l=0}^{n}\dfrac{(-1)^l\Gamma(n-l+\tfrac12)\Gamma(l+\tfrac12)}{\Gamma(l+1+\tfrac12\nu)(n-l)!}$ $\times\left(\dfrac{2a-\beta}{\beta}\right)^{2l} L_{2l}^{\nu}\left[\dfrac{ay^2}{2\beta(2a-\beta)}\right]$
(8)	$x^{\nu+1/2}\, e^{-ax^2}\, L_m^{\nu-\sigma}(ax^2)\, L_n^{\sigma}(ax^2)$ $\operatorname{Re}a>0, \quad \operatorname{Re}\nu>-1$	$(-1)^{m+n}(2a)^{-\nu-1}y^{\nu+1/2}\exp\left(-\dfrac{y^2}{4a}\right)$ $\times L_n^{\sigma-m+n}\left(\dfrac{y^2}{4a}\right)L_m^{\nu-\sigma+m-n}\left(\dfrac{y^2}{4a}\right)$
(9)	$x^{\nu+1/2}\, e^{-x^2}\, L_n^{\sigma}(x^2)\, L_n^{\nu-\sigma}(x^2)$ $\operatorname{Re}\nu>-1$	$2^{-\nu-1}\, y^{\nu+1/2}\, e^{-1/4 y^2}\, L_n^{\sigma}(\tfrac14 y^2)$ $\times L_n^{\nu-\sigma}(\tfrac14 y^2)$

Orthogonal polynomials (cont'd)

	$f(x)$	$\int_0^\infty f(x)\, J_\nu(xy)(xy)^{1/2}\, dx \qquad y>0$
(10)	$0 \qquad 0<x<a$ $x^{2n+1/2-\nu}(x^2-a^2)^{\nu-2n-1/2}$ $\times C_{2n}^{\nu-2n}(a/x) \qquad a<x<\infty$ $2n-1/2<\mathrm{Re}\,\nu<2n+1/2$	$(-1)^n\, 2^{2n-\nu+1}\,\Gamma(2\nu-2n)$ $\times[(2n)!\,\Gamma(\nu-2n)]^{-1} y^{-\nu+2n-1/2}$ $\times \cos(ay)$
(11)	$0 \qquad 0<x<a$ $x^{2n-\nu+3/2}(x^2-a^2)^{\nu-2n-3/2}$ $\times C_{2n+1}^{\nu-2n-1}(a/x) \qquad a<x<\infty$ $2n+1/2<\mathrm{Re}\,\nu<2n+3/2$	$(-1)^n 2^{2n-\nu+2}\,\Gamma(2\nu-2n-1)$ $\times[(2n+1)!\,\Gamma(\nu-2n-1)]^{-1}$ $\times y^{-\nu+2n+1/2}\sin(ay)$
(12)	$x^{\nu+1/2}(1-x^2)^{-1/2}\sin[a(1-x^2)^{1/2}]$ $\times C_{2n+1}^{\nu+1/2}[(1-x^2)^{1/2}] \qquad 0<x<1$ $0 \qquad 1<x<\infty$ $\mathrm{Re}\,\nu>-1/2$	$(-1)^n 2^{-1/2}\pi^{1/2} y^{\nu+1/2}(a^2+y^2)^{-1/2\nu-1/4}$ $\times C_{2n+1}^{\nu+1/2}[a(y^2+a^2)^{-1/2}]$ $\times J_{\nu+3/2+2n}[(a^2+y^2)^{1/2}]$
(13)	$x^{\nu+1/2}(1-x^2)^{-1/2}\cos[a(1-x^2)^{1/2}]$ $\times C_{2n}^{\nu+1/2}[(1-x^2)^{1/2}] \qquad 0<x<1$ $0 \qquad 1<x<\infty$ $\mathrm{Re}\,\nu>-1/2$	$(-1)^n 2^{-1/2}\pi^{1/2} y^{\nu+1/2}(a^2+y^2)^{-1/2\nu-1/4}$ $\times C_{2n}^{\nu+1/2}[a(y^2+a^2)^{-1/2}]$ $\times J_{\nu+1/2+2n}[(a^2+y^2)^{1/2}]$

8.10. Legendre functions

(1)	$(x^2+2)^{-1/2\nu-1/4}\, P_\mu^{-\nu-1/2}(x^2+1)$ $\mathrm{Re}\,\nu>-1$ $-3/2-\mathrm{Re}\,\nu<\mathrm{Re}\,\mu<\mathrm{Re}\,\nu+1/2$	$\dfrac{2^{1/2-\nu}\pi^{-1/2}[K_{\mu+1/2}(2^{-1/2}y)]^2}{\Gamma(\nu+\mu+3/2)\Gamma(\nu-\mu+1/2)}$
(2)	$0 \qquad 0<x<a$ $(x^2-a^2)^{1/2\mu-1/4}\mathrm{P}_{-1/2+\nu}^{1/2-\mu}(ax^{-1})$ $a<x<\infty$ $\lvert\mathrm{Re}\,\mu\rvert<1/2, \quad \mathrm{Re}\,\nu>-1$	$2^{1/2}\pi^{-1/2}y^{-\mu-1/2}\cos[ay+1/2(\nu-\mu)\pi]$

Legendre functions (cont'd)

	$f(x)$	$\int_0^\infty f(x)\, J_\nu(xy)(xy)^{1/2}\, dx \qquad y>0$
(3)	$x^{\nu-1/2}(1-x^2)^{1/2\nu+1/4}$ $\times P_\mu^{-\nu-1/2}(2x^{-2}-1) \quad 0<x<1$ $0 \quad 1<x<\infty$ $-3/2-\operatorname{Re}\nu<\operatorname{Re}\mu<\operatorname{Re}\nu+1/2$	$\dfrac{\Gamma(3/2+\mu+\nu)\,\Gamma(1/2+\nu-\mu)(2y)^{\nu+1/2}}{(2\pi)^{1/2}[\Gamma(3/2+\nu)]^2}$ $\times {}_1F_1(\nu+\mu+3/2;2\nu+2;iy)$ $\times {}_1F_1(\nu+\mu+3/2;2\nu+2;-iy)$
(4)	$x^{1/2}(a^2+x^2)^{-1/2\mu}$ $\times P_{\mu-1}^{-\nu}[a(a^2+x^2)^{-1/2}] \quad \operatorname{Re}a>0$ $\operatorname{Re}\nu>-1, \quad \operatorname{Re}\mu>1/2$	$\dfrac{y^{\mu-3/2}\,e^{-ay}}{\Gamma(\mu+\nu)}$
(5)	$x^{\nu+1/2}(x^2+a^2)^{1/2\nu}$ $\times P_\nu\left[\dfrac{x^2+2a^2}{2a(x^2+a^2)^{1/2}}\right]$ $\operatorname{Re}a>0, \quad -1<\operatorname{Re}\nu<0$	$\dfrac{(2a)^{\nu+1}\,y^{-\nu-1/2}}{\pi\,\Gamma(-\nu)}\,[K_{\nu+1/2}(1/2\,ay)]^2$
(6)	$x^{1/2-\nu}(x^2+a^2)^{-1/2\nu}$ $\times P_{\nu-1}\left[\dfrac{x^2+2a^2}{2a(x^2+a^2)^{1/2}}\right]$ $\operatorname{Re}a>0, \quad 0<\operatorname{Re}\nu<1$	$\dfrac{(2a)^{1-\nu}}{\Gamma(\nu)}\,y^{\nu-1/2}\,I_{\nu-1/2}(1/2\,ay)$ $\times K_{\nu-1/2}(1/2\,ay)$
(7)	$x^{1/2}\{P_\mu^{-1/2\nu}[(1+a^2x^2)^{1/2}]\}^2$ $\operatorname{Re}a>0$ $-3/4<\operatorname{Re}\mu<-1/4, \quad \operatorname{Re}\nu>-1$	$\dfrac{2[K_{\mu+1/2}(1/2\,a^{-1}y)]^2}{\pi a\Gamma(1+\mu+1/2\nu)\,\Gamma(1/2\nu-\mu)\,y^{1/2}}$
(8)	$x^{1/2}(1+a^2x^2)^{-1/2}$ $\times P_\mu^{-1/2\nu}[(1+a^2x^2)^{1/2}]$ $\times P_{\mu+1}^{-1/2\nu}[(1+a^2x^2)^{1/2}]$ $\operatorname{Re}\nu>-1, \quad \operatorname{Re}a>0$ $-7/4<\operatorname{Re}\mu<-1/4$	$\dfrac{y^{1/2}\,K_{\mu+1/2}(1/2\,a^{-1}y)\,K_{\mu+3/2}(1/2\,a^{-1}y)}{\pi a^2\,\Gamma(2+1/2\nu+\mu)\Gamma(1/2\nu-\mu)}$

Legendre functions (cont'd)

	$f(x)$	$\int_0^\infty f(x)\, J_\nu(xy)(xy)^{1/2}\,dx \qquad y>0$
(9)	$x^{1/2}(1+a^2x^2)^{-1/2}$ $\times P_\mu^{-1/2-1/2\nu}[(1+a^2x^2)^{1/2}]$ $\times P_\mu^{1/2-1/2\nu}[(1+a^2x^2)^{1/2}]$ $\operatorname{Re}\nu>-1, \quad \operatorname{Re}a>0$ $-5/4<\operatorname{Re}\mu<1/4$	$\dfrac{y^{1/2}[K_{\mu+1/2}(\tfrac{1}{2}a^{-1}y)]^2}{\pi a^2\,\Gamma(\nu/2+\mu+3/2)\,\Gamma(\nu/2-\mu+1/2)}$
(10)	$Q_{\nu-1/2}[(a^2+x^2)\,x^{-1}]$ $\operatorname{Re}\nu>-\tfrac{1}{2}$	$2^{-1/2}\,\pi y^{-1/2}\exp[-(a^2-\tfrac{1}{4})^{1/2}y]$ $\times J_\nu(\tfrac{1}{2}y)$
(11)	$x^{1/2-\mu}(1+a^2x^2)^{-1/2\mu-1/4}$ $\times Q_{\nu-1/2}^{\mu+1/2}(\pm i a x) \qquad \operatorname{Re}a>0$ $-\tfrac{3}{4}-\tfrac{1}{2}\operatorname{Re}\nu<\operatorname{Re}\mu<1+\operatorname{Re}\nu$	$i(2\pi)^{1/2}\,e^{i\pi(\mu\mp 1/2\nu\mp 1/4)}\,a^{-1}\,y^{\mu-1/2}$ $\times I_\nu(\tfrac{1}{2}a^{-1}y)\,K_\mu(\tfrac{1}{2}a^{-1}y)$
(12)	$(x^2+2)^{-1/2\nu-1/4}\,Q_\mu^{\nu+1/2}(x^2+1)$ $\operatorname{Re}\nu>-1$ $\operatorname{Re}(2\mu+\nu)>-5/2$	$2^{-\nu-1/2}\,\pi^{1/2}\,e^{(\nu+1/2)\pi i}\,y^{\nu+1/2}$ $\times K_{\mu+1/2}(2^{-1/2}y)\,I_{\mu+1/2}(2^{-1/2}y)$
(13)	$x^{-\nu-1/2}\,Q_{-1/2}^{\nu-1/2}(1+2a^2/x^2)$ $\operatorname{Re}a>0, \quad 0<\operatorname{Re}\nu<3/2$	$-ie^{i\pi\nu}\,\pi^{1/2}\,2^{-\nu}(y/a)^{\nu-1/2}$ $\times I_{\nu-1/2}(\tfrac{1}{2}ay)\,K_{\nu-1/2}(\tfrac{1}{2}ay)$
(14)	$x^{\nu-1/2}(a^2+x^2)^{1/4+1/2\nu}$ $\times Q_\mu^{\nu+1/2}(1+2a^2/x^2) \quad \operatorname{Re}a>0$ $\operatorname{Re}(\mu+\nu)>-3/2$ $\operatorname{Re}(\nu-\mu)>-1/2$	$-ie^{i\pi\nu}\pi^{-1/2}\,2^\nu[\Gamma(3/2+\mu+\nu)]^2$ $\times\Gamma(1/2+\nu-\mu)\,a^{\nu-1/2}y^{-\nu-3/2}$ $\times W_{-\mu-1/2,\nu+1/2}(ay)$ $\times\left[\dfrac{\cos(\mu\pi)}{\Gamma(2+2\nu)}M_{\mu+1/2,\nu+1/2}(ay)\right.$ $\left.+\dfrac{\sin(\pi\nu)}{\Gamma(\nu+\mu+3/2)}W_{\mu+1/2,\nu+1/2}(ay)\right]$

Legendre functions (cont'd)

	$f(x)$	$\int_0^\infty f(x)\, J_\nu(xy)(xy)^{1/2}\, dx \qquad y>0$
(15)	$x^{-\nu-1/2}(x^2+a^2)^{1/4-1/2\nu}$ $\times Q_\mu^{1/2-\nu}(1+2a^2/x^2)$ Re $a>0$ $0<\operatorname{Re}\nu<\operatorname{Re}\mu+3/2$	$\dfrac{ie^{-i\nu\pi}\pi^{1/2}\Gamma(3/2+\mu-\nu)}{2^\nu a^{\nu+1/2}\Gamma(2\nu)} y^{-\nu-3/2}$ $\times M_{\mu+1/2,\nu-1/2}(ay)\, W_{-\mu-1/2,\nu-1/2}(ay)$
(16)	$x^{1/2} P_\mu^{-1/2\nu}[(1+a^2x^2)^{1/2}]$ $\times Q_\mu^{-1/2\nu}[(1+a^2x^2)^{1/2}]$ Re $a>0$ Re $\mu>-3/4$, Re $\nu>-1$	$\dfrac{e^{-1/2\nu\pi i}\Gamma(1+\mu+1/2\nu)}{a\Gamma(1+\mu-1/2\nu)\, y^{1/2}} I_{\mu+1/2}\left(\dfrac{y}{2a}\right)$ $\times K_{\mu+1/2}\left(\dfrac{y}{2a}\right)$

8.11. Bessel functions of argument *kx*

(1)	$x^{-1/2} J_{\nu-1}(ax)$ $a>0$, Re $\nu>-1$	$0 \qquad 0<y<a$ $a^{\nu-1} y^{-\nu+1/2} \qquad a<y<\infty$
(2)	$x^{-3/2} J_\nu(ax)$ $a>0$, Re $\nu>0$	$\frac{1}{2}\nu^{-1} a^{-\nu} y^{\nu+1/2} \qquad 0<y\le a$ $\frac{1}{2}\nu^{-1} a^{\nu} y^{-\nu+1/2} \qquad a\le y<\infty$
(3)	$x^{-1/2} J_{\nu+1}(ax)$ $a>0$, Re $\nu>-3/2$	$a^{-\nu-1} y^{\nu+1/2} \qquad 0<y<a$ $0 \qquad a<y<\infty$
(4)	$x^{-2\lambda-1/2} J_\nu(ax) \qquad a>0$ Re $\nu+1/2>$ Re $\lambda>-1/2$	$\dfrac{a^\nu y^{\nu+1/2}\Gamma(\nu-\lambda+1/2)}{2^{2\lambda}(a+y)^{2\nu-2\lambda+1}\Gamma(\nu+1)\Gamma(\lambda+1/2)}$ $\times {}_2F_1\left[\nu-\lambda+\dfrac{1}{2}, \nu+\dfrac{1}{2}; 2\nu+1; \dfrac{4ay}{(a+y)^2}\right]$
(5)	$x^{-1/2} J_{\nu+2n+1}(ax)$ $a>0$, Re $\nu>-1-n$	$y^{\nu+1/2} a^{-\nu-1} P_n^{(\nu,0)}(1-2y^2/a^2)$ $0<y<a$ $0 \qquad a<y<\infty$

Bessel functions of kx (cont'd)

	$f(x)$	$\int_0^\infty f(x)\, J_\nu(xy)(xy)^{1/2}\, dx \qquad y>0$
(6)	$x^{-1/2} J_\mu(ax)$ $a>0, \quad \mathrm{Re}\,(\mu+\nu)>-1$	$y^{\nu+1/2}\, a^{-\nu-1} \dfrac{\Gamma(\frac12\mu+\frac12\nu+\frac12)}{\Gamma(\nu+1)\,\Gamma(\frac12\mu-\frac12\nu+\frac12)}$ $\times {}_2F_1\left(\dfrac{\mu+\nu+1}{2}, \dfrac{\nu-\mu+1}{2}; \nu+1; \dfrac{y^2}{a^2}\right)$ $0<y<a$ For $y>a$ interchange μ and ν.
(7)	$x^{\nu-\mu+1/2} J_\mu(ax)$ $a>0, \quad -1<\mathrm{Re}\,\nu<\mathrm{Re}\,\mu$	$\dfrac{2^{\nu-\mu+1}\, y^{\nu+1/2}}{\Gamma(\mu-\nu)\, a^\mu} (a^2-y^2)^{\mu-\nu-1}$ $0<y<a$ $0 \qquad a<y<\infty$
(8)	$x^{\mu-\nu+1/2} J_\mu(ax)$ $a>0, \quad \mathrm{Re}\,\nu>\mathrm{Re}\,\mu>-1$	$0 \qquad 0<y<a$ $\dfrac{2^{\mu-\nu+1}\, a^\mu}{\Gamma(\nu-\mu)\, y^{\nu-1/2}} (y^2-a^2)^{\nu-\mu-1}$ $a<y<\infty$
(9)	$x^{-\lambda-1/2} J_\mu(ax) \qquad a>0$ $\mathrm{Re}\,(\mu+\nu)+1>\mathrm{Re}\,\lambda>-1$	$\dfrac{\Gamma[\frac12(\mu+\nu-\lambda+1)]\, y^{\nu+1/2}}{2^\lambda a^{\nu-\lambda+1}\, \Gamma(\nu+1)\, \Gamma[\frac12(\lambda+\mu-\nu+1)]}$ $\times {}_2F_1\left(\dfrac{\mu+\nu-\lambda+1}{2}, \dfrac{\nu-\lambda-\mu+1}{2}; \nu+1; \dfrac{y^2}{a^2}\right)$ $0<y<a$ $\dfrac{\Gamma[\frac12(\mu+\nu-\lambda+1)]\, a^\mu}{2^\lambda y^{\mu-\lambda+1/2}\, \Gamma(\mu+1)\, \Gamma[\frac12(\lambda+\nu-\mu+1)]}$ $\times {}_2F_1\left(\dfrac{\mu+\nu-\lambda+1}{2}, \dfrac{\mu-\lambda-\nu+1}{2}; \mu+1; \dfrac{a^2}{y^2}\right)$ $a<y<\infty$

Bessel functions of kx (cont'd)

	$f(x)$	$\int_0^\infty f(x)\, J_\nu(xy)(xy)^{1/2}\, dx \qquad y > 0$
(10)	$x^{1/2}(x^2+\beta^2)^{-1} J_\nu(ax) \qquad a > 0$ $\operatorname{Re}\beta > 0, \quad \operatorname{Re}\nu > -1$	$y^{1/2} I_\nu(y\beta) K_\nu(a\beta) \qquad 0 < y < a$ $y^{1/2} I_\nu(a\beta) K_\nu(y\beta) \qquad a < y < \infty$
(11)	$x^{1/2-2n}(\beta^2+x^2)^{-1} J_\nu(ax)$ $a > 0, \quad \operatorname{Re}\beta > 0$ $\operatorname{Re}\nu > n-1, \quad n = 0, 1, 2, \cdots$	$(-1)^n \beta^{-2n} y^{1/2} I_\nu(y\beta) K_\nu(a\beta)$ $0 < y < a$ $(-1)^n \beta^{-2n} y^{1/2} I_\nu(a\beta) K_\nu(y\beta)$ $a < y < \infty$
(12)	$x^{\nu-\mu+1/2}(\beta^2+x^2)^{-1} J_\mu(ax)$ $a > 0, \quad \operatorname{Re}\beta > 0$ $1+\operatorname{Re}\mu > \operatorname{Re}\nu > -1$	$\beta^{\nu-\mu} y^{1/2} I_\mu(a\beta) K_\nu(y\beta)$ $a < y < \infty$
(13)	$x^{\nu-\mu+2n+1/2}(\beta^2+x^2)^{-1} J_\mu(ax)$ $a > 0, \quad \operatorname{Re}\beta > 0$ $\operatorname{Re}\mu - 2n + 1 > \operatorname{Re}\nu > -n-1$ n integer	$(-1)^n \beta^{\nu-\mu+2n} y^{1/2} I_\mu(a\beta) K_\nu(y\beta)$ $a < y < \infty$
(14)	$x^{\mu-\nu+1/2}(\beta^2+x^2)^{-1} J_\mu(ax)$ $a > 0, \quad \operatorname{Re}\beta > 0$ $1+\operatorname{Re}\nu > \operatorname{Re}\mu > -1$	$y^{1/2} \beta^{\mu-\nu} I_\nu(y\beta) K_\mu(a\beta)$ $0 < y < a$
(15)	$x^{\mu-\nu+2n+1/2}(\beta^2+x^2)^{-1} J_\mu(ax)$ $a > 0, \quad \operatorname{Re}\beta > 0$ $\operatorname{Re}\nu - 2n + 1 > \operatorname{Re}\mu > -n-1$ n integer	$(-1)^n \beta^{\mu-\nu+2n} y^{1/2} I_\nu(y\beta) K_\mu(a\beta)$ $0 < y < a$
(16)	$x^{1/2}(x^2+\beta^2)^{-1} J_{\nu-2n}(ax)$ $a > 0, \quad \operatorname{Re}\beta > 0$ $\operatorname{Re}\nu > n-1$	$(-1)^n y^{1/2} I_\nu(y\beta) K_{\nu-2n}(a\beta)$ $0 < y < a$ $(-1)^n y^{1/2} I_{\nu-2n}(a\beta) K_\nu(y\beta)$ $a < y < \infty$

Bessel functions of kx (cont'd)

	$f(x)$	$\int_0^\infty f(x)\, J_\nu(xy)(xy)^{1/2}\, dx \qquad y>0$
(17)	$x^{-1/2}\, e^{-\alpha x}\, J_\nu(\beta x)$ $\mathrm{Re}\,\alpha > \mathrm{Im}\,\beta > 0, \quad \mathrm{Re}\,\nu > -\tfrac{1}{2}$	$\pi^{-1}\,\beta^{-1/2}\, Q_{\nu-1/2}\left(\dfrac{\alpha^2+\beta^2+y^2}{2\beta y}\right)$
(18)	$x^{\mu-3/2}\, e^{-\alpha x}\, J_\nu(\beta x)$ $\mathrm{Re}\,\alpha > \lvert\mathrm{Im}\,\beta\rvert, \quad \mathrm{Re}(\mu+2\nu)>0$	$\dfrac{\beta^\nu y^{\nu+1/2}\,\Gamma(\mu+2\nu)}{\pi\alpha^{\mu+2\nu}\,\Gamma(2\nu+1)}$ $\times\displaystyle\int_0^\pi {}_2F_1\left(\frac{\mu}{2}+\nu,\ \frac{\mu+1}{2}+\nu;\ \nu+1;\ -\frac{u^2}{\alpha^2}\right)$ $\times(\sin\phi)^{2\nu}\,d\phi$ $u^2=\beta^2+y^2-2\beta y\cos\phi$
(19)	$x^{-1}\, e^{-xa\cos\phi\cos\psi}\, J_\mu(ax\sin\phi)$ $a>0, \quad 0<\phi,\ \psi<\tfrac{1}{2}\pi$ $\mathrm{Re}(\mu+\nu)>-\tfrac{1}{2}$	$\Gamma(\mu+\nu+\tfrac{1}{2})\,(\sin\psi)^{1/2}$ $\times\, P_{\nu-1/2}^{-\mu}(\cos\phi)\, P_{\mu-1/2}^{-\nu}(\cos\psi)$ $y=a\sin\psi$
(20)	$x^{-1/2}\, e^{-\beta x}\, J_\mu(ax)$ $\mathrm{Re}\,\beta > \lvert\mathrm{Im}\,\alpha\rvert$ $\mathrm{Re}(\mu+\nu+1)>0$	$2\pi^{-1}\,\alpha^\mu\,\beta\, y^{\nu+1/2}\displaystyle\int_0^{\frac{1}{2}\pi}(2\beta\sec\theta)^{\mu+\nu}$ $\times(\beta^2\sec^2\theta+y^2-\alpha^2+u)^{-\mu}$ $\times(\beta^2\sec^2\theta+\alpha^2-y^2+u)^{-\nu}$ $\times\sec^2\theta\cos[(\mu-\nu)\theta]\,u^{-1}\,d\theta$ $u^2=(\sec^2\theta b^2+\alpha^2+y^2)^2-4\alpha^2y^2$
(21)	$x^{\mu-\nu-1/2}\, e^{-\alpha x}\, J_\mu(\beta x)$ $\mathrm{Re}\,\alpha > \lvert\mathrm{Im}\,\beta\rvert, \quad \mathrm{Re}\,\mu > -\tfrac{1}{2}$	$\dfrac{\beta^\mu y^{\nu+1/2}\,\Gamma(\mu+\tfrac{1}{2})}{2^{\nu-\mu}\pi\,\Gamma(\nu+\tfrac{1}{2})}\displaystyle\int_0^\pi (\sin\phi)^{2\nu}$ $\times[(\alpha+iy\cos\phi)^2+\beta^2]^{-\mu-1/2}\,d\phi$
(22)	$x^{\lambda-3/2}\, e^{-\alpha x}\, J_\mu(\beta x)$ $\mathrm{Re}\,\alpha > \mathrm{Im}\,\beta > 0$ $\mathrm{Re}(\lambda+\mu+\nu)>0$	$y^{1/2}\displaystyle\sum_{m=0}^\infty \frac{\Gamma(\lambda+\mu+\nu+2m)}{m!\,\Gamma(\mu+m+1)}\left(-\frac{\beta^2}{4\alpha^2}\right)^m$ $\times\,{}_2F_1(-m,-\mu-m;\nu+1;y^2\beta^{-2})$

Bessel functions of kx (cont'd)

	$f(x)$	$\int_0^\infty f(x)\, J_\nu(xy)(xy)^{1/2}\, dx \qquad y>0$
(23)	$x^{1/2} e^{-\beta x^2} J_\nu(ax)$ $\operatorname{Re}\beta>0, \quad \operatorname{Re}\nu>-1$	$\dfrac{y^{1/2}}{2\beta}\exp\left(-\dfrac{a^2+y^2}{4\beta}\right) I_\nu\left(\dfrac{ay}{2\beta}\right)$
(24)	$x^{\lambda+1/2} e^{-ax^2} J_\mu(\beta x) \quad \operatorname{Re} a>0$ $\operatorname{Re}(\mu+\nu+\lambda)>-2$	$y^{1/2}\sum_{m=0}^\infty \dfrac{\Gamma(m+\frac{1}{2}\nu+\frac{1}{2}\mu+\frac{1}{2}\lambda)}{m!\,\Gamma(m+\mu+1)}\left(-\dfrac{\beta^2}{4a}\right)^m$ $\times\, {}_2F_1(-m,-\mu-m;\nu+1;y^2\beta^{-2})$
(25)	$x^\lambda J_\mu(ax) \begin{matrix}\cos(bx)\\ \sin(bx)\end{matrix}$	see under Fourier transforms.
(26)	$x^{1/2}\sin(ax^2) J_\nu(bx)$ $a>0, \quad b>0, \quad \operatorname{Re}\nu>-2$	$\dfrac{y^{1/2}}{2a}\cos\left(\dfrac{y^2+b^2}{4a}-\dfrac{\nu\pi}{2}\right) J_\nu\left(\dfrac{by}{2a}\right)$
(27)	$x^{1/2}\cos(ax^2) J_\nu(bx)$ $a>0, \quad b>0, \quad \operatorname{Re}\nu>-1$	$\dfrac{y^{1/2}}{2a}\sin\left(\dfrac{b^2+y^2}{4a}-\dfrac{\nu\pi}{2}\right) J_\nu\left(\dfrac{by}{2a}\right)$
(28)	$x^{-1/2}[J_0(\frac{1}{2}ax)]^2$ $a>0, \quad \operatorname{Re}\nu>-1$	$y^{-1/2}\{P_{\frac{1}{2}\nu-\frac{1}{2}}[(1-a^2/y^2)^{1/2}]\}^2$ $a<y<\infty$
(29)	$x^{1/2}[J_{\frac{1}{2}\nu}(\frac{1}{2}ax)]^2$ $a>0, \quad \operatorname{Re}\nu>-1$	$2\pi^{-1}y^{-1/2}(a^2-y^2)^{-1/2} \qquad 0<y<a$ $0 \qquad a<y<\infty$
(30)	$x^{1/2-\nu}[J_\nu(\frac{1}{2}ax)]^2$ $a>0, \quad \operatorname{Re}\nu>-\frac{1}{2}$	$\dfrac{2^{1-\nu}y^{\nu-1/2}(a^2-y^2)^{\nu-1/2}}{\pi^{1/2}a^{2\nu}\Gamma(\nu+\frac{1}{2})} \qquad 0<y<a$ $0 \qquad a<y<\infty$

Bessel functions of kx (cont'd)

	$f(x)$	$\int_0^\infty f(x)\, J_\nu(xy)(xy)^{1/2}\, dx \qquad y>0$
(31)	$x^{1/2-\nu} J_\nu(ax)\, J_\nu(bx)$ $a,\ b>0, \quad \mathrm{Re}\,\nu>-1/2$	$\dfrac{[y^2-(a-b)^2]^{\nu-1/2}[(a+b)^2-y^2]^{\nu-1/2}}{y^{\nu-1/2}2^{3\nu-1}\pi^{1/2}(ab)^\nu\Gamma(\nu+1/2)}$ $\lvert a-b\rvert<y<a+b$ 0 $0<y<\lvert a-b\rvert$ or $a+b<y<\infty$
(32)	$x^{1/2} J_{1/2(\nu+n)}(1/2\, ax)\, J_{1/2(\nu-n)}(1/2\, ax)$ $a>0, \quad \mathrm{Re}\,\nu>-1$	$2\pi^{-1}y^{-1/2}(a^2-y^2)^{-1/2}\, T_n(a^{-1}y)$ $0<y<a$ 0 $a<y<\infty$
(33)	$x^{-1/2} J_\mu^2(1/2\, ax) \qquad a>0$ $\mathrm{Re}\,\nu+\mathrm{Re}\,2\mu>-1$	$(1/2\, a)^{2\mu} y^{-2\mu-1/2}\dfrac{\Gamma(1/2+1/2\nu+\mu)}{[\Gamma(\mu+1)]^2\Gamma(1/2+1/2\nu-\mu)}$ $\times\{{}_2F_1[1/2-1/2\nu+\mu, 1/2+1/2\nu+\mu; \mu+1;$ $1/2-1/2(1-a^2/y^2)^{1/2}]\}^2$ $a<y<\infty$
(34)	$x^{1/2-\mu} J_\mu(ax)\, J_\nu(bx) \qquad a,\ b>0$ $\mathrm{Re}\,\nu>-1, \quad \mathrm{Re}\,\mu>-1/2$	$\dfrac{y^{\mu-1/2}(\sinh u)^{\mu-1/2}}{(1/2\,\pi^3)^{1/2}a^\mu b^{1-\mu}}e^{(\mu-1/2)\pi i}$ $\times\sin[(\nu-\mu)\pi]\, Q^{1/2-\mu}_{\nu-1/2}(\cosh u)$ $0<y<a-b$ $\dfrac{b^{\mu-1}y^{\mu-1/2}}{(2\pi)^{1/2}a^\mu}(\sin v)^{\mu-1/2}\, \mathrm{P}^{1/2-\mu}_{\nu-1/2}(\cos v)$ $\lvert a-b\rvert<y<a+b$ 0 $0<y<b-a$ or $a+b<y<\infty$ $2by\cosh u=a^2-b^2-y^2$ $2by\cos v=b^2+y^2-a^2$

Bessel functions of kx (cont'd)

	$f(x)$	$\int_0^\infty f(x)\, J_\nu(xy)(xy)^{1/2}\, dx \qquad y>0$
(35)	$x^{1/2-\nu} J_\mu(ax)\, J_\mu(bx) \qquad a,\, b>0$ $\operatorname{Re}\mu>-1, \quad \operatorname{Re}\nu>-1/2$	$0 \qquad 0<y<\lvert a-b\rvert$ $\dfrac{(ab)^{\nu-1}}{(2\pi)^{1/2}\, y^{\nu-1/2}} (\sin u)^{\nu-1/2}\, P^{1/2-\nu}_{\mu-1/2}(\cos u)$ $\lvert a-b\rvert<y<a+b$ $\dfrac{(ab)^{\nu-1}(\sinh v)^{\nu-1/2}}{(1/2\,\pi^3)^{1/2}\, y^{\nu-1/2}}\, e^{(\nu-1/2)\pi i}$ $\times\sin[(\mu-\nu)\pi]\, Q^{1/2-\nu}_{\mu-1/2}(\cosh v)$ $a+b<y<\infty$ $2ab\cos u=a^2+b^2-y^2$ $2ab\cosh v=y^2-a^2-b^2$
(36)	$x^{\rho-\mu-\nu+1/2} J_\mu(ax)\, J_\rho(bx)$ $b>a>0$ $\operatorname{Re}\rho>-1, \quad \operatorname{Re}(\rho-\mu-\nu)<1/2$	$0 \qquad 0<y<b-a$
(37)	$x^{\rho-\mu-\nu-3/2} J_\mu(ax)\, J_\rho(bx)$ $b>a>0$ $\operatorname{Re}\rho>0, \quad \operatorname{Re}(\rho-\mu-\nu)<5/2$	$\dfrac{2^{\rho-\mu-\nu-1}\, y^{\nu+1/2}\, a^\mu\, \Gamma(\rho)}{b^\rho\, \Gamma(\mu+1)\, \Gamma(\nu+1)}$ $0<y<b-a$
(38)	$x^{-1/2} J_\mu(xa\sin\phi\cos\psi)\, J_\rho(ax)$ $a>0, \quad 0<\phi,\ \psi<1/2\,\pi$ $\operatorname{Re}(\mu+\nu+\rho)>-1$	$\dfrac{a^{-1/2}\, \Gamma[1/2(1+\sigma+\rho)]}{\Gamma(\mu+1)\,\Gamma(\nu+1)\,\Gamma[1/2(1-\sigma+\rho)]}$ $\times(\sin\phi\cos\psi)^\mu(\sin\psi\cos\phi)^{\nu+1/2}$ $\times\, {}_2F_1\left(\dfrac{1+\sigma-\rho}{2}, \dfrac{1+\sigma+\rho}{2}; \mu+1; \sin^2\phi\right)$ $\times\, {}_2F_1\left(\dfrac{1+\sigma-\rho}{2}, \dfrac{1+\sigma+\rho}{2}; \nu+1; \sin^2\psi\right)$ $\sigma=\mu+\nu, \quad y=a\cos\phi\sin\psi$

Bessel functions of kx (cont'd)

	$f(x)$	$\int_0^\infty f(x)\, J_\nu(xy)(xy)^{1/2}\, dx \qquad y>0$
(39)	$x^{1/2} J_\mu(xa \sin\phi \cos\psi)\, J_{\nu-\mu}(ax)$ $a>0, \quad 0<\phi, \psi<\tfrac{1}{2}\pi$ $\operatorname{Re}\nu>-1$	$2\pi^{-1} a^{-3/2} \sin(\mu\pi)(\sin\phi)^{\mu}(\sin\psi)^{\nu+1/2}$ $\times(\cos\phi)^{1/2-\nu}(\cos\psi)^{-\mu}$ $\times[\cos(\phi+\psi)\cos(\phi-\psi)]^{-1}$ $y=a\cos\phi\sin\psi$
(40)	$x^{\lambda} J_\mu(ax)\, J_\rho(bx)$	see Bailey, W. N., 1936: *Proc. London Math. Soc.* (2), 40, 37-48.
(41)	$x^{2n-\mu-3/2}(x^2+c^2)^{-1}$ $\times J_\nu(ax)\, J_\mu(bx) \qquad a>b>0$ $\operatorname{Re}\nu>\tfrac{1}{2}-n$ $\operatorname{Re}\mu>2n-9/2$	$(-1)^{n+1} c^{2n-\mu-2} y^{1/2} I_\mu(bc)$ $\times I_\nu(yc)\, K_\nu(ac) \qquad 0<y<a-b$
(42)	$x^{\nu-M+1/2} \prod_{i=1}^{k} J_{\mu_i}(a_i x)$ $a_i>0, \quad \sum_{i=1}^{k}\mu_i=M$ $-1<\operatorname{Re}\nu<\operatorname{Re}M+\tfrac{1}{2}k-\tfrac{1}{2}$	$0 \qquad \sum_{i=1}^{k} a_i<y<\infty$
(43)	$x^{\nu-M-3/2} \prod_{i=1}^{k} J_{\mu_i}(a_i x)$ $a_i>0, \quad M=\sum_{i=1}^{k}\mu_i$ $0<\operatorname{Re}\nu<\operatorname{Re}M+k/2+3/2$	$2^{\nu-M-1} y^{1/2-\nu}\Gamma(\nu)\prod_{i=1}^{k}\dfrac{a_i^{\mu_i}}{\Gamma(1+\mu_i)}$ $\sum_{i=1}^{k} a_i<y<\infty$
(44)	$x^{-\lambda-1/2} Y_\mu(ax)$	see under Mellin transforms
(45)	$x^{1/2}(x^2+\beta^2)^{-1} Y_{\nu-2n-1}(ax)$ $a>0, \quad \operatorname{Re}\beta>0$ $\operatorname{Re}\nu>n-\tfrac{1}{2}$	$(-1)^n y^{1/2} I_\nu(y\beta)\, K_{\nu-2n-1}(a\beta)$ $0<y<a$

Bessel functions of kx (cont'd)

	$f(x)$	$\int_0^\infty f(x)\, J_\nu(xy)(xy)^{1/2}\,dx \qquad y>0$
(46)	$x^{1/2}(x^2+\beta^2)^{-1}$ $\times\{\cos[\tfrac{1}{2}(\nu-\mu)\pi]\,J_\mu(ax)$ $+\sin[\tfrac{1}{2}(\nu-\mu)\pi]\,Y_\mu(ax)\}$ $a>0,\quad \mathrm{Re}\,\beta>0$ $\mathrm{Re}(\nu\pm\mu)>-2$	$y^{1/2}\, I_\nu(\beta y)\, K_\mu(a\beta) \qquad 0<y\le a$
(47)	$x^{\rho+1/2}(\beta^2+x^2)^{-1}$ $\times\{\cos[\tfrac{1}{2}(\rho-\mu+\nu)\pi]\,J_\mu(ax)$ $+\sin[\tfrac{1}{2}(\rho-\mu+\nu)\pi]\,Y_\mu(ax)\}$ $a>0,\quad \mathrm{Re}\,\beta>0$ $\mathrm{Re}(\nu\pm\mu+\rho)>-2,\quad \mathrm{Re}\,\rho<1$	$\beta^\rho y^{1/2}\, I_\nu(\beta y)\, K_\mu(a\beta) \qquad 0<y<a$
(48)	$x^{1/2}\, J_{\frac{1}{2}\nu}(\tfrac{1}{2}ax)\, Y_{\frac{1}{2}\nu}(\tfrac{1}{2}ax)$ $a>0,\quad \mathrm{Re}\,\nu>-1$	$0 \qquad 0<y<a$ $-2\pi^{-1}y^{-1/2}(y^2-a^2)^{-1/2} \qquad a<y<\infty$
(49)	$x^{\nu+1/2}\, J_\nu(\tfrac{1}{2}ax)\, Y_\nu(\tfrac{1}{2}ax)$ $a>0,\quad \lvert\mathrm{Re}\,\nu\rvert<\tfrac{1}{2}$	$0 \qquad 0<y<a$ $-\dfrac{2^{\nu+1}a^{2\nu}y^{-\nu-1/2}(y^2-a^2)^{-\nu-1/2}}{\pi^{1/2}\,\Gamma(\tfrac{1}{2}-\nu)} \qquad a<y<\infty$
(50)	$x^{\rho+1/2}(x^2+\gamma^2)^{-1}\, J_\mu(bx)$ $\times\{\cos[\tfrac{1}{2}(\rho+\mu)\pi]\,J_\nu(ax)$ $+\sin[\tfrac{1}{2}(\rho+\mu)\pi]\,Y_\nu(ax)\}$ $a>b>0,\quad \mathrm{Re}\,\rho<3/2$ $\mathrm{Re}(\mu+\rho+2\nu)>-2$ $\mathrm{Re}(\mu+\rho)>-2$	$\gamma^\rho y^{1/2}\, I_\mu(b\gamma)\, I_\nu(y\gamma)\, K_\nu(a\gamma)$ $0<y<a-b$

Bessel functions of kx (cont'd)

	$f(x)$	$\int_0^\infty f(x)\, J_\nu(xy)(xy)^{1/2}\, dx \qquad y>0$
(51)	$x^{\rho+1/2}(x^2+\beta^2)^{-1} \prod_{i=1}^{k} [J_{\mu_i}(c_i x)]$ $\times\{\cos[\tfrac{1}{2}(\rho+M-\mu)\pi]\, J_\mu(ax)$ $+\sin[\tfrac{1}{2}(\rho+M-\mu)\pi]\, Y_\mu(ax)\}$ $a>0, \quad c_i>0$ $\nu+\sum_{i=1}^{k}\mu_i=M, \quad \operatorname{Re}\rho<(k+3)/2$ $\operatorname{Re}(\rho+M)>\lvert\operatorname{Re}\mu\rvert-2$	$\beta^\rho y^{1/2}\, I_\nu(\beta y)\, K_\mu(a\beta) \prod_{i=1}^{k} I_{\mu_i}(\beta c_i)$ $0<y<a-\sum_{i=1}^{k} c_i$

8.12. Bessel functions of other arguments

(1)	$x^{1/2}\, J_{\frac{1}{2}\nu}(\tfrac{1}{4}ax^2)$ $a>0, \quad \operatorname{Re}\nu>-1$	$2a^{-1}\, y^{1/2}\, J_{\frac{1}{2}\nu}(y^2/a)$
(2)	$x^{1/2}\exp(-\tfrac{1}{4}\alpha x^2)\, J_{\frac{1}{2}\nu}(\tfrac{1}{4}\beta x^2)$ $\operatorname{Re}\alpha>\lvert\operatorname{Im}\beta\rvert, \quad \operatorname{Re}\nu>-1$	$2(\alpha^2+\beta^2)^{-1/2}\, y^{1/2}\exp\left(-\dfrac{\alpha y^2}{\alpha^2+\beta^2}\right)$ $\times J_{\frac{1}{2}\nu}\left(\dfrac{\beta y^2}{\alpha^2+\beta^2}\right)$
(3)	$x^{(\frac{1}{2}-\nu)/3}\exp(\tfrac{1}{4}ax^2 i)$ $\times J_{(\nu-\frac{1}{2})/3}(\tfrac{1}{4}ax^2)$ $\operatorname{Im} a>0, \quad \operatorname{Re}\nu>-1$	$a^{(\nu-2)/3}\, y^{(\frac{1}{2}-\nu)/3}\exp\left(\dfrac{\nu+1}{6}\pi i-\dfrac{y^2}{4a}i\right)$ $\times J_{(\nu-\frac{1}{2})/3}\left(\dfrac{y^2}{4a}\right)$
(4)	$x^{(\frac{1}{2}-\nu)/3}\sin(\tfrac{1}{4}ax^2)$ $\times J_{(\nu-\frac{1}{2})/3}(\tfrac{1}{4}ax^2)$ $a>0, \quad \operatorname{Re}\nu>-5/2$	$a^{(\nu-2)/3}\, y^{(\frac{1}{2}-\nu)/3}\sin\left(\dfrac{\nu+1}{6}\pi-\dfrac{y^2}{4a}\right)$ $\times J_{(\nu-\frac{1}{2})/3}\left(\dfrac{y^2}{4a}\right)$

Bessel functions of other arguments (cont'd)

	$f(x)$	$\int_0^\infty f(x)\, J_\nu(xy)(xy)^{1/2}\, dx \qquad y>0$
(5)	$x^{(1/2-\nu)/3}\cos(\frac{1}{4}ax^2) \times J_{(\nu-1/2)/3}(\frac{1}{4}ax^2)$ $\quad a>0, \quad \mathrm{Re}\,\nu>-1$	$a^{(\nu-2)/3}\, y^{(1/2-\nu)/3}\cos\left(\frac{\nu+1}{6}\pi-\frac{y^2}{4a}\right) \times J_{(\nu-1/2)/3}\left(\frac{y^2}{4a}\right)$
(6)	$x^{1/2}[J_{1/4\nu}(\frac{1}{4}ax^2)]^2$ $\quad a>0, \quad \mathrm{Re}\,\nu>-1$	$-\frac{y^{1/2}}{a} J_{1/4\nu}\left(\frac{y^2}{4a}\right) Y_{1/4\nu}\left(\frac{y^2}{4a}\right)$
(7)	$x^{1/2} J_{1/4\nu}(\frac{1}{4}ax^2)\, J_{-1/4\nu}(\frac{1}{4}ax^2)$ $\quad a>0, \quad \mathrm{Re}\,\nu>-2$	$\frac{y^{1/2}}{a} J_{1/4\nu}\left(\frac{y^2}{4a}\right)\left[J_{1/4\nu}\left(\frac{y^2}{4a}\right)\sin\left(\frac{\pi\nu}{4}\right) - Y_{1/4\nu}\left(\frac{y^2}{4a}\right)\cos\left(\frac{\pi\nu}{4}\right)\right]$
(8)	$x^{1/2} J_{1/4\nu-\mu}(ax^2)\, J_{1/4\nu+\mu}(ax^2)$ $\quad a>0, \quad \mathrm{Re}\,\nu>-\frac{1}{2}$	$\frac{2}{\pi y^{3/2}}[e^{1/4\nu\pi i} W_{\mu,1/4\nu}(u) W_{-\mu,1/4\nu}(u) + e^{-1/4\nu\pi i} W_{\mu,1/4\nu}(v) W_{-\mu,1/4\nu}(v)]$ $\quad u=\frac{y^2}{8a}e^{1/2\pi i}, \quad v=\frac{y^2}{8a}e^{-1/2\pi i}$
(9)	$x^{-1/2} J_\nu(ax^{-1})$ $\quad a>0, \quad \mathrm{Re}\,\nu>-\frac{1}{2}$	$y^{-1/2} J_{2\nu}(2a^{1/2}y^{1/2})$
(10)	$x^{-5/2} J_\nu(ax^{-1})$ $\quad a>0, \quad \mathrm{Re}\,\nu>-\frac{1}{2}$	$a^{-1} y^{1/2} J_{2\nu}(2a^{1/2}y^{1/2})$
(11)	$x^{-3/2} J_{\nu-1}(ax^{-1})$ $\quad a>0, \quad \mathrm{Re}\,\nu>-\frac{1}{2}$	$a^{-1/2} J_{2\nu-1}(2a^{1/2}y^{1/2})$

Bessel functions of other arguments (cont'd)

	$f(x)$	$\int_0^\infty f(x)\, J_\nu(xy)(xy)^{1/2}\,dx \qquad y>0$
(12)	$x^{-2\nu} J_{1/2-\nu}(ax^{-1})$ $a>0, \quad -1/2<\mathrm{Re}\,\nu<3$	$-\frac{1}{2} i \csc(2\nu\pi)(y/a)^{\nu-1/2}$ $\times[e^{2\nu\pi i} J_{1-2\nu}(u)\, J_{2\nu-1}(v)$ $-e^{-2\nu\pi i} J_{2\nu-1}(u)\, J_{1-2\nu}(v)]$ $u=(\frac{1}{2}ay)^{1/2}\, e^{\frac{1}{4}\pi i}$ $v=(\frac{1}{2}ay)^{1/2}\, e^{-\frac{1}{4}\pi i}$
(13)	$x^{\rho-3/2} J_\mu(x^{-1})$ $-3/2-\mathrm{Re}\,\nu<\mathrm{Re}\,\rho<\mathrm{Re}\,\mu+3/2$	$\frac{1}{2}\pi \csc[\frac{1}{2}(\mu-\nu-\rho)\pi]y^{\nu+1/2}$ $\times\left[A\ {}_0F_3\left(1+\nu, 1+\frac{\rho-\mu+\nu}{2}, 1+\frac{\mu+\nu+\rho}{2}; \frac{y^2}{16}\right)\right.$ $\left.-y^\mu B\ {}_0F_3\left(1+\mu, 1+\frac{\mu+\nu-\rho}{2}, 1+\frac{\mu-\nu-\rho}{2}; \frac{y^2}{16}\right)\right]$ $A^{-1}=2^{\nu+\rho}\Gamma(1+\nu)\Gamma[1+\frac{1}{2}(\rho-\mu+\nu)]$ $\times\Gamma[1+\frac{1}{2}(\rho+\mu+\nu)]$ $B^{-1}=2^{2\mu-\rho}\Gamma(1+\mu)\Gamma[1+\frac{1}{2}(\mu+\nu-\rho)]$ $\times\Gamma[1+\frac{1}{2}(\mu-\nu-\rho)]$
(14)	$x^{1/2}(\beta^2+x^2)^{-1/2}\exp\left(-\frac{a^2\beta}{\beta^2+x^2}\right)$ $\times J_\nu\left(\frac{a^2 x}{\beta^2+x^2}\right)$ $\mathrm{Re}\,\beta>0, \quad \mathrm{Re}\,\nu>-1/2$	$y^{-1/2} e^{-\beta y} J_{2\nu}(2ay^{1/2})$
(15)	$J_{2\nu-1}(ax^{1/2}) \qquad \mathrm{Re}\,\nu>-1/2$	$\frac{1}{2}ay^{-3/2} J_{\nu-1}(\frac{1}{4}a^2y^{-1})$
(16)	$x^{-1/2} J_{2\nu}(ax^{1/2}) \qquad \mathrm{Re}\,\nu>-1/2$	$y^{-1/2} J_\nu(\frac{1}{4}a^2y^{-1})$
(17)	$x^{-1/2} e^{-\beta x} J_{2\nu}(2ax^{1/2})$ $\mathrm{Re}\,\beta>0, \quad \mathrm{Re}\,\nu>-1/2$	$y^{1/2}(y^2+\beta^2)^{-1/2}\exp\left(-\frac{a^2\beta}{\beta^2+y^2}\right)$ $\times J_\nu\left(\frac{a^2 y}{\beta^2+y^2}\right)$

Bessel functions of other arguments (cont'd)

	$f(x)$	$\int_0^\infty f(x)\, J_\nu(xy)(xy)^{1/2}\,dx \qquad y>0$
(18)	$x^{\nu+1/2}(x^2+\beta^2)^{-1/2\mu}$ $\times J_\mu[a(x^2+\beta^2)^{1/2}] \quad a>0$ $\operatorname{Re}\beta>0, \quad \operatorname{Re}\mu>\operatorname{Re}\nu>-1$	$a^{-\mu}y^{\nu+1/2}\beta^{-\mu+\nu+1}(a^2-y^2)^{1/2\mu-1/2\nu-1/2}$ $\times J_{\mu-\nu-1}[\beta(a^2-y^2)^{1/2}] \quad 0<y<a$ $0 \quad a<y<\infty$
(19)	$x^{\nu+1/2}(x^2+\beta^2)^{-1/2\mu-1}$ $\times J_{\mu-1}[a(x^2+\beta^2)^{1/2}] \quad a>0$ $\operatorname{Re}\beta>0, \ \operatorname{Re}(\mu+2)>\operatorname{Re}\nu>-1$	$(\tfrac{1}{2}a)^{\mu-1}\beta^\nu[\Gamma(\mu)]^{-1}y^{1/2}K_\nu(\beta y)$ $a<y<\infty$
(20)	$x^{\nu-3/2}(x^2+\beta^2)^{-\mu/2}$ $\times J_\mu[a(x^2+\beta^2)^{1/2}] \quad a>0$ $\operatorname{Re}\beta>0, \ \operatorname{Re}(\mu+2)>\operatorname{Re}\nu>0$	$\beta^{-\mu}2^{\nu-1}\Gamma(\nu)y^{1/2-\nu}J_\mu(a\beta)$ $a<y<\infty$
(21)	$x^{\nu+1/2}(x^2+\alpha^2)^{-1}(x^2+\beta^2)^{-1/2\mu}$ $\times J_\mu[c(x^2+\beta^2)^{1/2}]$ $\operatorname{Re}\alpha>0, \quad \operatorname{Re}\beta>0, \quad c>0$ $-1<\operatorname{Re}\nu<2+\operatorname{Re}\mu$	$\alpha^\nu y^{1/2}(\beta^2-\alpha^2)^{-1/2\mu}J_\mu[c(\beta^2-\alpha^2)^{1/2}]$ $\times K_\nu(\alpha y) \quad c\le y<\infty$
(22)	$x^{\nu+2n-3/2}(x^2+\alpha^2)^{-1}(x^2+\beta^2)^{-\mu/2}$ $\times J_\mu[c(x^2+\beta^2)^{1/2}]$ $\operatorname{Re}\alpha>0, \quad \operatorname{Re}\beta>0, \quad c>0$ $-n<\operatorname{Re}\nu<4-2n+\operatorname{Re}\mu$	$(-1)^{n+1}y^{1/2}\alpha^{\nu+2n-2}(\beta^2-\alpha^2)^{-1/2\mu}$ $\times J_\mu[c(\beta^2-\alpha^2)^{1/2}]K_\nu(\alpha y)$ $c<y<\infty$
(23)	$x^{\nu+1/2}(x^2+\beta^2)^{-1/2\nu-3/4}$ $\times C_{2n+1}^{\nu+1/2}[\beta(x^2+\beta^2)^{-1/2}]$ $\times J_{\nu+3/2+2n}[a(x^2+\beta^2)^{1/2}]$ $a>0, \quad \operatorname{Re}\beta>0, \quad \operatorname{Re}\nu>-1$	$(-1)^n 2^{1/2}\pi^{-1/2}a^{1/2-\nu}y^{\nu+1/2}(a^2-y^2)^{-1/2}$ $\times\sin[\beta(a^2-y^2)^{1/2}]$ $\times C_{2n+1}^{\nu+1/2}[(1-y^2/a^2)^{1/2}] \quad 0<y<a$ $0 \quad a<y<\infty$
(24)	$x^{\nu+1/2}(x^2+\beta^2)^{-1/2\nu-3/4}$ $\times C_{2n}^{\nu+1/2}[\beta(x^2+\beta^2)^{-1/2}]$ $\times J_{\nu+1/2+2n}[a(x^2+\beta^2)^{1/2}]$ $a>0, \quad \operatorname{Re}\beta>0, \quad \operatorname{Re}\nu>-1$	$(-1)^n 2^{1/2}\pi^{-1/2}a^{1/2-\nu}y^{\nu+1/2}(a^2-y^2)^{-1/2}$ $\times\cos[\beta(a^2-y^2)^{1/2}]$ $\times C_{2n}^{\nu+1/2}[(1-y^2/a^2)^{1/2}] \quad 0<y<a$ $0 \quad a<y<\infty$

Bessel functions of other arguments (cont'd)

	$f(x)$	$\int_0^\infty f(x)\, J_\nu(xy)(xy)^{1/2}\,dx \qquad y>0$
(25)	$x^{\nu-3/2}(x^2+\beta^2)^{-n\mu/2}$ $\times \prod_{i=1}^{n} J_\mu[a_i(x^2+\beta^2)^{1/2}]$ $a_i>0, \quad \operatorname{Re}\beta>0$ $\operatorname{Re}(n\mu+\tfrac{1}{2}n+\tfrac{1}{2})>\operatorname{Re}\nu>0$	$2^{\nu-1}\beta^{-n\mu}\,\Gamma(\nu)\,y^{1/2-\nu}\prod_{i=1}^{n} J_\mu(a_i\beta)$ $\sum_{i=1}^{n} a_i<y<\infty$
(26)	$x^{\nu+1/2}\prod_{i=1}^{n} z_i^{-\mu_i} J_{\mu_i}(a_i z_i)$ $a_i>0, \quad \operatorname{Re}\beta_i>0$ $z_i=(x^2+\beta_i^2)^{1/2}$ $\tfrac{1}{2}n+\sum_{i=1}^{n}\mu_i-\tfrac{1}{2}>\operatorname{Re}\nu>-1$	$0 \qquad \sum_{i=1}^{n} a_i<y<\infty$
(27)	$x^{\nu-3/2}\prod_{i=1}^{n} z_i^{-\mu_i} J_{\mu_i}(a_i z_i)$ $a_i>0, \quad \operatorname{Re}\beta_i>0$ $z_i=(x^2+\beta_i^2)^{1/2}$ $\tfrac{1}{2}n+\sum_{i=1}^{n}\mu_i+3/2>\operatorname{Re}\nu>0$	$2^{\nu-1}\,\Gamma(\nu)\,y^{1/2-\nu}$ $\times\prod_{i=1}^{n}[\beta_i^{-\mu_i} J_{\mu_i}(a_i\beta_i)]$ $\sum_{i=1}^{n} a_i<y<\infty$
(28)	$x^{\nu+1/2}(1-x^2)^{1/2\mu} J_\mu[a(1-x^2)^{1/2}] \qquad 0<x<1$ $0 \qquad 1<x<\infty$ $\operatorname{Re}\mu>-1, \quad \operatorname{Re}\nu>-1$	$a^\mu y^{\nu+1/2}(a^2+y^2)^{-1/2(\mu+\nu+1)}$ $\times J_{\mu+\nu+1}[(a^2+y^2)^{1/2}]$
(29)	$0 \qquad 0<x<c$ $x^{1/2-\nu}(x^2-c^2)^{1/2\mu} J_\mu[a(x^2-c^2)^{1/2}] \qquad c<x<\infty$ $a>0, \quad \operatorname{Re}\nu>\operatorname{Re}\mu>-1$	$0 \qquad 0<y<a$ $a^\mu c^{1+\mu-\nu} y^{-\nu+1/2}(y^2-a^2)^{1/2\nu-1/2\mu-1/2}$ $\times J_{\nu-\mu-1}[c(y^2-a^2)^{1/2}] \qquad a<y<\infty$

Bessel functions of other arguments (cont'd)

	$f(x)$	$\int_0^\infty f(x)\, J_\nu(xy)(xy)^{1/2}\, dx \qquad y>0$
(30)	$0 \qquad 0<x<c$ $x^{1/2-\nu}(x^2+\beta^2)^{-1}(x^2-c^2)^{1/2\mu}$ $\times J_\mu[a(x^2-c^2)^{1/2}] \qquad c<x<\infty$ $a>0, \quad \mathrm{Re}\,\beta>0$ $-1<\mathrm{Re}\,\mu<2+\mathrm{Re}\,\nu$	$\beta^{-\nu}(c^2+\beta^2)^{1/2\mu} y^{1/2} K_\mu[a(c^2+\beta^2)^{1/2}]$ $\times I_\nu(y\beta) \qquad 0<y<a$
(31)	$0 \qquad 0<x<c$ $x^{1/2-\nu}(x^2+\beta^2)^{-1}(x^2-c^2)^{1/2\mu+n-1}$ $\times J_\mu[a(x^2-c^2)^{1/2}] \quad c<x<\infty$ $a>0, \quad \mathrm{Re}\,\beta>0$ $-n<\mathrm{Re}\,\mu<4-2n+\mathrm{Re}\,\nu$	$(-1)^{n+1}\beta^{-\nu}(\beta^2+c^2)^{1/2\mu+n-1} y^{1/2}$ $\times K_\mu[a(\beta^2+c^2)^{1/2}]\, I_\nu(by)$ $0<y<a$
(32)	$x^{\nu+2n+1/2}(1-x^2)^{1/2\lambda+m}$ $\times J_\lambda[a(1-x^2)^{1/2}] \qquad 0<x<1$ $0 \qquad 1<x<\infty$ $a>0, \quad \mathrm{Re}\,\lambda>-1, \quad \mathrm{Re}\,\nu>-1$	$a^{-\lambda} y^{-\nu+1/2}\left(\frac{d}{a\,da}\right)^m\left(\frac{d}{y\,dy}\right)^n$ $\times\{a^{2\lambda+2m} y^{2\nu+2n}$ $\times(a^2+y^2)^{-1/2(\lambda+\nu+m+n+1)}$ $\times J_{\lambda+\nu+m+n+1}[(a^2+y^2)^{1/2}]\}$
(33)	$x^\rho(1-x^2)^\mu J_\lambda[a(1-x^2)^{1/2}]$ $0<x<1$ $0 \qquad 1<x<\infty$	see Bailey, W. N., 1938: *Quart. J. Math. Oxford Series* 9, 141-147.
(34)	$x^{1/2} J_{1/2\nu}\{1/2 a[(x^2+\beta^2)^{1/2}-\beta]\}$ $\times J_{1/2\nu}\{1/2 a[(x^2+\beta^2)^{1/2}+\beta]\}$ $a>0, \quad \mathrm{Re}\,\nu>-1$	$2\pi^{-1} y^{-1/2}(a^2-y^2)^{-1/2}$ $\times\cos[\beta(a^2-y^2)^{1/2}] \qquad 0<y<a$ $0 \qquad a<y<\infty$
(35)	$x^{1/2} Y_{1/2\nu}(1/4\, ax^2)$ $a>0, \quad \mathrm{Re}\,\nu>-1$	$-2a^{-1} y^{1/2}\, \mathbf{H}_{1/2\nu}(y^2/a)$
(36)	$x^{1/2} J_{1/4\nu}(1/4\, ax^2)\, Y_{1/4\nu}(1/4\, ax^2)$ $a>0, \quad \mathrm{Re}\,\nu>-1$	$-2a^{-1} y^{1/2}\left[J_{1/4\nu}\left(\frac{y^2}{4a}\right)\right]^2$

Bessel functions of other arguments (cont'd)

	$f(x)$	$\int_0^\infty f(x)\, J_\nu(xy)(xy)^{1/2}\, dx \qquad y>0$
(37)	$x^{-1/2}\, Y_\nu(ax^{-1})$ $a>0, \quad \lvert \mathrm{Re}\,\nu \rvert < 1/2$	$-2\pi^{-1} y^{-1/2} [K_{2\nu}(2a^{1/2} y^{1/2})$ $- 1/2\pi\, Y_{2\mu}(2a^{1/2} y^{1/2})]$
(38)	$x^{-5/2}\, Y_\nu(ax^{-1})$ $a>0, \quad \lvert \mathrm{Re}\,\nu \rvert < 1/2$	$2y^{1/2} a^{-1} \pi^{-1} [K_{2\nu}(2a^{1/2} y^{1/2})$ $+ 1/2\pi\, Y_{2\nu}(2a^{1/2} y^{1/2})]$
(39)	$x^{-1/2}\, Y_{2\nu}(2ax^{1/2})$ $a>0, \quad \mathrm{Re}\,\nu > -1/2$	$2\sec(\nu\pi)\, y^{-1/2} [1/2 \cos(\nu\pi)\, Y_\nu(a^2/y)$ $- Y_{-\nu}(a^2/y) + \mathbf{H}_{-\nu}(a^2/y)]$
(40)	$0 \qquad 0<x<c$ $x^{1/2-\nu}(x^2+\beta^2)^{-1}(x^2-c^2)^{1/2\mu+n-1/2}$ $\times Y_\mu[a(x^2-c^2)^{1/2}] \qquad c<x<\infty$ $a>0, \quad \mathrm{Re}\,\beta>0$ $-1/2 - n < \mathrm{Re}\,\mu < 3 - 2n + \mathrm{Re}\,\nu$	$(-1)^{n+1} \beta^{-\nu} y^{1/2} (\beta^2+c^2)^{1/2\mu+n-1/2}$ $\times K_\mu[a(\beta^2+c^2)^{1/2}]\, I_\nu(\beta y)$ $0<y<a$
(41)	$x^{1/2}\, J_{1/2\nu}\{1/2 a[(x^2+\beta^2)^{1/2}-\beta]\}$ $\times Y_{1/2\nu}\{1/2 a[(x^2+\beta^2)^{1/2}+\beta]\}$ $a>0, \quad \mathrm{Re}\,\nu>-1$	$2\pi^{-1} y^{-1/2} (a^2-y^2)^{-1/2}$ $\times \sin[\beta(a^2-y^2)^{1/2}] \qquad 0<y<a$ $-2\pi^{-1} y^{-1/2} (y^2-a^2)^{-1/2}$ $\times \exp[-\beta(y^2-a^2)^{1/2}] \qquad a<y<\infty$
(42)	$x^{1/2}[H^{(1)}_{1/4\nu+\mu}(ax^2)\, H^{(1)}_{1/4\nu-\mu}(ax^2)$ $- H^{(2)}_{1/4\nu+\mu}(ax^2)\, H^{(2)}_{1/4\nu-\mu}(ax^2)]$ $\mathrm{Re}\,\nu > -1/2$ $\mathrm{Re}(1/2 \pm \mu + 1/2\nu) > 0$	$\dfrac{8\Gamma(1/2-\mu+1/4\nu)\,\Gamma(1/2+\mu+1/4\nu)}{i\pi[\Gamma(1/2\nu+1)]^2\, y^{3/2}}$ $\times M_{\mu,1/4\nu}\left(\dfrac{y^2}{8a}\, e^{1/2\pi i}\right)$ $\times M_{\mu,1/4\nu}\left(\dfrac{y^2}{8a}\, e^{-1/2\pi i}\right)$

8.13. Modified Bessel functions of argument kx

	$f(x)$	$\int_0^\infty f(x)\, J_\nu(xy)(xy)^{1/2}\,dx \qquad y>0$
(1)	$x^{1/2} e^{-\beta x^2} I_\nu(ax)$ $\operatorname{Re}\beta>0, \quad \operatorname{Re}\nu>-1$	$\dfrac{y^{1/2}}{2\beta} \exp\left(\dfrac{a^2-y^2}{4\beta}\right) J_\nu\left(\dfrac{ay}{2\beta}\right)$
(2)	$x^{1/2} K_\nu(ax)$ $\operatorname{Re} a>0, \quad \operatorname{Re}\nu>-1$	$\dfrac{y^{\nu+1/2}}{a^\nu (y^2+a^2)}$
(3)	$x^{\mu+\nu+1/2} K_\mu(ax)$ $\operatorname{Re} a>0, \quad \operatorname{Re}(\nu+1)>\lvert\operatorname{Re}\mu\rvert$	$\dfrac{2^{\nu+\mu}\Gamma(\mu+\nu+1)\, y^{\nu+1/2}}{a^{-\mu}(y^2+a^2)^{\mu+\nu+1}}$
(4)	$x^{-\lambda-1/2} K_\mu(ax) \qquad \operatorname{Re} a>0$ $\operatorname{Re}(\nu-\lambda+1)>\lvert\operatorname{Re}\mu\rvert$	$\dfrac{\Gamma[\frac12(\nu-\lambda+\mu+1)]\,\Gamma[\frac12(\nu-\lambda-\mu+1)]}{2^{\lambda+1} a^{\nu-\lambda+1}\Gamma(\nu+1)\, y^{-\nu-1/2}}$ $\times\, {}_2F_1\left(\dfrac{\nu-\lambda+\mu+1}{2}, \dfrac{\nu-\lambda-\mu+1}{2}; \nu+1; -\dfrac{y^2}{a^2}\right)$
(5)	$x^\lambda K_\mu(ax) \begin{matrix}\cos(\beta x)\\ \sin(\beta x)\end{matrix}$	see under Fourier transforms
(6)	$x^{1/2} K_0(ax)\, J_\nu(\beta x)$ $\operatorname{Re}\nu>-1, \quad \operatorname{Re} a>\lvert\operatorname{Im}\beta\rvert$	$y^{1/2} r_1^{-1} r_2^{-1} (r_2-r_1)^\nu (r_2+r_1)^{-\nu}$ $r_1=[a^2+(\beta-y)^2]^{1/2}$ $r_2=[a^2+(\beta+y)^2]^{1/2}$
(7)	$x^{\nu+1/2} J_\nu(\tfrac12 ax)\, K_\nu(\tfrac12 ax)$ $\lvert\arg a\rvert<\pi/4, \quad \operatorname{Re}\nu>-\tfrac12$	$\dfrac{a^{2\nu}\, 2^\nu\, \Gamma(\nu+\frac12)\, y^{\nu+1/2}}{\pi^{1/2}(y^4+a^4)^{\nu+1/2}}$
(8)	$x^{\nu+1/2} J_\nu(ax)\, K_\nu(\beta x)$ $\operatorname{Re}\beta>\lvert\operatorname{Im} a\rvert, \quad \operatorname{Re}\nu>-\tfrac12$	$\dfrac{2^{3\nu}(a\beta)^\nu y^{\nu+1/2}\,\Gamma(\nu+\frac12)}{\pi^{1/2}[(a^2+\beta^2+y^2)^2-4a^2y^2]^{\nu+1/2}}$

Modified Bessel functions of kx (cont'd)

	$f(x)$	$\int_0^\infty f(x)\, J_\nu(xy)(xy)^{1/2}\, dx \qquad y>0$
(9)	$x^{\nu+1/2} J_{\nu-1}(ax) K_{\nu-1}(ax)$ $\lvert\arg a\rvert < \pi/4, \quad 0 < \mathrm{Re}\,\nu < 1/2$	$\dfrac{2^{3\nu-1} a^{2\nu-2}\Gamma(\nu+1/2)\, y^{\nu+5/2}}{\pi^{1/2}(y^4+a^4)^{\nu+1/2}}$
(10)	$x^{1/2} J_\mu(xa\sin\phi)$ $\times K_{\nu-\mu}(xa\cos\phi\cos\psi)$ $a>0, \quad 0<\phi,\ \psi<1/2\pi$ $\mathrm{Re}\,\mu>-1, \quad \mathrm{Re}\,\nu>-1$	$\dfrac{(\sin\phi)^\mu(\sin\psi)^{\nu+1/2}(\cos\phi)^{\nu-\mu}(\cos\psi)^{\mu-\nu}}{a^{3/2}(1-\sin^2\phi\sin^2\psi)}$ $y = a\sin\psi$
(11)	$x^{\nu+1/2} J_\mu(xa\sin\psi)$ $\times K_\mu(xa\cos\phi\cos\psi)$ $a>0, \quad 0<\phi,\ \psi<1/2\pi$ $\mathrm{Re}\,\nu>-1, \quad \mathrm{Re}(\mu+\nu)>-1$	$\dfrac{2^\nu\Gamma(\mu+\nu+1)[\sin\phi\cos^2(1/2\,a)]^{\nu+1/2}}{a^{\nu+3/2}(\cos\psi)^{2\nu+2}}$ $\times P_\nu^{-\mu}(\cos a) \qquad y = a\sin\phi$ $\tan(1/2\,a) = \tan\psi\cos\phi$
(12)	$x^{\mu+1/2} J_\nu(\beta x) K_\mu(ax)$ $\mathrm{Re}\,a > \lvert\mathrm{Im}\,\beta\rvert$ $\mathrm{Re}\,\nu>-1, \quad \mathrm{Re}(\mu+\nu)>-1$	$(2\pi)^{-1/2} a^\mu \beta^{-\mu-1} y^{-\mu-1/2} e^{-(\mu+1/2)\pi i}$ $\times(u^2-1)^{-1/2\mu-1/4}\, Q_{\nu-1/2}^{\mu+1/2}(u)$ $2\beta y u = a^2+\beta^2+y^2$
(13)	$x^{-1/2} J_\mu(xa\sin\phi)$ $\times K_\rho(xa\cos\phi\cos\psi)$ $a>0, \quad 0<\phi,\ \psi<1/2\pi$ $\mathrm{Re}(\mu+\nu+1) > \mathrm{Re}\,\rho$	$\dfrac{(\sin\phi)^\mu(\sin\psi)^{\nu+1/2}}{2a^{1/2}(\cos\phi\cos\psi)^\rho}$ $\times\dfrac{\Gamma[1/2(1+\mu+\nu-\rho)]\Gamma[1/2(1+\mu+\nu+\rho)]}{\Gamma(1+\mu)\Gamma(1+\nu)}$ $\times {}_2F_1\left(\dfrac{1+\mu+\nu-\rho}{2}, \dfrac{1+\mu-\nu-\rho}{2}; \mu+1; \sin^2\phi\right)$ $\times {}_2F_1\left(\dfrac{1+\mu+\nu-\rho}{2}, \dfrac{1+\nu-\mu-\rho}{2}; \nu+1; \sin^2\psi\right)$ $y = a\sin\psi$

Modified Bessel functions of kx (cont'd)

	$f(x)$	$\int_0^\infty f(x)\, J_\nu(xy)(xy)^{1/2}\, dx \qquad y > 0$
(14)	$x^{\rho+\nu-\mu+1/2} J_\mu(ax) K_\rho(\beta x)$ $\operatorname{Re}\beta > \lvert\operatorname{Im} a\rvert, \quad \operatorname{Re}\nu > -1$ $\operatorname{Re}\rho > -1, \quad \operatorname{Re}\mu > -1$ $\operatorname{Re}(\rho+\nu) > -1$	$2^{\rho+\nu-\mu-1}[\Gamma(\mu+1)]^{-1}\Gamma(\rho+\nu+1)$ $\times\Gamma(\rho+1)\Gamma(\nu+1)\, a^{\mu-\rho-\nu-2} y^{1/2}$ $\times(\cosh\sigma - \cos\theta)\, \mathrm{P}^{-\rho}_{\rho+\nu-\mu}(\cos\theta)$ $\times P^{-\nu}_{\rho+\nu-\mu}(\cosh\sigma)$ $y + i\beta = i a \operatorname{ctn}[\tfrac{1}{2}(\theta + i\sigma)]$
(15)	$x^\lambda J_\mu(ax) K_\rho(\beta x)$	see Bailey, W. N., 1936: *Proc. London Math. Soc.* (2), 40, 37-48.
(16)	$x^{1/2} I_{\frac{1}{2}\nu}(ax) K_{\frac{1}{2}\nu}(ax)$ $\operatorname{Re} a > 0, \quad \operatorname{Re}\nu > -1$	$y^{-1/2}(y^2 + 4a^2)^{-1/2}$
(17)	$x^{\nu+1/2} I_\nu(\tfrac{1}{2}ax) K_\nu(\tfrac{1}{2}ax)$ $\operatorname{Re} a > 0, \quad \lvert\operatorname{Re}\nu\rvert < \tfrac{1}{2}$	$\dfrac{2^\nu a^{2\nu}\Gamma(\nu+\frac{1}{2})}{\pi^{1/2}(y^3 + a^2 y)^{\nu+1/2}}$
(18)	$x^{\nu+1/2} I_\nu(ax) K_\nu(\beta x)$ $\operatorname{Re}\beta > \operatorname{Re} a, \quad \operatorname{Re}\nu > -\tfrac{1}{2}$	$\dfrac{2^{3\nu}(a\beta)^\nu y^{\nu+1/2}\Gamma(\nu+\frac{1}{2})}{\pi^{1/2}[(\beta^2 - a^2 + y^2)^2 + 4a^2y^2]^{\nu+1/2}}$
(19)	$x^{\nu-1/2} I_{\nu-1/2}(\tfrac{1}{2}ax) K_{\nu-1/2}(\tfrac{1}{2}ax)$ $\operatorname{Re} a > 0, \quad 0 < \operatorname{Re}\nu < 3/2$	$\dfrac{\Gamma(\nu)\,(2a)^{\nu-1}}{y^{\nu-1/2}}\, P_{-\nu}\left[\dfrac{2a^2 + y^2}{2a(a^2+y^2)^{1/2}}\right]$
(20)	$x^{-1/2} I_\mu(\tfrac{1}{2}ax) K_\mu(\tfrac{1}{2}ax)$ $\operatorname{Re} a > 0, \quad \operatorname{Re}\nu > -1$ $\operatorname{Re}(\nu + 2\mu) > -1$	$\dfrac{e^{\mu\pi i}\Gamma(\frac{1}{2}\nu + \mu + \frac{1}{2})}{\Gamma(\frac{1}{2}\nu - \mu + \frac{1}{2})\, y^{1/2}}$ $\times P^{-\mu}_{\frac{1}{2}\nu-\frac{1}{2}}[(1 + a^2/y^2)^{1/2}]$ $\times Q^{-\mu}_{\frac{1}{2}\nu-\frac{1}{2}}[(1 + a^2/y^2)^{1/2}]$
(21)	$x^{\mu+1/2} I_\nu(\tfrac{1}{2}ax) K_\mu(\tfrac{1}{2}ax)$ $\operatorname{Re} a > 0, \quad \operatorname{Re}\nu > -1$ $-\operatorname{Re}\nu - 1 < \operatorname{Re}\mu < \tfrac{1}{2}$	$(\tfrac{1}{2}\pi)^{-1/2} a^{-1} y^{-\mu-1/2} e^{-(\mu - \frac{1}{2}\nu + \frac{1}{4})\pi i}$ $\times(1 + y^2/a^2)^{-\frac{1}{2}\mu - \frac{1}{4}}\, Q^{\mu+1/2}_{\nu-1/2}(iy/a)$

Modified Bessel functions of kx (cont'd)

	$f(x)$	$\int_0^\infty f(x)\, J_\nu(xy)(xy)^{1/2}\,dx \qquad y>0$
(22)	$x^{\mu+1/2}\, I_\nu(ax)\, K_\mu(\beta x)$ $\operatorname{Re}\beta > \lvert\operatorname{Re} a\rvert$ $\operatorname{Re}\nu > -1, \quad \operatorname{Re}(\mu+\nu) > -1$	$(2\pi)^{-1/2} a^{-\mu-1} \beta^{\mu} y^{-\mu-1/2} e^{-(\mu-1/2\nu+1/4)\pi i}$ $\times (v^2+1)^{-1/2\mu-1/4}\, Q^{\mu+1/2}_{\nu-1/2}(iv)$ $2ayv = \beta^2 - a^2 + y^2$
(23)	$x^{1/2}\, I_{1/2(\nu-\mu)}(\tfrac{1}{2}ax) K_{1/2(\nu+\mu)}(\tfrac{1}{2}ax)$ $\operatorname{Re} a > 0, \quad \operatorname{Re}\nu > -1$ $\operatorname{Re}(\nu-\mu) > -2$	$a^{-\mu} y^{-1/2} (y^2+a^2)^{-1/2} [y + (y^2+a^2)^{1/2}]^{\mu}$
(24)	$x^{\nu+1/2}\, I_\mu(\beta x)\, K_\mu(ax)$ $\operatorname{Re} a > \lvert\operatorname{Re}\beta\rvert, \quad \operatorname{Re}\nu > -1$ $\operatorname{Re}(\mu+\nu) > -1$	$(2\pi)^{-1/2} (a\beta)^{-\nu-1} y^{\nu+1/2} e^{-(\nu+1/2)\pi i}$ $\times (u^2-1)^{-1/2\nu-1/4}\, Q^{\nu+1/2}_{\mu-1/2}(u)$ $2a\beta u = a^2 + \beta^2 + y^2$
(25)	$x^{\lambda}\, I_\mu(ax)\, K_\rho(\beta x)$	see Bailey, W. N., 1936: *Proc. London Math. Soc.* (2) 40, 37-48.
(26)	$x^{-\nu-1/2} [K_{\nu+1/2}(\tfrac{1}{2}ax)]^2$ $\operatorname{Re} a > 0, \quad -1 < \operatorname{Re}\nu < 0$	$\pi^{1/2} (2a)^{-\nu-1} \Gamma(-\nu) y^{\nu+1/2} (a^2+y^2)^{1/2\nu}$ $\times P_\nu \left[\dfrac{2a^2+y^2}{2a(a^2+y^2)^{1/2}}\right]$
(27)	$x^{1/2} [K_\mu(\tfrac{1}{2}ax)]^2 \qquad \operatorname{Re} a > 0$ $\operatorname{Re}(\tfrac{1}{2}\nu \pm \mu) > -1$	$\dfrac{e^{2\mu\pi i} y^{1/2}\, \Gamma(1+\tfrac{1}{2}\nu+\mu)}{(y^2+a^2)^{1/2}\, \Gamma(\tfrac{1}{2}\nu-\mu)}$ $\times Q^{-\mu}_{1/2\nu}[(1+a^2/y^2)^{1/2}]$ $\times Q^{-\mu}_{1/2\nu-1}[(1+a^2/y^2)^{1/2}]$
(28)	$x^{-1/2} [K_\mu(\tfrac{1}{2}ax)]^2 \qquad \operatorname{Re} a > 0$ $\operatorname{Re}(\tfrac{1}{2}\nu \pm \mu) > -\tfrac{1}{2}$	$\dfrac{e^{2\mu\pi i}\, \Gamma(\tfrac{1}{2}+\tfrac{1}{2}\nu+\mu)}{\Gamma(\tfrac{1}{2}+\tfrac{1}{2}\nu-\mu)\, y^{1/2}}$ $\times \{Q^{-\mu}_{1/2\nu-1/2}[(1+a^2/y^2)^{1/2}]\}^2$

Modified Bessel functions of kx (cont'd)

	$f(x)$	$\int_0^\infty f(x)\, J_\nu(xy)(xy)^{1/2}\, dx \qquad y>0$
(29)	$x^{1/2} K_{\mu-1/2}(\tfrac{1}{2}ax)\, K_{\mu+1/2}(\tfrac{1}{2}ax)$ $\operatorname{Re} a>0, \quad \operatorname{Re}\nu>-1$ $\lvert\operatorname{Re}\mu\rvert<1+\tfrac{1}{2}\operatorname{Re}\nu$	$\dfrac{e^{2\mu\pi i}\,\Gamma(\tfrac{1}{2}\nu+\mu+1)\,y^{1/2}}{\Gamma(\tfrac{1}{2}\nu-\mu)\,(y^2+a^2)^{1/2}}$ $\times Q^{-\mu+1/2}_{1/2\nu-1/2}[(1+a^2/y^2)^{1/2}]$ $\times Q^{-\mu-1/2}_{1/2\nu-1/2}[(1+a^2/y^2)^{1/2}]$
(30)	$x^{\nu+1/2} K_\mu(\alpha x)\, K_\mu(\beta x)$ $\operatorname{Re}\alpha>0, \quad \operatorname{Re}\beta>0$ $\operatorname{Re}(\nu\pm\mu)>-1, \quad \operatorname{Re}\nu>-1$	$\dfrac{\pi^{1/2}\,y^{\nu+1/2}\,\Gamma(\nu+\mu+1)\,\Gamma(\nu-\mu+1)}{2^{3/2}\,(\alpha\beta)^{\nu+1}\,(u^2-1)^{\nu/2+1/4}}$ $\times P^{-\nu-1/2}_{\mu-1/2}(u)$ $2\alpha\beta u=y^2+\beta^2+\alpha^2$
(31)	$x^\lambda K_\mu(\alpha x)\, K_\rho(\beta x)$	see Bailey, W. N., 1936: *Proc. London Math. Soc.* (2) 41, 215-220.

8.14. Modified Bessel functions of other arguments

(1)	$x^{1/2-\nu}\exp(-\tfrac{1}{4}a^2x^2)$ $\times I_\nu(\tfrac{1}{4}a^2x^2)$ $\lvert\arg a\rvert<\tfrac{1}{4}\pi, \quad \operatorname{Re}\nu>-\tfrac{1}{2}$	$(\tfrac{1}{2}\pi)^{-1/2}\,a^{-1}\,y^{\nu-1/2}\exp\left(-\dfrac{y^2}{4a^2}\right)$ $\times D_{-2\nu}\left(\dfrac{y}{a}\right)$
(2)	$x^{-\nu-3/2}\exp(-\tfrac{1}{4}a^2x^2)$ $\times I_{\nu+1}(\tfrac{1}{4}a^2x^2)$ $\lvert\arg a\rvert<\tfrac{1}{4}\pi, \quad \operatorname{Re}\nu>-1$	$(\tfrac{1}{2}\pi)^{-1/2}\,y^{\nu+1/2}\exp\left(-\dfrac{y^2}{4a^2}\right)$ $\times D_{-2\nu-3}\left(\dfrac{y}{a}\right)$
(3)	$x^{1/2}\exp(-\tfrac{1}{4}ax^2)\, I_{1/2\nu}(\tfrac{1}{4}ax^2)$ $\operatorname{Re} a>0, \quad \operatorname{Re}\nu>-1$	$(\tfrac{1}{2}\pi a y)^{-1/2}\exp\left(-\dfrac{y^2}{2a}\right)$

Modified Bessel functions (cont'd)

	$f(x)$	$\int_0^\infty f(x)\, J_\nu(xy)(xy)^{1/2}\, dx \qquad y>0$
(4)	$x^{\nu/3+1/6}\exp(-\tfrac{1}{4}ax^2)$ $\times I_{\nu/3+1/6}(\tfrac{1}{4}ax^2)$ $\mathrm{Re}\, a>0,\quad -1<\mathrm{Re}\,\nu<5/2$	$\pi^{-1}a^{-\nu/3-2/3}y^{\nu/3+1/6}\exp\left(\dfrac{y^2}{4a}\right)$ $\times K_{\nu/3+1/6}\left(\dfrac{y^2}{4a}\right)$
(5)	$x^{1/6-\nu/3}\exp(-\tfrac{1}{4}ax^2)$ $\times I_{\nu/3-1/6}(\tfrac{1}{4}ax^2)$ $\mathrm{Re}\, a>0,\quad \mathrm{Re}\,\nu>-1$	$a^{\nu/3-2/3}y^{1/6-\nu/3}\exp\left(-\dfrac{y^2}{4a}\right)$ $\times I_{\nu/3-1/6}\left(\dfrac{y^2}{4a}\right)$
(6)	$x^{1/2+2\mu-\nu}\exp(-\tfrac{1}{4}ax^2)$ $\times I_\mu(\tfrac{1}{4}ax^2)\qquad \mathrm{Re}\, a>0$ $\mathrm{Re}\,\nu>2\,\mathrm{Re}\,\mu+\tfrac{1}{2}>-\tfrac{1}{2}$	$2^{2\mu-\nu+1/2}(\pi a)^{-1/2}\,\Gamma(\tfrac{1}{2}+\mu)$ $\times[\Gamma(\tfrac{1}{2}-\mu+\nu)]^{-1}y^{\nu-2\mu-1/2}$ $\times\, {}_1F_1\left(\tfrac{1}{2}+\mu;\tfrac{1}{2}-\mu+\nu;-\dfrac{y^2}{2a}\right)$
(7)	$x^{1/2+\nu-2\mu}\exp(-\tfrac{1}{4}a^2x^2)$ $\times I_\mu(\tfrac{1}{4}a^2x^2)\qquad \lvert\arg a\rvert<\tfrac{1}{4}\pi$ $-1<\mathrm{Re}\,\nu<2\,\mathrm{Re}\,\mu+\tfrac{1}{2}$	$\pi^{-1/2}2^{1/4(3+2\nu-6\mu)}a^{-1/2-\nu+\mu}y^{\mu-1}$ $\times\exp\left(-\dfrac{y^2}{4a^2}\right)W_{k,m}\left(\dfrac{y^2}{2a^2}\right)$ $2k=\tfrac{1}{2}+\nu-3\mu,\quad 2m=-\tfrac{1}{2}+\mu-\nu$
(8)	$x^\lambda\exp(-\tfrac{1}{4}a^2x^2)\,I_\mu(\tfrac{1}{4}a^2x^2)$ $\lvert\arg a\rvert<\tfrac{1}{4}\pi$ $-3/2-\mathrm{Re}(2\mu+\nu)<\mathrm{Re}\,\lambda<0$	$(2\pi)^{-1/2}\left(\dfrac{2}{y}\right)^{\lambda+1}G^{21}_{23}\left(\dfrac{y^2}{2a^2}\middle\vert\begin{matrix}1-\mu,1+\mu\\ h,\tfrac{1}{2},k\end{matrix}\right)$ $h=\tfrac{3}{4}+\tfrac{1}{2}\lambda+\tfrac{1}{2}\nu$ $k=\tfrac{3}{4}+\tfrac{1}{2}\lambda-\tfrac{1}{2}\nu$
(9)	$x^{1/2}K_{1/2\nu}(\tfrac{1}{4}ax^2)$ $\mathrm{Re}\, a>0,\quad \mathrm{Re}\,\nu>-1$	$\pi a^{-1}y^{1/2}[I_{1/2\nu}(y^2/a)-\mathbf{L}_{1/2\nu}(y^2/a)]$
(10)	$x^{3/2}K_{1/2\nu+1/2}(\tfrac{1}{4}ax^2)$ $\mathrm{Re}\, a>0,\quad \mathrm{Re}\,\nu>-1$	$2\pi a^{-2}y^{3/2}[I_{1/2\nu-1/2}(y^2/a)$ $-\mathbf{L}_{1/2\nu-1/2}(y^2/a)]$

Modified Bessel functions (cont'd)

	$f(x)$	$\int_0^\infty f(x)\, J_\nu(xy)(xy)^{1/2}\, dx \qquad y>0$
(11)	$x^{\nu/3+1/6}\exp(-\frac{1}{4}ax^2)$ $\times K_{\nu/3+1/6}(\frac{1}{4}ax^2)$ $\mathrm{Re}\, a>0,\quad \mathrm{Re}\,\nu>-1$	$\pi a^{-\nu/3-2/3}y^{\nu/3+1/6}\exp\left(-\frac{y^2}{4a}\right)$ $\times I_{\nu/3+1/6}\left(\frac{y^2}{4a}\right)$
(12)	$x^{\nu/3+1/6}\exp(\frac{1}{4}ax^2)$ $\times K_{\nu/3+1/6}(\frac{1}{4}ax^2)$ $-1<\mathrm{Re}\,\nu<5/2$	$a^{-\nu/3-2/3}y^{\nu/3+1/6}\exp\left(\frac{y^2}{4a}\right)$ $\times K_{\nu/3+1/6}\left(\frac{y^2}{4a}\right)$
(13)	$x^{2\mu+\nu+1/2}\exp(-\frac{1}{4}a^2x^2)$ $\times K_\mu(\frac{1}{4}a^2x^2)\qquad \lvert\arg a\rvert<\frac{1}{4}\pi$ $\mathrm{Re}\,\nu>-1,\quad \mathrm{Re}(2\mu+\nu)>-1$	$\pi^{1/2}2^\mu a^{-2\mu-2\nu-2}y^{\nu+1/2}\Gamma(1+2\mu+\nu)$ $[\Gamma(\mu+\nu+3/2)]^{-1}$ $\times {}_1F_1\left(1+2\mu+\nu;\mu+\nu+\frac{3}{2};-\frac{y^2}{2a^2}\right)$
(14)	$x^{2\mu+\nu+1/2}\exp(\frac{1}{4}a^2x^2)$ $\times K_\mu(\frac{1}{4}a^2x^2)$ $\mathrm{Re}\,a>0,\quad \mathrm{Re}\,\nu>-1$ $-1<\mathrm{Re}(2\mu+\nu)<-\frac{1}{2}$	$\pi^{1/2}\Gamma(1+2\mu+\nu)[\Gamma(\frac{1}{2}-\mu)]^{-1}2^{3/2-k}$ $\times a^{-2m}y^{-\mu-1}\exp\left(\frac{y^2}{4a^2}\right)\, W_{k,m}\left(\frac{y^2}{2a^2}\right)$ $2k=-\frac{1}{2}-3\mu-\nu,\quad 2m=\frac{1}{2}+\mu+\nu$
(15)	$x^\lambda\exp(-\frac{1}{4}a^2x^2)K_\mu(\frac{1}{4}a^2x^2)$ $\lvert\arg a\rvert<\frac{1}{4}\pi$ $\mathrm{Re}(\lambda+\nu\pm2\mu)>-3/2$	$(\frac{1}{2}\pi)^{1/2}\left(\frac{2}{y}\right)^{\lambda+1}G^{12}_{23}\left(\frac{y^2}{2a^2}\middle\vert\begin{matrix}1-\mu,1+\mu\\ h,\frac{1}{2},k\end{matrix}\right)$ $h=\frac{3}{4}+\frac{1}{2}\lambda+\frac{1}{2}\nu,\quad k=\frac{3}{4}+\frac{1}{2}\lambda-\frac{1}{2}\nu$
(16)	$x^\lambda\exp(\frac{1}{4}a^2x^2)K_\mu(\frac{1}{4}a^2x^2)$ $\mathrm{Re}\,a>0$ $-3/2-\mathrm{Re}(\nu\pm2\mu)<\mathrm{Re}\,\lambda<0$	$(2\pi)^{-1/2}\cos(\mu\pi)(2/y)^{\lambda+1}$ $\times G^{22}_{23}\left(\frac{y^2}{2a^2}\middle\vert\begin{matrix}1-\mu,1+\mu\\ h,\frac{1}{2},k\end{matrix}\right)$ $h=\frac{3}{4}+\frac{1}{2}\lambda+\frac{1}{2}\nu$ $k=\frac{3}{4}+\frac{1}{2}\lambda-\frac{1}{2}\nu$

Modified Bessel functions (cont'd)

	$f(x)$	$\int_0^\infty f(x)\, J_\nu(xy)(xy)^{1/2}\, dx \qquad y>0$
(17)	$x^{1/2} I_{\frac14\nu}(\frac14 ax^2) K_{\frac14\nu}(\frac14 ax^2)$ $\operatorname{Re} a>0, \quad \operatorname{Re}\nu>-1$	$\dfrac{y^{1/2}}{a} I_{\frac14\nu}\left(\dfrac{y^2}{4a}\right) K_{\frac14\nu}\left(\dfrac{y^2}{4a}\right)$
(18)	$x^{1/2} I_{\frac14(\nu-\mu)}(\frac12 ax^2)$ $\times K_{\frac14(\nu+\mu)}(\frac12 ax^2) \qquad \operatorname{Re} a>0$ $\operatorname{Re}\nu>-1, \quad \operatorname{Re}(\nu-\mu)>-2$	$\dfrac{2\Gamma(\frac12+\frac14\nu-\frac14\mu)}{\Gamma(1+\frac12\nu)\, y^{3/2}} W_{\frac14\mu,\frac14\nu}\left(\dfrac{y^2}{4a}\right)$ $\times M_{-\frac14\mu,\frac14\nu}\left(\dfrac{y^2}{4a}\right)$
(19)	$x^{-5/2} K_\nu(ax^{-1})$ $\operatorname{Re} a>0, \quad \lvert\operatorname{Re}\nu\rvert<5/2$	$i a^{-1} y^{1/2} [e^{\frac12\nu\pi i} K_{2\nu}(2a^{1/2} e^{\frac14\pi i} y^{1/2})$ $-e^{-\frac12\nu\pi i} K_{2\nu}(2a^{1/2} e^{-\frac14\pi i} y^{1/2})]$
(20)	$x^{-2\nu-2} K_{\frac12-\nu}(ax^{-1})$ $\operatorname{Re} a>0, \quad -\frac12<\operatorname{Re}\nu<2$	$(2\pi)^{1/2} a^{-\nu-\frac12} y^{\nu+\frac12} K_{2\nu}(2^{1/2} a^{1/2} y^{1/2})$ $\times J_{2\nu}(2^{1/2} a^{1/2} y^{1/2})$
(21)	$K_{-2\nu-1}(2ax^{1/2})$ $\operatorname{Re} a>0, \quad \operatorname{Re}\nu>-1$	$-\frac14 \pi a \sec(\nu\pi)\, y^{-3/2} [\mathbf{H}_{-\nu-1}(a^2/y)$ $- Y_{-\nu-1}(a^2/y)]$
(22)	$x^{-1/2} K_{2\nu}(2ax^{1/2})$ $\operatorname{Re} a>0, \quad \operatorname{Re}\nu>-\frac12$	$\frac14\pi \sec(\nu\pi)\, y^{-1/2} [\mathbf{H}_{-\nu}(a^2/y)$ $- Y_{-\nu}(a^2/y)]$
(23)	$x^{1/2} J_\nu(2a^{1/2} x^{1/2}) K_\nu(2a^{1/2} x^{1/2})$ $\operatorname{Re} a>0, \quad \operatorname{Re}\nu>-1$	$\frac12 y^{-3/2} e^{-2a/y}$
(24)	$x^{\nu+\frac12} J_{2\nu}(2a^{1/2} x^{1/2}) K_{2\nu}(2a^{1/2} x^{1/2})$ $\operatorname{Re} a>0, \quad \operatorname{Re}\nu>-\frac12$	$\pi^{-1/2} 2^\nu a^{\nu+\frac12} y^{-2\nu-2} K_{\frac12-\nu}(2a/y)$
(25)	$x^{-\nu-\frac12} J_{2\nu+1}(2a^{1/2} x^{1/2})$ $\times K_{2\nu+1}(2a^{1/2} x^{1/2})$ $\operatorname{Re} a>0, \quad \operatorname{Re}\nu>-1$	$\pi^{1/2} 2^{-\nu-2} a^{-\nu-\frac12} y^{2\nu} [I_{\nu+\frac12}(2a/y)$ $- \mathbf{L}_{\nu+\frac12}(2a/y)]$

Modified Bessel functions (cont'd)

	$f(x)$	$\int_0^\infty f(x)\, J_\nu(xy)(xy)^{1/2}\,dx \qquad y>0$
(26)	$x^{-1/2} J_\mu(2a^{1/2}x^{1/2}) K_\mu(2a^{1/2}x^{1/2})$ $\operatorname{Re} a>0, \quad \operatorname{Re}\nu>-1$ $\operatorname{Re}(\nu+\mu)>-1$	$\dfrac{\Gamma(\frac{1}{2}+\frac{1}{2}\mu+\frac{1}{2}\nu)\,y^{1/2}}{4a\,\Gamma(1+\mu)} W_{-\frac{1}{2}\nu,\frac{1}{2}\mu}\left(\dfrac{2a}{y}\right)$ $\times M_{\frac{1}{2}\nu,\frac{1}{2}\mu}\left(\dfrac{2a}{y}\right)$
(27)	$x^{-1/2}[K_{2\nu}(2a^{1/2}x^{1/2})$ $-\frac{1}{2}\pi\, Y_{2\nu}(2a^{1/2}x^{1/2})]$ $a>0, \quad \operatorname{Re}\nu>-\frac{1}{2}$	$-\frac{1}{2}\pi y^{-1/2}\, Y_\nu(a/y)$
(28)	$x^{-1/2} K_\nu(2a^{1/2}x^{1/2})\; Y_\nu(2a^{1/2}x^{1/2})$ $\operatorname{Re} a>0, \quad \operatorname{Re}\nu>-\frac{1}{2}$	$-\frac{1}{4}a^{-1}y^{1/2}\, W_{\frac{1}{2}\nu,\frac{1}{2}\nu}(2a/y)$ $\times W_{-\frac{1}{2}\nu,\frac{1}{2}\nu}(2a/y)$
(29)	$x^{-1/2} K_\mu(2a^{1/2}x^{1/2})$ $\times\{\sin[\frac{1}{2}(\mu-\nu)\pi]\, J_\mu(2a^{1/2}x^{1/2})$ $+\cos[\frac{1}{2}(\mu-\nu)\pi]\, Y_\mu(2a^{1/2}x^{1/2})\}$ $\operatorname{Re} a>0, \quad \operatorname{Re}(\nu\pm\mu)>-1$	$-\frac{1}{4}a^{-1}y^{1/2}\, W_{\frac{1}{2}\nu,\frac{1}{2}\mu}(2a/y)$ $\times W_{-\frac{1}{2}\nu,\frac{1}{2}\mu}(2a/y)$
(30)	$x^{-1/2} K_\mu[(2ax)^{1/2} e^{\frac{1}{4}\pi i}]$ $\times K_\mu[(2ax)^{1/2} e^{-\frac{1}{4}\pi i}]$ $\operatorname{Re} a>0, \quad \operatorname{Re}(\nu\pm\mu)>-1$	$\frac{1}{4}a^{-1}y^{1/2}\,\Gamma\left(\dfrac{1+\mu+\nu}{2}\right)\Gamma\left(\dfrac{1-\mu+\nu}{2}\right)$ $\times W_{-\frac{1}{2}\nu,\frac{1}{2}\mu}(ay^{-1}e^{\frac{1}{2}\pi i})$ $\times W_{-\frac{1}{2}\nu,\frac{1}{2}\mu}(ay^{-1}e^{-\frac{1}{2}\pi i})$
(31)	$x^{-\nu-1/2} K_{2\nu+1}[(2ax)^{1/2} e^{\frac{1}{4}\pi i}]$ $\times K_{2\nu+1}[(2ax)^{1/2} e^{-\frac{1}{4}\pi i}]$ $\operatorname{Re} a>0, \; -1<\operatorname{Re}\nu<0$	$-2^{-5/2}\pi^{3/2}\csc(\nu\pi)\,a^{-\nu-1/2}y^{2\nu}$ $\times[\mathbf{H}_{\nu+1/2}(a/y)-Y_{\nu+1/2}[a/y)]$
(32)	$x^{\nu+1/2}(x^2+\beta^2)^{-\frac{1}{2}\nu-\frac{1}{4}}$ $\times K_{\nu+1/2}[a(x^2+\beta^2)^{1/2}]$ $\operatorname{Re} a>0, \quad \operatorname{Re}\beta>0$ $\operatorname{Re}\nu>-1$	$\pi^{1/2}2^{-1/2}a^{-\nu-1/2}y^{\nu+1/2}(a^2+y^2)^{-1/2}$ $\times e^{-\beta(a^2+y^2)^{1/2}}$

Modified Bessel functions (cont'd)

	$f(x)$	$\int_0^\infty f(x)\, J_\nu(xy)(xy)^{1/2}\, dx \qquad y>0$
(33)	$x^{\nu+1/2}(x^2+\beta^2)^{-1/2\nu-3/4}$ $\times K_{\nu+3/2}[\alpha(x^2+\beta^2)^{1/2}]$ $\mathrm{Re}\,\alpha>0, \quad \mathrm{Re}\,\beta>0$ $\mathrm{Re}\,\nu>-1$	$\pi^{1/2}2^{-1/2}\alpha^{-\nu-3/2}\beta^{-1}y^{\nu+1/2}$ $\times e^{-\beta(\alpha^2+y^2)^{1/2}}$
(34)	$x^{\nu+1/2}(x^2+\alpha^2)^{-1/4(\nu+1)}$ $\times K_{1/2(\nu+1)}[\alpha(x^2+\alpha^2)^{1/2}]$ $\mathrm{Re}\,\alpha>0, \quad \mathrm{Re}\,\nu>-1$	$y^{\nu+1/2}(y^2+\alpha^2)^{-1/4(\nu+1)}$ $\times K_{1/2(\nu+1)}[\alpha(y^2+\alpha^2)^{1/2}]$
(35)	$x^{\nu+1/2}(x^2+\beta^2)^{-1/2\mu}$ $\times K_\mu[\alpha(x^2+\beta^2)^{1/2}] \quad \mathrm{Re}\,\alpha>0$ $\mathrm{Re}\,\beta>0, \quad \mathrm{Re}\,\nu>-1$	$\alpha^{-\mu}\beta^{\nu+1-\mu}y^{\nu+1/2}(\alpha^2+y^2)^{1/2\mu-1/2\nu-1/2}$ $\times K_{\mu-\nu-1}[\beta(\alpha^2+y^2)^{1/2}]$
(36)	$x^{\nu+1/2}(b^2-x^2)^{1/2\mu}Y_\mu[a(b^2-x^2)^{1/2}]$ $0<x<b$ $-2\pi^{-1}x^{\nu+1/2}(x^2-b^2)^{1/2\mu}$ $\times K_\mu[a(x^2-b^2)^{1/2}]$ $b<x<\infty$ $\mathrm{Re}\,a>0, \ \mathrm{Re}\,\nu>-1, \ \mathrm{Re}\,\mu>-1$	$a^\mu b^{\mu+\nu+1}y^{\nu+1/2}(a^2+y^2)^{-1/2(\mu+\nu+1)}$ $\times Y_{\mu+\nu+1}[b(a^2+y^2)^{1/2}]$
(37)	$x^{1/2}\, I_{1/2\nu}\{1/2\beta[(\alpha^2+x^2)^{1/2}-\alpha]\}$ $\times K_{1/2\nu}\{1/2\beta[(\alpha^2+x^2)^{1/2}+\alpha]\}$ $\mathrm{Re}\,\alpha>0, \quad \mathrm{Re}\,\beta>0$ $\mathrm{Re}\,\nu>-1$	$y^{-1/2}(\beta^2+y^2)^{-1/2}\exp[-\alpha(\beta^2+y^2)^{1/2}]$

8.15. Functions related to Bessel functions

(1)	$\mathbf{H}_{\nu-1/2}(ax)$ $a>0, \quad -1<\mathrm{Re}\,\nu<3/2$	$2^{1/2}\pi^{-1/2}a^{\nu-1/2}y^{1/2-\nu}(a^2-y^2)^{-1/2}$ $\quad 0<y<a$ $0 \quad a<y<\infty$

Functions related to Bessel functions (cont'd)

	$f(x)$	$\int_0^\infty f(x)\, J_\nu(xy)(xy)^{1/2}\,dx \qquad y>0$
(2)	$x^\lambda \mathbf{H}_\mu(ax) \qquad a>0$ $-5/2 - \mathrm{Re}\,\nu < \mathrm{Re}(\lambda+\mu) < 0$	$2^{\lambda+1/2} y^{-\lambda-1}$ $\times G_{33}^{21}\left(\frac{y^2}{a^2}\middle\vert \begin{matrix} \frac{1-\mu}{2}, 1-\frac{\mu}{2}, 1+\frac{\mu}{2} \\ \frac{3}{4}+\frac{\lambda+\nu}{2}, \frac{1-\mu}{2}, \frac{3}{4}+\frac{\lambda-\nu}{2} \end{matrix}\right)$
(3)	$x^{1/2}\, \mathbf{H}_{1/2\nu}(\tfrac{1}{4}ax^2)$ $a>0, \quad -2<\mathrm{Re}\,\nu<3/2$	$-2a^{-1}y^{1/2}\, Y_{1/2\nu}(y^2/a)$
(4)	$x^\lambda \mathbf{H}_\mu(a/x)$ $a>0, \quad \mathrm{Re}(\lambda+\nu)>-2$ $-\mathrm{Re}\,\nu - 5/2 < \mathrm{Re}(\lambda-\mu) < 1$	$2^{\lambda+1/2} y^{-\lambda-1}$ $\times G_{15}^{21}\left(\frac{a^2y^2}{16}\middle\vert \begin{matrix} \frac{1+\mu}{2} \\ h, \frac{1+\mu}{2}, \frac{\mu}{2}, -\frac{\mu}{2}, k \end{matrix}\right)$ $h=\frac{3}{4}+\frac{\lambda+\nu}{2}, \qquad k=\frac{3}{4}+\frac{\lambda-\nu}{2}$
(5)	$x^{1/2}[\mathbf{H}_{-\nu}(ax) - Y_{-\nu}(ax)]$ $\lvert\arg a\rvert<\pi, \quad -\tfrac{1}{2}<\mathrm{Re}\,\nu$	$2a^{-\nu}\pi^{-1}\cos(\nu\pi)\,y^{\nu-1/2}(y+a)^{-1}$
(6)	$x^\lambda[\mathbf{H}_\mu(ax) - Y_\mu(ax)]$ $\lvert\arg a\rvert<\pi, \quad \mathrm{Re}(\lambda+\mu)<1$ $\mathrm{Re}(\lambda+\nu)+3/2>\lvert\mathrm{Re}\,\mu\rvert$	$2^{\lambda+1/2}\pi^{-2}\cos(\mu\pi)\,y^{-\lambda-1}$ $\times G_{33}^{23}\left(\frac{y^2}{a^2}\middle\vert \begin{matrix} \frac{1-\mu}{2}, 1-\frac{\mu}{2}, 1+\frac{\mu}{2} \\ \frac{3}{4}+\frac{\lambda+\nu}{2}, \frac{1-\mu}{2}, \frac{3}{4}+\frac{\lambda-\nu}{2} \end{matrix}\right)$
(7)	$x^{-1/2}[\mathbf{H}_{-\nu}(ax^{-1}) - Y_{-\nu}(ax^{-1})]$ $\lvert\arg a\rvert<\pi, \quad \lvert\mathrm{Re}\,\nu\rvert<\tfrac{1}{2}$	$4\pi^{-1}\cos(\nu\pi)\,y^{-1/2}K_{2\nu}(2a^{1/2}y^{1/2})$

Functions related to Bessel functions (cont'd)

	$f(x)$	$\int_0^\infty f(x)\, J_\nu(xy)(xy)^{1/2}\,dx \qquad y>0$
(8)	$x^{-3/2}[\mathbf{H}_{-\nu-1}(ax^{-1}) - Y_{-\nu-1}(ax^{-1})]$ $\lvert\arg a\rvert<\pi, \quad \lvert\operatorname{Re}\nu\rvert<1/2$	$-4\pi^{-1}a^{-1/2}\cos(\nu\pi)\,K_{-2\nu-1}(2a^{1/2}y^{1/2})$
(9)	$x^{2\nu}[\mathbf{H}_{\nu+1/2}(ax^{-1})-Y_{\nu+1/2}(ax^{-1})]$ $\lvert\arg a\rvert<\pi, \quad -1<\operatorname{Re}\nu<-1/6$	$-2^{5/2}\pi^{-3/2}a^{\nu+1/2}y^{-\nu-1/2}\sin(\nu\pi)$ $\times K_{2\nu+1}(2^{1/2}a^{1/2}e^{1/4\pi i}y^{1/2})$ $\times K_{2\nu+1}(2^{1/2}a^{1/2}e^{-1/4\pi i}y^{1/2})$
(10)	$x^{\lambda}[\mathbf{H}_{\mu}(ax^{-1})-Y_{\mu}(ax^{-1})]$ $\lvert\arg a\rvert<\pi, \quad \operatorname{Re}\lambda<-\lvert\operatorname{Re}\mu\rvert$ $\operatorname{Re}(\nu-\mu+\lambda)>-5/2$	$2^{\lambda+1/2}\pi^{-2}\cos(\mu\pi)\,y^{-\lambda-1}$ $\times G^{41}_{15}\left(\frac{a^2y^2}{16}\,\middle\vert\,\begin{matrix}\frac{1+\mu}{2}\\ h,\frac{1+\mu}{2},-\frac{\mu}{2},\frac{\mu}{2},k\end{matrix}\right)$ $h=\frac{3}{4}+\frac{\lambda+\nu}{2}, \quad k=\frac{3}{4}+\frac{\lambda-\nu}{2}$
(11)	$I_{\nu-1/2}(ax)-\mathbf{L}_{\nu-1/2}(ax)$ $\operatorname{Re}a>0, \quad \lvert\operatorname{Re}\nu\rvert<1/2$	$2^{1/2}\pi^{-1/2}a^{\nu-1/2}y^{1/2-\nu}(y^2+a^2)^{-1/2}$
(12)	$x^{1/2}[I_\nu(ax)-\mathbf{L}_\nu(ax)]$ $\operatorname{Re}a>0, \quad -1<\operatorname{Re}\nu<-1/2$	$2\pi^{-1}a^{\nu+1}y^{-\nu-1/2}(y^2+a^2)^{-1}$
(13)	$x^{\mu-\nu+1/2}[I_\mu(ax)-\mathbf{L}_\mu(ax)]$ $\operatorname{Re}a>0$ $-1<2\operatorname{Re}\mu+1<\operatorname{Re}\nu+1/2$	$\dfrac{2^{\mu-\nu+1}a^{\mu-1}y^{\nu-2\mu-1/2}}{\pi^{1/2}\,\Gamma(\nu-\mu+1/2)}$ $\times\,{}_2F_1(1,1/2;\nu-\mu+1/2;-y^2/a^2)$
(14)	$x^{\nu-\mu+1/2}[I_\mu(ax)-\mathbf{L}_\mu(ax)]$ $\operatorname{Re}a>0, \quad -1<\operatorname{Re}\nu<-1/2$	$\dfrac{2^{\nu-\mu+1}\Gamma(3/2+\nu)\,a^{\mu+1}}{\pi\Gamma(3/2+\mu)\,y^{\nu+5/2}}$ $\times\,{}_2F_1(1,3/2+\nu;3/2+\mu;-a^2/y^2)$

Functions related to Bessel functions (cont'd)

	$f(x)$	$\int_0^\infty f(x)\, J_\nu(xy)(xy)^{1/2}\,dx \qquad y>0$
(15)	$x^{\nu-\mu-1/2}[I_\mu(ax)-\mathbf{L}_\mu(ax)]$ $\operatorname{Re} a>0, \quad \lvert\operatorname{Re}\nu\rvert<1/2$	$\dfrac{2^{\nu-\mu}\Gamma(1/2+\nu)\,a^\mu}{\pi^{1/2}\Gamma(1+\mu)\,y^{\nu+1/2}}$ $\times {}_2F_1(1/2+\nu, 1/2; 1+\mu; -a^2/y^2)$
(16)	$x^\lambda[I_\mu(ax)-\mathbf{L}_\mu(ax)] \quad \operatorname{Re} a>0$ $-\operatorname{Re}\nu-3/2<\operatorname{Re}(\lambda+\mu)<0$	$2^{\lambda+1/2}\pi^{-1}y^{-\lambda-1}$ $\times G_{33}^{22}\left(\dfrac{y^2}{a^2}\,\middle\vert\, \begin{matrix} 1-\frac{\mu}{2}, \frac{1-\mu}{2}, 1+\frac{\mu}{2} \\ \frac{3}{4}+\frac{\lambda+\nu}{2}, \frac{1-\mu}{2}, \frac{3}{4}+\frac{\lambda-\nu}{2} \end{matrix}\right)$
(17)	$x^{1/2}[I_\nu(ax)-\mathbf{L}_{-\nu}(ax)]$ $\operatorname{Re} a>0, \quad \operatorname{Re}\nu>1/2$	$2\pi^{-1}a^{1-\nu}y^{\nu-1/2}\cos(\nu\pi)(y^2+a^2)^{-1}$
(18)	$x^{\mu-\nu+1/2}[I_\mu(ax)-\mathbf{L}_{-\mu}(ax)]$ $\operatorname{Re} a>0, \quad \operatorname{Re}\nu>-1/2, \quad \operatorname{Re}\mu>-1$	$\dfrac{2^{\mu-\nu+1}a^{-\mu-1}y^{\nu-1/2}}{\Gamma(1/2-\mu)\Gamma(1/2+\nu)}$ $\times {}_2F_1(1, 1/2+\mu; 1/2+\nu; -y^2/a^2)$
(19)	$x^{\nu-\mu+1/2}[I_\mu(ax)-\mathbf{L}_{-\mu}(ax)]$ $\operatorname{Re} a>0, \quad \operatorname{Re}\nu>-1/2$ $\operatorname{Re}(\nu-\mu)>-1, \quad \operatorname{Re}(\nu-2\mu)<1/2$	$2^{2+\nu-\mu}\pi^{-3/2}\cos(\mu\pi)\Gamma(3/2+\nu-\mu)$ $\times a^{1-\mu}y^{-5/2+2\mu-\nu}$ $\times {}_2F_1(3/2+\nu-\mu, 1; 3/2; -a^2/y^2)$
(20)	$x^{\mu+\nu-1/2}[I_\mu(ax)-\mathbf{L}_{-\mu}(ax)]$ $\operatorname{Re} a>0, \quad -1<\operatorname{Re}\nu<3/2$ $\operatorname{Re}(\mu+\nu)>-1/2$	$\dfrac{2^{\mu+\nu}\Gamma(1/2+\mu+\nu)}{\Gamma(1+\mu)\Gamma(1/2-\mu)}\,a^\mu y^{-1/2-2\mu-2\nu}$ $\times {}_2F_1(1/2+\mu+\nu, 1/2+\mu; 1+\mu; a^2/y^2)$

Functions related to Bessel functions (cont'd)

	$f(x)$	$\int_0^\infty f(x)\, J_\nu(xy)(xy)^{1/2}\,dx \qquad y>0$
(21)	$x^\lambda[I_\mu(ax)-\mathbf{L}_{-\mu}(ax)]$ $\mathrm{Re}\,a>0,\quad \mathrm{Re}(\mu+\nu+\lambda)>-3/2$ $-\mathrm{Re}\,\nu-5/2<\mathrm{Re}(\lambda-\mu)<1$	$2^{\lambda+1/2}\pi^{-1}\cos(\mu\pi)\,y^{-\lambda-1}$ $\times G_{33}^{22}\left(\frac{y^2}{a^2}\middle\vert \begin{matrix}\frac{1+\mu}{2},\, 1-\frac{\mu}{2},\, 1+\frac{\mu}{2}\\ \frac{3}{4}+\frac{\lambda+\nu}{2},\, \frac{1+\mu}{2},\, \frac{3}{4}+\frac{\lambda-\nu}{2}\end{matrix}\right)$
(22)	$x^{2\nu}[I_{\nu+1/2}(ax^{-1})-\mathbf{L}_{\nu+1/2}(ax^{-1})]$ $\mathrm{Re}\,a>0,\quad -1<\mathrm{Re}\,\nu<1/2$	$2^{3/2}\pi^{-1/2}y^{-\nu-1/2}a^{\nu+1/2}$ $\times J_{2\nu+1}[(2ay)^{1/2}]K_{2\nu+1}[(2ay)^{1/2}]$
(23)	$x^{\nu+1/2}\,U_{\nu+1}(2a^2\beta,\, ax)$ $a>0,\quad \mathrm{Re}\,\nu>-1$	$(2\beta)^{\nu+1}y^{\nu+1/2}\cos[\beta(a^2-y^2)]$ $\qquad 0<y<a$ 0 $\qquad a<y<\infty$
(24)	$x^{\nu+1/2}\,U_{\nu+2}(2a^2\beta,\, ax)$ $a>0,\quad \mathrm{Re}\,\nu>-1$	$(2\beta)^{\nu+1}y^{\nu+1/2}\sin[\beta(a^2-y^2)]$ $\qquad 0<y<a$ 0 $\qquad a<y<\infty$

8.16. Parabolic cylinder functions

(1)	$x^{\nu-1/2}\exp(-\frac{1}{4}x^2)\,D_{2\nu-1}(x)$ $\mathrm{Re}\,\nu>-1/2$	$-\frac{1}{2}\sec(\nu\pi)\,y^{\nu-1/2}\exp(-\frac{1}{4}y^2)$ $\times[D_{2\nu-1}(y)-D_{2\nu-1}(-y)]$
(2)	$x^{\nu-1/2}\exp(-\frac{1}{4}x^2)$ $\times\{[1-2\cos(\nu\pi)]\,D_{2\nu-1}(x)$ $-D_{2\nu-1}(-x)\}\qquad \mathrm{Re}\,\nu>-1/2$	$y^{\nu-1/2}\exp(-\frac{1}{4}y^2)$ $\times\{[1-2\cos(\nu\pi)]\,D_{2\nu-1}(y)$ $-D_{2\nu-1}(-y)\}$
(3)	$x^{\nu-1/2}\exp(-\frac{1}{4}x^2)$ $\times\{[1+2\cos(\nu\pi)]\,D_{2\nu-1}(x)$ $-D_{2\nu-1}(-x)\}\qquad \mathrm{Re}\,\nu>-1/2$	$-y^{\nu-1/2}\exp(-\frac{1}{4}y^2)$ $\times\{[1+2\cos(\nu\pi)]\,D_{2\nu-1}(y)$ $-D_{2\nu-1}(-y)\}$

Parabolic cylinder functions (cont'd)

	$f(x)$	$\int_0^\infty f(x)\, J_\nu(xy)(xy)^{1/2}\, dx \qquad y>0$
(4)	$x^{\nu-1/2} \exp(\tfrac14 x^2)\, D_{2\nu-1}(x)$ $-\tfrac12 < \operatorname{Re} \nu < \tfrac12$	$2^{1/2-\nu} \pi \sin(\nu\pi)\, \Gamma(2\nu)\, y^{1/2-\nu}$ $\times \exp(\tfrac14 y^2)\, K_\nu(\tfrac14 y^2)$
(5)	$x^{\nu-1/2} \exp(-\tfrac14 x^2)\, D_{2\nu+1}(x)$ $\operatorname{Re} \nu > -\tfrac12$	$\tfrac12 \sec(\nu\pi) \exp(-\tfrac14 y^2)\, y^{\nu+1/2}$ $\times [D_{2\nu}(y) + D_{2\nu}(-y)]$
(6)	$x^{\nu-1/2} \exp(-\tfrac14 x^2)$ $\times \{[1+2\cos(\nu\pi)]\, D_{2\nu+1}(x)$ $- D_{2\nu+1}(-x)\} \qquad \operatorname{Re} \nu > -\tfrac12$	$y^{\nu+1/2} \exp(-\tfrac14 y^2)$ $\times\{[1+2\cos(\nu\pi)]\, D_{2\nu}(x)$ $+ D_{2\nu}(-x)\}$
(7)	$x^{\nu-1/2} \exp(-\tfrac14 x^2)$ $\times \{[1-2\cos(\nu\pi)]\, D_{2\nu+1}(x)$ $- D_{2\nu+1}(-x)\} \qquad \operatorname{Re} \nu > -\tfrac12$	$-y^{\nu+1/2} \exp(-\tfrac14 y^2)$ $\times\{[1-2\cos(\nu\pi)]\, D_{2\nu}(y)$ $+ D_{2\nu}(-y)\}$
(8)	$x^{\nu-1/2} \exp(-\tfrac14 x^2)\, D_{-2\nu}(x)$ $\operatorname{Re} \nu > -\tfrac12$	$2^{-1/2} \pi^{1/2} y^{1/2-\nu} \exp(-\tfrac14 y^2)\, I_\nu(\tfrac14 y^2)$
(9)	$x^{\nu-1/2} \exp(\tfrac14 x^2)\, D_{-2\nu}(x)$ $\operatorname{Re} \nu > -\tfrac12$	$y^{\nu-1/2} \exp(\tfrac14 y^2)\, D_{-2\nu}(y)$
(10)	$x^{\nu-1/2} \exp(\tfrac14 x^2)\, D_{-2\nu-2}(x)$ $\operatorname{Re} \nu > -\tfrac12$	$(2\nu+1)^{-1} y^{\nu+1/2} \exp(\tfrac14 y^2) D_{-2\nu-1}(y)$
(11)	$x^{\nu-1/2} \exp(-\tfrac14 a^2 x^2) D_{2\mu}(ax)$ $\lvert\arg a\rvert < \tfrac14\pi, \quad \operatorname{Re} \nu > -\tfrac12$	$\dfrac{2^{\mu-1/2}\, \Gamma(\nu+\tfrac12)\, y^{\nu+1/2}}{\Gamma(\nu-\mu+1)\, a^{1+2\nu}}$ $\times {}_1F_1\left(\nu+\dfrac12;\ \nu-\mu+1;\ -\dfrac{y^2}{2a^2}\right)$

Parabolic cylinder functions (cont'd)

	$f(x)$	$\int_0^\infty f(x)\, J_\nu(xy)(xy)^{1/2}\, dx \qquad y>0$
(12)	$x^{\nu-1/2} \exp(\frac{1}{4}a^2x^2) D_{2\mu}(ax)$ $\lvert\arg a\rvert < \frac{1}{4}\pi$ $-\frac{1}{2} < \operatorname{Re}\nu < \operatorname{Re}(\frac{1}{2}-2\mu)$	$\dfrac{\Gamma(\frac{1}{2}+\nu)\, a^{2k}\, 2^{m+\mu}}{\Gamma(\frac{1}{2}-\mu)\, y^{\mu+1}} \exp\left(\dfrac{y^2}{4a^2}\right)$ $\times W_{k,m}\left(\dfrac{y^2}{2a^2}\right)$ $2k=\frac{1}{2}+\mu-\nu, \quad 2m=\frac{1}{2}+\mu+\nu$
(13)	$x^{\nu+1/2} \exp(-\frac{1}{4}x^2)\, D_{2\nu}(x)$ $\operatorname{Re}\nu > -1$	$\frac{1}{2}\sec(\nu\pi)\, y^{\nu-1/2} \exp(-\frac{1}{4}y^2)$ $\times[D_{2\nu+1}(y) - D_{2\nu+1}(-y)]$
(14)	$x^{\nu+1/2} \exp(-\frac{1}{4}x^2)$ $\times\{[1+2\cos(\nu\pi)]\, D_{2\nu}(x)$ $+D_{2\nu}(-x)\}$ $\quad \operatorname{Re}\nu > -1$	$y^{\nu-1/2} \exp(-\frac{1}{4}y^2)$ $\times\{[1+2\cos(\nu\pi)]\, D_{2\nu+1}(y)$ $-D_{2\nu+1}(-y)\}$
(15)	$x^{\nu+1/2} \exp(-\frac{1}{4}x^2)$ $\times\{[1-2\cos(\nu\pi)]\, D_{2\nu}(x)$ $+D_{2\nu}(-x)\}$ $\quad \operatorname{Re}\nu > -1$	$-y^{\nu-1/2} \exp(-\frac{1}{4}y^2)$ $\times\{[1-2\cos(\nu\pi)]\, D_{2\nu+1}(y)$ $-D_{2\nu+1}(-y)\}$
(16)	$x^{\nu+1/2} \exp(-\frac{1}{4}x^2)\, D_{2\nu+2}(x)$ $\operatorname{Re}\nu > -1$	$-\frac{1}{2}\sec(\nu\pi)\, y^{\nu+1/2} \exp(-\frac{1}{4}y^2)$ $\times[D_{2\nu+2}(y) + D_{2\nu+2}(-y)]$
(17)	$x^{\nu+1/2} \exp(-\frac{1}{4}x^2)$ $\times\{[1-2\cos(\nu\pi)]\, D_{2\nu+2}(x)$ $+D_{2\nu+2}(-x)\}$ $\quad \operatorname{Re}\nu > -1$	$y^{\nu+1/2} \exp(-\frac{1}{4}y^2)$ $\times\{[1-2\cos(\nu\pi)]\, D_{2\nu+2}(y)$ $+D_{2\nu+2}(-y)\}$
(18)	$x^{\nu+1/2} \exp(-\frac{1}{4}x^2)$ $\times\{[1+2\cos(\nu\pi)]\, D_{2\nu+2}(x)$ $+D_{2\nu+2}(-x)\}$ $\quad \operatorname{Re}\nu > -1$	$-y^{\nu+1/2} \exp(-\frac{1}{4}y^2)$ $\times\{[1+2\cos(\nu\pi)]\, D_{2\nu+2}(y)$ $+D_{2\nu+2}(-y)\}$
(19)	$x^{\nu+1/2} \exp(\frac{1}{4}x^2)\, D_{2\nu+2}(x)$ $-1 < \operatorname{Re}\nu < -5/6$	$\pi^{-1}\sin(\nu\pi)\Gamma(2\nu+3)\, y^{-\nu-3/2}$ $\times\exp(\frac{1}{4}y^2) K_{\nu+1}(\frac{1}{4}y^2)$

Parabolic cylinder functions (cont'd)

	$f(x)$	$\int_0^\infty f(x)\, J_\nu(xy)(xy)^{1/2}\,dx \qquad y>0$
(20)	$x^{\nu+1/2}\exp(\frac14 x^2)\,D_{-2\nu-1}(x)$ $\operatorname{Re}\nu>-\frac12$	$(2\nu+1)\,y^{\nu-1/2}\exp(\frac14 y^2)\,D_{-2\nu-2}(y)$
(21)	$x^{\nu+1/2}\exp(-\frac14 x^2)\,D_{-2\nu-3}(x)$ $\operatorname{Re}\nu>-1$	$2^{-1/2}\pi^{1/2}y^{-\nu-3/2}\exp(-\frac14 y^2)$ $\times I_{\nu+1}(\frac14 y^2)$
(22)	$x^{\nu+1/2}\exp(\frac14 x^2)\,D_{-2\nu-3}(x)$ $\operatorname{Re}\nu>-1$	$y^{\nu+1/2}\exp(\frac14 y^2)\,D_{-2\nu-3}(y)$
(23)	$x^{\nu+1/2}\exp(-\frac14 a^2x^2)\,D_{2\mu}(ax)$ $\lvert\arg a\rvert<\frac14\pi, \quad \operatorname{Re}\nu>-1$	$\dfrac{2^{\mu}\,\Gamma(\nu+3/2)\,y^{\nu+1/2}}{\Gamma(\nu-\mu+3/2)\,a^{2\nu+2}}$ $\times {}_1F_1\left(\nu+\dfrac32;\ \nu-\mu+\dfrac32;\ -\dfrac{y^2}{2a^2}\right)$
(24)	$x^{\nu+1/2}\exp(\frac14 a^2x^2)\,D_{2\mu}(ax)$ $\lvert\arg a\rvert<\frac34\pi$ $-1<\operatorname{Re}\nu<-\frac12-2\operatorname{Re}\mu$	$\dfrac{\Gamma(3/2+\nu)\,2^{1/2+m+\mu}\,a^{2k+1}}{\Gamma(-\mu)\,y^{\mu+3/2}}$ $\times\exp\left(\dfrac{y^2}{4a^2}\right)W_{k,m}\left(\dfrac{y^2}{2a^2}\right)$ $2k=\mu-\nu-1, \quad 2m=\mu+\nu+1$
(25)	$x^{\lambda}\exp(-\frac14 a^2x^2)\,D_{\mu}(ax)$ $\lvert\arg a\rvert<\frac14\pi$ $\operatorname{Re}(\lambda+\nu)>-3/2$	$\dfrac{2^{-c-3\nu/2}\,\pi^{1/2}\,\Gamma(2b)\,y^{\nu+1/2}}{\Gamma(\nu+1)\,\Gamma(c+\frac12)\,a^{2b}}$ $\times {}_2F_2\left(b, b+\dfrac12;\ \nu+1, c+\dfrac12;\ -\dfrac{y^2}{2a^2}\right)$ $2b=\lambda+\nu+3/2, \quad 2c=\lambda-\mu+3/2$

Parabolic cylinder functions (cont'd)

	$f(x)$	$\int_0^\infty f(x)\, J_\nu(xy)(xy)^{1/2}\,dx \qquad y>0$
(26)	$x^\lambda \exp(\tfrac14 a^2x^2)\,D_\mu(ax)$ $\lvert\arg a\rvert < \tfrac34\pi$ $\mathrm{Re}\,\mu < -\mathrm{Re}\,\lambda < \mathrm{Re}\,\nu + 3/2$	$\dfrac{2^{\lambda-\frac12}\mu^{-\frac12}\,\pi^{-\frac12}}{\Gamma(-\mu)y^{\lambda+1}}$ $\times G_{23}^{22}\left(\dfrac{y^2}{2a^2}\middle\vert\begin{matrix}\frac12, 1\\ \frac34+\frac{\lambda+\nu}{2}, -\frac{\mu}{2}, \frac34+\frac{\lambda-\nu}{2}\end{matrix}\right)$
(27)	$D_{-\frac12-\nu}(ae^{\frac14\pi i}x^{\frac12})$ $\times D_{-\frac12-\nu}(ae^{-\frac14\pi i}x^{\frac12})$ $\mathrm{Re}\,a>0, \quad \mathrm{Re}\,\nu > -\tfrac12$	$2^{-\nu}\pi^{\frac12}[\Gamma(\nu+\tfrac12)]^{-1}\,y^{-\nu-\frac12}$ $\times(a^2+2y)^{-\frac12}[(a^2+2y)^{\frac12}-a]^{2\nu}$
(28)	$x^{-\frac12}D_{-\frac12-\nu}(ae^{\frac14\pi i}x^{-\frac12})$ $\times D_{-\frac12-\nu}(ae^{-\frac14\pi i}x^{-\frac12})$ $\mathrm{Re}\,a>0, \quad \mathrm{Re}\,\nu > -\tfrac12$	$2^{\frac12}\pi^{\frac12}[\Gamma(\nu+\tfrac12)]^{-1}\,y^{-1}$ $\times \exp[-a(2y)^{\frac12}]$

8.17. Gauss' hypergeometric function

(1)	$x^{2a+\nu-\frac12}$ $\times {}_2F_1(a-\nu-\tfrac12, a; 2a; -\lambda^2x^2)$ $\mathrm{Re}\,\nu < -\tfrac12, \quad \mathrm{Re}\,\lambda>0$ $\mathrm{Re}(a+\nu) > -\tfrac12$	$\dfrac{i\Gamma(\frac12+a)\,\Gamma(\frac12+a+\nu)}{\pi\, 2^{1-\nu-2a}\,\lambda^{2a-1}}\,y^{-\nu-3/2}$ $\times W_{\frac12-a,-\frac12-\nu}(y/\lambda)$ $\times[W_{\frac12-a,-\frac12-\nu}(e^{-i\pi}y/\lambda)$ $-W_{\frac12-a,-\frac12-\nu}(e^{i\pi}y/\lambda)]$
(2)	$x^{2a-\nu-\frac12}$ $\times {}_2F_1(\nu+a-\tfrac12, a; 2a; -\lambda^2x^2)$ $\mathrm{Re}\,a > -\tfrac12, \quad \mathrm{Re}\,\nu > \tfrac12$ $\mathrm{Re}\,\lambda > 0$	$\dfrac{2^{2a-\nu}\Gamma(\frac12+a)}{\lambda^{2a-1}\Gamma(2\nu)}\,y^{\nu-3/2}$ $\times M_{a-\frac12,\nu-\frac12}(y/\lambda)W_{\frac12-a,\nu-\frac12}(y/\lambda)$

Gauss' hypergeometric function (cont'd)

	$f(x)$	$\int_0^\infty f(x)\, J_\nu(xy)\,(xy)^{1/2}\,dx \qquad y>0$
(3)	$x^{\nu+1/2}\,{}_2F_1(a,\beta;\nu+1;-\lambda^2x^2)$ $-1<\mathrm{Re}\,\nu$ $<2\max(\mathrm{Re}\,a,\ \mathrm{Re}\,\beta)-3/2$ $\mathrm{Re}\,\lambda>0$	$\dfrac{2^{\nu-a-\beta+2}\,\Gamma(\nu+1)}{\lambda^{a+\beta}\,\Gamma(a)\,\Gamma(\beta)}\,y^{a+\beta-\nu-3/2}$ $\times K_{a-\beta}(y/\lambda)$
(4)	$x^{\nu+1/2}\,{}_2F_1[a,\beta;\tfrac12(\beta+\nu)+1;-\lambda^2x^2]$ $-1<\mathrm{Re}\,\nu$ $<2\max(\mathrm{Re}\,a,\ \mathrm{Re}\,\beta)-3/2$	$\dfrac{\Gamma(\tfrac12\beta+\tfrac12\nu+1)}{\pi^{1/2}\,\Gamma(a)\,\Gamma(\beta)}\ \dfrac{y^{\beta-1/2}}{2^{\beta-1}\,\lambda^{\nu+\beta+1}}$ $\times\left[K_{\frac12(\nu-\beta+1)}\left(\dfrac{y}{2\lambda}\right)\right]^2$
(5)	$x^{\nu+1/2}\,{}_2F_1(a,\beta;\gamma;-\lambda^2x^2)$ $-1<\mathrm{Re}\,\nu$ $<2\max(\mathrm{Re}\,a,\ \mathrm{Re}\,\beta)-3/2$ $\mathrm{Re}\,\lambda>0$	$\dfrac{2^{\nu+1}\,\Gamma(\gamma)}{\Gamma(a)\,\Gamma(\beta)}\,y^{-\nu-3/2}$ $\times G_{13}^{30}\left(\dfrac{y^2}{4\lambda^2}\,\middle\vert\,\begin{matrix}\gamma\\ \nu+1,\ a,\ \beta\end{matrix}\right)$
(6)	$x^{\delta-1/2}\,{}_2F_1(a,\beta;\gamma;-\lambda^2x^2)$ $-\mathrm{Re}\,\nu-1<\mathrm{Re}\,\delta$ $<2\max(\mathrm{Re}\,a,\ \mathrm{Re}\,\beta)-\tfrac12$ $\mathrm{Re}\,\lambda>0$	$\dfrac{2^{\delta}\,\Gamma(\gamma)}{\Gamma(a)\,\Gamma(\beta)}\,y^{-\delta-1/2}$ $\times G_{24}^{31}\left(\dfrac{y^2}{4\lambda^2}\,\middle\vert\,\begin{matrix}1,\ \gamma\\ \dfrac{1+\delta+\nu}{2},\ a,\ \beta,\ \dfrac{1+\delta-\nu}{2}\end{matrix}\right)$
(7)	$x^{-2a-3/2}$ $\times{}_2F_1\left(\dfrac12+a,\ 1+a;\ 1+2a;-\dfrac{4\lambda^2}{x^2}\right)$ $\mathrm{Re}\,\nu>-1,\quad \mathrm{Re}\,\lambda>0$ $\mathrm{Re}\,a>-\tfrac12$	$\lambda^{-2a}\,y^{1/2}\,I_{\frac12\nu+a}(\lambda y)\,K_{\frac12\nu-a}(\lambda y)$
(8)	$x^{\nu-4a+1/2}$ $\times{}_2F_1(a,\ a+\tfrac12;\ \nu+1;-\lambda^2x^{-2})$ $\mathrm{Re}\,a-1<\mathrm{Re}\,\nu<4\,\mathrm{Re}\,a-3/2$ $\mathrm{Re}\,\lambda>0$	$\dfrac{\Gamma(\nu)}{\Gamma(2a)}\,2^{\nu}\lambda^{1-2a}y^{2a-\nu-1/2}\,I_\nu(\tfrac12\lambda y)$ $\times K_{2a-\nu-1}(\tfrac12\lambda y)$

Gauss' hypergeometric function (cont'd)

	$f(x)$	$\int_0^\infty f(x)\, J_\nu(xy)(xy)^{1/2}\, dx \qquad y>0$
(9)	$x^{\delta-1/2}\, {}_2F_1(a,\beta;\gamma;-\lambda^2 x^{-2})$ $-1-\mathrm{Re}\,\nu-2\min(\mathrm{Re}\, a, \mathrm{Re}\,\beta)$ $<\mathrm{Re}\,\delta<-1/2$ $\mathrm{Re}\,\lambda>0$	$\dfrac{2^\delta\,\Gamma(\gamma)}{\Gamma(a)\,\Gamma(\beta)}\, y^{-\delta-1/2}$ $\times G_{24}^{22}\left(\dfrac{\lambda^2 y^2}{4}\middle\vert\begin{matrix} 1-a, 1-\beta \\ \frac{1+\delta+\nu}{2}, 0, 1-\gamma, \frac{1+\delta-\nu}{2}\end{matrix}\right)$
(10)	$x^{\nu+1/2}(1+x)^{-2a}$ $\times\, {}_2F_1\left[a, \nu+\dfrac{1}{2}; 2\nu+1; \dfrac{4x}{(1+x)^2}\right]$ $-1<\mathrm{Re}\,\nu<2\mathrm{Re}\, a-3/2$	$\dfrac{\Gamma(\nu+1)\,\Gamma(\nu-a+1)}{\Gamma(a)}\, 2^{2\nu-2a+1}$ $\times y^{2a-2\nu-3/2}\, J_\nu(x)$

8.18. Confluent hypergeometric functions

(1)	$x^{-1}\exp(-1/2\, x^2)$ $\times M_{1/2\nu-1/4,\, 1/2\nu+1/4}(x^2)$ $\mathrm{Re}\,\nu>-1/2$	$(2\nu+1)\, 2^{-\nu}\, y^{\nu-1/2}\, \mathrm{Erfc}\,(1/2\, y)$
(2)	$x^{-3/2}\exp(-1/2\, x^2)$ $\times M_{1/2\nu+1/2,\, 1/2\nu+1/2}(x^2)$ $\mathrm{Re}\,\nu>-1$	$\dfrac{\Gamma(\nu+2)\, y^{\nu+1/2}}{\Gamma(\nu+3/2)\, 2^\nu}\, \mathrm{Erfc}\,(1/2\, y)$
(3)	$x^{2\mu-\nu-1/2}\exp(-1/4\, x^2)$ $\times M_{3\mu-\nu+1/2,\, \mu}(1/2\, x^2)$ $\mathrm{Re}\,\mu>-1/2, \quad \mathrm{Re}\,(4\mu-\nu)>-1/2$	$y^{2\mu-\nu-1/2}\exp(-1/4\, y^2)$ $\times M_{3\mu-\nu+1/2,\, \mu}(1/2\, y^2)$
(4)	$x^{\nu-2\mu-1/2}\exp(-1/4\, x^2)$ $\times M_{\nu-\mu,\, \mu}(1/2\, x^2)$ $\mathrm{Re}\,\nu>-1/2$	$\dfrac{\Gamma(2\mu+1)}{2^{\nu-\mu}\Gamma(\nu+1/2)}\, y^{\nu-1/2}\exp(-1/4\, y^2)$ $\times D_{2\nu-4\mu}(y)$

Confluent hypergeometric functions (cont'd)

	$f(x)$	$\int_0^\infty f(x)\, J_\nu(xy)(xy)^{1/2}\, dx \qquad y>0$
(5)	$x^{\nu-2\mu-1/2}\exp(-\frac{1}{4}x^2)$ $\times M_{\nu-\mu+1,\mu}(\frac{1}{2}x^2)$ $\operatorname{Re}\nu>-1$	$\dfrac{\Gamma(2\mu+1)}{\Gamma(\nu+3/2)\;2^{\nu-\mu+1}}\,y^{\nu+1/2}\exp(-\frac{1}{4}y^2)$ $\times D_{2\nu-4\mu+1}(y)$
(6)	$x^{-\lambda-1}\exp(-\frac{1}{4}x^2)$ $\times M_{\lambda+\mu,\mu}(\frac{1}{2}x^2)$ $\lambda=2\mu-\nu-\frac{1}{2}$ $-1<\operatorname{Re}\nu<4\operatorname{Re}\mu$	$\pi^{-1/2}\,2^{-5(\mu+\nu/3)}\,\dfrac{\Gamma(2\mu+1)}{\Gamma(4\mu-\nu)}\,y^{\lambda+2\mu}$ $\times\exp(-\frac{1}{4}y^2)\,K_\lambda(\frac{1}{4}y^2)$
(7)	$x^{-1/2}\exp(-\frac{1}{4}x^2)$ $\times M_{\kappa,\frac{1}{2}\nu}(\frac{1}{2}x^2)$ $\operatorname{Re}\nu>-1,\quad\operatorname{Re}\kappa<\frac{1}{2}$	$\dfrac{2^{-\kappa}\,\Gamma(\nu+1)}{\Gamma(\kappa+\frac{1}{2}\nu+\frac{1}{2})}\,y^{2\kappa-1/2}$ $\times\exp(-\frac{1}{2}y^2)$
(8)	$x^{2\mu-\nu-1/2}\exp(-\frac{1}{4}x^2)$ $\times M_{\kappa,\mu}(\frac{1}{2}x^2)$ $-\frac{1}{2}<\operatorname{Re}\mu<\operatorname{Re}(\kappa+\frac{1}{2}\nu)-\frac{1}{4}$	$\dfrac{\Gamma(2\mu+1)\,2^{\frac{1}{2}(\frac{1}{2}-\kappa+3\mu-\nu)}}{\Gamma(\frac{1}{2}+\kappa-\mu+\nu)\,y^{1-\kappa+\mu}}$ $\times\exp(-\frac{1}{4}y^2)\,M_{\alpha,\beta}(\frac{1}{2}y^2)$ $2\alpha=\frac{1}{2}+\kappa+3\mu-\nu$ $2\beta=-\frac{1}{2}+\kappa-\mu+\nu$
(9)	$x^{\nu-2\mu-1/2}\exp(-\frac{1}{4}x^2)$ $\times M_{\kappa,\mu}(\frac{1}{2}x^2)$ $-1<\operatorname{Re}\nu<2\operatorname{Re}(\kappa+\mu)-\frac{1}{2}$	$2^{\frac{1}{2}(\frac{1}{2}-\kappa-3\mu+\nu)}\,\dfrac{\Gamma(2\mu+1)}{\Gamma(\mu+\kappa+\frac{1}{2})}\,y^{\kappa+\mu-1}$ $\times\exp(-\frac{1}{4}y^2)\,W_{\alpha,\beta}(\frac{1}{2}y^2)$ $2\alpha=\kappa-3\mu+\nu+\frac{1}{2}$ $2\beta=\kappa+\mu-\nu-\frac{1}{2}$
(10)	$x^{2\rho-1/2}\exp(-\frac{1}{2}ax^2)\,M_{\kappa,\mu}(ax^2)$ $-1-\operatorname{Re}(\frac{1}{2}\nu+\mu)<\operatorname{Re}\rho<\operatorname{Re}\kappa-\frac{1}{4}$ $\operatorname{Re}a>0$	$\dfrac{\Gamma(2\mu+1)}{\Gamma(\mu+\kappa+\frac{1}{2})}\,2^{2\rho}\,y^{-2\rho-1/2}$ $\times G^{21}_{23}\left(\dfrac{y^2}{4a}\middle\vert\begin{matrix}\frac{1}{2}-\mu,\ \frac{1}{2}+\mu\\ \frac{1}{2}+\rho+\frac{1}{2}\nu,\ \kappa,\ \frac{1}{2}+\rho-\frac{1}{2}\nu\end{matrix}\right)$

Confluent hypergeometric functions (cont'd)

	$f(x)$	$\int_0^\infty f(x)\,J_\nu(xy)(xy)^{1/2}\,dx \qquad y>0$
(11)	$x^{\nu-2\mu-1/2}\exp(-\frac{1}{4}x^2)$ $\times W_{3\mu-\nu-1/2,\,\pm\mu}(\frac{1}{2}x^2)$ $\mathrm{Re}\,\nu>-1,\quad \mathrm{Re}(\nu-2\mu)>-1$	$\pi^{1/2}2^{\nu-3\mu}y^{4\mu-\nu-1/2}\exp(-\frac{1}{4}y^2)$ $\times I_{\nu-2\mu+1/2}(\frac{1}{4}y^2)$
(12)	$x^{\nu-2\mu-1/2}\exp(\frac{1}{4}x^2)$ $\times W_{\nu-3\mu+1/2,\,\pm\mu}(\frac{1}{2}x^2)$ $\mathrm{Re}\,\nu>-1,\quad \mathrm{Re}(\nu-2\mu)>-1$ $\mathrm{Re}(3\nu-8\mu)<-3/2$	$\pi^{-1/2}\,2^{\nu-3\mu}\dfrac{\Gamma(1+\nu-2\mu)}{\Gamma(4\mu-\nu)}\,y^{4\mu-\nu-1/2}$ $\times\exp(\frac{1}{4}y^2)\,K_{\nu-2\mu+1/2}(\frac{1}{4}y^2)$
(13)	$x^{\nu-2\mu-1/2}\exp(\frac{1}{4}x^2)$ $\times W_{3\mu-\nu-1/2,\,\pm\mu}(\frac{1}{2}x^2)$ $\mathrm{Re}\,\nu>-1,\quad \mathrm{Re}(\nu-2\mu)>-1$ $\mathrm{Re}(\nu-4\mu)>-\frac{1}{2}$	$y^{\nu-2\mu-1/2}\exp(\frac{1}{4}y^2)$ $\times W_{3\mu-\nu-1/2,\,\pm\mu}(\frac{1}{2}y^2)$
(14)	$x^{\nu-2\mu-1/2}\exp(-\frac{1}{4}x^2)$ $\times W_{\kappa,\,\pm\mu}(\frac{1}{2}x^2)$ $\mathrm{Re}\,\nu>-1,\quad \mathrm{Re}(\nu-2\mu)>-1$	$\dfrac{\Gamma(1+\nu-2\mu)}{\Gamma(1+2\beta)}\,2^{\beta-\mu}y^{\kappa+\mu-1}$ $\times\exp(-\frac{1}{4}y^2)\,M_{\alpha,\beta}(\frac{1}{2}y^2)$ $2\alpha=\frac{1}{2}+\kappa+\nu-3\mu$ $2\beta=\frac{1}{2}-\kappa+\nu-\mu$
(15)	$x^{\nu-2\mu-1/2}\exp(\frac{1}{4}x^2)$ $\times W_{\kappa,\,\pm\mu}(\frac{1}{2}x^2)$ $\mathrm{Re}\,\nu>-1,\quad \mathrm{Re}(\nu-2\mu)>-1$ $\mathrm{Re}(\kappa-\mu+\frac{1}{2}\nu)<-\frac{1}{4}$	$\dfrac{\Gamma(1+\nu-2\mu)}{\Gamma(\frac{1}{2}+\mu-\kappa)}\,2^{\frac{1}{2}(\frac{1}{2}+\kappa-3\mu+\nu)}y^{\mu-\kappa-1}$ $\times\exp(\frac{1}{4}y^2)\,W_{\alpha,\beta}(\frac{1}{2}y^2)$ $2\alpha=\kappa+3\mu-\nu-\frac{1}{2}$ $2\beta=\kappa-\mu+\nu+\frac{1}{2}$

Confluent hypergeometric functions (cont'd)

	$f(x)$	$\int_0^\infty f(x)\, J_\nu(xy)(xy)^{1/2}\, dx \qquad y>0$
(16)	$x^{2\rho-1/2}\exp(-\tfrac{1}{2}ax^2)$ $\times W_{\kappa,\mu}(ax^2) \qquad \mathrm{Re}\, a>0$ $\mathrm{Re}(\rho \pm \mu + \tfrac{1}{2}\nu) > -1$	$\dfrac{\Gamma(1+\mu+\nu/2+\rho)\Gamma(1-\mu+\nu/2+\rho)}{\Gamma(\nu+1)\Gamma(3/2-\kappa+\nu/2+\rho)}$ $\times 2^{-\nu-1} a^{-\frac{1}{2}\nu-\rho-\frac{1}{2}} y^{\nu+\frac{1}{2}}$ $\times {}_2F_2\left(\lambda+\mu, \lambda-\mu; \nu+1, \dfrac{1}{2}-\kappa+\lambda; -\dfrac{y^2}{4a}\right)$ $\lambda = 1+\tfrac{1}{2}\nu+\rho$
(17)	$x^{2\rho-1/2}\exp(\tfrac{1}{2}ax^2)$ $\times W_{\kappa,\mu}(ax^2) \qquad \lvert\arg a\rvert<\pi$ $-1-\mathrm{Re}(\tfrac{1}{2}\nu\pm\mu)$ $<\mathrm{Re}\,\rho<-\tfrac{1}{4}-\mathrm{Re}\,\kappa$	$\dfrac{2^{2\rho}y^{-2\rho-1/2}}{\Gamma(\tfrac{1}{2}+\mu-\kappa)\Gamma(\tfrac{1}{2}-\mu-\kappa)}$ $\times G^{22}_{23}\left(\dfrac{y^2}{4a}\middle\vert \begin{matrix}\tfrac{1}{2}-\mu, \tfrac{1}{2}+\mu \\ \tfrac{1}{2}+\rho+\tfrac{1}{2}\nu, -\kappa, \tfrac{1}{2}+\rho-\tfrac{1}{2}\nu\end{matrix}\right)$
(18)	$x^{-1/2} M_{\kappa,\frac{1}{2}\nu}(-iax) M_{-\kappa,\frac{1}{2}\nu}(-iax)$ $a>0, \quad \mathrm{Re}\,\nu>-1, \quad \lvert\mathrm{Re}\,\kappa\rvert<\tfrac{1}{4}$	$\dfrac{ae^{-\frac{1}{2}(\nu+1)\pi i}[\Gamma(1+\nu)]^2}{\Gamma(\tfrac{1}{2}+\kappa+\tfrac{1}{2}\nu)\Gamma(\tfrac{1}{2}-\kappa+\tfrac{1}{2}\nu)}\, y^{-\frac{1}{2}-2\kappa}$ $\times(a^2-y^2)^{-1/2}\{[a+(a^2-y^2)^{1/2}]^{2\kappa}$ $+[a-(a^2-y^2)^{1/2}]^{2\kappa}\} \qquad 0<y<a$ $0 \qquad a<y<\infty$
(19)	$x^{-1/2} M_{-\frac{1}{2}\mu,\frac{1}{2}\nu}(ax)$ $\times W_{\frac{1}{2}\mu,\frac{1}{2}\nu}(ax) \qquad \mathrm{Re}\,\nu>-1$ $\mathrm{Re}\,\mu<\tfrac{1}{2}, \quad \mathrm{Re}\,a>0$	$a\dfrac{\Gamma(\nu+1)}{\Gamma(\tfrac{1}{2}-\tfrac{1}{2}\mu+\tfrac{1}{2}\nu)}y^{-\mu-1/2}$ $\times[a+(a^2+y^2)^{1/2}]^{\mu}(a^2+y^2)^{-1/2}$
(20)	$x^{2\mu-\nu-1/2} W_{\kappa,\mu}(ax) M_{-\kappa,\mu}(ax)$ $\mathrm{Re}\,\mu>-\tfrac{1}{2}, \quad \mathrm{Re}\,a>0$ $\mathrm{Re}(2\mu+2\kappa-\nu)<\tfrac{1}{2}$	$2^{2\mu-\nu+2\kappa}a^{2\kappa}y^{\nu-2\mu-2\kappa-1/2}$ $\times\dfrac{\Gamma(2\mu+1)}{\Gamma(\nu-\kappa-\mu+\tfrac{1}{2})}$ $\times {}_3F_2(\tfrac{1}{2}-\kappa, 1-\kappa, \tfrac{1}{2}-\kappa+\mu;$ $1-2\kappa, \tfrac{1}{2}-\kappa-\mu+\nu; -y^2/a^2)$

Confluent hypergeometric functions (cont'd)

	$f(x)$	$\int_0^\infty f(x)\, J_\nu(xy)(xy)^{1/2}\,dx \qquad y>0$
(21)	$x^{2\rho-\nu-5/2} W_{\kappa,\mu}(ax)$ $\times M_{-\kappa,\mu}(ax) \qquad \operatorname{Re} a>0$ $\operatorname{Re}\rho>0, \quad \operatorname{Re}(\rho+\mu)>0$ $\operatorname{Re}(2\rho+2\kappa-\nu)<5/2$	$\dfrac{2^{2\rho-\nu-2}\,\Gamma(2\mu+1)}{\pi^{1/2}\,\Gamma(\frac12-\kappa+\mu)}\, y^{\nu-2\rho+3/2}$ $\times G_{44}^{23}\left(\dfrac{y^2}{a^2}\middle\vert \begin{matrix} \frac12, 0, \frac12-\mu, \frac12+\mu \\ \rho-\frac12, -\kappa, \kappa, \rho-\nu-\frac12 \end{matrix}\right)$
(22)	$x^{2\rho-\nu-5/2} W_{\kappa,\mu}(ax)$ $\times W_{-\kappa,\mu}(ax)$ $\operatorname{Re}\rho>\lvert\operatorname{Re}\mu\rvert, \quad \operatorname{Re} a>0$	$\dfrac{\Gamma(\rho+\mu)\,\Gamma(\rho-\mu)\,\Gamma(2\rho)}{\Gamma(\frac12+\kappa+\rho)\,\Gamma(\frac12-\kappa+\rho)\,\Gamma(1+\nu)}$ $\times 2^{-\nu-1} a^{1-2\rho} y^{\nu+\frac12}$ $\times {}_4F_3(\rho, \rho+\frac12, \rho+\mu, \rho-\mu;$ $\frac12+\kappa+\rho, \frac12-\kappa+\rho, 1+\nu; -y^2/a^2)$
(23)	$x^{2\rho-\nu-5/2} W_{\kappa,\mu}(iax)$ $\times W_{\kappa,\mu}(-iax) \qquad \operatorname{Re} a>0$ $\operatorname{Re}\rho>\lvert\operatorname{Re}\mu\rvert$ $\operatorname{Re}(2\rho+2\kappa-\nu)<5/2$	$\dfrac{2^{2\rho-\nu-2}\, y^{\nu-2\rho+3/2}}{\pi^{1/2}\,\Gamma(\frac12-\kappa+\mu)\,\Gamma(\frac12-\kappa-\mu)}$ $\times G_{44}^{24}\left(\dfrac{y^2}{a^2}\middle\vert \begin{matrix} \frac12, 0, \frac12-\mu, \frac12+\mu \\ \rho-\frac12, -\kappa, \kappa, \rho-\nu-\frac12 \end{matrix}\right)$
(24)	$x^{-3/2} M_{-\mu,\frac14\nu}(\frac12 x^2)$ $\times W_{\mu,\frac14\nu}(\frac12 x^2) \qquad \operatorname{Re}\nu>-1$	$\dfrac{\Gamma(1+\frac12\nu)\, y^{1/2}}{\Gamma(\frac12+\frac14\nu-\mu)}\, I_{\frac14\nu-\mu}(\frac14 y^2)\, K_{\frac14\nu+\mu}(\frac14 y^2)$
(25)	$x^{-3/2} M_{\alpha-\beta,\frac14\nu-\gamma}(\frac12 x^2)$ $\times W_{\alpha+\beta,\frac14\nu+\gamma}(\frac12 x^2)$ $\operatorname{Re}\beta<1/8, \quad \operatorname{Re}\nu>-1$ $\operatorname{Re}(\nu-4\gamma)>-2$	$\dfrac{\Gamma(1+\frac12\nu-2\gamma)}{\Gamma(1+\frac12\nu-2\beta)}\; y^{-3/2} M_{\alpha-\gamma,\frac14\nu-\beta}(\frac12 y^2)$ $\times W_{\alpha+\gamma,\frac14\gamma+\beta}(\frac12 y^2)$
(26)	$x^{1/2} M_{\frac12\nu,\mu}(2/x)\, W_{-\frac12\nu,\mu}(2/x)$ $\operatorname{Re}\nu>-1, \quad \operatorname{Re}\mu>-\frac14$	$\dfrac{4\Gamma(1+2\mu)}{\Gamma(\frac12+\frac12\nu+\mu)}\; y^{-1/2} J_{2\mu}(2y^{1/2}) K_{2\mu}(2y^{1/2})$

Confluent hypergeometric functions (cont'd)

	$f(x)$	$\int_0^\infty f(x)\, J_\nu(xy)(xy)^{1/2}\, dx \qquad y>0$
(27)	$x^{1/2}\, W_{\frac{1}{2}\nu,\mu}(2/x)\, W_{-\frac{1}{2}\nu,\mu}(2/x)$ $\mathrm{Re}(\nu \pm 2\mu) > -1$	$-4y^{-1/2}\{\sin[(\mu-\tfrac{1}{2}\nu)\pi]\, J_{2\mu}(2y^{1/2})$ $+\cos[(\mu-\tfrac{1}{2}\nu)\pi]\, Y_{2\mu}(2y^{1/2})\}\, K_{2\mu}(2y^{1/2})$
(28)	$x^{1/2}\, W_{-\frac{1}{2}\nu,\mu}(ia/x)$ $\times W_{-\frac{1}{2}\nu,\mu}(-ia/x) \qquad \mathrm{Re}\, a>0$ $\lvert\mathrm{Re}\,\mu\rvert < \tfrac{1}{2}, \qquad \mathrm{Re}\,\nu > -1$	$4ay^{-1/2}[\Gamma(\tfrac{1}{2}+\mu+\tfrac{1}{2}\nu)\Gamma(\tfrac{1}{2}-\mu+\tfrac{1}{2}\nu)]^{-1}$ $\times K_\mu[(2iay)^{1/2}]\, K_\mu[(-2iay)^{1/2}]$
(29)	$x^{-1/2} M_{-\mu,\frac{1}{2}\nu}\{a[(\beta^2+x^2)^{1/2}-\beta]\}$ $\times W_{\mu,\frac{1}{2}\nu}\{a[(\beta^2+x^2)^{1/2}+\beta]\}$ $\mathrm{Re}\,\nu>-1, \qquad \mathrm{Re}\,\mu<\tfrac{1}{4}$ $\mathrm{Re}\,a>0, \qquad \mathrm{Re}\,\beta>0$	$\dfrac{a\,\Gamma(1+\nu)\,[(a^2+y^2)^{1/2}+a]^{2\mu}}{\Gamma(\tfrac{1}{2}+\tfrac{1}{2}\nu-\mu)\, y^{\frac{1}{2}+2\mu}\,(a^2+y^2)^{1/2}}$ $\times \exp[-\beta(a^2+y^2)^{1/2}]$

8.19. Generalized hypergeometric series and miscellaneous functions

(1)	$x^{\nu+\frac{1}{2}}\ {}_1F_1(2a-\nu; a+1; -\tfrac{1}{2}x^2)$ $\mathrm{Re}\,\nu>-1, \qquad \mathrm{Re}(4a-3\nu)>\tfrac{1}{2}$	$\dfrac{2^{\nu-a+\frac{1}{2}}\,\Gamma(a+1)}{\pi^{1/2}\,\Gamma(2a-\nu)}\, y^{2a-\nu-\frac{1}{2}}$ $\times \exp(-\tfrac{1}{4}y^2)\, K_{a-\nu-\frac{1}{2}}(\tfrac{1}{4}y^2)$
(2)	$x^{a-\frac{1}{2}}\ {}_1F_1\left(a; \dfrac{1+a+\nu}{2}; -\dfrac{x^2}{2}\right)$ $\mathrm{Re}\,a>-\tfrac{1}{2}, \qquad \mathrm{Re}(a+\nu)>-1$	$y^{a-\frac{1}{2}}\ {}_1F_1\left(a; \dfrac{1+a+\nu}{2}; -\dfrac{y^2}{2}\right)$
(3)	$x^{\nu+\frac{1}{2}-2a}\ {}_1F_1(a; 1+\nu-a; -\tfrac{1}{2}x^2)$ $\mathrm{Re}\,a-1<\mathrm{Re}\,\nu<4\,\mathrm{Re}\,a-\tfrac{1}{2}$	$\dfrac{\pi^{1/2}\,\Gamma(1+\nu-a)}{2^{2a-\nu-\frac{1}{2}}\,\Gamma(a)}\, y^{2a-\nu-\frac{1}{2}}$ $\times \exp(-\tfrac{1}{4}y^2)\, I_{a-\frac{1}{2}}(\tfrac{1}{4}y^2)$

Miscellaneous functions (cont'd)

	$f(x)$	$\int_0^\infty f(x)\, J_\nu(xy)(xy)^{1/2}\, dx \qquad y>0$
(4)	$x^{\nu+1/2}\ {}_1F_1(a;\beta;-\lambda x^2)$ $-1<\mathrm{Re}\,\nu<2\,\mathrm{Re}\,a-\tfrac{1}{2}$ $\mathrm{Re}\,\lambda>0$	$\dfrac{2^{1-a}\,\Gamma(\beta)}{\Gamma(a)\,\lambda^{\frac{1}{2}a+\frac{1}{2}\nu}}\, y^{a-3/2}\exp\left(-\dfrac{y^2}{8\lambda}\right)$ $\times W_{\kappa,\mu}\left(\dfrac{y^2}{4\lambda}\right)$ $2\kappa=a-2\beta+\nu+2$ $2\mu=a-\nu-1$
(5)	$x^{2\beta-\nu-3/2}\ {}_1F_1(a;\beta;-\lambda x^2)$ $0<\mathrm{Re}\,\beta<\tfrac{3}{4}+\mathrm{Re}(a+\tfrac{1}{2}\nu)$ $\mathrm{Re}\,\lambda>0$	$\dfrac{2^{2\beta-2a-\nu-1}\,\Gamma(\beta)}{\Gamma(a-\beta+\nu+1)\,\lambda^{a}}\, y^{2a-2\beta+\nu+1/2}$ $\times\ {}_1F_1\left(a;\,1+a-\beta+\nu;\,-\dfrac{y^2}{4\lambda}\right)$
(6)	$x^{2\rho-1/2}\ {}_1F_1(a;\,\beta;\,-\lambda x^2)$ $-1-\mathrm{Re}\,\nu<2\,\mathrm{Re}\,\rho<\tfrac{1}{2}+2\,\mathrm{Re}\,a$ $\mathrm{Re}\,\lambda>0$	$\dfrac{2^{2\rho}\,\Gamma(\beta)}{\Gamma(a)\,y^{2\rho+1/2}}$ $\times G^{21}_{23}\left(\dfrac{y^2}{4\lambda}\,\middle\vert\,\begin{matrix}1,\ \beta\\ \tfrac{1}{2}+\rho+\tfrac{1}{2}\nu,\ a,\ \tfrac{1}{2}+\rho-\tfrac{1}{2}\nu\end{matrix}\right)$
(7)	$x^{\nu+1/2}\ {}_1F_2(a;\beta,\nu+1;-\tfrac{1}{4}x^2)$ $\mathrm{Re}\,\nu>-1,\quad \mathrm{Re}\,\beta>\mathrm{Re}\,a>0$	$\dfrac{\Gamma(\nu+1)\,\Gamma(\beta)}{\Gamma(a)\,\Gamma(\beta-a)}\,2^{\nu+1}\,y^{2a-\nu-3/2}$ $\times(1-y^2)^{\beta-a-1} \qquad 0<y<1$ $0 \qquad 1<y<\infty$
(8)	$x^{\nu+1/2}(1-x^2)^{\mu-\nu-1}$ $\times{}_1F_2[\mu+\tfrac{1}{2};\,2\mu+1,\,\mu-\nu;-a(1-x^2)]$ $0<x<1$ $0 \qquad 1<x<\infty$ $\mathrm{Re}\,\mu>\mathrm{Re}\,\nu>-1$	$2^{2\mu-\nu-1}\,a^{-\mu}\Gamma(1+\mu)\Gamma(\mu-\nu)y^{\nu+1/2}$ $\times J_\mu[\tfrac{1}{2}(y^2+4a^2)^{1/2}+\tfrac{1}{2}y]$ $\times J_\mu[\tfrac{1}{2}(y^2+4a^2)^{1/2}-\tfrac{1}{2}y]$

Miscellaneous functions (cont'd)

	$f(x)$	$\int_0^\infty f(x)\, J_\nu(xy)(xy)^{1/2}\, dx \qquad y>0$
(9)	$x^{\nu+1/2}(1-x^2)^{\mu-3/2} \times {}_1F_2\left[1;\frac{\mu}{2},\frac{\mu+1}{2};-a^2(1-x^2)^2\right]$ $\quad 0<x<1$ $0 \quad 1<x<\infty$ $\operatorname{Re}\nu>-1,\quad \operatorname{Re}\mu>0$	$2^{-\mu-2\nu-1}\,\Gamma(\mu)\, a^{-\mu-\nu} y^{\nu+1/2}\, U_{\nu+\mu}(4a,y)$
(10)	$x^{\rho-1/2} \times {}_2F_2\left(\rho,\frac{\rho+m+1}{2};\frac{\rho-m+1}{2},\frac{\rho+\nu+1}{2};\frac{x^2}{2}\right)$ $\operatorname{Re}(\rho+\nu+1)>0,\quad \operatorname{Re}\rho>1/2$ $\operatorname{Re}(\rho-m+1)>0$	$(-1)^m y^{\rho-1/2} \times {}_2F_2\left(\rho,\frac{\rho+m+1}{2};\frac{\rho-m+1}{2},\frac{\rho+\nu+1}{2};-\frac{y^2}{2}\right)$
(11)	$x^{\nu+1/2} \times {}_3F_2(a,a+\beta,a-\beta;a+1/2,\nu+1;-x^2)$ $\operatorname{Re}a>1/2\operatorname{Re}\nu+1/4>-1/4$ $\lvert\operatorname{Re}\beta\rvert<\operatorname{Re}a-1/2\operatorname{Re}\nu-1/4$	$\frac{2^{\nu-2a+2}\,\Gamma(a+1/2)\,\Gamma(\nu+1)}{\pi^{1/2}\,\Gamma(a)\,\Gamma(a+\beta)\,\Gamma(a-\beta)} \times y^{-2a-\nu-3/2}\,[K_\beta(1/2 y)]^2$
(12)	$x^{\nu+1/2} \times {}_3F_2(\beta+\rho,\beta-\rho,2\beta-\nu-1;\beta,\beta+1/2;-1/4 x^2)$ $\lvert\operatorname{Re}\rho\rvert<\operatorname{Re}\beta-1/2\operatorname{Re}\nu-1/4$ $\operatorname{Re}\nu>-1,\quad \operatorname{Re}(4\beta-3\nu)>5/2$	$\frac{2^{\nu+3-2\beta}\,\Gamma(2\beta)\, y^{2\beta-\nu-3/2}}{\Gamma(\beta+\rho)\,\Gamma(\beta-\rho)\,\Gamma(2\beta-\nu-1)} \times K_{\nu+1-\beta+\rho}(y)\, K_{\nu+1-\beta-\rho}(y)$

Miscellaneous functions (cont'd)

	$f(x)$	$\int_0^\infty f(x)\, J_\nu(xy)(xy)^{1/2}\,dx \qquad y>0$
(13)	$x^{\nu+1/2}$ $\times {}_3F_2(a, a+\tfrac{1}{2}, 2a-\nu-1;$ $a+\tfrac{1}{2}+\kappa, a+\tfrac{1}{2}-\kappa; -x^2)$ $-1<\operatorname{Re}\nu<2\operatorname{Re}a-1/2$ $\operatorname{Re}(4a-3\nu)>5/2$	$\dfrac{\Gamma(a+\tfrac{1}{2}+\kappa)\,\Gamma(a+\tfrac{1}{2}-\kappa)}{\Gamma(2a)\,\Gamma(2a-\nu-1)}\,2^{\nu+1}$ $\times y^{2a-\nu-5/2}\,W_{\kappa,a-\nu-1}(y)$ $\times W_{-\kappa,a-\nu-1}(y)$
(14)	$x^{\nu+1/2}$ $\times {}_3F_2(\nu+\tfrac{1}{2}, \nu+\tfrac{1}{2}+\mu, \nu+\tfrac{1}{2}-\mu;$ $\nu+1+\kappa, \nu+1-\kappa; -x^2)$ $\operatorname{Re}\nu>-\tfrac{1}{2}$ $\lvert\operatorname{Re}\mu\rvert<\tfrac{1}{2}\operatorname{Re}\nu+\tfrac{1}{4}$	$\dfrac{\Gamma(\nu+\kappa+1)\,\Gamma(\nu-\kappa+1)\,\pi^{1/2}\,2^{1-\nu}}{\Gamma(\nu+\tfrac{1}{2})\,\Gamma(\nu+\tfrac{1}{2}+\mu)\,\Gamma(\nu+\tfrac{1}{2}-\mu)}$ $\times y^{\nu-3/2}\,W_{\kappa,\mu}(y)\,W_{-\kappa,\mu}(y)$
(15)	$x^{\nu+1/2}$ $\times {}_3F_2\left(\nu+\dfrac{3}{2}, \nu+1+\mu, \nu+1-\mu;\ \nu+\dfrac{3}{2}+\kappa, \nu+\dfrac{3}{2}-\kappa; -x^2\right)$ $\operatorname{Re}\nu>-1$ $\lvert\operatorname{Re}\mu\rvert<\tfrac{1}{2}\operatorname{Re}\nu+\tfrac{3}{4}$	$\dfrac{\Gamma(\nu+\kappa+3/2)\,\Gamma(\nu-\kappa+3/2)\,\pi^{1/2}\,2^{-\nu}}{\Gamma(\nu+3/2)\,\Gamma(\nu+1+\mu)\,\Gamma(\nu+1-\mu)}$ $\times y^{\nu-1/2}\,W_{\kappa,\mu}(y)\,W_{-\kappa,\mu}(y)$
(16)	$x^{\nu+1/2}$ $\times {}_4F_3(a+\beta, a-\beta, a+\gamma, a-\gamma;$ $a, a+\tfrac{1}{2}, \nu+1; -x^2)$ $-1<\operatorname{Re}\nu<2\operatorname{Re}a-\tfrac{1}{2}$ $\lvert\operatorname{Re}\beta\rvert<\operatorname{Re}a-\tfrac{1}{2}\operatorname{Re}\nu-\tfrac{1}{4}$ $\lvert\operatorname{Re}\gamma\rvert<\operatorname{Re}a-\tfrac{1}{2}\operatorname{Re}\nu-\tfrac{1}{4}$	$\dfrac{2^{\nu-4a+3}\,\Gamma(2a)\,\Gamma(\nu+1)\,y^{2a-\nu-3/2}}{\Gamma(a+\beta)\,\Gamma(a-\beta)\,\Gamma(a-\gamma)\,\Gamma(a+\gamma)}$ $\times K_{\beta+\gamma}(\tfrac{1}{2}y)\,K_{\beta-\gamma}(\tfrac{1}{2}y)$

Miscellaneous functions (cont'd)

	$f(x)$	$\int_0^\infty f(x)\, J_\nu(xy)(xy)^{1/2}\, dx \qquad y > 0$
(17)	$x^{\nu+1/2} \times {}_4F_3(a, a+\frac{1}{2}, a+\mu, a-\mu; \frac{1}{2}+a+\kappa, \frac{1}{2}+a-\kappa; -x^2)$ $-1 < \operatorname{Re}\nu < 2\operatorname{Re}a - \frac{1}{2}$ $\lvert\operatorname{Re}\mu\rvert < \operatorname{Re}a - \frac{1}{2}\operatorname{Re}\nu - \frac{1}{4}$	$\dfrac{\Gamma(\frac{1}{2}+a+\kappa)\,\Gamma(\frac{1}{2}+a-\kappa)\,\Gamma(\nu+1)}{\Gamma(2a)\,\Gamma(a+\mu)\,\Gamma(a-\mu)} \times 2^{\nu+1} y^{2a-\nu-5/2} W_{\kappa,\mu}(y) W_{-\kappa,\mu}(y)$
(18)	$x^{2\rho-1/2} \times {}_pF_p(a_1, \ldots, a_p; \beta_1, \ldots, \beta_p; -\lambda x^2)$ $-1-\operatorname{Re}\nu < 2\operatorname{Re}\rho < \frac{1}{2} + 2\operatorname{Re}a_r$ $\operatorname{Re}\lambda > 0, \quad r = 1, \ldots, p$	$\dfrac{\Gamma(\beta_1)\cdots\Gamma(\beta_p)}{\Gamma(a_1)\cdots\Gamma(a_p)}\, 2^{2\rho} y^{-2\rho-1/2} \times G^{p+1,1}_{p+1,p+2}\left(\dfrac{y^2}{4\lambda}\,\middle\vert\, \begin{matrix} 1, \beta_1, \ldots, \beta_p \\ h, a_1, \ldots, a_p, k \end{matrix}\right)$ $h = \frac{1}{2} + \rho + \frac{1}{2}\nu, \quad k = \frac{1}{2} + \rho - \frac{1}{2}\nu$
(19)	$x^{2\rho-1/2} \times {}_{m+1}F_m(a_1, \ldots, a_{m+1}; \beta_1, \ldots, \beta_m; -\lambda^2 x^2)$ $\operatorname{Re}(2\rho+\nu) > -1, \quad \operatorname{Re}\lambda > 0$ $\operatorname{Re}(\rho - a_r) < \frac{1}{4}, \quad r = 1, \ldots, m+1$	$\dfrac{2^{2\rho}\Gamma(\beta_1)\cdots\Gamma(\beta_m)}{\Gamma(a_1)\cdots\Gamma(a_{m+1})\, y^{2\rho+1/2}} \times G^{m+2,1}_{m+1,m+3}\left(\dfrac{y^2}{4\lambda^2}\,\middle\vert\, \begin{matrix} 1, \beta_1, \ldots, \beta_m \\ h, a_1, \ldots, a_{m+1}, k \end{matrix}\right)$ $h = \frac{1}{2} + \rho + \frac{1}{2}\nu, \quad k = \frac{1}{2} + \rho - \frac{1}{2}\nu$
(20)	$x^{2\rho-1/2}\, G^{mn}_{pq}\left(\lambda x^2 \,\middle\vert\, \begin{matrix} a_1, \ldots a_p \\ \beta_1, \ldots, \beta_q \end{matrix}\right)$ $p + q < 2(m+n)$ $\lvert\arg\lambda\rvert < (m + n - \frac{1}{2}p - \frac{1}{2}q)\pi$ $\operatorname{Re}(\beta_j + \rho + \frac{1}{2}\nu) > -\frac{1}{2}$ $j = 1, \ldots, m$ $\operatorname{Re}(a_j + \rho) < \frac{3}{4}, \quad j = 1, \ldots, n$	$\dfrac{2^{2\rho}}{y^{2\rho+1/2}} \times G^{m,n+1}_{p+2,q}\left(\dfrac{4\lambda}{y^2}\,\middle\vert\, \begin{matrix} h, a_1, \ldots, a_p, k \\ \beta_1, \ldots, \beta_q \end{matrix}\right)$ $h = \frac{1}{2} - \rho - \frac{1}{2}\nu, \quad k = \frac{1}{2} - \rho + \frac{1}{2}\nu$

Miscellaneous functions (cont'd)

	$f(x)$	$\int_0^\infty f(x)\, J_\nu(xy)(xy)^{1/2}\, dx \qquad y>0$
(21)	$x^{\nu+2n-1}\,[\mathrm{B}(a+x,a-x)]^{-1}$ $-1<\mathrm{Re}\,\nu<2a-2n-7/2$	$0 \qquad \pi\le y<\infty$
(22)	$x^{\nu+1/2}\,\mathrm{Erfc}\,(ax)$ $\lvert\arg a\rvert<\tfrac{1}{4}\pi,\quad \mathrm{Re}\,\nu>-1$	$a^{-\nu}\dfrac{\Gamma(\nu+3/2)}{\Gamma(\nu+2)}\, y^{-3/2}\exp\left(-\dfrac{y^2}{8a^2}\right)$ $\times M_{\frac{1}{2}\nu+\frac{1}{2},\frac{1}{2}\nu+\frac{1}{2}}\left(\dfrac{y^2}{4a^2}\right)$
(23)	$x^{\nu-1/2}\,\mathrm{Erfc}\,(ax)$ $\lvert\arg a\rvert<\tfrac{1}{4}\pi,\quad \mathrm{Re}\,\nu>-\tfrac{1}{2}$	$2^{-1/2}\,a^{1/2-\nu}\dfrac{\Gamma(\nu+1/2)}{\Gamma(\nu+3/2)}\, y^{-1}\exp\left(-\dfrac{y^2}{8a^2}\right)$ $\times M_{\frac{1}{2}\nu-\frac{1}{4},\frac{1}{2}\nu+\frac{1}{4}}\left(\dfrac{y^2}{4a^2}\right)$
(24)	$x^{-\mu-1/2}\, s_{\nu+\mu,-\nu+\mu+1}(x)$ $\mathrm{Re}\,\mu>-1,\quad -1<\mathrm{Re}\,\nu<3/2$	$2^{\nu-1}\,\Gamma(\nu)\,y^{1/2-\nu}(1-y^2)^{\mu} \qquad 0<y<1$ $0 \qquad 1<y<\infty$

CHAPTER IX

Y-TRANSFORMS

We call

$$\mathfrak{Y}_{\nu}\{f(x); y\} = \int_0^{\infty} f(x)\, Y_{\nu}(xy)\,(xy)^{1/2}\, dx$$

the *Y-transform of order ν* of $f(x)$ and regard y as a positive real variable. The inversion formula 9.1(1) has been given by Titchmarsh (1937, p. 215). The reciprocal transform is the **H**-transform (see chapter XI).

From the transform pairs given in this chapter further transform pairs may be derived by means of the methods indicated in the introduction to volume I, and also by the general formulas of sec. 9.1. Moreover, Y-transforms and Hankel transforms are connected by the relation

$$\mathfrak{Y}_{\nu}\{f(x); y\} = \operatorname{ctn}(\nu\pi)\, \mathfrak{H}_{\nu}\{f(x); y\} - \csc(\nu\pi)\, \mathfrak{H}_{-\nu}\{f(x); y\}$$

which is an immediate consequence of the relation between Bessel functions of the first and second kind and may be used to evaluate Y-transforms by means of the table of Hankel transforms given in chapter VIII.

REFERENCE

Titchmarsh, E.C., 1937: *Introduction to the theory of Fourier integrals*, Oxford.

Y-TRANSFORMS

9.1. General formulas

	$f(x)$	$\int_0^\infty f(x)\, Y_\nu(xy)\,(xy)^{1/2}\,dx = g(y;\nu)$ $\quad y>0$
(1)	$\int_0^\infty g(y)\, \mathbf{H}_\nu(xy)(xy)^{1/2}\,dy$ $\quad -\tfrac{1}{2}<\operatorname{Re}\nu<\tfrac{1}{2}$	$g(y)$
(2)	$f(ax)$ $\quad a>0$	$a^{-1}\, g(a^{-1} y;\nu)$
(3)	$x^m\, f(x)$ $\quad m=0,1,2,\dots$	$y^{1/2-\nu}\left(\dfrac{d}{y\,dy}\right)^m [y^{\nu-1/2+m}\, g(y;\nu+m)]$
(4)	$x^m\, f(x)$ $\quad m=0,1,2,\dots$	$(-1)^m\, y^{1/2+\nu}\left(\dfrac{d}{y\,dy}\right)^m \times [y^{-1/2+m-\nu}\, g(y;\nu-m)]$
(5)	$2\nu\, x^{-1}\, f(x)$	$y\,g(y;\nu-1)+y\,g(y;\nu+1)$
(6)	$2\nu\, f'(x)$	$(\nu-\tfrac{1}{2})\,y\,g(y;\nu+1) - (\nu+\tfrac{1}{2})\,y\,g(y;\nu-1)$

General formulas (cont'd)

	$f(x)$	$\int_0^\infty f(x)\, Y_\nu(xy)\,(xy)^{1/2}\, dx = g(y;\nu) \qquad y>0$
(7)	$x^{1/2-\nu}\left(\dfrac{d}{x\,dx}\right)^m \times [x^{\nu+m-1/2} f(x)]$ $m=0,1,2,\ldots$	$y^m\, g(y;\nu+m)$
(8)	$x^{1/2+\nu}\left(\dfrac{d}{x\,dx}\right)^m \times [x^{m-\nu-1/2} f(x)]$ $m=0,1,2,\ldots$	$(-y)^m\, g(y;\nu-m)$
(9)	$x^{1/2-\nu}\int_0^x \xi^{\nu-\mu+1/2}(x^2-\xi^2)^{\mu-1} \times f(\xi)\, d\xi$ $\mathrm{Re}\,\nu+3/2>\mathrm{Re}\,\mu>0$	$2^{\mu-1}\,\Gamma(\mu)\, y^{-\mu}\, g(y;\nu-\mu)$
(10)	$x^{-\mu} f(x)$ $\mathrm{Re}\,\mu>0, \quad \mathrm{Re}\,\nu>-3/2$	$2^{1-\mu}[\Gamma(\mu)]^{-1}\, y^{\nu+1/2} \times \int_y^\infty \eta^{1/2-\mu-\nu}(\eta^2-y^2)^{\mu-1} \times g(\eta;\nu+\mu)\, d\eta$

9.2. Algebraic functions and powers with an arbitrary index

	$f(x)$	$\int_0^\infty f(x)\, Y_\nu(xy)\,(xy)^{1/2}\, dx \qquad y>0$
(1)	$x^{-1/2} \qquad -1<\mathrm{Re}\,\nu<1$	$-\tan(\tfrac{1}{2}\nu\pi)\, y^{-1/2}$

Algebraic functions (cont'd)

	$f(x)$	$\int_0^\infty f(x)\, Y_\nu(xy)(xy)^{1/2}\,dx \quad y>0$
(2)	$0 \quad 0<x<a$ $x^{1/2+\nu} \quad a<x<\infty$ $\operatorname{Re}\nu<-1/2$	$-a^{\nu+1}y^{-1/2}\,Y_{\nu+1}(ay)$
(3)	$x^\mu \quad \lvert\operatorname{Re}\nu\rvert-3/2<\mu<0$	$2^{\mu+1/2}\operatorname{ctn}[\tfrac{1}{2}(\nu+\tfrac{1}{2}-\mu)\pi]\,y^{-\mu-1}$ $\times\dfrac{\Gamma(3/4+\tfrac{1}{2}\nu+\tfrac{1}{2}\mu)}{\Gamma(1/4+\tfrac{1}{2}\nu-\tfrac{1}{2}\mu)}$
(4)	$0 \quad 0<x<a$ $x^\mu \quad a<x<\infty$ $\operatorname{Re}\mu<0$	$ay^{-\mu}[Y_{\nu-1}(ay)\,S_{\mu+1/2,\nu}(ay)$ $-(\mu+\nu-\tfrac{1}{2})\,Y_\nu(ay)$ $\times S_{\mu-1/2,\nu-1}(ay)]$
(5)	$x^{-1/2}(x+a)^{-1}$ $\lvert\arg a\rvert<\pi, \quad -1<\operatorname{Re}\nu<1$ $\nu\neq 0, \pm\tfrac{1}{2}$	$\pi y^{1/2}\csc(\nu\pi)\,[\mathbf{E}_\nu(ay)+Y_\nu(ay)]$ $+2\operatorname{ctn}(\nu\pi)\,[\mathbf{J}_\nu(ay)-J_\nu(ay)]$
(6)	$x^{\nu-1/2}(x+a)^{-1} \quad \lvert\arg a\rvert<\pi$ $-1/2<\operatorname{Re}\nu<3/2$	$-2^{\nu+1}\pi^{-1}a^\nu y^{1/2}\,\Gamma(\nu+1)$ $\times S_{-\nu-1,\nu}(ay)$
(7)	$x^{-\nu-1/2}(x+a)^{-1} \quad \lvert\arg a\rvert<\pi$ $-3/2<\operatorname{Re}\nu<1/2$	$a^{-\nu}y^{1/2}\{\tfrac{1}{2}\pi\tan(\nu\pi)[Y_\nu(ay)$ $-\mathbf{H}_\nu(ay)]-2^{1-\nu}\pi^{-1}\cos(\nu\pi)$ $\times\Gamma(1-\nu)\,S_{\nu-1,\nu}(ay)\}$

Algebraic functions (cont'd)

	$f(x)$	$\int_0^\infty f(x)\, Y_\nu(xy)(xy)^{1/2}\,dx \qquad y>0$
(8)	$x^{\mu-1/2}(x+a)^{-1} \qquad \lvert\arg a\rvert<\pi$ $\mathrm{Re}(\mu\pm\nu)>-1, \quad \mathrm{Re}\,\mu<3/2$	$(2a)^\mu \pi^{-1} y^{1/2}$ $\times\{\sin[\tfrac12\pi(\mu-\nu)]\Gamma(\tfrac12+\tfrac12\mu+\tfrac12\nu)$ $\times\Gamma(\tfrac12+\tfrac12\mu-\tfrac12\nu)\,S_{-\mu,\nu}(ay)$ $-2\cos[\tfrac12\pi(\mu-\nu)]\,\Gamma(1+\tfrac12\mu+\tfrac12\nu)$ $\times\Gamma(1+\tfrac12\mu-\tfrac12\nu)\,S_{-\mu-1,\nu}(ay)\}$
(9)	$x^{-1/2}(x-a)^{-1}$ $a>0, \quad -1<\mathrm{Re}\,\nu<1$	$\pi y^{1/2}\{\mathrm{ctn}(\nu\pi)\,[Y_\nu(ay)+\mathbf{E}_\nu(ay)]$ $+\mathbf{J}_\nu(ay)+2[\mathrm{ctn}(\nu\pi)]^2$ $\times[\mathbf{J}_\nu(ay)-J_\nu(ay)]\}$ The integral is a Cauchy Principal Value.
(10)	$x^{\nu-1/2}(x-a)^{-1}$ $a>0, \quad -1/2<\mathrm{Re}\,\nu<3/2$	$a^\nu y^{1/2}[\pi J_\nu(ay)-2^{\nu+1}\pi^{-1}\Gamma(\nu+1)$ $\times S_{-\nu-1,\nu}(ay)]$ The integral is a Cauchy Principal Value.
(11)	$x^{-\nu-1/2}(x-a)^{-1}$ $a>0, \quad -3/2<\mathrm{Re}\,\nu<1/2$	$a^{-\nu} y^{1/2}\{\tfrac12\pi\tan(\nu\pi)\,[\mathbf{H}_\nu(ay)$ $-Y_\nu(ay)]+\pi J_\nu(ay)$ $-2^{1-\nu}\pi^{-1}\cos(\nu\pi)\,\Gamma(1-\nu)$ $\times S_{\nu-1,\nu}(ay)\}$ The integral is a Cauchy Principal Value.

Algebraic functions (cont'd)

	$f(x)$	$\int_0^\infty f(x)\, Y_\nu(xy)(xy)^{1/2}\, dx \qquad y>0$
(12)	$x^{\mu-1/2}(x-a)^{-1} \qquad a>0$ $\mathrm{Re}(\mu\pm\nu)>-1, \quad \mathrm{Re}\,\mu>3/2$	$\pi a^\mu y^{1/2} J_\nu(ay)-(2a)^\mu \pi^{-1} y^{1/2}$ $\times\{\sin[\tfrac{1}{2}\pi(\mu-\nu)]\,\Gamma(\tfrac{1}{2}+\tfrac{1}{2}\mu+\tfrac{1}{2}\nu)$ $\times\Gamma(\tfrac{1}{2}+\tfrac{1}{2}\mu-\tfrac{1}{2}\nu)\, S_{-\mu,\nu}(ay)$ $+2\cos[\tfrac{1}{2}\pi(\mu-\nu)]\,\Gamma(1+\tfrac{1}{2}\mu+\tfrac{1}{2}\nu)$ $\times\Gamma(1+\tfrac{1}{2}\mu-\tfrac{1}{2}\nu)\, S_{-\mu-1,\nu}(ay)\}$ The integral is a Cauchy Principal Value.
(13)	$x^{-1/2}(x^2+a^2)^{-1}$ $\mathrm{Re}\,a>0, \quad -1<\mathrm{Re}\,\nu<1$	$\dfrac{y^{1/2}}{\cos\tfrac{1}{2}\nu\pi}\Big[-\dfrac{\pi}{2a}\tan\Big(\dfrac{\nu\pi}{2}\Big)I_\nu(ay)$ $-\dfrac{1}{a}K_\nu(ay)+\dfrac{y\sin(\tfrac{1}{2}\nu\pi)}{1-\nu^2}$ $\times\, {}_1F_2\Big(1;\dfrac{3-\nu}{2},\dfrac{3+\nu}{2};\dfrac{a^2y^2}{4}\Big)\Big]$
(14)	$x^{\nu-1/2}(x^2+a^2)^{-1}$ $\mathrm{Re}\,a>0, \quad -1/2<\mathrm{Re}\,\nu<5/2$	$-a^{\nu-1}y^{1/2}K_\nu(ay)$
(15)	$x^{\nu+3/2}(x^2+a^2)^{-1}$ $\mathrm{Re}\,a>0, \quad -3/2<\mathrm{Re}\,\nu<1/2$	$a^{\nu+1}y^{1/2}K_\nu(ay)$
(16)	$x^{-\nu-1/2}(x^2+a^2)^{-1} \qquad \mathrm{Re}\,a>0$ $-5/2<\mathrm{Re}\,\nu<1/2$	$a^{-\nu-1}y^{1/2}\{\tfrac{1}{2}\pi\tan(\nu\pi)\,[\mathbf{L}_\nu(ay)$ $-I_\nu(ay)]-\sec(\nu\pi)\,K_\nu(ay)\}$

Algebraic functions (cont'd)

	$f(x)$	$\int_0^\infty f(x)\, Y_\nu(xy)(xy)^{1/2}\, dx \qquad y>0$
(17)	$x^{\mu-3/2}(x^2+a^2)^{-1}$ $\quad \operatorname{Re} a>0$ $\lvert\operatorname{Re}\nu\rvert<\operatorname{Re}\mu<7/2$	$2^{\mu-3}\pi^{-1}y^{5/2-\mu}\cos[\tfrac{1}{2}\pi(\mu-\nu)]$ $\times\Gamma(\tfrac{1}{2}\mu+\tfrac{1}{2}\nu-1)\,\Gamma(\tfrac{1}{2}\mu-\tfrac{1}{2}\nu-1)$ $\times\,{}_1F_2\left(1;2-\dfrac{\mu+\nu}{2},2-\dfrac{\mu-\nu}{2};\dfrac{a^2y^2}{4}\right)$ $-\tfrac{1}{2}\pi a^{\mu-2}y^{1/2}\csc[\tfrac{1}{2}\pi(\mu+\nu)]$ $\times\operatorname{ctn}[\tfrac{1}{2}\pi(\mu-\nu)]\,I_\nu(ay)$ $-a^{\mu-2}y^{1/2}\csc[\tfrac{1}{2}\pi(\mu-\nu)]\,K_\nu(ay)$
(18)	$x^{-1/2}(x^2+a^2)^{-1/2}$ $\operatorname{Re} a>0,\quad -1<\operatorname{Re}\nu<1$	$-\pi^{-1}y^{1/2}\sec(\tfrac{1}{2}\nu\pi)\,K_{\frac{1}{2}\nu}(\tfrac{1}{2}ay)$ $\times[K_{\frac{1}{2}\nu}(\tfrac{1}{2}ay)+\pi\sin(\tfrac{1}{2}\nu\pi)$ $\times I_{\frac{1}{2}\nu}(\tfrac{1}{2}ay)]$
(19)	$x^{\frac{1}{2}+\nu}(x^2+a^2)^\mu$ $\quad \operatorname{Re} a>0$ $-1<\operatorname{Re}\nu<-2\operatorname{Re}\mu$	$2^{\nu-1}\pi^{-1}a^{2\mu+2}(1+\mu)^{-1}\Gamma(\nu)\,y^{\frac{1}{2}-\nu}$ $\times\,{}_1F_2(1;1-\nu,2+\mu;\tfrac{1}{4}a^2y^2)$ $-2^\mu a^{\mu+\nu+1}(\sin\nu\pi)^{-1}\Gamma(\mu+1)$ $\times y^{-\frac{1}{2}-\mu}[I_{\mu+\nu+1}(ay)$ $-2\cos(\mu\pi)\,K_{\mu+\nu+1}(ay)]$
(20)	$x^{\frac{1}{2}-\nu}(x^2+a^2)^\mu$ $\quad \operatorname{Re} a>0$ $\tfrac{1}{2}+2\operatorname{Re}\mu<\operatorname{Re}\nu<1$	$2^\mu a^{\mu-\nu+1}y^{-\frac{1}{2}-\mu}\{\pi^{-1}\cos(\nu\pi)$ $\times\Gamma(\mu+1)\,\Gamma(\nu)\,I_{\nu-\mu-1}(ay)$ $-2\csc(\nu\pi)[\Gamma(-\mu)]^{-1}K_{\nu-\mu-1}(ay)\}$ $-\dfrac{a^{2\mu+2}\operatorname{ctn}(\nu\pi)\,y^{\frac{1}{2}+\nu}}{2^{\nu+1}(\mu+1)\,\Gamma(\nu+1)}$ $\times\,{}_1F_2(1;\nu+1,\mu+2;\tfrac{1}{4}a^2y^2)$

Algebraic functions (cont'd)

	$f(x)$	$\int_0^\infty f(x)\, Y_\nu(xy)\,(xy)^{1/2}\,dx \qquad y>0$
(21)	$x^{-1/2}(x^2-a^2)^{-1}$ $a>0, \quad -1<\mathrm{Re}\,\nu<1$	$\tfrac12\pi a^{-1} y^{1/2}[J_\nu(ay)$ $+\tan(\tfrac12\nu\pi)\{\tan(\tfrac12\nu\pi)[\mathbf{J}_\nu(ay)$ $-J_\nu(ay)]-\mathbf{E}_\nu(ay)-Y_\nu(ay)\}]$ The integral is a Cauchy Principal Value.
(22)	$x^{\nu-1/2}(x^2-a^2)^{-1}$ $a>0, \ -1/2<\mathrm{Re}\,\nu<5/2$	$\tfrac12\pi a^{\nu-1} y^{1/2} J_\nu(ay)$ The integral is a Cauchy Principal Value.
(23)	$x^{\nu+1/2}(x^2-a^2)^{-1}$ $a>0, \quad -1<\mathrm{Re}\,\nu<3/2$	$\tfrac12\pi a^{\nu} y^{1/2} J_\nu(ay)$ $-2^{\nu+1}\pi^{-1}\Gamma(\nu+1)\,a^\nu y^{1/2}$ $\times S_{-\nu-1,\nu}(ay)$ The integral is a Cauchy Principal Value.
(24)	$x^{-\nu-1/2}(x^2-a^2)^{-1}$ $a>0, \quad -5/2<\mathrm{Re}\,\nu<1/2$	$\tfrac12\pi a^{-\nu-1} y^{1/2}\sec(\nu\pi)\,[J_{-\nu}(ay)$ $+\ \sin(\nu\pi)\,\mathbf{H}_\nu(ay)]$ The integral is a Cauchy Principal Value
(25)	$x^{\mu-3/2}(x^2-a^2)^{-1}$ $a>0, \quad \lvert\mathrm{Re}\,\nu\rvert<\mathrm{Re}\,\mu<7/2$	$\tfrac12\pi a^{\mu-2} y^{1/2} J_\nu(ay)$ $+2^{\mu-1}\pi^{-1}a^{\mu-2}y^{1/2}\cos[\tfrac12\pi(\mu-\nu)]$ $\times\Gamma\left(\frac{\mu-\nu}{2}\right)\Gamma\left(\frac{\mu+\nu}{2}\right)S_{1-\mu,\nu}(ay)$ The integral is a Cauchy Principal Value

Algebraic functions (cont'd)

	$f(x)$	$\int_0^\infty f(x)\, Y_\nu(xy)\,(xy)^{1/2}\,dx \qquad y>0$
(26)	$x^{-1/2}(a^2-x^2)^{-1/2} \quad 0<x<a$ $0 \quad a<x<\infty$ $\nu=0$	$\frac{1}{2}\pi y^{-1/2}\, J_0(\frac{1}{2}ay)\, Y_0(\frac{1}{2}ay)$
(27)	$0 \quad 0<x<a$ $x^{-1/2}(x^2-a^2)^{-1/2} \quad a<x<\infty$	$\frac{1}{4}\pi y^{1/2}\{[J_{\frac{1}{2}\nu}(\frac{1}{2}ay)]^2$ $-[Y_{\frac{1}{2}\nu}(\frac{1}{2}ay)]^2\}$
(28)	$x^{\nu+1/2}(a^2-x^2)^{-1/2} \quad 0<x<a$ $0 \quad a<x<\infty$ $\mathrm{Re}\,\nu>-1$	$(\frac{1}{2}\pi)^{1/2}\, a^{\nu+1/2}\csc(\nu\pi)$ $\times[\cos(\nu\pi)\, J_{\nu+1/2}(ay)$ $-\mathbf{H}_{-\nu-1/2}(y)]$
(29)	$0 \quad 0<x<a$ $x^{\nu+1/2}(x^2-a^2)^{-1/2} \quad a<x<\infty$ $\mathrm{Re}\,\nu<\frac{1}{2}$	$(\frac{1}{2}\pi)^{1/2}\, a^{\nu+1/2}\, J_{\nu+1/2}(ay)$
(30)	$x^{1/2-\nu}(a^2-x^2)^{-1/2} \quad 0<x<a$ $0 \quad a<x<\infty$ $\mathrm{Re}\,\nu<1$	$(\frac{1}{2}\pi)^{1/2}\, a^{1/2-\nu}\{\mathrm{ctn}(\nu\pi)[\mathbf{H}_{\nu-1/2}(ay)$ $-Y_{\nu-1/2}(ay)]-J_{\nu-1/2}(y)\}$
(31)	$x^{\nu-1/2}(a^2-x^2)^{\nu-1/2} \quad 0<x<a$ $0 \quad a<x<\infty$ $\mathrm{Re}\,\nu>-\frac{1}{2}$	$2^{\nu-1}\pi^{1/2}\, a^{2\nu}\, y^{1/2-\nu}\,\Gamma(\nu+\frac{1}{2})$ $\times J_\nu(\frac{1}{2}ay)\, Y_\nu(\frac{1}{2}ay)$

Algebraic functions (cont'd)

	$f(x)$	$\int_0^\infty f(x)\, Y_\nu(xy)\,(xy)^{1/2}\,dx \quad y>0$
(32)	$0 \quad 0<x<a$ $x^{\nu-1/2}(x^2-a^2)^{\nu-1/2} \quad a<x<\infty$ $-\tfrac{1}{2}<\operatorname{Re}\nu<\tfrac{1}{2}$	$2^{\nu-2}\pi^{1/2}a^{2\nu}y^{1/2-\nu}\Gamma(\nu+\tfrac{1}{2})$ $\times[J_\nu(\tfrac{1}{2}ay)\,J_{-\nu}(\tfrac{1}{2}ay)$ $-Y_\nu(\tfrac{1}{2}ay)\,Y_{-\nu}(\tfrac{1}{2}ay)]$
(33)	$0 \quad 0<x<a$ $x^{1/2-\nu}(x^2-a^2)^{\nu-1/2} \quad a<x<\infty$ $-\tfrac{1}{2}<\operatorname{Re}\nu<\tfrac{1}{2}$	$2^\nu\pi^{-1/2}y^{-1/2-\nu}\Gamma(\nu+\tfrac{1}{2})\sin(ay)$
(34)	$0 \quad 0<x<a$ $x^{-\nu-1/2}(x^2-a^2)^{-\nu-1/2}$ $a<x<\infty$ $-\tfrac{1}{2}<\operatorname{Re}\nu<\tfrac{1}{2}$	$2^{-\nu-2}\pi^{1/2}a^{-2\nu}y^{\nu+1/2}\Gamma(\tfrac{1}{2}-\nu)$ $\times\{[J_\nu(\tfrac{1}{2}ay)]^2-[Y_\nu(\tfrac{1}{2}ay)]^2\}$
(35)	$x^{\nu+1/2}(a^2-x^2)^\mu \quad 0<x<a$ $0 \quad a<x<\infty$ $\operatorname{Re}\mu>-1, \quad \operatorname{Re}\nu>-1$	$a^{\mu+\nu+1}y^{-\mu-1/2}[2^\mu\Gamma(\mu+1)$ $\times Y_{\mu+\nu+1}(ay)+2^{\nu+1}\pi^{-1}\Gamma(\nu+1)$ $\times S_{\mu-\nu,\,\mu+\nu+1}(ay)]$
(36)	$0 \quad 0<x<a$ $x^{\nu+1/2}(x^2-a^2)^\mu \quad a<x<\infty$ $-2<2\operatorname{Re}\mu<-\tfrac{1}{2}-\operatorname{Re}\nu$	$-2^\mu a^{\mu+\nu+1}y^{-\mu-1/2}\Gamma(\mu+1)$ $\times[\sin(\mu\pi)\,J_{\mu+\nu+1}(ay)$ $+\cos(\mu\pi)\,Y_{\mu+\nu+1}(ay)]$

Algebraic functions (cont'd)

	$f(x)$	$\int_0^\infty f(x)\, Y_\nu(xy)(xy)^{1/2}\, dx \qquad y>0$
(37)	$x^{1/2-\nu}(a^2-x^2)^\mu \quad 0<x<a$ $0 \quad a<x<\infty$ $\operatorname{Re}\mu>-1, \quad \operatorname{Re}\nu<1$	$a^{\mu-\nu+1}y^{-\mu-1/2}[2^{1-\nu}\pi^{-1}\cos(\nu\pi)$ $\times\Gamma(1-\nu)\, s_{\mu+\nu,\,\mu-\nu+1}(ay)$ $-2^\mu\csc(\nu\pi)\,\Gamma(\mu+1)$ $\times J_{\mu-\nu+1}(ay)]$
(38)	$0 \quad 0<x<a$ $x^{1/2-\nu}(x^2-a^2)^\mu \quad a<x<\infty$ $-1<\operatorname{Re}\mu<\tfrac{1}{2}\operatorname{Re}\nu-\tfrac{1}{4}$	$2^\mu a^{\mu-\nu+1}y^{-\mu-1/2}\,\Gamma(\mu+1)$ $\times Y_{\nu-\mu-1}(ay)$
(39)	$x^{2n+\nu+4\mu-1/2}(x^4+a^4)^{-\mu-1}$	see Watson, G.N., 1922: *Bessel Functions*, p. 432, Cambridge.
(40)	$x^{-1/2}(x^2+a^2)^{-1/2}$ $\times[(x^2+a^2)^{1/2}-x]^\mu$ $\operatorname{Re}a>0, \quad \operatorname{Re}\mu>-3/2$ $-1<\operatorname{Re}\nu<1$	$a^\mu y^{1/2}[\operatorname{ctn}(\nu\pi)\, I_{\frac{1}{2}\mu+\frac{1}{2}\nu}(\tfrac{1}{2}ay)$ $\times K_{\frac{1}{2}\mu-\frac{1}{2}\nu}(\tfrac{1}{2}ay)$ $-\csc(\nu\pi)\, I_{\frac{1}{2}\mu-\frac{1}{2}\nu}(\tfrac{1}{2}ay)$ $\times K_{\frac{1}{2}\mu+\frac{1}{2}\nu}(\tfrac{1}{2}ay)]$
(41)	$x^{-1/2}(x^2+a^2)^{-1/2}$ $\times\{[(x^2+a^2)^{1/2}+x]^\mu$ $+[(x^2+a^2)^{1/2}-x]^\mu\} \quad \nu=0$ $-3/2<\operatorname{Re}\mu<3/2$	$-2\pi^{-1}a^\mu y^{1/2}\cos(\tfrac{1}{2}\mu\pi)$ $\times[K_{\frac{1}{2}\mu}(\tfrac{1}{2}ay)]^2$

Algebraic functions (cont'd)

	$f(x)$	$\int_0^\infty f(x)\, Y_\nu(xy)(xy)^{1/2}\, dx \qquad y>0$
(42)	$x^{-1/2}(x^2+a^2)^{-1/2}$ $\times[(x^2+a^2)^{1/2}-a]^{2k}$ $\operatorname{Re} a>0, \quad \lvert\operatorname{Re}\nu\rvert<1/2+\operatorname{Re}k$	$-a^{-1}y^{-1/2}W_{-k,1/2\nu}(ay)$ $\times\left\{\dfrac{\Gamma(1/2+1/2\nu+k)}{\Gamma(\nu+1)}\tan[(1/2\nu-k)\pi]\right.$ $\times M_{k,1/2\nu}(ay)+\sec[(1/2\nu-k)\pi]$ $\left.\times W_{k,1/2\nu}(ay)\right\}$
(43)	$0 \qquad 0<x<a$ $x^{-1/2}(x^2-a^2)^{-1/2}$ $\times\{[x+(x^2-a^2)^{1/2}]^\mu$ $+[x-(x^2-a^2)^{1/2}]^\mu\}$ $a<x<\infty$ $-3/2<\operatorname{Re}\mu<3/2$	$1/2\pi a^\mu y^{1/2}[J_{1/2\nu+1/2\mu}(1/2ay)$ $\times J_{1/2\nu-1/2\mu}(1/2ay)-Y_{1/2\nu+1/2\mu}(1/2ay)$ $\times Y_{1/2\nu-1/2\mu}(1/2ay)]$

9.3. Other elementary functions

(1)	$x^{-1/2}e^{-ax}$ $\operatorname{Re}a>0, \quad -1<\operatorname{Re}\nu<1$	$y^{1/2}(y^2+a^2)^{-1/2}\csc(\nu\pi)$ $\times\{y^\nu[(y^2+a^2)^{1/2}+a]^{-\nu}\cos(\nu\pi)$ $-y^{-\nu}[(y^2+a^2)^{1/2}+a]^\nu\}$
(2)	$x^{\mu-3/2}e^{-ax}$ $a>0, \quad \operatorname{Re}\mu>\lvert\operatorname{Re}\nu\rvert$	$-2\pi^{-1}\Gamma(\mu+\nu)y^{1/2}(y^2+a^2)^{-1/2\mu}$ $\times Q_{\mu-1}^{-\nu}[a(y^2+a^2)^{-1/2}]$

Elementary functions (cont'd)

	$f(x)$	$\int_0^\infty f(x)\, Y_\nu(xy)(xy)^{1/2}\,dx \qquad y>0$
(3)	$x^{-1/2} e^{-ax^2}$ $\mathrm{Re}\, a>0, \quad -1<\mathrm{Re}\,\nu<1$	$-\frac{1}{2}\left(\frac{\pi y}{a}\right)^{1/2} \exp\left(-\frac{y^2}{8a}\right)$ $\times \left[\tan\frac{\nu\pi}{2}\, I_{1/2\nu}\left(\frac{y^2}{8a}\right)\right.$ $\left.+\pi^{-1}\sec\frac{\nu\pi}{2}\, K_{1/2\nu}\left(\frac{y^2}{8a}\right)\right]$
(4)	$x^{\mu-1/2} e^{-ax^2}$ $\mathrm{Re}\, a>0, \quad \mathrm{Re}\,\mu>\lvert\mathrm{Re}\,\nu\rvert-1$	$-a^{-1/2\mu} y^{-1/2} \sec[\tfrac{1}{2}(\nu-\mu)\pi]$ $\times \exp\left(-\frac{y^2}{8a}\right)$ $\times \left\{\frac{\Gamma(\frac{1}{2}+\frac{1}{2}\mu+\frac{1}{2}\nu)}{\Gamma(1+\nu)} \sin[\tfrac{1}{2}(\nu-\mu)\pi]\right.$ $\times M_{1/2\mu,\, 1/2\nu}\left(\frac{y^2}{4a}\right)$ $\left.+ W_{1/2\mu,\, 1/2\nu}\left(\frac{y^2}{4a}\right)\right\}$
(5)	$x^{-3/2} e^{-a/x} \qquad \mathrm{Re}\, a>0$	$2y^{1/2}\, Y_\nu[(2ay)^{1/2}]\, K_\nu[(2ay)^{1/2}]$
(6)	$x^{-1/2}(x^2+\beta^2)^{-1/2}$ $\times \exp[-a(x^2+\beta^2)^{-1/2}]$ $\mathrm{Re}\, a>0, \quad \mathrm{Re}\,\beta>0$ $-1<\mathrm{Re}\,\nu<1$	$-y^{1/2} \sec(\tfrac{1}{2}\nu\pi)$ $\times K_{1/2\nu}\{\tfrac{1}{2}\beta[(y^2+a^2)^{1/2}+a]\}$ $\times (\pi^{-1} K_{1/2\nu}\{\tfrac{1}{2}\beta[(y^2+a^2)^{1/2}-a]\}$ $+\sin(\tfrac{1}{2}\nu\pi)$ $\times I_{1/2\nu}\{\tfrac{1}{2}\beta[(y^2+a^2)^{1/2}-a]\})$

Elementary functions (cont'd)

	$f(x)$	$\int_0^\infty f(x)\, Y_\nu(xy)(xy)^{1/2}\, dx \qquad y>0$
(7)	$x^{-1/2} \sin ax^2$ $a>0, \quad -3<\mathrm{Re}\,\nu<3$	$-\frac{1}{4}\left(\frac{\pi y}{a}\right)^{1/2} \sec\left(\frac{\nu\pi}{2}\right)$ $\times \left[\cos\left(\frac{y^2}{8a} - \frac{3\nu+1}{4}\pi\right)\right.$ $\times J_{1/2\nu}\left(\frac{y^2}{8a}\right) - \sin\left(\frac{y^2}{8a}+\frac{\nu-1}{4}\pi\right)$ $\left.\times Y_{1/2\nu}\left(\frac{y^2}{8a}\right)\right]$
(8)	$x^{-1/2} \cos ax^2$ $a>0, \quad -1<\mathrm{Re}\,\nu<1$	$\frac{1}{4}\left(\frac{\pi y}{a}\right)^{1/2} \sec\left(\frac{\nu\pi}{2}\right)$ $\times \left[\sin\left(\frac{y^2}{8a} - \frac{3\nu+1}{4}\pi\right)\right.$ $\times J_{1/2\nu}\left(\frac{y^2}{8a}\right) + \cos\left(\frac{y^2}{8a}+\frac{\nu-1}{4}\pi\right)$ $\left.\times Y_{1/2\nu}\left(\frac{y^2}{8a}\right)\right]$
(9)	$x^{-1/2}(a^2-x^2)^{-1/2}$ $\times \sin[b(a^2-x^2)^{1/2}]$ $0<x<a$ $-x^{-1/2}(x^2-a^2)^{-1/2}$ $\times \exp[-b^2(x^2-a^2)^{1/2}]$ $a<x<\infty$ $b>0, \quad -1<\mathrm{Re}\,\nu<1$	$\frac{1}{4}\pi y^{1/2}\, Y^2_{1/2\nu}\{\frac{1}{2}a[(y^2+b^2)^{1/2}+b]\}$ $-\frac{1}{4}\pi y^{1/2}\, J^2_{1/2\nu}\{\frac{1}{2}a[(y^2+b^2)^{1/2}-b]\}$
	For other transforms containing trigonometric functions see the tables of Fourier transforms.	

9.4. Higher transcendental functions

	$f(x)$	$\int_0^\infty f(x)\, Y_\nu(xy)(xy)^{1/2}\, dx \qquad y>0$
(1)	$x^{1/2} P_n(1-2x^2) \qquad 0<x<1$ $0 \qquad 1<x<\infty$ $n=0, 1, 2, \ldots, \quad \nu=0$	$\pi^{-1} y^{-1/2} [S_{2n+1}(y)+\pi Y_{2n+1}(y)]$
(2)	$x^{\nu-2\mu+2n+3/2} \exp(x^2)\, \Gamma(\mu, x^2)$ n integer $\mathrm{Re}(\nu-\mu+n)>-3/2$ $\mathrm{Re}(-\mu+n)>-3/2$ $\mathrm{Re}\,\nu<\tfrac{1}{2}-2n$	$(-1)^n \dfrac{\Gamma(3/2-\mu+\nu+n)\Gamma(3/2-\mu+n)}{y^{1/2}\,\Gamma(1-\mu)}$ $\times \exp\left(\dfrac{y^2}{8}\right) W_{\mu-1/2\nu-n-1,\, 1/2\nu}\left(\dfrac{y^2}{4}\right)$
(3)	$0 \qquad 0<x<a$ $P_{\nu-1/2}(a^{-1}x) \qquad a<x<\infty$ $\mathrm{Re}\,\nu<\tfrac{1}{2}$	$\left(\dfrac{a}{2y}\right)^{1/2} [\cos(\tfrac{1}{2}ay)\, J_\nu(\tfrac{1}{2}ay)$ $-\sin(\tfrac{1}{2}ay)\, Y_\nu(\tfrac{1}{2}ay)]$
(4)	$0 \qquad 0<x<a$ $x^{-\mu}(x^2-a^2)^{-1/2\mu} P^{\mu}_{\nu-1/2}(x/a)$ $a<x<\infty$ $-\tfrac{1}{4}<\mathrm{Re}\,\mu<1$ $\mathrm{Re}(2\mu-\nu)>-\tfrac{1}{2}$	$2^{-3/2}\pi^{1/2} a^{1-\mu} y^{\mu} [J_\nu(\tfrac{1}{2}ay)$ $\times J_{\mu-1/2}(\tfrac{1}{2}ay) - Y_\nu(\tfrac{1}{2}ay)$ $\times Y_{\mu-1/2}(\tfrac{1}{2}ay)]$
(5)	$0 \qquad 0<x<a$ $(x^2-a^2)^{1/2\nu-1/4} P^{1/2-\nu}_{\mu}(2a^{-2}x^2-1)$ $a<x<\infty$ $\mathrm{Re}\,\nu>-\tfrac{1}{2}$ $\mathrm{Re}\,\nu+\lvert 2\,\mathrm{Re}\,\mu+1\rvert<3/2$	$\pi^{1/2} 2^{\nu-2} a y^{1/2-\nu} [J_{\mu+1/2}(\tfrac{1}{2}ay)$ $\times J_{-\mu-1/2}(\tfrac{1}{2}ay) - Y_{\mu+1/2}(\tfrac{1}{2}ay)$ $\times Y_{-\mu-1/2}(\tfrac{1}{2}ay)]$

Higher transcendental functions (cont'd)

	$f(x)$	$\int_0^\infty f(x)\, Y_\nu(xy)\,(xy)^{1/2}\,dx \quad y>0$
(6)	$x^\lambda J_\mu(ax)$	see under Mellin transforms
(7)	$\sin ax\, J_{\nu+1/2}(ax)$ $a>0, \quad \mathrm{Re}\,\nu > -3/2$	$(\pi\cos\theta)^{-1/2}(2a\sin\theta)^{-1}$ $\times\cos[(\nu+1)\theta]$ $y=2a\cos\theta, \quad 0<\theta<\tfrac{1}{2}\pi$ $0 \qquad 2a<y<\infty$
(8)	$x^{\nu+1/2}[J_\nu(ax)]^2$ $a>0, \quad -\tfrac{1}{2}<\mathrm{Re}\,\nu<\tfrac{1}{2}$	$0 \qquad 0<y<2a$ $\dfrac{2^{3\nu+1}a^{2\nu}}{\pi^{1/2}\,\Gamma(\tfrac{1}{2}-\nu)}\,y^{-\nu-1/2}(y^2-4a^2)^{-\nu-1/2}$ $2a<y<\infty$
(9)	$x^{1/2} J_{\frac{1}{2}\nu}(ax^2)$ $a>0, \quad \mathrm{Re}\,\nu>-1$	$\dfrac{y^{1/2}}{4a}\left[Y_{\frac{1}{2}\nu}\left(\dfrac{y^2}{4a}\right)\right.$ $-\tan\left(\dfrac{\nu\pi}{2}\right)J_{\frac{1}{2}\nu}\left(\dfrac{y^2}{4a}\right)$ $\left.+\sec\left(\dfrac{\nu\pi}{2}\right)\mathbf{H}_{-\frac{1}{2}\nu}\left(\dfrac{y^2}{4a}\right)\right]$
(10)	$x^{5/2} J_{\frac{1}{2}\nu-\frac{1}{2}}(ax^2)$ $a>0, \quad \mathrm{Re}\,\nu>-3/2$	$a^{-2}y^{1/2}J_{\frac{1}{2}\nu+\frac{1}{2}}\left(\dfrac{y^2}{4a}\right)$

Higher transcendental functions (cont'd)

	$f(x)$	$\int_0^\infty f(x)\, Y_\nu(xy)\,(xy)^{1/2}\,dx \qquad y>0$
(11)	$x^{1/2} J_{\frac{1}{4}\nu}(ax^2)\, J_{-\frac{1}{4}\nu}(ax^2)$ $a>0, \quad -2<\operatorname{Re}\nu<2$	$\dfrac{y^{1/2}\sec(\frac{1}{4}\nu\pi)}{16a}\Bigg\{[1+2\cos(\frac{1}{2}\nu\pi)]$ $\times\left[J_{\frac{1}{4}\nu}\left(\dfrac{y^2}{16a}\right)\right]^2+2\sin(\frac{1}{2}\nu\pi)$ $\times\, J_{\frac{1}{4}\nu}\left(\dfrac{y^2}{16a}\right) Y_{\frac{1}{4}\nu}\left(\dfrac{y^2}{16a}\right)$ $-\left[Y_{\frac{1}{4}\nu}\left(\dfrac{y^2}{16a}\right)\right]^2\Bigg\}$
(12)	$x^{-1/2} J_\nu(a^2x^{-1})$ $a>0, \quad -1/2<\operatorname{Re}\nu<3/2$	$y^{-1/2}[Y_{2\nu}(2ay^{1/2})$ $+2\pi^{-1}K_{2\nu}(2ay^{1/2})]$
(13)	$x^{-5/2} J_\nu(a^2x^{-1})$ $a>0, \quad -\frac{1}{2}<\operatorname{Re}\nu<\frac{1}{2}$	$a^{-2}y^{1/2}[Y_{2\nu}(2ay^{1/2})$ $-2\pi^{-1}K_{2\nu}(2ay^{1/2})]$
(14)	$x^{-1/2} Y_\nu(a^2x^{-1})$ $a>0, \quad -\frac{1}{2}<\operatorname{Re}\nu<\frac{1}{2}$	$-y^{-1/2}J_{2\nu}(2ay^{1/2})$
(15)	$x^{-5/2} Y_\nu(a^2x^{-1})$ $a>0, \quad -\frac{1}{2}<\operatorname{Re}\nu<\frac{1}{2}$	$-a^{-2}y^{1/2}J_{2\nu}(2ay^{1/2})$
(16)	$x^{-3/2} Y_{\nu+1}(a^2x^{-1})$ $a>0, \quad -3/2<\operatorname{Re}\nu<1/2$	$-a^{-1}J_{2\nu+1}(2ay^{1/2})$

Higher transcendental functions (cont'd)

	$f(x)$	$\int_0^\infty f(x)\, Y_\nu(xy)(xy)^{1/2}\,dx \quad y>0$
(17)	$J_{2\nu-1}(ax^{1/2})$ $a>0, \quad \operatorname{Re}\nu>-1/2$	$-\dfrac{a}{2y^{3/2}}\,\mathbf{H}_{\nu-1}\left(\dfrac{a^2}{4y}\right)$
(18)	$x^{-1/2} J_{2\nu}(ax^{1/2})$ $a>0, \quad \operatorname{Re}\nu>-1/2$	$-y^{-1/2}\,\mathbf{H}_\nu\left(\dfrac{a^2}{4y}\right)$
(19)	$x^{-1/2} Y_{2\nu}(ax^{1/2})$ $a>0, \quad -1/2<\operatorname{Re}\nu<1/2$	$\tfrac{1}{2}y^{-1/2}\Big[\sec(\nu\pi)\, J_{-\nu}\left(\dfrac{a^2}{4y}\right)$ $+\csc(\nu\pi)\,\mathbf{H}_{-\nu}\left(\dfrac{a^2}{4y}\right)$ $-2\operatorname{ctn}(2\nu\pi)\,\mathbf{H}_\nu\left(\dfrac{a^2}{4y}\right)\Big]$
(20)	$x^{\nu+2n-1/2}(x^2+\lambda^2)^{-1}$ $\times(x^2+a^2)^{-1/2\mu}$ $\times J_\mu[b(x^2+a^2)^{1/2}] \quad b>0$ $\operatorname{Re}\lambda>0, \quad n=0,1,2,\ldots$ $-1/2-n<\operatorname{Re}\mu<3-2n+\operatorname{Re}\nu$	$(-1)^{n+1}\lambda^{\nu+2n-1}y^{1/2}K_\nu(\lambda y)$ $\times(\lambda^2-a^2)^{-1/2\mu}I_\mu[b(\lambda^2-a^2)^{1/2}]$ $y>b$
(21)	$0 \quad 0<x<a$ $x^{\nu+1/2}(x^2-a^2)^{1/2\mu}$ $\times J_\mu[b(x^2-a^2)^{1/2}] \quad a<x<\infty$ $b>0$ $-1<\operatorname{Re}\mu<-\operatorname{Re}\nu$	$-2\pi^{-1}a^{\mu+\nu+1}b^\mu y^{\nu+\frac{1}{2}}\cos(\nu\pi)$ $\times(b^2-y^2)^{-1/2(\mu+\nu+1)}$ $\times K_{\mu+\nu+1}[a(b^2-y^2)^{1/2}]$ $0<y<b$

Higher transcendental functions (cont'd)

	$f(x)$	$\int_0^\infty f(x)\, Y_\nu(xy)\,(xy)^{1/2}\,dx \qquad y>0$
(22)	$0 \qquad 0<x<a$ $x^{\nu+1/2}(x^2-a^2)^{1/2\mu} \times J_\mu[b(x^2-a^2)^{1/2}] \qquad a<x<\infty$ $b>0$ $-1<\mathrm{Re}\,\mu<-\mathrm{Re}\,\nu$	$-a^{\mu+\nu+1}\, b^\mu\, y^{\nu+1/2}(y^2-b^2)^{-1/2(\mu+\nu+1)}$ $\times\{\sin(\mu\pi)\, J_{\mu+\nu+1}[a(y^2-b^2)^{1/2}]$ $+\cos(\mu\pi)\, Y_{\mu+\nu+1}[a(y^2-b^2)^{1/2}]\}$ $b<y<\infty$
(23)	$0 \qquad 0<x<a$ $x^{1/2-\nu}(x^2-a^2)^{1/2\mu} \times J_\mu[b(x^2-a^2)^{1/2} \qquad a<x<\infty$ $b>0, \quad -1<\mathrm{Re}\,\mu<\mathrm{Re}\,\nu$	$-2\pi^{-1}\, a^{\mu-\nu+1}\, b^\mu\, y^{1/2-\nu}$ $\times(b^2-y^2)^{-1/2(\mu-\nu+1)}$ $\times K_{\mu-\nu+1}[a(b^2-y^2)^{1/2}] \qquad 0<y<b$
(24)	$0 \qquad 0<x<a$ $x^{1/2-\nu}(x^2-a^2)^{1/2\mu} \times J_\mu[b(x^2-a^2)^{1/2}] \qquad a<x<\infty$ $b>0, \quad -1<\mathrm{Re}\,\mu<\mathrm{Re}\,\nu$	$a^{\mu-\nu+1}\, b^\mu\, y^{1/2-\nu}(y^2-b^2)^{1/2(\nu-\mu-1)}$ $\times Y_{\nu-\mu-1}[a(y^2-b^2)] \qquad b<y<\infty$
(25)	$x^{1/2}\, K_{1/2\nu}(ax^2)$ $\mathrm{Re}\,a>0, \quad -1<\mathrm{Re}\,\nu<1$	$\frac{\pi y^{1/2}}{4a}\Big[\csc(\nu\pi)\,\mathbf{L}_{-1/2\nu}\left(\frac{y^2}{4a}\right)$ $-\mathrm{ctn}(\nu\pi)\,\mathbf{L}_{1/2\nu}\left(\frac{y^2}{4a}\right)$ $-\tan(\tfrac12\nu\pi)\, I_{1/2\nu}\left(\frac{y^2}{4a}\right)$ $-\pi^{-1}\sec(\tfrac12\nu\pi)\, K_{1/2\nu}\left(\frac{y^2}{4a}\right)\Big]$

Higher transcendental functions (cont'd)

	$f(x)$	$\int_0^\infty f(x)\, Y_\nu(xy)(xy)^{1/2}\, dx \qquad y>0$
(26)	$x^{-1/2} \exp(\tfrac{1}{2} a x^2)\, K_0(\tfrac{1}{2} a x^2)$ $\nu = 0$	$-\tfrac{1}{2}\pi\, a^{-1/2}\, y^{1/2} \exp\left(\dfrac{y^2}{8a}\right) K_0\left(\dfrac{y^2}{8a}\right)$
(27)	$x^{-1/2-2\mu} \exp(\tfrac{1}{2} a x^2)$ $\times K_\mu(\tfrac{1}{2} a x^2)$ $\nu = 0, \quad -\tfrac{3}{4} < \operatorname{Re} \mu < \tfrac{1}{4}$	$-\dfrac{a^\mu \pi^{1/2}\, [\Gamma(\tfrac{1}{2} - 2\mu)]^2}{y^{1/2}\, \Gamma(1-2\mu)}$ $\times \exp\left(\dfrac{y^2}{8a}\right) W_{2\mu,0}\left(\dfrac{y^2}{4a}\right)$
(28)	$x^{-1/2} K_\nu(a x^{-1})$ $\operatorname{Re} a > 0, \quad -\tfrac{1}{2} < \operatorname{Re} \nu < \tfrac{1}{2}$	$-2y^{-1/2} [\sin(3\nu\pi/2) \operatorname{ker}_{2\nu}(2a^{1/2} y^{1/2})$ $+ \cos(3\nu\pi/2) \operatorname{kei}_{2\nu}(2a^{1/2} y^{1/2})]$
(29)	$x^{-5/2} K_\nu(a x^{-1})$ $\operatorname{Re} a > 0, \quad -5/2 < \operatorname{Re} \nu < 5/2$	$2a^{-1} y^{1/2} [\sin(3\pi\nu/2) \operatorname{kei}_{2\nu}(2a^{1/2} y^{1/2})$ $- \cos(3\pi\nu/2) \operatorname{ker}_{2\nu}(2a^{1/2} y^{1/2})]$
(30)	$x^{-2\nu} K_{\nu-1/2}(a x^{-1})$ $\operatorname{Re} a > 0, \quad \operatorname{Re} \nu > 1/6$	$(2\pi)^{1/2}\, a^{1/2-\nu}\, y^{\nu-1/2}\, Y_{2\nu-1}[(2ay)^{1/2}]$ $\times K_{2\nu-1}[(2ay)^{1/2}]$
(31)	$x^{-2\nu-2} K_{\nu-1/2}(a x^{-1})$ $\operatorname{Re} a > 0, \quad \operatorname{Re} \nu > -\tfrac{1}{2}$	$(2\pi)^{1/2}\, a^{-1/2-\nu}\, y^{1/2+\nu}\, Y_{2\nu}[(2ay)^{1/2}]$ $\times K_{2\nu}[(2ay)^{1/2}]$
(32)	$x^{2\nu-2} K_{\nu+1/2}(a x^{-1})$ $\operatorname{Re} a > 0, \quad \operatorname{Re} \nu < \tfrac{1}{2}$	$(\tfrac{1}{2}\pi)^{1/2} \csc(\nu\pi)\, a^{\nu-1/2}\, y^{1/2-\nu}$ $\times K_{2\nu}[(2ay)^{1/2}] \{J_{2\nu}[(2ay)^{1/2}]$ $- J_{-2\nu}[(2ay)^{1/2}]\}$

Higher transcendental functions (cont'd)

	$f(x)$	$\int_0^\infty f(x)\, Y_\nu(xy)(xy)^{1/2}\, dx \qquad y>0$
(33)	$x^{-1/2}[K_\mu(a^2x^{-1})]^2$ $\lvert\arg a\rvert < \frac{1}{4}\pi$ $-\frac{1}{4} < \mathrm{Re}\,\mu < \frac{1}{4}, \quad \nu = 0$	$2\pi y^{-1/2}[\cos(\mu\pi)\, J_{2\mu}(2ay^{1/2})$ $-\sin(\mu\pi)\, Y_{2\mu}(2ay^{1/2})]$
(34)	$x^{-1/2} K_{2\nu}(ax^{1/2})$ $\mathrm{Re}\, a > 0, \quad -\frac{1}{2} < \mathrm{Re}\,\nu < \frac{1}{2}$	$-\frac{1}{4}\pi y^{-1/2}\Big[\sec(\nu\pi)\, J_{-\nu}\left(\frac{a^2}{4y}\right)$ $-\csc(\nu\pi)\, \mathbf{H}_{-\nu}\left(\frac{a^2}{4y}\right)$ $+2\csc(2\nu\pi)\, \mathbf{H}_\nu\left(\frac{a^2}{4y}\right)\Big]$
(35)	$x^{\nu-1/2} J_{2\nu-1}(ax^{1/2})\, K_{2\nu-1}(ax^{1/2})$ $\lvert\arg a\rvert < \frac{1}{4}\pi, \quad \mathrm{Re}\,\nu > 0$	$2^{-\nu-1}\pi^{1/2} a^{2\nu-1} y^{-2\nu}\csc(\nu\pi)$ $\times\left[\mathbf{L}_{1/2-\nu}\left(\frac{a^2}{2y}\right) - I_{\nu-1/2}\left(\frac{a^2}{2y}\right)\right]$
(36)	$x^{-1/2}\, \mathbf{H}_{\nu-1}(ax)$ $a > 0, \quad -\frac{1}{2} < \mathrm{Re}\,\nu < \frac{1}{2}$	$-a^{\nu-1}y^{1/2-\nu} \qquad 0<y<a$ $0 \qquad a<y<\infty$
(37)	$x^{\nu-\mu+1/2}\, \mathbf{H}_\mu(ax)$ $a>0, \quad \mathrm{Re}\,\mu > \mathrm{Re}\,\nu$ $-3/2 < \mathrm{Re}\,\nu < 1/2$	$\frac{2^{\nu+1-\mu} y^{\nu+1/2}}{a^\mu\,\Gamma(\mu-\nu)}(a^2-y^2)^{\mu-\nu-1}$ $0<y<a$ $0 \qquad a<y<\infty$
(38)	$x^{-5/2} S_{-\nu-3,\nu}(a^2x^{-1}) \quad \mathrm{Re}\, a>0$ $-3/2 < \mathrm{Re}\,\nu < 1/2$	$\pi\, 2^{-\nu-2} a^{-2} y^{1/2} K_{2\nu}(2ay^{1/2})$ $\times[\Gamma(\nu+2)]^{-1}$

Higher transcendental functions (cont'd)

	$f(x)$	$\int_0^\infty f(x)\, Y_\nu(xy)\,(xy)^{1/2}\,dx \qquad y>0$
(39)	$x^{\nu-1/2}\exp(\tfrac14 a^2x^2)$ $\times D_{1/2\nu-1/2}(ax)$ $\lvert\arg a\rvert<\tfrac34\pi$ $-1/2<\operatorname{Re}\nu<2/3$	$-\pi^{-1}\,2^{3/4\nu+3/4}\,a^{-\nu}\,y^{-1/2}\,\Gamma(\nu+1)$ $\times\exp\left(\dfrac{y^2}{4a^2}\right)W_{-1/2\nu-1/2,1/2\nu}\left(\dfrac{y^2}{2a^2}\right)$
(40)	$D_{\nu-1/2}(ax^{-1/2})\,D_{-\nu-1/2}(ax^{-1/2})$ $\lvert\arg a\rvert<\tfrac14\pi$	$y^{-1}\exp(-ay^{1/2})$ $\times\sin[ay^{1/2}-\tfrac12(\nu-\tfrac12)\pi]$
(41)	$x^{\nu-m}\exp(-\tfrac14x^2)$ $\times M_{\kappa,1/4-1/2m}(\tfrac12x^2)\quad m$ integer $\operatorname{Re}(2\kappa-\nu)>-m\geq-1$ $\operatorname{Re}\nu>m-3/2$	$\dfrac{(-1)^m\,2^{1/2\nu-1/2m}\,\Gamma(3/2-m)}{\Gamma(3/4+\kappa-m/2)}(\tfrac12y^2)^\lambda$ $\times\exp(-\tfrac14y^2)\,W_{\alpha,\beta}(\tfrac12y^2)$ $\alpha=\kappa/2+m/4+\nu/2+5/8$ $\beta=\kappa/2+m/4-\nu/2-3/8$ $\lambda=\kappa/2+m/4-5/8$
(42)	$x^{-m}\exp(-\tfrac14x^2)$ $\times M_{\kappa,1/2\nu-1/2m+1/4}(\tfrac12x^2)$ m integer, $\quad 2\operatorname{Re}\kappa>-m\geq-1$ $\operatorname{Re}\nu>m-3/2$	$\dfrac{(-1)^m\,\Gamma(\nu-m+3/2)\,2^{-1/2m}}{\Gamma(\kappa+\nu/2-m/2+3/4)}(\tfrac12y^2)^\lambda$ $\times\exp(-\tfrac14y^2)\,W_{\alpha,\beta}(\tfrac12y^2)$ $\alpha=\kappa/2-3m/4+\nu/4+5/8$ $\beta=\kappa/2+m/4+\nu/4-3/8$ $\lambda=\kappa/2+m/4-\nu/4-5/8$

Higher transcendental functions (cont'd)

	$f(x)$	$\int_0^\infty f(x)\, Y_\nu(xy)(xy)^{1/2}\, dx \qquad y>0$
(43)	$x^{2\mu+\nu-1/2}\exp(-\tfrac{1}{4}x^2)$ $\times M_{\kappa,\mu}(\tfrac{1}{2}x^2)$ $-1<2\,\mathrm{Re}\,\mu<\mathrm{Re}(2\kappa-\nu)+\tfrac{1}{2}$ $\mathrm{Re}(2\mu+\nu)>-1$	$\pi^{-1}2^{\mu+\beta}y^{\kappa-\mu-1}\Gamma(2\mu+1)$ $\times\Gamma(\tfrac{1}{2}-\mu-\kappa)\exp(-\tfrac{1}{4}y^2)$ $\times\left\{\cos(2\mu\pi)\dfrac{\Gamma(2\mu+\nu+1)}{\Gamma(\mu+\nu-\kappa+3/2)}\right.$ $\times M_{\alpha,\beta}(\tfrac{1}{2}y^2)+\sin[(\mu-\kappa)\pi]$ $\left.\times W_{\alpha,\beta}(\tfrac{1}{2}y^2)\right\}$ $2\alpha=3\mu+\nu+\kappa+\tfrac{1}{2}$ $2\beta=\mu+\nu-\kappa+\tfrac{1}{2}$
(44)	$x^{2\mu-\nu-1/2}\exp(-\tfrac{1}{4}x^2)$ $\times M_{\kappa,\mu}(\tfrac{1}{2}x^2)$ $-1<2\,\mathrm{Re}\,\mu<\mathrm{Re}(2\kappa+\nu)+\tfrac{1}{2}$ $\mathrm{Re}(2\mu-\nu)>-1$	$\pi^{-1}2^{\mu+\beta}y^{\kappa-\mu-1}\exp(-\tfrac{1}{4}y^2)$ $\times\Gamma(2\mu+1)\Gamma(\tfrac{1}{2}-\kappa-\mu)$ $\times\left\{\cos[(\nu-2\mu)\pi]\dfrac{\Gamma(2\mu-\nu-1)}{\Gamma(2\beta+1)}\right.$ $\times M_{\alpha,\beta}(\tfrac{1}{2}y^2)-\sin[(\nu+\kappa-\mu)\pi]$ $\left.\times W_{\alpha,\beta}(\tfrac{1}{2}y^2)\right\}$ $2\alpha=3\mu-\nu+\kappa+\tfrac{1}{2}$ $2\beta=\mu-\nu-\kappa+\tfrac{1}{2}$
(45)	$x^{2\lambda}\exp(-\tfrac{1}{4}x^2)M_{\kappa,\mu}(\tfrac{1}{2}x^2)$ $\mathrm{Re}(\kappa-\lambda)>0$ $\mathrm{Re}(2\lambda+2\mu\pm\nu)>-5/2$	$\dfrac{2^\lambda\Gamma(2\mu+1)}{\Gamma(\tfrac{1}{2}+\kappa+\mu)}$ $\times G^{31}_{34}\left(\dfrac{y^2}{2}\,\middle\vert\,\begin{matrix}-\mu-\lambda,\ \mu-\lambda,\ l\\ h,\ k,\ \kappa-\lambda-\tfrac{1}{2},\ l\end{matrix}\right)$ $h=\tfrac{1}{4}+\tfrac{1}{2}\nu,\quad k=\tfrac{1}{4}-\tfrac{1}{2}\nu$ $l=-\tfrac{1}{4}-\tfrac{1}{2}\nu$

Higher transcendental functions (cont'd)

	$f(x)$	$\int_0^\infty f(x)\, Y_\nu(xy)\,(xy)^{1/2}\,dx \qquad y>0$
(46)	$x^{2\lambda}\exp(-\tfrac{1}{4}x^2)\, W_{\kappa,\mu}(\tfrac{1}{2}x^2)$ $\operatorname{Re}(2\lambda \pm 2\mu \pm \nu) > -5/2$	$(-1)^m\, 2^\lambda$ $\times G_{34}^{22}\left(\frac{y^2}{2}\,\middle\vert\, \begin{matrix} -\mu-\lambda,\ \mu-\lambda,\ l \\ h,\ k,\ \kappa-\lambda-\tfrac{1}{2},\ l \end{matrix}\right)$ $h = \tfrac{1}{4}+\tfrac{1}{2}\nu, \quad k = \tfrac{1}{4}-\tfrac{1}{2}\nu$ $l = -\tfrac{1}{4}-\tfrac{1}{2}\nu$
(47)	$x^{2\lambda}\exp(\tfrac{1}{4}x^2)\, W_{\kappa,\mu}(\tfrac{1}{2}x^2)$ $\operatorname{Re}(\kappa+\lambda) < 0$ $\operatorname{Re}(2\lambda \pm 2\mu \pm \nu) > -5/2$	$2^\lambda\,[\Gamma(\tfrac{1}{2}-\kappa+\mu)\,\Gamma(\tfrac{1}{2}-\kappa-\mu)]^{-1}$ $\times G_{34}^{32}\left(\frac{y^2}{2}\,\middle\vert\, \begin{matrix} -\mu-\lambda,\ \mu-\lambda,\ l \\ h,\ k,\ -\tfrac{1}{2}-\kappa-\lambda,\ l \end{matrix}\right)$ $h = \tfrac{1}{4}+\tfrac{1}{2}\nu, \quad k = \tfrac{1}{4}-\tfrac{1}{2}\nu$ $l = -\tfrac{1}{4}-\tfrac{1}{2}\nu$
(48)	$x^{1/2}\, W_{\frac{1}{2}\nu,\mu}(2/x)\; W_{-\frac{1}{2}\nu,\mu}(2/x)$ $-\tfrac{1}{4} < \operatorname{Re}\mu < \tfrac{1}{4}$	$4y^{-1/2}\, K_{2\mu}(2y^{1/2})$ $\times\{\cos[(\mu-\tfrac{1}{2}\nu)\pi]\, J_{2\mu}(2y^{1/2})$ $-\sin[(\mu-\tfrac{1}{2}\nu)\pi]\, Y_{2\mu}(2y^{1/2})\}$
(49)	$x^{\nu+3/2}$ $\times {}_2F_1\left(\frac{1}{2}, \frac{1}{2}-\nu; \frac{3}{2}; -a^2x^2\right)$ $\operatorname{Re} a > 0$ $-3/2 < \operatorname{Re}\nu < -1/2$	$\dfrac{2^\nu\, y^{-\nu-\frac{1}{2}}}{\pi^{1/2}\, a^2\, \Gamma(\tfrac{1}{2}-\nu)}\, K_\nu\left(\frac{y}{2a}\right)$ $\times K_{\nu+1}\left(\frac{y}{2a}\right)$
(50)	$x^{\nu+3/2}$ $\times {}_2F_1(1, 2\nu+3/2; \nu+2; -a^2x^2)$ $\operatorname{Re} a > 0, \quad -\tfrac{1}{2} < \operatorname{Re}\nu < \tfrac{1}{2}$	$\pi^{-1/2}\, 2^{-\nu}\, a^{-2\nu-3}\, \dfrac{\Gamma(\nu+2)}{\Gamma(2\nu+3/2)}$ $\times y^{\nu+\frac{1}{2}}\left[K_\nu\left(\frac{y}{2a}\right)\right]^2$

Higher transcendental functions (cont'd)

	$f(x)$	$\int_0^\infty f(x)\, Y_\nu(xy)(xy)^{1/2}\,dx \qquad y>0$
(51)	$x^{\nu+3/2}$ $\times {}_2F_1(1,\mu+\nu+3/2;3/2;-a^2x^2)$ $\operatorname{Re} a>0, \quad -3/2<\operatorname{Re}\nu<1/2$ $\operatorname{Re}(2\mu+\nu)>-3/2$	$\dfrac{\pi^{1/2}\, 2^{-\mu-\nu-1}\, a^{-\mu-2\nu-3}}{\Gamma(\mu+\nu+3/2)}$ $\times y^{\mu+\nu+1/2} K_\mu\left(\dfrac{y}{a}\right)$
(52)	$x^\sigma\, {}_2F_1(a,\beta;\gamma;-\lambda^2x^2)$ $\operatorname{Re}\lambda>0, \quad \operatorname{Re}\sigma>\lvert\operatorname{Re}\nu\rvert-3/2$ $\operatorname{Re}\sigma<2\operatorname{Re}a, \quad \operatorname{Re}\sigma<2\operatorname{Re}\beta$	$\dfrac{\lambda^{-\sigma-1}\,\Gamma(\gamma)}{2^{1/2}\,\Gamma(a)\,\Gamma(\beta)}$ $\times G^{41}_{35}\left(\dfrac{y^2}{4\lambda^2}\middle\vert\begin{matrix}1-p,\ \gamma-p,\ l\\ h,k,a-p,\beta-p,l\end{matrix}\right)$ $h=\tfrac14+\tfrac12\nu, \quad k=\tfrac14-\tfrac12\nu$ $l=-\tfrac14-\tfrac12\nu, \quad p=\tfrac12+\tfrac12\sigma$
(53)	$x^{\sigma-3/2}$ $\times {}_pF_{p-1}(a_1,\ldots,a_p;$ $\beta_1,\ldots,\beta_{p-1};-\lambda x^2)$ $\lvert\arg\lambda\rvert<\pi, \quad \operatorname{Re}\sigma>\lvert\operatorname{Re}\nu\rvert$ $\operatorname{Re}a_j>\tfrac12\operatorname{Re}\sigma-\tfrac34$	$\dfrac{y^{1/2}\,\Gamma(\beta_1)\cdots\Gamma(\beta_{p-1})}{2\lambda^{\frac12\sigma}\,\Gamma(a_1)\cdots\Gamma(a_p)}$ $\times G^{p+2,1}_{p+2,p+3}\left(\dfrac{y^2}{4\lambda}\middle\vert\begin{matrix}\beta_0^*,\ldots,\beta_{p-1}^*,\ l\\ h,k,a_1^*,\ldots,a_p^*,l\end{matrix}\right)$ $a_j^*=a_j-\dfrac{\sigma}{2}, \qquad j=1,\ldots,p$ $\beta_0^*=1-\dfrac{\sigma}{2},$ $\beta_j^*=\beta_j-\dfrac{\sigma}{2}, \qquad j=1,\ldots,p-1$ $h=\dfrac{\nu}{2}, \quad k=-\dfrac{\nu}{2}, \quad l=-\dfrac{1+\nu}{2}$

Higher transcendental functions (cont'd)

	$f(x)$	$\int_0^\infty f(x)\, Y_\nu(xy)(xy)^{1/2}\, dx \qquad y>0$
(54)	$x^{\sigma-3/2}$ $\times\, {}_pF_p(\alpha_1,\ldots,\alpha_p;\, \beta_1,\ldots,\beta_p;\, -\lambda x^2)$ $\operatorname{Re}\lambda>0, \quad \operatorname{Re}\sigma>\lvert\operatorname{Re}\nu\rvert$ $\operatorname{Re}\alpha_j>\tfrac{1}{2}\operatorname{Re}\sigma-\tfrac{3}{4}$ $j=1,\ldots,p$	$\dfrac{y^{1/2}}{2\lambda^{1/2\sigma}}\,\dfrac{\Gamma(\beta_1)\cdots\Gamma(\beta_p)}{\Gamma(\alpha_1)\cdots\Gamma(\alpha_p)}$ $\times\, G^{p+2,1}_{p+2,p+3}\left(\dfrac{y^2}{4\lambda}\,\middle\vert\,\begin{matrix}\beta_0^*,\ldots,\beta_p^*,\, l\\ h,k,\alpha_1^*,\ldots,\alpha_p^*,\, l\end{matrix}\right)$ $\beta_0^*=1-\dfrac{\sigma}{2}, \qquad \alpha_j^*=\alpha_j-\dfrac{\sigma}{2}$ $\beta_j^*=\beta_j-\dfrac{\sigma}{2}, \qquad j=1,\ldots,p$ $h=\dfrac{\nu}{2}, \quad k=-\dfrac{\nu}{2}, \quad l=-\dfrac{1+\nu}{2}$
(55)	$x^{\sigma-3/2}$ $\times\, {}_pF_q(\alpha_1,\ldots,\alpha_p;\, \beta_1,\ldots,\beta_q;\, -\lambda x^2)$ $p\le q-1, \quad \operatorname{Re}\sigma>\lvert\operatorname{Re}\nu\rvert$	$-\pi^{-1}\, 2^{\sigma-1}\, y^{1/2-\sigma}\cos[\tfrac{1}{2}\pi(\sigma-\nu)]$ $\times\,\Gamma\left(\dfrac{\sigma+\nu}{2}\right)\Gamma\left(\dfrac{\sigma-\nu}{2}\right)$ $\times\, {}_{p+2}F_q\left(\alpha_1,\ldots,\alpha_p,\, \dfrac{\sigma+\nu}{2},\, \dfrac{\sigma-\nu}{2};\right.$ $\left.\beta_1,\ldots,\beta_q;\, -\dfrac{4\lambda}{y^2}\right)$
(56)	$G^{mn}_{pq}\left(\lambda x^2\,\middle\vert\,\begin{matrix}\alpha_1,\ldots,\alpha_p\\ \beta_1,\ldots,\beta_q\end{matrix}\right)$ $p+q<2(m+n)$ $\lvert\arg\lambda\rvert<(m+n-\tfrac{1}{2}p-\tfrac{1}{2}q)\pi$ $\operatorname{Re}\alpha_j<1 \qquad j=1,\ldots,n$ $\operatorname{Re}(\beta_j\pm\tfrac{1}{2}\nu)>-\tfrac{3}{4}$ $j=1,\ldots,m$	$(2\lambda)^{-1/2}$ $\times\, G^{n+2,m}_{q+1,p+3}\left(\dfrac{y^2}{4\lambda}\,\middle\vert\,\begin{matrix}\tfrac{1}{2}-\beta_1,\ldots,\tfrac{1}{2}-\beta_q,\, l\\ h,k,\tfrac{1}{2}-\alpha_1,\ldots,\tfrac{1}{2}-\alpha_p,\, l\end{matrix}\right)$ $h=\tfrac{1}{4}+\tfrac{1}{2}\nu, \quad k=\tfrac{1}{4}-\tfrac{1}{2}\nu$ $l=-\tfrac{1}{4}-\tfrac{1}{2}\nu$

CHAPTER X

K-TRANSFORMS

We call

$$\mathfrak{K}_\nu\{f(x);\, y\} = \int_0^\infty f(x)\, K_\nu(xy)(xy)^{\frac{1}{2}}\, dx$$

the *K-transform of order* ν of $f(x)$ and regard y as a complex variable. This transformation was introduced by C.S. Meijer (1940) who gave the inversion formula 10.1 (1) and representation theorems: the transformation was further investigated by Boas (1942 a, 1942 b) and Erdélyi (1950-51).

By virtue of the connection between Bessel functions of the first and second kinds, and the modified Bessel function of the third kind K_ν, the $\mathfrak{K}_\nu$ transform may be expressed as a linear combination of any two of the transforms $\mathfrak{H}_\nu$, $\mathfrak{H}_{-\nu}$, $\mathfrak{Y}_\nu$, $\mathfrak{Y}_{-\nu}$. However, the variable y in the Hankel and Y-transforms occurring in these expressions is negative, and very few Hankel or Y-transforms converge for negative (or complex) values of y. Conversely,

$$\pi\,\mathfrak{H}_\nu\{f(x);\, y\} = e^{\frac{1}{2}i(\nu+\frac{1}{2})\pi}\,\mathfrak{K}_\nu\{f(x);\, iy\} + e^{-\frac{1}{2}i(\nu+\frac{1}{2})\pi}\,\mathfrak{K}_\nu\{f(x);\, -iy\}$$

$$\pi\,\mathfrak{Y}_\nu\{f(x);\, y\} = -\,e^{\frac{1}{2}i(\nu+\frac{1}{2})\pi}\,\mathfrak{K}_\nu\{f(x);\, iy\} - e^{-\frac{1}{2}i(\nu-\frac{1}{2})\pi}\,\mathfrak{K}_\nu\{f(x);\, -iy\},$$

and these relations enable us to evaluate Hankel and Y-transforms by means of a table of K-transforms, although in many cases the K-transforms involved are to be taken on the boundary of the half-plane of convergence, and additional restrictions on the parameters must be introduced to secure convergence. If $\nu = \pm \frac{1}{2}$, the K-transform reduces to the Laplace transform,

$$\mathfrak{K}_{\pm 1/2}\{f(x); y\} = (\tfrac{1}{2}\pi)^{1/2}\, \mathfrak{L}\{f(x); y\}$$

and the above relations become the expressions of Fourier's sine and cosine transforms in terms of Laplace integrals.

From the transform pairs given in this chapter further transform pairs may be derived by means of the methods indicated in the introduction to volume I, and also by the general formulas of sec. 10.1. The connection with the Laplace transformation in either of the two forms

$$\mathfrak{K}_\nu\{f(x); y\} = \frac{\pi^{1/2}\, 2^{-\nu}\, y^{\nu+1/2}}{\Gamma(\nu + \frac{1}{2})}$$

$$\times\, \mathfrak{L}\{\textstyle\int_0^x (x^2 - t^2)^{\nu-1/2}\, t^{1/2-\nu} f(t)\, dt; y\} \qquad \operatorname{Re}\nu > -\tfrac{1}{2}$$

$$\mathfrak{K}_\nu\{f(x); y\} = \frac{\pi^{1/2}\, 2^{-\nu}\, y^{1/2-\nu}}{\Gamma(\nu + \frac{1}{2})}$$

$$\times \int_y^\infty (t^2 - y^2)^{\nu-1/2}\, \mathfrak{L}\{x^{1/2+\nu} f(x); t\}\, dt \qquad \operatorname{Re}\nu > -\tfrac{1}{2}$$

may be used to evaluate K-transforms by means of the tables of Laplace transforms given in chapter IV.

REFERENCES

Boas, R.P., 1942a: *Proc. Nat. Acad. Sci. U.S.A.* 28, 21-24.

Boas, R.P., 1942b: *Bull. Amer. Math. Soc.* 48, 286-294.

Erdélyi, Arthur, 1950-51: *Rend. Sem. Mat. Univ. Torino* 10, 217-234.

Meijer, C.S., 1940: *Proc. Amsterdam Akad. Wet.* 43, 599-608 and 702-711.

K-TRANSFORMS

10.1. General formulas

	$f(x)$	$\int_0^\infty f(x)\, K_\nu(xy)\,(xy)^{1/2}\,dx = g(y;\,\nu)$
(1)	$\dfrac{1}{\pi i}\displaystyle\int_{c-i\infty}^{c+i\infty} g(y)\, I_\nu(xy)(xy)^{1/2}\,dy$	$g(y)$
(2)	$f(ax)$ $\qquad a>0$	$a^{-1}\, g(y/a;\,\nu)$
(3)	$x^m\, f(x)$ $\qquad m=0,\,1,\,2,\,\ldots$	$y^{1/2-\nu}\left(-\dfrac{d}{y\,dy}\right)^m [y^{m+\nu-1/2} g(y;\,\nu+m)]$
(4)	$x^m\, f(x)$ $\qquad m=0,\,1,\,2,\,\ldots$	$y^{1/2+\nu}\left(-\dfrac{d}{y\,dy}\right)^m [y^{m-\nu-1/2} g(y;\,\nu-m)]$
(5)	$2\nu x^{-1}\, f(x)$	$y\, g(y;\,\nu+1) - y\, g(y;\,\nu-1)$
(6)	$x^{-1}\, f(x)$	$y^{\nu+1/2}\int_y^\infty \eta^{-\nu-1/2} g(\eta;\,\nu+1)\,d\eta$

General formulas (cont'd)

	$f(x)$	$\int_0^\infty f(x)\, K_\nu(xy)(xy)^{1/2}\, dx = g(y;\nu)$
(7)	$x^{-\mu} f(x)$ Re $\mu > 0$	$2^{1-\mu}[\Gamma(\mu)]^{-1} y^{\nu+1/2} \times \int_y^\infty \eta^{1/2-\mu-\nu}(\eta^2-y^2)^{\mu-1} \times g(\eta;\nu+\mu)\, d\eta$
(8)	$2\nu f'(x)$	$(\nu-\tfrac{1}{2})\, y\, g(y;\nu+1) + (\nu+\tfrac{1}{2})\, y\, g(y;\nu-1)$
(9)	$x^{1/2-\nu}\left(\dfrac{d}{x\,dx}\right)^m \times [x^{m+\nu-1/2} f(x)]$ $m = 0, 1, 2, \ldots$	$y^m\, g(y;\nu+m)$
(10)	$x^{1/2+\nu}\left(\dfrac{d}{x\,dx}\right)^m \times [x^{m-\nu-1/2} f(x)]$ $m = 0, 1, 2, \ldots$	$y^m\, g(y;\nu-m)$
(11)	$x^{1/2-\nu}\int_0^x \xi^{\nu-\mu+1/2}(x^2-\xi^2)^{\mu-1} \times f(\xi)\, d\xi$ Re $\mu > 0$	$2^{\mu-1}\Gamma(\mu)\, y^{-\mu} g(y;\nu-\mu)$

10.2. Elementary functions

	$f(x)$	$\int_0^\infty f(x)\, K_\nu(xy)\, (xy)^{1/2}\, dx$
(1)	$x^{\rho-1}$ $\quad \operatorname{Re}\rho > \lvert\operatorname{Re}\nu\rvert - \tfrac{1}{2}$	$2^{\rho-3/2}\, y^{-\rho}\, \Gamma\left(\frac{\rho}{2}+\frac{\nu}{2}+\frac{1}{4}\right) \times \Gamma\left(\frac{\rho}{2}-\frac{\nu}{2}+\frac{1}{4}\right)$ $\quad \operatorname{Re} y > 0$
(2)	$0 \quad 0 < x < a$ $x^{\nu+1/2} \quad a < x < \infty$	$a^{\nu+1}\, y^{-1/2}\, K_{\nu+1}(ay)$ $\quad \operatorname{Re} y > 0$
(3)	$0 \quad 0 < x < a$ $x^{\sigma+1/2} \quad a < x < \infty$	$a y^{-1/2-\sigma}\, e^{-\frac{1}{2}\pi\sigma i}[K_{\nu-1}(ay)\, S_{\sigma+1,\nu}(iay)$ $+ i(\nu+\sigma)\, K_\nu(ay)\, S_{\sigma,\nu-1}(iay)]$ $\quad \operatorname{Re} y > 0$
(4)	$x^{\mu-1/2}(x+a)^{-1}$ $\quad \lvert\arg a\rvert < \pi$ $\operatorname{Re}\mu > \lvert\operatorname{Re}\nu\rvert - 1$	$2^{\mu-2}\,\Gamma\left(\frac{\mu}{2}+\frac{\nu}{2}\right)\Gamma\left(\frac{\mu}{2}-\frac{\nu}{2}\right) y^{1/2-\mu}$ $\times {}_1F_2\left(1; 1-\frac{\mu}{2}-\frac{\nu}{2}, 1-\frac{\mu}{2}+\frac{\nu}{2}; \frac{a^2y^2}{4}\right)$ $- 2^{\mu-3}\,\Gamma\left(\frac{\mu}{2}-\frac{\nu}{2}-\frac{1}{2}\right)\Gamma\left(\frac{\mu}{2}+\frac{\nu}{2}-\frac{1}{2}\right) a y^{3/2-\mu}$ $\times {}_1F_2\left(1; \frac{3}{2}-\frac{\mu}{2}-\frac{\nu}{2}, \frac{3}{2}-\frac{\mu}{2}+\frac{\nu}{2}; \frac{a^2y^2}{4}\right)$ $- \pi a^{\mu} y^{1/2} \csc[\pi(\mu-\nu)]\,\{K_\nu(ay)$ $+ \pi\cos(\mu\pi)\csc[\pi(\nu+\mu)]\, I_\nu(ay)\}$ $\quad \operatorname{Re} y > 0$

Elementary functions (cont'd)

	$f(x)$	$\int_0^\infty f(x)\, K_\nu(xy)(xy)^{\frac{1}{2}}\, dx$
(5)	$x^{-\frac{1}{2}}(x+a)^{-1}$ $\lvert\arg a\rvert < \pi, \quad -1 < \mathrm{Re}\,\nu < 1$	$\frac{1}{2}\pi^2[\csc(\nu\pi)]^2\, y^{\frac{1}{2}}\, [I_\nu(ay) + I_{-\nu}(ay) - e^{-\frac{1}{2}i\nu\pi}\, \mathbf{J}_\nu(iay) - e^{\frac{1}{2}i\nu\pi}\, \mathbf{J}_{-\nu}(iay)]$ $\quad \mathrm{Re}\, y > 0$
(6)	$x^{-\frac{1}{2}}(a^2+x^2)^{-\frac{1}{2}}$ $\mathrm{Re}\, a > 0, \quad -1 < \mathrm{Re}\,\nu < 1$	$\frac{1}{8}\pi^2 \sec(\frac{1}{2}\nu\pi)\, y^{\frac{1}{2}}\, \{[J_{\frac{1}{2}\nu}(\frac{1}{2}ay)]^2 + [Y_{\frac{1}{2}\nu}(\frac{1}{2}ay)]^2\}$ $\quad \mathrm{Re}\, y > 0$
(7)	$x^{-\frac{1}{2}-\nu}(x^2+a^2)^{-1}$ $\mathrm{Re}\, a > 0, \quad \mathrm{Re}\,\nu < \frac{1}{2}$	$\frac{1}{4}\pi^2 \sec(\nu\pi)\, a^{-\nu-1}\, y^{\frac{1}{2}}\, [\mathbf{H}_\nu(ay) - Y_\nu(ay)]$ $\quad \mathrm{Re}\, y > 0$
(8)	$x^{\frac{1}{2}+\nu}(x^2+a^2)^{\mu}$ $\mathrm{Re}\, a > 0, \quad \mathrm{Re}\,\nu > -1$	$2^\nu\, \Gamma(\nu+1)\, a^{\nu+\mu+1}\, y^{-\frac{1}{2}-\mu} \times S_{\mu-\nu,\, \mu+\nu+1}(ay)$ $\quad \mathrm{Re}\, y > 0$
(9)	$x^{\rho-3/2}(x^2+a^2)^{\mu}$ $\mathrm{Re}\, a > 0, \quad \mathrm{Re}\,\rho > \lvert\mathrm{Re}\,\nu\rvert$	$\frac{a^{\rho+2\mu}\, y^{\frac{1}{2}}}{4\Gamma(-\mu)}\, [f(\nu) + f(-\nu)] + 2^{2\mu+\rho-2} \times \Gamma\left(\frac{\rho}{2}+\mu-\frac{\nu}{2}\right)\Gamma\left(\frac{\rho}{2}+\mu+\frac{\nu}{2}\right) y^{\frac{1}{2}-\rho-2\mu} \times {}_1F_2\left(-\mu; 1-\mu, \frac{\rho}{2}-\frac{\nu}{2}, 1-\mu-\frac{\rho}{2}+\frac{\nu}{2}; -\frac{a^2y^2}{4}\right)$ Continued on the following page.

Elementary functions (cont'd)

	$f(x)$	$\int_0^\infty f(x)\,K_\nu(xy)(xy)^{1/2}\,dx$
(9)	Continued from the preceding page.	$f(\nu) = (\tfrac{1}{2}a)^\nu\,\Gamma(-\nu)\,\Gamma\left(\frac{\rho}{2}+\frac{\nu}{2}\right)$ $\times\Gamma\left(-\frac{\nu}{2}-\frac{\rho}{2}-\mu\right)\,y^\nu$ $\times\,{}_1F_2\left(\frac{\rho}{2}+\frac{\nu}{\nu};\frac{\rho}{2}+\mu+1+\frac{\nu}{2},\,1+\nu;\right.$ $\left.-\frac{a^2y^2}{4}\right)$ $\qquad\mathrm{Re}\,y>0$
(10)	$[x(a^2-x^2)]^{\nu-1/2}\quad 0<x<a$ $0\quad a<x<\infty$ $\mathrm{Re}\,\nu>-\tfrac{1}{2}$	$\pi^{1/2}\,2^{\nu-1}\,a^{2\nu}\,y^{1/2-\nu}\,\Gamma(\nu+\tfrac{1}{2})$ $\times\,I_\nu(\tfrac{1}{2}ay)\,K_\nu(\tfrac{1}{2}ay)$
(11)	$0\quad 0<x<a$ $[x(x^2-a^2)]^{\nu-1/2}\quad a<x<\infty$ $\mathrm{Re}\,\nu>-\tfrac{1}{2}$	$\pi^{-1/2}\,2^{\nu-1}\,a^{2\nu}\,y^{1/2-\nu}\,\Gamma(\nu+\tfrac{1}{2})$ $\times\,[K_\nu(\tfrac{1}{2}ay)]^2$ $\qquad\mathrm{Re}\,y>0$
(12)	$x^{1/2-\nu}(a^2-x^2)^\mu\quad 0<x<a$ $0\quad a<x<\infty$ $\mathrm{Re}\,\mu>-1,\quad\mathrm{Re}\,\nu<1$	$2^{-\nu-2}\,a^{2\mu+2}\,y^{\nu+1/2}\,(\mu+1)^{-1}\,\Gamma(-\nu)$ $\times\,{}_1F_2(1;\nu+1,\mu+2;\tfrac{1}{4}a^2y^2)$ $+\,\pi\,2^{\mu-1}\,a^{\mu-\nu+1}\,y^{-\mu-1/2}\csc(\nu\pi)$ $\times\,\Gamma(\mu+1)\,I_{\mu-\nu+1}(ay)$
(13)	$0\quad 0<x<a$ $x^{1/2-\nu}(x^2-a^2)^\mu\quad a<x<\infty$ $\mathrm{Re}\,\mu>-1$	$2^\mu\,a^{\mu-\nu+1}\,y^{-\mu-1/2}\,\Gamma(\mu+1)$ $\times\,K_{\mu-\nu+1}(ay)$ $\qquad\mathrm{Re}\,y>0$

Elementary functions (cont'd)

	$f(x)$	$\int_0^\infty f(x)\,K_\nu(xy)(xy)^{1/2}\,dx$
(14)	$x^{-1/2}(x^2+a^2)^{-1/2}$ $\times[(x^2+a^2)^{1/2}+x]^{-2\mu}$ $\mathrm{Re}\,a>0,\quad \nu=0$	$-\frac{1}{4}\pi\,a^{-2\mu}y^{1/2}\left\{J_\mu(\tfrac{1}{2}ay)\frac{\partial}{\partial\mu}\right.$ $\times[Y_\mu(\tfrac{1}{2}ay)]-Y_\mu(\tfrac{1}{2}ay)\frac{\partial}{\partial\mu}$ $\left.\times[J_\mu(\tfrac{1}{2}ay)]\right\}$ $\mathrm{Re}\,y>0$
(15)	$x^{-1/2}(x^2+a^2)^{-1/2}$ $\times[(x^2+a^2)^{1/2}+x]^{-2\mu}$ $\mathrm{Re}\,a>0$	$\frac{1}{4}\pi^2 a^{-2\mu}y^{1/2}\csc(\nu\pi)$ $\times[J_{\mu+\frac{1}{2}\nu}(\tfrac{1}{2}ay)\,Y_{\mu-\frac{1}{2}\nu}(\tfrac{1}{2}ay)$ $-Y_{\mu+\frac{1}{2}\nu}(\tfrac{1}{2}ay)\,J_{\mu-\frac{1}{2}\nu}(\tfrac{1}{2}ay)]$ $\mathrm{Re}\,y>0$
(16)	$x^{-1/2}(x^2+a^2)^{-1/2}$ $\times\{[(x^2+a^2)^{1/2}+x]^{2\mu}$ $+[(x^2+a^2)^{1/2}-x]^{2\mu}\}$ $\mathrm{Re}\,a>0,\quad \nu=0$	$\frac{1}{4}\pi^2 a^{2\mu}y^{1/2}\{[J_\mu(\tfrac{1}{2}ay)]^2$ $+[Y_\mu(\tfrac{1}{2}ay)]^2\}$ $\mathrm{Re}\,y>0$
(17)	$x^{-1/2}(x^2+a^2)^{-1/2}$ $\times\{[(x^2+a^2)^{1/2}+x]^{2\mu}$ $+\cos[(\tfrac{1}{2}\nu-\mu)\pi]$ $+[(x^2+a^2)^{1/2}-x]^{2\mu}$ $\times\cos[(\tfrac{1}{2}\nu+\mu)\pi]\}$ $\mathrm{Re}\,a>0$	$\frac{1}{4}\pi^2 a^{2\mu}y^{1/2}[J_{\frac{1}{2}\nu+\mu}(\tfrac{1}{2}ay)J_{\frac{1}{2}\nu-\mu}(\tfrac{1}{2}ay)$ $+Y_{\frac{1}{2}\nu+\mu}(\tfrac{1}{2}ay)\,Y_{\frac{1}{2}\nu-\mu}(\tfrac{1}{2}ay)]$ $\mathrm{Re}\,y>0$
(18)	$x^{-1/2-2\mu}(x^2+a^2)^{-1/2}$ $\times[(x^2+a^2)^{1/2}+a]^{2\mu}$ $\mathrm{Re}\,a>0$ $2\,\mathrm{Re}\,\mu+\lvert\mathrm{Re}\,\nu\rvert<1$	$\frac{1}{2}a^{-1}y^{-1/2}\,\Gamma\left(\frac{1+\nu}{2}-\mu\right)\Gamma\left(\frac{1-\nu}{2}-\mu\right)$ $\times W_{\mu,\frac{1}{2}\nu}(iay)\,W_{\mu,\frac{1}{2}\nu}(-iay)$ $\mathrm{Re}\,y>0$

Elementary functions (cont'd)

	$f(x)$	$\int_0^\infty f(x)\, K_\nu(xy)(xy)^{1/2}\, dx$
(19)	$0 \qquad 0<x<a$ $x^{-1/2}(x^2-a^2)^{-1/2}$ $\times \{[x+(x^2-a^2)^{1/2}]^{2\mu}$ $+ [x-(x^2-a^2)^{1/2}]^{2\mu}\}$ $a<x<\infty$	$a^{2\mu} y^{1/2} K_{1/2\nu+\mu}(\tfrac{1}{2}ay)$ $\times K_{1/2\nu-\mu}(\tfrac{1}{2}ay) \qquad \mathrm{Re}\, y>0$
(20)	$0 \qquad 0<x<a$ $x^{-1/2-2\mu}(x^2-a^2)^{-1/2}$ $\times \{[a+i(x^2-a^2)^{1/2}]^{2\mu}$ $+ [a-i(x^2-a^2)^{1/2}]^{2\mu}\}$ $a<x<\infty$	$\pi a^{-1} y^{-1/2} W_{\mu,1/2\nu}(ay)\, W_{-\mu,1/2\nu}(ay)$ $\mathrm{Re}\, y>0$
(21)	$x^{-1/2} e^{-\alpha x} \qquad \nu=0$	$y^{1/2}(y^2-\alpha^2)^{-1/2} \cos^{-1}(\alpha/y)$ $\mathrm{Re}(\alpha+y)>0$ $y(y^2-\alpha^2)^{-1/2}\cos^{-1}(\alpha/y) \to \tfrac{1}{2}\pi$ as $y\to\infty$.
(22)	$x^{-1/2} e^{-\alpha x} \qquad -1<\mathrm{Re}\,\nu<1$	$\dfrac{\pi y^{-1/2}}{\sin(\nu\pi)} \dfrac{\sin(\nu\theta)}{\sin\theta} \qquad \mathrm{Re}(\alpha+y)>0$ $\cos\theta=\alpha/y, \quad \theta\to\tfrac{1}{2}\pi$ as $y\to\infty$
(23)	$x^{\mu-1} e^{-\alpha x} \qquad \mathrm{Re}\,\mu>\lvert\mathrm{Re}\,\nu\rvert-\tfrac{1}{2}$	$\dfrac{\pi^{1/2} 2^\nu y^{\nu+1/2}}{(\alpha+y)^{\mu+\nu+1/2}} \dfrac{\Gamma(\mu+\nu+\tfrac{1}{2})\Gamma(\mu-\nu+\tfrac{1}{2})}{\Gamma(\mu+1)}$ $\times {}_2F_1\left(\mu+\nu+\tfrac{1}{2}, \nu+\tfrac{1}{2}; \mu+1; \dfrac{\alpha-y}{\alpha+y}\right)$ $\mathrm{Re}(\alpha+y)>0$

Elementary functions (cont'd)

	$f(x)$	$\int_0^\infty f(x)\, K_\nu(xy)(xy)^{1/2}\, dx$
(24)	$x^{-1/2} \exp(-ax^2)$ $\operatorname{Re} a > 0, \quad -1 < \operatorname{Re} \nu < 1$	$\frac{1}{4} \sec(\frac{1}{2}\nu\pi) \left(\frac{\pi y}{a}\right)^{1/2}$ $\times \exp\left(\frac{y^2}{8a}\right) K_{\frac{1}{2}\nu}\left(\frac{y^2}{8a}\right)$
(25)	$x^{-1/2-2\mu} \exp(-ax^2) \quad \operatorname{Re} a > 0$ $2\operatorname{Re} \mu < 1 - \lvert\operatorname{Re} \nu\rvert$	$\frac{1}{2} a^{\mu} y^{-1/2}\, \Gamma\left(\frac{1+\nu}{2} - \mu\right) \Gamma\left(\frac{1-\nu}{2} - \mu\right)$ $\times \exp\left(\frac{y^2}{8a}\right) W_{\mu, \frac{1}{2}\nu}\left(\frac{y^2}{4a}\right)$
(26)	$x^{-1/2} (x^2 + a^2)^{-1/2}$ $\times \exp[-\beta(x^2 + a^2)^{1/2}]$ $\operatorname{Re} a > 0, \quad \operatorname{Re} \beta > 0$ $-1 < \operatorname{Re} \nu < 1$	$\frac{1}{2} y^{1/2} \sec(\frac{1}{2}\nu\pi)$ $\times K_{\frac{1}{2}\nu}\{\frac{1}{2} a[\beta + (\beta^2 - y^2)^{1/2}]\}$ $\times K_{\frac{1}{2}\nu}\{\frac{1}{2} a[\beta - (\beta^2 - y^2)^{1/2}]\}$ $\operatorname{Re}(y + \beta) > 0$
(27)	$x^{-1} \cos(a x^{1/2})$ $-\frac{1}{2} < \operatorname{Re} \nu < \frac{1}{2}$	$\frac{1}{2}\pi \sec(\nu\pi) \Big[D_{\nu-1/2}\left(\frac{a}{2^{1/2} y^{1/2}}\right)$ $\times D_{-\nu-1/2}\left(-\frac{a}{2^{1/2} y^{1/2}}\right)$ $+ D_{\nu-1/2}\left(-\frac{a}{2^{1/2} y^{1/2}}\right) D_{-\nu-1/2}\left(\frac{a}{2^{1/2} y^{1/2}}\right)\Big]$ $\operatorname{Re} y > 0$

Elementary functions (cont'd)

	$f(x)$	$\int_0^\infty f(x)\, K_\nu(xy)(xy)^{1/2}\,dx$
(28)	$x^{-1}\exp(-ax^{1/2})$ $\times\cos(ax^{1/2}+\frac{1}{4}\pi-\frac{1}{2}\nu\pi)$ $-\frac{1}{2}<\mathrm{Re}\,\nu<\frac{1}{2}$	$(\frac{1}{2}\pi)^{1/2}\,\Gamma(\frac{1}{2}-\nu)\,D_{\nu-1/2}(ay^{-1/2}e^{\frac{1}{4}\pi i})$ $\times D_{\nu-1/2}(ay^{-1/2}e^{-\frac{1}{4}\pi i})$ $\mathrm{Re}\,y>0$
(29)	0 $0<x<a$ $x^{1/2-\nu}\sin[\beta(x^2-a^2)^{1/2}]$ $a<x<\infty$	$(\frac{1}{2}\pi)^{1/2}\,a^{3/2-\nu}\,\beta y^{1/2-\nu}(y^2+\beta^2)^{\nu/2-3/4}$ $\times K_{\nu-3/2}[a(y^2+\beta^2)^{1/2}]$ $\mathrm{Re}\,y>\lvert\mathrm{Im}\,\beta\rvert$
(30)	$x^{-1/2}(a^2-x^2)^{-1/2}$ $\times\cos[\beta(a^2-x^2)^{-1/2}]$ $0<x<a$ 0 $a<x<\infty$ $-1<\mathrm{Re}\,\nu<1$	$-\frac{1}{4}\pi^2 y^{1/2}\csc(\nu\pi)\,[J_{\frac{1}{2}\nu}(u)\,J_{\frac{1}{2}\nu}(v)$ $-J_{-\frac{1}{2}\nu}(u)\,J_{-\frac{1}{2}\nu}(v)]$ $u=\frac{1}{2}a[\beta+(\beta^2-y^2)^{1/2}]$ $v=\frac{1}{2}a[\beta-(\beta^2-y^2)^{1/2}]$
(31)	0 $0<x<a$ $x^{1/2-\nu}(x^2-a^2)^{-1/2}$ $\times\cos[\beta(x^2-a^2)^{1/2}]$ $a<x<\infty$	$(\frac{1}{2}\pi)^{1/2}\,a^{1/2-\nu}\,y^{1/2-\nu}(y^2+\beta^2)^{\frac{1}{2}\nu-\frac{1}{4}}$ $\times K_{\nu-1/2}[a(y^2+\beta^2)^{1/2}]$ $\mathrm{Re}\,y>\lvert\mathrm{Im}\,\beta\rvert$
(32)	$x^{-1/2}\sinh(ax)$ $-2<\mathrm{Re}\,\nu<2$	$\frac{1}{2}\pi\,y^{1/2}(y^2-a^2)^{-1/2}\csc(\frac{1}{2}\nu\pi)$ $\times\sin[\nu\sin^{-1}(a/y)]$ $\mathrm{Re}\,y>\lvert\mathrm{Re}\,a\rvert$

Elementary functions (cont'd)

	$f(x)$	$\int_0^\infty f(x)\, K_\nu(xy)(xy)^{1/2}\, dx$
(33)	$x^{-1/2} \cosh(ax)$ $-1 < \mathrm{Re}\, \nu < 1$	$\dfrac{\pi y^{1/2} \cos[\nu \sin^{-1}(a/y)]}{2(y^2 - a^2)^{1/2} \cos(\frac{1}{2}\nu\pi)}$ $\mathrm{Re}\, y > \lvert\mathrm{Re}\, a\rvert$
(34)	$x^{-3/2} \sinh(ax)$ $-1 < \mathrm{Re}\, \nu < 1$	$\dfrac{\pi \sec(\frac{1}{2}\nu\pi)}{2\nu y^{1/2}} \sin[\nu \sin^{-1}(a/y)]$ $\mathrm{Re}\, y \geq \lvert\mathrm{Re}\, a\rvert$
(35)	$x^{-1/2}(a^2 - x^2)^{-1/2}$ $\times \cosh[\beta(a^2 - x^2)^{1/2}]$ $0 < x < a$ $0 \qquad a < x < \infty$ $-1 < \mathrm{Re}\, \nu < 1$	$\frac{1}{4}\pi^2 y^{1/2} \csc(\frac{1}{2}\nu\pi) [I_{-\frac{1}{2}\nu}(u)\, I_{-\frac{1}{2}\nu}(v)$ $- I_{\frac{1}{2}\nu}(u)\, I_{\frac{1}{2}\nu}(v)]$ $u = \frac{1}{2}a[(\beta^2 + y^2)^{1/2} + \beta]$ $v = \frac{1}{2}a[(\beta^2 + y^2)^{1/2} - \beta]$

10.3. Higher transcendental functions

(1)	$x^{1/2} P_n(1 - 2x^2) \qquad 0 < x < 1$ $0 \qquad 1 < x < \infty$ $\nu = 0, \quad n = 0, 1, 2, \ldots$	$y^{-1/2}[(-1)^{n+1} K_{2n+1}(y) + \frac{1}{2}i\, S_{2n+1}(iy)]$
(2)	$0 \qquad 0 < x < a$ $x^{-1/2}(x^2 - a^2)^{-1/2} T_n(a/x)$ $a < x < \infty$ $n = 0, 1, 2, \ldots$	$\frac{1}{2}\pi\, a^{-1} y^{-1/2} W_{\frac{1}{2}n, \frac{1}{2}\nu}(ay)$ $\times W_{-\frac{1}{2}n, \frac{1}{2}\nu}(ay) \qquad \mathrm{Re}\, y > 0$

Higher transcendental functions (cont'd)

	$f(x)$	$\int_0^\infty f(x)\, K_\nu(xy)(xy)^{1/2}\, dx$
(3)	$0 \qquad 0 < x < a$ $x^{\mu}(x^2 - a^2)^{-\frac{1}{2}\mu}\, P^{\mu}_{\nu-\frac{1}{2}}(x/a)$ $a < x < \infty$ $\mathrm{Re}\,\mu < 1$	$(\frac{1}{2}\pi)^{\frac{1}{2}}\, y^{-1} e^{-\frac{1}{2}ay}\, W_{\mu,\nu}(ay)$ $\mathrm{Re}\, y > 0$
(4)	$0 \qquad 0 < x < a$ $x^{\mu-2}(x^2 - a^2)^{-\frac{1}{2}\mu}\, P^{\mu}_{\nu-\frac{1}{2}}(x/a)$ $a < x < \infty$ $\mathrm{Re}\,\mu < 1$	$(\frac{1}{2}\pi)^{\frac{1}{2}}\, a^{-1}\, e^{-\frac{1}{2}ay}\, W_{\mu-1,\nu}(ay)$ $\mathrm{Re}\, y > 0$
(5)	$0 \qquad 0 < x < a$ $x^{-\mu}(x^2 - a^2)^{-\frac{1}{2}\mu}\, P^{\mu}_{\nu-\frac{1}{2}}(x/a)$ $a < x < \infty$ $\mathrm{Re}\,\mu < 1$	$(2\pi)^{-\frac{1}{2}}\, a^{1-\mu}\, y^{\mu}\, K_\nu(\frac{1}{2}ay)$ $\times K_{\mu-\frac{1}{2}}(\frac{1}{2}ay) \qquad \mathrm{Re}\, y > 0$
(6)	$0 \qquad 0 < x < a$ $x^{\mu-1}(x^2 - a^2)^{-\frac{1}{2}\mu}\, P^{\mu}_{\nu-3/2}(x/a)$ $a < x < \infty$ $\mathrm{Re}\,\mu < 1$	$\left(\dfrac{\pi}{2ay}\right)^{\frac{1}{2}} e^{-\frac{1}{2}ay}\, W_{\mu-\frac{1}{2},\,\nu-\frac{1}{2}}(y)$
(7)	$x^{\frac{1}{2}}(x^2 + a^2)^{\frac{1}{2}\nu}\, P^{\nu}_{\mu}(1 + 2x^2 a^{-2})$ $\mathrm{Re}\, a > 0, \quad \mathrm{Re}\,\nu < 1$	$2^{-\nu}\, a y^{-\nu-\frac{1}{2}}\, S_{2\nu,\,2\mu+1}(ay)$ $\mathrm{Re}\, y > 0$

Higher transcendental functions (cont'd)

	$f(x)$	$\int_0^\infty f(x)\,K_\nu(xy)(xy)^{1/2}\,dx$
(8)	$x^{1/2}(x^2+a^2)^{\frac{1}{2}\nu}$ $\times[(\mu-\nu)P_\mu^\nu(1+2x^2a^{-2})$ $+(\mu+\nu)P_{-\mu}^\nu(1+2x^2a^{-2})]$ $\operatorname{Re} a>0,\quad \operatorname{Re}\nu<1$	$2^{1-\nu}\mu y^{-\nu-3/2}S_{2\nu+1,\,2\mu}(ay)$ $\operatorname{Re} y>0$
(9)	$x^{1/2}(x^2+a^2)^{\frac{1}{2}\nu-1}$ $\times[P_\mu^\nu(1+2x^2a^{-2})$ $+P_{-\mu}^\nu(1+2x^2a^{-2})]$ $\operatorname{Re} a>0,\quad \operatorname{Re}\nu<1$	$2^{1-\nu}y^{\frac{1}{2}-\nu}S_{2\nu-1,\,2\mu}(ay)$ $\operatorname{Re} y>0$
(10)	$0 \qquad 0<x<a$ $x^{1/2}(x^2-a^2)^{-\frac{1}{2}\nu}P_\mu^\nu(2x^2a^{-2}-1)$ $a<x<\infty$ $\operatorname{Re}\nu<1$	$2^{-\nu}ay^{\nu-\frac{1}{2}}K_{\mu+1}(ay) \qquad \operatorname{Re} y>0$
(11)	$0 \qquad 0<x<a$ $(x^2-a^2)^{\frac{1}{2}\nu-\frac{1}{4}}P_\mu^{\frac{1}{2}-\nu}(2x^2a^{-2}-1)$ $a<x<\infty$ $\operatorname{Re}\nu>-\frac{1}{2}$	$\pi^{-\frac{1}{2}}2^{\nu-1}ay^{\frac{1}{2}-\nu}[K_{\mu+\frac{1}{2}}(\frac{1}{2}ay)]^2$ $\operatorname{Re} y>0$
(12)	$x^{-\nu-\frac{1}{2}}(x^2+a^2)^{\frac{1}{4}-\frac{1}{2}\nu}$ $\times Q_{-\frac{1}{2}}^{\frac{1}{2}-\nu}(1+2a^2x^{-2})$ $\operatorname{Re} a>0,\quad \operatorname{Re}\nu<1$	$ie^{-i\pi\nu}\pi^{3/2}2^{-\nu-3}$ $\times a^{1/2-\nu}y^{\nu-1/2}[\Gamma(1-\nu)]^2$ $\times\{[J_{\nu-\frac{1}{2}}(\frac{1}{2}ay)]^2+[Y_{\nu-\frac{1}{2}}(\frac{1}{2}ay)]^2\}$ $\operatorname{Re} y>0$

Higher transcendental functions (cont'd)

	$f(x)$	$\int_0^\infty f(x)\, K_\nu(xy)(xy)^{1/2}\, dx$
(13)	$x^{-\nu-1/2}(x^2+a^2)^{1/4-1/2\nu}$ $\times Q_\mu^{1/2-\nu}(1+2a^2x^{-2})$ $\operatorname{Re} a>0, \quad \operatorname{Re}\mu>-3/2$ $\operatorname{Re}(\mu-\nu)>-3/2$	$ie^{-i\pi\nu}\pi^{1/2}2^{-\nu-1}a^{-\nu-1/2}$ $\times y^{\nu-3/2}[\Gamma(3/2+\mu-\nu)]^2$ $\times W_{-\mu-1/2,\nu-1/2}(iay)\, W_{-\mu-1/2,\nu-1/2}(-iay)$ $\operatorname{Re} y>0$
(14)	$x^{1/2} P_\mu^\nu[(1+x^2)^{1/2}] \qquad \operatorname{Re}\nu<1$	$y^{-1} S_{\nu+1/2,\mu+1/2}(y) \qquad \operatorname{Re} y>0$
(15)	$x^{1/2}(1+x^2)^{-1/2} P_\mu^\nu[(1+x^2)^{1/2}]$ $\operatorname{Re}\nu<1$	$S_{\nu-1/2,\mu+1/2}(y) \qquad \operatorname{Re} y>0$
(16)	$x^{\mu+\nu+1/2} J_\mu(ax)$ $\operatorname{Re}\mu>\lvert\operatorname{Re}\nu\rvert-1$	$2^{\mu+\nu}a^\mu y^{\nu+1/2}\Gamma(\mu+\nu+1)$ $\times(y^2+a^2)^{-\mu-\nu-1} \qquad \operatorname{Re} y>\lvert\operatorname{Im} a\rvert$
(17)	$x^{\sigma+1/2} J_\mu(ax)$ $\operatorname{Re}(\mu+\sigma)>\lvert\operatorname{Re}\nu\rvert-2$	$\dfrac{2^\sigma a^\mu}{\Gamma(\mu+1)} y^{-\sigma-\mu-3/2}$ $\times\Gamma\left(\dfrac{\mu+\nu+\sigma}{2}+1\right)\Gamma\left(\dfrac{\mu-\nu+\sigma}{2}+1\right)$ $\times {}_2F_1\left(\dfrac{\mu+\nu+\sigma}{2}+1, \dfrac{\mu-\nu+\sigma}{2}+1; \mu+1; -\dfrac{a^2}{y^2}\right)$ $\operatorname{Re} y>\lvert\operatorname{Im} a\rvert$

Higher transcendental functions (cont'd)

	$f(x)$	$\int_0^\infty f(x)\,K_\nu(xy)(xy)^{1/2}\,dx$
(18)	$x^{-1/2}\,[J_\mu(ax)]^2$ $2\,\mathrm{Re}\,\mu > \lvert\mathrm{Re}\,\nu\rvert - 1$	$\tfrac{1}{2}\Gamma(\mu+\tfrac{1}{2}\nu+\tfrac{1}{2})\,\Gamma(\mu-\tfrac{1}{2}\nu+\tfrac{1}{2})\,y^{-1/2}$ $\times\{P^{-\mu}_{\frac{1}{2}\nu-\frac{1}{2}}[(1+4a^2y^{-2})^{1/2}]\}^2$ $\mathrm{Re}\,y > 2\,\lvert\mathrm{Im}\,a\rvert$
(19)	$x^{1/2}\,[J_\mu(ax)]^2$ $2\,\mathrm{Re}\,\mu > \lvert\mathrm{Re}\,\nu\rvert - 2$	$\Gamma(\mu+\tfrac{1}{2}\nu+1)\,\Gamma(\mu-\tfrac{1}{2}\nu+1)$ $\times\,y^{-3/2}\,(1+4a^2y^{-2})^{-1/2}$ $\times\,P^{-\mu}_{\frac{1}{2}\nu}[(1+4a^2y^{-2})^{1/2}]$ $\times\,P^{-\mu}_{\frac{1}{2}\nu-1}[(1+4a^2y^{-2})^{1/2}]$ $\mathrm{Re}\,y > 2\,\lvert\mathrm{Im}\,a\rvert$
(20)	$x^{1/2}\,J_\mu(ax)\,J_{\mu+1}(ax)$ $2\,\mathrm{Re}\,\mu > \lvert\mathrm{Re}\,\nu\rvert - 3$	$\Gamma\left(\mu+\dfrac{3+\nu}{2}\right)\,\Gamma\left(\mu+\dfrac{3-\nu}{2}\right)$ $\times\,y^{-3/2}\,(1+4a^2y^{-2})^{1/2}$ $\times\,P^{-\mu}_{\frac{1}{2}\nu-\frac{1}{2}}[(1+4a^2y^{-2})^{1/2}]$ $\times\,P^{-\mu-1}_{\frac{1}{2}\nu-\frac{1}{2}}[(1+4a^2y^{-2})^{1/2}]$ $\mathrm{Re}\,y > 2\,\lvert\mathrm{Im}\,a\rvert$
(21)	$x^{-1/2}\,J_\mu(ax)\,J_{-\mu}(ax)$ $-1 < \mathrm{Re}\,\nu < 1$	$\tfrac{1}{2}\pi\,y^{-1/2}\,\sec(\tfrac{1}{2}\nu\pi)$ $\times\,P^{\mu}_{\frac{1}{2}\nu-\frac{1}{2}}[(1+4a^2y^{-2})^{1/2}]$ $\times\,P^{-\mu}_{\frac{1}{2}\nu-\frac{1}{2}}[(1+4a^2y^{-2})^{1/2}]$ $\mathrm{Re}\,y > 2\,\lvert\mathrm{Im}\,a\rvert$

Higher transcendental functions (cont'd)

	$f(x)$	$\int_0^\infty f(x)\, K_\nu(xy)(xy)^{1/2}\, dx$
(22)	$x^{1/2} J_\mu(ax) J_{-\mu}(ax)$ $-2 < \operatorname{Re} \nu < 2$	$-\tfrac{1}{2}\pi y^{-3/2} z^{-1} \csc(\tfrac{1}{2}\nu\pi)$ $\times[(\mu-\tfrac{1}{2}\nu) P^{\mu}_{\frac{1}{2}\nu}(z) P^{-\mu}_{\frac{1}{2}\nu-1}(z)$ $-(\tfrac{1}{2}\nu+\mu) P^{\mu}_{\frac{1}{2}\nu-1}(z) P^{-\mu}_{\frac{1}{2}\nu}(z)]$ $z=(1+4a^2y^{-2})^{1/2}, \quad \operatorname{Re} y > 2\lvert\operatorname{Im} a\rvert$
(23)	$x^{1/2} J_\mu(ax) J_{1-\mu}(ax)$ $-3 < \operatorname{Re} \nu < 3$	$\dfrac{a\,\Gamma(\frac{3+\nu}{2})\,\Gamma(\frac{3-\nu}{2})}{y^{5/2}\,\Gamma(2-\mu)\,\Gamma(1+\mu)}$ $\times\, {}_4F_3\left(\dfrac{3+\nu}{2}, \dfrac{3-\nu}{2}, 1, \dfrac{3}{2};\ 2-\mu, 1+\mu, 2; -\dfrac{4a^2}{y^2}\right)$ $\operatorname{Re} y > 2\lvert\operatorname{Im} a\rvert$
(24)	$x^{1/2} J_\mu(ax) J_{-\mu-1}(ax)$ $-1 < \operatorname{Re} \nu < 1$	$\tfrac{1}{2}\pi y^{-3/2} z^{-1} \sec(\tfrac{1}{2}\nu\pi)$ $\times[P^{-\mu}_{\frac{1}{2}\nu-\frac{1}{2}}(z) P^{\mu+1}_{\frac{1}{2}\nu-\frac{1}{2}}(z)$ $+(\tfrac{1}{2}\nu-\tfrac{1}{2}-\mu)(\tfrac{1}{2}\nu+\tfrac{1}{2}+\mu)$ $\times P^{-\mu-1}_{\frac{1}{2}\nu-\frac{1}{2}}(z) P^{\mu}_{\frac{1}{2}\nu-\frac{1}{2}}(z)]$ $-\tfrac{1}{4}\pi^{-1} y^{1/2} \sin(\mu\pi) \sec(\tfrac{1}{2}\nu\pi)$ $z=(1+4a^2y^{-2})^{1/2}; \quad \operatorname{Re} y > 2\lvert\operatorname{Im} a\rvert$

Higher transcendental functions (cont'd)

	$f(x)$	$\int_0^\infty f(x)\, K_\nu(xy)(xy)^{1/2}\, dx$
(25)	$x^{\sigma+1/2} J_\mu(ax) J_\lambda(ax)$ $\mathrm{Re}(\sigma+\mu+\lambda) > \lvert\mathrm{Re}\,\nu\rvert - 2$	$\dfrac{2^\sigma a^{\mu+\lambda} y^{-\mu-\lambda-\sigma-3/2}}{\Gamma(1+\mu)\Gamma(1+\lambda)}$ $\times \Gamma\left(\dfrac{\mu+\lambda+\nu+\sigma}{2}+1\right)$ $\times \Gamma\left(\dfrac{\mu+\lambda+\sigma-\nu}{2}+1\right)$ $\times {}_4F_3\left(\dfrac{\mu+\lambda+1}{2}, \dfrac{\mu+\lambda}{2}+1, \dfrac{\mu+\lambda+\nu+\sigma}{2}+1, \dfrac{\mu+\lambda-\nu+\sigma}{2}+1; 1+\mu, 1+\lambda, 1+\mu+\lambda; -\dfrac{4a^2}{y^2}\right)$ $\mathrm{Re}\, y > \lvert\mathrm{Im}\, a\rvert$
(26)	$x^{\sigma+1/2} J_\mu(ax) J_\lambda(\beta x)$	see Bailey, W.N., 1936: *Proc. London Math. Soc.* 40, 37-48; *J. London Math. Soc.* 11, 16-20.
(27)	$x^{1/2} J_{1/2\nu}(ax^2)$ $a > 0, \quad \mathrm{Re}\,\nu > -1$	$\dfrac{\pi y^{1/2}}{8a\cos(\frac{1}{2}\nu\pi)} \left[\mathbf{H}_{-1/2\nu}\left(\dfrac{y^2}{4a}\right) - Y_{-1/2\nu}\left(\dfrac{y^2}{4a}\right)\right]$ $\mathrm{Re}\, y > 0$

Higher transcendental functions (cont'd)

	$f(x)$	$\int_0^\infty f(x)\, K_\nu(xy)(xy)^{1/2}\, dx$
(28)	$x^{1/2}\, Y_{\frac{1}{2}\nu}(ax^2)$ $a>0, \quad -1<\operatorname{Re}\nu<1$	$\dfrac{\pi y^{1/2}}{4a\sin(\nu\pi)}\left[\cos(\tfrac{1}{2}\nu\pi)\, \mathbf{H}_{-\frac{1}{2}\nu}\left(\dfrac{y^2}{4a}\right) - \sin(\tfrac{1}{2}\nu\pi)\, J_{-\frac{1}{2}\nu}\left(\dfrac{y^2}{4a}\right) - \mathbf{H}_{\frac{1}{2}\nu}\left(\dfrac{y^2}{4a}\right)\right]$ $\operatorname{Re} y>0$
(29)	$x^{1/2}\, J_{\frac{1}{4}\nu}(ax^2)\, J_{-\frac{1}{4}\nu}(ax^2)$ $a>0, \quad -2<\operatorname{Re}\nu<2$	$\dfrac{\pi y^{1/2}}{32a\cos(\frac{1}{4}\nu\pi)}\left\{\left[J_{\frac{1}{4}\nu}\left(\dfrac{y^2}{16a}\right)\right]^2 + \left[Y_{\frac{1}{4}\nu}\left(\dfrac{y^2}{16a}\right)\right]^2\right\}$ $\operatorname{Re} y>0$
(30)	$x^{1/2}\, J_{\mu+\frac{1}{4}\nu}(ax^2)\, J_{\mu-\frac{1}{4}\nu}(ax^2)$ $a>0, \quad 4\operatorname{Re}\mu>\lvert\operatorname{Re}\nu\rvert-2$	$\pi^{-1} y^{-3/2}\, \Gamma(\mu+\tfrac{1}{4}\nu+\tfrac{1}{2})\, \Gamma(\mu-\tfrac{1}{4}\nu+\tfrac{1}{2}) \times W_{-\mu,\frac{1}{4}\nu}\left(\dfrac{y^2}{8a}\, e^{\frac{1}{2}i\pi}\right) \times W_{-\mu,\frac{1}{4}\nu}\left(\dfrac{y^2}{8a}\, e^{-\frac{1}{2}i\pi}\right)$ $\operatorname{Re} y>0$
(31)	$x^{-1/2}\, J_\nu(a/x)$ $\quad a>0$ $-5/2<\operatorname{Re}\nu<5/2$	$y^{-1/2}\, e^{\frac{1}{2}i(\nu+1)\pi}\, K_{2\nu}[2(ay)^{1/2}\, e^{\frac{1}{4}i\pi}] + y^{-1/2}\, e^{-\frac{1}{2}i(\nu+1)\pi}\, K_{2\nu}[2(ay)^{1/2}\, e^{-\frac{1}{4}i\pi}]$ $\operatorname{Re} y>0$

Higher transcendental functions (cont'd)

	$f(x)$	$\int_0^\infty f(x)\, K_\nu(xy)\,(xy)^{1/2}\, dx$
(32)	$x^{-5/2}\, J_\nu(a/x)$ $a > 0, \quad -1/2 < \operatorname{Re} \nu < 1/2$	$a^{-1}\, y^{1/2}\, e^{1/2 i\nu\pi} K_{2\nu}[2(ay)^{1/2}\, e^{1/4 i\pi}]$ $+ a^{-1}\, y^{1/2}\, e^{-1/2 i\nu\pi} K_{2\nu}[2(ay)^{1/2}\, e^{-1/4 i\pi}]$ $\operatorname{Re} y > 0$
(33)	$x^{2\nu-2}\, J_{\nu+1/2}(a/x)$ $a > 0, \quad \operatorname{Re} \nu > -1/3$	$(2\pi)^{1/2}\, (y/a)^{-\nu+1/2}\, J_{2\nu}[(2ay)^{1/2}]$ $\times K_{2\nu}[(2ay)^{1/2}] \qquad \operatorname{Re} y > 0$
(34)	$x^{-2\nu}\, J_{\nu-1/2}(a/x)$ $a > 0, \quad \operatorname{Re} \nu < 1$	$(2\pi)^{1/2}\, (y/a)^{\nu-1/2}\, K_{2\nu-1}[(2ay)^{1/2}]$ $\times \{\sin(\nu\pi)\, J_{2\nu-1}[(2ay)^{1/2}]$ $+ \cos(\nu\pi)\, Y_{2\nu-1}[(2ay)^{1/2}]\}$ $\operatorname{Re} y > 0$
(35)	$x^{2\nu}\, J_{1/2+\nu}(a/x)$ $a > 0, \quad \operatorname{Re} \nu > -1$	$(2\pi)^{1/2}\, (y/a)^{-\nu-1/2}\, J_{1+2\nu}[(2ay)^{1/2}]$ $\times K_{1+2\nu}[(2ay)^{1/2}] \qquad \operatorname{Re} y > 0$
(36)	$x^{\sigma-1}\, J_\mu(a/x)$ $a > 0, \quad \operatorname{Re} \sigma > \lvert\operatorname{Re} \nu\rvert - 2$	$2^{-\sigma-3/2}\, a^\sigma$ $\times G^{30}_{04}\left(\frac{a^2 y^2}{4} \middle\vert \frac{\mu-\sigma}{2}, \frac{1}{4}+\frac{\nu}{2}, \frac{1}{4}-\frac{\nu}{2}, -\frac{\mu+\sigma}{2}\right)$ $\operatorname{Re} y > 0$

Higher transcendental functions (cont'd)

	$f(x)$	$\int_0^\infty f(x)\, K_\nu(xy)(xy)^{1/2}\, dx$
(37)	$x^{-1/2}\, Y_\nu(a/x)$ $a>0, \quad -5/2<\operatorname{Re}\nu<5/2$	$-y^{-1/2}\, e^{\frac{1}{2}i\nu\pi} K_{2\nu}[2(ay)^{1/2} e^{\frac{1}{4}i\pi}]$ $-y^{-1/2}\, e^{-\frac{1}{2}i\nu\pi} K_{2\nu}[2(ay)^{1/2} e^{-\frac{1}{4}i\pi}]$ $\operatorname{Re} y>0$
(38)	$x^{-5/2}\, Y_\nu(a/x)$ $a>0, \quad -\frac{1}{2}<\operatorname{Re}\nu<\frac{1}{2}$	$a^{-1} y^{1/2} e^{\frac{1}{2}i(\nu+1)\pi}$ $\times K_{2\nu}[2(ay)^{1/2} e^{\frac{1}{4}i\pi}]$ $+a^{-1} y^{1/2} e^{-\frac{1}{2}i(\nu+1)\pi}$ $\times K_{2\nu}[2(ay)^{1/2} e^{-\frac{1}{4}i\pi}] \quad \operatorname{Re} y>0$
(39)	$x^{2\nu-2}\, Y_{\nu+\frac{1}{2}}(a/x)$ $a>0, \quad \operatorname{Re}\nu>-1/3$	$(2\pi)^{1/2} (y/a)^{\frac{1}{2}-\nu}\, Y_{2\nu}[(2ay)^{1/2}]$ $\times K_{2\nu}[(2ay)^{1/2}] \quad \operatorname{Re} y>0$
(40)	$x^{-2\nu}\, Y_{\nu-\frac{1}{2}}(a/x)$ $a>0, \quad \operatorname{Re}\nu<1$	$-(\frac{1}{2}\pi)^{1/2} (y/a)^{\nu-\frac{1}{2}} \sec(\nu\pi)$ $\times K_{2\nu-1}[(2ay)^{1/2}]\{J_{2\nu-1}[(2ay)^{1/2}]$ $-J_{1-2\nu}[(2ay)^{1/2}]\}$
(41)	$x^{2\nu}\, Y_{\nu+\frac{1}{2}}(a/x)$ $a>0, \quad \operatorname{Re}\nu<-1$	$(2\pi)^{1/2} (y/a)^{-\nu-\frac{1}{2}}\, Y_{2\nu+1}[(2ay)^{1/2}]$ $\times K_{2\nu+1}[(2ay)^{1/2}] \quad \operatorname{Re} y>0$
(42)	$x^{-1/2}\, J_\mu(a/x)\, Y_\mu(a/x)$ $a>0, \quad \nu=0$	$-2y^{-1/2}\, J_{2\mu}(2a^{1/2} y^{1/2})$ $\times K_{2\mu}(2a^{1/2} y^{1/2}) \quad \operatorname{Re} y>0$

Higher transcendental functions (cont'd)

	$f(x)$	$\int_0^\infty f(x)\, K_\nu(xy)(xy)^{1/2}\, dx$
(43)	$x^{-1/2}\{[J_\mu(a/x)]^2 - [Y_\mu(a/x)]^2\}$ $a>0, \quad \nu=0$	$4y^{-1/2}\, Y_{2\mu}(2a^{1/2}y^{1/2})\, K_{2\mu}(2a^{1/2}y^{1/2})$ $\operatorname{Re} y>0$
(44)	$J_{2\nu-1}(ax^{1/2}) \qquad \operatorname{Re}\nu>-1/2$	$\dfrac{\pi a}{4y^{3/2}}\left[I_{\nu-1}\left(\dfrac{a^2}{4y}\right) - \mathbf{L}_{\nu-1}\left(\dfrac{a^2}{4y}\right)\right]$ $\operatorname{Re} y>0$
(45)	$x^{-1/2}\, J_{2\nu}(ax^{1/2}) \qquad \operatorname{Re}\nu>-1/2$	$\dfrac{\pi}{2y^{1/2}}\left[I_\nu\left(\dfrac{a^2}{4y}\right) - \mathbf{L}_\nu\left(\dfrac{a^2}{4y}\right)\right]$ $\operatorname{Re} y>0$
(46)	$x^{-1/2}\, Y_{2\nu}(ax^{1/2})$ $-1/2<\operatorname{Re}\nu<1/2$	$\dfrac{\pi}{2y^{1/2}}\left[\csc(2\nu\pi)\,\mathbf{L}_{-\nu}\left(\dfrac{a^2}{4y}\right) - \operatorname{ctn}(2\nu\pi)\,\mathbf{L}_\nu\left(\dfrac{a^2}{4y}\right) - \tan(\nu\pi)\, I_\nu\left(\dfrac{a^2}{4y}\right) - \dfrac{\sec(\nu\pi)}{\pi}\, K_\nu\left(\dfrac{a^2}{4y}\right)\right]$ $\operatorname{Re} y>0$
(47)	$0 \qquad 0<x<a$ $x^{1/2-\nu}(x^2-a^2)^{1/2\mu} \times J_\mu[\beta(x^2-a^2)^{1/2}]$ $a<x<\infty$ $\operatorname{Re}\mu>-1$	$a^{\mu-\nu+1}\beta^\mu y^{1/2-\nu}(y^2+\beta^2)^{1/2(\nu-\mu-1)} \times K_{\nu-\mu-1}[a(y^2+\beta^2)^{1/2}]$ $\operatorname{Re} y>\operatorname{Im}\beta$

Higher transcendental functions (cont'd)

	$f(x)$	$\int_0^\infty f(x)\, K_\nu(xy)(xy)^{1/2}\, dx$
(48)	$x^{1/2} K_\nu(ax) \qquad -1 < \operatorname{Re}\nu < 1$	$\dfrac{\pi\, a^{-\nu} y^{1/2}\ \nu}{2\sin(\nu\pi)} \quad \dfrac{a^{2\nu} - y^{2\nu}}{a^2 - y^2}$ $\operatorname{Re}(y+a) > 0$
(49)	$x^{\sigma - 3/2} K_\mu(ax)$ $\operatorname{Re}\sigma > \lvert\operatorname{Re}\mu\rvert + \lvert\operatorname{Re}\nu\rvert$	$\dfrac{2^{\sigma-3} a^{-\nu-\sigma}}{\Gamma(\sigma)}\, \Gamma\left(\dfrac{\sigma+\mu+\nu}{2}\right) \Gamma\left(\dfrac{\sigma-\mu+\nu}{2}\right)$ $\times\, \Gamma\left(\dfrac{\sigma+\mu-\nu}{2}\right) \Gamma\left(\dfrac{\sigma-\mu-\nu}{2}\right) y^{\nu+1/2}$ $\times\, {}_2F_1\left(\dfrac{\sigma+\mu+\nu}{2}, \dfrac{\sigma-\mu+\nu}{2}; \sigma; 1 - \dfrac{y^2}{a^2}\right)$ $\operatorname{Re}(y+a) > 0$
(50)	$x^{1/2} [2\pi^{-1} K_0(ax) - Y_0(ax)]$ $\nu = 0$	$2\pi^{-1} y^{1/2} [(y^2 + a^2)^{-1} + (y^2 - a^2)^{-1}]$ $\times \log(y/a)$ $\operatorname{Re} y > \lvert\operatorname{Im} a\rvert, \quad \operatorname{Re}(y+a) > 0$
(51)	$x^{\sigma+1/2} J_\mu(ax) K_\lambda(\beta x)$ $x^{\sigma+1/2} K_\mu(ax) K_\lambda(\beta x)$	see Bailey, W.N., 1936: *J. London Math. Soc.* 11, 16-20; *Proc. London Math. Soc.* 40, 37-48.

Higher transcendental functions (cont'd)

	$f(x)$	$\int_0^\infty f(x)\, K_\nu(xy)(xy)^{1/2}\, dx$
(52)	$x^{1/2} K_{1/2\nu}(ax^2)$ $\mathrm{Re}\, a > 0, \quad -1 < \mathrm{Re}\, \nu < 1$	$\dfrac{\pi y^{1/2}}{8a} \left\{ \sec(\tfrac{1}{2}\nu\pi)\, K_{1/2\nu}\left(\dfrac{y^2}{4a}\right) + \pi \csc(\nu\pi) \left[\mathbf{L}_{-1/2\nu}\left(\dfrac{y^2}{4a}\right) - \mathbf{L}_{1/2\nu}\left(\dfrac{y^2}{4a}\right) \right] \right\}$
(53)	$x^{2\mu+\nu+1/2} \exp(-\tfrac{1}{2} a x^2)$ $\times I_\mu(\tfrac{1}{2} a x^2) \qquad \mathrm{Re}\, a > 0$ $\mathrm{Re}\, \mu > -\tfrac{1}{2}, \quad \mathrm{Re}(2\mu+\nu) > -1$	$\pi^{-1/2}\, 2^{\mu-1/2}\, a^{-1/2\mu-1/2\nu-1/4}\, y^{-\mu-1}$ $\times \Gamma(2\mu+\nu+1)\, \Gamma(\mu+\tfrac{1}{2})$ $\times \exp\left(\dfrac{y^2}{8a}\right) W_{k,m}\left(\dfrac{y^2}{4a}\right)$ $2k = -3\mu - \nu - \tfrac{1}{2}$ $2m = \mu + \nu + \tfrac{1}{2}$
(54)	$x^{-1/2} K_\nu(a/x) \qquad \mathrm{Re}\, a > 0$	$\pi y^{-1/2} K_{2\nu}(2a^{1/2} y^{1/2}) \qquad \mathrm{Re}\, y > 0$
(55)	$x^{-5/2} K_\nu(a/x) \qquad \mathrm{Re}\, a > 0$	$\pi a^{-1} y^{1/2} K_{2\nu}(2a^{1/2} y^{1/2}) \qquad \mathrm{Re}\, y > 0$
(56)	$x^{2\nu} K_{\nu+1/2}(a/x) \qquad \mathrm{Re}\, a > 0$	$(2\pi)^{1/2} (y/a)^{-\nu-1/2} K_{2\nu+1}[(2ay)^{1/2} e^{1/4 i\pi}]$ $\times K_{2\nu+1}[(2ay)^{1/2} e^{-1/4 i\pi}] \qquad \mathrm{Re}\, y > 0$
(57)	$x^{2\nu-2} K_{\nu+1/2}(a/x) \qquad \mathrm{Re}\, a > 0$	$(2\pi)^{1/2} (y/a)^{1/2-\nu} K_{2\nu}[(2ay)^{1/2} e^{1/4 i\pi}]$ $\times K_{2\nu}[(2ay)^{1/2} e^{-1/4 i\pi}] \qquad \mathrm{Re}\, y > 0$

Higher transcendental functions (cont'd)

	$f(x)$	$\int_0^\infty f(x)\, K_\nu(xy)(xy)^{1/2}\, dx$
(58)	$x^{\sigma-1} K_\mu(a/x)$ Re $a>0$	$2^{-\sigma-5/2} a^\sigma \times G_{04}^{40}\left(\frac{a^2 y^2}{4} \middle\vert \frac{\mu-\sigma}{2}, \frac{1}{4}+\frac{\nu}{2}, \frac{1}{4}-\frac{\nu}{2}, -\frac{\mu+\sigma}{2}\right)$
(59)	$x^{-1/2}[K_\mu(a/x)]^2$ Re $a>0$, $\nu=0$	$2\pi y^{-1/2} K_{2\mu}(2a^{1/2} y^{1/2} e^{1/4 i\pi}) \times K_{2\mu}(2a^{1/2} y^{1/2} e^{-1/4 i\pi})$ Re $y>0$
(60)	$x^{-1/2} I_{2\nu}(ax^{1/2})$ Re $\nu>-1/2$	$\frac{\pi}{2y^{1/2}}\left[I_\nu\left(\frac{a^2}{4y}\right)+\mathbf{L}_\nu\left(\frac{a^2}{4y}\right)\right]$ Re $y>0$
(61)	$x^{-1/2}[J_{2\nu}(ax^{1/2})+I_{2\nu}(ax^{1/2})]$ Re $\nu>-1/2$	$\frac{\pi}{y^{1/2}} I_\nu\left(\frac{a^2}{4y}\right)$ Re $y>0$
(62)	$x^{-1/2}[I_{2\nu}(ax^{1/2})-J_{2\nu}(ax^{1/2})]$ Re $\nu>-1/2$	$\frac{\pi}{y^{1/2}} \mathbf{L}_\nu\left(\frac{a^2}{4y}\right)$ Re $y>0$
(63)	$x^{-1/2} K_{2\nu}(ax^{1/2})$ $-1/2<$ Re $\nu<1/2$	$\frac{\pi y^{-1/2}}{4\cos(\nu\pi)}\left\{K_\nu\left(\frac{a^2}{4y}\right) + \frac{\pi}{2\sin(\nu\pi)}\left[\mathbf{L}_{-\nu}\left(\frac{a^2}{4y}\right) - \mathbf{L}_\nu\left(\frac{a^2}{4y}\right)\right]\right\}$ Re $y>0$

Higher transcendental functions (cont'd)

	$f(x)$	$\int_0^\infty f(x)\, K_\nu(xy)(xy)^{1/2}\, dx$
(64)	$x^{\nu+1/2}\, I_{2\nu}(ax^{1/2})\, J_{2\nu}(ax^{1/2})$ $\quad \mathrm{Re}\,\nu > -1/2$	$\pi^{1/2}\, 2^{-\nu-1}\, a^{2\nu+1}\, y^{-2\nu-2} \times J_{\nu-1/2}\left(\frac{a^2}{2y}\right)$ $\quad \mathrm{Re}\,y > 0$
(65)	$x^{\nu-1/2}\, I_{2\nu-1}(ax^{1/2})\, J_{2\nu-1}(ax^{1/2})$ $\quad \mathrm{Re}\,\nu > 0$	$\pi^{1/2}\, 2^{-\nu}\, a^{2\nu-1}\, y^{-2\nu} \times J_{\nu-1/2}\left(\frac{a^2}{2y}\right)$ $\quad \mathrm{Re}\,y > 0$
(66)	$x^{\nu-1/2}\, I_{2\nu-1}(ax^{1/2})\, Y_{2\nu-1}(ax^{1/2})$ $\quad \mathrm{Re}\,\nu > 0$	$\pi^{1/2}\, 2^{-\nu-1}\, a^{2\nu-1}\, y^{-2\nu}\csc(\nu\pi) \times \left[\mathbf{H}_{1/2-\nu}\left(\frac{a^2}{2y}\right) + \cos(\nu\pi)\, J_{\nu-1/2}\left(\frac{a^2}{2y}\right) + \sin(\nu\pi)\, Y_{\nu-1/2}\left(\frac{a^2}{2y}\right)\right]$ $\quad \mathrm{Re}\,y > 0$
(67)	$x^{\nu-1/2}\, J_{2\nu-1}(ax^{1/2})\, K_{2\nu-1}(ax^{1/2})$ $\quad \mathrm{Re}\,\nu > 0$	$\pi^{3/2}\, 2^{-\nu-2}\, a^{2\nu-1}\, y^{-2\nu}\csc(\nu\pi) \times \left[\mathbf{H}_{1/2-\nu}\left(\frac{a^2}{2y}\right) - Y_{1/2-\nu}\left(\frac{a^2}{2y}\right)\right]$ $\quad \mathrm{Re}\,y > 0$

Higher transcendental functions (cont'd)

	$f(x)$	$\int_0^\infty f(x)\,K_\nu(xy)(xy)^{1/2}\,dx$
(68)	$x^{-\nu-1/2}\, I_{2\nu+1}(ax^{1/2})$ $\times J_{-2\nu-1}(ax^{1/2})$ $\quad \mathrm{Re}\,\nu < 1/2$	$-\pi^{1/2}\, 2^{\nu}\, a^{-2\nu-1}\, y^{2\nu}$ $\times \left[\cos(\nu\pi)\,\mathbf{H}_{\nu+1/2}\left(\frac{a^2}{2y}\right)\right.$ $\left. + \sin(\nu\pi)\, J_{\nu+1/2}\left(\frac{a^2}{2y}\right)\right]$ $\quad \mathrm{Re}\,y > 0$
(69)	$x^{-\nu-1/2}\, I_{-2\nu-1}(ax^{1/2})$ $\times J_{2\nu+1}(ax^{1/2})$ $\quad \mathrm{Re}\,\nu < 1/2$	$\pi^{1/2}\, 2^{\nu}\, a^{-2\nu-1}\, y^{2\nu}$ $\times \left[\cos(\nu\pi)\,\mathbf{H}_{\nu+1/2}\left(\frac{a^2}{2y}\right)\right.$ $\left. - \sin(\nu\pi)\, J_{\nu+1/2}\left(\frac{a^2}{2y}\right)\right]$ $\quad \mathrm{Re}\,y > 0$
(70)	$x^{1/2-\nu}[I_{2\nu}(ax^{1/2})\, J_{-2\nu}(ax^{1/2})$ $- J_{2\nu}(ax^{1/2})\, I_{-2\nu}(ax^{1/2})]$ $\quad \mathrm{Re}\,\nu < 3/2$	$-\pi^{1/2}\, 2^{\nu}\, a^{1-2\nu}\sin(\nu\pi)\, y^{2\nu-2}$ $\times J_{\nu+1/2}\left(\frac{a^2}{2y}\right)$ $\quad \mathrm{Re}\,y > 0$
(71)	$x^{-1/2}\, K_\mu(ax^{1/2})$ $\times [\sin(\tfrac{1}{2}\mu\pi)\, J_\mu(ax^{1/2})$ $+ \cos(\tfrac{1}{2}\mu\pi)\, Y_\mu(ax^{1/2})]$ $\quad -1 < \mathrm{Re}\,\mu < 1, \quad \nu = 0$	$-\frac{\pi^2 y^{-1/2}}{16\cos(\frac{1}{2}\mu\pi)}\, H^{(1)}_{\frac{1}{2}\mu}\left(\frac{a^2}{4y}\right)$ $\times H^{(2)}_{\frac{1}{2}\mu}\left(\frac{a^2}{4y}\right)$ $\quad \mathrm{Re}\,y > 0$

Higher transcendental functions (cont'd)

	$f(x)$	$\int_0^\infty f(x)\,K_\nu(xy)(xy)^{1/2}\,dx$
(72)	$x^{-1/2} K_\mu(ax^{1/2})$ $\times \{\sin[\frac{1}{2}(\mu-\nu)\pi]\, J_\mu(ax^{1/2})$ $+ \cos[\frac{1}{2}(\mu-\nu)\pi]\, Y_\mu(ax^{1/2})\}$ $\lvert\operatorname{Re}\mu\rvert + \lvert\operatorname{Re}\nu\rvert < 1$	$-\frac{1}{2}a^{-2}y^{1/2}\,\Gamma\left(\frac{1+\mu-\nu}{2}\right)\Gamma\left(\frac{1-\mu-\nu}{2}\right)$ $\times W_{\frac{1}{2}\nu,\frac{1}{2}\mu}\left(\frac{a^2}{2y}e^{\frac{1}{2}i\pi}\right)$ $\times W_{\frac{1}{2}\nu,\frac{1}{2}\mu}\left(\frac{a^2}{2y}\,e^{-\frac{1}{2}i\pi}\right)$ $\operatorname{Re} y > 0$
(73)	$x^{1/2}\,\mathbf{H}_\nu(ax)$ $\operatorname{Re}\nu > -3/2$	$a^{\nu+1}\,y^{-\nu-1/2}(y^2+a^2)^{-1}$ $\operatorname{Re} y > \lvert\operatorname{Im} a\rvert$
(74)	$x^{\mu+\nu+1/2}\,\mathbf{H}_\mu(ax)$ $\operatorname{Re}\mu > -3/2$ $\operatorname{Re}(\mu+\nu) > -3/2$	$\pi^{-1/2}\,2^{\mu+\nu+1}\,a^{\mu+1}$ $\times y^{-2\mu-\nu-5/2}\,\Gamma(\mu+\nu+3/2)$ $\times {}_2F_1\left(\mu+\nu+\frac{3}{2},\,1;\frac{3}{2};-\frac{a^2}{y^2}\right)$ $\operatorname{Re} y > \lvert\operatorname{Im} a\rvert$
(75)	$x^{1/2}\,\mathbf{H}_{\frac{1}{2}\nu}(ax^2)$ $a > 0,\quad \operatorname{Re}\nu > -2$	$\frac{y^{1/2}\,\Gamma(1+\frac{1}{2}\nu)}{2^{1-\frac{1}{2}\nu}\,a\pi}\,S_{-\frac{1}{2}\nu-1,\,\frac{1}{2}\nu}\left(\frac{y^2}{4a}\right)$ $\operatorname{Re} y > 0$
(76)	$x^{3/2}\,\mathbf{H}_{\frac{1}{2}\nu+\frac{1}{2}}(ax^2)$ $a > 0,\quad \operatorname{Re}\nu > -3$	$\frac{2^{\nu/2+1/2}\,y^{3/2}}{a^2\,\pi}\,\Gamma\left(\frac{3+\nu}{2}\right)$ $\times S_{-\frac{\nu+5}{2},\,\frac{\nu-1}{2}}\left(\frac{y^2}{4a}\right)$ $\operatorname{Re} y > 0$

Higher transcendental functions (cont'd)

	$f(x)$	$\int_0^\infty f(x)\, K_\nu(xy)(xy)^{1/2}\, dx$
(77)	$x^{5/2}\, \mathbf{H}_{\frac{1}{2}\nu}(ax^2)$ $a>0, \quad \mathrm{Re}\,\nu > -3$	$\dfrac{y^{5/2}\,\Gamma(2+\frac{1}{2}\nu)}{2^{1-\frac{1}{2}\nu}\, a^3\, \pi}\; S_{-3-\frac{1}{2}\nu,\,\frac{1}{2}\nu}\left(\dfrac{y^2}{4a}\right)$ $\mathrm{Re}\, y > 0$
(78)	$x^{1/2}\, s_{\mu,\frac{1}{2}\nu}(ax^2) \qquad a>0$ $\mathrm{Re}\,\mu > \frac{1}{2}\lvert\mathrm{Re}\,\nu\rvert - 2$	$\dfrac{y^{1/2}}{4a}\,\Gamma(\mu+\frac{1}{2}\nu+1)\,\Gamma(\mu-\frac{1}{2}\nu+1)$ $\times S_{-\mu-1,\,\frac{1}{2}\nu}\left(\dfrac{y^2}{4a}\right) \qquad \mathrm{Re}\, y>0$
(79)	$x^{3/2}\, s_{\mu,\frac{1}{2}\nu+\frac{1}{2}}(ax^2) \qquad a>0$ $2\,\mathrm{Re}\,\mu > \lvert\mathrm{Re}\,\nu\rvert - 5$	$\dfrac{y^{3/2}}{8a^2}\left(\mu+\dfrac{3-\nu}{2}\right)$ $\times\Gamma\left(\mu+\dfrac{1-\nu}{2}\right)\,\Gamma\left(\mu+\dfrac{3+\nu}{2}\right)$ $\times S_{-\mu-2,\,\frac{1}{2}\nu-\frac{1}{2}}\left(\dfrac{y^2}{4a}\right) \qquad \mathrm{Re}\, y>0$
(80)	$x^{5/2}\, s_{\mu,\frac{1}{2}\nu}(ax^2) \qquad a>0$ $\mathrm{Re}\,\mu > \frac{1}{2}\lvert\mathrm{Re}\,\nu\rvert - 3$	$\dfrac{y^{5/2}}{16a^3}\,(2+\mu+\frac{1}{2}\nu)\,(2+\mu-\frac{1}{2}\nu)$ $\times\Gamma(1+\mu+\frac{1}{2}\nu)\,\Gamma(1+\mu-\frac{1}{2}\nu)$ $\times S_{-\mu-3,\,\frac{1}{2}\nu}\left(\dfrac{y^2}{4a}\right) \qquad \mathrm{Re}\, y>0$
(81)	$D_{\nu-\frac{1}{2}}(ax^{-\frac{1}{2}})\,D_{-\nu-\frac{1}{2}}(ax^{-\frac{1}{2}})$ $\lvert\arg a\rvert < \frac{1}{4}\pi$	$\dfrac{\pi}{2y}\;\exp[-a(2y)^{1/2}] \qquad \mathrm{Re}\, y>0$

Higher transcendental functions (cont'd)

	$f(x)$	$\int_0^\infty f(x)\,K_\nu(xy)(xy)^{1/2}\,dx$
(82)	$x^{2\mu+\nu-1/2}\exp(-\tfrac{1}{2}ax^2)$ $\times M_{\kappa,\mu}(ax^2)$ $\quad \mathrm{Re}\,a>0$ $\mathrm{Re}\,\mu>-\tfrac{1}{2},\ \mathrm{Re}(2\mu+\nu)>-1$	$2^{\mu-\kappa-1/2}\,a^{1/4-1/2(\mu+\nu+\kappa)}\,y^{\kappa-\mu-1}$ $\times\Gamma(2\mu+1)\,\Gamma(2\mu+\nu+1)$ $\times\exp\left(\frac{y^2}{8a}\right)W_{k,m}\left(\frac{y^2}{4a}\right)$ $2k=-3\mu-\nu-\kappa-\tfrac{1}{2}$ $2m=\mu+\nu-\kappa+\tfrac{1}{2}$ $\mathrm{Re}\,y>0$
(83)	$x^{-3/2}M_{\kappa,0}(iax^2)\,M_{\kappa,0}(-iax^2)$ $a>0,\quad \nu=0$	$\frac{\pi y^{1/2}}{16}\left\{\left[J_\kappa\left(\frac{y^2}{8a}\right)\right]^2+\left[Y_\kappa\left(\frac{y^2}{8a}\right)\right]^2\right\}$
(84)	$x^{-3/2}M_{\kappa,\mu}(iax^2)M_{\kappa,\mu}(-iax^2)$ $a>0,\quad \mathrm{Re}\,\mu>-\tfrac{1}{2},\quad \nu=0$	$ay^{-3/2}[\Gamma(2\mu+1)]^2\,W_{-\mu,\kappa}\left(\frac{iy^2}{4a}\right)$ $\times W_{-\mu,\kappa}\left(-\frac{iy^2}{4a}\right)$ $\quad \mathrm{Re}\,y>0$
(85)	$x^{1/2}\,W_{1/2\nu,\mu}(a/x)\,W_{-1/2\nu,\mu}(a/x)$ $\mathrm{Re}\,a>0$	$2ay^{-1/2}\,K_{2\mu}[(2ay)^{1/2}e^{1/4 i\pi}]$ $\times K_{2\mu}[(2ay)^{1/2}e^{-1/4 i\pi}]\quad \mathrm{Re}\,y>0$
(86)	$x^{\nu+1/2}\ {}_2F_1(\alpha,\beta;\nu+1;-\lambda^2x^2)$ $\mathrm{Re}\,\lambda>0,\quad \mathrm{Re}\,\nu>-1$	$2^{\nu+1}\lambda^{-\alpha-\beta}y^{\alpha+\beta-\nu-3/2}\,\Gamma(\nu+1)$ $\times S_{1-\alpha-\beta,\,\alpha-\beta}(y/\lambda)\quad \mathrm{Re}\,y>0$

Higher transcendental functions (cont'd)

	$f(x)$	$\int_0^\infty f(x)\,K_\nu(xy)(xy)^{1/2}\,dx$
(87)	$x^{\nu+2\gamma-3/2}$ $\times\, {}_3F_2(1,\alpha,\beta;\gamma,\gamma+\nu;-\lambda^2x^2)$ $\operatorname{Re}\lambda>0,\quad \operatorname{Re}\gamma>0$ $\operatorname{Re}(\gamma+\nu)>0$	$2^{\nu+2\gamma-2}\lambda^{-\alpha-\beta}y^{\alpha+\beta-2\gamma-\nu+1/2}$ $\times\,\Gamma(\gamma)\,\Gamma(\gamma+\nu)\,S_{1-\alpha-\beta,\alpha-\beta}(y/\lambda)$ $\operatorname{Re}y>0$
(88)	$x^{\mu-3/2}$ $\times\, {}_pF_q(\alpha_1,\ldots,\alpha_p;\beta_1,\ldots\beta_q;-\lambda x^2)$ $p\le q-1,\quad \operatorname{Re}\mu>\lvert\operatorname{Re}\nu\rvert$	$2^{\mu-2}y^{1/2-\mu}\Gamma\left(\dfrac{\mu+\nu}{2}\right)\Gamma\left(\dfrac{\mu-\nu}{2}\right)$ $\times\, {}_{p+2}F_q\left(\alpha_1,\ldots,\alpha_p,\dfrac{\mu+\nu}{2},\dfrac{\mu-\nu}{2};\ \beta_1,\ldots,\beta_q;\dfrac{4\lambda}{y^2}\right)$ $\operatorname{Re}y>0$
(89)	$x^{\mu-3/2}$ $\times E(\alpha_1,\ldots,\alpha_p:\rho_1,\ldots,\rho_q:ax^{-2})$ $a>0,\quad \operatorname{Re}\mu>\lvert\operatorname{Re}\nu\rvert$	$2^{\mu-2}a^{-\mu}y^{1/2}$ $\times E(\alpha_1,\ldots,\alpha_{p+2}:\rho_1,\ldots,\rho_q:\tfrac{1}{4}ay^2)$ $\alpha_{p+1}=\dfrac{\mu+\nu}{2},\quad \alpha_{p+2}=\dfrac{\mu-\nu}{2}$ $\operatorname{Re}y>0$
(90)	$G^{mn}_{pq}\left(\lambda x^2\,\middle\vert\,\begin{matrix}\alpha_1,\ldots,\alpha_p\\ \beta_1,\ldots,\beta_q\end{matrix}\right)$ $p+q<2(m+n)$ $\lvert\arg\lambda\rvert<(m+n-\tfrac{1}{2}p-\tfrac{1}{2}q)\pi$ $\operatorname{Re}\beta_j>\tfrac{1}{2}\lvert\operatorname{Re}\nu\rvert-\tfrac{3}{4}$ $j=1,\ldots,m$	$2^{-3/2}\lambda^{-1/2}$ $\times\, G^{n+2,m}_{q,p+2}\left(\dfrac{y^2}{4\lambda}\,\middle\vert\,\begin{matrix}\tfrac{1}{2}-\beta_1,\ldots,\tfrac{1}{2}-\beta_q\\ h,k,\tfrac{1}{2}-\alpha_1,\ldots,\tfrac{1}{2}-\alpha_p\end{matrix}\right)$ $h=\tfrac{1}{4}+\tfrac{1}{2}\nu,\quad k=\tfrac{1}{4}-\tfrac{1}{2}\nu$ $\operatorname{Re}y>0$

CHAPTER XI

H-TRANSFORMS

We call

$$\int_0^\infty f(x)\,\mathbf{H}_\nu(xy)(xy)^{1/2}\,dx$$

the **H**-*transform of order* ν of $f(x)$ and regard y as a positive real variable. The inversion formula 11.1(1) was given by Titchmarsh (1937, p. 215). The **H**-transform is the reciprocal of the Y-transform (see chapter IX).

From the transform pairs given in this chapter further transform pairs may be derived by means of the methods indicated in the introduction to vol. I, and also by means of the general formulas of sec. 11.1. Moreover, **H**-transforms being reciprocal to Y-transforms when $-\frac{1}{2} < \operatorname{Re}\nu < \frac{1}{2}$, many further formulas may be obtained from the tables in chapter IX: the extension of such formulas by means of analytic continuation to a wider range of $\operatorname{Re}\nu$ (the range of absolute convergence of the integral) is immediate.

REFERENCE

Titchmarsh, E.C., 1937: *Introduction to the theory of Fourier integrals*. Oxford.

$\mathbf{H}$-TRANSFORMS

11.1. General formulas

	$f(x)$	$\int_0^\infty f(x)\,\mathbf{H}_\nu(xy)(xy)^{1/2}\,dx = g(y;\nu) \qquad y>0$
(1)	$\int_0^\infty g(y;\nu)\,Y_\nu(xy)(xy)^{1/2}\,dy$ $-1/2<\mathrm{Re}\,\nu<1/2$	$g(y;\nu)$
(2)	$f(ax) \qquad a>0$	$a^{-1}\,g(a^{-1}y;\nu)$
(3)	$x^m f(x) \qquad m=0,1,2,\ldots$	$y^{1/2-\nu}\left(\dfrac{d}{y\,dy}\right)^m [y^{\nu-1/2+m}\,g(y;\nu+m)]$
(4)	$x^{1/2+\nu}\left(\dfrac{d}{x\,dx}\right)^m [x^{m-\nu-1/2}f(x)]$ $m=0,1,2,\ldots$	$(-y)^m\,g(y;\nu-m)$
(5)	$x^{\nu+1/2}\int_x^\infty \xi^{1/2-\nu-\mu}(\xi^2-x^2)^{\mu-1}$ $\times f(\xi)\,d\xi$ $\mathrm{Re}\,\mu>0, \quad \mathrm{Re}\,\nu>-3/2$	$2^{\mu-1}\,\Gamma(\mu)\,y^{-\mu}\,g(y;\nu+\mu)$
(6)	$x^{-\mu}f(x)$ $\mathrm{Re}\,\nu+3/2>\mathrm{Re}\,\mu>0$	$2^{1-\mu}[\Gamma(\mu)]^{-1}\,y^{1/2-\nu}$ $\times\int_0^y \eta^{1/2-\mu+\nu}(y^2-\eta^2)^{\mu-1}$ $\times\,g(\eta;\nu-\mu)\,d\eta$

11.2. Elementary functions

	$f(x)$	$\int_0^\infty f(x)\,\mathbf{H}_\nu(xy)(xy)^{1/2}\,dx \qquad y>0$
(1)	$x^{-1/2} \qquad -2<\operatorname{Re}\nu<0$	$-\operatorname{ctn}(\tfrac{1}{2}\nu\pi)\,y^{-1/2}$
(2)	$x^{\nu+1/2} \qquad 0<x<a$ $0 \qquad a<x<\infty$ $\operatorname{Re}\nu>-3/2$	$a^{\nu+1}y^{-1/2}\,\mathbf{H}_{\nu+1}(ay)$
(3)	$x^{1/2-\nu} \qquad 0<x<a$ $0 \qquad a<x<\infty$	$\dfrac{ay^{\nu-1/2}}{2^{\nu-1}\pi^{1/2}\,\Gamma(\nu+\tfrac{1}{2})}$ $-a^{1-\nu}y^{-1/2}\,\mathbf{H}_{\nu-1}(ay)$
(4)	$x^{\lambda-1/2}$ $\operatorname{Re}\lambda<\tfrac{1}{2}$ $-2<\operatorname{Re}(\lambda+\nu)<0$	$2^{\lambda}y^{-\lambda-1/2}\tan[\tfrac{1}{2}(\lambda+\nu+1)\pi]$ $\times\dfrac{\Gamma(\tfrac{1}{2}+\tfrac{1}{2}\lambda+\tfrac{1}{2}\nu)}{\Gamma(\tfrac{1}{2}-\tfrac{1}{2}\lambda+\tfrac{1}{2}\nu)}$
(5)	$x^{\lambda-1/2} \qquad 0<x<a$ $0 \qquad a<x<\infty$ $\operatorname{Re}(\lambda+\nu)>-2$	$\dfrac{a^{\lambda+\nu+2}\,y^{\nu+3/2}}{2^{\nu}\pi^{1/2}\,\Gamma(\nu+3/2)\,(\lambda+\nu+2)}$ $\times\,{}_2F_3\left(1,\dfrac{\lambda+\nu}{2}+1;\ \dfrac{3}{2},\right.$ $\left.\nu+\dfrac{3}{2},\dfrac{\lambda+\nu}{2}+2;-\dfrac{a^2y^2}{4}\right)$
(6)	$x^{-1/2}(x^2+a^2)^{-1}$ $\operatorname{Re}a>0, \qquad \nu=1$	$\dfrac{\pi y^{1/2}}{2a}[I_1(ay)-\mathbf{L}_1(ay)]$

Elementary functions (cont'd)

	$f(x)$	$\int_0^\infty f(x)\,\mathbf{H}_\nu(xy)(xy)^{1/2}\,dx \qquad y>0$
(7)	$x^{-1/2}(x^2+a^2)^{-1}$ $\operatorname{Re} a>0, \quad -2<\operatorname{Re}\nu<2$	$-\dfrac{\pi y^{1/2}}{2a\sin(\tfrac12\nu\pi)}\;\mathbf{L}_\nu(ay)$ $+\dfrac{y^{3/2}\operatorname{ctn}(\tfrac12\nu\pi)}{1-\nu^2}$ $\times\;{}_1F_2\left(1;\dfrac{3-\nu}{2},\dfrac{3+\nu}{2};\dfrac{a^2y^2}{4}\right)$
(8)	$x^{\nu+1/2}(x^2+a^2)^{\mu-1}$ $\operatorname{Re} a>0, \quad \operatorname{Re}\nu>-3/2$ $\operatorname{Re}(\mu+\nu)<1/2$ $\operatorname{Re}(2\mu+\nu)<3/2$	$\dfrac{2^{\mu-1}\,\pi a^{\mu+\nu}\,y^{1/2-\mu}}{\Gamma(1-\mu)\cos[(\mu+\nu)\pi]}$ $\times[I_{-\mu-\nu}(ay)-\mathbf{L}_{\mu+\nu}(ay)]$
(9)	$x^{1/2-\nu}(x^2+a^2)^{\mu-1}$ $\operatorname{Re} a>0, \quad \operatorname{Re}\mu<1/2$ $\operatorname{Re}(2\mu-\nu)<3/2$	$\dfrac{2^{\mu-1}\,\pi a^{\mu-\nu}\,y^{1/2-\mu}}{\Gamma(1-\mu)\cos(\mu\pi)}\;I_{\nu-\mu}(ay)$ $+\dfrac{a^{2\mu+1}\,y^{\nu+3/2}\,\Gamma(-1/2-\mu)}{2^{\nu+2}\,\Gamma(1-\mu)\,\Gamma(\nu+3/2)}$ $\times\;{}_1F_2\left(1;\mu+\dfrac32,\;\nu+\dfrac32;\dfrac{a^2y^2}{4}\right)$
(10)	$x^{\lambda-1/2}(x^2+a^2)^{\mu-1}$ $\operatorname{Re} a>0, \;\operatorname{Re}(\lambda+\nu)>-2$ $\operatorname{Re}(\lambda+2\mu)<5/2$ $\operatorname{Re}(\lambda+2\mu+\nu)<2$	$2^{-1/2}[\Gamma(1-\mu)]^{-1}\,a^{\lambda+2\mu-3/2}$ $\times\,G^{22}_{24}\left(\dfrac{a^2y^2}{4}\,\middle\vert\,\begin{matrix} l,\,m \\ l,\,m-\mu,\,h,\,k\end{matrix}\right)$ $h=\dfrac14+\dfrac{\nu}{2}, \quad k=\dfrac14-\dfrac{\nu}{2}$ $l=\dfrac34+\dfrac{\nu}{2}, \quad m=\dfrac34-\dfrac{\lambda}{2}$

Elementary functions (cont'd)

	$f(x)$	$\int_0^\infty f(x)\,\mathbf{H}_\nu(xy)(xy)^{1/2}\,dx \qquad y>0$
(11)	$(x^2+a^2)^{-1/2}$ $\times[x+(x^2+a^2)^{1/2}]^{\nu+1}$ $\mathrm{Re}\,a>0, \quad -2<\mathrm{Re}\,\nu<0$	$\dfrac{\pi^{1/2}\,a^{\nu+1/2}}{y^{1/2}\sin(\nu\pi)}\,[\sinh(\tfrac{1}{2}ay)\,I_{\nu+1/2}(\tfrac{1}{2}ay)$ $-\cosh(\tfrac{1}{2}ay)\,I_{-\nu-1/2}(\tfrac{1}{2}ay)]$
(12)	$x^{\nu+1/2}(a^2-x^2)^{\mu-1} \quad 0<x<a$ $0 \quad a<x<\infty$ $\mathrm{Re}\,\mu>0, \quad \mathrm{Re}\,\nu>-3/2$	$2^{\mu-1}\,a^{\mu+\nu}\,y^{1/2-\mu}\,\Gamma(\mu)\,\mathbf{H}_{\mu+\nu}(ay)$
(13)	$x^{\lambda-1/2}(a^2-x^2)^{\mu-1} \quad 0<x<a$ $0 \quad a<x<\infty$ $\mathrm{Re}\,\mu>0, \quad \mathrm{Re}(\lambda+\nu)>-2$	$\dfrac{a^{2\mu+\nu+\lambda}\,y^{\nu+3/2}\,\Gamma(\mu)\,\Gamma(\frac{\lambda+\nu}{2}+1)}{2^{\nu+1}\,\pi^{1/2}\,\Gamma(\nu+3/2)\Gamma(\frac{\lambda+\nu}{2}+\mu+1)}$ $\times\,{}_2F_3\left(1, \dfrac{\lambda+\nu}{2}+1; \dfrac{3}{2}, \nu+\dfrac{3}{2}, \dfrac{\lambda+\nu}{2}+\mu+1; -\dfrac{a^2y^2}{4}\right)$
(14)	$0 \quad 0<x<a$ $x^{-\nu-1/2}(x^2-a^2)^{-\nu-1/2}$ $a<x<\infty$ $-\tfrac{1}{2}<\mathrm{Re}\,\nu<1$	$2^{-\nu-1}\,\pi^{1/2}\,a^{-2\nu}\,y^{\nu+1/2}$ $\times\,\Gamma(\tfrac{1}{2}-\nu)\,[J_\nu(\tfrac{1}{2}ay)]^2$
(15)	$0 \quad 0<x<a$ $x^{\nu+1/2}(x^2-a^2)^m \quad a<x<\infty$ $m=0, 1, 2, \ldots,$ $\mathrm{Re}\,\nu<-2m-\tfrac{1}{2}$	$(-1)^{m+1}\,2^m\,a^{m+\nu+1}\,y^{-m-1/2}\,m!$ $\times\,\mathbf{H}_{\nu+m+1}(ay)$

Elementary functions (cont'd)

	$f(x)$	$\int_0^\infty f(x)\,\mathbf{H}_\nu(xy)(xy)^{1/2}\,dx \qquad y>0$
(16)	$0 \qquad 0<x<a$ $x^{\nu+1/2}(x^2-a^2)^{\mu-1} \qquad a<x<\infty$ $\operatorname{Re}\mu>0, \quad \operatorname{Re}(\mu+\nu)<1/2$ $\operatorname{Re}(2\mu+\nu)<3/2$	$2^{\mu-1}a^{\mu+\nu}y^{1/2-\mu}\Gamma(\mu)\sec[(\mu+\nu)\pi]$ $\times[\sin(\mu\pi)\,J_{-\mu-\nu}(ay)$ $+\cos(\nu\pi)\,\mathbf{H}_{\mu+\nu}(ay)]$
(17)	$0 \qquad 0<x<a$ $x^{-\nu-1/2}(x^2-a^2)^{\mu} \qquad a<x<\infty$ $-1<\operatorname{Re}\mu<0$ $\operatorname{Re}\nu>2\operatorname{Re}\mu-1/2$	$-\pi\,2^{2\mu-\nu}y^{\nu-2\mu-1/2}$ $\times[\Gamma(1/2-\mu)\,\Gamma(1/2+\nu-\mu)\sin(\mu\pi)]^{-1}$ $\times\,{}_1F_2(-\mu;\,1/2-\mu,\,1/2+\nu-\mu;\,-1/4\,a^2y^2)$
(18)	$0 \qquad 0<x<a$ $x^{\lambda-1/2}(x^2-a^2)^{\mu-1} \qquad a<x<\infty$ $\operatorname{Re}\mu>0, \quad \operatorname{Re}(\lambda+2\mu)<5/2$ $\operatorname{Re}(\lambda+2\mu+\nu)<2$	$2^{-1/2}\Gamma(\mu)\,a^{2\mu+\lambda-3/2}$ $\times G^{21}_{24}\left(\frac{a^2y^2}{4}\middle\vert\begin{matrix} l,\ m \\ l,\ m-\mu,\ h,\ k\end{matrix}\right)$ $h=\frac{1}{4}+\frac{\nu}{2}, \qquad k=\frac{1}{4}-\frac{\nu}{2}$ $l=\frac{3}{4}+\frac{\nu}{2}, \qquad m=\frac{3}{4}-\frac{\lambda}{2}$
(19)	$x^{\lambda-1/2}e^{-\alpha x}$ $\operatorname{Re}\alpha>0, \quad \operatorname{Re}(\lambda+\nu)>-2$	$\frac{y^{\nu+3/2}\,\Gamma(\lambda+\nu+2)}{2^\nu\alpha^{\lambda+\nu+2}\,\pi^{1/2}\,\Gamma(\nu+3/2)}$ $\times\,{}_3F_2\left(1,\ \frac{\lambda+\nu}{2}+1,\ \frac{\lambda+\nu+3}{2};\right.$ $\left.\frac{3}{2},\ \nu+\frac{3}{2};\ -\frac{y^2}{\alpha^2}\right)$

Elementary functions (cont'd)

	$f(x)$	$\int_0^\infty f(x)\,\mathbf{H}_\nu(xy)(xy)^{1/2}\,dx \qquad y>0$
(20)	$x^{\lambda+1/2}\exp(-ax^2)$ $\operatorname{Re} a>0,\quad \operatorname{Re}(\lambda+\nu)>-3$	$2^{-\nu-1}\pi^{-1/2}a^{-(\lambda+\nu+3)/2}y^{\nu+3/2}$ $\times\dfrac{\Gamma(\frac{\lambda+\nu+3}{2})}{\Gamma(\nu+3/2)}$ $\times\,{}_2F_2\left(1,\dfrac{\lambda+\nu+3}{2};\dfrac{3}{2},\nu+\dfrac{3}{2};-\dfrac{y^2}{4a}\right)$
(21)	$x^{-\nu-1/2}\sin(ax)$ $a>0,\quad \operatorname{Re}\nu>-\tfrac{1}{2}$	$0 \qquad 0<y<a$ $\pi^{1/2}2^{-\nu}y^{1/2-\nu}[\Gamma(\nu+\tfrac{1}{2})]^{-1}$ $\times(y^2-a^2)^{\nu-1/2} \qquad a<y<\infty$
(22)	$x^{1/2}\cos[(\nu+1)\theta]/\sin\theta$ $0<x<a$ $0 \qquad a<x<\infty$ $0<\theta<\tfrac{1}{2}\pi,\quad x=a\cos\theta$ $\operatorname{Re}\nu>-2$	$\pi^{1/2}a^{1/2}\sin(\tfrac{1}{2}ay)\,J_{\nu+1/2}(\tfrac{1}{2}ay)$

11.3. Higher transcendental functions

(1)	$J_{\nu+1/2}(ax)$ $a>0,\quad -3/2<\operatorname{Re}\nu<1$	$0 \qquad 0<y<a$ $\left(\dfrac{2}{\pi}\right)^{1/2}\left(\dfrac{y}{a}\right)^{\nu+1/2}(y^2-a^2)^{-1/2}$ $a<y<\infty$

Higher transcendental functions (cont'd)

	$f(x)$	$\int_0^\infty f(x)\,\mathbf{H}_\nu(xy)(xy)^{1/2}\,dx \qquad y>0$
(2)	$x^{-1/2}\,Y_{\nu+1}(ax)$ $a>0, \quad -3/2<\operatorname{Re}\nu<3/2$	$0 \qquad 0<y<a$ $-a^{-\nu-1}\,y^{1/2+\nu} \qquad a<y<\infty$
(3)	$x^{\mu-\nu+1/2}\,Y_\mu(ax)$ $a>0, \quad \operatorname{Re}(\nu-\mu)>0$ $-3/2<\operatorname{Re}\mu<1/2$	$0 \qquad 0<y<a$ $\dfrac{2^{1+\mu-\nu}\,a^\mu}{\Gamma(\nu-\mu)}\,y^{1/2-\nu}\,(y^2-a^2)^{\nu-\mu-1}$ $a<y<\infty$
(4)	$x^{1/2-\mu}[\sin(\mu\pi)\,J_{\mu+\nu}(ax)$ $+\cos(\mu\pi)\,Y_{\mu+\nu}(ax)]$ $a>0, \quad 1<\operatorname{Re}\mu<3/2$ $\operatorname{Re}\nu>-3/2, \quad \operatorname{Re}(\nu-\mu)<1/2$	$0 \qquad 0<y<a$ $\dfrac{y^{1/2+\nu}(y^2-a^2)^{\mu-1}}{2^{\mu-1}\,a^{\mu+\nu}\,\Gamma(\mu)} \qquad a<y<\infty$
(5)	$x^{\nu+1/2}\,J_\nu(ax)\,Y_\nu(ax)$ $a>0, \quad -3/4<\operatorname{Re}\nu<0$	$\dfrac{\Gamma(2\nu+3/2)\,y^{\nu+3/2}}{\pi^{3/2}\,2^{\nu+2}\,a^{2\nu+3}\,\Gamma(\nu+2)}$ $\times\,{}_2F_1\left(1,\,2\nu+\dfrac{3}{2};\,\nu+2;\,\dfrac{y^2}{4a^2}\right)$ $0<y<2a$
(6)	$x^{\nu+1/2}\{[J_\nu(ax)]^2-[Y_\nu(ax)]^2\}$ $a>0, \quad -3/4<\operatorname{Re}\nu<0$	$0 \qquad 0<y<2a$ $\dfrac{2^{3\nu+2}\,a^{2\nu}\,y^{-\nu-1/2}}{\pi^{1/2}\,\Gamma(1/2-\nu)}\,(y^2-4a^2)^{-\nu-1/2}$ $2a<y<\infty$

Higher transcendental functions (cont'd)

	$f(x)$	$\int_0^\infty f(x)\, \mathbf{H}_\nu(xy)(xy)^{1/2}\, dx \qquad y>0$
(7)	$x^{1/2}\{[J_{1/2\nu}(ax)]^2-[Y_{1/2\nu}(ax)]^2\}$ $a>0, \quad -3/2<\operatorname{Re}\nu<0$	$0 \qquad 0<y<2a$ $4\pi^{-1} y^{-1/2}(y^2-4a^2)^{-1/2} \qquad 2a<y<\infty$
(8)	$x^{1/2}[J_{1/2\nu+1/2\mu}(ax)\, J_{1/2\nu-1/2\mu}(ax)$ $-Y_{1/2\nu+1/2\mu}(ax)\, Y_{1/2\nu-1/2\mu}(ax)]$ $a>0, \quad -3/2<\operatorname{Re}\nu<0$	$0 \qquad 0<y<2a$ $4\pi^{-1} y^{-1/2}(y^2-4a^2)^{-1/2}\cosh(\mu u)$ $y=2a\cosh u, \quad u>0$
(9)	$J_{2\nu+1}(ax^{1/2})$ $a>0, \quad -3/2<\operatorname{Re}\nu<1/4$	$-\dfrac{a}{2y^{3/2}}\, Y_{\nu+1}\left(\dfrac{a^2}{4y}\right)$
(10)	$x^{-1/2} J_{2\nu}(ax^{1/2})$ $a>0, \quad -1<\operatorname{Re}\nu<5/4$	$-y^{-1/2}\, Y_\nu\left(\dfrac{a^2}{4y}\right)$
(11)	$x^{1/2}[\{J_{1/2\nu}[b(z-a)]\}^2$ $-\{Y_{1/2\nu}[b(z+a)]\}^2]$ $\operatorname{Re} a>0, \quad b>0$ $-3/2<\operatorname{Re}\nu<1$ $z=(x^2+a^2)^{1/2}$	$-\dfrac{4\sin[a(4b^2-y^2)^{1/2}]}{\pi y^{1/2}(4b^2-y^2)^{1/2}} \qquad 0<y<2b$ $\dfrac{4\exp[-a(y^2-4b^2)^{1/2}]}{\pi y^{1/2}(y^2-4b^2)^{1/2}} \qquad 2b<y<\infty$
(12)	$x^{1/2} K_\nu(ax)$ $\operatorname{Re} a>0, \quad \operatorname{Re}\nu>-3/2$	$a^{-\nu-1} y^{\nu+3/2}(y^2+a^2)^{-1}$

Higher transcendental functions (cont'd)

	$f(x)$	$\int_0^\infty f(x)\,\mathbf{H}_\nu(xy)(xy)^{1/2}\,dx \qquad y>0$
(13)	$x^{\mu+\nu+1/2}\,K_\mu(ax)$ $\mathrm{Re}\,a>0, \quad \mathrm{Re}\,\nu>-3/2$ $\mathrm{Re}(\mu+\nu)>-3/2$	$2^{\mu+\nu+1}\pi^{-1/2}\,a^{-\mu-2\nu-3}$ $\times\, y^{\nu+3/2}\,\Gamma(\mu+\nu+3/2)$ $\times\,{}_2F_1\left(1,\,\mu+\nu+\frac{3}{2};\,\frac{3}{2};\,-\frac{y^2}{a^2}\right)$
(14)	$x^{\mu-\nu+1/2}\,K_\mu(ax)$ $\mathrm{Re}\,a>0, \quad \mathrm{Re}\,\mu>-3/2$	$\dfrac{2^{\mu-\nu}\,y^{\nu+3/2}\,\Gamma(\mu+3/2)}{a^{\mu+3}\,\Gamma(\nu+3/2)}$ $\times\,{}_2F_1\left(1,\,\mu+\frac{3}{2};\,\nu+\frac{3}{2};\,-\frac{y^2}{a^2}\right)$
(15)	$x^{\sigma-1/2}\,K_\mu(ax) \qquad \mathrm{Re}\,a>0$ $\mathrm{Re}(\sigma+\nu)>\|\mathrm{Re}\,\mu\|-2$	$2^\sigma\,\pi^{-1/2}\,a^{-\nu-\sigma-2}\,y^{\nu+3/2}$ $\times\dfrac{\Gamma(1+\frac{\nu+\sigma+\mu}{2})\,\Gamma(1+\frac{\nu+\sigma-\mu}{2})}{\Gamma(\nu+3/2)}$ $\times\,{}_3F_2\left(1,\,1+\frac{\nu+\sigma+\mu}{2},\,1+\frac{\nu+\sigma-\mu}{2};\right.$ $\left.\frac{3}{2},\,\nu+\frac{3}{2};\,-\frac{y^2}{a^2}\right)$
(16)	$x^{\sigma-1/2}\,e^{-ax}\,K_\nu(ax)$	see Mohan, Brij, 1942: *Bull. Calcutta Math. Soc.* 34, 55-59.
(17)	$x^{\nu+1/2}\,[K_\nu(ax)]^2$ $\mathrm{Re}\,a>0, \quad \mathrm{Re}\,\nu>-3/4$	$\pi^{1/2}\,2^{-\nu-3}\,a^{-2\nu-3}\,y^{\nu+3/2}$ $\times\,\Gamma(2\nu+3/2)\,[\Gamma(\nu+2)]^{-1}$ $\times\,{}_2F_1\left(1,\,2\nu+\frac{3}{2};\,\nu+2;\,-\frac{y^2}{4a^2}\right)$

Higher transcendental functions (cont'd)

	$f(x)$	$\int_0^\infty f(x)\, \mathbf{H}_\nu(xy)(xy)^{1/2}\,dx \qquad y>0$
(18)	$x^{1/2}[K_\mu(ax)]^2 \qquad \nu=0$ $\mathrm{Re}\,a>0, \quad -3/2<\mathrm{Re}\,\mu<3/2$	$-\pi\, 2^{-\mu-1} a^{-2\mu} y^{-1/2} z^{-1} \sec(\mu\pi)$ $\times[(z+y)^{2\mu}+(z-y)^{2\mu}]$ $z=(y^2+4a^2)^{1/2}$
(19)	$x^{-\nu-1/2} K_0(ax) K_1(ax)$ $\mathrm{Re}\,a>0$	$\dfrac{\pi^{3/2} y^{\nu+3/2}}{2^{\nu+3} a^2 \Gamma(\nu+3/2)}$ $\times {}_2F_1\left(\dfrac{1}{2},\dfrac{1}{2};\nu+\dfrac{3}{2};-\dfrac{y^2}{4a^2}\right)$
(20)	$x^{-\nu-1/2} K_\nu(ax) K_{\nu+1}(ax)$ $\mathrm{Re}\,a>0, \quad \mathrm{Re}\,\nu<1/2$	$\pi^{1/2} 2^{-\nu-2} a^{-2} y^{\nu+3/2} \Gamma(1/2-\nu)$ $\times {}_2F_1\left(\dfrac{1}{2},\dfrac{1}{2}-\nu;\dfrac{3}{2};-\dfrac{y^2}{4a^2}\right)$
(21)	$x^{\sigma-5/2} K_\lambda(ax) K_\mu(ax)$ $\mathrm{Re}\,a>0$ $\mathrm{Re}(\sigma+\nu)>\lvert\mathrm{Re}\,\lambda\rvert+\lvert\mathrm{Re}\,\mu\rvert$	$\dfrac{2^{\sigma-3} y^{\nu+3/2}\Gamma\left(\frac{\sigma+\nu+\lambda+\mu}{2}\right)}{\pi^{1/2} a^{\sigma+\nu}\Gamma(\nu+3/2)\Gamma(\sigma+\nu)}$ $\times\Gamma\left(\dfrac{\sigma+\nu+\lambda-\mu}{2}\right)\Gamma\left(\dfrac{\sigma+\nu-\lambda+\mu}{2}\right)$ $\times\Gamma\left(\dfrac{\sigma+\nu-\lambda-\mu}{2}\right)$ $\times {}_5F_4\left(1,\dfrac{\sigma+\nu+\lambda+\mu}{2},\dfrac{\sigma+\nu+\lambda-\mu}{2},\right.$ $\dfrac{\sigma+\nu-\lambda+\mu}{2},\dfrac{\sigma+\nu-\lambda-\mu}{2};$ $\left.\dfrac{3}{2},\nu+\dfrac{3}{2},\dfrac{\sigma+\nu}{2},\dfrac{\sigma+\nu+1}{2};-\dfrac{y^2}{4a^2}\right)$

Higher transcendental functions (cont'd)

	$f(x)$	$\int_0^\infty f(x)\,\mathbf{H}_\nu(xy)(xy)^{1/2}\,dx \qquad y>0$
(22)	$x^{\nu+3/2}K_\nu(ax)K_{\nu+1}(ax)$ $\operatorname{Re} a>0, \quad \operatorname{Re}\nu>-5/4$	$\pi^{1/2}2^{-\nu-3}a^{-2\nu-4}y^{\nu+3/2}$ $\times\Gamma(2\nu+5/2)\,[\Gamma(\nu+2)]^{-1}$ $\times\,{}_2F_1\left(1,\,2\nu+\frac{5}{2};\,\nu+2;\,-\frac{y^2}{4a^2}\right)$
(23)	$x^{\sigma-5/2}\exp(-\tfrac{1}{2}a^2x^2)$ $\times K_\mu(\tfrac{1}{2}a^2x^2)$ $\lvert\arg a\rvert<\tfrac{1}{4}\pi$ $\operatorname{Re}(\sigma+\nu)>2\lvert\operatorname{Re}\mu\rvert$	$\pi^{1/2}2^{-\nu-2}a^{-\nu-\sigma}y^{\nu+3/2}$ $\times\dfrac{\Gamma(\frac{\nu+\sigma}{2}+\mu)\,\Gamma(\frac{\nu+\sigma}{2}-\mu)}{\Gamma(\frac{3}{2})\,\Gamma(\nu+\frac{3}{2})\,\Gamma(\frac{\nu+\sigma}{2})}$ $\times\,{}_3F_3\left(1,\frac{\nu+\sigma}{2}+\mu,\frac{\nu+\sigma}{2}-\mu;\frac{3}{2},\right.$ $\left.\nu+\frac{3}{2},\frac{\nu+\sigma}{2};-\frac{y^2}{4a^2}\right)$
(24)	$x^{1/2}\exp\left(\frac{a^2x^2}{8}\right)K_{\frac{1}{2}\nu}\left(\frac{a^2x^2}{8}\right)$ $\lvert\arg a\rvert<\tfrac{3}{4}\pi$ $-3/2<\operatorname{Re}\nu<0$	$2\pi^{-1/2}a^{-\frac{1}{2}\nu-1}y^{\frac{1}{2}\nu-\frac{1}{2}}$ $\times\cos(\tfrac{1}{2}\nu\pi)\,\Gamma(-\tfrac{1}{2}\nu)$ $\times\exp\left(\frac{y^2}{2a^2}\right)W_{k,m}\left(\frac{y^2}{a^2}\right)$ $k=\tfrac{1}{4}\nu, \quad m=\tfrac{1}{2}+\tfrac{1}{4}\nu$

Higher transcendental functions (cont'd)

	$f(x)$	$\int_0^\infty f(x)\,\mathbf{H}_\nu(xy)(xy)^{1/2}\,dx \qquad y>0$
(25)	$x^\sigma \exp(ax^2) K_\mu(ax^2)$ $\lvert\arg a\rvert < 3\pi/2, \quad \mathrm{Re}\,\sigma<1$ $\lvert\mathrm{Re}\,\mu\rvert - 5/2 < \mathrm{Re}(\sigma+\nu) < 1/2$	$\pi^{-1/2}\, 2^{-1-\frac{1}{2}\sigma}\, a^{-\frac{1}{2}-\frac{1}{2}\sigma} \cos(\mu\pi)$ $\times G^{23}_{34}\left(\frac{y^2}{8a}\middle\vert \begin{matrix} l,\ \frac{1-\sigma}{2}+\mu,\ \frac{1-\sigma}{2}-\mu \\ l,\ -\frac{\sigma}{2},\ h,\ k\end{matrix}\right)$ $h=\frac{1}{4}+\frac{\nu}{2}, \quad k=\frac{1}{4}-\frac{\nu}{2}, \quad l=\frac{3}{4}+\frac{\nu}{2}$
(26)	$K_{2\nu-1}(2ax^{1/2})$ $\mathrm{Re}\,a>0, \quad \mathrm{Re}\,\nu>-1$	$2^{\nu+1}\pi^{-1} a\, y^{-3/2}\,\Gamma(\nu+1)$ $\times S_{-\nu-2,\,\nu-1}(a^2/y)$
(27)	$x^{-1/2} K_{2\nu}(2ax^{1/2})$ $\mathrm{Re}\,a>0, \quad \mathrm{Re}\,\nu>-1$	$2^\nu \pi^{-1} y^{-1/2}\,\Gamma(\nu+1)\, S_{-\nu-1,\nu}(a^2/y)$
(28)	$x^{1/2} K_{2\nu}(2ax^{1/2})$ $\mathrm{Re}\,a>0, \quad \mathrm{Re}\,\nu>-2$	$2^\nu\pi^{-1} a^2 y^{-5/2}\,\Gamma(\nu+2)$ $\times S_{-\nu-3,\nu}(a^2/y)$
(29)	$x^\sigma K_\mu(2ax^{1/2}) \qquad \mathrm{Re}\,a>0$ $2\,\mathrm{Re}(\sigma+\nu) > \lvert\mathrm{Re}\,\mu\rvert - 5$	$2^{2\sigma-\frac{1}{2}}\, a^{-2\sigma-2}\,\pi^{-1}$ $\times G^{15}_{53}\left(\frac{4y^2}{a^4}\middle\vert \begin{matrix} l,\ \beta_1,\ \dots,\ \beta_4 \\ l,\ h,\ k\end{matrix}\right)$ $h=\frac{1}{4}+\frac{\nu}{2}, \quad k=\frac{1}{4}-\frac{\nu}{2}, \quad l=\frac{3}{4}+\frac{\nu}{2}$ $2\beta_1 = 1-\sigma+\frac{1}{2}\mu, \quad 2\beta_2 = 1-\sigma-\frac{1}{2}\mu$ $2\beta_3 = -\sigma+\frac{1}{2}\mu, \quad 2\beta_4 = -\sigma-\frac{1}{2}\mu$

Higher transcendental functions (cont'd)

	$f(x)$	$\int_0^\infty f(x)\,\mathbf{H}_\nu(xy)(xy)^{1/2}\,dx \qquad y>0$
(30)	$x^{-1/2}[2\pi^{-1}K_{2\nu}(2ax^{1/2}) + Y_{2\nu}(2ax^{1/2})]$ $a>0, \quad -\tfrac{1}{2}<\operatorname{Re}\nu<\tfrac{1}{2}$	$y^{-1/2}J_\nu(a^2/y)$
(31)	$x^{1/2-\nu}[J_{2\nu}(ax^{1/2}) - J_{-2\nu}(ax^{1/2})]K_{2\nu}(ax^{1/2})$ $\lvert\arg a\rvert<\tfrac{1}{4}\pi$ $-3/2<\operatorname{Re}\nu<3/2$	$2^\nu\pi^{-1/2}a^{1-2\nu}y^{2\nu-2} \times \sin(\nu\pi)K_{\nu+1/2}\left(\dfrac{a^2}{2y}\right)$
(32)	$x^{1/2}Y_\nu(ax^{1/2})K_\nu(ax^{1/2})$ $\lvert\arg a\rvert<\tfrac{1}{4}\pi, \quad \operatorname{Re}\nu>-3/2$	$\dfrac{1}{2y^{3/2}}\exp\left(-\dfrac{a^2}{2y}\right)$
(33)	$x^{\nu-1/2}Y_{2\nu-1}(ax^{1/2})K_{2\nu-1}(ax^{1/2})$ $\lvert\arg a\rvert<\tfrac{1}{4}\pi, \quad \operatorname{Re}\nu>-\tfrac{1}{4}$	$\dfrac{a^{2\nu-1}}{\pi^{1/2}2^\nu y^{2\nu}}K_{\nu-1/2}\left(\dfrac{a^2}{2y}\right)$
(34)	$x^{\nu+1/2}Y_{2\nu}(ax^{1/2})K_{2\nu}(ax^{1/2})$ $\lvert\arg a\rvert<\tfrac{1}{4}\pi, \quad \operatorname{Re}\nu>-\tfrac{3}{4}$	$\dfrac{a^{2\nu+1}}{\pi^{1/2}2^{\nu+1}y^{2\nu+2}}K_{\nu-1/2}\left(\dfrac{a^2}{2y}\right)$
(35)	$x^{-1/2}\{\cos[\tfrac{1}{2}(\mu-\nu)\pi]J_\mu(ax^{1/2}) - \sin[\tfrac{1}{2}(\mu-\nu)\pi]Y_\mu(ax^{1/2})\} \times K_\mu(ax^{1/2})$ $\lvert\arg a\rvert<\tfrac{1}{4}\pi$ $\operatorname{Re}\nu>\lvert\operatorname{Re}\mu\rvert-2$	$a^{-2}y^{1/2}W_{1/2\nu,1/2\mu}\left(\dfrac{a^2}{2y}\right) \times W_{-1/2\nu,1/2\mu}\left(\dfrac{a^2}{2y}\right)$

Higher transcendental functions (cont'd)

	$f(x)$	$\int_0^\infty f(x)\,\mathbf{H}_\nu(xy)(xy)^{1/2}\,dx \qquad y>0$
(36)	$x^{\nu-1/2} K_{2\nu-1}(ax^{1/2} e^{1/4 i\pi}) \times K_{2\nu-1}(ax^{1/2} e^{-1/4 i\pi})$ $\operatorname{Re} a>0, \quad \operatorname{Re}\nu>-1/4$	$\pi^{-1/2} 2^{3\nu-1} y^{-\nu-1/2} \times \Gamma(\nu+1)\,\Gamma(2\nu+1/2) \times S_{-3\nu-1/2,\nu-1/2}\left(\frac{a}{2^{1/2} y^{1/2}}\right)$
(37)	$x^{-1/2}\,\mathbf{H}_\nu(a^2/x)$ $a>0, \quad \operatorname{Re}\nu>-3/2$	$-y^{-1/2} J_{2\nu}(2ay^{1/2})$
(38)	$x^{-3/2}\,\mathbf{H}_{\nu-1}(a^2/x)$ $a>0, \quad \operatorname{Re}\nu>-1/2$	$-a^{-1} J_{2\nu-1}(2ay^{1/2})$
(39)	$x^{-1/2}[J_{-\nu}(a^2/x) + \sin(\nu\pi)\,\mathbf{H}_\nu(a^2/x)]$ $a>0, \quad -3/2<\operatorname{Re}\nu<0$	$y^{-1/2}[2\pi^{-1} K_{2\nu}(2ay^{1/2}) - Y_{2\nu}(2ay^{1/2})]$
(40)	$x^\sigma S_{\lambda,\mu}(ax)$ $\lvert\arg a\rvert<\pi, \quad \operatorname{Re}(\lambda+\sigma)<1$ $\operatorname{Re}(\sigma+\nu)>\lvert\operatorname{Re}\mu\rvert-5/2$ $-3/2<\operatorname{Re}(\lambda+\nu+\sigma)<1/2$	$\frac{2^{\lambda+\sigma-1/2} a^{-\sigma-1}}{\Gamma(1/2-1/2\lambda-1/2\mu)\,\Gamma(1/2-1/2\lambda+1/2\mu)} \times G^{24}_{44}\left(\frac{y^2}{a^2}\middle\vert \begin{matrix} l, -\frac{\lambda+\sigma}{2}, \frac{1-\sigma+\mu}{2}, \frac{1-\sigma-\mu}{2} \\ l, -\frac{\lambda+\sigma}{2}, h, k \end{matrix}\right)$ $h=\frac{1}{4}+\frac{\nu}{2}, \quad k=\frac{1}{4}-\frac{\nu}{2}, \quad l=\frac{3}{4}+\frac{\nu}{2}$

Higher transcendental functions (cont'd)

	$f(x)$	$\int_0^\infty f(x)\,\mathbf{H}_\nu(xy)(xy)^{1/2}\,dx \qquad y>0$
(41)	$x^{-\nu-1/2}\exp(-\frac{1}{4}x^2)$ $\times[D_\mu(x)-D_\mu(-x)]$ $\mathrm{Re}(\mu+\nu)>-3/2$ $\mathrm{Re}\,\mu>-1$	$\dfrac{2^{3/2}\,\Gamma(\frac{1}{2}\mu+\frac{1}{2})}{\Gamma(\frac{1}{2}\mu+\nu+1)}\,y^{\mu+\nu+1/2}\sin(\frac{1}{2}\mu\pi)$ $\times\,{}_1F_1(\frac{1}{2}\mu+\frac{1}{2};\,\frac{1}{2}\mu+\nu+1;\,-\frac{1}{2}y^2)$
(42)	$x^{2\lambda}\exp(-\frac{1}{4}x^2)\,M_{\kappa,\mu}(\frac{1}{2}x^2)$ $\mathrm{Re}(2\lambda+2\mu+\nu)>-7/2$ $\mathrm{Re}(\kappa-\lambda)>0$ $\mathrm{Re}(2\lambda-2\kappa+\nu)<-\frac{1}{2}$	$\dfrac{2^{-\lambda}\,\Gamma(2\mu+1)}{\Gamma(\frac{1}{2}+\kappa+\mu)}$ $\times\,G^{22}_{34}\left(\dfrac{y^2}{2}\middle\vert\begin{matrix}l,\,-\mu-\lambda,\,\mu-\lambda\\ l,\,\kappa-\lambda-\frac{1}{2},\,h,\,k\end{matrix}\right)$ $h=\dfrac{1}{4}+\dfrac{\nu}{2},\quad k=\dfrac{1}{4}-\dfrac{\nu}{2},\quad l=\dfrac{3}{4}+\dfrac{\nu}{2}$
(43)	$x^{2\lambda}\exp(-\frac{1}{4}x^2)\,W_{\kappa,\mu}(\frac{1}{2}x^2)$ $\mathrm{Re}(2\lambda+\nu)>2\lvert\mathrm{Re}\,\mu\rvert-7/2$	$2^{1/4-\lambda-\nu/2}\,\pi^{-1/2}\,y^{\nu+3/2}$ $\times\dfrac{\Gamma(\frac{7}{4}+\frac{\nu}{2}+\lambda+\mu)\,\Gamma(\frac{7}{4}+\frac{\nu}{2}+\lambda-\mu)}{\Gamma(\nu+\frac{3}{2})\,\Gamma(\frac{9}{4}+\lambda-\kappa-\frac{\nu}{2})}$ $\times\,{}_3F_3\left(1,\dfrac{7}{4}+\dfrac{\nu}{2}+\lambda+\mu,\dfrac{7}{4}+\dfrac{\nu}{2}+\lambda-\mu;\right.$ $\left.\dfrac{3}{2},\,\nu+\dfrac{3}{2},\,\dfrac{9}{4}+\lambda-\kappa+\dfrac{\nu}{2};\,-\dfrac{y^2}{2}\right)$
(44)	$x^{-1/2}\exp(\frac{1}{2}x^2)$ $\times\,W_{-\frac{1}{2}\nu-\frac{1}{2},\frac{1}{2}\nu}(x^2)$ $\mathrm{Re}\,\nu>-1$	$2^{-\nu-1}\,y^{\nu+1/2}\,\pi\exp(\frac{1}{4}y^2)\,\mathrm{Erfc}(\frac{1}{2}y)$

Higher transcendental functions (cont'd)

	$f(x)$	$\int_0^\infty f(x)\, \mathbf{H}_\nu(xy)(xy)^{1/2}\,dx \qquad y>0$
(45)	$x^{-1/2}\exp(\frac{1}{4}x^2)\, W_{\kappa,\frac{1}{2}\nu}(\frac{1}{2}x^2)$ $-3/2<\operatorname{Re}\nu<-2\operatorname{Re}\kappa$ $\operatorname{Re}\kappa<\frac{1}{4}$	$\dfrac{2^{\frac{1}{2}\kappa-\frac{1}{4}\nu}\, y^{\frac{1}{2}\nu-\kappa-\frac{1}{2}}\,\Gamma(-\kappa-\frac{1}{2}\nu)}{\Gamma(\frac{1}{2}-\kappa+\frac{1}{2}\nu)\,\Gamma(\frac{1}{2}-\kappa-\frac{1}{2}\nu)}$ $\times\exp(\frac{1}{4}y^2)\, W_{k,m}(\frac{1}{2}y^2)$ $2k=\kappa+\frac{1}{2}\nu,\quad 2m=\kappa+\frac{1}{2}\nu+1$
(46)	$x^{2\lambda}\exp(\frac{1}{4}x^2)\, W_{\kappa,\lambda}(\frac{1}{2}x^2)$ $\operatorname{Re}(2\lambda+\nu)>2\lvert\operatorname{Re}\mu\rvert-7/2$ $\operatorname{Re}(2\kappa+2\lambda+\nu)<-1/2$ $\operatorname{Re}(\kappa+\lambda)<0$	$[2^\lambda\,\Gamma(\frac{1}{2}-\kappa+\mu)\,\Gamma(\frac{1}{2}-\kappa-\mu)]^{-1}$ $\times G_{34}^{23}\left(\frac{y^2}{2}\,\middle\vert\,\begin{matrix} l,\ -\mu-\lambda,\ \mu-\lambda \\ l,\ -\kappa-\lambda-\frac{1}{2},\ h,\ k\end{matrix}\right)$ $h=\dfrac{1}{4}+\dfrac{\nu}{2},\quad k=\dfrac{1}{4}-\dfrac{\nu}{2},\quad l=\dfrac{3}{4}+\dfrac{\nu}{2}$
(47)	$G_{pq}^{mn}\left(\lambda x^2\,\middle\vert\,\begin{matrix} a_1,\ldots,a_p \\ \beta_1,\ldots,\beta_q\end{matrix}\right)$ $p+q<2(m+n)$ $\lvert\arg\lambda\rvert<(m+n-\frac{1}{2}p-\frac{1}{2}q)\pi$ $\operatorname{Re}a_j<\min(1,\frac{3}{4}-\frac{1}{2}\nu)$ $j=1,\ldots,n$ $\operatorname{Re}(2\beta_j+\nu)>-5/2$ $j=1,\ldots,m$	$(2\lambda)^{-1/2}$ $\times G_{q+1,\,p+3}^{n+1,\,m+1}\left(\frac{y^2}{4\lambda}\,\middle\vert\,\begin{matrix} l,\frac{1}{2}-\beta_1,\ldots,\frac{1}{2}-\beta_q \\ l,\frac{1}{2}-a_1,\ldots,\frac{1}{2}-a_p,h,k\end{matrix}\right)$ $h=\dfrac{1}{4}+\dfrac{\nu}{2},\quad k=\dfrac{1}{4}-\dfrac{\nu}{2},\quad l=\dfrac{3}{4}+\dfrac{\nu}{2}$

CHAPTER XII

KONTOROVICH-LEBEDEV TRANSFORMS

The pair of reciprocal formulas

$$g(y) = \int_0^\infty f(x)\, K_{ix}(y)\, dx$$

$$f(x) = 2\pi^{-2} x \sinh(\pi x) \int_0^\infty g(y)\, K_{ix}(y)\, y^{-1}\, dy$$

was given by Kontorovich and Lebedev (1938, 1939) who used these formulas in the solution of certain boundary value problems. Further applications to boundary value problems were given by Lebedev and Kontorovich, and the mathematical theory was developed by Lebedev (1946, 1949). It should be noted that $K_{ix}(y)$ is real when x is real and y is positive. Alternative forms of this inversion were stated in the papers referred to above. See also Erdélyi, et al. (1953, p. 75).

In this chapter we give a short list of integrals corresponding to the first of the above formulas; integrals corresponding to the second formula may be evaluated by means of the tables given in chapter X. We take y to be a positive real variable, although some of the integrals given below are valid for complex y.

REFERENCES

Erdélyi, Arthur, Wilhelm Magnus, Fritz Oberhettinger, and F.G. Tricomi, 1953: *Higher transcendental functions*, vol. II, McGraw-Hill.

Kontorovich, M.J. and N.N. Lebedev, 1938: *J. Exper. Theor. Phys. USSR*, 8, 1192-1206.

Kontorovich, M.J. and N.N. Lebedev, 1939: *Acad. Sci. USSR, J. Phys.* 1, 229-241.

Lebedev, N.N. and M.J. Kontorovich, 1939: *J. Exper. Theor. Phys. USSR*, 9, 729-741.

Lebedev, N.N., 1946: *Acad. Sci. USSR, Doklady* 52, 655-658.

Lebedev, N.N., 1949: *Acad. Sci. USSR, Doklady* 65, 621-624.

KONTOROVICH-LEBEDEV TRANSFORMS

12.1. Formulas

	$f(x)$	$\int_0^\infty f(x)\, K_{ix}(y)\, dx \qquad y > 0$
(1)	$x \sin(ax) \qquad \lvert \operatorname{Im} a \rvert < \frac{1}{2}\pi$	$\frac{1}{2}\pi y \sinh a \exp(-y \cosh a)$
(2)	$\cos ax \qquad \lvert \operatorname{Im} a \rvert < \frac{1}{2}\pi$	$\frac{1}{2}\pi \exp(-y \cosh a)$
(3)	$x \tanh(\pi x)\, P_{-\frac{1}{2}+ix}(z)$	$(\frac{1}{2}\pi y)^{\frac{1}{2}}\, e^{-zy}$
(4)	$x \tanh(\pi x)\, K_{ix}(\beta)$ $\lvert \arg \beta \rvert < \pi$	$\frac{1}{2}\pi (\beta y)^{\frac{1}{2}} (\beta + y)^{-1} \exp(-\beta - y)$
(5)	$x \sinh(\pi x)\, K_{2ix}(a)$ $\lvert \arg a \rvert < \frac{1}{4}\pi$	$\dfrac{\pi^{3/2} a}{2^{5/2} y^{1/2}} \exp\left(-y - \dfrac{a^2}{8y}\right)$
(6)	$x \sin(\frac{1}{2}\pi x)\, K_{\frac{1}{2}ix}(a)$ $\lvert \arg a \rvert < \frac{1}{2}\pi$	$\dfrac{\pi^{3/2} y}{2^{1/2} a^{1/2}} \exp\left(-a - \dfrac{y^2}{8a}\right)$
(7)	$\cosh(ax)\, K_{ix}(\beta)$ $\lvert \operatorname{Re} a \rvert + \lvert \arg \beta \rvert < \pi$	$\frac{1}{2}\pi K_0[(y^2 + \beta^2 + 2\beta y \cos a)^{\frac{1}{2}}]$

Formulas (cont'd)

	$f(x)$	$\int_0^\infty f(x)\,K_{ix}(y)\,dx \qquad y>0$
(8)	$x(x^2+n^2)^{-1}\sinh(\pi x)$ $\times K_{ix}(a)$ $a>0, \quad n=0,1,2,3,\ldots$	$\tfrac{1}{2}\pi^2 I_n(y)\,K_n(a) \qquad 0<y<a$ $\tfrac{1}{2}\pi^2 I_n(a)\,K_n(y) \qquad a<y<\infty$
(9)	$x\sinh(\pi x)\,K_{ix}(\alpha)\,K_{ix}(\beta)$ $\lvert\arg\alpha\rvert+\lvert\arg\beta\rvert<\tfrac{1}{2}\pi$	$\dfrac{\pi^2}{4}\exp\left[-\dfrac{y}{2}\left(\dfrac{\alpha}{\beta}+\dfrac{\beta}{\alpha}+\dfrac{\alpha\beta}{y^2}\right)\right]$
(10)	$x\sinh(\tfrac{1}{2}\pi x)\,K_{\frac{1}{2}ix}(\alpha)\,K_{\frac{1}{2}ix}(\beta)$ $\lvert\arg\alpha\rvert+\lvert\arg\beta\rvert<\pi$	$\dfrac{\pi^2 y}{2z}\exp\left[-\dfrac{(\alpha+\beta)z}{2(\alpha\beta)^{\frac{1}{2}}}\right]$ $z=(y^2+4\alpha\beta)^{\frac{1}{2}}$
(11)	$x\sinh(\pi x)\,K_{\frac{1}{2}ix+\lambda}(a)$ $\times K_{\frac{1}{2}ix-\lambda}(a) \qquad a>0$	$0 \qquad 0<y<2a$ $\dfrac{\pi^2 y}{2^{2\lambda+1}a^{2\lambda}z}\left[(y+z)^{2\lambda}+(y-z)^{2\lambda}\right]$ $2a<y<\infty$ $z=(y^2-4a^2)^{\frac{1}{2}}$
(12)	$x\sinh(\pi x)\,\Gamma(\lambda+ix)$ $\times\Gamma(\lambda-ix)\,K_{ix}(a)$ $\lvert\arg a\rvert<\pi, \quad \mathrm{Re}\,\lambda>0$	$2^{\nu-1}\pi^{3/2}(ay)^{\lambda}(y+a)^{-\lambda}$ $\times\Gamma(\lambda+\tfrac{1}{2})\,K_{\lambda}(y+a)$
(13)	$x\sinh(2\pi x)\,\Gamma(\lambda+ix)$ $\times\Gamma(\lambda-ix)\,K_{ix}(a)$ $a>0, \quad 0<\mathrm{Re}\,\lambda<\tfrac{1}{2}$	$\dfrac{2^{\lambda}\pi^{5/2}}{\Gamma(\tfrac{1}{2}-\lambda)}\left(\dfrac{ay}{\lvert y-a\rvert}\right)^{\lambda}K_{\lambda}(\lvert y-a\rvert)$

Formulas (cont'd)

	$f(x)$	$\int_0^\infty f(x)\,K_{ix}(y)\,dx \qquad y>0$
(14)	$x\sinh(\pi x)\,\Gamma(\lambda+\tfrac{1}{2}ix)$ $\times\,\Gamma(\lambda-\tfrac{1}{2}ix)\,K_{ix}(a)$ $\lvert\arg a\rvert<\tfrac{1}{2}\pi, \quad \operatorname{Re}\lambda>0$	$2\pi^2\left(\dfrac{ay}{2z}\right)^{2\lambda} K_{2\lambda}(z)$ $z=(y^2+a^2)^{1/2}$
(15)	$\dfrac{x\tanh(\pi x)\,K_{ix}(a)}{\Gamma(\tfrac{3}{4}+\tfrac{1}{2}ix)\;\Gamma(\tfrac{3}{4}-\tfrac{1}{2}ix)}$ $\lvert\arg a\rvert<\tfrac{1}{2}\pi$	$\dfrac{1}{2}\left(\dfrac{\pi a y}{y^2+a^2}\right)^{1/2}\exp[-(y^2+a^2)^{1/2}]$
(16)	$x\sinh(\pi x)\,\Gamma(\lambda+ix)\,\Gamma(\lambda-ix)$ $\times\,P^{1/2-\lambda}_{-1/2+ix}(\beta)\,K_{ix}(a)$ $\lvert\arg a\rvert<\tfrac{1}{2}\pi$ $\lvert\arg(\beta-1)\rvert<\pi$ $\operatorname{Re}\lambda>0$	$2^{-1/2}\,\pi^{3/2}(ay/z)^{\lambda}\,(\beta^2-1)^{\frac{1}{2}\lambda-\frac{1}{4}}$ $\times\,K_{\lambda}(z)$ $z=(y^2+a^2+2a\beta y)^{1/2}$

MISCELLANEOUS TRANSFORMS

CHAPTER XIII

FRACTIONAL INTEGRALS

We call

$$g(y;\mu) = \mathfrak{R}_\mu\{f(x); y\} = \frac{1}{\Gamma(\mu)} \int_0^y f(x)\,(y-x)^{\mu-1}\,dx$$

the *Riemann-Liouville* (fractional) *integral of order* μ, and

$$h(y;\mu) = \mathfrak{W}_\mu\{f(x); y\} = \frac{1}{\Gamma(\mu)} \int_y^\infty f(x)\,(x-y)^{\mu-1}\,dx$$

the *Weyl* (fractional) *integral of order* μ, of $f(x)$. In general, μ and y are envisaged as complex numbers, the path of integration being the segment $x = yt$, $0 < t < 1$ in g, and one of the rays $x = yt$, $t > 1$ or $x = y + t$, $t > 0$ in h.

Many authors denote $g(y;\mu)$ by $I^\mu f$ or $I^\mu_+ f$, and $h(y;\mu)$ by $K^\mu f$ or $K^\mu_- f$. The integral

$$\frac{1}{\Gamma(\mu)} \int_y^a f(x)\,(x-y)^{\mu-1}\,dx,$$

sometimes denoted by $I^\mu_- f$, may be expressed as $\mathfrak{R}_\mu\{f(a-x); a-y\}$ by a change of variables; alternatively it may be written as $\mathfrak{W}_\mu\{f(x); y\}$ by adopting the convention that $f(x) = 0$ when $x > a$.

Fractional derivatives of order a may be defined by the formulas

$$D_0^\alpha f(x) = \frac{d^n}{dx^n}\, \mathfrak{R}_{n-\alpha}\{f(t); x\} \qquad n-1 < \operatorname{Re}\alpha < n$$

$$D_\infty^\alpha f(x) = \frac{d^n}{dx^n}\, \mathfrak{W}_{n-\alpha}\{f(t); x\} \qquad n-1 < \operatorname{Re}\alpha < n$$

so that tables of fractional integrals may be used to evaluate fractional derivatives.

On p. 184 we give a brief selection of books and papers which contain information on the theory of fractional integrals and derivatives. Hardy and Littlewood (1928) give further references. As far as we know, there is no extensive table of fractional integrals although numerous integrals of this kind occur in almost any table of integrals.

An extension of the operators $\mathfrak{R}_\mu$ and $\mathfrak{W}_\mu$ was introduced by Kober (1940) and Erdélyi (1940). Kober (1941b) also discussed fractional integrals of imaginary order. Fractional integration by parts over a finite interval is expressed by the formula

$$\int_0^a g_1(x;\mu)\, f_2(a-x)\, dx = \int_0^a f_1(a-x)\, g_2(x;\mu)\, dx,$$

and was discussed by Young and Love (1938). For the infinite interval the formula is

$$\int_0^\infty f_1(x)\, g_2(x;\mu)\, dx = \int_0^\infty h_1(x;\mu)\, f_2(x)\, dx,$$

and was discussed by Kober (1940). In these formulas $g_{1,2} = \mathfrak{R} f_{1,2}$, and $h_1 = \mathfrak{W} f_1$.

The operators $\mathfrak{R}_\mu$, $\mathfrak{W}_\mu$ are connected with differentiation and integration, and with each other, by a number of relations. We list a few here, others being given in the list of general formulas in sections 13.1 and 13.2.

$$g(x;1) = \int_0^x f(t)\, dt, \qquad h(x;1) = \int_x^\infty f(t)\, dt$$

$$\frac{d}{dx} g(x;\mu) = g(x;\mu-1), \qquad -\frac{d}{dx} h(x;\mu) = h(x;\mu-1)$$

$$\mathfrak{R}_\mu \mathfrak{R}_\nu = \mathfrak{R}_{\mu+\nu}, \qquad \mathfrak{W}_\mu \mathfrak{W}_\nu = \mathfrak{W}_{\mu+\nu}.$$

The functions $g(x;\mu)$ and $h(x;\mu)$ may be regarded as μ times repeated indefinite integrals of $f(x)$, the fixed limit being 0 in the case of g, and ∞ in the case of h.

The connection of fractional integrals with other integral transforms may be seen from the following formulas.

$$\mathfrak{L}\{g(t;\mu);p\} = p^{-\mu}\,\mathfrak{L}\{f(t);p\}$$

$$\mathfrak{F}_e\{h(x;\mu);y\} = e^{\frac{1}{2}\mu\pi i}\, y^{-\mu}\,\mathfrak{F}_e\{f(x);y\}$$

$$\mathfrak{M}\{g(x;\mu); s\} = \frac{\Gamma(1-s-\mu)}{\Gamma(1-s)} \ \mathfrak{M}\{f(x); s+\mu\}$$

$$\mathfrak{M}\{h(x;\mu); s\} = \frac{\Gamma(s)}{\Gamma(s+\mu)} \ \mathfrak{M}\{f(x); s+\mu\}$$

which may be used in conjunction with the tables of vol. I to evaluate fractional integrals by means of tables of Fourier, Laplace, Mellin transforms and their inversions. These formulas may also be used to derive from a known pair of, say, Fourier transforms a new pair by means of integration of fractional order.

The connection of fractional integrals and Laplace transforms is discussed in Doetsch (1937, p. 293-305) and Widder (1941, p. 70-75). Doetsch also discusses Abel's integral equation, $g = \mathfrak{R}_{\mu} f$. For the connection of fractional integrals and Fourier transforms see Kober (1941a, Lemma 3). For the connection of fractional integrals and Mellin transforms see Kober (1940). For the connection of fractional integrals and Hankel transforms see Erdélyi and Kober (1940) and Erdélyi (1940); see also 8.1 (13) to 8.1 (16). For fractional integrals in the theory of Fourier series see Zygmund (1935, p. 222 ff.).

Fractional integrals occur in the solution by definite integrals of linear differential equations. In this context fractional integrals are often called *Euler transforms* (see, for instance, Ince, 1927, p. 191 ff.), $\mathfrak{R}_{\mu} f$ being the Euler transform of the first kind, and $\mathfrak{W}_{\mu} f$ the Euler transform of the second kind, of f.

M. Riesz (1949) has developed a theory of fractional integrals of functions of several variables: this theory has been applied by Riesz and others to the solution of partial differential equations. (See, for instance, Baker and Copson, 1950, Chap. I, § 7.)

From the fractional integrals given in the tables, further fractional integrals may be derived by the general methods enumerated in the introduction to vol. I, by means of the general formulas stated above and in sections 13.1 and 13.2, and by means of the connection, also stated above, between fractional integrals and other integral transforms.

REFERENCES

Baker, B.B. and E.T. Copson, 1950: *The mathematical theory of Huygens' principle*. Oxford, Clarendon Press.

Doetsch, Gustav, 1937: *Theorie und Anwendung der Laplace-Transformation*. Berlin, Springer.

Erdélyi, Arthur, 1940: *Quart. J. Math., Oxford Ser.* 11, 293-303.

Erdélyi, Arthur and Hermann Kober, 1940: *Quart J. Math.*, Oxford *Ser.* 11, 212-221.

Hardy, G.H., 1918: *Messenger of Math.* 47, 145-150.

Hardy, G.H. and J.E. Littlewood, 1925: *Proc. London Math. Soc.* (2) 24, xxxvii-xli.

Hardy, G.H. and J.E. Littlewood, 1928: *Math. Z.* 27, 565-606.

Hardy, G.H. and J.E. Littlewood, 1932: *Math. Z.* 34, 403-439.

Hardy, G.H., Littlewood, J.E. and G. Polya, 1934: *Inequalities*, Cambridge.

Ince, E.L., 1927: *Ordinary differential equations*. Longmans, Green & Co., London.

Kober, Hermann, 1940: *Quart. J. Math., Oxford Ser.* 11, 193-211.

Kober, Hermann, 1941a: *Quart. J. Math., Oxford Ser.*, 12, 78-85.

Kober, Hermann, 1941b: *Trans. Amer. Math. Soc.* 50, 160-174.

Kuttner, Brian, 1953: *Proc. London Math. Soc.* (3) 3, 480-497.

Love, E.R., 1938: *Proc. London Math. Soc.* (2) 44, 363-397.

Riesz, Marcel, 1949: *Acta Math.* 81, 1-223.

Weyl, Hermann, 1917: *Vierteljschr. Naturforsch. Ges. Zürich*, 62, 296-302.

Widder, D.V., 1941: *The Laplace transform*. Princeton University Press, Princeton, New Jersey.

Young, L.C. and E.R. Love, 1938: *Proc. London Math. Soc.* (2) 44, 1-28.

Zygmund, Antoni, 1935: *Trigonometrical series*. Warszawa-Lwow.

FRACTIONAL INTEGRALS

13.1. Riemann-Liouville fractional integrals

	$f(x)$	$[\Gamma(\mu)]^{-1} \int_0^y f(x)(y-x)^{\mu-1}\,dx = g(y;\mu)$
(1)	$f(ax)$	$a^{-\mu} g(ay;\mu)$
(2)	$f(a/x)$	$a y^{\mu-1} \mathfrak{W}_{\mu}\{t^{-\mu-1} f(t); a/y\}$ For tables see sec 13.2.
(3)	$f'(x)$	$g(y;\mu-1) - f(0)\, y^{\mu-1}/\Gamma(\mu)$
(4)	$\int_0^x f(t)\,dt$	$g(y;\mu+1)$
(5)	$g(x;\nu)$	$g(y;\mu+\nu)$
(6)	1 $\quad$ $\operatorname{Re}\mu > 0$	$\dfrac{y^{\mu}}{\Gamma(\mu+1)}$
(7)	$x^{\nu-1}$ $\quad$ $\operatorname{Re}\mu > 0, \quad \operatorname{Re}\nu > 0$	$\dfrac{\Gamma(\nu)}{\Gamma(\mu+\nu)}\, y^{\mu+\nu-1}$
(8)	$(x+a)^{\nu}$ $\quad$ $\operatorname{Re}\mu > 0$	$\dfrac{a^{\nu} y^{\mu}}{\Gamma(\mu+1)}\, {}_2F_1(1,-\nu;1+\mu;-y/a)$ $\lvert\arg(y/a)\rvert < \pi$

Riemann-Liouville integrals (cont'd)

	$f(x)$	$[\Gamma(\mu)]^{-1} \int_0^y f(x)\,(y-x)^{\mu-1}\,dx$
(9)	$x^{\nu-1}(x+a)^{\lambda}$ $\operatorname{Re}\mu>0, \quad \operatorname{Re}\nu>0$	$\dfrac{a^{\lambda} y^{\mu+\nu-1}\Gamma(\nu)}{\Gamma(\mu+\nu)}$ $\times\ {}_2F_1(-\lambda, \nu; \mu+\nu; -y/a)$ $\lvert\arg y/a\rvert<\pi$
(10)	$x^{\nu-1}(x^2+a^2)^{\lambda}$ $\operatorname{Re}\mu>0, \quad \operatorname{Re}\nu>0$	$\dfrac{a^{2\lambda} y^{\mu+\nu-1}\,\Gamma(\nu)}{\Gamma(\mu+\nu)}$ $\times\ {}_3F_2\left(-\lambda, \dfrac{\nu}{2}, \dfrac{\nu+1}{2}; \dfrac{\mu+\nu}{2}, \dfrac{\mu+\nu+1}{2}; -\dfrac{y^2}{a^2}\right)$ $\operatorname{Re}(y/a)>0$
(11)	$x^{\nu-1}(x^k+a^k)^{\lambda}$ $k=1, 2, \ldots,$ $\operatorname{Re}\mu>0, \quad \operatorname{Re}\nu>0$	$\dfrac{a^{k\lambda}\, y^{\mu+\nu+1}\,\Gamma(\nu)}{\Gamma(\mu+\nu)}$ $\times\ {}_{k+1}F_k\left(-\lambda, \dfrac{\nu}{k}, \dfrac{\nu+1}{k}, \ldots, \dfrac{\nu+k-1}{k}; \dfrac{\mu+\nu}{k}, \dfrac{\mu+\nu+1}{k}, \ldots, \dfrac{\mu+\nu+k-1}{k}; -\dfrac{y^k}{a^k}\right)$ $\lvert\arg(y/a)\rvert<\pi/k$
(12)	$x^{-\frac{1}{2}}(x+2)^{-\frac{1}{2}}\{[(x+2)^{\frac{1}{2}} + x^{\frac{1}{2}}]^{2\nu} + [(x+2)^{\frac{1}{2}} - x^{\frac{1}{2}}]^{2\nu}\}$ $\operatorname{Re}\mu>0$	$2^{\mu+\frac{1}{2}}\,\pi^{\frac{1}{2}}\,[y(y+2)]^{\frac{1}{2}\mu-\frac{1}{4}}$ $\times P^{\frac{1}{2}-\mu}_{\nu-\frac{1}{2}}(y+1) \qquad \lvert\arg y\rvert<\pi$

Riemann-Liouville integrals (cont'd)

	$f(x)$	$[\Gamma(\mu)]^{-1} \int_0^y f(x)(y-x)^{\mu-1}\,dx$
(13)	$x^{\mu-1} e^{\alpha x}$ $\quad$ $\operatorname{Re}\mu > 0$	$\pi^{1/2}(y/\alpha)^{\mu-1/2} \exp(\tfrac{1}{2}\alpha y) \times I_{\mu-1/2}(\tfrac{1}{2}\alpha y)$
(14)	$x^{\nu-1} e^{\alpha x}$ $\quad$ $\operatorname{Re}\mu > 0,\quad \operatorname{Re}\nu > 0$	$\dfrac{\Gamma(\nu)}{\Gamma(\mu+\nu)}\, y^{\mu+\nu-1}\, {}_1F_1(\nu;\, \mu+\nu;\, \alpha y)$
(15)	$x^{\nu-1}\exp(\alpha x^k)$ $\quad$ $\operatorname{Re}\mu > 0,\quad \operatorname{Re}\nu > 0$, $k = 2, 3, 4, \ldots$	$\dfrac{\Gamma(\nu)}{\Gamma(\mu+\nu)}\, y^{\mu+\nu-1} \times {}_kF_k\left(\dfrac{\nu}{k}, \dfrac{\nu+1}{k}, \ldots, \dfrac{\nu+k-1}{k};\ \dfrac{\mu+\nu}{k}, \dfrac{\mu+\nu+1}{k}, \ldots, \dfrac{\mu+\nu+k-1}{k};\ \alpha y^k\right)$
(16)	$x^{-\mu-1}\exp(-\alpha/x)$ $\quad$ $\operatorname{Re}\mu > 0$	$\alpha^{-\mu} y^{\mu-1}\exp(-\alpha/y)$ $\quad$ $\lvert\arg y\rvert < \pi$
(17)	$x^{-2\mu}\exp(-\alpha/x)$ $\quad$ $\operatorname{Re}\mu > 0$	$(\pi y)^{-1/2}\, \alpha^{1/2-\mu} \exp\left(-\dfrac{\alpha}{2y}\right) \times K_{\mu-1/2}\left(\dfrac{\alpha}{2y}\right)$ $\quad$ $\operatorname{Re}(\alpha/y) > 0$
(18)	$x^{\nu-1}\exp(-\alpha/x)$ $\quad$ $\operatorname{Re}\mu > 0$	$\alpha^{\frac{1}{2}\nu-\frac{1}{2}}\, y^{-\kappa} \exp\left(-\dfrac{\alpha}{2y}\right) \times W_{\kappa,\frac{1}{2}\nu}\left(\dfrac{\alpha}{y}\right)$, $\kappa = \tfrac{1}{2} - \mu - \tfrac{1}{2}\nu,\quad \operatorname{Re}(\alpha/y) > 0$

Riemann-Liouville integrals (cont'd)

	$f(x)$	$[\Gamma(\mu)]^{-1} \int_0^y f(x)(y-x)^{\mu-1}\,dx$
(19)	$\exp(ax^{1/2})$ $\quad \operatorname{Re}\mu > 0$	$\dfrac{y^{\mu}}{\Gamma(\mu+1)} + \pi^{1/2}(\tfrac{1}{2}a)^{1/2-\mu} y^{1/2\mu+1/4} \times [I_{\mu+1/2}(ay^{1/2}) + \mathbf{L}_{\mu+1/2}(ay^{1/2})]$
(20)	$x^{-1/2}\exp(ax^{1/2})$	$\pi^{1/2}(\tfrac{1}{2}a)^{1/2-\mu} y^{1/2\mu-1/4} \times [I_{\mu-1/2}(ay^{1/2}) + \mathbf{L}_{\mu-1/2}(ay^{1/2})]$
(21)	$x^{\nu-1}\exp(ax^{1/2})$ $\quad \operatorname{Re}\mu > 0, \quad \operatorname{Re}\nu > 0$	$\dfrac{y^{\mu+\nu-1}\Gamma(\nu)}{\Gamma(\mu+\nu)} \times {}_1F_2\left(\nu; \dfrac{1}{2}, \mu+\nu; \dfrac{a^2 y}{4}\right) + \dfrac{ay^{\mu+\nu-1/2}\Gamma(\nu+\tfrac{1}{2})}{\Gamma(\mu+\nu+\tfrac{1}{2})} \times {}_1F_2\left(\nu+\dfrac{1}{2}; \dfrac{3}{2}, \mu+\nu+\dfrac{1}{2}; \dfrac{a^2y}{4}\right)$
(22)	$x^{-3/2}\exp(-ax^{-1/2})$ $\quad \mu = 1$	$2a^{-1}\exp(-ay^{-1/2})$
(23)	$x^{-\mu-1/2}\exp(-ax^{-1/2})$ $\quad \operatorname{Re}\mu > 0$	$2^{\mu+1/2}\pi^{-1/2} a^{1/2-\mu} y^{1/2\mu-3/4} \times K_{\mu-1/2}(ay^{-1/2}) \quad \operatorname{Re}(ay^{-1/2}) > 0$
(24)	$x^{\nu-1}\log x$ $\quad \operatorname{Re}\mu > 0, \quad \operatorname{Re}\nu > 0$	$\dfrac{y^{\mu+\nu-1}\Gamma(\nu)}{\Gamma(\mu+\nu)} \times [\log y + \psi(\nu) - \psi(\mu+\nu)]$

Riemann-Liouville integrals (cont'd)

	$f(x)$	$[\Gamma(\mu)]^{-1} \int_0^y f(x)(y-x)^{\mu-1}\,dx$
(25)	$x^{\mu-1} \sin(ax)$ $\qquad \operatorname{Re}\mu > 0$	$\pi^{1/2} (y/a)^{\mu-1/2} \sin(\tfrac{1}{2}ay)\, J_{\mu-1/2}(\tfrac{1}{2}ay)$
(26)	$x^{\nu-1} \sin(ax)$ $\qquad \operatorname{Re}\mu > 0, \quad \operatorname{Re}\nu > -1$	$\dfrac{y^{\mu+\nu-1}\,\Gamma(\nu)}{2i\,\Gamma(\mu+\nu)} [{}_1F_1(\nu;\mu+\nu;iay) - {}_1F_1(\nu;\mu+\nu;-iay)]$
(27)	$\sin(ax^{1/2})$ $\qquad \operatorname{Re}\mu > 0$	$2^{\mu-1/2}\pi^{1/2} a^{1/2-\mu} y^{1/2\mu+1/4} J_{\mu+1/2}(ay^{1/2})$
(28)	$x^{-1/2} \sin(ax^{1/2})$ $\qquad \operatorname{Re}\mu > 0$	$\pi^{1/2} 2^{\mu-1/2} a^{1/2-\mu} y^{1/2\mu-1/4} \mathbf{H}_{\mu-1/2}(ay^{1/2})$
(29)	$x^{\nu-1} \sin(ax^{1/2})$ $\qquad \operatorname{Re}\mu > 0, \quad \operatorname{Re}\nu > -\tfrac{1}{2}$	$\dfrac{ay^{\mu+\nu-1/2}\,\Gamma(\nu+\tfrac{1}{2})}{\Gamma(\mu+\nu+\tfrac{1}{2})} \times {}_1F_2\left(\nu+\dfrac{1}{2};\dfrac{3}{2},\mu+\nu+\dfrac{1}{2};-\dfrac{a^2y}{4}\right)$
(30)	$x^{-\mu-1/2} \sin(ax^{-1/2})$ $\qquad 0 < \operatorname{Re}\mu < 1, \quad a > 0$	$2^{\mu-1/2}\pi^{1/2} a^{1/2-\mu} y^{1/2\mu-3/4} J_{1/2-\mu}(ay^{-1/2})$ $\qquad \lvert\arg y\rvert < \pi$
(31)	$x^{\mu-1} \cos(ax)$ $\qquad \operatorname{Re}\mu > 0$	$\pi^{1/2} (y/a)^{\mu-1/2} \cos(\tfrac{1}{2}ay)\, J_{\mu-1/2}(\tfrac{1}{2}ay)$
(32)	$x^{\nu-1} \cos(ax)$ $\qquad \operatorname{Re}\mu > 0, \quad \operatorname{Re}\nu > 0$	$\dfrac{y^{\mu+\nu-1}\,\Gamma(\nu)}{2\Gamma(\mu+\nu)} [{}_1F_1(\nu;\mu+\nu;iay) + {}_1F_1(\nu;\mu+\nu;-iay)]$

Riemann-Liouville integrals (cont'd)

	$f(x)$	$[\Gamma(\mu)]^{-1} \int_0^y f(x)\,(y-x)^{\mu-1}\,dx$
(33)	$\cos(ax^{1/2})$ $\quad \operatorname{Re}\mu > 0$	$\dfrac{y^{\mu}}{\Gamma(\mu+1)} - 2^{\mu-1/2}\, a^{1/2-\mu}\, \pi^{1/2}$ $\times y^{1/2\mu+1/4}\, \mathbf{H}_{\mu+1/2}(ay^{1/2})$
(34)	$x^{-1/2}\cos(ay^{1/2})$ $\quad \operatorname{Re}\mu > 0$	$2^{\mu-1/2}\,\pi^{1/2}\, a^{1/2-\mu}\, y^{1/2\mu-1/4}\, J_{\mu-1/2}(ay^{1/2})$
(35)	$x^{\nu-1}\cos(ax^{1/2})$ $\quad \operatorname{Re}\mu > 0, \quad \operatorname{Re}\nu > 0$	$\dfrac{y^{\mu+\nu-1}\,\Gamma(\nu)}{\Gamma(\mu+\nu)}$ $\times\, {}_1F_2\left(\nu; \dfrac{1}{2}, \mu+\nu; -\dfrac{a^2 y}{4}\right)$
(36)	$x^{-\mu-1/2}\cos(ax^{-1/2})$ $\quad 0 < \operatorname{Re}\mu < 1, \quad a > 0$	$-2^{\mu-1/2}\,\pi^{1/2}\, a^{1/2-\mu}\, y^{1/2\mu-3/4}\, Y_{1/2-\mu}(ay^{-1/2})$ $\quad \lvert\arg y\rvert < \pi$
(37)	$P_n(1-\gamma x)$ $\quad \operatorname{Re}\mu > 0$	$\dfrac{n!\,y^{\mu}}{\Gamma(\mu+n+1)}\, P_n^{(\mu,-\mu)}(1-\gamma y)$
(38)	$x^{\nu-1}P_n(1-\gamma x)$ $\quad \operatorname{Re}\mu > 0, \quad \operatorname{Re}\nu > 0$	$\dfrac{\Gamma(\nu)\,y^{\mu+\nu-1}}{\Gamma(\mu+\nu)}$ $\times\, {}_3F_2(-n, n+1, \nu; 1, \mu+\nu; \tfrac{1}{2}\gamma y)$
(39)	$x^{\lambda-1/2}\,C_n^{\lambda}(1-\gamma x)$ $\quad \operatorname{Re}\lambda > -1, \quad \lambda \neq 0, -\tfrac{1}{2}$ $\quad \operatorname{Re}\mu > 0$	$\dfrac{(2\lambda)_n\,\Gamma(\lambda+\tfrac{1}{2})}{\Gamma(\lambda+\mu+n+\tfrac{1}{2})}$ $\times\, y^{\lambda+\mu-1/2}\, P_n^{(\alpha,\beta)}(1-\gamma y)$ $\quad \alpha = \lambda+\mu-\tfrac{1}{2}, \quad \beta = \lambda-\mu-\tfrac{1}{2}$

Riemann-Liouville integrals (cont'd)

	$f(x)$	$[\Gamma(\mu)]^{-1} \int_0^y f(x)(y-x)^{\mu-1}\,dx$
(40)	$x^{\nu-1} C_n^{\lambda}(1-\gamma x)$ $2\lambda \neq 0, -1, -2, \ldots$ $\operatorname{Re}\mu > 0, \quad \operatorname{Re}\nu > 0$	$\dfrac{(2\lambda)_n \Gamma(\nu)}{n!\,\Gamma(\mu+\nu)} y^{\mu+\nu-1}$ $\times\, {}_3F_2(-n, n+2\lambda, \nu; \lambda+\tfrac{1}{2}; \mu+\nu; \tfrac{1}{2}\gamma y)$
(41)	$x^{\nu-1} C_{2n}^{\lambda}(\gamma x^{1/2})$ $\operatorname{Re}\mu > 0, \quad \operatorname{Re}\nu > 0$	$(-1)^n \dfrac{(\lambda)_n \Gamma(\nu)}{n!\,\Gamma(\mu+\nu)} y^{\mu+\nu-1}$ $\times\, {}_3F_2(-n, n+\lambda, \nu; \tfrac{1}{2}, \mu+\nu; \gamma^2 y)$
(42)	$x^{\nu-1} C_{2n+1}^{\lambda}(\gamma x^{1/2})$ $\operatorname{Re}\mu > 0, \quad \operatorname{Re}\nu > -\tfrac{1}{2}$	$2(-1)^n \gamma y^{\mu+\nu-1/2}$ $\times \dfrac{(\lambda)_{n+1} \Gamma(\nu+\tfrac{1}{2})}{n!\;\Gamma(\mu+\nu+\tfrac{1}{2})}$ $\times\, {}_3F_2\left(-n,\; n+\lambda+1,\; \nu+\dfrac{1}{2};\right.$ $\left.\dfrac{3}{2},\; \mu+\nu+\dfrac{1}{2};\; \gamma^2 y\right)$
(43)	$x^{\alpha} P_n^{(\alpha,\beta)}(1-\gamma x)$ $\operatorname{Re}\alpha > -1, \quad \operatorname{Re}\mu > 0$	$\dfrac{\Gamma(\alpha+n+1)}{\Gamma(\alpha+\mu+n+1)} y^{\alpha+\mu}$ $\times\, P_n^{(\alpha+\mu,\,\beta-\mu)}(1-\gamma y)$
(44)	$x^{\beta} P_n^{(\alpha,\beta)}(\gamma x - 1)$ $\operatorname{Re}\beta > -1, \quad \operatorname{Re}\mu > 0$	$\dfrac{\Gamma(\beta+n+1)}{\Gamma(\beta+\mu+n+1)} y^{\beta+\mu}$ $\times\, P_n^{(\alpha-\mu,\,\beta+\mu)}(\gamma y - 1)$

Riemann-Liouville integrals (cont'd)

	$f(x)$	$[\Gamma(\mu)]^{-1} \int_0^y f(x)\,(y-x)^{\mu-1}\,dx$
(45)	$x^{-\beta-\mu-n-1}(1-\tfrac{1}{2}\gamma x)^{\beta}$ $\times P_n^{(\alpha,\beta)}(1-\gamma x)$ $0<\mathrm{Re}\,\mu<-\mathrm{Re}\,\beta-n$	$\dfrac{\Gamma(-\beta-\mu-n)}{\Gamma(-\beta-\mu)}\, y^{-\beta-n-1}(1-\tfrac{1}{2}\gamma y)^{\beta+\mu}$ $\times P_n^{(\alpha,\beta+\mu)}(1-\gamma y)$
(46)	$x^{\lambda-1} P_n^{(\alpha,\beta)}(1-\gamma x)$ $\mathrm{Re}\,\lambda>0, \quad \mathrm{Re}\,\mu>0$	$\dfrac{\Gamma(\alpha+n+1)\,\Gamma(\lambda)}{n!\,\Gamma(\alpha+1)\,\Gamma(\lambda+\mu)}\, y^{\lambda+\mu-1}$ $\times {}_3F_2(-n, n+\alpha+\beta+1, \lambda; \alpha+1, \lambda+\mu; \tfrac{1}{2}\gamma y)$
(47)	$x^{\lambda-1} P_n^{(\alpha,\beta)}(\gamma x-1)$ $\mathrm{Re}\,\lambda>0, \quad \mathrm{Re}\,\mu>0$	$(-1)^n \dfrac{\Gamma(\beta+n+1)\,\Gamma(\lambda)}{n!\,\Gamma(\beta+1)\,\Gamma(\lambda+\mu)}\, y^{\lambda+\mu-1}$ $\times {}_3F_2(-n, n+\alpha+\beta+1, \lambda; \beta+1, \lambda+\mu; \tfrac{1}{2}\gamma y)$
(48)	$x^{\lambda-1}(1-\tfrac{1}{2}\gamma x)^{\beta} P_n^{(\alpha,\beta)}(1-\gamma x)$ $\mathrm{Re}\,\lambda>0, \quad \mathrm{Re}\,\mu>0$	$\dfrac{\Gamma(n+\alpha+1)\,\Gamma(\lambda)}{n!\,\Gamma(\alpha+1)\,\Gamma(\lambda+\mu)}\, y^{\lambda+\mu-1}$ $\times {}_3F_2(\alpha+n+1, -\beta-n, \lambda; \alpha+1, \lambda+\mu; \tfrac{1}{2}\gamma y)$
(49)	$x^{\alpha} L_n^{\alpha}(\beta x)$ $\mathrm{Re}\,\alpha>-1, \mathrm{Re}\,\mu>0$	$\dfrac{\Gamma(\alpha+n+1)}{\Gamma(\alpha+\mu+n+1)}\, y^{\alpha+\mu} L_n^{\alpha+\mu}(\beta y)$
(50)	$x^{\lambda-1} L_n^{\alpha}(\beta x)$ $\mathrm{Re}\,\lambda>0, \quad \mathrm{Re}\,\mu>0$	$\dfrac{\Gamma(\alpha+n+1)\,\Gamma(\lambda)}{n!\,\Gamma(\alpha+1)\,\Gamma(\lambda+\mu)}\, y^{\lambda+\mu-1}$ $\times {}_2F_2(-n, \lambda; \alpha+1, \lambda+\mu; \beta y)$

Riemann-Liouville integrals (cont'd)

	$f(x)$	$[\Gamma(\mu)]^{-1} \int_0^y f(x)(y-x)^{\mu-1}\,dx$
(51)	$x^{\lambda-1} e^{-\beta x} L_n^{\alpha}(\beta x)$ $\operatorname{Re}\lambda > 0, \quad \operatorname{Re}\mu > 0$	$\dfrac{\Gamma(\alpha+n+1)\,\Gamma(\lambda)}{n!\,\Gamma(\alpha+1)\,\Gamma(\lambda+\mu)}\, y^{\lambda+\mu-1}$ $\times\, {}_2F_2(\alpha+n+1, \lambda;\, \alpha+1, \lambda+\mu;\, -\beta y)$
(52)	$[x(1+\tfrac{1}{2}\gamma x)]^{-\frac{1}{2}\lambda} P_\nu^{\lambda}(1+\gamma x)$ $\operatorname{Re}\lambda < 1, \quad \operatorname{Re}\mu > 0$	$(2/\gamma)^{\frac{1}{2}\mu}[y(1+\tfrac{1}{2}\gamma y)]^{\frac{1}{2}\mu-\frac{1}{2}\lambda}\cdot$ $\times\, P_\nu^{\lambda-\mu}(1+\gamma y) \qquad \lvert\arg \gamma y\rvert < \pi$
(53)	$x^{\kappa+\frac{1}{2}\lambda-1}(1+\tfrac{1}{2}\gamma x)^{-\frac{1}{2}\lambda}$ $\times\, P_\nu^{\lambda}(1+\gamma x)$ $\operatorname{Re}\kappa > 0, \quad \operatorname{Re}\mu > 0$	$\dfrac{(\tfrac{1}{2}\gamma)^{-\frac{1}{2}\lambda}\,\Gamma(\kappa)}{\Gamma(1-\lambda)\,\Gamma(\kappa+\mu)}\, y^{\kappa+\mu-1}$ $\times\, {}_3F_2(-\nu, 1+\nu, \kappa;\, 1-\lambda, \kappa+\mu;\, -\tfrac{1}{2}\gamma y)$ $\lvert\gamma y\rvert < 1$
(54)	$[x(1-x)]^{-\frac{1}{2}\lambda}\, \mathrm{P}_\nu^{\lambda}(1-2x)$ $\operatorname{Re}\lambda < 1, \quad \operatorname{Re}\mu > 0$	$[y(1-y)]^{\frac{1}{2}\mu-\frac{1}{2}\lambda}\, \mathrm{P}_\nu^{\lambda-\mu}(1-2y)$ $0 < y < 1$
(55)	$x^{\kappa+\frac{1}{2}\lambda-1}(1-x)^{-\frac{1}{2}\lambda}\, \mathrm{P}_\nu^{\lambda}(1-2x)$ $\operatorname{Re}\kappa > 0, \quad \operatorname{Re}\mu > 0$	$\dfrac{\Gamma(\kappa)\, y^{\kappa+\mu-1}}{\Gamma(\kappa+\mu)\,\Gamma(1-\lambda)}$ $\times\, {}_3F_2(-\nu, 1+\nu, \kappa;\, 1-\lambda, \kappa+\mu;\, y)$ $0 < y < 1$
(56)	$x^{\lambda-1} J_\nu(ax)$ $\operatorname{Re}\mu > 0, \quad \operatorname{Re}(\lambda+\nu) > 0$	$\dfrac{\Gamma(\lambda+\nu)}{\Gamma(\nu+1)\,\Gamma(\lambda+\mu+\nu)}\,(\tfrac{1}{2}a)^{\nu} y^{\lambda+\mu+\nu-1}$ $\times\, {}_2F_3\left(\dfrac{\lambda+\nu}{2}, \dfrac{\lambda+\nu+1}{2};\, \nu+1,\right.$ $\left.\dfrac{\lambda+\mu+\nu}{2}, \dfrac{\lambda+\mu+\nu+1}{2};\, -\dfrac{a^2y^2}{4}\right)$

Riemann-Liouville integrals (cont'd)

	$f(x)$	$[\Gamma(\mu)]^{-1} \int_0^y f(x)\,(y-x)^{\mu-1}\,dx$
(57)	$x^{\nu} e^{\pm iax} J_{\nu}(ax)$ $\operatorname{Re}\mu > 0, \quad \operatorname{Re}\nu > -\tfrac{1}{2}$	$\dfrac{(2a)^{\nu} y^{\mu+2\nu} \Gamma(\nu+\tfrac{1}{2})}{\pi^{\frac{1}{2}} \Gamma(\mu+2\nu+1)}$ $\times\ {}_1F_1(\nu+\tfrac{1}{2};\ \mu+2\nu+1;\ \pm 2iay)$
(58)	$x^{\lambda-\nu-1} e^{\pm iax} J_{\nu}(ax)$ $\operatorname{Re}\lambda > 0, \quad \operatorname{Re}\mu > 0$	$\dfrac{2^{-\nu} a^{\nu} y^{\lambda+\mu-1} \Gamma(\lambda)}{\Gamma(\lambda+\mu)\Gamma(\nu+1)}$ $\times\ {}_2F_2(\lambda,\ \nu+\tfrac{1}{2};\ \lambda+\mu,\ 2\nu+1;\ \pm 2iay)$
(59)	$x^{-\frac{1}{2}} J_{2\nu}(ax^{\frac{1}{2}})$ $\mu = \tfrac{1}{2}, \quad \operatorname{Re}\nu > -\tfrac{1}{2}$	$\pi^{\frac{1}{2}} [J_{\nu}(\tfrac{1}{2}ay^{\frac{1}{2}})]^2$
(60)	$x^{\frac{1}{2}\nu-\frac{1}{2}} J_{\nu}(ax^{\frac{1}{2}})$ $\mu = \nu + \tfrac{1}{2}, \quad \operatorname{Re}\nu > -\tfrac{1}{2}$	$\pi^{\frac{1}{2}} \left(\dfrac{2y}{a}\right)^{\nu} [J_{\nu}(\tfrac{1}{2}ay^{\frac{1}{2}})]^2$
(61)	$x^{\frac{1}{2}\nu-\frac{1}{2}} J_{\nu}(ax^{\frac{1}{2}})$ $\mu = \nu - \tfrac{1}{2}, \quad \operatorname{Re}\nu > \tfrac{1}{2}$	$(\tfrac{1}{2}a)^{1-\nu} \pi^{\frac{1}{2}} y^{\nu-\frac{1}{2}} J_{\nu}(\tfrac{1}{2}ay^{\frac{1}{2}})$ $\times J_{\nu-1}(\tfrac{1}{2}ay^{\frac{1}{2}})$
(62)	$x^{-\frac{1}{2}\nu-\frac{1}{2}} J_{\nu}(ax^{\frac{1}{2}})$ $\mu = \tfrac{1}{2} - \nu, \quad \operatorname{Re}\nu < \tfrac{1}{2}$	$\pi^{\frac{1}{2}} \left(\dfrac{a}{2y}\right)^{\nu} J_{\nu}(\tfrac{1}{2}ay^{\frac{1}{2}}) J_{-\nu}(\tfrac{1}{2}ay^{\frac{1}{2}})$
(63)	$x^{\frac{1}{2}\nu} J_{\nu}(ax^{\frac{1}{2}})$ $\operatorname{Re}\mu > 0, \quad \operatorname{Re}\nu > -1$	$2^{\mu} a^{-\mu} y^{\frac{1}{2}\mu+\frac{1}{2}\nu} J_{\mu+\nu}(ay^{\frac{1}{2}})$
(64)	$x^{-\frac{1}{2}\nu} J_{\nu}(ax^{\frac{1}{2}}) \qquad \operatorname{Re}\mu > 0$	$\dfrac{2^{2-\nu} a^{-\mu} y^{\frac{1}{2}\mu-\frac{1}{2}\nu}}{\Gamma(\mu)\Gamma(\nu)} s_{\mu+\nu-1,\,\mu-\nu}(ay^{\frac{1}{2}})$

Riemann-Liouville integrals((cont'd)

	$f(x)$	$[\Gamma(\mu)]^{-1} \int_0^y f(x)\,(y-x)^{\mu-1}\,dx$
(65)	$x^{\lambda-\frac{1}{2}\nu-1} J_\nu(ax^{\frac{1}{2}})$ $\operatorname{Re}\lambda > 0, \quad \operatorname{Re}\mu > 0$	$\dfrac{a^\nu\, y^{\lambda+\mu-1}\,\Gamma(\lambda)}{2^\nu\,\Gamma(\lambda+\mu)\,\Gamma(\nu+1)}$ $\times\, {}_1F_2(\lambda;\, \nu+1,\, \lambda+\mu;\, -\tfrac{1}{4}a^2 y)$
(66)	$x^{\lambda-\nu-1}\,[J_\nu(ax^{\frac{1}{2}})]^2$ $\operatorname{Re}\lambda > 0, \quad \operatorname{Re}\mu > 0$ For several particular cases see Bailey, W.N., 1938: *Quart. J. Math. Oxford Ser.*, 9, 141-147.	$\dfrac{(\frac{1}{2}a)^{2\nu}\, y^{\lambda+\mu-1}\,\Gamma(\lambda)}{[\Gamma(\nu+1)]^2\,\Gamma(\lambda+\mu)}$ $\times\, {}_2F_3(\lambda,\, \nu+\tfrac{1}{2};\, \lambda+\mu,\, \nu+1,\, 2\nu+1;\, -a^2 y)$
(67)	$x^{\lambda-1}\, J_\nu(ax^{\frac{1}{2}})\, J_{-\nu}(ax^{\frac{1}{2}})$ $\operatorname{Re}\lambda > 0, \quad \operatorname{Re}\mu > 0$	$\dfrac{\Gamma(\lambda)\sin(\nu\pi)}{\nu\pi\,\Gamma(\lambda+\mu)}\, y^{\lambda+\mu-1}$ $\times\, {}_2F_3(\tfrac{1}{2},\, \lambda;\, 1+\nu,\, 1-\nu,\, \lambda+\mu;\, -a^2 y)$
(68)	$x^{-\frac{1}{2}}\, Y_\nu(ax^{\frac{1}{2}})$ $\mu = \tfrac{1}{2}, \quad -1 < \operatorname{Re}\nu < 1$	$\pi^{\frac{1}{2}}\operatorname{ctn}(\nu\pi)\,[J_{\frac{1}{2}\nu}(\tfrac{1}{2}ay^{\frac{1}{2}})]^2$ $-\,\pi^{\frac{1}{2}}\csc(\nu\pi)\,[J_{-\frac{1}{2}\nu}(\tfrac{1}{2}ay^{\frac{1}{2}})]^2$
(69)	$x^{\frac{1}{2}\nu}\, Y_\nu(ax^{\frac{1}{2}})$ $\mu = \tfrac{1}{2}, \quad \operatorname{Re}\nu > -1$	$2^{\frac{1}{2}}\, a^{-\frac{1}{2}}\, y^{\frac{1}{2}\nu+\frac{1}{4}}\csc(\nu\pi)$ $\times\,[\cos(\nu\pi)\, J_{\nu+\frac{1}{2}}(ay^{\frac{1}{2}})$ $-\,\mathbf{H}_{-\nu-\frac{1}{2}}(ay^{\frac{1}{2}})]$
(70)	$x^{-\frac{1}{2}\nu}\, Y_\nu(ax^{\frac{1}{2}})$ $\mu = \tfrac{1}{2}, \quad \operatorname{Re}\nu < 1$	$2^{\frac{1}{2}}\, a^{-\frac{1}{2}}\, y^{\frac{1}{4}-\frac{1}{2}\nu}\csc(\nu\pi)$ $\times\,[\cos(\nu\pi)\,\mathbf{H}_{\nu-\frac{1}{2}}(ay^{\frac{1}{2}})$ $-\,J_{\frac{1}{2}-\nu}(ay^{\frac{1}{2}})]$

Riemann-Liouville integrals (cont'd)

	$f(x)$	$[\Gamma(\mu)]^{-1} \int_0^y f(x)\,(y-x)^{\mu-1}\,dx$
(71)	$x^{\frac{1}{2}\nu-\frac{1}{2}}\, Y_\nu(ax^{\frac{1}{2}})$ $\mu=\nu+\frac{1}{2}, \quad \mathrm{Re}\,\nu>-\frac{1}{2}$	$\pi^{\frac{1}{2}}\left(\frac{2y}{a}\right)^{\nu} J_\nu(\tfrac{1}{2}ay^{\frac{1}{2}})\, Y_\nu(\tfrac{1}{2}ay^{\frac{1}{2}})$.
(72)	$x^{-\frac{1}{2}\nu-\frac{1}{2}}\, Y_\nu(ax^{\frac{1}{2}})$ $\mu=\frac{1}{2}-\nu, \quad \mathrm{Re}\,\nu<\frac{1}{2}$	$\pi^{\frac{1}{2}}\left(\frac{a}{2y}\right)^{\nu} J_{-\nu}(\tfrac{1}{2}ay^{\frac{1}{2}})\, Y_\nu(\tfrac{1}{2}ay^{\frac{1}{2}})$
(73)	$x^{\frac{1}{2}\nu}\, Y_\nu(ax^{\frac{1}{2}})$ $\mathrm{Re}\,\mu>0, \quad \mathrm{Re}\,\nu>-1$	$\left(\frac{2}{a}\right)^{\mu} y^{\frac{1}{2}\mu+\frac{1}{2}\nu}\,\mathrm{ctn}\,(\nu\pi)\, J_{\mu+\nu}(ay^{\frac{1}{2}})$ $+\ \frac{2^{\nu+2}\, y^{\frac{1}{2}\mu+\frac{1}{2}\nu}\,\Gamma(\nu+1)}{\pi a^{\mu}\,\Gamma(\mu)}$ $\times\, s_{\mu-\nu-1,\,\mu+\nu}(ay^{\frac{1}{2}})$
(74)	$x^{\frac{1}{2}\nu}\, Y_\nu(ax^{\frac{1}{2}})$ $\mathrm{Re}\,\mu>0, \quad \mathrm{Re}\,\nu>-1$	$\left(\frac{2}{a}\right)^{\mu} y^{\frac{1}{2}\mu+\frac{1}{2}\nu}\, Y_{\mu+\nu}(ay^{\frac{1}{2}})$ $+\ \frac{2^{\nu+2}\, y^{\frac{1}{2}\mu+\frac{1}{2}\nu}\,\Gamma(\nu+1)}{\pi\, a^{\mu}\,\Gamma(\mu)}$ $\times\, S_{\mu-\nu-1,\,\mu+\nu}(ay^{\frac{1}{2}})$
(75)	$x^{-\frac{1}{2}\nu}\, Y_\nu(ax^{\frac{1}{2}})$ $\mathrm{Re}\,\mu>0, \quad \mathrm{Re}\,\nu<1$	$\frac{y^{\frac{1}{2}\mu-\frac{1}{2}\nu}\,\mathrm{ctn}\,(\nu\pi)}{2^{\nu-2}\, a^{\mu}\,\Gamma(\mu)\,\Gamma(\nu)}\, s_{\mu+\nu-1,\mu-\nu}(ay^{\frac{1}{2}})$ $-\ (2/a)^{\mu}\, y^{\frac{1}{2}\mu-\frac{1}{2}\nu}\,\csc\,(\nu\pi)$ $\times\, J_{\mu-\nu}(ay^{\frac{1}{2}})$

Riemann-Liouville integrals (cont'd)

	$f(x)$	$[\Gamma(\mu)]^{-1} \int_0^y f(x)\,(y-x)^{\mu-1}\,dx$
(76)	$x^{\lambda-1}\, Y_\nu(a x^{1/2})$ $\operatorname{Re}\lambda > \tfrac12 \lvert\operatorname{Re}\nu\rvert, \quad \operatorname{Re}\mu > 0$	$2^{-\nu} a^{\nu} y^{\lambda+\mu+\frac12\nu-1} \operatorname{ctn}(\nu\pi)$ $\times \dfrac{\Gamma(\lambda+\frac12\nu)}{\Gamma(1+\nu)\,\Gamma(\lambda+\mu+\frac12\nu)}$ $\times\, {}_1F_2(\lambda+\tfrac12\nu;\ 1+\nu,\ \lambda+\mu+\tfrac12\nu;\ -\tfrac14 a^2 y)$ $-\, 2^{\nu} a^{-\nu} y^{\lambda+\mu-\frac12\nu-1} \csc(\nu\pi)$ $\times \dfrac{\Gamma(\lambda-\frac12\nu)}{\Gamma(1-\nu)\,\Gamma(\lambda+\mu-\frac12\nu)}$ $\times\, {}_1F_2(\lambda-\tfrac12\nu;\ 1-\nu,\ \lambda+\mu-\tfrac12\nu;\ -\tfrac14 a^2 y)$
(77)	$x^{\nu} e^{\pm a x} I_\nu(a x)$ $\operatorname{Re}\mu > 0, \quad \operatorname{Re}\nu > -\tfrac12$	$\dfrac{(2a)^{\nu} y^{\mu+2\nu}\,\Gamma(\nu+\frac12)}{\pi^{1/2}\,\Gamma(\mu+2\nu+1)}$ $\times\, {}_1F_1(\nu+\tfrac12;\ \mu+2\nu+1;\ \pm 2ay)$
(78)	$x^{\lambda-1} e^{\pm a x} I_\nu(a x)$ $\operatorname{Re}\mu > 0, \quad \operatorname{Re}(\lambda+\nu) > 0$	$\dfrac{(\frac12 a)^{\nu} y^{\lambda+\mu+\nu-1}\,\Gamma(\lambda+\nu)}{\Gamma(\nu+1)\,\Gamma(\lambda+\mu+\nu)}$ $\times\, {}_2F_2(\nu+\tfrac12,\ \lambda+\nu;\ 2\nu+1,\ \mu+\lambda+\nu;\ \pm 2ay)$
(79)	$x^{-1/2} I_{2\nu}(a x^{1/2})$ $\mu = \tfrac12, \quad \operatorname{Re}\nu > -\tfrac12$	$\pi^{1/2} [I_\nu(\tfrac12 a y^{1/2})]^2$
(80)	$x^{\frac12\nu-\frac12} I_\nu(a x^{1/2})$ $\mu = \nu+\tfrac12, \quad \operatorname{Re}\nu > -\tfrac12$	$\pi^{1/2} \left(\dfrac{2y}{a}\right)^{\nu} [I_\nu(\tfrac12 a y^{1/2})]^2$

Riemann-Liouville integrals (cont'd)

	$f(x)$	$[\Gamma(\mu)]^{-1}\int_0^y f(x)\,(y-x)^{\mu-1}\,dx$
(81)	$x^{\frac{1}{2}\nu-\frac{1}{2}}\, I_\nu(ax^{\frac{1}{2}})$ $\mu=\nu-\frac{1}{2}, \quad \mathrm{Re}\,\nu>\frac{1}{2}$	$(\frac{1}{2}a)^{1-\nu}\,\pi^{\frac{1}{2}}\,y^{\nu-\frac{1}{2}}\,I_\nu(\frac{1}{2}ay^{\frac{1}{2}})$ $\times I_{\nu-1}(\frac{1}{2}ay^{\frac{1}{2}})$
(82)	$x^{-\frac{1}{2}\nu-\frac{1}{2}}\, I_\nu(ax^{\frac{1}{2}})$ $\mu=\frac{1}{2}-\nu, \quad \mathrm{Re}\,\nu<\frac{1}{2}$	$\pi^{\frac{1}{2}}\left(\dfrac{a}{2y}\right)^{\nu} I_\nu(\frac{1}{2}ay^{\frac{1}{2}})\,I_{-\nu}(\frac{1}{2}ay^{\frac{1}{2}})$
(83)	$x^{\frac{1}{2}\nu}\, I_\nu(ax^{\frac{1}{2}})$ $\mathrm{Re}\,\mu>0, \quad \mathrm{Re}\,\nu>-1$	$2^\mu\,a^{-\mu}\,y^{\frac{1}{2}\mu+\frac{1}{2}\nu}\,I_{\mu+\nu}(ay^{\frac{1}{2}})$
(84)	$x^{\lambda-\frac{1}{2}\nu-1}\, I_\nu(ax^{\frac{1}{2}})$ $\mathrm{Re}\,\lambda>0, \quad \mathrm{Re}\,\mu>0$	$\dfrac{a^\nu\,y^{\lambda+\mu-1}\,\Gamma(\lambda)}{2^\nu\,\Gamma(\nu+1)\,\Gamma(\lambda+\mu)}$ $\times\, {}_1F_2(\lambda;\, \nu+1,\, \lambda+\mu;\, \frac{1}{4}a^2y)$
(85)	$x^{-\frac{1}{2}}\, K_{2\nu}(ax^{\frac{1}{2}})$ $\mu=\frac{1}{2}, \quad -\frac{1}{2}<\mathrm{Re}\,\nu<\frac{1}{2}$	$\frac{1}{2}\sec(\nu\pi)\,[I_\nu(\frac{1}{2}ay^{\frac{1}{2}})$ $+\,I_{-\nu}(\frac{1}{2}ay^{\frac{1}{2}})]\,K_\nu(\frac{1}{2}ay^{\frac{1}{2}})$
(86)	$x^{\frac{1}{2}\nu-\frac{1}{2}}\, K_\nu(ax^{\frac{1}{2}})$ $\mu=\nu+\frac{1}{2}, \quad \mathrm{Re}\,\nu>-\frac{1}{2}$	$\pi^{\frac{1}{2}}(2y/a)^\nu\,I_\nu(\frac{1}{2}ay^{\frac{1}{2}})\,K_\nu(\frac{1}{2}ay^{\frac{1}{2}})$
(87)	$x^{\lambda-1}\, K_\nu(ax^{\frac{1}{2}})$ $\mathrm{Re}\,\lambda>\frac{1}{2}\lvert\mathrm{Re}\,\nu\rvert, \quad \mathrm{Re}\,\mu>0$	$2^{\nu-1}\,a^{-\nu}\,y^{\lambda+\mu-\frac{1}{2}\nu-1}\,\dfrac{\Gamma(\nu)\,\Gamma(\lambda-\frac{1}{2}\nu)}{\Gamma(\lambda+\mu-\frac{1}{2}\nu)}$ $\times\, {}_1F_2\left(\lambda-\dfrac{\nu}{2};\, 1-\nu,\, \lambda+\mu-\dfrac{\nu}{2};\, \dfrac{a^2y}{4}\right)$ $+\,2^{1-\nu}\,a^\nu\,y^{\lambda+\mu+\frac{1}{2}\nu-1}\,\dfrac{\Gamma(-\nu)\,\Gamma(\lambda+\frac{1}{2}\nu)}{\Gamma(\lambda+\mu+\frac{1}{2}\nu)}$ $\times\, {}_1F_2\left(\lambda+\dfrac{\nu}{2};\, 1+\nu,\, \lambda+\mu+\dfrac{\nu}{2};\, \dfrac{a^2y}{4}\right)$

Riemann-Liouville integrals (cont'd)

	$f(x)$	$[\Gamma(\mu)]^{-1} \int_0^y f(x)\,(y-x)^{\mu-1}\,dx$
(88)	$x^{\frac{1}{2}\nu}\,\mathbf{H}_\nu(\gamma x^{\frac{1}{2}})$ $\operatorname{Re}\nu > -3/2, \quad \operatorname{Re}\mu > 0$	$(\tfrac{1}{2}\gamma)^{-\mu}\,y^{\frac{1}{2}\mu+\frac{1}{2}\nu}\,\mathbf{H}_{\mu+\nu}(\gamma y^{\frac{1}{2}})$
(89)	$x^{\lambda-\nu/2-3/2}\,\mathbf{H}_\nu(\gamma x^{\frac{1}{2}})$ $\operatorname{Re}\lambda > 0, \quad \operatorname{Re}\mu > 0$	$\dfrac{\Gamma(\lambda)\,\gamma^{\nu+1}\,y^{\lambda+\mu-1}}{2^\nu\,\pi^{\frac{1}{2}}\,\Gamma(\nu+3/2)\,\Gamma(\lambda+\mu)} \times {}_2F_3\left(1,\,\lambda;\,\frac{3}{2},\,\nu+\frac{3}{2},\,\lambda+\mu;\,-\frac{\gamma^2 y}{4}\right)$
(90)	$x^{\frac{1}{2}\nu}\,\mathbf{L}_\nu(\gamma x^{\frac{1}{2}})$ $\operatorname{Re}\mu > 0, \quad \operatorname{Re}\nu > -3/2$	$(\tfrac{1}{2}\gamma)^{-\mu}\,y^{\frac{1}{2}\mu+\frac{1}{2}\nu}\,\mathbf{L}_{\mu+\nu}(\gamma y^{\frac{1}{2}})$
(91)	$x^{\lambda-\nu/2-3/2}\,\mathbf{L}_\nu(\gamma x^{\frac{1}{2}})$ $\operatorname{Re}\lambda > 0, \quad \operatorname{Re}\mu > 0$	$\dfrac{\gamma^{\nu+1}\,y^{\lambda+\mu-1}\,\Gamma(\lambda)}{2^\nu\,\pi^{\frac{1}{2}}\,\Gamma(\nu+3/2)\,\Gamma(\lambda+\mu)} \times {}_2F_3\left(1,\,\lambda;\,\frac{3}{2},\,\nu+\frac{3}{2},\,\lambda+\mu;\,\frac{\gamma^2 y}{4}\right)$
(92)	$x^{\lambda-\frac{1}{2}\kappa-\frac{1}{2}}\,s_{\kappa,\nu}(ax^{\frac{1}{2}})$ $\operatorname{Re}\lambda > -1, \quad \operatorname{Re}\mu > 0$	$\dfrac{a^\kappa\,y^{\lambda+\mu}\,\Gamma(\lambda+1)}{(\kappa-\nu+1)\,(\kappa+\nu+1)\,\Gamma(\lambda+\mu+1)} \times {}_2F_3\left(1,\,\lambda+1;\,\frac{\kappa-\nu+3}{2},\,\frac{\kappa+\nu+3}{2},\,\lambda+\mu+1;\,-\frac{a^2 y}{4}\right)$

Riemann-Liouville integrals (cont'd)

	$f(x)$	$[\Gamma(\mu)]^{-1} \int_0^y f(x)\,(y-x)^{\mu-1}\,dx$
(93)	$x^{\kappa-\mu-1}\, e^{-\frac{1}{2}ax}\, W_{\kappa,\lambda}(ax)$ $0 < \mathrm{Re}\,\mu < \mathrm{Re}\,\kappa - \lvert\mathrm{Re}\,\lambda\rvert + \frac{1}{2}$	$y^{\kappa-1}\, e^{-\frac{1}{2}ay} \sec[(\kappa-\mu-\lambda)\pi]$ $\times \left\{ \sin(\mu\pi) \dfrac{\Gamma(\kappa-\mu+\lambda+\frac{1}{2})}{\Gamma(2\lambda+1)} \right.$ $\times M_{\kappa-\mu,\lambda}(ay)$ $\left. + \cos[(\kappa-\lambda)\pi]\, W_{\kappa-\mu,\lambda}(ay) \right\}$
(94)	$x^{\nu-1}\ {}_pF_q(a_1, \ldots, a_p;$ $\nu, b_2, \ldots, b_q; ax)$ $p \le q+1$ $\mathrm{Re}\,\mu > 0, \quad \mathrm{Re}\,\nu > 0$	$\dfrac{y^{\mu+\nu-1}\,\Gamma(\nu)}{\Gamma(\mu+\nu)}$ $\times\ {}_pF_q(a_1, \ldots, a_p; \mu+\nu, b_2, \ldots, b_q; ay)$ $\lvert ay\rvert < 1$ if $p = q+1$
(95)	$x^{\nu-1}\ {}_pF_q(a_1, \ldots, a_p;$ $b_1, \ldots, b_q; ax)$ $p \le q+1$ $\mathrm{Re}\,\mu > 0, \quad \mathrm{Re}\,\nu > 0$	$\dfrac{y^{\mu+\nu-1}\,\Gamma(\nu)}{\Gamma(\mu+\nu)}$ $\times\ {}_{p+1}F_{q+1}(\nu, a_1, \ldots, a_p;$ $\mu+\nu, b_1, \ldots, b_q; ay)$ $\lvert ay\rvert < 1$ if $p = q+1$
(96)	$G_{pq}^{mn}\left(ax \middle\vert \begin{matrix} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{matrix}\right)$ $p \le q, \quad \mathrm{Re}\,\mu > 0$ $\mathrm{Re}\,b_j > -1 \quad j = 1, \ldots, m$	$y^{\mu}\, G_{p+1,\,q+1}^{m,\,n+1}\left(ay \middle\vert \begin{matrix} 0, a_1, \ldots, a_p \\ b_1, \ldots, b_q, -\mu \end{matrix}\right)$ $\lvert ay\rvert < 1$ if $p = q$
(97)	$G_{pq}^{mn}\left(ax \middle\vert \begin{matrix} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{matrix}\right)$ $p+q < 2(m+n), \quad \mathrm{Re}\,\mu > 0$ $\mathrm{Re}\,b_j > -1 \quad j = 1, \ldots, m$	$y^{\mu}\, G_{p+1,\,q+1}^{m,\,n+1}\left(ay \middle\vert \begin{matrix} 0, a_1, \ldots, a_p \\ b_1, \ldots, b_q, -\mu \end{matrix}\right)$ $\lvert \arg ay\rvert < (m+n-\frac{1}{2}p-\frac{1}{2}q)\pi$

13.2. Weyl fractional integrals

	$f(x)$	$[\Gamma(\mu)]^{-1}\int_y^\infty f(x)\,(x-y)^{\mu-1}\,dx = h(y;\mu)$
(1)	$f(ax)$	$a^{-\mu}h(ay;\mu)$
(2)	$f(a/x)$	$a y^{\mu-1}\,\Re_\mu\{t^{-\mu-1}f(t);\,a/y\}$ For tables see sec. 13.1.
(3)	$f'(x)$	$-h(y;\mu-1)$
(4)	$\int_x^\infty f(t)\,dt$	$h(y;\mu+1)$
(5)	$h(x;\nu)$	$h(y;\mu+\nu)$
(6)	$x^{-\lambda}$ $\quad 0<\mathrm{Re}\,\mu<\mathrm{Re}\,\lambda$	$\dfrac{\Gamma(\lambda-\mu)}{\Gamma(\lambda)}y^{\mu-\lambda}$
(7)	$(x+a)^{-\lambda}$ $\quad 0<\mathrm{Re}\,\mu<\mathrm{Re}\,\lambda$	$\dfrac{\Gamma(\lambda-\mu)}{\Gamma(\lambda)}(y+a)^{\mu-\lambda}$ $\quad \lvert\arg(y/a)\rvert<\pi$
(8)	$x^{-\lambda}(x+a)^{\nu}$ $\quad 0<\mathrm{Re}\,\mu<\mathrm{Re}(\lambda-\nu)$	$y^{\mu+\nu-\lambda}\dfrac{\Gamma(\lambda-\mu-\nu)}{\Gamma(\lambda-\nu)}$ $\times\,{}_2F_1(-\nu,\lambda-\mu-\nu;\lambda-\nu;-a/y)$ $\quad \lvert\arg(a/y)\rvert<\pi$ or $\lvert a/y\rvert<1$

Weyl integrals (cont'd)

	$f(x)$	$[\Gamma(\mu)]^{-1} \int_y^\infty f(x)(x-y)^{\mu-1}\,dx$
(9)	$x^{-\lambda}(x^2+a^2)^{\nu}$ $0 < \mathrm{Re}\,\mu < \mathrm{Re}(\lambda - 2\nu)$	$\dfrac{\Gamma(\lambda-\mu-2\nu)}{\Gamma(\lambda-\mu)} y^{\mu-\lambda+2\nu}$ $\times {}_3F_2\left(-\nu, \dfrac{\lambda-\mu}{2}-\nu, \dfrac{1+\lambda-\mu}{2}-\nu; \dfrac{\lambda}{2}-\nu, \dfrac{1+\lambda}{2}-\nu; -\dfrac{a^2}{y^2}\right)$ $\lvert y\rvert > \lvert a\rvert$ or $\mathrm{Re}(a/y) > 0$
(10)	$(x^2-1)^{-\frac{1}{2}}\,[(x+1)^{\frac{1}{2}}-(x-1)^{\frac{1}{2}}]^{2\nu}$ $0 < \mathrm{Re}\,\mu < 1 + \mathrm{Re}\,\nu$	$2^{\nu+\frac{1}{2}}\,\pi^{-\frac{1}{2}}\,e^{(\mu-\frac{1}{2})\pi i}$ $\times (y^2-1)^{\frac{1}{2}\mu-\frac{1}{4}}\,Q^{\frac{1}{2}-\mu}_{\nu-\frac{1}{2}}(y)$ $\lvert\arg(y-1)\rvert < \pi$
(11)	e^{-ax} $\mathrm{Re}\,\mu > 0$	$a^{-\mu}\,e^{-ay}$ $\mathrm{Re}(ay) > 0$
(12)	$x^{\mu-1}\,e^{-ax}$ $\mathrm{Re}\,\mu > 0$	$\pi^{-\frac{1}{2}}(y/a)^{\mu-\frac{1}{2}}\exp(-\tfrac{1}{2}ay)$ $\times K_{\mu-\frac{1}{2}}(\tfrac{1}{2}ay)$ $\mathrm{Re}(ay) > 0$
(13)	$x^{-\lambda}\,e^{-ax}$ $\mathrm{Re}\,\mu > 0$	$a^{\nu-\frac{1}{2}}y^{-\nu-\frac{1}{2}}\,e^{-\frac{1}{2}ay}\,W_{\kappa,\nu}(ay)$ $2\kappa = 1-\lambda-\mu, \quad 2\nu = \lambda-\mu$ $\mathrm{Re}(ay) > 0$
(14)	$x^{-2\mu}\exp(a/x)$ $\mathrm{Re}\,\mu > 0$	$(\pi/y)^{\frac{1}{2}}\,a^{\frac{1}{2}-\mu}\exp\left(\dfrac{a}{2y}\right)$ $\times I_{\mu-\frac{1}{2}}\left(\dfrac{a}{2y}\right)$

Weyl integrals (cont'd)

	$f(x)$	$[\Gamma(\mu)]^{-1} \int_y^\infty f(x)\,(x-y)^{\mu-1}\,dx$
(15)	$x^{-\lambda} \exp(a/x)$ $\quad 0 < \operatorname{Re}\mu < \operatorname{Re}\lambda$	$\dfrac{\Gamma(\lambda-\mu)}{\Gamma(\lambda)}\, y^{\mu-\lambda}\, {}_1F_1(\lambda-\mu;\, \lambda;\, a/y)$
(16)	$\exp(-a x^{1/2})$ $\quad \operatorname{Re}\mu > 0$	$2^{\mu+1/2}\, \pi^{-1/2}\, a^{1/2-\mu}\, y^{1/2\mu+1/4}\, K_{\mu+1/2}(a y^{1/2})$ $\quad \operatorname{Re}(a y^{1/2}) > 0$
(17)	$x^{-1/2} \exp(-a x^{1/2})$ $\quad \operatorname{Re}\mu > 0$	$2^{\mu+1/2}\, \pi^{-1/2}\, a^{1/2-\mu}\, y^{1/2\mu-1/4}\, K_{\mu-1/2}(a y^{1/2})$ $\quad \operatorname{Re}(a y^{1/2}) > 0$
(18)	$x^{-\lambda} \log x$ $\quad 0 < \operatorname{Re}\mu < \operatorname{Re}\lambda$	$\dfrac{\Gamma(\lambda-\mu)}{\Gamma(\lambda)}\, y^{\mu-\lambda}[\log y + \psi(\lambda) - \psi(\lambda-\mu)]$
(19)	$\sin(ax)$ $\quad a > 0, \quad 0 < \operatorname{Re}\mu < 1$	$a^{-\mu} \sin(ay + \tfrac{1}{2}\mu\pi)$
(20)	$x^{\mu-1} \sin(ax)$ $\quad a > 0, \quad 0 < \operatorname{Re}\mu < \tfrac{1}{2}$	$\tfrac{1}{2}\pi^{1/2}(y/a)^{\mu-1/2}[\cos(\tfrac{1}{2}ay)\, J_{1/2-\mu}(\tfrac{1}{2}ay) - \sin(\tfrac{1}{2}ay)\, Y_{1/2-\mu}(\tfrac{1}{2}ay)]$
(21)	$x^{-2\mu} \sin(a/x)$ $\quad \operatorname{Re}\mu > 0$	$\left(\dfrac{\pi}{y}\right)^{1/2} a^{1/2-\mu} \sin\left(\dfrac{a}{2y}\right) \times J_{\mu-1/2}\left(\dfrac{a}{2y}\right)$
(22)	$\sin(a x^{1/2})$ $\quad a > 0, \quad 0 < \operatorname{Re}\mu < \tfrac{1}{2}$	$2^{\mu-1/2}\, \pi^{1/2}\, a^{1/2-\mu}\, y^{1/2\mu+1/4}\, Y_{-1/2-\mu}(a y^{1/2})$
(23)	$x^{-1/2} \sin(a x^{1/2})$ $\quad a > 0, \quad 0 < \operatorname{Re}\mu < 1$	$2^{\mu-1/2}\, \pi^{1/2}\, a^{1/2-\mu}\, y^{1/2\mu-1/4}\, J_{1/2-\mu}(a y^{1/2})$

Weyl integrals (cont'd)

	$f(x)$	$[\Gamma(\mu)]^{-1} \int_y^\infty f(x)(x-y)^{\mu-1}\, dx$
(24)	$\cos(ax) \quad a>0, \quad 0<\operatorname{Re}\mu<1$	$a^{-\mu}\cos(ay+\tfrac{1}{2}\mu\pi)$
(25)	$x^{\mu-1}\cos(ay)$ $a>0, \quad 0<\operatorname{Re}\mu<\tfrac{1}{2}$	$-\tfrac{1}{2}\pi^{1/2}(y/a)^{\mu-1/2}[\sin(\tfrac{1}{2}ay)J_{1/2-\mu}(\tfrac{1}{2}ay)$ $+\cos(\tfrac{1}{2}ay)\,Y_{1/2-\mu}(\tfrac{1}{2}ay)]$
(26)	$x^{-2\mu}\cos(a/x) \qquad \operatorname{Re}\mu>0$	$(\pi/y)^{1/2}\,a^{1/2-\mu}\cos(a/2y)\,J_{\mu-1/2}(a/2y)$
(27)	$\cos(ax^{1/2})$ $a>0, \quad 0<\operatorname{Re}\mu<\tfrac{1}{2}$	$2^{\mu-1/2}\,\pi^{1/2}\,a^{1/2-\mu}\,y^{1/2\mu+1/4}\,J_{-1/2-\mu}(ay^{1/2})$
(28)	$x^{-1/2}\cos(ax^{1/2})$ $a>0, \quad 0<\operatorname{Re}\mu<1$	$-2^{\mu-1/2}\,\pi^{1/2}\,a^{1/2-\mu}\,y^{1/2\mu-1/4}\,Y_{1/2-\mu}(ay^{1/2})$
(29)	$Q_\nu(x) \qquad 0<\operatorname{Re}\mu<1+\operatorname{Re}\nu$	$e^{\mu\pi i}(y^2-1)^{1/2\mu}\,Q_\nu^{-\mu}(y)$ $\lvert\arg(y-1)\rvert<\pi$
(30)	$(x^2-1)^{1/2\lambda}\,Q_\nu^{-\lambda}(x)$ $0<\operatorname{Re}\mu<1+\operatorname{Re}(\nu-\lambda)$	$e^{\mu\pi i}(y^2-1)^{1/2\lambda+1/2\mu}\,Q_\nu^{-\lambda-\mu}(y)$ $\lvert\arg(y-1)\rvert<\pi$
(31)	$x^{-\nu}\,e^{iax}\,J_\nu(ax)$ $a>0, \quad 0<\operatorname{Re}\mu<\tfrac{1}{2}+\operatorname{Re}\nu$	$\dfrac{e^{1/2 i\mu\pi}(2a)^{\nu-\mu}\,\Gamma(\tfrac{1}{2}-\mu+\nu)}{\pi^{1/2}\,\Gamma(1-\mu+2\nu)}$ $\times\,{}_1F_1(\tfrac{1}{2}-\mu+\nu;\,1-\mu+2\nu;\,2aiy)$ $y>0$
(32)	$x^{1/2\nu-1/2}\,J_\nu(ax^{1/2}) \qquad a>0$ $\mu=\nu+\tfrac{1}{2}, \quad -\tfrac{1}{2}<\operatorname{Re}\nu<\tfrac{1}{2}$	$-\dfrac{\pi^{1/2}}{2}\left(\dfrac{2y}{a}\right)^{\nu}\left[J_\nu\left(\dfrac{ay^{1/2}}{2}\right)Y_{-\nu}\left(\dfrac{ay^{1/2}}{2}\right)\right.$ $\left.+\,J_{-\nu}\left(\dfrac{ay^{1/2}}{2}\right)Y_\nu\left(\dfrac{ay^{1/2}}{2}\right)\right] \quad y>0$

Weyl integrals (cont'd)

	$f(x)$	$[\Gamma(\mu)]^{-1} \int_y^\infty f(x)\,(x-y)^{\mu-1}\,dx$
(33)	$x^{\frac{1}{2}\nu-\frac{1}{2}} J_{-\nu}(ax^{\frac{1}{2}})$ $\quad a>0$ $\mu=\nu+\frac{1}{2}, \quad -\frac{1}{2}<\mathrm{Re}\,\nu<\frac{1}{2}$	$-\pi^{\frac{1}{2}}\left(\frac{2y}{a}\right)^{\nu} J_{-\nu}\left(\frac{ay^{\frac{1}{2}}}{2}\right) \times Y_{-\nu}\left(\frac{ay^{\frac{1}{2}}}{2}\right)$ $\quad y>0$
(34)	$x^{-\frac{1}{2}\nu} J_{\nu}(ax^{\frac{1}{2}})$ $\quad a>0$ $0<\mathrm{Re}\,\mu<\frac{1}{2}\mathrm{Re}\,\nu+\frac{3}{4}$	$2^{\mu} a^{-\mu} y^{\frac{1}{2}\mu-\frac{1}{2}\nu} J_{\nu-\mu}(ay^{\frac{1}{2}})$ $\quad y>0$
(35)	$x^{-\frac{1}{2}\nu} J_{-\nu}(ax^{\frac{1}{2}})$ $\quad a>0$ $0<\mathrm{Re}\,\mu<\frac{1}{2}\mathrm{Re}\,\nu+\frac{3}{4}$	$2^{\mu} a^{-\mu} y^{\frac{1}{2}\mu-\frac{1}{2}\nu}[\cos(\nu\pi)\, J_{\nu-\mu}(ay^{\frac{1}{2}}) - \sin(\nu\pi)\, Y_{\nu-\mu}(ay^{\frac{1}{2}})]$ $\quad y>0$
(36)	$x^{\lambda} J_{\nu}(ax^{\frac{1}{2}})$ $\quad a>0$ $0<\mathrm{Re}\,\mu<\frac{1}{4}-\mathrm{Re}\,\lambda$	$2^{2\lambda} a^{-2\lambda} y^{\mu} \times G_{13}^{20}\left(\frac{a^2 y}{4}\middle\vert\begin{matrix} 0 \\ -\mu,\ \lambda+\frac{1}{2}\nu,\ \lambda-\frac{1}{2}\nu \end{matrix}\right)$ $\quad y>0$
(37)	$x^{-\nu}[J_{\nu}(ax^{\frac{1}{2}})]^2$ $\quad a>0$ $\mu=\nu-\frac{1}{2}, \quad \mathrm{Re}\,\nu>\frac{1}{2}$	$\pi^{-\frac{1}{2}} a^{-\nu} y^{-\frac{1}{2}\nu-\frac{1}{2}}\, \mathbf{H}_{\nu}(2ay^{\frac{1}{2}})$ $\quad y>0$
(38)	$x^{\frac{1}{2}\nu-\frac{1}{2}} Y_{\nu}(ax^{\frac{1}{2}})$ $\quad a>0$ $\mu=\nu+\frac{1}{2}, \quad -\frac{1}{2}<\mathrm{Re}\,\nu<\frac{1}{2}$	$\frac{\pi^{\frac{1}{2}}}{2}\left(\frac{2y}{a}\right)^{\nu}\left[J_{\nu}\left(\frac{ay^{\frac{1}{2}}}{2}\right)J_{-\nu}\left(\frac{ay^{\frac{1}{2}}}{2}\right) - Y_{\nu}\left(\frac{ay^{\frac{1}{2}}}{2}\right)Y_{-\nu}\left(\frac{ay^{\frac{1}{2}}}{2}\right)\right]$ $\quad y>0$

Weyl integrals (cont'd)

	$f(x)$	$[\Gamma(\mu)]^{-1} \int_y^\infty f(x)\,(x-y)^{\mu-1}\,dx$
(39)	$x^{\frac{1}{2}\nu}\, Y_\nu(ax^{\frac{1}{2}})$ $\quad a>0$ $0<\operatorname{Re}\mu<\frac{3}{4}-\frac{1}{2}\operatorname{Re}\nu$	$2^\mu a^{-\mu} y^{\frac{1}{2}\mu+\frac{1}{2}\nu}$ $\times[\cos(\nu\pi)\, Y_{-\mu-\nu}(ay^{\frac{1}{2}})$ $-\sin(\nu\pi)\, J_{-\mu-\nu}(ay^{\frac{1}{2}})]$ $\quad y>0$
(40)	$x^{-\frac{1}{2}\nu}\, Y_\nu(ax^{\frac{1}{2}})$ $\quad a>0$ $0<\operatorname{Re}\mu<\frac{1}{2}\operatorname{Re}\nu+\frac{3}{4}$	$2^\mu a^{-\mu} y^{\frac{1}{2}\mu-\frac{1}{2}\nu}\, Y_{\nu-\mu}(ay^{\frac{1}{2}})$ $\quad y>0$
(41)	$x^{-\nu}\, J_\nu(ax^{\frac{1}{2}})\, Y_\nu(ax^{\frac{1}{2}})$ $\quad a>0$ $\mu=\nu-\frac{1}{2},\quad \operatorname{Re}\nu>\frac{1}{2}$	$-\pi^{-\frac{1}{2}} a^{-\nu} y^{-\frac{1}{2}\nu-\frac{1}{2}}\, J_\nu(2ay^{\frac{1}{2}})$
(42)	$x^{-\nu}[Y_\nu(ax^{\frac{1}{2}})]^2$ $\quad a>0$ $\mu=\nu-\frac{1}{2},\quad \operatorname{Re}\nu>\frac{1}{2}$	$\pi^{-\frac{1}{2}} a^{-\nu} y^{-\frac{1}{2}\nu-\frac{1}{2}}[\mathbf{H}_\nu(2ay^{\frac{1}{2}})$ $-2Y_\nu(2ay^{\frac{1}{2}})]$ $\quad y>0$
(43)	$x^{\frac{1}{2}\nu-\frac{1}{2}}\, H_\nu^{(1)}(ax^{\frac{1}{2}})$ $\mu=\nu+\frac{1}{2},\quad \operatorname{Re}\nu>-\frac{1}{2}$	$\dfrac{\pi^{\frac{1}{2}} i}{2}\left(\dfrac{2y}{a}\right)^\nu H_\nu^{(1)}\left(\dfrac{ay^{\frac{1}{2}}}{2}\right)$ $\times H_{-\nu}^{(1)}\left(\dfrac{ay^{\frac{1}{2}}}{2}\right)$ $\quad \operatorname{Im}(ay^{\frac{1}{2}})>0$
(44)	$x^{\frac{1}{2}\nu-\frac{1}{2}}\, H_{-\nu}^{(1)}(ax^{\frac{1}{2}})$ $\mu=\nu+\frac{1}{2},\quad \operatorname{Re}\nu>-\frac{1}{2}$	$\dfrac{\pi^{\frac{1}{2}} i}{2}\left(\dfrac{2y}{a}\right)^\nu \left[H_{-\nu}^{(1)}\left(\dfrac{ay^{\frac{1}{2}}}{2}\right)\right]^2$ $\operatorname{Im}(ay^{\frac{1}{2}})>0$
(45)	$x^{-\frac{1}{2}\nu}\, H_\nu^{(1)}(ax^{\frac{1}{2}})$ $\quad \operatorname{Re}\mu>0$	$2^\mu a^{-\mu} y^{\frac{1}{2}\mu-\frac{1}{2}\nu}\, H_{\nu-\mu}^{(1)}(ay^{\frac{1}{2}})$ $\operatorname{Im}(ay^{\frac{1}{2}})>0$
(46)	$x^{\frac{1}{2}\nu-\frac{1}{2}}\, H_\nu^{(2)}(ax^{\frac{1}{2}})$ $\mu=\nu+\frac{1}{2},\quad \operatorname{Re}\nu>-\frac{1}{2}$	$\dfrac{\pi^{\frac{1}{2}} i}{2}\left(\dfrac{2y}{a}\right)^\nu H_\nu^{(2)}\left(\dfrac{ay^{\frac{1}{2}}}{2}\right)$ $\times H_{-\nu}^{(2)}\left(\dfrac{ay^{\frac{1}{2}}}{2}\right)$ $\quad \operatorname{Im}(ay^{\frac{1}{2}})<0$

Weyl integrals (cont'd)

	$f(x)$	$[\Gamma(\mu)]^{-1}\int_y^\infty f(x)\,(x-y)^{\mu-1}\,dx$
(47)	$x^{\frac{1}{2}\nu-\frac{1}{2}}\,H_{-\nu}^{(2)}(ax^{\frac{1}{2}})$ $\mu=\nu+\frac{1}{2},\quad \mathrm{Re}\,\nu>-\frac{1}{2}$	$\dfrac{\pi^{\frac{1}{2}}i}{2}\left(\dfrac{2y}{a}\right)^{\nu}\left[H_{-\nu}^{(2)}\left(\dfrac{ay^{\frac{1}{2}}}{2}\right)\right]^2$ $\mathrm{Im}\,(ay^{\frac{1}{2}})<0$
(48)	$x^{-\frac{1}{2}\nu}\,H_{\nu}^{(2)}(ax^{\frac{1}{2}})\qquad \mathrm{Re}\,\mu>0$	$2^{\mu}\,a^{-\mu}\,y^{\frac{1}{2}\mu-\frac{1}{2}\nu}\,H_{\nu-\mu}^{(2)}(ay^{\frac{1}{2}})$ $\mathrm{Im}\,(ay^{\frac{1}{2}})<0$
(49)	$x^{-\nu}\,e^{-ax}\,I_{\nu}(ax)$ $0<\mathrm{Re}\,\mu<\frac{1}{2}+\mathrm{Re}\,\nu$	$\dfrac{(2a)^{\nu-\mu}\,\Gamma(\frac{1}{2}-\mu+\nu)}{\pi^{\frac{1}{2}}\,\Gamma(1-\mu+2\nu)}$ $\times\,{}_1F_1(\frac{1}{2}-\mu+\nu;\;1-\mu+2\nu;\;-2ay)$ $\mathrm{Re}\,(ay)>0$
(50)	$x^{-\lambda}\,e^{-ax}\,I_{\nu}(ax)$ $0<\mathrm{Re}\,\mu<\frac{1}{2}+\mathrm{Re}\,\lambda$	$\pi^{-\frac{1}{2}}\,(2a)^{\lambda}y^{\mu}$ $\times\,G_{23}^{21}\left(2ay\,\middle\vert\,\begin{matrix}\frac{1}{2}-\lambda,\,0\\ -\mu,\,\nu-\lambda,\,-\nu-\lambda\end{matrix}\right)$ $\mathrm{Re}\,(ay)>0$
(51)	$x^{-\mu-\frac{1}{2}}\,e^{-ax}\,K_{\nu}(ax)\qquad \mathrm{Re}\,\mu>0$	$\pi^{\frac{1}{2}}\,(2a)^{-\frac{1}{2}}\,y^{-1}\,e^{-ay}\,W_{-\mu,\nu}(2ay)$ $\mathrm{Re}\,(ay)>0$
(52)	$x^{-\nu}\,e^{ax}\,K_{\nu}(ax)$ $0<\mathrm{Re}\,\mu<\frac{1}{2}+\mathrm{Re}\,\nu$	$\dfrac{\pi^{\frac{1}{2}}\,y^{\frac{1}{2}\mu-\nu-\frac{1}{2}}\,\Gamma(\frac{1}{2}-\mu+\nu)}{(2a)^{\frac{1}{2}\mu+\frac{1}{2}}\,\Gamma(\frac{1}{2}+\nu)}$ $\times\,e^{ay}\,W_{\frac{1}{2}\mu,\,\nu-\frac{1}{2}\mu}(2ay)$ $\lvert\arg(ay)\rvert<3\pi/2$

Weyl integrals (cont'd)

	$f(x)$	$[\Gamma(\mu)]^{-1}\int_y^\infty f(x)\,(x-y)^{\mu-1}\,dx$
(53)	$x^{-\nu}e^{-ax}K_\nu(ax)$ $\quad\mathrm{Re}\,\mu>0$	$\pi^{1/2}(2a)^{-1/2\mu-1/2}y^{1/2\mu-\nu-1/2}$ $\times e^{-ay}W_{-1/2\mu,\,\nu-1/2\mu}(2ay)$ $\quad\mathrm{Re}(ay)>0$
(54)	$x^{-\lambda}e^{ax}K_\nu(ax)$ $\quad 0<\mathrm{Re}\,\mu<1/2+\mathrm{Re}\,\lambda$	$\pi^{-1/2}(2a)^\lambda y^\mu\cos(\nu\pi)$ $\times G_{23}^{31}\left(2ay\,\middle\vert\,\begin{matrix}1/2-\lambda,\,0\\-\mu,\,\nu-\lambda,\,-\nu-\lambda\end{matrix}\right)$ $\quad\lvert\arg(ay)\rvert<3\pi/2$
(55)	$x^{-\lambda}e^{-ax}K_\nu(ax)$ $\quad\mathrm{Re}\,\mu>0$	$\pi^{1/2}(2a)^\lambda y^\mu$ $\times G_{23}^{30}\left(2ay\,\middle\vert\,\begin{matrix}0,\,1/2-\lambda\\-\mu,\,\nu-\lambda,\,-\nu-\lambda\end{matrix}\right)$ $\quad\mathrm{Re}(ay)>0$
(56)	$x^{-1/2}K_{2\nu}(ax^{1/2})$ $\quad\mu=1/2$	$\pi^{-1/2}[K_\nu(1/2\,ay^{1/2})]^2$ $\quad\mathrm{Re}(ay^{1/2})>0$
(57)	$x^{1/2\nu-1/2}K_\nu(ax^{1/2})$ $\quad\mu=\nu+1/2,\quad\mathrm{Re}\,\nu>-1/2$	$\pi^{-1/2}\left(\dfrac{2y}{a}\right)^\nu\left[K_\nu\left(\dfrac{ay^{1/2}}{2}\right)\right]^2$ $\quad\mathrm{Re}(ay^{1/2})>0$
(58)	$x^{1/2\nu-1/2}K_\nu(ax^{1/2})$ $\quad\mu=\nu-1/2,\quad\mathrm{Re}\,\nu>1/2$	$\pi^{-1/2}(2/a)^{\nu-1}y^{\nu-1/2}$ $\times K_\nu(1/2\,ay^{1/2})\,K_{\nu-1}(1/2\,ay^{1/2})$ $\quad\mathrm{Re}(ay^{1/2})>0$

Weyl integrals (cont'd)

	$f(x)$	$[\Gamma(\mu)]^{-1} \int_y^\infty f(x)\,(x-y)^{\mu-1}\,dx$
(59)	$x^{-\frac{1}{2}\nu} K_\nu(ax^{\frac{1}{2}})$ $\mathrm{Re}\,\mu > 0$	$2^\mu a^{-\mu} y^{\frac{1}{2}\mu-\frac{1}{2}\nu} K_{\nu-\mu}(ay^{\frac{1}{2}})$ $\mathrm{Re}(ay^{\frac{1}{2}}) > 0$
(60)	$x^{-\lambda} K_\nu(ax^{\frac{1}{2}})$ $\mathrm{Re}\,\mu > 0$	$2^{-2\lambda-1} a^{2\lambda} y^\mu \times G^{30}_{13}\left(\frac{a^2 y}{4}\middle\vert\begin{matrix} 0 \\ -\mu,\ \frac{1}{2}\nu-\lambda,\ -\frac{1}{2}\nu-\lambda\end{matrix}\right)$ $\mathrm{Re}(ay^{\frac{1}{2}}) > 0$
(61)	$x^{-\nu} I_\nu(ax^{\frac{1}{2}}) K_\nu(ax^{\frac{1}{2}})$ $\mu = \nu - \frac{1}{2}$, $\mathrm{Re}\,\nu > \frac{1}{2}$	$\frac{1}{2}\pi^{\frac{1}{2}} a^{-\nu} y^{-\frac{1}{2}\nu-\frac{1}{2}} \times [I_\nu(2ay^{\frac{1}{2}}) - \mathbf{L}_\nu(2ay^{\frac{1}{2}})]$ $\mathrm{Re}(ay^{\frac{1}{2}}) > 0$
(62)	$x^{-\nu} [K_\nu(ax^{\frac{1}{2}})]^2$ $\mu = \nu - \frac{1}{2}$, $\mathrm{Re}\,\nu > \frac{1}{2}$	$\pi^{\frac{1}{2}} a^{-\nu} y^{-\frac{1}{2}\nu-\frac{1}{2}} K_\nu(2ay^{\frac{1}{2}})$ $\mathrm{Re}(ay^{\frac{1}{2}}) > 0$
(63)	$x^{\frac{1}{2}\mu} \mathbf{H}_{-\mu}(ax^{\frac{1}{2}})$ $a > 0$, $0 < \mathrm{Re}\,\mu < \frac{1}{2}$	$\left(\frac{2y}{a}\right)^\mu \left[Y_{-2\mu}(ay^{\frac{1}{2}}) + \frac{2}{\pi} S_{0,\,2\mu}(ay^{\frac{1}{2}})\right]$ $y > 0$
(64)	$x^{\frac{1}{2}\nu-\frac{1}{2}} \mathbf{H}_{-\nu}(ax^{\frac{1}{2}})$ $a > 0$ $\mu = \nu + \frac{1}{2}$, $-\frac{1}{2} < \mathrm{Re}\,\nu < \frac{1}{2}$	$\pi^{\frac{1}{2}} \left(\frac{2y}{a}\right)^\nu \left[J_{-\nu}\left(\frac{ay^{\frac{1}{2}}}{2}\right)\right]^2$ $y > 0$

Weyl integrals (cont'd)

	$f(x)$	$[\Gamma(\mu)]^{-1} \int_y^\infty f(x)\,(x-y)^{\mu-1}\,dx$
(65)	$x^{\frac{1}{2}\nu}\,\mathbf{H}_\nu(ax^{\frac{1}{2}})$ $\quad a>0$ $\operatorname{Re}\mu>0, \quad \operatorname{Re}(\mu+\nu)<\frac{1}{2}$ $\operatorname{Re}(\mu+\frac{1}{2}\nu)<\frac{3}{4}$	$\dfrac{(2/a)^\mu\, y^{\frac{1}{2}\nu+\frac{1}{2}\mu}}{\cos[(\mu+\nu)\pi]}\,[\cos(\nu\pi)\,\mathbf{H}_{\mu+\nu}(ay^{\frac{1}{2}})$ $+\sin(\mu\pi)\,J_{-\mu-\nu}(ay^{\frac{1}{2}})] \quad y>0$
(66)	$x^{\frac{1}{2}\mu}[\mathbf{H}_{-\mu}(ax^{\frac{1}{2}})-Y_{-\mu}(ax^{\frac{1}{2}})]$ $0<\operatorname{Re}\mu<\frac{1}{2}$	$\dfrac{2}{\pi}\left(\dfrac{2y}{a}\right)^\mu S_{0,2\mu}(ay^{\frac{1}{2}})$ $\lvert\arg(ay^{\frac{1}{2}})\rvert<\pi$
(67)	$x^{\frac{1}{2}\nu-\frac{1}{2}}[\mathbf{H}_{-\nu}(ax^{\frac{1}{2}})-Y_{-\nu}(ax^{\frac{1}{2}})]$ $\mu=\nu+\frac{1}{2}, \quad -\frac{1}{2}<\operatorname{Re}\nu<\frac{1}{2}$	$\dfrac{\pi^{\frac{1}{2}}}{2}\left(\dfrac{2y}{a}\right)^\nu \left\{\left[J_\nu\left(\dfrac{ay^{\frac{1}{2}}}{2}\right)\right]^2\right.$ $\left.+\left[Y_\nu\left(\dfrac{ay^{\frac{1}{2}}}{2}\right)\right]^2\right\}$ $\lvert\arg(ay^{\frac{1}{2}})\rvert<\pi$
(68)	$x^{\frac{1}{2}\nu}[\mathbf{H}_\nu(ax^{\frac{1}{2}})-Y_\nu(ax^{\frac{1}{2}})]$ $0<\operatorname{Re}\mu<\frac{1}{2}-\operatorname{Re}\nu$	$\dfrac{(2/a)^\mu\cos(\nu\pi)}{\cos[(\mu+\nu)\pi]}\,y^{\frac{1}{2}\nu+\frac{1}{2}\mu}$ $\times[\mathbf{H}_{\mu+\nu}(ay^{\frac{1}{2}})-Y_{\mu+\nu}(ay^{\frac{1}{2}})]$ $\lvert\arg(ay^{\frac{1}{2}})\rvert<\pi$
(69)	$x^{\frac{1}{2}\nu-\frac{1}{2}}[I_{-\nu}(ax^{\frac{1}{2}})-\mathbf{L}_{-\nu}(ax^{\frac{1}{2}})]$ $\mu=\nu+\frac{1}{2}, \quad -\frac{1}{2}<\operatorname{Re}\nu<\frac{1}{2}$	$\dfrac{2}{\pi^{\frac{1}{2}}}\left(\dfrac{2y}{a}\right)^\nu I_{-\nu}\left(\dfrac{ay^{\frac{1}{2}}}{2}\right)$ $\times K_\nu\left(\dfrac{ay^{\frac{1}{2}}}{2}\right) \quad \operatorname{Re}(ay^{\frac{1}{2}})>0$

Weyl integrals (cont'd)

	$f(x)$	$[\Gamma(\mu)]^{-1} \int_y^\infty f(x)\,(x-y)^{\mu-1}\,dx$
(70)	$x^{\frac{1}{2}\nu}[I_{-\nu}(ax^{\frac{1}{2}}) - \mathbf{L}_\nu(ax^{\frac{1}{2}})]$ $0 < \operatorname{Re}\mu < \frac{1}{2} - \operatorname{Re}\nu$	$\dfrac{\cos(\nu\pi)}{\cos[(\mu+\nu)\pi]}\left(\dfrac{2}{a}\right)^{\mu} y^{\frac{1}{2}\mu+\frac{1}{2}\nu}$ $\times [I_{-\mu-\nu}(ay^{\frac{1}{2}}) - \mathbf{L}_{\mu+\nu}(ay^{\frac{1}{2}})]$ $\operatorname{Re}(ay^{\frac{1}{2}}) > 0$
(71)	$x^{\frac{1}{2}\nu} S_{\lambda,\nu}(ax^{\frac{1}{2}})$ $0 < 2\operatorname{Re}\mu < 1 - \operatorname{Re}(\lambda+\nu)$	$\dfrac{\Gamma(\frac{1}{2} - \frac{1}{2}\lambda - \mu - \frac{1}{2}\nu)}{\Gamma(\frac{1}{2} - \frac{1}{2}\lambda - \frac{1}{2}\nu)}\, a^{-\mu} y^{\frac{1}{2}\mu+\frac{1}{2}\nu}$ $\times S_{\lambda+\mu,\,\mu+\nu}(ay^{\frac{1}{2}})$ $\lvert\arg(ay^{\frac{1}{2}})\rvert < \pi$
(72)	$x^{\lambda-\frac{1}{2}}\, e^{\frac{1}{2}ax}\, W_{\kappa,\lambda}(ax)$ $0 < \operatorname{Re}\mu < \frac{1}{2} - \operatorname{Re}(\kappa+\lambda)$	$\dfrac{\Gamma(\frac{1}{2} - \kappa - \lambda - \mu)}{\Gamma(\frac{1}{2} - \kappa - \lambda)}\, a^{-\frac{1}{2}\mu} y^{\frac{1}{2}\mu+\lambda-\frac{1}{2}}$ $\times e^{\frac{1}{2}ay}\, W_{\kappa+\frac{1}{2}\mu,\,\lambda+\frac{1}{2}\mu}(ay)$ $\lvert\arg(ay)\rvert < 3\pi/2$
(73)	$x^{\kappa-\mu-1}\, e^{-\frac{1}{2}ax}\, W_{\kappa,\lambda}(ax)$ $\operatorname{Re}\mu > 0$	$y^{\kappa-1}\, e^{-\frac{1}{2}ay}\, W_{\kappa-\mu,\lambda}(ay)$ $\operatorname{Re}(ay) > 0$
(74)	$x^{\lambda-\frac{1}{2}}\, e^{-\frac{1}{2}ax}\, W_{\kappa,\lambda}(ax)$ $\operatorname{Re}\mu > 0$	$a^{-\frac{1}{2}\mu} y^{\frac{1}{2}\mu+\lambda-\frac{1}{2}}\, e^{-\frac{1}{2}ay}$ $\times W_{\kappa-\frac{1}{2}\mu,\,\lambda-\frac{1}{2}\mu}(ay) \quad \operatorname{Re}(ay) > 0$

Weyl integrals (cont'd)

	$f(x)$	$[\Gamma(\mu)]^{-1} \int_y^\infty f(x)\,(x-y)^{\mu-1}\,dx$
(75)	$x^{-\rho} e^{\frac{1}{2}ax} W_{\kappa,\lambda}(ax)$ $0 < \mathrm{Re}\,\mu < \mathrm{Re}(\rho-\kappa)$	$\dfrac{y^{\mu-\rho}}{\Gamma(\frac{1}{2}+\lambda-\kappa)\,\Gamma(\frac{1}{2}-\lambda-\kappa)}$ $\times G^{31}_{23}\left(ay \middle\vert \begin{matrix} 1+\kappa,\ \rho \\ \rho-\mu,\ \frac{1}{2}+\lambda,\ \frac{1}{2}-\lambda \end{matrix}\right)$ $\lvert\arg(ay)\rvert < 3\pi/2$
(76)	$x^{-\rho} e^{-\frac{1}{2}ax} W_{\kappa,\lambda}(ax)$ $\mathrm{Re}\,\mu > 0$	$y^{\mu-\rho} G^{30}_{23}\left(ay \middle\vert \begin{matrix} \rho,\ 1-\kappa \\ \rho-\mu,\ \frac{1}{2}+\lambda,\ \frac{1}{2}-\lambda \end{matrix}\right)$ $\mathrm{Re}(ay) > 0$
(77)	$x^{-\lambda}\,{}_pF_q(a_1, \dots, a_p;$ $b_1, \dots, b_q; -a/x)$ $0 < \mathrm{Re}\,\mu < \mathrm{Re}\,\lambda, \quad p \le q+1$	$\dfrac{\Gamma(\lambda-\mu)}{\Gamma(\lambda)} y^{\mu-\lambda}$ $\times\ {}_{p+1}F_{q+1}(\lambda-\mu, a_1, \dots, a_p;$ $\lambda, b_1, \dots, b_q; -a/y)$ $\lvert y\rvert > \lvert a\rvert$ or $\lvert\arg(a/y)\rvert < \pi$ if $p = q+1$
(78)	$G^{mn}_{pq}\left(ax \middle\vert \begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix}\right) \quad p \ge q$ $0 < \mathrm{Re}\,\mu < 1 - \mathrm{Re}\,a_j$ $j = 1, \dots, n$	$y^\mu G^{m+1,n}_{p+1,q+1}\left(ay \middle\vert \begin{matrix} a_1, \dots, a_p, 0 \\ -\mu, b_1, \dots, b_q \end{matrix}\right)$ $\lvert ay\rvert > 1$ if $p = q$
(79)	$G^{mn}_{pq}\left(ax \middle\vert \begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix}\right)$ $p+q < 2(m+n)$ $0 < \mathrm{Re}\,\mu < 1 - \mathrm{Re}\,a_j$ $j = 1, \dots, n$	$y^\mu G^{m+1,n}_{p+1,q+1}\left(ay \middle\vert \begin{matrix} a_1, \dots, a_p, 0 \\ -\mu, b_1, \dots, b_q \end{matrix}\right)$ $\lvert\arg(ay)\rvert < (m+n-\frac{1}{2}p-\frac{1}{2}q)\pi$

CHAPTER XIV

STIELTJES TRANSFORMS

We call

$$g(y) = \mathfrak{S}\{f(x); y\} = \int_0^\infty f(x)\,(x+y)^{-1}\,dx$$

the *Stieltjes transform* of $f(x)$. Here integration is over the positive real x-axis, and y is a complex variable ranging over the complex y-plane cut along the negative real axis.

Stieltjes transforms are iterated Laplace transforms,

$$\mathfrak{S}\{f(x); y\} = \mathfrak{L}\{\mathfrak{L}\{f(x); t\}; y\}$$

and accordingly, information about Stieltjes transforms is found in works on Laplace transforms, in particular in Widder (1941, Chapter VIII) and Titchmarsh (1937, sections 11.8, 11.9). Stieltjes transforms are also connected with the moment problem for the semi-infinite interval (Shohat and Tamarkin, 1943) and hence with certain continued fractions.

We also give a brief list of *generalized Stieltjes transforms* of order ρ

$$g(y; \rho) = \mathfrak{S}_\rho\{f(x); y\} = \int_0^\infty f(x)\,(x+y)^{-\rho}\,dx$$

where x and y are as before, and ρ is a complex parameter. For the theory see Widder (1941, Chapter VIII). Generalized Stieltjes transforms of different orders are connected with each other, and with Stieltjes transforms, by fractional integration according to the formulas

$$\Gamma(\rho)\,\mathfrak{W}_{\rho-1}\,\mathfrak{S}_\rho = \mathfrak{S}$$

$$\Gamma(\rho)\,\mathfrak{W}_\mu\,\mathfrak{S}_\rho = \Gamma(\rho-\mu)\,\mathfrak{S}_{\rho-\mu}.$$

From the transform pairs given in the tables, further integrals may be derived by the methods mentioned in the introduction to vol. I, by the general formulas given in the tables, and by using the above formulas in connection with tables of Laplace transforms and fractional integrals.

REFERENCES

Shohat, J.A. and J.D. Tamarkin, 1943: *The problem of moments*. Amer. Math. Soc. New York.

Titchmarsh, E.C., 1937: *Introduction to the theory of Fourier integrals*. Oxford.

Widder, D.V., 1941: *The Laplace transform*. Princeton University Press, Princeton, N.J.

STIELTJES TRANSFORMS

14.1. General formulas

	$f(x)$	$\int_0^\infty f(x)\,(x+y)^{-1}\,dx \qquad \lvert\arg y\rvert < \pi$
(1)	$f(x)$	$g(y)$
(2)	$x f(x)$	$\int_0^\infty f(x)\,dx - y\,g(y)$
(3)	$(x+a)^{-1} f(x) \qquad \lvert\arg a\rvert < \pi$	$(y-a)^{-1}[g(a) - g(y)]$
(4)	$\dfrac{f(x)-f(a)}{x-a} \qquad a > 0$	$(y+a)^{-1}[\tfrac{1}{2} g(ae^{i\pi}) + \tfrac{1}{2} g(ae^{-i\pi})$ $- f(a)\log(y/a) - g(y)]$
(5)	$g(xe^{i\pi}) - g(xe^{-i\pi})$	$2\pi i\, g(y)$
(6)	$f(ax) \qquad a > 0$	$g(ay)$
(7)	$x^{-1} f(a/x) \qquad a > 0$	$y^{-1} g(a/y)$
(8)	$f(x^{1/2})$	$g(iy^{1/2}) + g(-iy^{1/2})$
(9)	$f'(x)$	$-y^{-1} f(0) - g'(y)$

14.2. Elementary functions

	$f(x)$	$\int_0^\infty f(x)\,(x+y)^{-1}\,dx \qquad \lvert\arg y\rvert < \pi$
(1)	$-1 \qquad 2n < x < 2n+1$ $1 \qquad 2n+1 < x < 2n+2$ $n = 0, 1, 2, \ldots$	$\log\{\tfrac{1}{2}y\,[\Gamma(\tfrac{1}{2}y)/\Gamma(\tfrac{1}{2}y+\tfrac{1}{2})]^2\}$
(2)	$(a+x)^{-1} \qquad \lvert\arg a\rvert < \pi$	$(a-y)^{-1}\log(a/y)$
(3)	$\dfrac{1}{a^2+x^2} \qquad \mathrm{Re}\, a > 0$	$\dfrac{1}{a^2+y^2}\left[\dfrac{\pi y}{2a} - \log\left(\dfrac{y}{a}\right)\right]$
(4)	$\dfrac{x}{a^2+x^2} \qquad \mathrm{Re}\, a > 0$	$\dfrac{1}{a^2+y^2}\left[\dfrac{\pi a}{2} + y\log\left(\dfrac{y}{a}\right)\right]$
(5)	$x^\nu \qquad -1 < \mathrm{Re}\,\nu < 0$	$-\pi\, y^\nu \csc(\pi\nu)$
(6)	$\dfrac{x^\nu}{a+x}$ $\lvert\arg a\rvert < \pi, \quad -1 < \mathrm{Re}\,\nu < 1$	$\dfrac{\pi(a^\nu - y^\nu)}{(a-y)\sin(\nu\pi)}$
(7)	$\dfrac{x^\nu}{a^2+x^2}$ $\mathrm{Re}\, a > 0, \quad -1 < \mathrm{Re}\,\nu < 2$	$\dfrac{\pi}{a^2+y^2}\left[\dfrac{a^{\nu-1}y}{2\cos(\tfrac{1}{2}\nu\pi)} + \dfrac{a^\nu}{2\sin(\tfrac{1}{2}\nu\pi)} - \dfrac{y^\nu}{\sin(\nu\pi)}\right]$
(8)	$\dfrac{x^\nu - a^\nu}{x-a} \qquad -1 < \mathrm{Re}\,\nu < 1$	$\dfrac{\pi}{a+y}\left[\dfrac{y^\nu}{\sin(\nu\pi)} - a^\nu \operatorname{ctn}(\nu\pi) + \dfrac{a^\nu}{\pi}\log\left(\dfrac{a}{y}\right)\right]$

Elementary functions (cont'd)

	$f(x)$	$\int_0^\infty f(x)\,(x+y)^{-1}\,dx \qquad \lvert\arg y\rvert < \pi$
(9)	$x^{\nu-1}(a+x)^{1-\mu}$ $\lvert\arg a\rvert < \pi, \quad 0 < \operatorname{Re}\nu < \operatorname{Re}\mu$	$\dfrac{\Gamma(\nu)\,\Gamma(\mu-\nu)\,y^{\nu-1}}{\Gamma(\mu)\,a^{\mu-1}}$ $\times\ {}_2F_1(\mu-1,\ \nu;\ \mu;\ 1-y/a)$
(10)	$x^{-\rho}\,(a+x)^{-\sigma} \qquad \lvert\arg a\rvert < \pi$ $-\operatorname{Re}\sigma < \operatorname{Re}\rho < 1$	$\pi\csc(\rho\pi)\,y^{-\rho}\,(a-y)^{-\sigma}\,I_{1-y/a}(\sigma,\ \rho)$
(11)	$e^{-ax} \qquad \operatorname{Re}a > 0$	$-\,e^{ay}\operatorname{Ei}(-ay)$
(12)	$e^{-ax} \qquad 0 < x < b$ $0 \qquad b < x < \infty$	$e^{ay}\,[\operatorname{Ei}(-ab-ay) - \operatorname{Ei}(-ay)]$
(13)	$0 \qquad 0 < x < b$ $e^{-ax} \qquad b < x < \infty$ $\operatorname{Re}a > 0$	$-\,e^{ay}\ \operatorname{Ei}(-ab-ay)$
(14)	$x^n\,e^{-ax} \qquad \operatorname{Re}a > 0$	$(-1)^{n+1}\,y^n\,e^{ay}\operatorname{Ei}(-ay)$ $+\ \sum_{r=1}^{n}(-1)^{n-r}(r-1)!\,a^{-r}y^{n-r}$
(15)	$x^{-1/2}\,e^{-ax} \qquad \operatorname{Re}a > 0$	$\pi y^{-1/2}\,e^{ay}\operatorname{Erfc}(a^{1/2}\,y^{1/2})$
(16)	$x^{1/2}\,e^{-ax} \qquad \operatorname{Re}a > 0$	$\pi^{1/2}\,a^{-1/2} - \pi y^{1/2}\,e^{ay}\operatorname{Erfc}(a^{1/2}\,y^{1/2})$
(17)	$x^{-\nu}e^{-ax}$ $\operatorname{Re}a > 0, \quad \operatorname{Re}\nu < 1$	$\Gamma(1-\nu)\,y^{-\nu}\,e^{ay}\,\Gamma(\nu,\ ay)$
(18)	$x^{-1}(1-e^{-ax}) \qquad \operatorname{Re}a > 0$	$y^{-1}\,[\log(a\gamma y) - e^{ay}\operatorname{Ei}(-ay)]$

Elementary functions (cont'd)

	$f(x)$	$\int_0^\infty f(x)\,(x+y)^{-1}\,dx \qquad \lvert\arg y\rvert < \pi$
(19)	$x^{\nu-1}\,e^{-a/x}$ $\mathrm{Re}\,a > 0, \quad \mathrm{Re}\,\nu < 1$	$\Gamma(1-\nu)\,y^{\nu-1}\,e^{a/y}\,\Gamma(\nu;\,a/y)$
(20)	$\exp(-ax^{1/2}) \qquad \mathrm{Re}\,a > 0$	$2\cos(ay^{1/2})\,\mathrm{ci}(ay^{1/2})$ $-\,2\sin(ay^{1/2})\,\mathrm{si}(ay^{1/2})$
(21)	$x^{-1/2}\exp(-ax^{1/2}) \qquad \mathrm{Re}\,a > 0$	$-\,2y^{-1/2}[\sin(ay^{1/2})\,\mathrm{ci}(ay^{1/2})$ $+\cos(ay^{1/2})\,\mathrm{si}(ay^{1/2})]$
(22)	$x^{\lambda}\exp(-ax^{1/2})$ $\mathrm{Re}\,a > 0, \quad \mathrm{Re}\,\lambda > -1$	$\Gamma(2\lambda+1)\,y^{\lambda}\,[\exp(iay^{1/2}+\lambda\pi i)$ $\times\,\Gamma(-2\lambda,\,iay^{1/2})$ $+\exp(-iay^{1/2}-\lambda\pi i)\,\Gamma(-2\lambda,\,-iay^{1/2})]$
(23)	$[\exp(ax^{1/2})-1]^{-1} \qquad \mathrm{Re}\,a > 0$	$\log(ay^{1/2}) - (2ay^{1/2})^{-1} - \psi(ay^{1/2})$
(24)	$(a+x)^{-1}\log x \qquad \lvert\arg a\rvert < \pi$	$\tfrac{1}{2}(y-a)^{-1}[(\log y)^2 - (\log a)^2]$
(25)	$(a+x)^{-1}\log(x/a)$ $\lvert\arg a\rvert < \pi$	$\dfrac{1}{2(y-a)}\left[\log\left(\dfrac{y}{a}\right)\right]^2$
(26)	$(x-a)^{-1}\log(x/a) \qquad a > 0$	$\tfrac{1}{2}(y+a)^{-1}\{\pi^2 + [\log\,(y/a)]^2\}$
(27)	$x^{-1/2}\log(ax+\beta)$ $\mathrm{Re}\,a > 0, \quad \mathrm{Re}\,\beta > 0$	$2\pi y^{-1/2}\log(a^{1/2}y^{1/2}+\beta^{1/2})$
(28)	$x^{\nu}\log x \qquad -1 < \mathrm{Re}\,\nu < 0$	$-\pi\,y^{\nu}\csc(\nu\pi)\,[\log y - \pi\,\mathrm{ctn}\,(\nu\pi)]$

Elementary functions (cont'd)

	$f(x)$	$\int_0^\infty f(x)(x+y)^{-1}\,dx \qquad \lvert\arg y\rvert < \pi$
(29)	$x^\nu (a+x)^{-1} \log x$ $\lvert\arg a\rvert < \pi, \quad -1 < \operatorname{Re} \nu < 1$	$-\pi \csc(\nu\pi)(y-a)^{-1}[a^\nu \log a$ $-y^\nu \log y - \pi \operatorname{ctn}(\nu\pi)(a^\nu - y^\nu)]$
(30)	$x^\nu (a+x)^{-1} \log(x/a)$ $\lvert\arg a\rvert < \pi, \quad -1 < \operatorname{Re} \nu < 1$	$\pi \csc(\nu\pi)(y-a)^{-1}[y^\nu \log(y/a)$ $+\pi \operatorname{ctn}(\nu\pi)(a^\nu - y^\nu)]$
(31)	$\sin(ax) \qquad a > 0$	$-\sin(ay)\operatorname{ci}(ay) - \cos(ay)\operatorname{si}(ay)$
(32)	$x^{1/2} \sin(ax) \qquad a > 0$	$\pi y^{1/2}[\sin(ay) - 2^{1/2}\sin(ay+\frac{1}{4}\pi)C(ay)$ $+2^{1/2}\cos(ay+\frac{1}{4}\pi)S(ay)]$ $-2^{-1/2}\pi^{1/2}a^{-1/2}$
(33)	$x^{-1/2} \sin(ax) \qquad a > 0$	$\pi y^{-1/2}[2^{1/2}\sin(ay+\frac{1}{4}\pi)C(ay)$ $-2^{1/2}\cos(ay+\frac{1}{4}\pi)S(ay) - \sin(ay)]$
(34)	$x^{-\nu} \sin(ax)$ $a > 0, \quad -1 < \operatorname{Re} \nu < 2$	$\frac{1}{2} i\,\Gamma(1-\nu) y^{-\nu}[e^{-iay}\Gamma(\nu, -iay)$ $-e^{iay}\Gamma(\nu, iay)]$
(35)	$0 \qquad 0 < x < a$ $(x^2-a^2)^{-1/2}\sin(bx) \qquad a < x < \infty$	See Erdélyi, Arthur, 1939: *Proc. Edinburgh Math. Soc.* (2) 6, 94-104.
(36)	$\sin(ax^{1/2}) \qquad a > 0$	$\pi \exp(-ay^{1/2})$
(37)	$x^{-1}\sin(ax^{1/2}) \qquad a > 0$	$\pi y^{-1}[1 - \exp(-ay^{1/2})]$
(38)	$x^{-1/2}\sin(ax^{1/2}) \qquad a > 0$	$y^{-1/2}[\exp(-ay^{1/2})\overline{\operatorname{Ei}}(ay^{1/2})$ $-\exp(ay^{1/2})\operatorname{Ei}(-ay^{1/2})]$

Elementary functions (cont'd)

	$f(x)$	$\int_0^\infty f(x)(x+y)^{-1}\,dx \qquad \lvert\arg y\rvert < \pi$
(39)	$x^{\lambda}\sin(ax^{1/2})$ $a>0, \quad -3/2<\operatorname{Re}\lambda<1/2$	$-\pi y^{\lambda}\sec(\lambda\pi)\sinh(ay^{1/2})$ $-a^{-2\lambda}\Gamma(2\lambda)\sin(\lambda\pi)$ $\times[{}_1F_1(1;1-2\lambda;ay^{1/2})$ $+{}_1F_1(1;1-2\lambda;-ay^{1/2})]$
(40)	$(x+\beta)^{-1}\sin(ax^{1/2})$ $a>0, \quad \lvert\arg\beta\rvert<\pi$	$\pi(y-\beta)^{-1}[\exp(-a\beta^{1/2})-\exp(-ay^{1/2})]$
(41)	$x^{-\beta}\sin(ax^{1/2}+\beta\pi)$ $a>0, \quad -1/2<\operatorname{Re}\beta<1$	$\pi y^{-\beta}\exp(-ay^{1/2})$
(42)	$\sin(ax^{1/2}-bx^{-1/2}) \qquad a,\, b>0$	$\pi\exp(-ay^{1/2}-by^{-1/2})$
(43)	$x^{-1/2}[\sin(ax^{1/2})]^2 \qquad a>0$	$1/2\,\pi y^{-1/2}[1-\exp(-2ay^{1/2})]$
(44)	$x^{-1/2}\sin(ax^{1/2})\sin(bx^{1/2})$ $a>0, \quad b>0$	$1/2\,\pi y^{-1/2}\{\exp(-\lvert a-b\rvert y^{1/2})$ $-\exp[-(a+b)y^{1/2}]\}$
(45)	$x^{-1/2}\sin(ax^{1/2})\sin(bx^{1/2})$ $a\geq b>0$	$\pi y^{-1/2}\exp(-ay^{1/2})\sinh(by^{1/2})$
(46)	$\log(\beta x)\sin(ax^{1/2})$ $a>0, \quad \lvert\arg\beta\rvert<\pi$	$\pi[\log(\beta y)\exp(-ay^{1/2})$ $-\exp(ay^{1/2})\operatorname{Ei}(-ay^{1/2})$ $-\exp(-ay^{1/2})\operatorname{Ei}(ay^{1/2})]$
(47)	$x^{-1/2}\log\lvert\sin(ax^{1/2})\rvert \qquad a>0$	$\pi y^{-1/2}\log[1/2-1/2\exp(-2ay^{1/2})]$

Elementary functions (cont'd)

	$f(x)$	$\int_0^\infty f(x)\,(x+y)^{-1}\,dx \qquad \lvert\arg y\rvert<\pi$
(48)	$\cos(ax) \qquad a>0$	$\cos(ay)\,\mathrm{ci}(ay)-\sin(ay)\,\mathrm{si}(ay)$
(49)	$x^{-1}[\cos(bx)-\cos(ax)]$ $a,\ b>0$	$y^{-1}[-\mathrm{ci}(by)\cos(by)+\mathrm{si}(by)\sin(by)$ $+\mathrm{ci}(ay)\cos(ay)-\mathrm{si}(ay)\sin(ay)$ $+\log(ab^{-1})]$
(50)	$x^{1/2}\cos(ax) \qquad a>0$	$2^{-1/2}\pi^{1/2}a^{-1/2}-\pi y^{1/2}[\cos(ay)$ $-2^{1/2}\cos(ay+\tfrac14\pi)\,C(ay)$ $-2^{1/2}\sin(ay+\tfrac14\pi)\,S(ay)]$
(51)	$x^{-1/2}\cos(ax) \qquad a>0$	$\pi y^{-1/2}[\cos(ay)-2^{1/2}\cos(ay+\tfrac14\pi)\,C(ay)$ $-2^{1/2}\sin(ay+\tfrac14\pi)\,S(ay)]$
(52)	$x^{-\nu}\cos(ax)$ $a>0, \quad -1<\mathrm{Re}\,\nu<1$	$\tfrac12\Gamma(1-\nu)\,y^{-\nu}[e^{iay}\,\Gamma(\nu,\,iay)$ $+e^{-iay}\,\Gamma(\nu,\,-iay)]$
(53)	$0 \qquad 0<x<a$ $(x^2-a^2)^{-1/2}\cos(bx) \quad a<x<\infty$	See Erdélyi, Arthur, 1939: *Proc. Edinburgh Math. Soc.* (2) 6, 94-104.
(54)	$\cos(ax^{1/2}) \qquad a>0$	$-\exp(-ay^{1/2})\,\overline{\mathrm{Ei}}(ay^{1/2})$ $-\exp(ay^{1/2})\,\mathrm{Ei}(-ay^{1/2})$
(55)	$x^{-1/2}\cos(ax^{1/2}) \qquad a>0$	$\pi y^{-1/2}\exp(-ay^{1/2})$
(56)	$x^{\lambda}\cos(ax^{1/2})$ $a>0, \quad -1<\mathrm{Re}\,\lambda<\tfrac12$	$-\pi y^{\lambda}\csc(\lambda\pi)\cosh(ay^{1/2})$ $-a^{-2\lambda}\cos(\lambda\pi)\,\Gamma(2\lambda)$ $\times[{}_1F_1(1;\,1-2\lambda;\,ay^{1/2})$ $+\,{}_1F_1(1;\,1-2\lambda;\,-ay^{1/2})]$

Elementary functions (cont'd)

	$f(x)$	$\int_0^\infty f(x)\,(x+y)^{-1}\,dx \qquad \lvert\arg y\rvert < \pi$
(57)	$x^{-1/2}(x+\beta)^{-1}\cos(ax^{1/2})$ $a>0, \quad \lvert\arg\beta\rvert<\pi$	$\pi(y-\beta)^{-1}[\beta^{-1/2}\exp(-a\beta^{1/2})$ $-y^{-1/2}\exp(-ay^{1/2})]$
(58)	$x^{-1/2}\cos(ax^{1/2}-bx^{-1/2})$ $a,\ b>0$	$\pi y^{-1/2}\exp(-ay^{1/2}-by^{-1/2})$
(59)	$x^{-3/2}[\cos(ax^{1/2})-\cos(bx^{1/2})]$ $a>0, \quad b>0$	$\pi y^{-3/2}[(b-a)y^{1/2}$ $+\exp(-by^{1/2})-\exp(-ay^{1/2})]$
(60)	$x^{-1/2}[\cos(ax^{1/2})]^2 \qquad a>0$	$\tfrac{1}{2}\pi y^{-1/2}[1-\exp(-2ay^{1/2})]$
(61)	$x^{-1/2}\cos(ax^{1/2})\cos(bx^{1/2})$ $a>0, \quad b>0$	$\tfrac{1}{2}\pi y^{-1/2}\{\exp(-\lvert a-b\rvert y^{1/2})$ $+\exp[-(a+b)y^{1/2}]\}$
(62)	$x^{-1/2}\cos(ax^{1/2})\cos(bx^{1/2})$ $a\geq b>0$	$\pi y^{-1/2}\exp(-ay^{1/2})\cosh(by^{1/2})$
(63)	$\dfrac{x^{-1/2}}{[\beta\sin(ax^{1/2})]^2+[\gamma\cos(ax^{1/2})]^2}$ $\lvert\arg(\beta/\gamma)\rvert<\pi$	$\dfrac{\pi y^{-1/2}[\tfrac{1}{2}(\beta/\gamma-\gamma/\beta)\sinh(2ay^{1/2})-1]}{[\beta\sinh(ay^{1/2})]^2-[\gamma\cosh(ay^{1/2})]^2}$
(64)	$\dfrac{\sin(2ax^{1/2})}{[\beta\sin(ax^{1/2})]^2+[\gamma\cos(ax^{1/2})]^2}$ $\lvert\arg(\beta/\gamma)\rvert<\pi$	$\dfrac{\pi[(\beta-\gamma)/(\beta+\gamma)-\exp(-2ay^{1/2})]}{[\beta\sinh(ay^{1/2})]^2-[\gamma\cosh(ay^{1/2})]^2}$
(65)	$x^{-1/2}\log(\beta x)\cos(ax^{1/2})$ $a>0, \quad \lvert\arg\beta\rvert<\pi$	$\pi y^{-1/2}[\log(\beta y)\exp(-ay^{1/2})$ $+\exp(ay^{1/2})\,\mathrm{Ei}(-ay^{1/2})$ $-\exp(-ay^{1/2})\,\overline{\mathrm{Ei}}(ay^{1/2})]$

Elementary functions (cont'd)

	$f(x)$	$\int_0^\infty f(x)(x+y)^{-1}\,dx \qquad \lvert\arg y\rvert < \pi$
(66)	$x^{-1/2} \log\lvert\cos(ax^{1/2})\rvert \qquad a>0$	$\pi y^{-1/2} \log[\tfrac{1}{2}+\tfrac{1}{2}\exp(-2ay^{1/2})]$
(67)	$x^{-1/2} \log[1+2\beta\cos(ax^{1/2})+\beta^2]$ $\quad a>0, \quad \lvert\beta\rvert<1$	$2\pi y^{-1/2} \log[1+\beta\exp(-ay^{1/2})]$
(68)	$x^{-1/2} \log\{[b\sin(ax^{1/2})]^2 + [c\cos(ax^{1/2})]^2\} \quad a,\, b,\, c>0$	$2\pi y^{-1/2} \log[b\sinh(ay^{1/2}) + c\cosh(ay^{1/2})] - 2\pi a$
(69)	$[\cos(ax^{1/2})]^n \sin(nax^{1/2})$ $\quad a>0, \quad n=1, 2, \ldots$	$2^{-n}\pi\{[1+\exp(-2ay^{1/2})]^n - 1\}$
(70)	$x^{-1/2} \log\lvert\tan(ax^{1/2})\rvert \qquad a>0$	$\pi y^{-1/2} \log[\tanh(ay^{1/2})]$
(71)	$x^{-1/2} \log\{1+[b\tan(ax^{1/2})]^2\}$ $\quad a,\, b>0$	$2\pi y^{-1/2} \log[1+b\tanh(ay^{1/2})]$
(72)	$x^{-1/2} \log\{1+[b\,\mathrm{ctn}(ax^{1/2})]^2\}$ $\quad a,\, b>0$	$2\pi y^{-1/2} \log[1+b\,\mathrm{ctnh}(ay^{1/2})]$
(73)	$\mathrm{csch}(\pi x^{1/2})$	$-y^{-1/2} + \psi(\tfrac{1}{2}y^{1/2}+\tfrac{1}{2}) - \psi(\tfrac{1}{2}y^{1/2})$
(74)	$x^{-1/2}\,\mathrm{sech}(\pi x^{1/2})$	$y^{-1/2}[\psi(\tfrac{1}{2}y^{1/2}+\tfrac{3}{4}) - \psi(\tfrac{1}{2}y^{1/2}+\tfrac{1}{4})]$
(75)	$\dfrac{x^{-1/2}\sin(ax^{1/2})}{\sinh(bx^{1/2})}$ $\dfrac{x^{-1/2}\cos(ax^{1/2})}{\cosh(bx^{1/2})}$ $\dfrac{x^{-1/2}\cos(ax^{1/2})}{c+\cosh(bx^{1/2})}$	See Ramanujan, Srinivasa, 1914: *Messenger of Math.* 44, 75-85.

14.3. Higher transcendental functions

	$f(x)$	$\int_0^\infty f(x)\,(x+y)^{-1}\,dx \qquad \lvert\arg y\rvert < \pi$
(1)	$\operatorname{ci}(ax) \qquad a>0$	$\tfrac{1}{2}[\operatorname{ci}(ay)]^2 + \tfrac{1}{2}[\operatorname{si}(ay)]^2$
(2)	$J_\nu(ax) \qquad a>0, \quad \operatorname{Re}\nu > -1$	$\pi \csc(\nu\pi)\,[\mathbf{J}_\nu(ay) - J_\nu(ay)]$
(3)	$x^\nu J_\nu(ax)$ $a>0, \quad -1/2 < \operatorname{Re}\nu < 3/2$	$\tfrac{1}{2}\pi y^\nu \sec(\nu\pi)\,[\mathbf{H}_{-\nu}(ay) - Y_{-\nu}(ay)]$
(4)	$x^{\nu+1} J_\nu(ax)$ $a>0, \quad -1 < \operatorname{Re}\nu < \tfrac{1}{2}$	$2^\nu \pi^{-1/2} a^{-\nu-1}\,\Gamma(\nu+\tfrac{1}{2})$ $+\tfrac{1}{2}\pi \sec(\nu\pi)\, y^{\nu+1}$ $\times [Y_{-\nu}(ay) - \mathbf{H}_{-\nu}(ay)]$
(5)	$x^{-\nu} J_\nu(ax)$ $a>0, \quad \operatorname{Re}\nu > -3/2$	$\tfrac{1}{2}\pi y^{-\nu}[\mathbf{H}_\nu(ay) - Y_\nu(ay)]$ $-\dfrac{2^{1-\nu} y^{-\nu}}{\Gamma(\nu)}\, S_{\nu-1,\nu}(ay)$
(6)	$x^\lambda J_\nu(ax) \quad a>0, \quad \operatorname{Re}\lambda < 3/2$ $\operatorname{Re}(\lambda+\nu) > -1$	$-\pi y^\lambda \csc[(\lambda+\nu)\pi]\, J_\nu(ay)$ $+\dfrac{2^{\lambda-1} a^{-\lambda}\,\Gamma(\tfrac{1}{2}\lambda + \tfrac{1}{2}\nu)}{\Gamma(1 - \tfrac{1}{2}\lambda + \tfrac{1}{2}\nu)}$ $\times {}_1F_2\left(1; 1 - \dfrac{\lambda+\nu}{2}, 1 - \dfrac{\lambda-\nu}{2}; -\dfrac{a^2y^2}{4}\right)$ $-\dfrac{2^{\lambda-2} a^{1-\lambda} y\, \Gamma(\tfrac{1}{2}\lambda + \tfrac{1}{2}\nu - \tfrac{1}{2})}{\Gamma(3/2 - \tfrac{1}{2}\lambda + \tfrac{1}{2}\nu)}$ $\times {}_1F_2\left(1; \dfrac{3-\lambda-\nu}{2}, \dfrac{3-\lambda+\nu}{2}; -\dfrac{a^2y^2}{4}\right)$

Higher transcendental functions (cont'd)

	$f(x)$	$\int_0^\infty f(x)\,(x+y)^{-1}\,dx \qquad \lvert\arg y\rvert < \pi$
(7)	$x^\nu \sin(ax)\, J_\nu(ax)$ $a>0, \quad -1<\operatorname{Re}\nu<\tfrac12$	$\tfrac12\pi\, y^\nu \sec(\nu\pi)\,[\cos(ay-\nu\pi)\, J_\nu(ay)$ $+\sin(ay-\nu\pi)\, Y_\nu(ay)]$
(8)	$x^\nu \cos(ax)\, J_\nu(ax)$ $a>0, \quad -\tfrac12<\operatorname{Re}\nu<\tfrac12$	$\tfrac12\pi\, y^\nu \sec(\nu\pi)\,[\sin(ay-\nu\pi)\, J_\nu(ay)$ $-\cos(ay-\nu\pi)\, Y_\nu(ay)]$
(9)	$x^\nu \cos(ax+\beta)\, J_\nu(ax)$ $a>0, \quad -\tfrac12<\operatorname{Re}\nu<\tfrac12$	$\tfrac12\pi\, y^\nu \sec(\nu\pi)\,[\sin(ay-\nu\pi-\beta)\, J_\nu(ay)$ $-\cos(ay-\nu\pi-\beta)\, Y_\nu(ay)]$
(10)	$x^{\frac12\nu+k} J_\nu(ax^{\frac12})$ $a>0, \quad k=0,1,2$ $-k-1<\operatorname{Re}\nu<-2k+3/2$	$2(-1)^k\, y^{\frac12\nu+k} K_\nu(ay^{\frac12})$
(11)	$x^{\frac12\nu+k-\frac12} J_\nu(ax^{\frac12})$ $a>0, \quad k=0,1,2$ $-k-\tfrac12<\operatorname{Re}\nu<-2k+5/2$	$(-1)^k\,\pi\sec(\nu\pi)\, y^{\frac12\nu+k-\frac12}$ $\times[I_\nu(ay^{\frac12})-\mathbf{L}_{-\nu}(ay^{\frac12})]$
(12)	$x^{k-\frac12\nu-\frac12} J_\nu(ax^{\frac12})$ $a>0, \quad k=0,1,2,\ldots$ $\operatorname{Re}\nu>2k-5/2$	$\pi\, y^{k-\frac12\nu-\frac12}\,[I_\nu(ay^{\frac12})-\mathbf{L}_\nu(ay^{\frac12})]$
(13)	$x^\lambda J_\nu(ax^{\frac12})$ $a>0, \quad \operatorname{Re}(\lambda+\tfrac12\nu)>-1$ $\operatorname{Re}\lambda<\tfrac34$	$\left(\dfrac{2}{a}\right)^{2\lambda} \dfrac{\Gamma(\lambda+\frac12\nu)}{\Gamma(1-\lambda+\frac12\nu)}$ $\times\, {}_1F_2\left(1;1-\lambda-\dfrac{\nu}{2},1-\lambda+\dfrac{\nu}{2};\dfrac{a^2y}{4}\right)$ $-\pi\csc[(\lambda+\tfrac12\nu)\pi]\, y^\lambda I_\nu(ay^{\frac12})$

Higher transcendental functions (cont'd)

	$f(x)$	$\int_0^\infty f(x)(x+y)^{-1}\,dx \qquad \lvert\arg y\rvert < \pi$
(14)	$\sin(ax^{1/2})\,J_0(bx^{1/2}) \qquad 0<b<a$	$\pi\exp(-ay^{1/2})\,I_0(by^{1/2})$
(15)	$x^{-1/2}\sin(ax^{1/2})\,J_0(bx^{1/2})$ $\qquad 0<a<b$	$2y^{-1/2}\sinh(ay^{1/2})\,K_0(by^{1/2})$
(16)	$\cos(ax^{1/2})\,J_0(bx^{1/2}) \qquad 0<a<b$	$2\cosh(ay^{1/2})\,K_0(by^{1/2})$
(17)	$x^{-1/2}\cos(ax^{1/2})\,J_0(bx^{1/2})$ $\qquad 0<b<a$	$\pi y^{-1/2}\exp(-ay^{1/2})\,I_0(by^{1/2})$
(18)	$x^{\frac{1}{2}\nu-\frac{1}{2}}\sin(ax^{1/2})\,J_\nu(bx^{1/2})$ $\qquad 0<a<b, \quad -1<\mathrm{Re}\,\nu<3/2$	$2y^{\frac{1}{2}\nu-\frac{1}{2}}\sinh(ay^{1/2})\,K_\nu(by^{1/2})$
(19)	$x^{-\frac{1}{2}\nu}\sin(ax^{1/2})\,J_\nu(bx^{1/2})$ $\qquad 0<b<a, \quad \mathrm{Re}\,\nu>-\tfrac{1}{2}$	$\pi y^{-\frac{1}{2}\nu}\exp(-ay^{1/2})\,I_\nu(by^{1/2})$
(20)	$x^{\frac{1}{2}\nu}\cos(ax^{1/2})\,J_\nu(bx^{1/2})$ $\qquad 0<a<b, \quad -1<\mathrm{Re}\,\nu<\tfrac{1}{2}$	$2y^{\frac{1}{2}\nu}\cosh(ay^{1/2})\,K_\nu(by^{1/2})$
(21)	$x^{-\frac{1}{2}\nu-\frac{1}{2}}\cos(ax^{1/2})\,J_\nu(bx^{1/2})$ $\qquad 0<b<a, \quad \mathrm{Re}\,\nu>-3/2$	$\pi y^{-\frac{1}{2}\nu-\frac{1}{2}}\exp(-ay^{1/2})\,I_\nu(by^{1/2})$
(22)	$[J_\nu(ax)]^2 \qquad a>0$	$2\,I_\nu(ay^{1/2})\,K_\nu(ay^{1/2})$
(23)	$J_\nu(ax^{1/2})\,J_\nu(bx^{1/2})$ $\qquad a, b>0, \quad \mathrm{Re}\,\nu>-1$	$2\,I_\nu(ay^{1/2})\,K_\nu(by^{1/2}) \qquad b>a$ $2\,I_\nu(by^{1/2})\,K_\nu(ay^{1/2}) \qquad b<a$

Higher transcendental functions (cont'd)

	$f(x)$	$\int_0^\infty f(x)(x+y)^{-1}\,dx \qquad \lvert\arg y\rvert < \pi$
(24)	$x^{\frac{1}{2}\nu-\frac{1}{2}\mu} J_\nu(bx^{\frac{1}{2}}) J_\mu(ax^{\frac{1}{2}})$ $0 < a < b$ $2 + \operatorname{Re}\mu > \operatorname{Re}\nu > -1$	$2y^{\frac{1}{2}\nu-\frac{1}{2}\mu} I_\mu(ay^{\frac{1}{2}}) K_\nu(by^{\frac{1}{2}})$
(25)	$x^\lambda J_\mu(ax^{\frac{1}{2}}) J_\nu(ax^{\frac{1}{2}})$ $a > 0, \quad \operatorname{Re}\lambda < 1$ $\operatorname{Re}(2\lambda+\mu+\nu) > -2$	$a^{-2\lambda}\pi^{-\frac{1}{2}} G^{23}_{35}\left(a^2y\,\middle\vert\,\begin{matrix}0, \lambda, \lambda+\frac{1}{2}\\ 0, p, q, r, s\end{matrix}\right)$ $p = \lambda+\frac{1}{2}\mu+\frac{1}{2}\nu, \quad q = \lambda+\frac{1}{2}\mu-\frac{1}{2}\nu$ $r = \lambda-\frac{1}{2}\mu+\frac{1}{2}\nu, \quad s = \lambda-\frac{1}{2}\mu-\frac{1}{2}\nu$
(26)	$x^{\frac{1}{2}\mu+n}(x+\gamma)^{-\frac{1}{2}\nu} J_\mu(ax^{\frac{1}{2}})$ $\times J_\nu[b(x+\gamma)^{\frac{1}{2}}]$ $a > b > 0, \quad n = 0, 1, 2, \ldots$ $-1-n < \operatorname{Re}\mu < 2-2n+\operatorname{Re}\nu$	$2(-1)^n y^{\frac{1}{2}\mu+n}(y-\gamma)^{-\frac{1}{2}\nu}$ $\times K_\mu(ay^{\frac{1}{2}}) I_\mu[b(y-\gamma)^{\frac{1}{2}}]$
(27)	$x^{-\frac{1}{2}}[\sin(ax) J_\nu(ax)$ $+ \cos(ax) Y_\nu(ax)]$ $a > 0, \quad -\frac{1}{2} < \operatorname{Re}\nu < \frac{1}{2}$	$\pi\sec(\nu\pi) y^{-\frac{1}{2}}[-\sin(ay) J_\nu(ay)$ $+ \cos(ay) Y_\nu(ay)]$
(28)	$x^{-\frac{1}{2}}[\cos(ax) J_\nu(ax)$ $- \sin(ax) Y_\nu(ax)]$ $a > 0, \quad -\frac{1}{2} < \operatorname{Re}\nu < \frac{1}{2}$	$\pi\sec(\nu\pi) y^{-\frac{1}{2}}[\cos(ay) J_\nu(ay)$ $+ \sin(ay) Y_\nu(ay)]$

Higher transcendental functions (cont'd)

	$f(x)$	$\int_0^\infty f(x)\,(x+y)^{-1}\,dx \qquad \lvert\arg y\rvert < \pi$
(29)	$x^\lambda Y_\nu(ax) \qquad a>0$ $-1+\lvert\mathrm{Re}\,\nu\rvert < \mathrm{Re}\,\lambda < 3/2$	$-\dfrac{\pi\,\mathrm{ctn}(\nu\pi)\,y^\lambda}{\sin[(\nu+\lambda)\pi]}\,J_\nu\left(\dfrac{ay}{2}\right)$ $-\dfrac{\pi y^\lambda}{\sin(\nu\pi)\,\sin[(\nu-\lambda)\pi]}\,J_{-\nu}\left(\dfrac{ay}{2}\right)$ $-2^{\lambda-1}\,\pi^{-1}\,a^{-\lambda}\cos[\tfrac{1}{2}(\lambda-\nu)\pi]$ $\times\Gamma\left(\dfrac{\lambda-\nu}{2}\right)\Gamma\left(\dfrac{\lambda+\nu}{2}\right)$ $\times\,{}_1F_2\left(1;\,1-\dfrac{\lambda-\nu}{2},\,1-\dfrac{\lambda+\nu}{2};\,-\dfrac{a^2y^2}{4}\right)$ $+\,2^{\lambda-2}\,\pi^{-1}\,a^{1-\lambda}y\,\sin\left(\dfrac{\lambda-\nu}{2}\pi\right)$ $\times\Gamma\left(\dfrac{\lambda-\nu-1}{2}\right)\Gamma\left(\dfrac{\lambda+\nu-1}{2}\right)$ $\times\,{}_1F_2\left(1;\dfrac{3-\lambda+\nu}{2},\,\dfrac{3-\lambda-\nu}{2};\,-\dfrac{a^2y^2}{4}\right)$
(30)	$x^{\frac{1}{2}\nu-\frac{1}{2}}\,Y_\nu(ax^{\frac{1}{2}})$ $a>0, \quad -1/2<\mathrm{Re}\,\nu<5/2$	$-\,2y^{\frac{1}{2}\nu-\frac{1}{2}}\,K_\nu(ay^{\frac{1}{2}})$
(31)	$x^{\frac{1}{2}\nu+\frac{1}{2}}\,Y_\nu(ax^{\frac{1}{2}})$ $a>0, \quad -3/2<\mathrm{Re}\,\nu<1/2$	$2y^{\frac{1}{2}\nu+\frac{1}{2}}\,K_\nu(ay^{\frac{1}{2}})$
(32)	$x^\lambda\,Y_\nu(ax^{\frac{1}{2}}) \qquad a>0$ $-1+\tfrac{1}{2}\lvert\mathrm{Re}\,\nu\rvert<\mathrm{Re}\,\lambda<\tfrac{3}{4}$	$y^\lambda\,G^{31}_{24}\left(\dfrac{a^2y}{4}\,\middle\vert\,\begin{matrix}-\lambda,\,-\dfrac{\nu+1}{2}\\ -\lambda,\,-\dfrac{\nu}{2},\,\dfrac{\nu}{2},\,-\dfrac{\nu+1}{2}\end{matrix}\right)$

Higher transcendental functions (cont'd)

	$f(x)$	$\int_0^\infty f(x)\,(x+y)^{-1}\,dx \qquad \lvert\arg y\rvert < \pi$
(33)	$x^{\lambda-1}\{\cos[(\lambda-\tfrac{1}{2}\nu)\pi]\,J_\nu(ax^{1/2})$ $+\sin[(\lambda-\tfrac{1}{2}\nu)\pi]\,Y_\nu(ax^{1/2})\}$ $a>0, \quad \lvert\operatorname{Re}\nu\rvert < 2\operatorname{Re}\lambda < 7/2$	$-2y^{\lambda-1}K_\nu(ay^{1/2})$
(34)	$x^{\frac{1}{2}\mu+n-\frac{1}{2}}(x+\gamma)^{-\frac{1}{2}\nu}\,Y_\mu(ax^{1/2})$ $\times J_\nu[b(x+\gamma)^{1/2}]$ $a>b>0, \quad n=0,1,2,\ldots$ $-\tfrac{1}{2}-n < \operatorname{Re}\mu < 3-2n+\operatorname{Re}\nu$	$2(-1)^{n+1}y^{\frac{1}{2}\mu+n-\frac{1}{2}}(y-\gamma)^{-\frac{1}{2}\nu}$ $\times K_\mu(ay^{1/2})\,I_\nu[b(y-\gamma)^{1/2}]$
(35)	$x^{\lambda-1}(x+\gamma)^{-\frac{1}{2}\mu}J_\mu[b(x+\gamma)^{1/2}]$ $\times\{\cos[(\lambda-\tfrac{1}{2}\nu)\pi]\,J_\nu(ax^{1/2})$ $+\sin[(\lambda-\tfrac{1}{2}\nu)\pi]\,Y_\nu(ax^{1/2})\}$ $a>b>0$ $\lvert\operatorname{Re}\nu\rvert < 2\operatorname{Re}\lambda < 4+\operatorname{Re}\mu$	$-2y^{\lambda-1}(y-\gamma)^{-\frac{1}{2}\mu}$ $\times I_\mu[b(y-\gamma)^{1/2}]\,K_\nu(ay^{1/2})$
(36)	$x^\nu e^{-ax}I_\nu(ax)$ $\operatorname{Re}a>0, \quad -\tfrac{1}{2}<\operatorname{Re}\nu<\tfrac{1}{2}$	$y^\nu\sec(\nu\pi)\,e^{ay}K_\nu(ay)$
(37)	$x^\lambda e^{-ax}I_\nu(ax)$ $\operatorname{Re}a>0, \quad \operatorname{Re}\lambda<\tfrac{1}{2}$ $\operatorname{Re}(\lambda+\nu)>-1$	$\pi^{-1/2}y^\lambda G_{23}^{22}\left(2ay\,\middle\vert\,\begin{matrix}-\lambda,\ \tfrac{1}{2}\\ -\lambda,\ \nu,\ -\nu\end{matrix}\right)$
(38)	$x^\lambda e^{ax}K_\nu(ax)$ $\lvert\arg a\rvert<3\pi/2$ $\operatorname{Re}\lambda-\lvert\operatorname{Re}\nu\rvert>-1$	$\pi^{-1/2}\cos(\nu\pi)\,y^\lambda$ $\times G_{23}^{32}\left(2ay\,\middle\vert\,\begin{matrix}-\lambda,\ \tfrac{1}{2}\\ -\lambda,\ \nu,\ -\nu\end{matrix}\right)$

Higher transcendental functions (cont'd)

	$f(x)$	$\int_0^\infty f(x)\,(x+y)^{-1}\,dx \qquad \lvert\arg y\rvert < \pi$
(39)	$x^{-1/2}\,e^{-ax}\,K_\nu(ax)$ $\mathrm{Re}\,a>0, \quad -1/2<\mathrm{Re}\,\nu<1/2$	$\pi\sec(\nu\pi)\,y^{-1/2}\,e^{ay}\,K_\nu(ay)$
(40)	$x^{\lambda}e^{-ax}\,K_\nu(ax)$ $\mathrm{Re}\,a>0, \quad \mathrm{Re}\,\lambda-\lvert\mathrm{Re}\,\nu\rvert>-1$	$\pi^{1/2}\,y^{\lambda}\,G^{31}_{23}\left(2ay\,\middle\vert\,\begin{matrix}-\lambda,\ 1/2\\ -\lambda,\ \nu,\ -\nu\end{matrix}\right)$
(41)	$x^{-1/2\nu-1/2}\,K_\nu(ax^{1/2})$ $\mathrm{Re}\,a>0, \quad \mathrm{Re}\,\nu<1/2$	$1/2\,\pi^2\,y^{-1/2\nu-1/2}\sec(\nu\pi)$ $\times[\mathbf{H}_\nu(ay^{1/2})-Y_\nu(ay^{1/2})]$
(42)	$x^{\lambda}K_\nu(ax^{1/2})$ $\mathrm{Re}\,a>0, \quad \mathrm{Re}\,\lambda>1/2\lvert\mathrm{Re}\,\nu\rvert-1$	$2^{2\lambda+1}\,y^{\lambda}\,\Gamma(1+\lambda+1/2\nu)\Gamma(1+\lambda-1/2\nu)$ $\times S_{-2\lambda-1,\,\nu}(ay^{1/2})$
(43)	$x^{-1/2}[2\pi^{-1}K_0(ax^{1/4})-Y_0(ax^{1/4})]$ $\lvert\arg a\rvert<1/4\,\pi$	$4y^{-1/4}\,\mathrm{ker}(ay^{1/4})$
(44)	$x^{1/2\nu}\,\mathbf{H}_\nu(ax^{1/2})$ $a>0, \quad -1/2<\mathrm{Re}\,\nu<1/2$	$\pi\sec(\nu\pi)\,y^{1/2\nu}[I_{-\nu}(ay^{1/2})-\mathbf{L}_\nu(ay^{1/2})]$
(45)	$x^{-1/2\nu}\,\mathbf{H}_\nu(ax^{1/2})$ $a>0, \quad \mathrm{Re}\,\nu>-3/2$	$\pi\,y^{-1/2\nu}[I_\nu(ay^{1/2})-\mathbf{L}_\nu(ay^{1/2})]$
(46)	$x^{\lambda}\,\mathbf{H}_\nu(ax^{1/2}) \quad a>0, \quad \mathrm{Re}\,\lambda<3/4$ $-1/2<\mathrm{Re}(\lambda+1/2\nu)<1/2$	$\dfrac{\pi}{\cos[(\lambda+1/2\nu)\pi]}\Bigg[\dfrac{(2/a)^{2\lambda}}{\Gamma(1-\lambda+1/2\nu)\Gamma(1-\lambda-1/2\nu)}$ $\times\,{}_1F_2(1;1-\lambda+1/2\nu,1-\lambda-1/2\nu;1/4a^2y)$ $-y^{\lambda}\,\mathbf{L}_\nu(ay^{1/2})\Bigg]$

Higher transcendental functions (cont'd)

	$f(x)$	$\int_0^\infty f(x)\,(x+y)^{-1}\,dx \qquad \lvert\arg y\rvert < \pi$
(47)	$x^{-1/2}[\cos(\tfrac{1}{2}\nu\pi)\,J_\nu(ax^{1/2}) + \sin(\tfrac{1}{2}\nu\pi)\,\mathbf{H}_\nu(ax^{1/2})]$ $a>0, \quad -\tfrac{1}{2}<\operatorname{Re}\nu<2$	$\pi y^{-1/2}[I_\nu(ay^{1/2}) - \mathbf{L}_\nu(ay^{1/2})]$
(48)	$x^{-1/2}[I_\nu(ax^{1/2}) - \mathbf{L}_\nu(ax^{1/2})]$ $\operatorname{Re} a>0, \quad -1<\operatorname{Re}\nu<2$	$\pi a^{-1}\csc(\tfrac{1}{2}\nu\pi)[\mathbf{H}_\nu(ay^{1/2}) + \mathbf{E}_\nu(ay^{1/2})]$
(49)	$x^{\lambda}[I_\nu(ax^{1/2}) - \mathbf{L}_\nu(ax^{1/2})]$ $\operatorname{Re} a>0$ $-2<\operatorname{Re}(2\lambda+\nu)<1$	$\dfrac{y^{\lambda}}{\pi}\,G^{32}_{24}\left(\dfrac{a^2y}{4}\,\middle\vert\,\begin{matrix}-\lambda,\ \tfrac{1}{2}\nu+\tfrac{1}{2}\\ -\lambda,\ \tfrac{1}{2}\nu,\ \tfrac{1}{2}\nu+\tfrac{1}{2},\ -\tfrac{1}{2}\nu\end{matrix}\right)$
For other integrals with Bessel functions see Watson, G.N., 1922: *A treatise on the theory of Bessel functions*, Cambridge, in particular sections 13.5 to 13.6.		
(50)	$x^{\mu-1/2}e^{-1/2\,ax}M_{\kappa,\mu}(ax)$ $\operatorname{Re} a>0$ $-\tfrac{1}{2}<\operatorname{Re}\mu<\operatorname{Re}\kappa+\tfrac{1}{2}$	$\Gamma(2\mu+1)\,\Gamma(\kappa-\mu+\tfrac{1}{2})\,y^{\mu-1/2}$ $\times e^{1/2\,ax}\,W_{-\kappa,\mu}(ax)$
(51)	$x^{\lambda}e^{-1/2\,ax}M_{\kappa,\mu}(ax) \qquad \operatorname{Re} a>0$ $-3/2-\operatorname{Re}\mu<\operatorname{Re}\lambda<\operatorname{Re}\kappa$	$\dfrac{\Gamma(2\mu+1)\,y^{\lambda}}{\Gamma(\kappa+\mu+\tfrac{1}{2})}\,G^{22}_{23}\left(ax\,\middle\vert\,\begin{matrix}-\lambda,\ 1-\kappa\\ -\lambda,\ \mu+\tfrac{1}{2},\ \tfrac{1}{2}-\mu\end{matrix}\right)$
(52)	$x^{\lambda}e^{1/2\,ax}W_{\kappa,\mu}(ax)$ $\lvert\arg a\rvert<3\pi/2$ $\operatorname{Re}(\kappa+\lambda)<0$ $\operatorname{Re}\lambda>\lvert\operatorname{Re}\mu\rvert-3/2$	$y^{\lambda}[\Gamma(\tfrac{1}{2}+\mu-\kappa)\,\Gamma(\tfrac{1}{2}-\mu-\kappa)]^{-1}$ $\times G^{32}_{23}\left(ay\,\middle\vert\,\begin{matrix}-\lambda,\ \kappa+1\\ -\lambda,\ \tfrac{1}{2}+\mu,\ \tfrac{1}{2}-\mu\end{matrix}\right)$

Higher transcendental functions (cont'd)

	$f(x)$	$\int_0^\infty f(x)\,(x+y)^{-1}\,dx \qquad \lvert\arg y\rvert < \pi$
(53)	$x^{\kappa-1}\,e^{-\frac{1}{2}ax}\,W_{\kappa,\mu}(ax)$ $\operatorname{Re} a > 0, \quad \operatorname{Re}\kappa > \lvert\operatorname{Re}\mu\rvert - \frac{1}{2}$	$\Gamma(\kappa+\mu+\frac{1}{2})\,\Gamma(\kappa-\mu+\frac{1}{2})\,y^{\kappa-1}$ $\times\, e^{\frac{1}{2}ax}\,W_{-\kappa,\mu}(ax)$
(54)	$x^{\lambda}\,e^{-\frac{1}{2}ax}\,W_{\kappa,\mu}(ax) \qquad \operatorname{Re} a > 0$ $\operatorname{Re}\lambda > \lvert\operatorname{Re}\mu\rvert - 3/2$	$y^{\lambda}\,G^{31}_{23}\left(ay\,\middle\vert\,\begin{matrix}-\lambda,\ 1-\kappa\\ -\lambda,\ \frac{1}{2}+\mu,\ \frac{1}{2}-\mu\end{matrix}\right)$
(55)	$G^{mn}_{pq}\left(ax\,\middle\vert\,\begin{matrix}a_1,\ldots,a_p\\ b_1,\ldots,b_q\end{matrix}\right)$ $p+q < 2(m+n)$ $\lvert\arg a\rvert < (m+n-\frac{1}{2}p-\frac{1}{2}q)\pi$ $\operatorname{Re} a_j < 1 \qquad j=1,\ldots,n$ $\operatorname{Re} b_j > -1 \qquad j=1,\ldots,m$	$G^{m+1,\,n+1}_{p+1,\,q+1}\left(ay\,\middle\vert\,\begin{matrix}0,\ a_1,\ldots,a_p\\ 0,\ b_1,\ldots,b_q\end{matrix}\right)$
(56)	$G^{mn}_{pq}\left(ax^2\,\middle\vert\,\begin{matrix}a_1,\ldots,a_p\\ b_1,\ldots,b_q\end{matrix}\right)$ $p+q < 2(m+n)$ $\lvert\arg a\rvert < (m+n-\frac{1}{2}p-\frac{1}{2}q)\pi$ $\operatorname{Re} a_j < 1 \qquad j=1,\ldots,n$ $\operatorname{Re} b_j > -\frac{1}{2} \qquad j=1,\ldots,m$	$\frac{1}{2\pi}\,G^{m+2,\,n+2}_{p+2,q+2}\left(a\,y^2\,\middle\vert\,\begin{matrix}0,\ \frac{1}{2},\ a_1,\ldots,a_p\\ 0,\ \frac{1}{2},\ b_1,\ldots,b_q\end{matrix}\right)$

14.4. Generalized Stieltjes transforms

	$f(x)$	$\int_0^\infty f(x)\,(x+y)^{-\rho}dx \qquad \lvert\arg y\rvert < \pi$
(1)	$f(x)$	$g\,(y;\,\rho)$
(2)	$x\,f(x)$	$g\,(y;\,\rho-1) - yg\,(y;\,\rho)$
(3)	$f\,(ax) \qquad a>0$	$a^{\rho-1}\,g\,(ay;\,\rho)$
(4)	$x^{\rho-2}\,f(a/x) \qquad a>0$	$a^{\rho-1}\,y^{-\rho}\,g\,(a/y)$
(5)	$f'(x)$	$\rho\,g\,(y;\,\rho+1) - y^{-\rho}\,f(0)$
(6)	$\int_0^x f(t)\,dt$	$(\rho-1)^{-1}\,g\,(y;\,\rho-1) \qquad \mathrm{Re}\,\rho>1$
(7)	$[\Gamma(\mu)]^{-1}\int_0^x f(t)(x-t)^{\mu-1}\,dt$ $0<\mathrm{Re}\,\mu<\mathrm{Re}\,\rho$	$\dfrac{\Gamma(\rho-\mu)}{\Gamma(\rho)}\,g\,(y;\,\rho-\mu)$
(8)	$x^{\nu-1} \qquad \mathrm{Re}\,\nu>0$	$\dfrac{\Gamma(\nu)\,\Gamma(\rho-\nu)}{\Gamma(\rho)}\,y^{\nu-\rho} \qquad \mathrm{Re}\,\rho>\mathrm{Re}\,\nu$
(9)	$x^{\nu-1}(a+x)^{-\mu}$ $\lvert\arg a\rvert<\pi, \quad \mathrm{Re}\,\nu>0$	$\dfrac{\Gamma(\nu)\,\Gamma(\mu-\nu+\rho)\,y^{\nu-\rho}}{\Gamma(\mu+\rho)\,a^{\mu}}$ $\times\,{}_2F_1(\mu,\,\nu;\,\mu+\rho;\,1-y/a)$ $\mathrm{Re}\,\rho>\mathrm{Re}(\nu-\mu)$
(10)	$e^{-\alpha x} \qquad \mathrm{Re}\,\alpha>0$	$\alpha^{\rho-1}\,e^{\alpha y}\,\Gamma(1-\rho,\,\alpha y)$
(11)	$x^{-\rho}\,e^{-\alpha x}$ $\mathrm{Re}\,\alpha>0, \quad \mathrm{Re}\,\rho<1$	$\pi^{-1/2}\,\Gamma(1-\rho)(\alpha/y)^{\rho-1/2}\,e^{1/2\,\alpha y}$ $\times\,K_{\rho-1/2}(1/2\,\alpha y)$

Generalized Stieltjes transforms (cont'd)

	$f(x)$	$\int_0^\infty f(x)\,(x+y)^{-\rho}\,dx \qquad \lvert\arg y\rvert < \pi$
(12)	$x^{\lambda} e^{-ax}$ $\mathrm{Re}\, a > 0, \quad \mathrm{Re}\, \lambda > -1$	$\Gamma(\lambda+1)\, a^{\frac{1}{2}\rho-\frac{1}{2}\lambda-1}\, y^{\frac{1}{2}\lambda-\frac{1}{2}\rho}$ $\times e^{\frac{1}{2}ay}\, W_{k,m}(ay)$ $2k = -\lambda-\rho, \quad 2m = \lambda-\rho+1$
(13)	$x^{\lambda} \exp(-a/x) \qquad \mathrm{Re}\, a > 0$	$\Gamma(\rho-\lambda-1)\, a^{\frac{1}{2}\lambda} y^{-\frac{1}{2}\lambda-1}$ $\times \exp\left(\dfrac{a}{2y}\right) W_{k,m}\left(\dfrac{a}{y}\right)$ $k = \frac{1}{2}\lambda - \rho + 1, \quad m = \frac{1}{2}\lambda + \frac{1}{2}$ $\mathrm{Re}\, \rho > \mathrm{Re}\, \lambda + 1$
(14)	$x^{-\frac{1}{2}} \exp(-a x^{\frac{1}{2}}) \qquad \mathrm{Re}\, a > 0$	$\pi^{\frac{1}{2}} (2y^{\frac{1}{2}}/a)^{\frac{1}{2}-\rho}\, \Gamma(1-\rho)$ $\times [\mathbf{H}_{\frac{1}{2}-\rho}(ay^{\frac{1}{2}}) - Y_{\frac{1}{2}-\rho}(ay^{\frac{1}{2}})]$
(15)	$x^{\lambda} \exp(-a x^{\frac{1}{2}})$ $\mathrm{Re}\, a > 0, \quad \mathrm{Re}\, \lambda > -1$	$\dfrac{y^{\lambda-\rho+1}}{\pi^{\frac{1}{2}}\,\Gamma(\rho)}\, G^{31}_{13}\left(\dfrac{a^2 y}{4}\,\middle\vert\, \begin{matrix} -\lambda \\ \rho-\lambda-1,\, 0,\, \frac{1}{2} \end{matrix}\right)$
(16)	$\sin(ax^{\frac{1}{2}}) \qquad a > 0$	$\dfrac{2\pi^{\frac{1}{2}}\, y^{\frac{1}{2}}}{\Gamma(\rho)} \left(\dfrac{2y^{\frac{1}{2}}}{a}\right)^{\frac{1}{2}-\rho} K_{\rho-3/2}(ay^{\frac{1}{2}})$ $\mathrm{Re}\, \rho > \frac{1}{2}$
(17)	$x^{\lambda} \sin(ax^{\frac{1}{2}})$ $a > 0, \quad \mathrm{Re}\, \lambda > -\frac{1}{2}$	$\dfrac{\pi^{\frac{1}{2}}\, y^{\lambda-\rho+1}}{\Gamma(\rho)}\, G^{21}_{13}\left(\dfrac{a^2 y}{4}\,\middle\vert\, \begin{matrix} -\lambda \\ \rho-\lambda-1,\, \frac{1}{2},\, 0 \end{matrix}\right)$ $\mathrm{Re}\, \rho > \mathrm{Re}\, \lambda + \frac{1}{2}$
(18)	$x^{-\frac{1}{2}} \cos(ax^{\frac{1}{2}}) \qquad a > 0$	$\dfrac{2\pi^{\frac{1}{2}}}{\Gamma(\rho)} \left(\dfrac{2y^{\frac{1}{2}}}{a}\right)^{\frac{1}{2}-\rho} K_{\rho-\frac{1}{2}}(ay^{\frac{1}{2}})$ $\mathrm{Re}\, \rho > 0$

Generalized Stieltjes transforms (cont'd)

	$f(x)$	$\int_0^\infty f(x)\,(x+y)^{-\rho}\,dx \qquad \lvert\arg y\rvert < \pi$
(19)	$x^{\lambda}\cos(ax^{1/2})$ $a>0, \quad \mathrm{Re}\,\lambda > -1$	$\dfrac{\pi^{1/2}\,y^{\lambda-\rho+1}}{\Gamma(\rho)}\,G_{13}^{21}\left(\dfrac{a^2y}{4}\,\middle\vert\,\begin{matrix}-\lambda\\ \rho-\lambda-1,\ 0,\ 1/2\end{matrix}\right)$ $\mathrm{Re}\,\rho > \mathrm{Re}\,\lambda + 1/2$
(20)	$x^{1/2\nu}J_{\nu}(ax^{1/2})$ $a>0, \quad \mathrm{Re}\,\nu > -1$	$\dfrac{a^{\rho-1}}{2^{\rho}\,\Gamma(\rho)}\,y^{1/2\nu+1/2-1/2\rho}\,K_{\nu-\rho+1}(ay^{1/2})$ $\mathrm{Re}\,\rho > 1/2\,\mathrm{Re}\,\nu + 1/4$
(21)	$x^{\lambda}J_{\nu}(ax^{1/2})$ $a>0, \quad \mathrm{Re}(\lambda+1/2\nu) > -1$	$\dfrac{2^{2\lambda}\,y^{1-\rho}}{a^{2\lambda}\,\Gamma(\rho)}\,G_{13}^{21}\left(\dfrac{a^2y}{4}\,\middle\vert\,\begin{matrix}0\\ \rho-1,\ \lambda+1/2\nu,\ \lambda-1/2\nu\end{matrix}\right)$ $\mathrm{Re}\,\rho > \mathrm{Re}\,\lambda + 1/4$
(22)	$x^{\nu}e^{-ax}I_{\nu}(ax)$ $\mathrm{Re}\,a>0, \quad \mathrm{Re}\,\nu > -1/2$	$\dfrac{\Gamma(\nu+1/2)\,\Gamma(\rho-\nu-1/2)}{\pi^{1/2}\,\Gamma(\rho)}\,(2a)^{1/2\rho-1}$ $\times\, y^{\nu-1/2\rho}\,e^{ay}\,W_{k,m}(2ay)$ $k = 1/2 - 1/2\rho, \qquad m = 1/2 - 1/2\rho + \nu$ $\mathrm{Re}\,\rho > \mathrm{Re}\,\nu + 1/2$
(23)	$x^{\lambda}e^{-ax}I_{\nu}(ax)$ $\mathrm{Re}\,a>0, \quad \mathrm{Re}(\lambda+\nu) > -1$	$\dfrac{y^{\lambda+1-\rho}}{\pi^{1/2}\,\Gamma(\rho)}\,G_{23}^{22}\left(2ay\,\middle\vert\,\begin{matrix}-\lambda,\ 1/2\\ \rho-\lambda-1,\ \nu,\ -\nu\end{matrix}\right)$ $\mathrm{Re}\,\rho > \mathrm{Re}\,\lambda + 1/2$
(24)	$x^{\lambda}e^{ax}K_{\nu}(ax) \quad \lvert\arg a\rvert < 3\pi/2$ $\mathrm{Re}\,\lambda > \lvert\mathrm{Re}\,\nu\rvert - 1$	$\dfrac{\cos(\nu\pi)}{\pi^{1/2}\,\Gamma(\rho)}\,y^{\lambda+1-\rho}$ $\times\, G_{23}^{32}\left(2ay\,\middle\vert\,\begin{matrix}-\lambda,\ 1/2\\ \rho-\lambda-1,\ \nu,\ -\nu\end{matrix}\right)$ $\mathrm{Re}\,\rho > \mathrm{Re}\,\lambda + 1/2$

Generalized Stieltjes transforms (cont'd)

	$f(x)$	$\int_0^\infty f(x)\,(x+y)^{-\rho}\,dx \qquad \lvert\arg y\rvert < \pi$
(25)	$x^{\rho-3/2}\,e^{-\alpha x}\,K_\nu(\alpha x)$ $\operatorname{Re}\alpha>0, \quad \operatorname{Re}\rho>\lvert\operatorname{Re}\nu\rvert+\tfrac{1}{2}$	$\dfrac{\pi^{1/2}\,\Gamma(\rho+\nu-\tfrac{1}{2})\,\Gamma(\rho-\nu-\tfrac{1}{2})}{(2\alpha)^{1/2}\,y\,\Gamma(\rho)}$ $\times\, e^{\alpha y}\,W_{1-\rho,\nu}(2\alpha y)$
(26)	$x^\lambda\,e^{-\alpha x}\,K_\nu(\alpha x)$ $\operatorname{Re}\alpha>0, \quad \operatorname{Re}\lambda>\lvert\operatorname{Re}\nu\rvert-1$	$\dfrac{\pi^{1/2}}{\Gamma(\rho)}\,y^{\lambda+1-\rho}$ $\times\, G^{31}_{23}\left(2\alpha y\,\middle\vert\,\begin{matrix}-\lambda,\ \tfrac{1}{2}\\ \rho-\lambda-1,\ \nu,\ -\nu\end{matrix}\right)$
(27)	$x^\lambda\,K_\nu(\alpha x^{1/2}) \qquad \operatorname{Re}\alpha>0$ $\operatorname{Re}\lambda>\tfrac{1}{2}\lvert\operatorname{Re}\nu\rvert-1$	$\dfrac{y^{\lambda-\rho+1}}{2\Gamma(\rho)}\,G^{31}_{13}\left(\dfrac{\alpha^2 y}{4}\,\middle\vert\,\begin{matrix}-\lambda\\ \rho-\lambda-1,\ \tfrac{1}{2}\nu,\ -\tfrac{1}{2}\nu\end{matrix}\right)$
(28)	$x^{\mu-1/2}\,e^{-1/2\,\alpha x}\,M_{\kappa,\mu}(\alpha x)$ $\operatorname{Re}\alpha>0, \quad \operatorname{Re}\mu>-\tfrac{1}{2}$	$\Gamma(2\mu+1)\,\Gamma(\kappa+\rho-\mu-\tfrac{1}{2})\,[\Gamma(\rho)]^{-1}$ $\times\,\alpha^{1/2\rho-1/2}\,y^{\lambda+1/2-1/2\rho}\,e^{1/2\,\alpha y}\,W_{k,m}(\alpha y)$ $k=\tfrac{1}{2}-\tfrac{1}{2}\rho-\kappa, \qquad m=\tfrac{1}{2}-\tfrac{1}{2}\rho+\mu$ $\operatorname{Re}\rho>\operatorname{Re}(\mu-\kappa)+\tfrac{1}{2}$
(29)	$x^\lambda\,e^{-1/2\,\alpha x}\,M_{\kappa,\mu}(\alpha x)$ $\operatorname{Re}\alpha>0, \quad \operatorname{Re}(\lambda+\mu)>-3/2$	$\dfrac{\Gamma(2\mu+1)\,y^{\lambda+1-\rho}}{\Gamma(\rho)\,\Gamma(\kappa+\mu+\tfrac{1}{2})}$ $\times\, G^{22}_{23}\left(\alpha y\,\middle\vert\,\begin{matrix}-\lambda,\ 1-\kappa\\ \rho-\lambda-1,\ \tfrac{1}{2}+\mu,\ \tfrac{1}{2}-\mu\end{matrix}\right)$ $\operatorname{Re}\rho>\operatorname{Re}(\lambda-\kappa)+1$

Generalized Stieltjes transforms (cont'd)

	$f(x)$	$\int_0^\infty f(x)\,(x+y)^{-\rho}\,dx \qquad \lvert\arg y\rvert < \pi$
(30)	$x^{\lambda} e^{\frac{1}{2}\alpha x} W_{\kappa,\mu}(\alpha x)$ $\lvert\arg \alpha\rvert < 3\pi/2$ $\mathrm{Re}\,\lambda > \lvert\mathrm{Re}\,\mu\rvert - 3/2$	$\dfrac{y^{\lambda+1-\rho}}{\Gamma(\rho)\,\Gamma(\frac{1}{2}-\kappa+\mu)\,\Gamma(\frac{1}{2}-\kappa-\mu)}$ $\times\, G^{32}_{23}\left(\alpha y \,\middle\vert\, \begin{matrix} -\lambda,\ 1+\kappa \\ \rho-\lambda-1,\ \frac{1}{2}+\mu,\ \frac{1}{2}-\mu \end{matrix}\right)$ $\mathrm{Re}\,\rho > \mathrm{Re}\,(\lambda+\kappa)+1$
(31)	$x^{\kappa+\rho-2} e^{-\frac{1}{2}\alpha x} W_{\kappa,\mu}(\alpha x)$ $\mathrm{Re}\,\alpha > 0$ $\mathrm{Re}\,\rho > \lvert\mathrm{Re}\,\mu\rvert - \mathrm{Re}\,\kappa + \frac{1}{2}$	$\Gamma(\kappa+\rho+\mu-\frac{1}{2})\,\Gamma(\kappa+\rho-\mu-\frac{1}{2})\,[\Gamma(\rho)]^{-1}$ $\times\, y^{\kappa-1} e^{\frac{1}{2}\alpha y} W_{1-\kappa-\rho,\,\mu}(\alpha y)$
(32)	$x^{\lambda} e^{-\frac{1}{2}\alpha x} W_{\kappa,\mu}(\alpha x) \quad \mathrm{Re}\,\alpha > 0$ $\mathrm{Re}\,\lambda > \lvert\mathrm{Re}\,\mu\rvert - 3/2$	$\dfrac{y^{\lambda+1-\rho}}{\Gamma(\rho)}\, G^{31}_{23}\left(\alpha y \,\middle\vert\, \begin{matrix} -\lambda,\ 1-\kappa \\ \rho-\lambda-1,\ \frac{1}{2}+\mu,\ \frac{1}{2}-\mu \end{matrix}\right)$
(33)	$G^{mn}_{pq}\left(\alpha x \,\middle\vert\, \begin{matrix} a_1,\ldots,a_p \\ b_1,\ldots,b_q \end{matrix}\right)$ $p+q < 2(m+n)$ $\lvert\arg \alpha\rvert < (m+n-\frac{1}{2}p-\frac{1}{2}q)\pi$ $\mathrm{Re}\,b_j > -1 \qquad j=1,\ldots,m$	$\dfrac{y^{1-\rho}}{\Gamma(\rho)}\, G^{m+1,\,n+1}_{p+1,\,q+1}\left(\alpha y \,\middle\vert\, \begin{matrix} 0,\ a_1,\ldots,a_p \\ \rho-1,\ b_1,\ldots,b_q \end{matrix}\right)$ $\mathrm{Re}\,\rho > \mathrm{Re}\,a_j \qquad j=1,\ldots,n$

CHAPTER XV

HILBERT TRANSFORMS

We call

$$g(y) = \pi^{-1} ⨍_{-\infty}^{\infty} f(x)\,(x-y)^{-1}\,dx$$

the *Hilbert transform* of $f(x)$. Here x and y are real variables, and

$$⨍_{-\infty}^{\infty} = \lim_{\epsilon \to +0} \left(\int_{-\infty}^{y-\epsilon} + \int_{y+\epsilon}^{\infty} \right)$$

is the Cauchy Principal Value of $\int_{-\infty}^{\infty}$.

For the theory of Hilbert transforms see chapter V of Titchmarsh's book (1937) and the references given there. Additional references to papers which appeared after the publication of Titchmarsh's book are given below. The finite Hilbert transform,

$$g(y) = \pi^{-1} \int_a^b f(x)\,(x-y)^{-1}\,dx$$

and its application to airfoil theory was discussed recently by Tricomi (1951 a, b) and Nickel (1951, 1953); the latter author gives references to earlier work on this subject.

In the above relation, $g(x)$ is said to be *conjugate* to $f(x)$: the relationship is *skew-reciprocal*, i.e., $-f(x)$ is conjugate to $g(x)$. For the relation of Hilbert transforms to Fourier integrals see Titchmarsh (1937) and Kober (1942, 1943 a, b). The connection with Laplace transforms may be expressed by stating that, formally, the imaginary part of a Laplace transform evaluated on a line parallel to the imaginary axis is conjugate to the real part of that Laplace transform evaluated on the same line. Hilbert transforms may be evaluated by means of tables of Stieltjes transforms (Chapter XIV) using the formulas

$$g(y) = \pi^{-1}\, \mathfrak{S}\{f(x)\,;\, -y\} - (2\pi)^{-1}\, \mathfrak{S}\{f(-x)\,;\, |y|\, e^{i\pi}\}$$
$$- (2\pi)^{-1}\, \mathfrak{S}\{f(-x)\,;\, |y|\, e^{-i\pi}\} \qquad -\infty < y < 0$$

$$g(y) = (2\pi)^{-1}\, \mathfrak{S}\{f(x)\,;\, ye^{i\pi}\} + (2\pi)^{-1}\, \mathfrak{S}\{f(x)\,;\, ye^{-i\pi}\}$$
$$- \pi^{-1}\, \mathfrak{S}\{f(-x)\,;\, y\} \qquad 0 < y < \infty$$

Related transforms are

$$⨍_{-\pi}^{\pi} f(x)\, \operatorname{ctn}[\tfrac{1}{2}(x-y)]\, dx$$

$$⨍_{0}^{\pi} f(x)\, (\cos x - \cos y)^{-1}\, dx.$$

These can be reduced to Hilbert transforms by a change of the variables of integration.

From the transform pairs given in the tables, further transform pairs may be derived by the methods mentioned in the introduction to vol. I, by the general formulas given in sec. 15.1, and by exploiting the connection with other transforms (see above).

REFERENCES

Cossar, James, 1939: *Proc. London Math. Soc.* (2) 45, 369-381.

Kober, Hermann, 1942: *Bull. Amer. Math. Soc.* 48, 421-426.

Kober, Hermann, 1943a: *J. London Math. Soc.* 18, 66-71.

Kober, Hermann, 1943b: *Quart. J. Math. Oxford Ser.* 14, 49-54.

Nickel, Karl, 1951: *Math. Z.* 54, 81-96.

Nickel, Karl, 1953: *Math. Z.* 58, 49-62.

Titchmarsh, E.C., 1937: *Introduction to the theory of Fourier integrals.* Oxford.

Tricomi, F.G., 1951a: *Quart. J. Math. Oxford Ser.* (2) 2, 199-211.

Tricomi, F.G., 1951b: *Z. Angew. Math. Physik* 2, 402-406.

HILBERT TRANSFORMS

15.1. General formulas

	$f(x)$		$\pi^{-1} \int_{-\infty}^{\infty} f(x)(x-y)^{-1}\,dx$ *
(1)	$f(x)$		$g(y)$
(2)	$g(x)$		$-f(y)$
(3)	$f(a+x)$	a real	$g(a+y)$
(4)	$f(ax)$	$a>0$	$g(ay)$
(5)	$f(-ax)$	$a>0$	$-g(-ay)$
(6)	$x\,f(x)$		$y\,g(y) + \pi^{-1} \int_{-\infty}^{\infty} f(x)\,dx$
(7)	$(x+a)\,f(x)$		$(y+a)\,g(y) + \pi^{-1} \int_{-\infty}^{\infty} f(x)\,dx$
(8)	$f'(x)$		$g'(y)$

15.2. Elementary functions

(1)	1		0
(2)	0 1 0	$-\infty < x < a$ $a < x < b$ $b < x < \infty$	$\dfrac{1}{\pi} \log \left\| \dfrac{b-y}{a-y} \right\|$

* y is real, and the integral is a Cauchy Principal Value.

Elementary functions (cont'd)

	$f(x)$		$\pi^{-1} \int_{-\infty}^{\infty} f(x)(x-y)^{-1}\,dx$
(3)	0 x^{-1}	$-\infty < x < a$ $a < x < \infty$ $a > 0$	$\dfrac{1}{\pi y} \log \left\lvert \dfrac{a}{a-y} \right\rvert$ $y \neq 0, \quad y \neq a$
(4)	x^{-1} 0 x^{-1}	$-\infty < x < a$ $a < x < b$ $b < x < \infty$ $a < 0 < b$	$\dfrac{1}{\pi y} \log \left\lvert \dfrac{(y-a)\,b}{a\,(b-y)} \right\rvert$ $y \neq 0,\ a,\ b$
(5)	0 x^{-2}	$-\infty < x < a$ $a < x < \infty$ $a > 0$	$\dfrac{1}{\pi y^2} \log \left\lvert \dfrac{a}{a-y} \right\rvert - \dfrac{1}{\pi a y}$ $y \neq 0, \quad y \neq a$
(6)	$(x+a)^{-1}$	$\operatorname{Im} a > 0$	$i\,(y+a)^{-1}$
(7)	$(x+a)^{-1}$	$\operatorname{Im} a < 0$	$-i\,(y+a)^{-1}$
(8)	0 $(ax+b)^{-1}$	$-\infty < x < 0$ $0 < x < \infty$ $a,\ b > 0$	$\dfrac{1}{\pi\,(ay+b)} \log \left\lvert \dfrac{b}{ay} \right\rvert$ $y \neq -b/a, \quad y \neq 0$
(9)	0 $(ax+b)^{-2}$	$-\infty < x < 0$ $0 < x < \infty$ $a,\ b > 0$	$\dfrac{1}{\pi\,(ay+b)^2} \log \left\lvert \dfrac{b}{ay} \right\rvert - \dfrac{1}{\pi b\,(ay+b)}$ $y \neq 0, \quad y \neq -b/a$

y is real, and the integral is a Cauchy Principal Value.

Elementary functions (cont'd)

	$f(x)$	$\pi^{-1} \fint_{-\infty}^{\infty} f(x)(x-y)^{-1}dx$
(10)	$(x^2+a^2)^{-1}$ $\operatorname{Re} a > 0$	$-\dfrac{y}{a(y^2+a^2)}$
(11)	$\dfrac{x}{x^2+a^2}$ $\operatorname{Re} a > 0$	$\dfrac{a}{y^2+a^2}$
(12)	$\dfrac{\lambda x+\mu a}{x^2+a^2}$ $\operatorname{Re} a > 0$	$\dfrac{\lambda a-\mu y}{y^2+a^2}$
(13)	0 $-\infty < x < 0$ $\dfrac{cx+d}{(ax+b)^2}$ $0 < x < \infty$ $a,\ b > 0$	$\dfrac{cy+d}{\pi(ay+b)^2} \log\left\lvert\dfrac{b}{ay}\right\rvert$ $-\dfrac{ad-bc}{\pi ab(ay+b)}$ $y \neq 0, \quad y \neq -b/a$
(14)	$(a-x)^{1/2}-(b-x)^{1/2}$ $-\infty < x < a$ $-(b-x)^{1/2}$ $a < x < b$ 0 $b < x < \infty$	0 $-\infty < y < a$ $(y-a)^{1/2}$ $a < y < b$ $(y-a)^{1/2}-(y-b)^{1/2}$ $b < y < \infty$
(15)	0 $-\infty < x < a$ $(x-a)^{1/2}$ $a < x < b$ $(x-a)^{1/2}-(x-b)^{1/2}$ $b < x < \infty$	$(b-y)^{1/2}-(a-y)^{1/2}$ $-\infty < y < a$ $(b-y)^{1/2}$ $a < y < b$ 0 $b < y < \infty$
(16)	$\lvert a-x\rvert^{1/2}-\lvert b-x\rvert^{1/2}$ $a > 0, \quad b > 0$	$(b-y)^{1/2}-(a-y)^{1/2}$ $-\infty < y < a$ $(b-y)^{1/2}+(y-a)^{1/2}$ $a < y < b$ $(y-a)^{1/2}-(y-b)^{1/2}$ $b < y < \infty$

y is real, and the integral is a Cauchy Principal Value.

Elementary functions (cont'd)

	$f(x)$	$\pi^{-1} \int_{-\infty}^{\infty} f(x)(x-y)^{-1}\,dx$
(17)	$0 \qquad -\infty < x < 0$ $(ax+b)^{-1/2} \qquad 0 < x < \infty$ $a,\ b > 0$	$2\pi^{-1}(-ay-b)^{-1/2}$ $\times \tan^{-1}\{[-(ay+b)/b]^{1/2}\}$ $-\infty < y < -b/a$ $2\pi^{-1} b^{-1} \qquad y = -b/a$ $\dfrac{1}{(ay+b)^{1/2}} \log \left\| \dfrac{b^{1/2} + (ay+b)^{1/2}}{b^{1/2} - (ay+b)^{1/2}} \right\|$ $-b/a < y < \infty$
(18)	$0 \qquad -\infty < x < 0$ $(a^2 - x^2)^{1/2} \qquad 0 < x < a$ $0 \qquad a < x < \infty$	$-\pi^{-1} a - \tfrac{1}{2}y - \pi^{-1}(y^2 - a^2)^{1/2}$ $\times \cos^{-1}(-a/y) \qquad -\infty < y < -a$ $-\pi^{-1} a - \tfrac{1}{2}y + \pi^{-1}(a^2 - y^2)^{1/2}$ $\times \log \left\| \dfrac{a + (a^2 - y^2)^{1/2}}{-y} \right\| \quad -a < y < a$ $-\pi^{-1} a - \tfrac{1}{2}y + \pi^{-1}(y^2 - a^2)^{1/2}$ $\times \cos^{-1}(-a/y) \qquad a < y < \infty$ $0 < \cos^{-1} < \pi$
(19)	$0 \qquad -\infty < x < -a$ $(a^2 - x^2)^{1/2} \qquad -a < x < a$ $0 \qquad a < x < \infty$	$-y - (y^2 - a^2)^{1/2} \qquad -\infty < y < -a$ $-y \qquad -a < y < a$ $-y + (y^2 - a^2)^{1/2} \qquad a < y < \infty$

y is real, and the integral is a Cauchy Principal Value.

Elementary functions (cont'd)

	$f(x)$		$\pi^{-1} ⨍_{-\infty}^{\infty} f(x)(x-y)^{-1}\,dx$	
(20)	0	$-\infty < x < 0$	$\dfrac{\cos^{-1}(-a/y)}{\pi(y^2-a^2)^{1/2}}$	$-\infty < y < -a$
	$(a^2-x^2)^{-1/2}$	$0 < x < a$	$\dfrac{1}{\pi(a^2-y^2)^{1/2}} \log\left\vert\dfrac{a+(a^2-y^2)^{1/2}}{-y}\right\vert$	$-a < y < a$
	0	$a < x < \infty$	$-\dfrac{\cos^{-1}(-a/y)}{\pi(y^2-a^2)^{1/2}}$	$a < y < \infty$
				$0 < \cos^{-1} < \pi$
(21)	0	$-\infty < x < -a$	$(y^2-a^2)^{-1/2}$	$-\infty < y < -a$
	$(a^2-x^2)^{-1/2}$	$-a < x < a$	0	$-a < y < a$
	0	$a < x < \infty$	$-(y^2-a^2)^{-1/2}$	$a < y < \infty$
(22)	0	$-\infty < x < a$	$\dfrac{1}{\pi(y^2-a^2)^{1/2}} \log\left\vert\dfrac{-y+(y^2-a^2)^{1/2}}{a}\right\vert$	$-\infty < y < -a$
	$(x^2-a^2)^{-1/2}$	$a < x < \infty$	$\dfrac{1}{\pi(a^2-y^2)^{1/2}} \cos^{-1}\left(\dfrac{-y}{a}\right)$	$-a < y < a$
		$a > 0$	$\dfrac{1}{\pi(y^2-a^2)^{1/2}} \log\left\vert\dfrac{-y+(y^2-a^2)^{1/2}}{a}\right\vert$	$a < y < \infty$
				$0 < \cos^{-1} < \pi$

y is real, and the integral is a Cauchy Principal Value.

Elementary functions (cont'd)

	$f(x)$	$\pi^{-1} \fint_{-\infty}^{\infty} f(x)\,(x-y)^{-1}\,dx$
(23)	$-(x^2-a^2)^{-1/2}$ $\quad -\infty < x < -a$ 0 $\quad -a < x < a$ $(x^2-a^2)^{-1/2}$ $\quad a < x < \infty$	0 $\quad -\infty < y < -a$ $(a^2-y^2)^{-1/2}$ $\quad -a < y < a$ 0 $\quad a < y < \infty$
(24)	0 $\quad -\infty < x < 0$ $(a-x)^{1/2}\,(a+x)^{-1/2}$ $\quad 0 < x < a$ 0 $\quad a < x < \infty$	$-\dfrac{1}{2}+\dfrac{1}{\pi}\left\lvert\dfrac{a-y}{a+y}\right\rvert^{1/2} \cos^{-1}\left(-\dfrac{a}{y}\right)$ $-\infty < y < -a$ $-\dfrac{1}{2}+\dfrac{1}{\pi}\left(\dfrac{a-y}{a+y}\right)^{1/2} \log\left\lvert\dfrac{a+(a^2-y^2)^{1/2}}{-y}\right\rvert$ $-a < y < a$ $-\dfrac{1}{2}+\dfrac{1}{\pi}\left(\dfrac{y-a}{y+a}\right)^{1/2} \cos^{-1}\left(-\dfrac{a}{y}\right)$ $a < y < \infty$ $0 < \cos^{-1} < \pi$
(25)	0 $\quad -\infty < x < -a$ $(a-x)^{1/2}\,(a+x)^{-1/2}$ $\quad -a < x < a$ 0 $\quad a < x < \infty$	$-1+(a-y)^{1/2}\,\lvert y+a\rvert^{-1/2}$ $-\infty < y < -a$ -1 $\quad -a < y < a$ $-1+(y-a)^{1/2}\,(y+a)^{-1/2}$ $a < y < \infty$

y is real, and the integral is a Cauchy Principal Value.

Elementary functions (cont'd)

	$f(x)$	$\pi^{-1} \fint_{-\infty}^{\infty} f(x)\,(x-y)^{-1}\,dx$
(26)	$0 \qquad -\infty < x < 0$ $(a+x)^{1/2}\,(a-x)^{-1/2} \qquad 0 < x < a$ $0 \qquad a < x < \infty$	$\frac{1}{2} - \frac{1}{\pi}\left\lvert\frac{a+y}{a-y}\right\rvert^{1/2} \cos^{-1}\left(-\frac{a}{y}\right)$ $-\infty < y < -a$ $\frac{1}{2} + \frac{1}{\pi}\left(\frac{a+y}{a-y}\right)^{1/2} \log\left\lvert\frac{a+(a^2-y^2)^{1/2}}{-y}\right\rvert$ $-a < y < a$ $\frac{1}{2} - \frac{1}{\pi}\left(\frac{y+a}{y-a}\right)^{1/2} \cos^{-1}\left(-\frac{a}{y}\right)$ $a < y < \infty$ $0 < \cos^{-1} < \pi$
(27)	$0 \qquad -\infty < x < -a$ $x\,(a-x)^{1/2}\,(a+x)^{-1/2}$ $-a < x < a$ $0 \qquad a < x < \infty$	$a - y + y\left\lvert\frac{a-y}{a+y}\right\rvert^{1/2} \qquad -\infty < y < -a$ $a - y \qquad -a < y < a$ $a - y + y\left(\frac{y-a}{y+a}\right)^{1/2} \qquad a < y < \infty$
(28)	$0 \qquad -\infty < x < 0$ $x^{\nu-1} \qquad 0 < x < \infty$ $0 < \mathrm{Re}\,\nu < 1$	$\csc(\nu\pi)\,(-y)^{\nu-1} \qquad -\infty < y < 0$ $-\operatorname{ctn}(\nu\pi)\,y^{\nu-1} \qquad 0 < y < \infty$
(29)	$\lvert x\rvert^{\nu-1} \qquad 0 < \mathrm{Re}\,\nu < 1$	$-\operatorname{ctn}(\tfrac{1}{2}\nu\pi)\,\operatorname{sgn} y\,\lvert y\rvert^{\nu-1}$
(30)	$\operatorname{sgn} x\,\lvert x\rvert^{\nu-1} \qquad 0 < \mathrm{Re}\,\nu < 1$	$\tan(\tfrac{1}{2}\nu\pi)\,\lvert y\rvert^{\nu-1}$

y is real, and the integral is a Cauchy Principal Value.

Elementary functions (cont'd)

	$f(x)$	$\pi^{-1} \,⨍_{-\infty}^{\infty} f(x)\,(x-y)^{-1}\,dx$
(31)	$0 \qquad -\infty < x < a$ $(x-a)^{\nu}(b-x)^{-\nu} \qquad a < x < b$ $0 \qquad b < x < \infty$ $\lvert\operatorname{Re}\nu\rvert < 1$	$\csc(\nu\pi)\left[1-\left(\frac{a-y}{b-y}\right)^{\nu}\right]$ $-\infty < y < a$ $\csc(\nu\pi)\left[1-\cos(\nu\pi)\left(\frac{y-a}{b-y}\right)^{\nu}\right]$ $a < y < b$ $\csc(\nu\pi)\left[1-\left(\frac{y-a}{y-b}\right)^{\nu}\right]_{b < y < \infty}$
(32)	$0 \qquad -\infty < x < a$ $(x-a)^{\nu-1}(b-x)^{-\nu} \qquad a < x < b$ $0 \qquad b < x < \infty$ $0 < \operatorname{Re}\nu < 1$	$\frac{\csc(\nu\pi)}{b-y}\left\lvert\frac{a-y}{b-y}\right\rvert^{\nu-1}$ $-\infty < y < a \quad \text{or} \quad b < y < \infty$ $-(y-a)^{\nu-1}(b-y)^{-\nu}\operatorname{ctn}(\nu\pi)$ $a < y < b$
(33)	$0 \qquad -\infty < x < a$ $(x-a)^{\rho-1}(b-x)^{\sigma-1}$ $a < x < b$ $0 \qquad b < x < \infty$ $\operatorname{Re}\rho > 0, \quad \operatorname{Re}\sigma > 0$	$\frac{\Gamma(\rho)\Gamma(\sigma)(b-a)^{\rho+\sigma-1}}{(b-y)\,\pi\Gamma(\rho+\sigma)}$ $\times\, {}_2F_1\left(1,\sigma;\rho+\sigma;\frac{b-a}{b-y}\right)$ $-\infty < y < a \quad \text{or} \quad b < y < \infty$ $(y-a)^{\rho-1}(b-y)^{\sigma-1}\operatorname{ctn}(\sigma\pi)$ $-\frac{\Gamma(\rho)\Gamma(\sigma-1)}{\pi\Gamma(\rho+\sigma-1)}(b-a)^{\rho+\sigma-2}$ $\times\, {}_2F_1\left(2-\rho-\sigma,1;2-\sigma;\frac{b-y}{b-a}\right)$ $a < y < b$

y is real, and the integral is a Cauchy Principal Value.

Elementary functions (cont'd)

	$f(x)$	$\pi^{-1} \fint_{-\infty}^{\infty} f(x)\,(x-y)^{-1}\,dx$
(34)	$0 \qquad -\infty < x < 0$ $x^{\nu-1}(x+a)^{1-\mu} \qquad 0 < x < \infty$ $a > 0, \quad 0 < \operatorname{Re}\nu < \operatorname{Re}\mu$	$\dfrac{\Gamma(\mu-\nu)\,\Gamma(\nu)\,(-y)^{\nu-1}}{\pi\,\Gamma(\mu)\,a^{\mu-1}}$ $\times\,{}_2F_1(\mu-1, \nu; \mu; 1+y/a)$ $-\infty < y < 0$ $y^{\nu-1}(y+a)^{1-\mu}\operatorname{ctn}[(\mu-\nu)\pi]$ $-\dfrac{\Gamma(\mu-\nu-1)\,\Gamma(\nu)\,a^{1-\mu+\nu}}{(y+a)\,\pi\,\Gamma(\mu-1)}$ $\times\,{}_2F_1\left(2-\mu, 1; 2-\mu+\nu; \dfrac{a}{y+a}\right)$ $0 < y < \infty$
(35)	$\exp(-a\lvert x\rvert) \qquad a > 0$	$\pi^{-1}\operatorname{sgn} y\,[\exp(a\lvert y\rvert)\,\mathrm{Ei}(-a\lvert y\rvert)$ $-\exp(-a\lvert y\rvert)\,\overline{\mathrm{Ei}}(a\lvert y\rvert)]$
(36)	$\operatorname{sgn} x\,\exp(-a\lvert x\rvert) \qquad a > 0$	$-\pi^{-1}[\exp(a\lvert y\rvert)\,\mathrm{Ei}(-a\lvert y\rvert)$ $+\exp(-a\lvert y\rvert)\,\overline{\mathrm{Ei}}(a\lvert y\rvert)]$
(37)	$0 \qquad -\infty < x < a$ $e^{-bx} \qquad a < x < \infty$ $b > 0$	$-\pi^{-1}e^{-by}\,\mathrm{Ei}(by-ab) \qquad -\infty < y < a$ $-\pi^{-1}e^{-by}\,\overline{\mathrm{Ei}}(by-ab) \qquad a < y < \infty$
(38)	$e^{iax} \qquad a > 0$	ie^{iay}

y is real, and the integral is a Cauchy Principal Value.

Elementary functions (cont'd)

	$f(x)$	$\pi^{-1} \displaystyle⨍_{-\infty}^{\infty} f(x)(x-y)^{-1}\,dx$
(39)	$0 \qquad -\infty < x < 0$ $\exp(-ax^{1/2}) \qquad 0 < x < \infty$ $a > 0$	$2\pi^{-1}\cos(a\lvert y\rvert^{1/2})\,\mathrm{ci}(a\lvert y\rvert^{1/2}) - 2\pi^{-1}\sin(a\lvert y\rvert^{1/2})\,\mathrm{si}(a\lvert y\rvert^{1/2}) \qquad -\infty < y < 0$ $-\pi^{-1}\exp(ay^{1/2})\,\mathrm{Ei}(-ay^{1/2}) - \pi^{-1}\exp(-ay^{1/2})\,\overline{\mathrm{Ei}}(ay^{1/2}) \qquad 0 < y < \infty$
(40)	$\log\left\lvert\dfrac{b-x}{x-a}\right\rvert \qquad a < b$	$0 \qquad -\infty < y < a$ $-\pi \qquad a < y < b$ $0 \qquad b < y < \infty$
(41)	$\dfrac{1}{x}\log\left\lvert\dfrac{1+ax}{1-bx}\right\rvert \qquad a > 0, \quad b > 0$	$-\pi y^{-1} \qquad -\infty < y < -a^{-1}$ $0 \qquad -a^{-1} < y < b^{-1}$ $-\pi y^{-1} \qquad -b^{-1} < y < \infty$
(42)	$\log\left\lvert\dfrac{x^2-a^2}{x^2-b^2}\right\rvert \qquad 0 < a < b$	$-\pi \qquad -b < y < -a$ $\pi \qquad a < y < b$ $0 \qquad$ elsewhere
(43)	$\sin(ax) \qquad a > 0$	$\cos(ay)$
(44)	$\dfrac{\sin(ax)}{x} \qquad a > 0$	$\dfrac{\cos(ay)-1}{y}$
(45)	$0 \qquad -\infty < x < 0$ $\sin(ax^{1/2}) \qquad 0 < x < \infty$ $a > 0$	$\exp(-a\lvert y\rvert^{1/2}) \qquad -\infty < y < 0$ $\cos(ay^{1/2}) \qquad 0 < y < \infty$

y is real, and the integral is a Cauchy Principal Value.

Elementary functions (cont'd)

	$f(x)$		$\pi^{-1} \fint_{-\infty}^{\infty} f(x)(x-y)^{-1}\,dx$
(46)	$\operatorname{sgn} x \sin(a\lvert x\rvert^{1/2})$	$a > 0$	$\cos(a\lvert y\rvert^{1/2}) + \exp(-a\lvert y\rvert^{1/2})$
(47)	$\cos(ax)$	$a > 0$	$-\sin(ay)$
(48)	$\dfrac{1-\cos(ax)}{x}$	$a > 0$	$\dfrac{\sin(ay)}{y}$

15.3. Higher transcendental functions

(1)	$e^{-ax}\,\mathrm{Ei}(ax)$ $e^{-ax}\,\overline{\mathrm{Ei}}(ax)$	$-\infty < x < 0$ $0 < x < \infty$	0 πe^{-ay}	$-\infty < y < 0$ $0 < y < \infty$
(2)	$\mathrm{ci}(a\lvert x\rvert)$	$a > 0$	$\operatorname{sgn} y\ \mathrm{si}(a\lvert y\rvert)$	
(3)	$\operatorname{sgn} x\ \mathrm{si}(a\lvert x\rvert)$	$a > 0$	$\mathrm{Ci}(a\lvert y\rvert)$	
(4)	$\cos(ax)\,\mathrm{ci}(a\lvert x\rvert)$ $-\sin(a\lvert x\rvert)\,\mathrm{si}(a\lvert x\rvert)$	$a > 0$	$\operatorname{sgn} y \cos(ay)\,\mathrm{si}(a\lvert y\rvert)$ $+\sin(ay)\,\mathrm{ci}(a\lvert y\rvert)$	
(5)	$\sin(ax)\,\mathrm{ci}(a\lvert x\rvert)$ $+\operatorname{sgn} x \cos(ax)\,\mathrm{si}(a\lvert x\rvert)$	$a > 0$	$\sin(a\lvert y\rvert)\,\mathrm{si}(a\lvert y\rvert) - \cos(ay)\,\mathrm{ci}(a\lvert y\rvert)$	
(6)	0 $\mathrm{P}_n(x)$	$-\infty < x < -1,\quad 1 < x < \infty$ $-1 < x < 1$ $n = 0, 1, 2, \ldots$	$-2\pi^{-1} Q_n(y)$ $-2\pi^{-1}\,\mathrm{Q}_n(y)$	$-\infty < y < -1,\quad 1 < y < \infty$ $-1 < y < 1$
(7)	0 $(1-x^2)^{-1/2}\,T_n(x)$	$-\infty < x < -1,\quad 1 < x < \infty$ $-1 < x < 1$ $n = 1, 2, \ldots$	$U_{n-1}(y)$	$-1 < y < 1$

y is real, and the integral is a Cauchy Principal Value.

Higher transcendental functions (cont'd)

	$f(x)$	$\pi^{-1} \fint_{-\infty}^{\infty} f(x)(x-y)^{-1}\,dx$
(8)	$0 \qquad -\infty < x < -1, \quad 1 < x < \infty$ $(1-x^2)^{1/2}\, U_n(x) \qquad -1 < x < 1$ $n = 0, 1, 2, \ldots$	$-T_{n+1}(y) \qquad -1 < y < 1$
(9)	$0 \qquad -\infty < x < -1, \quad 1 < x < \infty$ $(1-x)^{\alpha}(1+x)^{\beta} P_n^{(\alpha,\beta)}(x)$ $-1 < x < 1$ $\operatorname{Re} \alpha > -1, \quad \operatorname{Re} \beta > -1$	$-2\pi^{-1}(y-1)^{\alpha}(y+1)^{\beta} Q_n^{(\alpha,\beta)}(y)$ $-\infty < y < -1, \quad 1 < y < \infty$ $-2\pi^{-1}(1-y)^{\alpha}(1+y)^{\beta} \mathrm{Q}_n^{(\alpha,\beta)}(y)$ $-1 < y < 1$
(10)	$0 \qquad -\infty < x < 0$ $J_{\nu}(ax) \qquad 0 < x < \infty$ $a > 0, \quad \operatorname{Re} \nu > -1$	$\csc(\nu\pi)\,[\mathbf{J}_{\nu}(-ay) - J_{\nu}(-ay)]$ $-\infty < y < 0$ $\csc(\nu\pi)\,[\mathbf{J}_{\nu}(-ay) - \cos(\nu\pi)\, J_{\nu}(ay)]$ $0 < y < \infty$
(11)	$-J_{-\nu}(-ax) \qquad -\infty < x < 0$ $J_{\nu}(ax) \qquad 0 < x < \infty$ $a > 0, \quad -1 < \operatorname{Re} \nu < 1$	$-Y_{-\nu}(-ay) \qquad -\infty < y < 0$ $-Y_{\nu}(ay) \qquad 0 < y < \infty$
(12)	$0 \qquad -\infty < x < 0$ $x^{\nu} J_{\nu}(ax) \qquad 0 < x < \infty$ $a > 0, \quad -1/2 < \operatorname{Re} \nu < 3/2$	$\tfrac{1}{2}\lvert y\rvert^{\nu}\,[\tan(\nu\pi)\operatorname{sgn} y\, J_{\nu}(a\lvert y\rvert)$ $-Y_{\nu}(a\lvert y\rvert) - \sec(\nu\pi)\operatorname{sgn} y\, \mathbf{H}_{-\nu}(a\lvert y\rvert)]$
(13)	$\lvert x\rvert^{\nu} J_{\nu}(a\lvert x\rvert)$ $a > 0, \quad -1/2 < \operatorname{Re} \nu < 3/2$	$\operatorname{sgn} y\, \lvert y\rvert^{\nu}\,[\tan(\nu\pi)\, J_{\nu}(a\lvert y\rvert)$ $-\sec(\nu\pi)\, \mathbf{H}_{-\nu}(a\lvert y\rvert)]$
(14)	$\operatorname{sgn} x\, \lvert x\rvert^{\nu} J_{\nu}(a\lvert x\rvert)$ $a > 0, \quad -1/2 < \operatorname{Re} \nu < 3/2$	$-\lvert y\rvert^{\nu}\, Y_{\nu}(a\lvert y\rvert)$

y is real, and the integral is a Cauchy Principal Value.

Higher transcendental functions (cont'd)

	$f(x)$	$\pi^{-1} \int_{-\infty}^{\infty} f(x)(x-y)^{-1}\,dx$
(15)	$\lvert x\rvert^{-\nu} J_\nu(a\lvert x\rvert)$ $a>0, \quad \mathrm{Re}\,\nu > -3/2$	$-\operatorname{sgn} y\, \lvert y\rvert^{-\nu}\, \mathbf{H}_\nu(a\lvert y\rvert)$
(16)	$0 \qquad -\infty < x < 0$ $x^\lambda J_\nu(ax) \qquad 0 < x < \infty$ $a > 0$ $-1 - \mathrm{Re}\,\nu < \mathrm{Re}\,\lambda < 3/2$	$\dfrac{2^{\lambda-1}\Gamma(\frac{1}{2}\lambda+\frac{1}{2}\nu)}{\pi a^\lambda \Gamma(1-\frac{1}{2}\lambda+\frac{1}{2}\nu)}$ $\times\, {}_1F_2\left(1;\, 1-\dfrac{\lambda+\nu}{2},\, 1-\dfrac{\lambda-\nu}{2};\, -\dfrac{a^2y^2}{4}\right)$ $+\dfrac{2^{\lambda-2}\, y\, \Gamma(\frac{1}{2}\lambda+\frac{1}{2}\nu-\frac{1}{2})}{\pi a^{\lambda-1}\Gamma(3/2-\frac{1}{2}\lambda+\frac{1}{2}\nu)}$ $\times\, {}_1F_2\left(1;\, \dfrac{3-\lambda-\nu}{2},\, \dfrac{3-\lambda+\nu}{2};\, -\dfrac{a^2y^2}{4}\right)$ $-h(y)\, \lvert y\rvert^\lambda J_\nu(a\lvert y\rvert)$ $h(y) = \begin{cases} \csc[(\lambda+\nu)\pi] & -\infty < y < 0 \\ \operatorname{ctn}[(\lambda+\nu)\pi] & 0 < y < \infty \end{cases}$
(17)	$\sin(ax)\, J_1(ax) \qquad a > 0$	$\cos(ay)\, J_1(ay)$
(18)	$\sin(ax)\, J_n(by)$ $0 < b < a, \quad n = 0, 1, 2, \ldots$	$\cos(ay)\, J_n(by)$
(19)	$\cos(ax)\, J_1(ax) \qquad a > 0$	$-\sin(ay)\, J_1(ay)$
(20)	$\cos(ax)\, J_n(bx)$ $0 < b < a, \quad n = 0, 1, 2, \ldots$	$-\sin(ay)\, J_n(by)$

y is real, and the integral is a Cauchy Principal Value.

Higher transcendental functions (cont'd)

	$f(x)$	$\pi^{-1} \fint_{-\infty}^{\infty} f(x)\,(x-y)^{-1}\,dx$
(21)	$\operatorname{sgn} x\, \lvert x\rvert^{\nu} \sin(a\lvert x\rvert - \pi\nu)$ $\times J_{\nu}(a\lvert x\rvert)$ $a>0, \quad -\tfrac{1}{2} < \operatorname{Re}\nu < \tfrac{1}{2}$	$\lvert y\rvert^{\nu} \cos(a\lvert y\rvert - \pi\nu)\, J_{\nu}(a\lvert y\rvert)$
(22)	$\lvert x\rvert^{\nu} \cos(a\lvert x\rvert - \pi\nu)\, J_{\nu}(a\lvert x\rvert)$ $a>0, \quad -\tfrac{1}{2} < \operatorname{Re}\nu < \tfrac{1}{2}$	$-\operatorname{sgn} y\, \lvert y\rvert^{\nu} \sin(a\lvert y\rvert - \pi\nu)\, J_{\nu}(a\lvert y\rvert)$
(23)	$\lvert x\rvert^{-\nu} \sin(ax)\, J_{\nu}(a\lvert x\rvert)$ $a>0, \ \operatorname{Re}\nu > -\tfrac{1}{2}$	$\lvert y\rvert^{-\nu} \cos(ay)\, J_{\nu}(a\lvert y\rvert)$
(24)	$\lvert x\rvert^{-\nu} \cos(ax)\, J_{\nu}(a\lvert x\rvert)$ $a>0, \quad \operatorname{Re}\nu > -\tfrac{1}{2}$	$-\lvert y\rvert^{-\nu} \sin(ay)\, J_{\nu}(a\lvert y\rvert)$
(25)	$\lvert x\rvert^{\frac{1}{2}} J_{\nu-\frac{1}{4}}(a\lvert x\rvert)\, J_{-\nu-\frac{1}{4}}(a\lvert x\rvert)$ $a>0$	$-\operatorname{sgn} y\, \lvert y\rvert^{\frac{1}{2}} J_{\frac{1}{4}+\nu}(a\lvert y\rvert)\, J_{\frac{1}{4}-\nu}(a\lvert y\rvert)$
(26)	$\operatorname{sgn} x\, \lvert x\rvert^{\frac{1}{2}} J_{\frac{1}{4}+\nu}(a\lvert x\rvert)$ $\times J_{\frac{1}{4}-\nu}(a\lvert x\rvert) \quad a>0$	$\lvert y\rvert^{\frac{1}{2}} J_{\nu-\frac{1}{4}}(a\lvert y\rvert)\, J_{-\nu-\frac{1}{4}}(a\lvert y\rvert)$
(27)	$0 \quad -\infty < x < 0$ $x^{\frac{1}{2}\nu} J_{\nu}(ax^{\frac{1}{2}}) \quad 0 < x < \infty$ $a>0, \quad -1 < \operatorname{Re}\nu < 3/2$	$2\pi^{-1}(-y)^{\frac{1}{2}\nu} K_{\nu}[a(-y)^{\frac{1}{2}}] \quad -\infty < y < 0$ $-y^{\frac{1}{2}\nu} Y_{\nu}(ay^{\frac{1}{2}}) \quad 0 < y < \infty$
(28)	$\lvert x\rvert^{\frac{1}{2}\nu} J_{\nu}(a\lvert x\rvert^{\frac{1}{2}})$ $a>0, \quad -1 < \operatorname{Re}\nu < 3/2$	$-\operatorname{sgn} y\, \lvert y\rvert^{\frac{1}{2}\nu} [2\pi^{-1} K_{\nu}(a\lvert y\rvert^{\frac{1}{2}})$ $+ Y_{\nu}(a\lvert y\rvert^{\frac{1}{2}})]$
(29)	$\operatorname{sgn} x\, \lvert x\rvert^{\frac{1}{2}\nu} J_{\nu}(a\lvert x\rvert^{\frac{1}{2}})$ $a>0, \quad -1 < \operatorname{Re}\nu < 3/2$	$2\pi^{-1} \lvert y\rvert^{\frac{1}{2}\nu} K_{\nu}(a\lvert y\rvert^{\frac{1}{2}})$ $- \lvert y\rvert^{\frac{1}{2}\nu} Y_{\nu}(a\lvert y\rvert^{\frac{1}{2}})$

y is real, and the integral is a Cauchy Principal Value.

Higher transcendental functions (cont'd)

	$f(x)$	$\pi^{-1} \int_{-\infty}^{\infty} f(x)\,(x-y)^{-1}\,dx$
(30)	$0 \quad -\infty < x < 0$ $x^{\frac{1}{2}\nu-\frac{1}{2}} J_\nu(ax^{\frac{1}{2}}) \quad 0 < x < \infty$ $a > 0, \quad -1/2 < \mathrm{Re}\,\nu < 5/2$	$\lvert y\rvert^{\frac{1}{2}\nu-\frac{1}{2}} \sec(\nu\pi)\,[I_\nu(a\lvert y\rvert^{\frac{1}{2}})$ $- \mathbf{L}_{-\nu}(a\lvert y\rvert^{\frac{1}{2}})] \quad -\infty < y < 0$ $y^{\frac{1}{2}\nu-\frac{1}{2}}[\tan(\nu\pi)\,J_\nu(ay^{\frac{1}{2}})$ $- \sec(\nu\pi)\,\mathbf{H}_{-\nu}(ay^{\frac{1}{2}})] \quad 0 < y < \infty$
(31)	$\lvert x\rvert^{\frac{1}{2}\nu-\frac{1}{2}} J_\nu(a\lvert x\rvert^{\frac{1}{2}})$ $a > 0, \quad -1/2 < \mathrm{Re}\,\nu < 5/2$	$\mathrm{sgn}\,y\,\lvert y\rvert^{\frac{1}{2}\nu-\frac{1}{2}}\{\tan(\nu\pi)\,J_\nu(a\lvert y\rvert^{\frac{1}{2}})$ $+ \sec(\nu\pi)\,[\mathbf{L}_{-\nu}(a\lvert y\rvert^{\frac{1}{2}}) - \mathbf{H}_{-\nu}(a\lvert y\rvert^{\frac{1}{2}})$ $- I_\nu(a\lvert y\rvert^{\frac{1}{2}})]\}$
(32)	$\mathrm{sgn}\,x\,\lvert x\rvert^{\frac{1}{2}\nu-\frac{1}{2}} J_\nu(a\lvert x\rvert^{\frac{1}{2}})$ $a > 0, \quad -1/2 < \mathrm{Re}\,\nu < 5/2$	$\lvert y\rvert^{\frac{1}{2}\nu-\frac{1}{2}}\{\tan(\nu\pi)\,J_\nu(a\lvert y\rvert^{\frac{1}{2}})$ $+ \sec(\nu\pi)\,[I_\nu(a\lvert y\rvert^{\frac{1}{2}})$ $- \mathbf{L}_{-\nu}(a\lvert y\rvert^{\frac{1}{2}}) - \mathbf{H}_{-\nu}(a\lvert y\rvert^{\frac{1}{2}})]\}$
(33)	$0 \quad -\infty < x < 0$ $x^{-\frac{1}{2}\nu-\frac{1}{2}} J_\nu(ax^{\frac{1}{2}}) \quad 0 < x < \infty$ $a > 0, \quad \mathrm{Re}\,\nu > -5/2$	$\lvert y\rvert^{-\frac{1}{2}\nu-\frac{1}{2}}[I_\nu(a\lvert y\rvert^{\frac{1}{2}}) - \mathbf{L}_\nu(a\lvert y\rvert^{\frac{1}{2}})]$ $-\infty < y < 0$ $- y^{-\frac{1}{2}\nu-\frac{1}{2}} \mathbf{H}_\nu(ay^{\frac{1}{2}}) \quad 0 < y < \infty$
(34)	$\lvert x\rvert^{-\frac{1}{2}\nu-\frac{1}{2}} J_\nu(a\lvert x\rvert^{\frac{1}{2}})$ $a > 0, \quad \mathrm{Re}\,\nu > -5/2$	$\mathrm{sgn}\,y\,\lvert y\rvert^{-\frac{1}{2}\nu-\frac{1}{2}}[\mathbf{L}_\nu(a\lvert y\rvert^{\frac{1}{2}})$ $- \mathbf{H}_\nu(a\lvert y\rvert^{\frac{1}{2}}) - I_\nu(a\lvert y\rvert^{\frac{1}{2}})]$
(35)	$\mathrm{sgn}\,x\,\lvert x\rvert^{-\frac{1}{2}\nu-\frac{1}{2}} J_\nu(a\lvert x\rvert^{\frac{1}{2}})$ $a > 0, \quad \mathrm{Re}\,\nu > -5/2$	$\lvert y\rvert^{-\frac{1}{2}\nu-\frac{1}{2}}[I_\nu(a\lvert y\rvert^{\frac{1}{2}})$ $- \mathbf{L}_\nu(a\lvert y\rvert^{\frac{1}{2}}) - \mathbf{H}_\nu(a\lvert y\rvert^{\frac{1}{2}})]$

y is real,aand the integral is a Cauchy Principal Value.

Higher transcendental functions (cont'd)

	$f(x)$	$\pi^{-1} \fint_{-\infty}^{\infty} f(x)\,(x-y)^{-1}\,dx$
(36)	$0 \qquad -\infty < x < 0$ $x^{\lambda} J_{\nu}(ax^{1/2}) \qquad 0 < x < \infty$ $a > 0$ $-1 - \tfrac{1}{2}\operatorname{Re}\nu < \operatorname{Re}\lambda < \tfrac{3}{4}$	$\dfrac{2^{2\lambda}\Gamma(\lambda+\tfrac{1}{2}\nu)}{\pi a^{2\lambda}\Gamma(1-\lambda+\tfrac{1}{2}\nu)}$ $\times {}_1F_2\left(1; 1-\lambda-\dfrac{\nu}{2}, 1-\lambda+\dfrac{\nu}{2}; -\dfrac{a^2 y}{4}\right) - h(y)$ $h(y) = \lvert y\rvert^{\lambda}\csc[(\lambda+\tfrac{1}{2}\nu)\pi]\, I_{\nu}(a\lvert y\rvert^{1/2})$ $-\infty < y < 0$ $h(y) = y^{\lambda}\operatorname{ctn}[(\lambda+\tfrac{1}{2}\nu)\pi]\, J_{\nu}(a\lvert y\rvert^{1/2})$ $0 < y < \infty$
(37)	$0 \qquad -a < x < a$ $\operatorname{sgn} x\,(x^2-a^2)^{\frac{1}{2}\nu}$ $\times J_{\nu}[b(x^2-a^2)^{1/2}]$ $-\infty < x < -a \quad \text{or} \quad a < x < \infty$ $a > 0, \quad b > 0$ $-1 < \operatorname{Re}\nu < 3/2$	$2\pi^{-1}(a^2-y^2)^{\frac{1}{2}\nu} K_{\nu}[b(a^2-y^2)^{1/2}]$ $-a < y < a$ $-(y^2-a^2)^{\frac{1}{2}\nu} Y_{\nu}[b(y^2-a^2)^{1/2}]$ $-\infty < y < -a \quad \text{or} \quad a < y < \infty$
(38)	$Y_{-\nu}(-ax) \qquad -\infty < x < 0$ $Y_{\nu}(ax) \qquad 0 < x < \infty$ $a > 0, \quad -1 < \operatorname{Re}\nu < 1$	$-J_{-\nu}(-ay) \qquad -\infty < y < 0$ $J_{\nu}(ay) \qquad 0 < y < \infty$
(39)	$\sin(\tfrac{1}{2}\nu\pi)\, J_{\nu}(a\lvert x\rvert)$ $+\cos(\tfrac{1}{2}\nu\pi)\, Y_{\nu}(a\lvert x\rvert)$ $a > 0, \quad -1 < \operatorname{Re}\nu < 1$	$\operatorname{sgn} y\,[\cos(\tfrac{1}{2}\nu\pi)\, J_{\nu}(a\lvert y\rvert)$ $-\sin(\tfrac{1}{2}\nu\pi)\, Y_{\nu}(a\lvert y\rvert)]$
(40)	$\operatorname{sgn} x\,[\sin(\tfrac{1}{2}\nu\pi)\, Y_{\nu}(a\lvert x\rvert)$ $-\cos(\tfrac{1}{2}\nu\pi)\, J_{\nu}(a\lvert x\rvert)]$ $a > 0, \quad -1 < \operatorname{Re}\nu < 1$	$\cos(\tfrac{1}{2}\nu\pi)\, Y_{\nu}(a\lvert y\rvert)$ $+\sin(\tfrac{1}{2}\nu\pi)\, J_{\nu}(a\lvert y\rvert)$

y is real, and the integral is a Cauchy Principal Value.

Higher transcendental functions (cont'd)

	$f(x)$	$\pi^{-1} ⨍_{-\infty}^{\infty} f(x)\,(x-y)^{-1}\,dx$
(41)	$\lvert x\rvert^{\nu}\, Y_{\nu}(a\lvert x\rvert)$ $a>0, \quad -1/2<\operatorname{Re}\nu<3/2$	$\operatorname{sgn} y\, \lvert y\rvert^{\nu}\, J_{\nu}(a\lvert y\rvert)$
(42)	$\operatorname{sgn} x\, \lvert x\rvert^{-\mu}\{\sin[\tfrac{1}{2}(\mu+\nu)\pi]$ $\times Y_{\nu}(a\lvert x\rvert)$ $-\cos[\tfrac{1}{2}(\mu+\nu)\pi]\, J_{\nu}(a\lvert x\rvert)\}$ $a>0$ $-\tfrac{1}{2}<\operatorname{Re}\mu<1-\lvert\operatorname{Re}\nu\rvert$	$\lvert y\rvert^{-\mu}\{\cos[\tfrac{1}{2}(\mu+\nu)\pi]\, Y_{\nu}(a\lvert y\rvert)$ $+\sin[\tfrac{1}{2}(\mu+\nu)\pi]\, J_{\nu}(a\lvert y\rvert)\}$
(43)	$\lvert x\rvert^{-\mu}\{\sin[\tfrac{1}{2}(\mu+\nu)\pi]\, J_{\nu}(a\lvert x\rvert)$ $+\cos[\tfrac{1}{2}(\mu+\nu)\pi]\, Y_{\nu}(a\lvert x\rvert)\}$ $a>0$ $-3/2<\operatorname{Re}\mu<1-\lvert\operatorname{Re}\nu\rvert$	$\operatorname{sgn} y\, \lvert y\rvert^{-\mu}\{\cos[\tfrac{1}{2}(\mu+\nu)\pi]\, J_{\nu}(a\lvert y\rvert)$ $-\sin[\tfrac{1}{2}(\mu+\nu)\pi]\, Y_{\nu}(a\lvert y\rvert)\}$
(44)	$\operatorname{sgn} x\, \lvert x\rvert^{-\mu}\{\cos[a\lvert x\rvert$ $-\tfrac{1}{2}(\mu+\nu)\pi]\, J_{\nu}(b\lvert x\rvert)$ $+\sin[a\lvert x\rvert-\tfrac{1}{2}(\mu+\nu)\pi]\, Y_{\nu}(b\lvert x\rvert)\}$ $a<b$ $-3/2<\operatorname{Re}\mu<1-\lvert\operatorname{Re}\nu\rvert$	$\lvert y\rvert^{-\mu}\{\sin[a\lvert y\rvert-\tfrac{1}{2}(\mu+\nu)\pi]\, J_{\nu}(b\lvert y\rvert)$ $-\cos[a\lvert y\rvert-\tfrac{1}{2}(\mu+\nu)\pi]\, Y_{\nu}(b\lvert y\rvert)\}$
(45)	$\lvert x\rvert^{-\mu}\{\cos[a\lvert x\rvert$ $-\tfrac{1}{2}(\mu+\nu)\pi]\, Y_{\nu}(b\lvert x\rvert)$ $-\sin[a\lvert x\rvert-\tfrac{1}{2}(\mu+\nu)\pi]\, J_{\nu}(b\lvert x\rvert)\}$ $a<b$ $-3/2<\operatorname{Re}\mu<1-\lvert\operatorname{Re}\nu\rvert$	$\operatorname{sgn} y\, \lvert y\rvert^{-\mu}\{\sin[a\lvert y\rvert-\tfrac{1}{2}(\mu+\nu)\pi]$ $\times Y_{\nu}(b\lvert y\rvert)+\cos[a\lvert y\rvert-\tfrac{1}{2}(\mu+\nu)\pi]$ $\times J_{\nu}(b\lvert y\rvert)\}$

y is real, and the integral is a Cauchy Principal Value.

Higher transcendental functions (cont'd)

	$f(x)$	$\pi^{-1} \int_{-\infty}^{\infty} f(x)\,(x-y)^{-1}\,dx$
(46)	$0 \qquad -\infty < x < 0$ $x^{\mu}\{\cos[(\mu-\tfrac{1}{2}\nu)\pi]\, J_{\nu}(ax^{1/2})$ $+ \sin[(\mu-\tfrac{1}{2}\nu)\pi]\, Y_{\nu}(ax^{1/2})]$ $0 < x < \infty$ $a > 0, \quad \lvert\mathrm{Re}\,\nu\rvert - 1 < \mathrm{Re}\,\mu < \tfrac{1}{2}$	$2\pi^{-1}\, \lvert y\rvert^{\mu} K_{\nu}(a\lvert y\rvert^{1/2}) \qquad -\infty < y < 0$ $y^{\mu}\{\sin[(\mu-\tfrac{1}{2}\nu)\pi]\, J_{\nu}(ay^{1/2})$ $- \cos[(\mu-\tfrac{1}{2}\nu)\pi]\, Y_{\nu}(ay^{1/2})\}$ $0 < y < \infty$
(47)	$0 \qquad -\infty < x < 0$ $e^{-ax}\, I_0(ax) \qquad 0 < x < \infty$ $a > 0$	$\pi^{-1}\, e^{-ay}\, K_0(a\lvert y\rvert)$
(48)	$\exp(-a\lvert x\rvert)\, I_0(ax) \qquad a > 0$	$-2\pi^{-1}\sinh(ay)\, K_0(a\lvert y\rvert)$
(49)	$\mathrm{sgn}\, x\, \exp(-a\lvert x\rvert)\, I_0(ax)$ $a > 0$	$2\pi^{-1}\cosh(ay)\, K_0(a\lvert y\rvert)$
(50)	$0 \qquad -\infty < x < -a$ $(a^2-x^2)^{\frac{1}{2}\nu} e^{-bx} J_{\nu}[b(a^2-x^2)^{1/2}]$ $-a < x < a$ $2(x^2-a^2)^{\frac{1}{2}\nu}\cos(\nu\pi)\, e^{-bx}$ $\times I_{\nu}[b(x^2-a^2)^{1/2}] \qquad a < x < \infty$ $a > 0, \quad b > 0$ $-1 < \mathrm{Re}\,\nu < \tfrac{1}{2}$	$2\pi^{-1}(y^2-a^2)^{\frac{1}{2}\nu} e^{-by} K_{\nu}[b(y^2-a^2)^{1/2}]$ $-\infty < y < -a$ $-(a^2-y^2)^{\frac{1}{2}\nu} e^{-by}\, Y_{\nu}[b(a^2-y^2)^{1/2}]$ $-a < y < a$ $2(y^2-a^2)^{\frac{1}{2}\nu} e^{-by}\{\pi^{-1}K_{\nu}[b(y^2-a^2)^{1/2}]$ $+ \sin(\nu\pi)\, I_{\nu}[b(y^2-a^2)^{1/2}]\}$ $a < y < \infty$
(51)	$e^{ax} K_0(a\lvert x\rvert) \qquad a > 0$	$\pi\, e^{ay}\, I_0(ay) \qquad -\infty < y < 0$ $0 \qquad 0 < y < \infty$

y is real, and the integral is a Cauchy Principal Value.

Higher transcendental functions (cont'd)

	$f(x)$	$\pi^{-1} \fint_{-\infty}^{\infty} f(x)\,(x-y)^{-1}\,dx$
(52)	$\sinh(ax)\, K_0(a\lvert x\rvert) \qquad a>0$	$\tfrac{1}{2}\pi \exp(-a\lvert y\rvert)\, I_0(ay)$
(53)	$\cosh(ax)\, K_0(a\lvert x\rvert) \qquad a>0$	$-\tfrac{1}{2}\pi \operatorname{sgn} y \exp(-a\lvert y\rvert)\, I_0(ay)$
(54)	$\lvert x\rvert^{-\nu} e^{ax} K_\nu(a\lvert x\rvert)$ $a>0, \quad -\tfrac{1}{2} < \operatorname{Re}\nu < \tfrac{1}{2}$	$\tfrac{1}{2}\pi \sec(\nu\pi)\, \lvert y\rvert^{-\nu} e^{ay}$ $\times [I_\nu(a\lvert y\rvert) + I_{-\nu}(a\lvert y\rvert)]$ $-\infty < y < 0$ $-\pi \tan(\nu\pi)\, y^{-\nu} e^{ay} K_\nu(ay)$ $0 < y < \infty$
(55)	$\lvert x\rvert^{-\nu} \sinh(ax)\, K_\nu(a\lvert x\rvert)$ $a>0, \quad -\tfrac{1}{2} < \operatorname{Re}\nu < 1$	$\lvert y\rvert^{-\nu} [\tfrac{1}{2}\pi \sec(\nu\pi) \exp(-a\lvert y\rvert)\, I_\nu(a\lvert y\rvert)$ $-\tan(\nu\pi) \sinh(a\lvert y\rvert)\, K_\nu(a\lvert y\rvert)]$
(56)	$\lvert x\rvert^{-\nu} \cosh(ax)\, K_\nu(a\lvert x\rvert)$ $a>0, \quad -\tfrac{1}{2} < \operatorname{Re}\nu < \tfrac{1}{2}$	$-\lvert y\rvert^{-\nu} \operatorname{sgn} y\, [\tfrac{1}{2}\pi \sec(\nu\pi) \exp(-a\lvert y\rvert)$ $\times I_\nu(a\lvert y\rvert) + \tan(\nu\pi) \cosh(ay) K_\nu(a\lvert y\rvert)]$
(57)	$\lvert x\rvert^{2\nu} \exp(-ax^2)\, [K_\nu(ax^2)$ $+ \pi \sin(\nu\pi)\, I_\nu(ax^2)]$ $a>0, \quad -\tfrac{1}{4} < \operatorname{Re}\nu < \tfrac{1}{4}$	$-\pi \cos(\nu\pi) \operatorname{sgn} y\, \lvert y\rvert^{2\nu}$ $\times \exp(-ay^2)\, I_\nu(ay^2)$
(58)	$\operatorname{sgn} x\, \lvert x\rvert^{2\nu} \exp(-ax^2)\, I_\nu(ax^2)$ $a>0, \quad -\tfrac{1}{4} < \operatorname{Re}\nu < \tfrac{1}{2}$	$\lvert y\rvert^{2\nu} \exp(-ay^2)$ $\times [\pi^{-1} \sec(\nu\pi)\, K_\nu(ay^2)$ $+ \tan(\nu\pi)\, I_\nu(ay^2)]$
(59)	$\operatorname{sgn} x\, \lvert x\rvert^{-\nu} \mathbf{H}_\nu(a\lvert x\rvert)$ $a>0, \quad \operatorname{Re}\nu > -3/2$	$\lvert y\rvert^{-\nu} J_\nu(a\lvert y\rvert)$

y is real, and the integral is a Cauchy Principal Value.

Higher transcendental functions (cont'd)

	$f(x)$	$\pi^{-1} \int_{-\infty}^{\infty} f(x)\,(x-y)^{-1}\,dx$
(60)	$0 \qquad -\infty < x < 0$ $G_{pq}^{mn}\left(ax \middle\vert \begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix}\right)$ $0 < x < \infty$ $p + q < 2(m+n)$ $\lvert\arg a\rvert < (m + n - \tfrac{1}{2}p - \tfrac{1}{2}q)\pi$ $\operatorname{Re} a_j < 1 \qquad j = 1, \dots, n$ $\operatorname{Re} b_j > -1 \qquad j = 1, \dots, m$	$\pi^{-1} G_{p+1,q+1}^{m+1,n+1}\left(a\lvert y\rvert \middle\vert \begin{matrix} 0, a_1, \dots, a_p \\ 0, b_1, \dots, b_q \end{matrix}\right)$ $-\infty < y < 0$ $(-1)^k G_{p+2,q+2}^{m+1,n+1}\left(ay \middle\vert \begin{matrix} 0, a_1, \dots, a_p, \tfrac{1}{2}+k \\ 0, b_1, \dots, b_q, \tfrac{1}{2}+k \end{matrix}\right)$ $0 < y < \infty$ k integer
(61)	$G_{pq}^{mn}\left(ax^2 \middle\vert \begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix}\right)$ $p + q < 2(m+n)$ $\lvert\arg a\rvert < (m + n - \tfrac{1}{2}p - \tfrac{1}{2}q)\pi$ $\operatorname{Re} a_j < 1 \qquad j = 1, \dots, n$ $\operatorname{Re} b_j > -\tfrac{1}{2} \qquad j = 1, \dots, m$	$\operatorname{sgn} y\, G_{p+2,q+2}^{m+1,n+1}\left(ay^2 \middle\vert \begin{matrix} \tfrac{1}{2}, a_1, \dots, a_p, 1 \\ \tfrac{1}{2}, b_1, \dots, b_q, 1 \end{matrix}\right)$
(62)	$x\, G_{pq}^{mn}\left(ax^2 \middle\vert \begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix}\right)$ $p + q < 2(m+n)$ $\lvert\arg a\rvert < (m + n - \tfrac{1}{2}p - \tfrac{1}{2}q)\pi$ $\operatorname{Re} a_j < \tfrac{1}{2} \qquad j = 1, \dots, n$ $\operatorname{Re} b_j > -1 \qquad j = 1, \dots, m$	$\lvert y\rvert\, G_{p+2,q+2}^{m+1,n+1}\left(ay^2 \middle\vert \begin{matrix} -\tfrac{1}{2}, a_1, \dots, a_p, 0 \\ -\tfrac{1}{2}, b_1, \dots, b_q, 0 \end{matrix}\right)$

y is real, and the integral is a Cauchy Principal Value.

INTEGRALS OF HIGHER TRANSCENDENTAL FUNCTIONS

This part contains mostly integrals which have not been listed in the tables of Chapters I to XV.

CHAPTER XVI

ORTHOGONAL POLYNOMIALS

In this chapter we list integrals involving the classical orthogonal polynomials. For the theory of these polynomials see H.T.F. vol. II, Chapter X and the literature quoted there, especially Szegö's book. The notation used in the present compilation for Hermite polynomials differs from that used in H.T.F.

Further integrals may be evaluated by the methods mentioned in the introduction to vol. I, by the use of Rodrigues' formula and its analogues (given below) followed by repeated integrations by parts, by using generating functions (see below), and also by utilizing the relations (see below) between the various systems of orthogonal polynomials and between these polynomials and Legendre functions, hypergeometric series, confluent hypergeometric functions in conjunction with tables given in other chapters of this book.

Tchebichef polynomials

$$T_n(x) = (-1)^n\, T_n(-x) = \cos(n\,\theta) \qquad x = \cos\theta$$

$$= \frac{(1-x^2)^{1/2}}{2^n (1/2)_n} \left(-\frac{d}{dx}\right)^n [(1-x^2)^{n-1/2}]$$

$$= {}_2F_1(-n, n; 1/2; 1/2 - 1/2\, x)$$

$$= 1/2\, n\, C_n^0(x) = \frac{n!}{(1/2)_n} P_n^{(-1/2, -1/2)}(x)$$

$$U_n(x) = (-1)^n\, U_n(-x) = \frac{\sin[(n+1)\theta]}{\sin\theta} \qquad x = \cos\theta$$

$$= \frac{(n+1)(1-x^2)^{-1/2}}{2^{n+1} (1/2)_{n+1}} \left(-\frac{d}{dx}\right)^n [(1-x^2)^{n+1/2}]$$

$$U_n(x) = (n+1)\ {}_2F_1\left(-n,\ n+1; \frac{3}{2}; \frac{1-x}{2}\right)$$

$$= C_n^{1}(x) = \frac{(n+1)!}{2(\tfrac{1}{2})_{n+1}}\ P_n^{(\frac{1}{2},\frac{1}{2})}(x)$$

$$1 + 2\sum_{n=1}^{\infty} T_n(x)\, z^n = \frac{1-z^2}{1-2xz+z^2}$$

$$\sum_{n=0}^{\infty} U_n(x)\, z^n = (1-2xz+z^2)^{-1}$$

For other generating functions see H.T.F. vol II, p. 186.

Legendre polynomials

$$P_n(x) = (-1)^n P_n(-x) = \frac{1}{2^n\, n!}\left(\frac{d}{dx}\right)^n [(x^2-1)^n]$$

$$= {}_2F_1(-n,\ n+1;\ 1;\ \tfrac{1}{2} - \tfrac{1}{2}x)$$

$$= \frac{2^n (\tfrac{1}{2})_n}{n!}\ x^n\ {}_2F_1(-\tfrac{1}{2}n,\ \tfrac{1}{2} - \tfrac{1}{2}n;\ \tfrac{1}{2};\ x^{-2})$$

$$= C_n^{\frac{1}{2}}(x) = P_n^{(0,0)}(x)$$

$$\sum_{n=0}^{\infty} P_n(x)\, z^n = (1-2xz+z^2)^{-\frac{1}{2}}$$

For the connection with Legendre functions see H.T.F. vol. I, p. 150 ff; for additional hypergeometric series representing Legendre polynomials see H.T.F. vol. I, p. 124-131 ($\mu = 0$, $\nu = n$), and vol. II, p. 180; and for other generating functions see H.T.F. vol II, p. 182.

For the definition of associated Legendre polynomials, and their properties see H.T.F. vol. I, p. 148 ff. and below.

Gegenbauer polynomials

These polynomials are also called *ultraspherical polynomials* and are denoted by $P_n^{(\nu)}(x)$.

$$C_n^\nu(x) = (-1)^n C_n^\nu(-x)$$

$$= \frac{2^{\nu-\frac{1}{2}}\,\Gamma(2\nu+n)\,\Gamma(\nu+\frac{1}{2})}{n!\,\Gamma(2\nu)}\,(x^2-1)^{\frac{1}{4}-\frac{1}{2}\nu}\,P^{\frac{1}{2}-\nu}_{n+\nu-\frac{1}{2}}(x)$$

$$= \frac{(2\nu)_n\,(1-x^2)^{\frac{1}{2}-\nu}}{2^n\,n!\,(\nu+\frac{1}{2})_n}\left(-\frac{d}{dx}\right)^n[(1-x^2)^{n+\nu-\frac{1}{2}}]$$

$$= \frac{(2\nu)_n}{n!}\,{}_2F_1(-n,\,n+2\nu;\,\nu+\tfrac{1}{2};\,\tfrac{1}{2}-\tfrac{1}{2}x)$$

$$= \frac{2^n\,(\nu)_n}{n!}(x-1)^n\,{}_2F_1\left(-n,\frac{1}{2}-n-\nu;\,1-2n-2\nu;\,\frac{2}{1-x}\right)$$

$$= \frac{(2\nu)_n}{(\nu+\frac{1}{2})_n}\,P_n^{(\nu-\frac{1}{2},\,\nu-\frac{1}{2})}(x)$$

$$C_{2n}^\nu(x) = (-1)^n\frac{(\nu)_n}{n!}\,{}_2F_1(-n,\,n+\nu;\,\tfrac{1}{2};\,x^2)$$

$$= \frac{(\nu)_n}{(\frac{1}{2})_n}\,P_n^{(\nu-\frac{1}{2},\,-\frac{1}{2})}(2x^2-1)$$

$$C_{2n+1}^\nu(x) = (-1)^n\frac{(\nu)_{n+1}}{n!}\,2x\;{}_2F_1(-n,\,n+\nu+1;\,3/2;\,x^2)$$

$$= \frac{(\nu)_{n+1}}{(\frac{1}{2})_{n+1}}\,x\,P_n^{(\nu-\frac{1}{2},\,\frac{1}{2})}(2x^2-1)$$

$$\sum_{n=0}^{\infty} C_n^\nu(x)\,z^n = (1-2xz+z^2)^{-\nu}$$

For the connection with Legendre functions, and for additional hypergeometric expansions see H.T.F. vol. I, p. 175 ff, p. 124-131, Vol. II, p. 176; for other generating functions see H.T.F. vol. II, p. 177.

Jacobi polynomials

$$P_n^{(\alpha,\beta)}(x) = (-1)^n P_n^{(\beta,\alpha)}(-x)$$

$$= \frac{(1-x)^{-\alpha}(1+x)^{-\beta}}{2^n\, n!}\left(-\frac{d}{dx}\right)^n [(1-x)^{\alpha+n}(1+x)^{\beta+n}]$$

$$= \binom{n+\alpha}{n}\, {}_2F_1\left(-n,\, n+\alpha+\beta+1;\, \alpha+1;\, \frac{1-x}{2}\right)$$

$$= (-1)^n \binom{n+\beta}{n}\, {}_2F_1\left(-n,\, n+\alpha+\beta+1;\, \beta+1;\, \frac{1+x}{2}\right)$$

$$= \binom{n+\alpha}{n}\left(\frac{1+x}{2}\right)^n {}_2F_1\left(-n,\, -n-\beta;\, \alpha+1;\, \frac{x-1}{x+1}\right)$$

$$= \binom{n+\beta}{n}\left(\frac{x-1}{2}\right)^n {}_2F_1\left(-n,\, -n-\alpha;\, \beta+1;\, \frac{x+1}{x-1}\right)$$

$$\sum_{n=0}^{\infty} P_n^{(\alpha,\beta)}(x)\, z^n = 2^{\alpha+\beta} R^{-1} (1-z+R)^{-\alpha}(1+z+R)^{-\beta}$$

$$R = (1-2xz+z^2)^{1/2}$$

Other expansions may be obtained from those given above by means of the transformations given in H.T.F. vol. I, sec. 2.9.

Hermite polynomials

$$\mathrm{He}_n(x) = (-1)^n \mathrm{He}_n(-x) = 2^{-\frac{1}{2}n} H_n(2^{-\frac{1}{2}} x)$$

$$= e^{\frac{1}{2}x^2}\left(-\frac{d}{dx}\right)^n [e^{-\frac{1}{2}x^2}]$$

$$= x^n\, {}_2F_0\left(-\frac{n}{2}, \frac{1-n}{2}; -\frac{2}{x^2}\right)$$

$$= 2^{\frac{1}{2}n+\frac{1}{4}} x^{-\frac{1}{2}} e^{\frac{1}{4}x^2} W_{\frac{1}{2}n+\frac{1}{4},\, -\frac{1}{4}}\left(\frac{x^2}{2}\right)$$

$$= e^{\frac{1}{4}x^2} D_n(x)$$

$$\mathrm{He}_{2n}(x) = (-2)^n (\tfrac{1}{2})_n \; {}_1F_1(-n\,;\, \tfrac{1}{2};\, \tfrac{1}{2}x^2)$$
$$= (-2)^n n!\, L_n^{-1/2}(\tfrac{1}{2}x^2)$$

$$\mathrm{He}_{2n+1}(x) = (-2)^n (3/2)_n\, x \; {}_1F_1(-n;\, 3/2;\, \tfrac{1}{2}x^2)$$
$$= (-2)^n n!\, x\, L_n^{1/2}(\tfrac{1}{2}x^2)$$

$$\sum_{n=0}^{\infty} \mathrm{He}_n(x) \frac{z^n}{n!} = \exp(-\tfrac{1}{2}z^2 + xz)$$

For other generating functions see H.T.F. vol. II, p. 194.

Laguerre polynomials

$$L_n^{\alpha}(x) = \frac{1}{n\,!}\, e^{x} x^{-\alpha} \left(\frac{d}{dx}\right)^n [e^{-x} x^{n+\alpha}]$$

$$= \binom{n+\alpha}{n} \; {}_1F_1(-n;\, \alpha+1\,;\, x)$$

$$= \frac{(-1)^n}{n\,!}\, x^n \; {}_2F_0(-n,\, -\alpha-n\,;\, -1/x)$$

$$= \frac{(-1)^n}{n!}\, x^{-\frac{1}{2}\alpha-\frac{1}{2}}\, e^{\frac{1}{2}x}\, W_{\frac{1}{2}+\frac{1}{2}\alpha+n,\, \frac{1}{2}\alpha}(x)$$

$$\sum_{n=0}^{\infty} L_n^{\alpha}(x)\, z^n = (1-z)^{-\alpha-1} \exp \frac{xz}{z-1}$$

For other generating functions see H.T.F. vol. II, p. 189.

ORTHOGONAL POLYNOMIALS

16.1. Tchebichef polynomials

The integrals in this section may also be expressed as integrals of trigonometric functions.

In this section m and n are non-negative integers.

(1)	$\int_{-1}^{1} (1-x)^{-\frac{1}{2}} (1+x)^{\alpha} T_n(x)\,dx = \dfrac{2^{\alpha+2n+\frac{1}{2}} \pi^{\frac{1}{2}} (n!)^2\, \Gamma(\alpha+1)\, \Gamma(\alpha+3/2)}{(2n)!\, \Gamma(\alpha+n+3/2)\, \Gamma(\alpha-n+3/2)}$ $\operatorname{Re} \alpha > -1$
(2)	$\int_{-1}^{1} (1-x)^{\alpha}(1+x)^{\beta} T_n(x)\,dx = \dfrac{2^{\alpha+\beta+2n+1} (n!)^2\, \Gamma(\alpha+1)\, \Gamma(\beta+1)}{(2n)!\, \Gamma(\alpha+\beta+2)}$ $\times\, {}_3F_2(-n, n, \alpha+1; \tfrac{1}{2}, \alpha+\beta+2; 1)$ $\operatorname{Re} \alpha > -1, \quad \operatorname{Re} \beta > -1$
(3)	$\fint_{-1}^{1} (x-y)^{-1} (1-x^2)^{-\frac{1}{2}} T_n(x)\,dx = \pi\, U_{n-1}(y)$ $-1 < y < 1$
(4)	$\int_{-1}^{1} \sin(xyz) \cos[(1-x^2)^{\frac{1}{2}} (1-y^2)^{\frac{1}{2}} z]\, T_{2n+1}(x)\,dx$ $= (-1)^n \pi\, T_{2n+1}(y)\, J_{2n+1}(z)$
(5)	$\int_{-1}^{1} \cos(xyz) \cos[(1-x^2)^{\frac{1}{2}} (1-y^2)^{\frac{1}{2}} z]\, T_{2n}(x)\,dx$ $= (-1)^n \pi\, T_{2n}(y)\, J_{2n}(z)$
(6)	$\int_{-1}^{1} [T_n(x)]^2\,dx = 1 - (4n^2-1)^{-1}$

Tchebichef polynomials (cont'd) $m, n = 0, 1, 2, \ldots$

(7)	$\int_{-1}^{1} (1-x^2)^{-1/2} [T_0(x)]^2 \, dx = \pi$	
(8)	$\int_{-1}^{1} (1-x^2)^{-1/2} [T_n(x)]^2 \, dx = \tfrac{1}{2}\pi$	$n \neq 0$
(9)	$\int_{-1}^{1} (1-x^2)^{-1/2} T_m(x) \, T_n(x) \, dx = 0$	$m \neq n$
(10)	$\int_{-1}^{1} (1-x)^{-1/2} (1+x)^{m-n-3/2} T_m(x) \, T_n(x) \, dx = 0$	$m > n$
(11)	$\int_{-1}^{1} (1-x)^{-1/2} (1+x)^{m+n-3/2} T_m(x) \, T_n(x) \, dx$ $= \dfrac{\pi(2m+2n-2)!}{2^{m+n}(2m-1)!\,(2n-1)!}$	$m + n \neq 0$
(12)	$\int_{-1}^{1} (1+x)^{-1/2} (1-x)^{\alpha-1} T_m(x) \, T_n(x) \, dx$ $= \dfrac{\pi^{1/2} \, 2^{\alpha-1/2} \, \Gamma(\alpha) \, \Gamma(n-\alpha+\tfrac{1}{2})}{\Gamma(\tfrac{1}{2}-\alpha) \, \Gamma(\alpha+n+\tfrac{1}{2})}$ $\times \, {}_4F_3(-m, m, \alpha, \alpha+\tfrac{1}{2}; \tfrac{1}{2}, \alpha+n+\tfrac{1}{2}, \alpha-n+\tfrac{1}{2}; 1)$	$\mathrm{Re}\, \alpha > 0$
(13)	$\int_0^1 x^{-1/2} (1-x^2)^{-1/2} e^{-2\alpha/x} T_n(x) \, dx = \pi^{1/2} D_{n-1/2}(2\alpha^{1/2}) \, D_{-n-1/2}(2\alpha^{1/2})$	$\mathrm{Re}\, \alpha > 0$
(14)	$\int_{-1}^{1} (1-x^2)^{-1/2} T_n(1-x^2 y) \, dx = \tfrac{1}{2}\pi [P_n(1-y) + P_{n-1}(1-y)]$	
(15)	$\int_0^{\infty} (1+x^2)^{-n} \operatorname{sech}(\tfrac{1}{2}\pi x) \, T_{2n}[(1+x^2)^{-1/2}] \, dx$ $= (-1)^{n+1} \dfrac{2\pi^{2n}}{(2n)!} (2^{n-1} - 1) B_{2n}$	

Tchebichef polynomials (cont'd) $m, n = 0, 1, 2, \ldots$

(16)	$\int_0^\infty (1+x^2)^{-\frac{1}{2}n} \operatorname{sech}(\tfrac{1}{2}\pi x)\, T_n[(1+x^2)^{-\frac{1}{2}}]\, dx = 2^{1-n}(1-2^{1-n})\zeta(n)$
(17)	$\int_0^\infty (1+x^2)^{\frac{1}{2}-n} [\cosh(\tfrac{1}{2}\pi x)]^{-2}\, T_{2n-1}[(1+x^2)^{-\frac{1}{2}}]\, dx$ $= 2(-1)^{n+1}\pi^{2n-1} \dfrac{2n-1}{(2n)!} B_{2n}$
(18)	$\int_0^\infty (1+x^2)^{-\frac{1}{2}n} [\cosh(\tfrac{1}{2}\pi x)]^{-2}\, T_n[(1+x^2)^{-\frac{1}{2}}]\, dx$ $= \pi^{-1} n\, 2^{1-n} \zeta(n+1)$
(19)	$\int_0^\infty (a^2+x^2)^{-\frac{1}{2}n} \operatorname{sech}(\tfrac{1}{2}\pi x)\, T_n[a(a^2+x^2)^{-\frac{1}{2}}]\, dx$ $= 2^{1-2n}\left[\zeta\left(n, \dfrac{a+1}{4}\right) - \zeta\left(n, \dfrac{a+3}{4}\right)\right] = 2^{1-n}\,\Phi\left(-1, n, \dfrac{a+1}{2}\right)$ $\operatorname{Re} a > 0$
(20)	$\int_0^\infty (a^2+x^2)^{-\frac{1}{2}n} [\cosh(\tfrac{1}{2}\pi x)]^{-2}\, T_n[a(a^2+x^2)^{-\frac{1}{2}}]\, dx$ $= \pi^{-1} n\, 2^{1-n} \zeta\left(n+1, \dfrac{a+1}{2}\right)$ $\qquad \operatorname{Re} a > 0$
(21)	$\int_{-1}^1 (1-x)^{\frac{1}{2}} (1+x)^{a}\, U_n(x)\, dx = \dfrac{\pi^{1/2}\, 2^{a+2n+3/2}[(n+1)!]^2\,\Gamma(a+\frac{1}{2})\Gamma(a+1)}{(2n+2)!\,\Gamma(a+n+5/2)\,\Gamma(a-n+1/2)}$ $\operatorname{Re} a > -1$
(22)	$\int_{-1}^1 (1-x)^{a}(1+x)^{\beta}\, U_n(x)\, dx = \dfrac{2^{a+\beta+2n+2}[(n+1)!]^2\,\Gamma(a+1)\,\Gamma(\beta+1)}{(2n+2)!\,\Gamma(a+\beta+2)}$ $\times\, {}_3F_2(-n, n+1, a+1; 3/2, a+\beta+2; 1)$ $\operatorname{Re} a > -1, \quad \operatorname{Re} \beta > -1$

Tchebichef polynomials (cont'd) $m, n = 0, 1, 2, \ldots$

(23)	$⨍_{-1}^{1} (x-y)^{-1} (1-x^2)^{-1/2} U_n(x)\,dx = -\pi\, T_{n+1}(y)$	$-1 < y < 1$
(24)	$\int_{-1}^{1} \cos(xyz) \sin[(1-x^2)^{1/2} (1-y^2)^{1/2} z]\, U_{2n}(x)\,dx$ $= (-1)^n \pi (1-y^2)^{1/2} U_{2n}(y) J_{2n+1}(z)$	
(25)	$\int_{-1}^{1} \sin(xyz) \sin[(1-x^2)^{1/2} (1-y^2)^{1/2} z]\, U_{2n+1}(x)\,dx$ $= (-1)^n \pi (1-y^2)^{1/2} U_{2n+1}(y) J_{2n+2}(z)$	
(26)	$\int_{-1}^{1} (1-x)^{-1/2} (1+x)^{1/2} [U_n(x)]^2\,dx = (n+1)\pi$	
(27)	$\int_{-1}^{1} (1-x^2)^{1/2} [U_n(x)]^2\,dx = \tfrac{1}{2}\pi$	
(28)	$\int_{-1}^{1} (1-x^2)^{1/2} U_m(x)\, U_n(x)\,dx = 0$	$m \neq n$
(29)	$\int_{-1}^{1} (1-x)(1+x)^{1/2} U_m(x)\, U_n(x)\,dx$ $= \dfrac{2^{5/2}(m+1)(n+1)}{(m+n+3/2)(m+n+5/2)[1-4(m-n)^2]}$	
(30)	$\int_{-1}^{1} (1-x)^{1/2} (1+x)^{m-n-1/2} U_m(x)\, U_n(x)\,dx = 0$	$m > n$
(31)	$\int_{-1}^{1} (1-x)^{1/2} (1+x)^{m+n+3/2} U_m(x)\, U_n(x)\,dx$ $\times \dfrac{\pi(2m+2n+2)!}{2^{m+n+2}(2m+1)!\,(2n+1)!}$	

Tchebichef polynomials (cont'd) $m, n = 0, 1, 2, \ldots$

(32)	$\int_{-1}^{1} (1+x)^{1/2}(1-x)^{a-1} U_m(x) U_n(x)\, dx = \dfrac{\pi^{1/2}\, 2^{a-1/2}\, (m+1)(n+1)\, \Gamma(a)\, \Gamma(n-a+3/2)}{\Gamma(3/2-a)\, \Gamma(3/2+a+n)} \times {}_4F_3(-m, m+2, a, a-1/2; 3/2, a+n+3/2, a-n-1/2; 1)$ Re $a > 0$
(33)	$\int_{-1}^{1} (1-x^2)^{-1/2}\, U_{2n}(xz)\, dx = \pi P_n(2z^2-1)$
(34)	$\int_{-1}^{1} U_n[x(1-y^2)^{1/2}(1-z^2)^{1/2} + yz]\, dx = \dfrac{2}{n+1} U_n(y)\, U_n(z)$
(35)	$\displaystyle\int_0^\infty \frac{x\, U_{2n-1}[(1+x^2)^{-1/2}]}{(1+x^2)^{n+1/2}(e^{\pi x}+1)}\, dx = \frac{1}{2(2n-1)} + \frac{(-1)^n \pi^{2n}}{2(2n)!} B_{2n}$
(36)	$\displaystyle\int_0^\infty \frac{x\, U_n[(1+x^2)^{-1/2}]}{(1+x^2)^{\frac{1}{2}n+1}(e^{\pi x}+1)}\, dx = \frac{1}{2n} - 2^{-n-1}\zeta(n+1)$
(37)	$\displaystyle\int_0^\infty \frac{x\, U_{2n-1}[(1+x^2)^{-1/2}]}{(1+x^2)^{n+1/2}(e^{2\pi x}-1)}\, dx = \frac{(-1)^{n+1}(2\pi)^{2n}}{4(2n)!} B_{2n} - \frac{1}{4} - \frac{1}{4n-2}$
(38)	$\displaystyle\int_0^\infty \frac{x\, U_n[(1+x^2)^{-1/2}]}{(1+x^2)^{\frac{1}{2}n+1}(e^{2\pi x}-1)}\, dx = \frac{1}{2}\zeta(n+1) - \frac{1}{4} - \frac{1}{2n}$
(39)	$\displaystyle\int_0^\infty \frac{x\, U_n[a(a^2+x^2)^{-1/2}]}{(a^2+x^2)^{\frac{1}{2}n+1}(e^{\pi x}+1)}\, dx = \frac{a^{-n}}{2n} - 2^{-n-1}\zeta\left(n+1, \frac{a+1}{2}\right)$ Re $a > 0$

Tchebichef polynomials (cont'd) $m, n = 0, 1, 2, \ldots$

(40)	$\int_0^\infty \frac{x U_n[a(a^2+x^2)^{-1/2}]}{(a^2+x^2)^{\frac{1}{2}n+1}(e^{2\pi x}-1)} dx = \frac{1}{2}\zeta(n+1,a) - \frac{a^{-n-1}}{4} - \frac{a^{-n}}{2n}$ Re $a > 0$

16.2. Legendre polynomials

See also under Gegenbauer polynomials, Legendre functions, hypergeometric series.

In this section m and n are non-negative integers.

(1)	$\int_0^1 x^\lambda P_{2m}(x)\,dx = \frac{(-1)^m(-\frac{1}{2}\lambda)_m}{2(\frac{1}{2}+\frac{1}{2}\lambda)_{m+1}}$ Re $\lambda > -1$
(2)	$\int_0^1 x^\lambda P_{2m+1}(x)\,dx = \frac{(-1)^m(\frac{1}{2}-\frac{1}{2}\lambda)_m}{2(1+\frac{1}{2}\lambda)_{m+1}}$ Re $\lambda > -2$
(3)	$\int_{-1}^1 (1-x)^{-1/2} P_n(x)\,dx = \frac{2^{3/2}}{2n+1}$
(4)	$\int_{-1}^1 (1-x^2)^{-1/2} P_{2m}(x)\,dx = \pi\left[\frac{(\frac{1}{2})_m}{m!}\right]^2$
(5)	$\int_{-1}^1 x(1-x^2)^{-1/2} P_{2m+1}(x)\,dx = \pi\frac{(\frac{1}{2})_m(\frac{1}{2})_{m+1}}{m!(m+1)!}$
(6)	$\int_{-1}^1 (1-x)^{\alpha-1}(1+x)^{\beta-1} P_n(x)\,dx = \frac{2^{\alpha+\beta-1}\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$ $\times {}_3F_2(-n, 1+n, \alpha; 1, \alpha+\beta; 1)$ Re $\alpha > 0$, Re $\beta > 0$

Legendre polynomials (cont'd) $m, n = 0, 1, 2, \ldots$

(7)	$\int_{-1}^{1} (z-x)^{-1} P_n(x)\,dx = 2Q_n(z)$	z in the cut plane
(8)	$\int_{-1}^{1} x\,(z-x)^{-1} P_0(x)\,dx = 2Q_1(z)$	z in the cut plane
(9)	$\int_{-1}^{1} (z-x)^{-1} x^{n+1} P_n(x)\,dx = 2z^{n+1} Q_n(z) - \dfrac{2^{n+1}(n!)^2}{(2n+1)!}$	z in the cut plane
(10)	$\int_{-1}^{1} (z-x)^{-1} x^{m} P_n(x)\,dx = 2z^{m} Q_m(z)$	$m \le n$, z in the cut plane
(11)	$\int_{-1}^{1} (a^2+b^2-2abx)^{-\frac{1}{2}} \sin[\lambda(a^2+b^2-2abx)^{\frac{1}{2}}] P_n(x)\,dx$ $= \pi(ab)^{-\frac{1}{2}} J_{n+\frac{1}{2}}(a\lambda) J_{n+\frac{1}{2}}(b\lambda)$	$a, b > 0$
(12)	$\int_{-1}^{1} (a^2+b^2-2abx)^{-\frac{1}{2}} \cos[\lambda(a^2+b^2-2abx)^{\frac{1}{2}}] P_n(x)\,dx$ $= \pi(ab)^{-\frac{1}{2}} J_{n+\frac{1}{2}}(a\lambda) Y_{n+\frac{1}{2}}(b\lambda)$	$0 \le a \le b$
(13)	$\int_{-1}^{1} [P_n(x)]^2\,dx = (n+\frac{1}{2})^{-1}$	
(14)	$\int_{-1}^{1} P_m(x) P_n(x)\,dx = 0$	$m \neq n$
(15)	$\int_{-1}^{1} (1+x)^{m+n} P_m(x) P_n(x)\,dx = \dfrac{2^{m+n+1}[(m+n)!]^4}{(m!\,n!)^2\,(2m+2n+1)!}$	

The complex z-plane is cut along the real axis from -1 to 1.

Legendre polynomials (cont'd) $m, n = 0, 1, 2, \ldots$

(16)	$\int_{-1}^{1} (1+x)^{m-n-1} P_m(x) P_n(x)\,dx = 0$	$m > n$
(17)	$\int_{-1}^{1} (1-x)^{a-1} P_m(x) P_n(x)\,dx = \dfrac{2^a \Gamma(a)\Gamma(n-a+1)}{\Gamma(1-a)\Gamma(n+a+1)}$ $\times\, {}_4F_3(-m, m+1, a, a; 1, a+n+1, a-n; 1)$	$\mathrm{Re}\, a > 0$
(18)	$\int_{-1}^{1} (z-x)^{-1} P_m(x) P_n(x)\,dx = 2P_m(x) Q_n(z)$	$m \le n$, z in the cut plane
(19)	$\int_{-1}^{1} (z-x)^{-1} P_n(x) P_{n+1}(x)\,dx = 2P_{n+1}(z) Q_n(z) - 2(n+1)^{-1}$	z in the cut plane
(20)	$\int_{-1}^{1} x(z-x)^{-1} [P_n(x)]^2\,dx = 2z\, P_n(z) Q_n(z) - 2(2n+1)^{-1}$	z in the cut plane
(21)	$\int_{-1}^{1} x(z-x)^{-1} P_m(x) P_n(x)\,dx = 2z\, P_m(z) Q_n(z)$	$m < n$, z in the cut plane
	For other similar integrals see MacRobert, T.M., 1948: *Proc. Glasgow Math. Assoc.* 1, 10-12.	
(22)	$\int_{0}^{1} x^{2\mu-1} P_n(1-2x^2)\,dx = \dfrac{(-1)^n [\Gamma(\mu)]^2}{2\Gamma(\mu+n)\Gamma(\mu-n)}$	$\mathrm{Re}\, \mu > 0$
(23)	$\int_{0}^{1} x(a^2+x^2)^{-1/2} P_n(1-2x^2)\,dx = \dfrac{[a+(a^2+1)^{1/2}]^{-2n-1}}{2n+1}$	$\mathrm{Re}\, a > 0$

The complex z-plane is cut along the real axis from -1 to 1.

Legendre polynomials (cont'd) $k, l, m, n = 0, 1, 2, \ldots$

(24)	$\int_{-1}^{1} (1-x)^{\alpha}(1+x)^{\beta}\,\mathrm{P}_n^m(x)\,dx$ See Shabde, N.G., 1940: *Bull. Calcutta Math. Soc.* 32, 121-128.
(25)	$\int_{-1}^{1} (z-x)^{-1}(1-x^2)^{\frac{1}{2}m}\,\mathrm{P}_n^m(x)\,dx = (-2)^m (z^2-1)^{\frac{1}{2}m} Q_n^m(z)$ $m \le n$, z in the cut plane
(26)	$\int_{-1}^{1} x^k (z-x)^{-1}(1-x^2)^{\frac{1}{2}m}\,\mathrm{P}_n^m(x)\,dx = (-2)^m z^k (z^2-1)^{\frac{1}{2}m} Q_n^m(z)$ $m \le n$, $k = 0, 1, \ldots, n-m$, z in the cut plane
(27)	$\int_{-1}^{1} [\mathrm{P}_n^m(x)]^2\,dx = \dfrac{2}{2n+1}\,\dfrac{(n+m)!}{(n-m)!}$ $\qquad m \le n$
(28)	$\int_{-1}^{1} \mathrm{P}_n^m(x)\,\mathrm{P}_k^m(x)\,dx = 0$ $\qquad k \ne n$
(29)	$\int_{-1}^{1} \mathrm{P}_n^m(x)\,\mathrm{Q}_k^m(x)\,dx = (-1)^m \dfrac{1-(-1)^{k+n}}{(n-k)(n+k+1)}\,\dfrac{(k+m)!}{(k-m)!}$
(30)	$\int_{-1}^{1} (1-x^2)^{-1}\,\mathrm{P}_n^m(x)\,\mathrm{P}_n^k(x)\,dx = 0$ $\qquad k \ne m$
(31)	$\int_{-1}^{1} (1-x^2)^{-1}[\mathrm{P}_n^m(x)]^2\,dx = \dfrac{(n+m)!}{m(n-m)!}$

The complex z-plane is cut along the real axis from -1 to 1.

Gegenbauer polynomials (cont'd) $m, n = 0, 1, 2, \ldots$

(4)	$\int_{-1}^{1} (1-x)^{\alpha}(1+x)^{\beta} C_n^{\nu}(x)\,dx = \dfrac{2^{\alpha+\beta+1}\,\Gamma(\alpha+1)\,\Gamma(\beta+1)\,\Gamma(n+2\nu)}{n!\,\Gamma(2\nu)\,\Gamma(\alpha+\beta+2)}$ $\times\, {}_3F_2(-n, n+2\nu, \alpha+1; \nu+\tfrac{1}{2}, \alpha+\beta+2; 1)$ $\operatorname{Re}\alpha > -1, \quad \operatorname{Re}\beta > -1$
(5)	$\int_{-1}^{1} x^m (z-x)^{-1} (1-x^2)^{\nu-\frac{1}{2}} C_n^{\nu}(x)\,dx$ $= \dfrac{\pi^{1/2}\,2^{3/2-\nu}}{\Gamma(\nu)}\, e^{-(\nu-\frac{1}{2})\pi i} z^m (z^2-1)^{\frac{1}{2}\nu-\frac{1}{4}}\, Q_{n+\nu-\frac{1}{2}}^{\nu-\frac{1}{2}}(z)$ $m \le n, \quad \operatorname{Re}\nu > -\tfrac{1}{2}, \quad z$ in the cut plane
(6)	$\int_{-1}^{1} x^{n+1} (z-x)^{-1} (1-x^2)^{\nu-\frac{1}{2}} C_n^{\nu}(x)\,dx$ $= \dfrac{\pi^{1/2}\,2^{3/2-\nu}}{\Gamma(\nu)}\, e^{-(\nu-\frac{1}{2})\pi i} z^{n+1} (z^2-1)^{\frac{1}{2}\nu-\frac{1}{4}}\, Q_{n+\nu-\frac{1}{2}}^{\nu-\frac{1}{2}}(z)$ $- \dfrac{\pi\, 2^{1-2\nu-n}\, n!}{\Gamma(\nu)\Gamma(\nu+n+1)}$ $\operatorname{Re}\nu > -\tfrac{1}{2}, \quad z$ in the cut plane
(7)	$\int_{-1}^{1} (1-x^2)^{\nu-\frac{1}{2}} e^{iax} C_n^{\nu}(x)\,dx = \dfrac{\pi\, 2^{1-\nu}\, i^n\, \Gamma(2\nu+n)}{n!\,\Gamma(\nu)}\, a^{-\nu} J_{\nu+n}(a)$ $\operatorname{Re}\nu > -\tfrac{1}{2}$
(8)	$\int_{-1}^{1} (1-x^2)^{\nu-\frac{1}{2}} [C_n^{\nu}(x)]^2\,dx = \dfrac{\pi\, 2^{1-2\nu}\,\Gamma(2\nu+n)}{n!\,(n+\nu)\,[\Gamma(\nu)]^2}$ $\operatorname{Re}\nu > -\tfrac{1}{2}$
(9)	$\int_{-1}^{1} (1-x)^{\nu-3/2} (1+x)^{\nu-1/2} [C_n^{\nu}(x)]^2\,dx = \dfrac{\pi^{\frac{1}{2}}\,\Gamma(\nu-\frac{1}{2})\,\Gamma(2\nu+n)}{n!\,\Gamma(\nu)\,\Gamma(2\nu)}$ $\operatorname{Re}\nu > \tfrac{1}{2}$

The complex z-plane is cut along the real axis from -1 to 1.

Gegenbauer polynomials (cont'd) $m,\ n = 0,\ 1,\ 2,\ \ldots$

(10)	$\int_{-1}^{1} (1-x)^{\nu-\frac12} (1+x)^{2\nu-1} [C_n^{\nu}(x)]^2\, dx = \dfrac{2^{3\nu-\frac12} [\Gamma(2\nu+n)]^2\, \Gamma(2n+\nu+\frac12)}{(n!)^2\, \Gamma(2\nu)\, \Gamma(3\nu+2n+\frac12)}$ Re $\nu > 0$
(11)	$\int_{-1}^{1} (1-x)^{3\nu+2n-3/2} (1+x)^{\nu-1/2} [C_n^{\nu}(x)]^2\, dx = \dfrac{\pi^{\frac12} [\Gamma(\nu+\frac12)]^2\, \Gamma(\nu+2n+\frac12)\, \Gamma(2\nu+2n)\, \Gamma(3\nu+2n-\frac12)}{2^{2\nu+2n} [n!\, \Gamma(\nu+n+\frac12)\, \Gamma(2\nu)]^2\, \Gamma(2\nu+2n+\frac12)}$ Re $\nu > 1/6$
(12)	$\int_{-1}^{1} (1-x^2)^{\nu-\frac12}\, C_m^{\nu}(x)\, C_n^{\nu}(x)\, dx = 0$ $m \neq n$, Re $\nu > -\frac12$
(13)	$\int_{-1}^{1} (1-x)^{\nu-1/2} (1+x)^{\nu+m-n-3/2}\, C_m^{\nu}(x)\, C_n^{\nu}(x)\, dx = (-1)^m \dfrac{2^{2-2\nu-m+n}\, \pi^{3/2}\, \Gamma(2\nu+n)\Gamma(\nu-\frac12+m-n)\Gamma(\frac12-\nu+m-n)}{m!\,(n-m)!\,[\Gamma(\nu)]^2\,\Gamma(\frac12+\nu+m)\Gamma(\frac12-\nu-n)\Gamma(\frac12+m-n)}$ Re $\nu > \frac12$
(14)	$\int_{-1}^{1} (1-x)^{2\nu-1} (1+x)^{\nu-\frac12}\, C_m^{\nu}(x)\, C_n^{\nu}(x)\, dx = \dfrac{2^{3\nu-\frac12}\,\Gamma(\nu+\frac12)\Gamma(2\nu+m)\Gamma(2\nu+n)\Gamma(\nu+\frac12+m+n)\Gamma(\frac12-\nu+n-m)}{m!\,n!\,\Gamma(2\nu)\Gamma(\frac12-\nu)\Gamma(\nu+\frac12+n-m)\,\Gamma(3\nu+\frac12+m+n)}$ Re $\nu > \frac12$
(15)	$\int_{-1}^{1} (1-x)^{\nu-1/2} (1+x)^{3\nu+m+n-3/2}\, C_m^{\nu}(x)\, C_n^{\nu}(x)\, dx = \dfrac{2^{4\nu+m+n-1}\,[\Gamma(\nu+\frac12)\Gamma(2\nu+m+n)]^2\,\Gamma(\nu+m+n+\frac12)\Gamma(3\nu+m+n-\frac12)}{\Gamma(\nu+m+\frac12)\Gamma(\nu+n+\frac12)\Gamma(2\nu+m)\Gamma(2\nu+n)\Gamma(4\nu+2m+2n)}$ Re $\nu > 1/6$

Gegenbauer polynomials (cont'd) $m, n = 0, 1, 2, \ldots$

(16)	$\int_{-1}^{1} (1-x)^{a}(1+x)^{\nu-\frac{1}{2}}\, C_m^{\mu}(x)\, C_n^{\nu}(x)\, dx$ $= \dfrac{2^{a+\nu+\frac{1}{2}}\,\Gamma(a+1)\Gamma(\nu+\frac{1}{2})\Gamma(\nu-a+n-\frac{1}{2})\Gamma(2\mu+m)\Gamma(2\nu+n)}{m!\,n!\,\Gamma(\nu-a-1/2)\Gamma(\nu-a+n+3/2)\Gamma(2\mu)\Gamma(2\nu)}$ $\times\ {}_4F_3\left(-m, m+2\mu, a+1, a-\nu+\frac{3}{2};\ \mu+\frac{1}{2}, \nu+a+n+\frac{3}{2}, a-\nu-n+\frac{3}{2};\ 1\right)$ $\mathrm{Re}\, a > -1, \quad \mathrm{Re}\,\nu > -\frac{1}{2}$
(17)	$\int_{-1}^{1} (z-x)^{-1}(1-x^2)^{\nu-\frac{1}{2}}\, C_m^{\nu}(x)\, C_n^{\nu}(x)\, dx$ $= \dfrac{\pi^{\frac{1}{2}}\, 2^{\frac{1}{2}-\nu}}{\Gamma(\nu)}\, e^{-(\nu-\frac{1}{2})\pi i}(z^2-1)^{\frac{1}{2}\nu-\frac{1}{4}}\, C_m^{\nu}(z)\, Q_{n+\nu-\frac{1}{2}}^{\nu-\frac{1}{2}}(z)$ $m \le n, \quad \mathrm{Re}\,\nu > -\frac{1}{2}, \quad z$ in the cut plane
(18)	$\int_{-1}^{1} (1-x^2)^{\nu-\frac{1}{2}}\, C_m^{\nu}(x)\, C_n^{\nu}(x)\, C_k^{\nu}(x)\, dx$ See Hsü, Hsien-Yü, 1938: *Duke Math. J.* 4, 374-383.
(19)	$\int_{-1}^{1} (1-x^2)^{\frac{1}{2}\nu-1}\, C_{2n}^{\nu}(ax)\, dx = \dfrac{\pi^{\frac{1}{2}}\,\Gamma(\frac{1}{2}\nu)}{\Gamma(\frac{1}{2}\nu+\frac{1}{2})}\, C_n^{\frac{1}{2}\nu}(2a^2-1) \qquad \mathrm{Re}\,\nu > 0$
(20)	$\int_{-1}^{1} (1-x^2)^{\nu-1}\, C_n^{\nu}(\cos a\cos\beta + x\sin a\sin\beta)\, dx$ $= \dfrac{2^{2\nu-1}\, n!\,[\Gamma(\nu)]^2}{\Gamma(2\nu+n)}\, C_n^{\nu}(\cos a)\, C_n^{\nu}(\cos\beta) \qquad \mathrm{Re}\,\nu > 0$
(21)	$\int_{0}^{1} x^{2\nu}(1-x^2)^{\sigma-1}\, C_n^{\nu}(1-x^2y)\, dx = \dfrac{(2\nu)_n\,\Gamma(\nu+\frac{1}{2})\Gamma(\sigma)}{2\Gamma(n+\nu+\sigma+\frac{1}{2})}\, P_n^{(a,\beta)}(1-y)$ $\mathrm{Re}\,\nu > -\frac{1}{2}, \quad \mathrm{Re}\,\sigma > 0, \quad a = \nu+\sigma-\frac{1}{2}, \quad \beta = \nu-\sigma-\frac{1}{2}$

The complex z-plane is cut along the real axis from -1 to 1.

16.4. Jacobi polynomials

See also under hypergeometric series.
In this section m and n are non-negative integers.

(1)	$\int_{-1}^{1}(1-x)^{\alpha}(1+x)^{\sigma}P_n^{(\alpha,\beta)}(x)\,dx$ $= \dfrac{2^{\alpha+\sigma+1}\,\Gamma(\sigma+1)\,\Gamma(\alpha+n+1)\,\Gamma(\sigma-\beta+1)}{\Gamma(\sigma-\beta-n+1)\,\Gamma(\alpha+\sigma+n+2)}$ $\operatorname{Re}\alpha>-1, \quad \operatorname{Re}\sigma>-1$
(2)	$\int_{-1}^{1}(1-x)^{\rho}(1+x)^{\beta}P_n^{(\alpha,\beta)}(x)\,dx$ $= \dfrac{2^{\beta+\rho+1}\,\Gamma(\rho+1)\,\Gamma(\beta+n+1)\,\Gamma(\alpha-\rho+n)}{n!\,\Gamma(\alpha-\rho)\,\Gamma(\beta+\rho+n+2)}$ $\operatorname{Re}\rho>-1, \quad \operatorname{Re}\beta>-1$
(3)	$\int_{-1}^{1}(1-x)^{\rho}(1+x)^{\sigma}P_n^{(\alpha,\beta)}(x)\,dx = \dfrac{2^{\rho+\sigma+1}\,\Gamma(\rho+1)\,\Gamma(\sigma+1)}{\Gamma(\rho+\sigma+2)}$ $\times\ {}_3F_2(-n,\ \alpha+\beta+n+1,\ \rho+1;\ \alpha+1,\ \rho+\sigma+2;\ 1)$ $\operatorname{Re}\rho>-1, \quad \operatorname{Re}\sigma>-1$
(4)	$\int_{-1}^{1}(z-x)^{-1}(1-x)^{\alpha}(1+x)^{\beta}P_n^{(\alpha,\beta)}(x)\,dx$ $= \dfrac{2^{\alpha+\beta+n+1}\,\Gamma(\alpha+n+1)\,\Gamma(\beta+n+1)}{\Gamma(\alpha+\beta+2n+2)\,(z-1)^{n+1}}$ $\times\ {}_2F_1\left(n+1,\alpha+n+1;\ \alpha+\beta+2n+2;\ \dfrac{2}{z-1}\right)$ $\operatorname{Re}\alpha>-1, \quad \operatorname{Re}\beta>-1, \quad z$ in the cut plane

The complex z-plane is cut along the real axis from -1 to 1.

Jacobi polynomials (cont'd) $m, n = 0, 1, 2, \ldots$

(5)	$\int_{-1}^{1} (1-x)^{\alpha}(1+x)^{\beta}[P_n^{(\alpha,\beta)}(x)]^2\,dx = \dfrac{2^{\alpha+\beta+1}\,\Gamma(\alpha+n+1)\,\Gamma(\beta+n+1)}{n!\,(\alpha+\beta+2n+1)\Gamma(\alpha+\beta+n+1)}$ $\operatorname{Re}\alpha > -1, \quad \operatorname{Re}\beta > -1$
(6)	$\int_{-1}^{1} (1-x)^{\alpha-1}(1+x)^{\beta}[P_n^{(\alpha,\beta)}(x)]^2\,dx = \dfrac{2^{\alpha+\beta}\,\Gamma(\alpha+n+1)\,\Gamma(\beta+n+1)}{n!\,\alpha\,\Gamma(\alpha+\beta+n+1)}$ $\operatorname{Re}\alpha > 0, \quad \operatorname{Re}\beta > -1$
(7)	$\int_{-1}^{1} (1-x)^{2\alpha}(1+x)^{\beta}[P_n^{(\alpha,\beta)}(x)]^2\,dx$ $= \dfrac{2^{4\alpha+\beta+1}\,\Gamma(\alpha+\frac{1}{2})\,[\Gamma(\alpha+n+1)]^2\,\Gamma(\beta+2n+1)}{\pi^{\frac{1}{2}}\,(n!)^2\,\Gamma(\alpha+1)\,\Gamma(2\alpha+\beta+2n+2)}$ $\operatorname{Re}\alpha > -\frac{1}{2}, \quad \operatorname{Re}\beta > -1$
(8)	$\int_{-1}^{1} (1-x)^{2\alpha+\beta+2n}(1+x)^{\beta}[P_n^{(\alpha,\beta)}(x)]^2\,dx$ $= \dfrac{2^{2\alpha+2\beta+2n+1}\,\Gamma(\beta+2n+1)[\Gamma(\alpha+\beta+2n+1)]^2\,\Gamma(2\alpha+\beta+2n+1)}{[n!\,\Gamma(\alpha+\beta+n+1)]^2\,\Gamma(2\alpha+2\beta+4n+2)}$ $\operatorname{Re}\beta > -1, \quad \operatorname{Re}(2\alpha+\beta) > -1$
(9)	$\int_{-1}^{1} (1-x)^{\alpha}(1+x)^{\beta}\,P_n^{(\alpha,\beta)}(x)\,P_m^{(\alpha,\beta)}(x)\,dx = 0$ $m \neq n, \quad \operatorname{Re}\alpha > -1, \quad \operatorname{Re}\beta > -1$
(10)	$\int_{-1}^{1} (1-x)^{\rho}(1+x)^{\beta}\,P_n^{(\alpha,\beta)}(x)\,P_n^{(\rho,\beta)}(x)\,dx$ $= \dfrac{2^{\rho+\beta+1}\,\Gamma(\rho+n+1)\,\Gamma(\beta+n+1)\,\Gamma(\alpha+\beta+2n+1)}{n!\,\Gamma(\rho+\beta+2n+2)\,\Gamma(\alpha+\beta+n+1)}$ $\operatorname{Re}\rho > -1, \quad \operatorname{Re}\beta > -1$

Jacobi polynomials (cont'd) $m,\ n = 0,\ 1,\ 2,\ \ldots$

(11)	$\int_{-1}^{1} (1-x)^{\rho-1} (1+x)^{\beta} P_n^{(\alpha,\beta)}(x)\, P_n^{(\rho,\beta)}(x)\, dx = \dfrac{2^{\rho+\beta}\,\Gamma(\alpha+n+1)\,\Gamma(\beta+n+1)\,\Gamma(\rho)}{n!\,\Gamma(\alpha+1)\,\Gamma(\rho+\beta+n+1)}$ $\operatorname{Re}\beta > -1, \quad \operatorname{Re}\rho > 0$
(12)	$\int_{-1}^{1} (1-x)^{\alpha} (1+x)^{\sigma} P_n^{(\alpha,\beta)}(x)\, P_m^{(\alpha,\sigma)}(x)\, dx = \dfrac{2^{\alpha+\sigma+1}\,\Gamma(\alpha+n+1)\Gamma(\alpha+\beta+m+n+1)\Gamma(\sigma+m+1)\Gamma(\sigma-\beta+1)}{m!\,(n-m)!\,\Gamma(\alpha+\beta+n+1)\Gamma(\alpha+\sigma+m+n+2)\Gamma(\sigma-\beta+m+1)}$ $\operatorname{Re}\alpha > -1, \quad \operatorname{Re}\sigma > -1$
(13)	$\int_{-1}^{1} (1-x)^{\alpha} (1+x)^{\beta+\sigma} P_n^{(\alpha,\beta)}(x)\, P_m^{(\alpha,\sigma)}(x)\, dx = \dfrac{2^{\alpha+\beta+\sigma+1}\,\Gamma(\alpha+m+n+1)\,\Gamma(\beta+n+1)\,\Gamma(\beta+\sigma+1)\,\Gamma(\sigma+m+1)}{m!\,n!\,\Gamma(\alpha+\beta+\sigma+m+n+2)\,\Gamma(\beta-m+n+1)\,\Gamma(\sigma+m-n+1)}$ $\operatorname{Re}\alpha > -1, \quad \operatorname{Re}(\beta+\sigma) > -1$
(14)	$\int_{-1}^{1} (1-x)^{\alpha} (1+x)^{\alpha+\beta+\sigma+m+n} P_n^{(\alpha,\beta)}(x)\, P_m^{(\alpha,\sigma)}(x)\, dx = \dfrac{2^{2\alpha+\beta+\sigma+m+n+1}\,\Gamma(\alpha+\beta+\sigma+m+n+1)\,\Gamma(\alpha+\sigma+m+n+1)}{m!\,n!\,\Gamma(\alpha+\beta+n+1)\,\Gamma(\alpha+\sigma+n+1)} \times \dfrac{\Gamma(\alpha+m+n+1)\,\Gamma(\alpha+\beta+m+n+1)}{\Gamma(2\alpha+\beta+\sigma+2m+2n+2)}$ $\operatorname{Re}\alpha > -1, \quad \operatorname{Re}(\alpha+\beta+\sigma) > -1$
(15)	$\int_{-1}^{1} (1-x)^{\alpha} (1+x)^{\sigma+m-n-1} P_n^{(\alpha,\beta)}(x)\, P_m^{(\alpha,\sigma)}(x)\, dx = \dfrac{2^{\alpha+\sigma+m-n}\,\Gamma(\alpha+n+1)\,\Gamma(\beta+n+1)\,\Gamma(\sigma+m-n)\,\Gamma(\sigma-\beta+m-n)}{n!(n-m)!\Gamma(\alpha+\sigma+m+1)\Gamma(\beta-m+n+1)\,\Gamma(\sigma-\beta+2m-2n)}$ $\operatorname{Re}\alpha > 0, \quad \operatorname{Re}\sigma > n-m$

Jacobi polynomials (cont'd) $m, n = 0, 1, 2, \ldots$

(16)	$\int_{-1}^{1} (1-x)^{\rho} (1+x)^{\beta} P_n^{(\alpha,\beta)}(x) P_m^{(\rho,\beta)}(x)\,dx$ $= \dfrac{2^{\beta+\rho+1}\Gamma(\alpha+\beta+m+n+1)\Gamma(\beta+n+1)\Gamma(\rho+m+1)\Gamma(\rho-\alpha-m+n)}{n!(n-m)!\Gamma(\alpha+\beta+n+1)\Gamma(\beta+\rho+m+n+2)\Gamma(\rho-\alpha)}$ Re $\beta > -1$, Re $\rho > -1$
(17)	$\int_{-1}^{1} (1-x)^{\alpha+\rho} (1+x)^{\beta} P_n^{(\alpha,\beta)}(x) P_m^{(\rho,\beta)}(x)\,dx$ $= \dfrac{(-1)^{m+n} 2^{\alpha+\beta+\rho+1}\Gamma(\alpha+n+1)\Gamma(\alpha+\rho+1)\Gamma(\beta+m+n+1)\Gamma(\rho+m+1)}{m!n!\Gamma(\alpha-m+n+1)\Gamma(\alpha+\beta+\rho+m+n+2)\Gamma(\rho+m-n+1)}$ Re $(\alpha+\rho) > -1$, Re $\beta > -1$
(18)	$\int_{-1}^{1} (1-x)^{\alpha+\beta+\rho+m+n} (1+x)^{\beta} P_n^{(\alpha,\beta)}(x) P_m^{(\rho,\beta)}(x)\,dx$ $= \dfrac{(-1)^{m+n} 2^{\alpha+\beta+\rho+m+n+1}\Gamma(\alpha+\beta+m+n+1)}{m!\,n!\,\Gamma(\alpha+\beta+n+1)}$ $\times \dfrac{\Gamma(\alpha+\beta+\rho+m+n+1)\Gamma(\beta+m+n+1)\Gamma(\beta+\rho+m+n+1)}{\Gamma(\alpha+2\beta+\rho+2m+2n+2)\Gamma(\beta+\rho+m+1)}$ Re $\beta > -1$, Re $(\alpha+\beta+\rho) > -1$
(19)	$\int_{-1}^{1} (1-x)^{\rho+m-n-1} (1+x)^{\beta} P_n^{(\alpha,\beta)}(x) P_m^{(\rho,\beta)}(x)\,dx$ $= \dfrac{2^{\beta+\rho+m-n}\Gamma(\alpha+n+1)\Gamma(\beta+n+1)\Gamma(\rho+m-n)\Gamma(\alpha-\rho-2m+2n+1)}{n!(n-m)!\Gamma(\alpha-m+n+1)\Gamma(\alpha-\rho-m+n+1)\Gamma(\beta+\rho+m+1)}$ Re $\beta > -1$, Re $\rho > n-m$

Hermite polynomials (cont'd) $m, n = 0, 1, 2, \ldots$

(6)	$\int_0^\infty \exp(-\frac{1}{2}x^2)\cos(\beta x)\,\mathrm{He}_{2n}(x)\,dx = (-1)^n(\frac{1}{2}\pi)^{\frac{1}{2}}\beta^{2n}\exp(-\frac{1}{2}\beta^2)$
(7)	$\int_0^\infty \exp(-\frac{1}{2}x^2)\sinh(\beta x)\,\mathrm{He}_{2n+1}(x)\,dx = (\frac{1}{2}\pi)^{\frac{1}{2}}\beta^{2n+1}\exp(\frac{1}{2}\beta^2)$
(8)	$\int_0^\infty \exp(-\frac{1}{2}x^2)\cosh(\beta x)\,\mathrm{He}_{2n}(x)\,dx = (\frac{1}{2}\pi)^{\frac{1}{2}}\beta^{2n}\exp(\frac{1}{2}\beta^2)$
(9)	$\int_{-\infty}^\infty \exp(-\frac{1}{2}x^2)[\mathrm{He}_n(x)]^2\,dx = (2\pi)^{\frac{1}{2}}\,n!$
(10)	$\int_{-\infty}^\infty e^{-x^2}\,\mathrm{He}_m(x)\,\mathrm{He}_n(x)\,dx = (-1)^{\frac{1}{2}m-\frac{1}{2}n}\,\Gamma\left(\frac{m+n+1}{2}\right)$ $m+n$ even
(11)	$\int_{-\infty}^\infty \exp(-\frac{1}{2}x^2)\,\mathrm{He}_m(x)\,\mathrm{He}_n(x)\,dx = 0$ $\quad m \neq n$
(12)	$\int_{-\infty}^\infty \exp(-a^2x^2)\,\mathrm{He}_m(x)\,\mathrm{He}_n(x)\,dx$ $= a^{-m-n-1}(1-2a^2)^{\frac{1}{2}m+\frac{1}{2}n}\,\Gamma\left(\frac{m+n+1}{2}\right)$ $\times\ {}_2F_1\left(-m, -n; \frac{1-m-n}{2}; \frac{a^2}{2a^2-1}\right)$ $\quad \mathrm{Re}\,a^2 > 0, \quad m+n$ even
(13)	$\int_{-\infty}^\infty \exp[-\frac{1}{2}(x-y)^2]\,\mathrm{He}_m(x)\,\mathrm{He}_n(x)\,dx = (2\pi)^{\frac{1}{2}}\,m!\,y^{n-m}L_n^{n-m}(-y^2)$ $m \leq n$

Hermite polynomials (cont'd) $k, m, n = 0, 1, 2, \ldots$

(14)	$\int_{-\infty}^{\infty} \exp(-x^2)\, \mathrm{He}_k(x)\, \mathrm{He}_m(x)\, \mathrm{He}_n(x)\, dx$ $= \pi^{-1}\, \Gamma(s-k)\, \Gamma(s-m)\, \Gamma(s-n)$ $k + m + n$ even, $2s = k + m + n + 1$
(15)	$\int_{-\infty}^{\infty} \exp(-\tfrac{1}{2}x^2)\, \mathrm{He}_k(x)\, \mathrm{He}_m(x)\, \mathrm{He}_n(x)\, dx = \dfrac{(2\pi)^{1/2}\, k!\, m!\, n!}{(s-k)!\,(s-m)!\,(s-n)!}$ $k + m + n = 2s$ even
(16)	$\int_{-\infty}^{\infty} \exp(-a^2 x^2)\, \mathrm{He}_m(x)\, \mathrm{He}_n(x)\, \mathrm{He}_k(x) \ldots dx$ See Busbridge, I.W., 1948: *J. London Math. Soc.* 23, 135-141.
(17)	$\int_{-\infty}^{\infty} \exp[-\tfrac{1}{2}(x-y)^2]\, \mathrm{He}_n(ax)\, dx = (2\pi)^{1/2}\, (1-a^2)^{1/2 n}\, \mathrm{He}_n\left[\dfrac{ay}{(1-a^2)^{1/2}}\right]$
(18)	$\int_0^{\infty} \exp(-\tfrac{1}{2}x^2)\, \sin(\beta x)\, \mathrm{He}_{2n+1}(ax)\, dx$ $= (-1)^n (\tfrac{1}{2}\pi)^{1/2} (a^2-1)^{n+1/2} \exp(-\tfrac{1}{2}\beta^2)\, \mathrm{He}_{2n+1}\left[\dfrac{a\beta}{(a^2-1)^{1/2}}\right]$
(19)	$\int_0^{\infty} \exp(-\tfrac{1}{2}x^2)\, \cos(\beta x)\, \mathrm{He}_{2n}(ax)\, dx$ $= (\tfrac{1}{2}\pi)^{1/2}\, (1-a^2)^n \exp(-\tfrac{1}{2}\beta^2)\, \mathrm{He}_{2n}\left[\dfrac{a\beta}{(a^2-1)^{1/2}}\right]$
(20)	$\int_{-\infty}^{\infty} \exp(-\tfrac{1}{2}x^2)\, H_m(ax)\, H_n(x) = 0$ $m < n$

Hermite polynomials (cont'd) $m, n = 0, 1, 2, \ldots$

(21)	$\int_{-\infty}^{\infty} \exp(-\tfrac{1}{2}x^2)\, H_{2m+n}(ax)\, H_n(x)\, dx = \pi^{1/2} \dfrac{(2m+n)!}{m!\, 2^{m-1/2}} (a^2 - 1)^m\, a^n$
(22)	$\int_{-\infty}^{\infty} \exp[-\tfrac{1}{2}(a^2 + \beta^2)\, x^2]\, \mathrm{He}_m(ax)\, \mathrm{He}_n(\beta x)\, dx$ $= (-1)^{1/2 m - 1/2 n}\, 2^{1/2 m + 1/2 n + 1/2}\, \Gamma\left(\dfrac{m+n+1}{2}\right) a^n\, \beta^m\, (a^2 + \beta^2)^{-m-n-1/2}$ $\mathrm{Re}(a^2 + \beta^2) > 0, \quad m + n$ even
(23)	$\int_{-\infty}^{\infty} \exp(-\tfrac{1}{2}x^2)\, \mathrm{He}_m(ax)\, \mathrm{He}_n(\beta x)\, dx$ $= 2^{1/2 - 1/2 m - 1/2 n}\, \pi^{1/2}\, n!\,(a^2-1)^{-1/2 m}\, (\beta^2-1)^{-1/2 n}\, (a^2+\beta^2-1)^{1/2 m + 1/2 n} P^n_{m+n}(z)$ $\dfrac{1}{z^2} = \dfrac{1}{a^2} + \dfrac{1}{\beta^2} - \dfrac{1}{a^2\beta^2}$
(24)	$\int_{-\infty}^{\infty} \exp(-\lambda^2 x^2)\, \mathrm{He}_m(ax)\, \mathrm{He}_n(\beta x)\, dx = 0 \qquad m + n$ odd
(25)	$\int_{-\infty}^{\infty} x^{\rho} \exp(-x^2)\, \mathrm{He}_{2n}(ax)\, \mathrm{He}_{2n}(\beta x)\, dx$ See Buchholz, Herbert, 1953: *Die konfluente hypergeometrische Funktion*. Springer-Verlag, Berlin, Göttinger, Heidelberg, Sec. 13.
(26)	$\int_{-\infty}^{\infty} \exp[-\tfrac{1}{2}(x - y)^2]\, \mathrm{He}_m(ax)\, \mathrm{He}_n(ax)\, dx$ $= (2\pi)^{1/2} \sum_{k=0}^{\min(m,n)} k! \binom{m}{k}\binom{n}{k} (1 - a^2)^{1/2 m + 1/2 n - k}$ $\times\, \mathrm{He}_{m+n-2k}\left[\dfrac{ay}{(1 - a^2)^{1/2}}\right]$

Hermite polynomials (cont'd) $k, m, n = 0, 1, 2, \ldots$

(27)	$\int_{-\infty}^{\infty} \exp(-\lambda^2 x^2) \operatorname{He}_m(\alpha x) \operatorname{He}_n(\beta x) \operatorname{He}_k(\gamma x)\, dx = 0$	$m + n + k$ odd
(28)	$\int_{-\infty}^{\infty} \exp(-\lambda^2 x^2) \operatorname{He}_k(\alpha x) \operatorname{He}_m(\beta x) \operatorname{He}_n(\gamma x) \ldots dx$ See Bailey, W.N., 1948: *J. London Math. Soc.* 23, 291-297. Lord, R.D., 1949: *J. London Math. Soc.* 24, 101-112.	
(29)	$\int_0^{\infty} x^{\rho-1} \exp(-\lambda^2 x^2) \operatorname{He}_k(\alpha x) \operatorname{He}_m(\beta x) \operatorname{He}_n(\gamma x) \ldots dx$ See Appell, Paul and M.J. Kampé de Fériet, 1926: *Fonctions hypergéométriques et hypersphériques. Polynomes d' Hermite.* Gauthier-Villars, p. 343. Erdélyi, Arthur, 1936: *Math. Z.* 40, 693-702.	
(30)	$\int_{-\infty}^{\infty} \exp(-\tfrac{1}{2} x^2) \operatorname{He}_m(x+y) \operatorname{He}_n(x+z)\, dx$ $= (2\pi)^{1/2}\, m!\, z^{n-m} L_m^{n-m}(-yz)$	$m \leq n$
(31)	$\int_0^{\pi} (\cos x)^n \operatorname{He}_{2n}[\alpha(1-\sec x)^{1/2}]\, dx = \dfrac{(-1)^n \pi\, (2n)!}{2^n (n!)^2} [\operatorname{He}_n(\alpha)]^2$	

16.6. Laguerre polynomials

See also confluent hypergeometric functions
In this section m and n are non-negative integers.

(1)	$\int_0^{\infty} x^{\beta-1} e^{-x} L_n^{\alpha}(x)\, dx = \dfrac{\Gamma(\alpha-\beta+n+1)\,\Gamma(\beta)}{n!\,\Gamma(\alpha-\beta+1)}$	$\operatorname{Re}\beta > 0$
(2)	$\int_0^{\infty} x^{\alpha} e^{-x} [L_n^{\alpha}(x)]^2\, dx = \dfrac{\Gamma(\alpha+n+1)}{n!}$	$\operatorname{Re}\alpha > 0$

Laguerre polynomials (cont'd) $m, n = 0, 1, 2, \ldots$

(3)	$\int_0^\infty x^\alpha e^{-x} L_m^\alpha(x) L_n^\alpha(x)\, dx = 0 \qquad m \neq n, \quad \operatorname{Re} \alpha > -1$
(4)	$\int_0^\infty x^{\alpha+\beta} e^{-x} L_m^\alpha(x) L_n^\beta(x)\, dx = (-1)^{m+n} \binom{\alpha+m}{n} \binom{\beta+n}{m}$ $\operatorname{Re}(\alpha+\beta) > -1$
(5)	$\int_0^1 x^\alpha (1-x)^{\beta-\alpha-1} L_n^\alpha(xy)\, dx = \dfrac{\Gamma(\alpha+n+1)\,\Gamma(\beta-\alpha)}{\Gamma(\beta+n+1)} L_n^\beta(y)$ $\operatorname{Re}\beta > \operatorname{Re}\alpha > -1$
(6)	$\int_0^\infty x^\nu e^{-x} L_n^\alpha(\lambda x) L_n^\alpha(\mu x)\, dx$ See Buchholz, Herbert, 1953: *Die konfluente hypergeometrische Funktion*, Springer Verlag. Berlin, Göttingen, Heidelberg. Sec. 12.
(7)	$\int_0^1 x^\alpha (1-x)^\beta L_m^\alpha(xy) L_n^\beta[(1-x)y]\, dx$ $= \dfrac{(m+n)!\,\Gamma(\alpha+m+1)\,\Gamma(\beta+n+1)}{m!\,n!\,\Gamma(\alpha+\beta+m+n+2)} L_{m+n}^{\alpha+\beta+1}(y)$ $\operatorname{Re}\alpha > -1, \quad \operatorname{Re}\beta > -1$
(8)	$\int_{-\infty}^\infty x^{m-n} \exp[-\tfrac{1}{2}(x-y)^2] L_n^{m-n}(x^2)\, dx$ $= \dfrac{(2\pi)^{1/2}}{n!} i^{n-m} \operatorname{He}_n(iy) \operatorname{He}_m(iy)$
(9)	$\int_0^\infty \exp(-\tfrac{1}{2}x^2) [L_n^{-1/4}(\tfrac{1}{2}x^2)]^2 \cos(xy)\, dx$ $= (\tfrac{1}{2}\pi)^{1/2} \exp(-\tfrac{1}{2}y^2) [L_n^{-1/4}(\tfrac{1}{2}y^2)]^2$

Laguerre polynomials (cont'd)) $m, n = 0, 1, 2, \ldots$

(10)	$\int_0^\infty x \exp(-\tfrac{1}{2}x^2)\,[L_n^{1/4}(\tfrac{1}{2}x^2)]^2 \sin(xy)\,dx$ $= (\tfrac{1}{2}\pi)^{1/2}\, y \exp(-\tfrac{1}{2}y^2)\,[L_n^{1/4}(\tfrac{1}{2}y^2)]^2$
(11)	$\int_0^\infty x \exp(-\tfrac{1}{2}x^2)\, L_n^{\alpha}(\tfrac{1}{2}x^2)\, L_n^{1/2-\alpha}(\tfrac{1}{2}x^2) \sin(xy)\,dx$ $= (\tfrac{1}{2}\pi)^{1/2}\, y \exp(-\tfrac{1}{2}y^2)\, L_n^{\alpha}(\tfrac{1}{2}y^2)\, L_n^{1/2-\alpha}(\tfrac{1}{2}y^2)$
(12)	$\int_0^\infty \exp(-\tfrac{1}{2}x^2)\, L_n^{\alpha}(\tfrac{1}{2}x^2)\, L_n^{-1/2-\alpha}(\tfrac{1}{2}x^2) \cos(xy)\,dx$ $= (\tfrac{1}{2}\pi)^{1/2} \exp(-\tfrac{1}{2}y^2)\, L_n^{\alpha}(\tfrac{1}{2}y^2)\, L_n^{-\alpha-1/2}(\tfrac{1}{2}y^2)$
(13)	$\int_0^\infty \exp(-\tfrac{1}{2}x^2)\, L_n(\tfrac{1}{2}x^2)\, \mathrm{He}_{2n+1}(\tfrac{1}{2}x) \sin(xy)\,dx$ $= (\tfrac{1}{2}\pi)^{1/2} \exp(-\tfrac{1}{2}y^2)\, L_n(\tfrac{1}{2}y^2)\, \mathrm{He}_{2n+1}(\tfrac{1}{2}y)$
(14)	$\int_0^\infty \exp(-\tfrac{1}{2}x^2)\, L_n(\tfrac{1}{2}x^2)\, \mathrm{He}_{2n}(\tfrac{1}{2}x) \cos(xy)\,dx$ $= (\tfrac{1}{2}\pi)^{1/2} \exp(-\tfrac{1}{2}y^2)\, L_n(\tfrac{1}{2}y^2)\, \mathrm{He}_{2n}(\tfrac{1}{2}y)$
(15)	$\int_0^\infty x^{\rho-1} e^{-x} L_{m_1}^{\alpha_1}(\lambda_1 x) \cdots L_{m_n}^{\alpha_n}(\lambda_n x)\,dx$ See Erdélyi, Arthur, 1936: *Math. Z.* 40, 693-702.

CHAPTER XVII

GAMMA FUNCTION, INCOMPLETE GAMMA FUNCTIONS, AND RELATED FUNCTIONS

For these functions see H.T.F. vol. I, Chapter I and vol. II, Chapter IX. The expressions, given below, of incomplete gamma functions and related functions in terms of confluent hypergeometric functions will assist in the evaluation of integrals involving these functions. For this reason, only a small selection of integrals involving incomplete gamma functions and their particular cases is given here.

Error functions and Fresnel integrals

$$\begin{aligned}\operatorname{Erf}(x) &= \pi^{-1/2}\,\gamma(1/2, x^2)\\ &= 2\pi^{-1/2}\,x\;{}_1F_1(1/2; 3/2; -x^2)\\ &= 2\pi^{-1/2}\,xe^{-x^2}\;{}_1F_1(1; 3/2; x^2)\\ &= 2\pi^{-1/2}\,x^{-1/2}\,e^{-1/2 x^2}\,M_{-1/4,1/4}(x^2)\\ &= 1-\operatorname{Erfc}(x)\end{aligned}$$

$$\begin{aligned}\operatorname{Erfc}(x) &= \pi^{-1/2}\,\Gamma(1/2, x^2)\\ &= 2^{1/2}\,\pi^{-1/2}\,e^{-1/2 x^2}\,D_{-1}(2^{1/2}x)\\ &= \pi^{-1/2}\,x^{-1}\,e^{-x^2}\;{}_2F_0(1, 1/2; -x^{-2})\\ &= \pi^{-1/2}\,x^{-1/2}\,e^{-1/2 x^2}\,W_{-1/4,1/4}(x^2)\\ &= 1-\operatorname{Erf}(x)\end{aligned}$$

$$C(x) \pm i S(x) = e^{\pm \frac{1}{4}\pi i} \operatorname{Erf}(e^{\mp \frac{1}{4}\pi i} x^{\frac{1}{2}})$$
$$= 2\pi^{-\frac{1}{2}} x^{\frac{1}{2}} {}_1F_1(1/2; 3/2; \pm ix)$$

Exponential integral and related functions

$$-\operatorname{Ei}(-x) = E_1(x) = \Gamma(0, x)$$
$$= x^{-1} e^{-x} {}_2F_0(1, 1; -x^{-1})$$
$$= x^{-\frac{1}{2}} e^{-\frac{1}{2}x} W_{-\frac{1}{2},0}(x)$$

$$\overline{\operatorname{Ei}}(x) = \tfrac{1}{2}[\operatorname{Ei}(x + i0) + \operatorname{Ei}(x - i0)]$$
$$= x^{-1} e^{x} {}_2F_0(1, 1; x^{-1})$$

$$\operatorname{Ci}(x) \pm i \operatorname{si}(x) = -\operatorname{ci}(x) \pm i \operatorname{si}(x)$$
$$= \operatorname{Ei}(\pm ix) = -\Gamma(0, \mp ix)$$
$$= \mp ix^{-1} e^{\pm ix} {}_2F_0(1, 1; \mp ix^{-1})$$
$$= -x^{-\frac{1}{2}} e^{\pm \frac{1}{4}\pi i \pm \frac{1}{2}xi} W_{-\frac{1}{2},0}(\mp ix)$$

Incomplete gamma functions

$$\gamma(a, x) = a^{-1} x^{a} {}_1F_1(a; a+1; -x)$$
$$= a^{-1} x^{\frac{1}{2}a-\frac{1}{2}} e^{-\frac{1}{2}x} M_{\frac{1}{2}a-\frac{1}{2}, \frac{1}{2}a}(x)$$
$$= \Gamma(a) - \Gamma(a, x)$$

$$\Gamma(a, x) = x^{a-1} e^{-x} {}_2F_0(1, 1-a; -x^{-1})$$
$$= x^{\frac{1}{2}a-\frac{1}{2}} e^{-\frac{1}{2}x} W_{\frac{1}{2}a-\frac{1}{2}, \frac{1}{2}a}(x)$$
$$= \Gamma(a) - \gamma(a, x)$$

GAMMA FUNCTION, INCOMPLETE GAMMA FUNCTIONS, AND RELATED FUNCTIONS

17.1. The gamma function

(1)	$\int_{-\infty}^{\infty} \Gamma(\alpha + x)\,\Gamma(\beta - x)\,dx = 0$ $\mathrm{Re}(\alpha+\beta) < 1$ and either $\mathrm{Im}\,\alpha < 0 < \mathrm{Im}\,\beta$ or $\mathrm{Im}\,\beta < 0 < \mathrm{Im}\,\alpha$
(2)	$\int_{-\infty}^{\infty} \Gamma(\alpha + x)\,\Gamma(\beta - x)\,dx = i\pi\, 2^{1-\alpha-\beta}\,\Gamma(\alpha+\beta)$ $\mathrm{Re}(\alpha+\beta) < 1, \quad \mathrm{Im}\,\alpha,\ \mathrm{Im}\,\beta < 0$
(3)	$\int_{-\infty}^{\infty} \Gamma(\alpha + x)\,\Gamma(\beta - x)\,dx = -i\pi\, 2^{1-\alpha-\beta}\,\Gamma(\alpha+\beta)$ $\mathrm{Re}(\alpha+\beta) < 1, \quad \mathrm{Im}\,\alpha,\ \mathrm{Im}\,\beta > 0$
(4)	$\int_{-\infty}^{\infty} \frac{\Gamma(\alpha+x)}{\Gamma(\beta+x)}\,dx = 0 \qquad \mathrm{Im}\,\alpha \neq 0, \quad \mathrm{Re}(\alpha-\beta) < -1$
(5)	$\int_{-\infty}^{\infty} \frac{dx}{\Gamma(\alpha+x)\,\Gamma(\beta-x)} = \frac{2^{\alpha+\beta-2}}{\Gamma(\alpha+\beta-1)} \qquad \mathrm{Re}(\alpha+\beta) > 1$
(6)	$\int_{-\infty}^{\infty} \frac{\Gamma(\alpha+x)}{\Gamma(\beta+x)}\exp[(2\pi n + \pi - 2\theta)xi]\,dx = 0 \qquad \mathrm{Re}(\beta-\alpha) > 0$ $-\tfrac{1}{2}\pi < \theta < \tfrac{1}{2}\pi, \quad n$ integer, $\quad (n+\tfrac{1}{2})\,\mathrm{Im}(\alpha) > 0$

The gamma function (cont'd)

(7)	$\int_{-\infty}^{\infty}\Gamma(\alpha+x)\,\Gamma(\beta-x)\,\exp[2(\pi n+\theta)xi]\,dx$ $=2\pi i\,\Gamma(\alpha+\beta)(2\cos\theta)^{-\alpha-\beta}\exp[(\beta-\alpha)\theta i]$ $\times[\eta_n(\beta)\exp(2n\pi\beta i)-\eta_n(-\alpha)\exp(-2n\pi\alpha i)]$ $\mathrm{Re}(\alpha+\beta)<1,\quad -\tfrac{1}{2}\pi<\theta<\tfrac{1}{2}\pi,\quad n$ integer $\eta_n(\zeta)=0$ if $(\tfrac{1}{2}-n)\,\mathrm{Im}\,\zeta>0$ $\eta_n(\zeta)=\mathrm{sgn}(\tfrac{1}{2}-n)$ if $(\tfrac{1}{2}-n)\,\mathrm{Im}\,\zeta<0$
(8)	$\int_{-\infty}^{\infty}\frac{\Gamma(\alpha+x)}{\Gamma(\beta+x)}\exp[(2\pi n+\pi-2\theta)xi]\,dx$ $=2\pi i\,\mathrm{sgn}(n+\tfrac{1}{2})\,\frac{(2\cos\theta)^{\beta-\alpha-1}}{\Gamma(\beta-\alpha)}\exp[-(2\pi n+\pi-\theta)\alpha i+\theta(\beta-1)i]$ $\mathrm{Re}(\beta-\alpha)>0,\quad -\tfrac{1}{2}\pi<\theta<\tfrac{1}{2}\pi,\quad n$ integer, $\quad (n+\tfrac{1}{2})\,\mathrm{Im}\,\alpha<0$
(9)	$\int_{-\infty}^{\infty}\frac{\sin(cx)\,dx}{\Gamma(\alpha+x)\,\Gamma(\beta-x)}=0\qquad \mathrm{Re}(\alpha+\beta)>1,\quad c>\pi$
(10)	$\int_{-\infty}^{\infty}\frac{\sin(cx)\,dx}{\Gamma(\alpha+x)\,\Gamma(\beta-x)}=\frac{[2\cos(\tfrac{1}{2}c)]^{\alpha+\beta-2}}{\Gamma(\alpha+\beta-1)}\sin[\tfrac{1}{2}c(\beta-\alpha)]$ $\mathrm{Re}(\alpha+\beta)>1,\quad 0<c<\pi$
(11)	$\int_{-\infty}^{\infty}\frac{\sin(2n\pi x)}{\sin(\pi x)}\,\frac{dx}{\Gamma(\alpha+x)\,\Gamma(\beta-x)}=0$ $\mathrm{Re}(\alpha+\beta)>1,\quad n$ integer
(12)	$\int_{-\infty}^{\infty}\frac{\sin[(2n+1)\pi x]}{\sin(\pi x)}\,\frac{dx}{\Gamma(\alpha+x)\,\Gamma(\beta-x)}=\frac{2^{\alpha+\beta-2}}{\Gamma(\alpha+\beta-1)}$ $\mathrm{Re}(\alpha+\beta)>1,\quad n$ integer

The gamma function (cont'd)

(13)	$\int_{-\infty}^{\infty} \frac{\cos(cx)\,dx}{\Gamma(a+x)\,\Gamma(\beta-x)} = 0 \qquad \mathrm{Re}(a+\beta) > 1, \quad c > \pi$
(14)	$\int_{-\infty}^{\infty} \frac{\cos(cx)\,dx}{\Gamma(a+x)\,\Gamma(\beta-x)} = \frac{[2\cos(\frac{1}{2}c)]^{a+\beta-2}}{\Gamma(a+\beta-1)} \cos[\tfrac{1}{2}c(\beta-a)]$ $\mathrm{Re}(a+\beta) > 1, \quad 0 \le c < \pi$
(15)	$\int_{-\infty}^{\infty} \frac{P(x)\,e^{icx}\,dx}{\Gamma(a+x)\,\Gamma(\beta-x)}$, $P(x)$ polynomial See Ramanujan, Srinivasa, 1920: *Quart. J. Math.* 48, 294-310.
(16)	$\int_{-\infty}^{\infty} \frac{\Phi(x)\exp[(2\pi n+\theta)xi]\,dx}{\Gamma(a+x)\,\Gamma(\beta-x)}$ $= \frac{[2\cos(\frac{1}{2}\theta)]^{a+\beta-2}}{\Gamma(a+\beta-1)} \exp[\tfrac{1}{2}\theta(\beta-a)i] \int_0^1 \Phi(t)\exp(2\pi nti)\,dt$ $\mathrm{Re}(a+\beta) > 1, \quad -\pi < \theta < \pi, \quad n$ integer, $\quad \Phi(x+1) = \Phi(x)$
(17)	$\int_{-\infty}^{\infty} \frac{\Phi(x)\,e^{ixc}\,dx}{\Gamma(a+x)\,\Gamma(\beta-x)}$, $\quad \Phi(x)$ periodic, period real See Ramanujan, Srinivasa, 1920: *Quart. J. Math.* 48, 294-310.
(18)	$\int_{-\infty}^{\infty} \frac{\Gamma(\gamma+x)\,\Gamma(\delta+x)}{\Gamma(a+x)\,\Gamma(\beta+x)}\,dx = 0$ $\mathrm{Re}(a+\beta-\gamma-\delta) > 1, \quad \mathrm{Im}\,\gamma,\ \mathrm{Im}\,\delta > 0$

The gamma function (cont'd)

(19)	$\displaystyle\int_{-\infty}^{\infty}\frac{\Gamma(\gamma+x)\,\Gamma(\delta+x)}{\Gamma(\alpha+x)\,\Gamma(\beta+x)}\,dx$ $\displaystyle=\frac{\pm 2\pi^2 i\,\Gamma(\alpha+\beta-\gamma-\delta-1)}{\sin[\pi(\gamma-\delta)]\,\Gamma(\alpha-\gamma)\,\Gamma(\alpha-\delta)\,\Gamma(\beta-\gamma)\,\Gamma(\beta-\delta)}$ $\mathrm{Re}(\alpha+\beta-\gamma-\delta)>1,\quad \mathrm{Im}\,\gamma,\ \mathrm{Im}\,\delta<0$ $\pm$ according as $\mathrm{Im}\,\gamma \gtrless \mathrm{Im}\,\delta$
(20)	$\displaystyle\int_{-\infty}^{\infty}\frac{\Gamma(\alpha-\beta-\gamma+x+1)\,dx}{\Gamma(\alpha+x)\,\Gamma(\beta-x)\,\Gamma(\gamma+x)}$ $\displaystyle=\frac{\pi\exp[\pm\tfrac{1}{2}\pi(\delta-\gamma)i]}{\Gamma(\beta+\gamma-1)\,\Gamma(\frac{\alpha+\beta}{2})\,\Gamma(\frac{\gamma-\delta+1}{2})}$ $\mathrm{Re}(\beta+\gamma)>1,\quad \delta=\alpha-\beta-\gamma+1,\quad \mathrm{Im}\,\delta\neq 0$ $\pm$ according as $\mathrm{Im}\,\delta\gtrless 0$
(21)	$\displaystyle\int_{-\infty}^{\infty}\frac{dx}{\Gamma(\alpha+x)\,\Gamma(\beta-x)\,\Gamma(\gamma+x)\,\Gamma(\delta-x)}$ $\displaystyle=\frac{\Gamma(\alpha+\beta+\gamma+\delta-3)}{\Gamma(\alpha+\beta-1)\,\Gamma(\beta+\gamma-1)\,\Gamma(\gamma+\delta-1)\,\Gamma(\delta+\alpha-1)}$ $\mathrm{Re}\,(\alpha+\beta+\gamma+\delta)>3$
(22)	$\displaystyle\int_{-\infty}^{\infty}\frac{\sin(\pi x)\,dx}{\Gamma(\alpha+x)\,\Gamma(\beta-x)\,\Gamma(\gamma+x)\,\Gamma(\delta-x)}$ $\displaystyle=\frac{\sin[\tfrac{1}{2}\pi(\beta-\alpha)]}{2\Gamma(\frac{\alpha+\beta}{2})\,\Gamma(\frac{\gamma+\delta}{2})\,\Gamma(\alpha+\delta-1)}$ $\alpha+\delta=\beta+\gamma,\quad \mathrm{Re}(\alpha+\beta+\gamma+\delta)>2$

The gamma function (cont'd)

(23) $$\int_{-\infty}^{\infty} \frac{\cos(\pi x)\,dx}{\Gamma(a+x)\,\Gamma(\beta-x)\,\Gamma(\gamma+x)\,\Gamma(\delta-x)} = \frac{\cos[\tfrac{1}{2}\pi(\beta-a)]}{2\Gamma(\frac{a+\beta}{2})\,\Gamma(\frac{\gamma+\delta}{2})\,\Gamma(a+\delta-1)}$$

$a+\delta=\beta+\gamma, \quad \mathrm{Re}(a+\beta+\gamma+\delta)>2$

(24) $$\int_{-\infty}^{\infty} \frac{\Phi(x)\,dx}{\Gamma(a+x)\,\Gamma(\beta-x)\,\Gamma(\gamma+x)\,\Gamma(\delta-x)} = \frac{\Gamma(a+\beta+\gamma+\delta-3)}{\Gamma(a+\beta-1)\,\Gamma(\beta+\gamma-1)\,\Gamma(\gamma+\delta-1)\,\Gamma(\delta+a-1)} \int_0^1 \Phi(t)\,dt$$

$\mathrm{Re}(a+\beta+\gamma+\delta)>3, \quad \Phi(x+1)=\Phi(x)$

(25) $$\int_{-\infty}^{\infty} \frac{\Phi(x)\,dx}{\Gamma(a+x)\,\Gamma(\beta-x)\,\Gamma(\gamma+x)\,\Gamma(\delta-x)} = \frac{\int_0^1 \Phi(t)\cos[\tfrac{1}{2}\pi(2t+a-\beta)]\,dt}{\Gamma(\frac{a+\beta}{2})\,\Gamma(\frac{\gamma+\delta}{2})\,\Gamma(a+\delta-1)}$$

$a+\delta=\beta+\gamma, \quad \mathrm{Re}(a+\beta+\gamma+\delta)>2, \quad \Phi(x+1)=-\Phi(x)$

(26) $$\int_{-\infty}^{\infty} [\Gamma(a+x)\Gamma(\beta-x)\Gamma(\gamma+kx)\Gamma(\delta-kx)]^{-1}\exp(\pi cxi)\,dx = 0$$

$\mathrm{Re}(a+\beta+\gamma+\delta)>2, \quad c,\ k$ real, $\quad |c|>|k|+1$

Fur further similar integrals see Ramanujan, Srinivasa, 1920: *Quart. J. Math.* 48, 294-310.

The gamma function (cont'd)

(27)	$\int_0^\infty \lvert\Gamma(a+ix)\,\Gamma(b+ix)\rvert^2\,dx = \tfrac{1}{2}\pi^{\frac{1}{2}}\,\Gamma(a)\,\Gamma(a+\tfrac{1}{2})\,\Gamma(b)\,\Gamma(b+\tfrac{1}{2}) \times \Gamma(a+b)/\Gamma(a+b+\tfrac{1}{2})$ $a>0, \quad b>0$
(28)	$\int_0^\infty \left\lvert \dfrac{\Gamma(a+ix)}{\Gamma(b+ix)} \right\rvert^2 dx = \dfrac{\tfrac{1}{2}\pi^{\frac{1}{2}}\,\Gamma(a)\,\Gamma(a+\tfrac{1}{2})\,\Gamma(b-a-\tfrac{1}{2})}{\Gamma(b)\,\Gamma(b-\tfrac{1}{2})\,\Gamma(b-a)}$ $0<a<b-\tfrac{1}{2}$
(29)	$(2\pi i)^{-1}\int_{-i\infty}^{i\infty}\Gamma(s-\kappa-\lambda)\,\Gamma(\lambda+\mu-s+\tfrac{1}{2})\,\Gamma(\lambda-\mu-s+\tfrac{1}{2})\,z^s\,ds$ $=\Gamma(\tfrac{1}{2}-\kappa-\mu)\,\Gamma(\tfrac{1}{2}-\kappa+\mu)\,z^\lambda e^{\frac{1}{2}z}\,W_{\kappa,\mu}(z)$ $\mathrm{Re}(\kappa+\lambda)<0, \quad \mathrm{Re}\,\lambda>\lvert\mathrm{Re}\,\mu\rvert-\tfrac{1}{2}, \quad \lvert\arg z\rvert<3\pi/2$
(30)	$\dfrac{1}{2\pi i}\int_{-i\infty}^{i\infty}\dfrac{\Gamma(\lambda+\mu-s+\tfrac{1}{2})\Gamma(\lambda-\mu-s+\tfrac{1}{2})}{\Gamma(\lambda-\kappa-s+1)}\,z^s\,ds = z^\lambda e^{-\frac{1}{2}z}\,W_{\kappa,\mu}(z)$ $\mathrm{Re}\,\lambda>\lvert\mathrm{Re}\,\mu\rvert-\tfrac{1}{2}, \quad \lvert\arg z\rvert<\tfrac{1}{2}\pi$
(31)	$\dfrac{1}{(2\pi i)}\int_{-i\infty}^{i\infty}\dfrac{\Gamma(\kappa-\lambda+s)\,\Gamma(\lambda+\mu-s+\tfrac{1}{2})}{\Gamma(\mu-\lambda+s+\tfrac{1}{2})}\,z^s\,ds$ $=\dfrac{\Gamma(\kappa+\mu+\tfrac{1}{2})}{\Gamma(2\mu+1)}\,z^\lambda e^{-\frac{1}{2}z}\,M_{\kappa,\mu}(z)$ $\mathrm{Re}(\kappa-\lambda)>0, \quad \mathrm{Re}(\lambda+\mu)>-\tfrac{1}{2}, \quad \lvert\arg z\rvert<\tfrac{1}{2}\pi$
(32)	$\dfrac{1}{2\pi i}\int_{-i\infty}^{i\infty}\Gamma(a+s)\,\Gamma(\beta+s)\,\Gamma(\gamma-s)\,\Gamma(\delta-s)\,ds$ $=\dfrac{\Gamma(a+\gamma)\,\Gamma(a+\delta)\,\Gamma(\beta+\gamma)\,\Gamma(\beta+\delta)}{\Gamma(a+\beta+\gamma+\delta)}$ $\mathrm{Re}\,a, \quad \mathrm{Re}\,\beta, \quad \mathrm{Re}\,\gamma, \quad \mathrm{Re}\,\delta>0$

The gamma function (cont'd)

(33)	$$\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\left[\frac{\Gamma(\tfrac{1}{2}-s)}{\Gamma(s)}\right]^2 z^s ds = z^{\frac{1}{2}}\left[2\pi^{-1}K_0(4z^{\frac{1}{4}})-Y_0(4z^{\frac{1}{4}})\right]$$ $z>0$
(34)	$$\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\frac{\prod_{j=1}^{m}\Gamma(b_j-s)\prod_{j=1}^{n}\Gamma(1-a_j+s)}{\prod_{j=m+1}^{q}\Gamma(1-b_j+s)\prod_{j=n+1}^{p}\Gamma(a_j-s)}z^s\,ds$$ $$=G_{pq}^{mn}\left(z\,\middle\vert\,\begin{matrix}a_1,\dots,a_p\\ b_1,\dots,b_q\end{matrix}\right)$$ $p+q<2(m+n)$, $\lvert\arg z\rvert<(m+n-\tfrac{1}{2}p-\tfrac{1}{2}q)\pi$ $\operatorname{Re} a_j<1$ $j=1,\dots,n$, $\operatorname{Re} b_j>0$ $j=1,\dots,m$
(35)	$$\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\frac{\prod_{j=1}^{m}\Gamma(b_j-s)\prod_{j=1}^{n}\Gamma(1-a_j+s)}{\prod_{j=m+1}^{q}\Gamma(1-b_j+s)\prod_{j=n+1}^{p}\Gamma(a_j-s)}z^s\,ds$$ $$=G_{pq}^{mn}\left(z\,\middle\vert\,\begin{matrix}a_1,\dots,a_p\\ b_1,\dots,b_q\end{matrix}\right)$$ $p+q\le 2(m+n)$, $\lvert\arg z\rvert\le(m+n-\tfrac{1}{2}p-\tfrac{1}{2}q)\pi$ $\operatorname{Re} a_j<1$ $j=1,\dots,n$, $\operatorname{Re} b_j>0$ $j=1,\dots,m$ $\operatorname{Re}\left(\sum_{j=1}^{p}a_j-\sum_{j=1}^{q}b_j\right)>\tfrac{1}{2}p-\tfrac{1}{2}q+1$
	For further integrals of this type see sec. 7.3.

An empty product is interpreted as 1.

The gamma function (cont'd)

(36)	$\int_0^1 \sin(2\pi n x) \log[\Gamma(a+x)]\,dx$ $= -(2n\pi)^{-1}[\log a + \cos(2n\pi a)\operatorname{ci}(2n\pi a) - \sin(2n\pi a)\operatorname{si}(2n\pi a)]$ $a>0, \quad n=1,2,3,\ldots$
(37)	$\int_0^1 \cos(2\pi n x) \log[\Gamma(a+x)]\,dx$ $= -(2n\pi)^{-1}[\sin(2n\pi a)\operatorname{ci}(2n\pi a) + \cos(2n\pi a)\operatorname{si}(2n\pi a)]$ $a>0, \quad n=1,2,3,\ldots$
(38)	$\int_0^1 \exp(2\pi n x i) \log[\Gamma(a+x)]\,dx$ $= (2n\pi i)^{-1}[\log a - \exp(-2\pi n a i)\operatorname{Ei}(2n\pi i a)]$ $a>0, \quad n=\pm 1, \pm 2, \ldots$
(39)	$\int_0^1 \log[\Gamma(x)]\,dx = \tfrac{1}{2}\log(2\pi)$
(40)	$\int_0^1 \log[\Gamma(a+x)] = a\log a - a + \tfrac{1}{2}\log(2\pi)$ $a \geq 0$
(41)	$\int_0^n \log[\Gamma(a+x)]\,dx = \sum_{k=0}^{n-1}(a+k)\log(a+k) - na$ $+ \tfrac{1}{2}n\log(2\pi) - \tfrac{1}{2}n(n-1)$ $a \geq 0, \quad n=1,2,3,\ldots$
(42)	$\int_0^1 \sin(2\pi n x)\log[\Gamma(x)]\,dx = (2\pi n)^{-1}\log(2\pi\gamma n)$ $n=1,2,\ldots$

The gamma function (cont'd)

(43)	$\int_0^1 \sin[(2n+1)\pi x] \log[\Gamma(x)]\, dx$ $= \frac{1}{(2n+1)\pi}\left[\log\left(\frac{\pi}{2}\right) + 2\left(1 + \frac{1}{3} + \cdots + \frac{1}{2n-1}\right) + \frac{1}{2n+1}\right]$ $n = 0, 1, 2, \ldots$
(44)	$\int_0^1 \cos(2\pi n x) \log[\Gamma(x)]\, dx = \frac{1}{4n}$ $\quad n = 1, 2, 3, \ldots$

17.2. The ψ function

(1)	$\int_0^1 \psi(a+x)\, dx = \log a$ $\quad a > 0$
(2)	$\int_0^1 e^{2n\pi xi} \psi(a+x)\, dx = e^{-2n\pi ai} \operatorname{Ei}(2n\pi ai)$ $a > 0, \quad n = \pm 1, \pm 2, \ldots$
(3)	$\int_0^1 \sin(2n\pi x)\, \psi(x)\, dx = -\frac{1}{2}\pi$ $\quad n = 1, 2, \ldots$
(4)	$\int_0^1 \sin(2n\pi x)\, \psi(a+x)\, dx = \sin(2n\pi a) \operatorname{ci}(2n\pi a)$ $+ \cos(2n\pi a) \operatorname{si}(2n\pi a)$ $\quad a \geq 0, \quad n = 1, 2, \ldots$
(5)	$\int_0^1 \cos(2n\pi x)\, \psi(a+x)\, dx = \sin(2n\pi a) \operatorname{si}(2n\pi a)$ $- \cos(2n\pi a) \operatorname{ci}(2n\pi a)$ $\quad a > 0, \quad n = 1, 2, \ldots$
(6)	$\int_0^\infty x^{-a}[C + \psi(1+x)] = -\pi \csc(\pi a)\, \zeta(a)$ • $\quad 1 < \operatorname{Re} a < 2$

The ψ function (cont'd)

(7)	$\int_0^\infty x^{-a}[\log x - \psi(1+x)]\,dx = \pi \csc(\pi a)\,\zeta(a)$	$0 < \operatorname{Re} a < 1$
(8)	$\int_0^\infty x^{-a}[\log(1+x) - \psi(1+x)]\,dx = \pi \csc(\pi a)\,[\zeta(a) - (a-1)^{-1}]$	$0 < \operatorname{Re} a < 1$
(9)	$\int_0^\infty x^{-a}[(1+x)^{-1} - \psi'(1+x)]\,dx = -\pi a \csc(\pi a)\,[\zeta(1+a) - a^{-1}]$	$-1 < \operatorname{Re} a < 1$
(10)	$\int_0^\infty x^{-a}[x^{-1} - \psi'(1+x)]\,dx = -\pi a \csc(a\pi)\,\zeta(1+a)$	$-2 < \operatorname{Re} a < 0$
(11)	$\int_0^\infty x^{-a}\psi^{(n)}(1+x)\,dx = (-1)^{n-1}\dfrac{\pi\Gamma(a+n)}{\Gamma(a)\sin\pi a}\,\zeta(a+n)$	$n = 1, 2, \ldots,\quad \psi^{(n)}(z) = \dfrac{d^n\psi}{dz^n},\quad 0 < \operatorname{Re} a < 1$
(12)	$\int_0^\infty [\psi(x+1) - \log x]\cos(2\pi xy)\,dx = \tfrac{1}{2}[\psi(y+1) - \log y]$	

17.3. Incomplete gamma functions and related functions

(1)	$\int_0^\infty x^{\rho-1}\operatorname{Erfc}(ax)\,dx = \dfrac{\Gamma(\frac{1}{2}\rho + \frac{1}{2})}{\pi^{1/2}\rho a^\rho}$	$\operatorname{Re} a > 0,\quad \operatorname{Re}\rho > 0$
(2)	$\int_0^\infty x^{\nu-1}\exp(\beta^2x^2)\operatorname{Erfc}(ax)\,dx = \dfrac{\Gamma(\frac{\nu+1}{2})}{\pi^{1/2}\nu a^\nu}\,{}_2F_1\left(\dfrac{\nu}{2}, \dfrac{\nu+1}{2}; \dfrac{\nu}{2}+1; \dfrac{\beta^2}{a^2}\right)$	$\operatorname{Re}\nu > 0,\quad 0, \operatorname{Re}\beta^2 < \operatorname{Re} a^2$

Incomplete gamma functions etc. (cont'd)

(3)	$\int_0^\infty x^{\nu-1} \sin(\beta x) \operatorname{Erfc}(ax)\, dx$ $= \dfrac{\Gamma(1+\frac{1}{2}\nu)\, b}{\pi^{\frac{1}{2}} (\nu+1)\, a^{\nu+1}}\; {}_2F_2\left(\dfrac{\nu+1}{2}, \dfrac{\nu}{2}+1; \dfrac{3}{2}, \dfrac{\nu+3}{2}; -\dfrac{\beta^2}{4a^2}\right)$ Re $a > 0$, Re $\nu > -1$
(4)	$\int_0^\infty x^{\nu-1} \cos(\beta x) \operatorname{Erfc}(ax)\, dx$ $= \dfrac{\Gamma(\frac{1}{2}+\frac{1}{2}\nu)}{\pi^{\frac{1}{2}}\, \nu a^\nu}\; {}_2F_2\left(\dfrac{\nu}{2}, \dfrac{\nu+1}{2}; \dfrac{1}{2}, \dfrac{\nu}{2}+1; -\dfrac{\beta^2}{4a^2}\right)$ Re $\nu > 0$, Re $a > 0$
(5)	$\int_0^\infty e^{\beta x} \operatorname{Erfc}(a^{\frac{1}{2}} x^{\frac{1}{2}})\, dx = \dfrac{1}{\beta}\left[\dfrac{a^{\frac{1}{2}}}{(a-\beta)^{\frac{1}{2}}} - 1\right]$ 0, Re $\beta <$ Re a
(6)	$\int_0^\infty \sin(\beta x) \operatorname{Erfc}(a^{\frac{1}{2}} x^{\frac{1}{2}})\, dx = \dfrac{1}{\beta} - \left(\dfrac{\frac{1}{2}a}{a^2+\beta^2}\right)^{\frac{1}{2}} [(a^2+\beta^2)^{\frac{1}{2}} - a]^{-\frac{1}{2}}$ Re $a > \lvert \operatorname{Im} \beta \rvert$
(7)	$\int_0^\infty \cos(\beta x) \operatorname{Erfc}(a^{\frac{1}{2}} x^{\frac{1}{2}})\, dx = \left(\dfrac{\frac{1}{2}a}{a^2+\beta^2}\right)^{\frac{1}{2}} [(a^2+\beta^2)^{\frac{1}{2}} + a]^{-\frac{1}{2}}$ Re $a > \lvert \operatorname{Im} \beta \rvert$
(8)	$\int_0^\infty \sin(bx) \operatorname{Erfc}(a^{\frac{1}{2}} x^{-\frac{1}{2}})\, dx = b^{-1} \exp[-(2ab)^{\frac{1}{2}}] \cos[(2ab)^{\frac{1}{2}}]$ Re $a > 0$, $b > 0$
(9)	$\int_0^\infty \cos(bx) \operatorname{Erfc}(a^{\frac{1}{2}} x^{-\frac{1}{2}})\, dx = -b^{-1} \exp[-(2ab)^{\frac{1}{2}}] \sin[(2ab)^{\frac{1}{2}}]$ Re $a > 0$, $b > 0$

Incomplete gamma functions etc. (cont'd)

(10)	$\int_0^\infty \cosh(2\nu t) \exp[(a \cosh t)^2] \operatorname{Erfc}(a \cosh t)\, dt$ $= \tfrac{1}{2}\sec(\nu\pi) \exp(\tfrac{1}{2}a^2) K_\nu(a^2) \qquad \operatorname{Re} a > 0, \quad -\tfrac{1}{2} < \operatorname{Re}\nu < \tfrac{1}{2}$
	For integrals involving products of error functions and other confluent hypergeometric functions see Bock, Philipp, 1939: *Compositio Math.* 7, 123-134. Note that the function denoted by Erfc by Bock is $\tfrac{1}{2}\pi^{1/2}$ Erfc in the notation used in the present work.
(11)	$\int_0^a e^x \operatorname{Ei}(-x)\, dx = -\log(a\gamma) + e^a \operatorname{Ei}(-a)$
(12)	$\int_0^c e^{-\beta x} \operatorname{Ei}(-ax)\, dx = -\beta^{-1}\{e^{-\beta c} \operatorname{Ei}(-ac) + \log(1+\beta/a)$ $- \operatorname{Ei}[-(a+\beta)c]\}$
(13)	$\int_0^\infty x^\nu e^x \operatorname{Ei}(-x)\, dx = \pi \csc(\pi\nu)\, \Gamma(\nu+1) \qquad -1 < \operatorname{Re}\nu < 0$
(14)	$\int_0^\infty x^{\nu-1} e^{-\beta x} \operatorname{Ei}(-ax)\, dx = -\dfrac{\Gamma(\nu)}{\nu(a+\beta)^\nu}\, {}_2F_1\left(1, \nu; \nu+1; \dfrac{\beta}{a+\beta}\right)$ $\lvert\arg a\rvert < \pi, \quad \operatorname{Re}(a+\beta) > 0, \quad \operatorname{Re}\nu > 0$
	For further integrals involving $-\operatorname{Ei}(-x) = E_1(x)$ see LeCaine, J., 1948: *National Research Council of Canada, Division of Atomic Energy*, Document No. MT-131(NRC 1553), 45 pp.. and Busbridge, I.W., 1950: *Quart. J. Math. Oxford Ser.* (2) 1, 176-184.
(15)	$\int_0^\infty x^{\mu-1} e^{-\beta x} \gamma(\nu, ax)\, dx = \dfrac{a^\nu \Gamma(\mu+\nu)}{\nu(a+\beta)^{\mu+\nu}}\, {}_2F_1\left(1, \mu+\nu; \nu+1; \dfrac{a}{a+\beta}\right)$ $\operatorname{Re}(a+\beta) > 0, \quad \operatorname{Re}\beta > 0, \quad \operatorname{Re}(\mu+\nu) > 0$

Incomplete gamma functions etc. (cont'd)

(16)	$\int_0^\infty x^{\mu-1} e^{-\beta x}\, \Gamma(\nu, ax)\, dx = \dfrac{a^\nu \Gamma(\mu+\nu)}{\mu(a+\beta)^{\mu+\nu}}\ {}_2F_1\left(1, \mu+\nu; \mu+1; \dfrac{\beta}{a+\beta}\right)$ $\operatorname{Re}(a+\beta) > 0, \quad \operatorname{Re}\mu > 0, \quad \operatorname{Re}(\mu+\nu) > 0$
(17)	$\int_0^\infty e^{-\beta x}\, \gamma(\nu, ax^2)\, dx = 2^{1-\nu} \beta^{-1}\, \Gamma(2\nu) \exp\left(\dfrac{\beta^2}{8a}\right) D_{-2\nu}\left[\dfrac{\beta}{(2a)^{1/2}}\right]$ $\operatorname{Re} a > 0, \quad \operatorname{Re}\beta > 0, \quad \nu \neq 0, \quad \operatorname{Re}\nu > -\tfrac{1}{2}$
(18)	$\int_0^\infty x^{1-2\nu} \exp(ax^2) \sin(bx)\, \Gamma(\nu, ax^2)\, dx$ $= \pi^{1/2}\, 2^{-\nu} a^{\nu-1}\, \Gamma\left(\dfrac{3}{2} - \nu\right) \exp\left(\dfrac{b^2}{8a}\right) D_{2\nu-2}\left[\dfrac{b}{(2a)^{1/2}}\right]$ $\lvert\arg a\rvert < 3\pi/2, \quad 0 < \operatorname{Re}\nu < 1$
	For other integrals involving $E_n(x) = x^{n-1}\, \Gamma(1-n, x)$ see LeCaine, J., 1948: *National Research Council of Canada, Division of Atomic Energy*, Document No. MT-131 (NRC 1553), 45 pp., and Busbridge, I.W., 1950: *Quart. J. Math. Oxford Ser.* (2) 1, 176-184.
(19)	$\int_0^\infty e^{-\beta x}\, \gamma(\nu, ax^{1/2})\, dx = 2^{-\frac{1}{2}\nu} a^\nu \beta^{-\frac{1}{2}\nu-1}\, \Gamma(\nu) \exp\left(\dfrac{a^2}{8\beta}\right)$ $\times D_{-\nu}\left[\dfrac{a}{(2\beta)^{1/2}}\right] \qquad \operatorname{Re}\beta > 0, \quad \operatorname{Re}\nu > 0$

CHAPTER XVIII

LEGENDRE FUNCTIONS

For the theory of these functions see H.T.F. vol. I, Chapter III and the literature quoted there, especially the books by Hobson, MacRobert, Whittaker and Watson. Numerous expansions of Legendre functions in hypergeometric series are listed in H.T.F. vol. I, p. 124-139, and these may be used to reduce integrals involving Legendre functions to integrals involving hypergeometric series.

LEGENDRE FUNCTIONS

18.1. Legendre functions of variable $\alpha x + \beta$: finite intervals

(1)	$\int_0^1 x^{\lambda-1} P_\nu(x)\,dx = \dfrac{\pi^{1/2}\, 2^{-\lambda}\,\Gamma(\lambda)}{\Gamma(\frac{1}{2}+\frac{1}{2}\lambda-\frac{1}{2}\nu)\,\Gamma(1+\frac{1}{2}\lambda+\frac{1}{2}\nu)}$	$\operatorname{Re}\lambda > 0$
(2)	$\int_0^1 x^{\lambda-1} P_\nu^m(x)\,dx = \dfrac{(-1)^m\,\pi^{1/2}\,2^{-2m-1}\,\Gamma(\frac{1}{2}\lambda)\,\Gamma(1+m+\nu)}{\Gamma(\frac{1}{2}+\frac{1}{2}m)\,\Gamma(1+\frac{1}{2}\lambda+m)\,\Gamma(1-m+\nu)}$ $\times\, {}_3F_2\left(\dfrac{m+\nu+1}{2}, \dfrac{m-\nu}{2}, \dfrac{m}{2}+1; m+1, \dfrac{\lambda+m}{2}+1; 1\right)$	$\operatorname{Re}\lambda > 0, \quad m = 0, 1, 2, \ldots$
(3)	$\int_0^1 x^{\lambda-1} P_\nu^\mu(x)\,dx = \dfrac{\pi^{1/2}\,2^{2\mu-1}\,\Gamma(\frac{1}{2}\lambda)}{\Gamma(\frac{1}{2}-\frac{1}{2}\mu)\,\Gamma(1+\frac{1}{2}\lambda-\frac{1}{2}\mu)}$ $\times\, {}_3F_2\left(\dfrac{\nu-\mu+1}{2}, -\dfrac{\mu+\nu}{2}, 1-\dfrac{\mu}{2}; 1-\mu, \dfrac{\lambda-\mu}{2}+1; 1\right)$	$\operatorname{Re}\lambda > 0, \quad \operatorname{Re}\mu < 2$
(4)	$\int_0^1 x^{\lambda-1}(1-x^2)^{\frac{1}{2}m} P_\nu^m(x)\,dx$ $= \dfrac{(-1)^m\,\pi^{1/2}\,2^{-\lambda-m}\,\Gamma(\lambda)\,\Gamma(1+m+\nu)}{\Gamma(\frac{1}{2}+\frac{1}{2}\lambda+\frac{1}{2}m-\frac{1}{2}\nu)\,\Gamma(1+\frac{1}{2}\lambda+\frac{1}{2}m+\frac{1}{2}\nu)\,\Gamma(1-m+\nu)}$	$\operatorname{Re}\lambda > 0, \quad m = 0, 1, 2, \ldots$

Variable $\alpha x + \beta$: finite intervals (cont'd)

(5)	$\int_0^1 x^{\lambda-1}(1-x^2)^{-\frac{1}{2}\mu}\,\mathrm{P}_\nu^\mu(x)\,dx$ $= \dfrac{\pi^{\frac{1}{2}}\,2^{\mu-\lambda}\,\Gamma(\lambda)}{\Gamma(\frac{1}{2}+\frac{1}{2}\lambda-\frac{1}{2}\mu-\frac{1}{2}\nu)\,\Gamma(1+\frac{1}{2}\lambda-\frac{1}{2}\mu+\frac{1}{2}\nu)}$ $\mathrm{Re}\,\lambda>0, \quad \mathrm{Re}\,\mu<1$
(6)	$\int_0^1 x^{\lambda-1}(1-x^2)^{\kappa}\,\mathrm{P}_\nu^\mu(x)\,dx = \dfrac{2^{\mu-1}\,\Gamma(1+\kappa-\frac{1}{2}\mu)\,\Gamma(\frac{1}{2}\lambda)}{\Gamma(1-\mu)\,\Gamma(1+\kappa+\frac{1}{2}\lambda-\frac{1}{2}\mu)}$ $\times\,{}_3F_2\left(\dfrac{\nu-\mu+1}{2}, -\dfrac{\mu+\nu}{2}, 1+\kappa-\dfrac{\mu}{2}; 1-\mu, 1+\dfrac{\lambda-\mu}{2}+\kappa; 1\right)$ $\mathrm{Re}\,(\kappa-\frac{1}{2}\mu)>-1, \quad \mathrm{Re}\,\lambda>0$
(7)	$\int_0^1 x^{\lambda-1}(1-x^2)^{-\frac{1}{2}\mu}\sin(ax)\,P_\nu^\mu(x)\,dx$ $= \dfrac{\pi^{\frac{1}{2}}\,2^{\mu-\lambda-1}\,\Gamma(\lambda+1)\,a}{\Gamma[1+\frac{1}{2}(\lambda-\mu-\nu)]\,\Gamma[\frac{1}{2}(3+\lambda-\mu+\nu)]}$ $\times\,{}_2F_3\left(\dfrac{1+\lambda}{2}, 1+\dfrac{\lambda}{2}; \dfrac{3}{2}, 1+\dfrac{\lambda-\mu-\nu}{2}, \dfrac{3+\lambda-\mu+\nu}{2}; -\dfrac{a^2}{4}\right)$ $\mathrm{Re}\,\lambda>-1, \quad \mathrm{Re}\,\mu<1$
(8)	$\int_0^1 x^{\lambda-1}(1-x^2)^{-\frac{1}{2}\mu}\cos(ax)\,P_\nu^\mu(x)\,dx$ $= \dfrac{\pi^{\frac{1}{2}}\,2^{\mu-\lambda}\,\Gamma(\lambda)}{\Gamma[1+\frac{1}{2}(\lambda-\mu+\nu)]\,\Gamma[\frac{1}{2}(1+\lambda-\mu-\nu)]}$ $\times\,{}_2F_3\left(\dfrac{\lambda}{2}, \dfrac{\lambda+1}{2}; \dfrac{1}{2}, \dfrac{1+\lambda-\mu-\nu}{2}, 1+\dfrac{\lambda-\mu+\nu}{2}; -\dfrac{a^2}{4}\right)$ $\mathrm{Re}\,\lambda>0, \quad \mathrm{Re}\,\mu<1$

Variable $\alpha x + \beta$: finite intervals (cont'd)

(9)	$\int_0^1 (1 - x^2)^{-1} [\mathrm{P}_\nu^{n-\nu}(x)]^2\, dx = -\dfrac{n!}{2(n-\nu)\,\Gamma(1-n+2\nu)}$ $n = 0, 1, 2, \ldots, \quad \mathrm{Re}\,\nu > n$
(10)	$\int_0^1 \mathrm{P}_\lambda(x)\, \mathrm{P}_\nu(x)\, dx = 2[\pi(\lambda-\nu)(\lambda+\nu+1)]^{-1}$ $\times [A \sin(\tfrac{1}{2}\lambda\pi)\cos(\tfrac{1}{2}\nu\pi) - A^{-1}\cos(\tfrac{1}{2}\lambda\pi)\sin(\tfrac{1}{2}\nu\pi)]$ $A = \dfrac{\Gamma(\tfrac{1}{2}+\tfrac{1}{2}\nu)\,\Gamma(1+\tfrac{1}{2}\lambda)}{\Gamma(\tfrac{1}{2}+\tfrac{1}{2}\lambda)\,\Gamma(1+\tfrac{1}{2}\nu)}$
(11)	$\int_0^1 \mathrm{P}_\nu(x)\, \mathrm{Q}_\lambda(x)\, dx = [(\lambda-\nu)(\lambda+\nu+1)]^{-1}\{A^{-1}\cos[\tfrac{1}{2}(\nu-\lambda)\pi] - 1\}$ $A = \dfrac{\Gamma(1+\tfrac{1}{2}\lambda)\,\Gamma(\tfrac{1}{2}+\tfrac{1}{2}\nu)}{\Gamma(\tfrac{1}{2}+\tfrac{1}{2}\lambda)\,\Gamma(1+\tfrac{1}{2}\nu)}$
(12)	$\int_0^1 \mathrm{Q}_\lambda(x)\, \mathrm{Q}_\nu(x)\, dx = [(\lambda-\nu)(\lambda+\nu+1)]^{-1}\{\psi(\nu+1) - \psi(\lambda+1)$ $-\tfrac{1}{2}\pi(A - A^{-1})\sin[\tfrac{1}{2}(\lambda+\nu)\pi] + \tfrac{1}{2}\pi(A + A^{-1})\sin[\tfrac{1}{2}(\lambda-\nu)\pi]\}$ $A = \dfrac{\Gamma(1+\tfrac{1}{2}\lambda)\,\Gamma(\tfrac{1}{2}+\tfrac{1}{2}\nu)}{\Gamma(\tfrac{1}{2}+\tfrac{1}{2}\lambda)\,\Gamma(1+\tfrac{1}{2}\nu)}$
(13)	$\int_0^1 [P_\nu^\mu(x)]^2\, dx, \qquad \int_0^1 P_\nu^\mu(x)\, Q_\nu^\mu(x)\, dx.$ See Barnes, E.W., 1908: *Quart. J. Math.* 39, 97-204. Note that Barnes' definition of Legendre functions of the second kind differs from the one used in this book.
(14)	$\int_0^1 P_\nu^m(x)\, P_\lambda^m(x)\, dx$ See Shabde, N.G., 1937: *Bull. Calcutta Math. Soc.* 29, 33-40.

Variable $\alpha x + \beta$: finite intervals (cont'd)

(15)	$\int_{-1}^{1}(1+x)^{\lambda-1}\,\mathrm{P}_\nu(x)\,dx = \dfrac{2^\lambda[\Gamma(\lambda)]^2}{\Gamma(\lambda+\nu+1)\,\Gamma(\lambda-\nu)}$ Re $\lambda > 0$
(16)	$\int_{-1}^{1}(1-x^2)^{\lambda-1}\,\mathrm{P}_\nu^\mu(x)\,dx$ $= \dfrac{\pi\, 2^\mu\,\Gamma(\lambda+\frac{1}{2}\mu)\,\Gamma(\lambda-\frac{1}{2}\mu)}{\Gamma(\lambda+\frac{1}{2}\nu+1)\Gamma(\lambda-\frac{1}{2}\nu)\,\Gamma(-\frac{1}{2}\mu+\frac{1}{2}\nu+1)\,\Gamma(-\frac{1}{2}\mu-\frac{1}{2}\nu+\frac{1}{2})}$ $2\,\mathrm{Re}\,(\lambda) > \lvert\mathrm{Re}\,\mu\rvert$
(17)	$\int_{-1}^{1}(1-x^2)^{-\frac{1}{2}\mu}(z-x)^{-1}\,\mathrm{P}_{\mu+n}^{\mu}(x)\,dx = 2e^{-i\mu\pi}(z^2-1)^{-\frac{1}{2}\mu}Q_{\mu+n}^{\mu}(z)$ $n = 0, 1, 2, \ldots,$ Re $\mu + n > -1$, z in the cut complex plane
(18)	$\int_{-1}^{1}(1-x)^{-\frac{1}{2}\mu}(1+x)^{\frac{1}{2}\mu-\frac{1}{2}}(z+x)^{\mu-\frac{1}{2}}\,\mathrm{P}_\nu^\mu(x)\,dx$ $= \dfrac{2e^{-2\mu\pi i}\Gamma(\frac{1}{2}+\mu)}{\pi^{\frac{1}{2}}\,\Gamma(\mu-\nu)\,\Gamma(\mu+\nu+1)}(z-1)^\mu \times Q_\nu^\mu\left[\left(\dfrac{1+z}{2}\right)^{\frac{1}{2}}\right] Q_{-\nu-1}^{\mu}\left[\left(\dfrac{1+z}{2}\right)^{\frac{1}{2}}\right]$ $-\frac{1}{2} < \mathrm{Re}\,\mu < 1$, z in the cut complex plane
(19)	$\int_{-1}^{1}(1-x)^{-\frac{1}{2}\mu-1}(1+x)^{\frac{1}{2}\mu-\frac{1}{2}}(z+x)^{\mu-\frac{1}{2}}[(\nu-\mu)\,\mathrm{P}_\nu^\mu(x)-(\nu+\mu)\,\mathrm{P}_{\nu-1}^{\mu}(x)]\,dx$ $= \dfrac{e^{-2\mu\pi i}\Gamma(\mu+\frac{1}{2})(z-1)^\mu}{(\frac{1}{2}\pi)^{\frac{1}{2}}\,\Gamma(\mu+\nu)\,\Gamma(\mu-\nu)(z+1)^{\frac{1}{2}}}\left\{Q_\nu^\mu\left[\left(\dfrac{1+z}{2}\right)^{\frac{1}{2}}\right]Q_{-\nu}^{\mu}\left[\left(\dfrac{1+z}{2}\right)^{\frac{1}{2}}\right] + Q_{\nu-1}^{\mu}\left[\left(\dfrac{1+z}{2}\right)^{\frac{1}{2}}\right]Q_{-\nu-1}^{\mu}\left[\left(\dfrac{1+z}{2}\right)^{\frac{1}{2}}\right]\right\}$ $-\frac{1}{2} < \mathrm{Re}\,\mu < 0$, z in the cut complex plane

The complex z-plane is cut along the real axis from -1 to 1.

Variable $\alpha x+\beta$: finite intervals (cont'd)

(20)	$\int_{-1}^{1}(1-x)^{-\frac{1}{2}\mu}(1+x)^{\frac{1}{2}\mu-\frac{1}{2}}(z+x)^{\mu-3/2}\,\mathrm{P}_{\nu}^{\mu}(x)\,dx$ $= -\dfrac{\Gamma(\mu-\frac{1}{2})(z-1)^{\mu-\frac{1}{2}}(z+1)^{-\frac{1}{2}}}{\pi^{\frac{1}{2}}\,e^{2\mu\pi i}\,\Gamma(\mu+\nu)\,\Gamma(\mu-\nu-1)}\left\{Q_{\nu}^{\mu}\left[\left(\dfrac{1+z}{2}\right)^{\frac{1}{2}}\right]Q_{-\nu-1}^{\mu-1}\left[\left(\dfrac{1+z}{2}\right)^{\frac{1}{2}}\right]\right.$ $\left.+\,Q_{\nu}^{\mu-1}\left[\left(\dfrac{1+z}{2}\right)^{\frac{1}{2}}\right]\,Q_{-\nu-1}^{\mu}\left[\left(\dfrac{1+z}{2}\right)^{\frac{1}{2}}\right]\right\}$ $-\frac{1}{2}<\mathrm{Re}\,\mu<1$, z in the cut complex plane
(21)	$\int_{-1}^{1}(1-x)^{-\frac{1}{2}\mu}(1+x)^{\frac{1}{2}\mu+\nu-1}\exp\left(-\dfrac{1-x}{1+x}y\right)\,\mathrm{P}_{\nu}^{\mu}(x)\,dx$ $=2^{\nu}y^{\frac{1}{2}\mu+\nu-\frac{1}{2}}\,e^{\frac{1}{2}y}\,W_{\frac{1}{2}\mu-\nu-\frac{1}{2},\,\frac{1}{2}\mu}(y)$ $\mathrm{Re}\,y>0$
(22)	$\int_{-1}^{1}\mathrm{P}_{\nu}(x)\,\mathrm{P}_{\lambda}(x)\,dx$ $=\dfrac{4\sin(\nu\pi)\sin(\lambda\pi)\,[\psi(\nu+1)-\psi(\lambda+1)]+2\pi\sin[(\lambda-\nu)\pi]}{\pi^{2}(\lambda-\nu)(\lambda+\nu+1)}$
(23)	$\int_{-1}^{1}[\mathrm{P}_{\nu}(x)]^{2}\,dx=\dfrac{\pi^{2}-2[\sin(\nu\pi)]^{2}\,\psi'(\nu+1)}{\pi^{2}(\nu+\frac{1}{2})}$
(24)	$\int_{-1}^{1}\mathrm{P}_{\nu}(x)\,\mathrm{P}_{\lambda}(x)(1+x)^{\lambda+\nu}\,dx=\dfrac{2^{\lambda+\nu+1}[\Gamma(\lambda+\nu+1)]^{4}}{[\Gamma(\lambda+1)\Gamma(\nu+1)]^{2}\,\Gamma(2\lambda+2\nu+2)}$ $\mathrm{Re}(\lambda+\nu)>-1$
(25)	$\int_{-1}^{1}\mathrm{P}_{\nu}(x)\,\mathrm{Q}_{\nu}(x)\,dx=-\dfrac{\sin(2\pi\nu)\,\psi'(\nu+1)}{\pi(2\nu+1)}$

The complex z-plane is cut along the real axis from -1 to 1.

Variable $\alpha x + \beta$: finite intervals (cont'd)

(26)	$\int_{-1}^{1} P_\nu(x)\, Q_\lambda(x)\, dx = [(\nu-\lambda)(\nu+\lambda+1)]^{-1}\{1-\cos[(\lambda-\nu)\pi]$ $- 2\pi^{-1}\sin(\nu\pi)\cos(\lambda\pi)\,[\psi(\nu+1)-\psi(\lambda+1)]\}$
(27)	$\int_{-1}^{1} [Q_\nu(x)]^2\, dx = \dfrac{\frac{1}{2}\pi^2 - \{1-[\cos(\nu\pi)]^2\}\,\psi'(\nu+1)}{2\nu+1}$
(28)	$\int_{-1}^{1} Q_\nu(x)\, Q_\lambda(x)\, dx = [(\lambda-\nu)(\lambda+\nu+1)]^{-1}\{\tfrac{1}{2}\pi\sin[(\lambda-\nu)\pi]$ $+ [\psi(\nu+1)-\psi(\lambda+1)]\,[1+\cos(\lambda\pi)\cos(\nu\pi)]\}$
(29)	$\int_{-1}^{1} (1-x^2)^{\frac{1}{2}m-M-\frac{1}{2}} \left(\dfrac{1-x}{1+x}\right)^{\frac{1}{2}\nu} P_\mu^M(x)\, P_\lambda^M(x)\, dx$ See Shabde, N.G., 1940: *Bull. Calcutta Math. Soc.* 32, 121-128.
(30)	$\int_{-1}^{1} P_\lambda^\rho(x)\, P_\mu^\sigma(x)\, P_\nu^\tau(x)\, dx$ See Gaunt, J.A., 1929: *Philos. Trans. Royal Soc.* 228, 151-196.
(31)	$\int_{-1}^{1} (1-x^2)^{\lambda-1}(1-a^2x^2)^{\frac{1}{2}\mu} P_\nu(ax)\, dx$ $= \dfrac{\pi\, 2^\mu\, \Gamma(\lambda)}{\Gamma(\frac{1}{2}+\lambda)\,\Gamma(\frac{1}{2}-\frac{1}{2}\mu-\frac{1}{2}\nu)\,\Gamma(1-\frac{1}{2}\mu+\frac{1}{2}\nu)}$ $\times\, {}_2F_1\left(-\dfrac{\mu+\nu}{2}, \dfrac{1-\mu+\nu}{2}; \dfrac{1}{2}+\lambda; a^2\right)$ $\mathrm{Re}\,\lambda > 0, \quad -1 < a < 1$

Variable $\alpha x + \beta$: finite intervals (cont'd)

(32)	$\int_0^1 x^{-\frac{1}{2}\mu-\frac{1}{2}} (1-x)^{-\mu-\frac{1}{2}} (1+\alpha x)^{\frac{1}{2}\mu} P_\nu^\mu(1+2\alpha x)\, dx$ $= \pi^{\frac{1}{2}} \Gamma(\tfrac{1}{2}-\mu)\, \alpha^{\frac{1}{2}\mu} \{P_\nu^\mu[(1+\alpha)^{\frac{1}{2}}]\}^2 \qquad \mathrm{Re}\,\mu < \tfrac{1}{2}, \quad \lvert\arg\alpha\rvert < \pi$
(33)	$\int_0^1 x^{-\mu/2-1/2} (1-x)^{-\mu-3/2} (1+\alpha x)^{\mu/2} P_\nu^\mu(1+2\alpha x)\, dx$ $= \pi^{\frac{1}{2}} \Gamma(-\mu-\tfrac{1}{2})(1+\alpha)^{-\frac{1}{2}} \alpha^{\frac{1}{2}\mu+\frac{1}{2}} P_\nu^{\mu+1}[(1+\alpha)^{\frac{1}{2}}]\, P_\nu^\mu[(1+\alpha)^{\frac{1}{2}}]$ $\mathrm{Re}\,\mu < -\tfrac{1}{2}, \quad \lvert\arg\alpha\rvert < \pi$
(34)	$\int_0^1 x^{\frac{1}{2}\mu-\frac{1}{2}} (1-x)^{\mu-\frac{1}{2}} (1+\alpha x)^{-\frac{1}{2}\mu} P_\nu^\mu(1+2\alpha x)\, dx$ $= \pi^{\frac{1}{2}} \Gamma(\tfrac{1}{2}+\mu)\, \alpha^{-\frac{1}{2}\mu} P_\nu^\mu[(1+\alpha)^{\frac{1}{2}}]\, P_\nu^{-\mu}[(1+\alpha)^{\frac{1}{2}}]$ $\mathrm{Re}\,\mu > -\tfrac{1}{2}, \quad \lvert\arg\alpha\rvert < \pi$
(35)	$\int_0^1 x^{\mu/2-1/2} (1-x)^{\mu-3/2} (1+\alpha x)^{-\mu/2} P_\nu^\mu(1+2\alpha x)\, dx$ $= \tfrac{1}{2}\pi^{\frac{1}{2}} \Gamma(\mu-\tfrac{1}{2})(1+\alpha)^{-\frac{1}{2}} \alpha^{\frac{1}{2}-\frac{1}{2}\mu} \{P_\nu^{1-\mu}[(1+\alpha)^{\frac{1}{2}}] P_\nu^\mu[(1+\alpha)^{\frac{1}{2}}]$ $+ (\mu+\nu)(1-\mu+\nu)\, P_\nu^{-\mu}[(1+\alpha)^{\frac{1}{2}}]\, P_\nu^\mu[(1+\alpha)^{\frac{1}{2}}]\}$ $\mathrm{Re}\,\mu > \tfrac{1}{2}, \quad \lvert\arg\alpha\rvert < \pi$
(36)	$\int_0^1 x^{-\frac{1}{2}\mu-1} (1-x)^{-\mu-\frac{1}{2}} (1+\alpha x)^{\frac{1}{2}\mu-\frac{1}{2}} [(1-\nu-\mu)\, P_{\nu-1}^{\mu-1}(1+2\alpha x)$ $+ (1+\nu-\mu) P_\nu^{\mu-1}(1+2\alpha x)]\, dx = 2\pi^{\frac{1}{2}} \Gamma(\tfrac{1}{2}-\mu)(1+\alpha)^{-\frac{1}{2}} \alpha^{\frac{1}{2}\mu+\frac{1}{2}}$ $\times P_\nu^\mu[(1+\alpha)^{\frac{1}{2}}]\, P_{\nu-1}^\mu[(1+\alpha)^{\frac{1}{2}}] \qquad \mathrm{Re}\,\mu < \tfrac{1}{2}, \quad \lvert\arg\alpha\rvert < \pi$
(37)	$\int_0^1 x^{-\frac{1}{2}\mu-1} (1-x)^{-\mu-\frac{1}{2}} (1+\alpha x)^{\frac{1}{2}\mu-\frac{1}{2}} [P_{\nu-1}^{1-\mu}(1+2\alpha x)$ $- P_\nu^{1-\mu}(1+2\alpha x)]\, dx = \pi^{\frac{1}{2}} \Gamma(\tfrac{1}{2}-\mu)(1+\alpha)^{-\frac{1}{2}} \alpha^{\frac{1}{2}\mu+\frac{1}{2}}$ $\times \{(\mu-\nu) P_\nu^\mu[(1+\alpha)^{\frac{1}{2}}]\, P_{\nu-1}^{-\mu}[(1+\alpha)^{\frac{1}{2}}]$ $- (\mu+\nu) P_{\nu-1}^\mu[(1+\alpha)^{\frac{1}{2}}]\, P_\nu^{-\mu}[(1+\alpha)^{\frac{1}{2}}]\} \qquad \mathrm{Re}\,\mu < \tfrac{1}{2}, \quad \lvert\arg\alpha\rvert < \pi$

Variable $\alpha x + \beta$: finite intervals (cont'd)

(38)	$\int_0^1 x^{-\frac{1}{2}\mu-\frac{1}{2}} (1-x)^{-\mu-\frac{1}{2}} (1+ax)^{\frac{1}{2}\mu} Q_\nu^\mu(1+2ax)\,dx$ $= \pi^{\frac{1}{2}} \Gamma(\tfrac{1}{2}-\mu)\, a^{\frac{1}{2}\mu} P_\nu^\mu[(1+a)^{\frac{1}{2}}]\, Q_\nu^\mu[(1+a)^{\frac{1}{2}}]$ $\operatorname{Re}\mu < \tfrac{1}{2}, \quad \lvert\arg a\rvert < \pi$
(39)	$\int_0^1 x^{-\mu/2-1/2}(1-x)^{-\mu-3/2} (1+ax)^{\mu/2} Q_\nu^\mu(1+2ax)\,dx$ $= \tfrac{1}{2}\pi^{\frac{1}{2}} \Gamma(-\mu-\tfrac{1}{2}) (1+a)^{-\frac{1}{2}} a^{\frac{1}{2}\mu+\frac{1}{2}}$ $\times \{P_\nu^{\mu+1}[(1+a)^{\frac{1}{2}}]\, Q_\nu^\mu[(1+a)^{\frac{1}{2}}]$ $+ P_\nu^\mu[(1+a)^{\frac{1}{2}}]\, Q_\nu^{\mu+1}[(1+a)^{\frac{1}{2}}]\}$ $\operatorname{Re}\mu < -\tfrac{1}{2}, \quad \lvert\arg a\rvert < \pi$

18.2. Legendre functions of variable $\alpha x + \beta$: infinite intervals

(1)	$\int_0^\infty (x^2-1)^{\frac{1}{2}\mu} \sin(ax)\, P_\nu^\mu(x)\,dx$ $= \dfrac{2^\mu \pi^{\frac{1}{2}} a^{-\mu-\frac{1}{2}}}{\Gamma(\tfrac{1}{2}-\tfrac{1}{2}\mu-\tfrac{1}{2}\nu)\,\Gamma(1-\tfrac{1}{2}\mu+\tfrac{1}{2}\nu)} S_{\mu+\frac{1}{2},\,\nu+\frac{1}{2}}(a)$ $a>0, \quad \operatorname{Re}\mu < 3/2, \quad \operatorname{Re}(\mu+\nu) < 1$
(2)	$\int_1^\infty (x^2-1)^{\lambda-1} P_\nu^\mu(x)\,dx = \dfrac{2^{\mu-1}\Gamma(\lambda-\tfrac{1}{2}\mu)\Gamma(1-\lambda+\tfrac{1}{2}\nu)\Gamma(\tfrac{1}{2}-\lambda-\tfrac{1}{2}\nu)}{\Gamma(1-\tfrac{1}{2}\mu+\tfrac{1}{2}\nu)\Gamma(\tfrac{1}{2}-\tfrac{1}{2}\mu-\tfrac{1}{2}\nu)\Gamma(1-\lambda-\tfrac{1}{2}\mu)}$ $\operatorname{Re}\lambda > \operatorname{Re}\mu, \quad \operatorname{Re}(1-2\lambda-\nu) > 0, \quad \operatorname{Re}(2-2\lambda+\nu) > 0$
(3)	$\int_1^\infty x^{-\rho}(x^2-1)^{-\frac{1}{2}\mu} P_\nu^\mu(x)\,dx = \dfrac{2^{\rho+\mu-2}\,\Gamma\left(\frac{\rho+\mu+\nu}{2}\right)\Gamma\left(\frac{\rho+\mu-\nu-1}{2}\right)}{\pi^{\frac{1}{2}}\Gamma(\rho)}$ $\operatorname{Re}\mu < 1, \quad \operatorname{Re}(\rho+\mu+\nu) > 0, \quad \operatorname{Re}(\rho+\mu-\nu) > 1$

Variable $\alpha x + \beta$: infinite intervals (cont'd)

(4)	$\int_1^\infty (x-1)^{\lambda-1} (x^2-1)^{\frac{1}{2}\mu} P_\nu^\mu(x)\, dx$ $= \dfrac{2^{\lambda+\mu}\Gamma(\lambda)\Gamma(-\lambda-\mu-\nu)\Gamma(1-\lambda-\mu+\nu)}{\Gamma(1-\mu+\nu)\Gamma(-\mu-\nu)\Gamma(1-\lambda-\mu)}$ $\mathrm{Re}\,\lambda > 0, \quad \mathrm{Re}(\lambda+\mu+\nu) < 0, \quad \mathrm{Re}(\lambda+\mu-\nu) < 1$
(5)	$\int_1^\infty (x-1)^{\lambda-1} (x^2-1)^{-\frac{1}{2}\mu} P_\nu^\mu(x)\, dx$ $= -\dfrac{2^{\lambda-\mu}\sin(\nu\pi)\Gamma(\lambda-\mu)\Gamma(-\lambda+\mu-\nu)\Gamma(1-\lambda+\mu+\nu)}{\pi\Gamma(1-\lambda)}$ $\mathrm{Re}(\lambda-\mu) > 0, \quad \mathrm{Re}(\mu-\lambda-\nu) > 0, \quad \mathrm{Re}(\mu-\lambda+\nu) > -1$
(6)	$\int_1^\infty (x-1)^{-\frac{1}{2}\mu} (x+1)^{\frac{1}{2}\mu-\frac{1}{2}} (z+x)^{\mu-\frac{1}{2}} P_\nu^\mu(x)\, dx$ $= \pi^{\frac{1}{2}} \dfrac{\Gamma(-\mu-\nu)\Gamma(1-\mu+\nu)}{\Gamma(\frac{1}{2}-\mu)} (z-1)^\mu \left\{ P_\nu^\mu\left[\left(\dfrac{1+z}{2}\right)^{\frac{1}{2}}\right]\right\}^2$ $\mathrm{Re}(\mu+\nu) < 0, \quad \mathrm{Re}(\mu-\nu) < 1, \quad \lvert\arg(z+1)\rvert < \pi$
(7)	$\int_1^\infty (x-1)^{-\frac{1}{2}\mu} (x+1)^{\frac{1}{2}\mu-\frac{1}{2}} (z+x)^{\mu-3/2} P_\nu^\mu(x)\, dx$ $= \pi^{\frac{1}{2}} \dfrac{\Gamma(1-\mu-\nu)\Gamma(2-\mu+\nu)}{\Gamma(3/2-\mu)} (z-1)^{\mu-\frac{1}{2}} (z+1)^{-\frac{1}{2}}$ $\times P_\nu^\mu\left[\left(\dfrac{1+z}{2}\right)^{\frac{1}{2}}\right] P_\nu^{\mu-1}\left[\left(\dfrac{1+z}{2}\right)^{\frac{1}{2}}\right]$ $\mathrm{Re}\,\mu < 1, \quad \mathrm{Re}(\mu+\nu) < 1, \quad \mathrm{Re}(\mu-\nu) < 2, \quad \lvert\arg(1+z)\rvert < \pi$

Variable $\alpha x+\beta$: infinite intervals (cont'd)

(8)	$\int_1^\infty (x-1)^{-\frac{1}{2}\mu-1}(x+1)^{\frac{1}{2}\mu-\frac{1}{2}}(z+x)^{\mu-\frac{1}{2}}[(\nu-\mu)P_\nu^\mu(x)$ $-(\nu+\mu)P_{\nu-1}^\mu(x)]dx=(2\pi)^{\frac{1}{2}}\dfrac{\Gamma(1-\mu-\nu)\Gamma(1-\mu+\nu)}{\Gamma(\frac{1}{2}-\mu)}(z-1)^\mu(z+1)^{-\frac{1}{2}}$ $\times P_\nu^\mu\left[\left(\dfrac{1+z}{2}\right)^{\frac{1}{2}}\right]P_{\nu-1}^\mu\left[\left(\dfrac{1+z}{2}\right)^{\frac{1}{2}}\right]$ $\mathrm{Re}\,\mu<0,\quad \mathrm{Re}\,\mu<1-\lvert\mathrm{Re}\,\nu\rvert,\quad \lvert\arg(z+1)\rvert<\pi$
(9)	$\int_1^\infty (x-1)^{\lambda-1}(x^2-1)^{\frac{1}{2}\mu}(x+z)^{-\rho}P_\nu^\mu(x)\,dx$ $=\dfrac{2^{\lambda+\mu-\rho}\Gamma(\lambda-\rho)\Gamma(\rho-\lambda-\mu-\nu)\Gamma(\rho-\lambda-\mu+\nu+1)}{\Gamma(1-\mu+\nu)\Gamma(-\mu-\nu)\Gamma(1+\rho-\lambda-\mu)}$ $\times\,{}_3F_2(\rho,\rho-\lambda-\mu-\nu,\rho-\lambda-\mu+\nu+1;\rho-\lambda+1,\rho-\lambda-\mu+1;\tfrac{1}{2}+\tfrac{1}{2}z)$ $+\dfrac{\Gamma(\rho-\lambda)\Gamma(\lambda)}{\Gamma(\rho)\Gamma(1-\mu)}2^\mu(z+1)^{\lambda-\rho}$ $\times\,{}_3F_2(\lambda,-\mu-\nu,1-\mu+\nu;1-\mu,1-\rho+\lambda;\tfrac{1}{2}+\tfrac{1}{2}z)\qquad \mathrm{Re}\,\lambda>0$ $\mathrm{Re}(\rho-\lambda-\mu-\nu)>0,\quad \mathrm{Re}(\rho-\lambda-\mu+\nu+1)>0,\quad \lvert\arg(z+1)\rvert<\pi$
(10)	$\int_1^\infty (x-1)^{\lambda-1}(x^2-1)^{-\frac{1}{2}\mu}(x+z)^{-\rho}P_\nu^\mu(x)\,dx$ $=-\dfrac{\sin(\nu\pi)\Gamma(\lambda-\mu-\rho)\Gamma(\rho-\lambda+\mu-\nu)\Gamma(\rho-\lambda+\mu+\nu+1)}{2^{\rho-\lambda+\mu}\pi\Gamma(1+\rho-\lambda)}$ $\times\,{}_3F_2(\rho,\rho-\lambda+\mu-\nu,\rho-\lambda+\mu+\nu+1;1+\rho-\lambda,1+\rho-\lambda+\mu;\tfrac{1}{2}+\tfrac{1}{2}z)$ $+\dfrac{\Gamma(\lambda-\mu)\Gamma(\rho-\lambda+\mu)}{\Gamma(\rho)\Gamma(1-\mu)}(z+1)^{\lambda-\rho-\mu}$ $\times\,{}_3F_2(\lambda-\mu,-\nu,\nu+1;1+\lambda-\mu-\rho,1-\mu;\tfrac{1}{2}+\tfrac{1}{2}z)\qquad \mathrm{Re}(\lambda-\mu)>0$ $\mathrm{Re}(\rho-\lambda+\mu-\nu)>0,\quad \mathrm{Re}(\rho-\lambda+\mu+\nu+1)>0,\quad \lvert\arg(z+1)\rvert<\pi$

Variable $ax+\beta$: infinite intervals (cont'd)

(11)	$\int_1^\infty e^{-ax}(x^2-1)^{-\frac{1}{2}\mu} P_\nu^\mu(x)\,dx = 2^{\frac{1}{2}}\pi^{-\frac{1}{2}} a^{\mu-\frac{1}{2}} K_{\nu+\frac{1}{2}}(a)$ $\operatorname{Re} a>0, \quad \operatorname{Re}\mu<1$
(12)	$\int_1^\infty \left(\frac{x+1}{x-1}\right)^{\frac{1}{2}\mu} e^{-ax} P_\nu^\mu(x)\,dx = a^{-1} W_{\mu,\nu+\frac{1}{2}}(2a)$ $\operatorname{Re} a>0, \quad \operatorname{Re}\mu<1$
(13)	$\int_1^\infty (x-1)^{\lambda-1}(x^2-1)^{\frac{1}{2}\mu} e^{-ax} P_\nu^\mu(x)\,dx$ $= \frac{a^{-\lambda-\mu}e^{-a}}{\Gamma(1-\mu+\nu)\Gamma(-\mu-\nu)} G_{23}^{31}\left(2a\,\middle\vert\,\begin{matrix}1+\mu,\ 1\\ \lambda+\mu,\ -\nu,\ 1+\nu\end{matrix}\right)$ $\operatorname{Re} a>0, \quad \operatorname{Re}\lambda>0$
(14)	$\int_1^\infty (x-1)^{\lambda-1}(x^2-1)^{-\frac{1}{2}\mu} e^{-ax} Q_\nu^\mu(x)\,dx$ $= \frac{1}{2}e^{\mu\pi i} a^{\mu-\lambda} e^{-a} G_{23}^{22}\left(2a\,\middle\vert\,\begin{matrix}1-\mu,\ 1\\ \lambda-\mu,\ \nu+1,\ -\nu\end{matrix}\right)$ $\operatorname{Re} a>0, \quad \operatorname{Re}\lambda>0, \quad \operatorname{Re}(\lambda-\mu)>0$
(15)	$\int_1^\infty (x-1)^{\lambda-1}(x^2-1)^{-\frac{1}{2}\mu} e^{-ax} P_\nu^\mu(x)\,dx$ $= -\pi^{-1}\sin(\nu\pi)\, a^{\mu-\lambda} e^{-a} G_{23}^{31}\left(2a\,\middle\vert\,\begin{matrix}1,\ 1-\mu\\ \lambda-\mu,\ 1+\nu,\ -\nu\end{matrix}\right)$ $\operatorname{Re} a>0, \quad \operatorname{Re}\lambda>\operatorname{Re}\mu$
(16)	$\int_1^\infty (\alpha^2+\beta^2+2\alpha\beta x)^{-\frac{1}{2}} \exp[-(\alpha^2+\beta^2+2\alpha\beta x)^{\frac{1}{2}}] P_\nu(x)\,dx$ $= 2\pi^{-1}(\alpha\beta)^{-\frac{1}{2}} K_{\nu+\frac{1}{2}}(\alpha) K_{\nu+\frac{1}{2}}(\beta) \qquad \operatorname{Re}\alpha>0, \quad \operatorname{Re}\beta>0$

Variable $\alpha x + \beta$: infinite intervals (cont'd)

(17)	$\int_1^\infty (x^2-1)^{-\frac{1}{2}\mu} \exp(a^2x^2)\,\mathrm{Erfc}(ax)\,P_\nu^\mu(x)\,dx$ $= \pi^{-1}2^{\mu-1}\Gamma\left(\frac{1+\mu+\nu}{2}\right)\Gamma\left(\frac{\mu-\nu}{2}\right)a^{\mu-3/2}\exp\left(\frac{a^2}{2}\right)$ $\times W_{\frac{1}{4}-\frac{1}{2}\mu,\frac{1}{4}+\frac{1}{2}\nu}(a^2)$ $\mathrm{Re}\,a>0,\quad \mathrm{Re}\,\mu<1,\quad \mathrm{Re}(\mu+\nu)>-1,\quad \mathrm{Re}(\mu-\nu)>0$
(18)	$\int_1^\infty Q_\nu(x)\,dx = [\nu(\nu+1)]^{-1}$ $\mathrm{Re}\,\nu>0$
(19)	$\int_1^\infty P_\nu(x)\,Q_\lambda(x)\,dx = [(\lambda-\nu)(\lambda+\nu+1)]^{-1}$ $\mathrm{Re}(\lambda-\nu)>0,\quad \mathrm{Re}(\lambda+\nu)>-1$
(20)	$\int_1^\infty [Q_\nu(x)]^2\,dx = (2\nu+1)^{-1}\psi'(\nu+1)$ $\mathrm{Re}\,\nu>-\frac{1}{2}$
(21)	$\int_1^\infty Q_\nu(x)\,Q_\lambda(x)\,dx = \frac{\psi(\lambda+1)-\psi(\nu+1)}{(\lambda-\nu)(\lambda+\nu+1)}$ $\mathrm{Re}(\lambda+\nu)>-1$
(22)	$\int_1^\infty [Q_\nu^\mu(x)]^2\,dx$ See Barnes, E.W., 1908: *Quart. J. Math.* 39, 97-204. Note that Barnes' definition of Legendre functions of the second kind differs from the one used in this book.
(23)	$\int_1^\infty (x^2-1)^{\lambda-1}Q_\nu^\mu(x)\,dx$ $= e^{\mu\pi i}\frac{\Gamma(\frac{1}{2}+\frac{1}{2}\nu+\frac{1}{2}\mu)\Gamma(1-\lambda+\frac{1}{2}\nu)\Gamma(\lambda+\frac{1}{2}\mu)\Gamma(\lambda-\frac{1}{2}\mu)}{2^{2\lambda-\mu}\Gamma(1+\frac{1}{2}\nu-\frac{1}{2}\mu)\Gamma(\frac{1}{2}+\lambda+\frac{1}{2}\nu)}$ $\lvert\mathrm{Re}\,\mu\rvert<2\,\mathrm{Re}\,\lambda<\mathrm{Re}\,\nu+2$

Variable $\alpha x + \beta$: infinite intervals (cont'd)

(24)	$\int_1^\infty (x-1)^{\lambda-1}(x^2-1)^{\frac{1}{2}\mu} e^{-\alpha x} Q_\nu^\mu(x)\,dx$ $= \dfrac{\Gamma(\nu+\mu+1)}{2\Gamma(\nu-\mu+1)} e^{\mu\pi i} \alpha^{-\lambda-\mu} e^{-\alpha} G_{23}^{22}\left(2\alpha\middle\vert\begin{matrix}1+\mu,\ 1\\ \lambda+\mu,\ \nu+1,\ -\nu\end{matrix}\right)$ $\operatorname{Re}\alpha>0,\quad \operatorname{Re}\lambda>0,\quad \operatorname{Re}(\lambda+\mu)>0$
(25)	$\int_1^\infty (x^2-1)^{\lambda-1}(a^2x^2-1)^{\frac{1}{2}\mu} P_\nu^\mu(ax)\,dx$ $= \dfrac{\Gamma(\lambda)\Gamma(1-\lambda-\frac{1}{2}\mu+\frac{1}{2}\nu)\Gamma(\frac{1}{2}-\lambda-\frac{1}{2}\mu-\frac{1}{2}\nu)}{\Gamma(1-\frac{1}{2}\mu+\frac{1}{2}\nu)\Gamma(\frac{1}{2}-\frac{1}{2}\nu-\frac{1}{2}\mu)\Gamma(1-\lambda-\mu)}$ $\times 2^{\mu-1} a^{\mu-\nu-1} {}_2F_1\left(\dfrac{1-\mu+\nu}{2}, 1-\lambda-\dfrac{\mu-\nu}{2}; 1-\lambda-\mu; 1-\dfrac{1}{a^2}\right)$ $\operatorname{Re}a>0,\quad \operatorname{Re}\lambda>0,\quad \operatorname{Re}(\nu-\mu-2\lambda)>-2,\quad \operatorname{Re}(2\lambda+\mu+\nu)<1$
(26)	$\int_1^\infty x^{\mu-1} Q_\nu(ax)\,dx = e^{\mu\pi i}\Gamma(\mu) a^{-\mu}(a^2-1)^{\frac{1}{2}\mu} Q_\nu^{-\mu}(a)$ $\lvert\arg(a-1)\rvert<\pi,\quad \operatorname{Re}\mu>0,\quad \operatorname{Re}(\nu-\mu)>-1$
(27)	$\int_1^\infty (x^2-1)^{\lambda-1}(a^2x^2-1)^{-\frac{1}{2}\mu} Q_\nu^\mu(ax)\,dx$ $= \dfrac{\Gamma(\frac{\mu+\nu+1}{2})\Gamma(\lambda)\Gamma(1-\lambda+\frac{\mu+\nu}{2})}{\Gamma(\nu+\frac{3}{2})} 2^{\mu-2} e^{\mu\pi i} a^{-\mu-\nu-1}$ $\times {}_2F_1\left(\dfrac{\mu+\nu+1}{2}, 1-\lambda+\dfrac{\mu+\nu}{2}; \nu+\dfrac{3}{2}; a^{-2}\right)$ $\lvert\arg(a-1)\rvert<\pi,\quad \operatorname{Re}\lambda>0,\quad \operatorname{Re}(2\lambda-\mu-\nu)<2$
(28)	$\int_1^\infty x^{-\frac{1}{2}\mu-\frac{1}{2}}(x-1)^{-\mu-\frac{1}{2}}(1+ax)^{\frac{1}{2}\mu} Q_\nu^\mu(1+2ax)\,dx$ $= \pi^{-\frac{1}{2}} e^{-\mu\pi i}\Gamma(\frac{1}{2}-\mu) a^{\frac{1}{2}\mu}\{Q_\nu^\mu[(1+a)^{\frac{1}{2}}]\}^2$ $\lvert\arg a\rvert<\pi,\quad \operatorname{Re}\mu<\frac{1}{2},\quad \operatorname{Re}(\mu+\nu)>-1$

Variable $\alpha x + \beta$: infinite intervals (cont'd)

(29)	$\int_1^\infty x^{-\mu/2-1/2}(x-1)^{-\mu-3/2}(1+ax)^{\mu/2}\, Q_\nu^\mu(1+2ax)\,dx$ $= -\pi^{-1/2} e^{-\mu\pi i}\Gamma(-\mu-\tfrac{1}{2})\, a^{\frac{1}{2}\mu+\frac{1}{2}}(1+a^2)^{-1/2}$ $\times Q_\nu^{\mu+1}[(1+a)^{1/2}]\, Q_\nu^\mu[(1+a)^{1/2}]$ $\lvert\arg a\rvert<\pi,\quad \operatorname{Re}\mu<-\tfrac{1}{2},\quad \operatorname{Re}(\mu+\nu+2)>0$
(30)	$\int_1^\infty (x-1)^{\mu-1} P_\nu(ax)\, Q_\lambda(ax)\,dx,\quad \int_1^\infty (x-1)^{\mu-1} Q_\nu(ax) Q_\lambda(ax)\,dx$ See Shabde, N.G., 1937: *Bull. Calcutta Math. Soc.* 29,33-40.

18.3. Legendre functions of other variables

(1)	$\int_a^\infty P_\nu(2x^2a^{-2}-1)\sin(bx)\,dx$ $= -\dfrac{\pi a}{4\cos(\nu\pi)}\left\{\left[J_{\nu+1/2}\left(\dfrac{ab}{2}\right)\right]^2-\left[J_{-\nu-1/2}\left(\dfrac{ab}{2}\right)\right]^2\right\}$ $a,\ b>0,\quad -1<\operatorname{Re}\nu<0$
(2)	$\int_a^\infty P_\nu(2x^2a^{-2}-1)\cos(bx)\,dx$ $= -\tfrac{1}{4}\pi a\left[J_{\nu+1/2}(\tfrac{1}{2}ab)\, J_{-\nu-1/2}(\tfrac{1}{2}ab) - Y_{\nu+1/2}(\tfrac{1}{2}ab)\, Y_{-\nu-1/2}(\tfrac{1}{2}ab)\right]$ $a,\ b>0,\quad -1<\operatorname{Re}\nu<0$
(3)	$\int_0^\infty (a+x)^{-\mu-\nu-2} P_\mu\left(\dfrac{a-x}{a+x}\right) P_\nu\left(\dfrac{a-x}{a+x}\right)dx$ $= \dfrac{a^{-\mu-\nu-1}[\Gamma(\mu+\nu+1)]^4}{[\Gamma(\mu+1)\Gamma(\nu+1)]^2\,\Gamma(2\mu+2\nu+2)}$ $\lvert\arg a\rvert<\pi,\quad \operatorname{Re}(\mu+\nu)>-1$

Other variables (cont'd)

(4)	$\int_0^1 x^{-1} \cos(ax)\, P_\nu(2x^{-2}-1)\, dx$ $= -\tfrac{1}{2}\pi \csc(\nu\pi)\ {}_1F_1(\nu+1;\, 1;\, ai)\ {}_1F_1(\nu+1;\, 1;\, -ai)$ $a>0, \quad -1<\operatorname{Re}\nu<0$
(5)	$\int_0^\infty x^{-1} e^{-\beta x} Q_{-1/2}(1+2x^{-2})\, dx = \dfrac{\pi^2}{8}\{[J_0(\tfrac{1}{2}\beta)]^2 + [Y_0(\tfrac{1}{2}\beta)]^2\}$ $\operatorname{Re}\beta>0$
(6)	$\int_0^\infty x^{-1} e^{-ax} Q_\nu(1+2x^{-2})\, dx = \tfrac{1}{2}[\Gamma(\nu+1)]^2 a^{-1}$ $\times W_{-\nu-1/2,0}(ai)\, W_{-\nu-1/2,0}(-ai) \qquad \operatorname{Re}a>0, \quad \operatorname{Re}\nu>-1$
(7)	$\int_0^\infty x^{\lambda-1}(x^2+a^2)^{\frac{1}{2}\nu} e^{-\beta x}\, \mathrm{P}_\nu^\mu\left[\dfrac{x}{(x^2+a^2)^{1/2}}\right] dx$ $= \dfrac{2^{-\nu-2} a^{\lambda+\nu}}{\pi\,\Gamma(-\mu-\nu)}\, G_{24}^{32}\left(\dfrac{a^2\beta^2}{4}\,\middle\vert\, \begin{matrix} 1-\frac{\lambda}{2},\ \frac{1-\lambda}{2} \\ 0,\ \frac{1}{2},\ -\frac{\lambda+\mu+\nu}{2},\ -\frac{\lambda-\mu+\nu}{2}\end{matrix}\right)$ $a>0, \quad \operatorname{Re}\beta>0, \quad \operatorname{Re}\lambda>0$
(8)	$\int_0^\infty x^{1/2} \sin(bx)\, [P_\nu^{-1/4}(y)]^2\, dx = \dfrac{(\tfrac{1}{2}\pi)^{-1/2} a^{-1} b^{-1/2}}{\Gamma(5/4+\nu)\Gamma(1/4-\nu)}\left[K_{\nu+1/2}\left(\dfrac{b}{2a}\right)\right]^2$ $\operatorname{Re}a>0, \quad b>0, \quad -5/4<\operatorname{Re}\nu<1/4$
(9)	$\int_0^\infty x^{1/2} \sin(bx)\, P_\nu^{-1/4}(y)\, Q_\nu^{-1/4}(y)\, dx$ $= \dfrac{(\tfrac{1}{2}\pi)^{1/2} e^{-\frac{1}{4}\pi i}\,\Gamma(\nu+5/4)}{ab^{1/2}\,\Gamma(\nu+3/4)}\, I_{\nu+1/2}\left(\dfrac{b}{2a}\right) K_{\nu+1/2}\left(\dfrac{b}{2a}\right)$ $\operatorname{Re}a>0, \quad b>0, \quad \operatorname{Re}\nu>-5/4$

$y = (1+a^2x^2)^{1/2}$

Other variables (cont'd)

(10)	$\int_0^\infty x^{1/2} y^{-1} \sin(bx)\, P_\nu^{-1/4}(y)\, P_{\nu-1}^{-1/4}(y)\, dx$ $= \dfrac{(2\pi)^{-1/2} a^{-2} b^{1/2}}{\Gamma(5/4+\nu)\,\Gamma(5/4-\nu)}\, K_{\nu-1/2}\left(\dfrac{b}{2a}\right) K_{\nu+1/2}\left(\dfrac{b}{2a}\right)$ $\mathrm{Re}\, a > 0, \quad b > 0, \quad -5/4 < \mathrm{Re}\, \nu < 5/4$
(11)	$\int_0^\infty x^{1/2} y^{-1} \sin(bx)\, P_\nu^{1/4}(y)\, P_\nu^{-3/4}(y)\, dx$ $= \dfrac{(2\pi)^{-1/2} a^{-2} b^{1/2}}{\Gamma(7/4+\nu)\,\Gamma(3/4-\nu)} \left[K_{\nu+1/2}\left(\dfrac{b}{2a}\right)^2\right]$ $\mathrm{Re}\, a > 0, \quad b > 0, \quad -7/4 < \mathrm{Re}\, \nu < 3/4$
(12)	$\int_0^\infty x^{1/2} \cos(bx)\, [P_\nu^{1/4}(y)]^2\, dx = \dfrac{a^{-1} (\frac{1}{2}\pi b)^{-1/2}}{\Gamma(\frac{3}{4}+\nu)\,\Gamma(-\frac{1}{4}-\nu)} \left[K_{\nu+1/2}\left(\dfrac{b}{2a}\right)\right]^2$ $\mathrm{Re}\, a > 0, \quad b > 0, \quad -\frac{3}{4} < \mathrm{Re}\, \nu < -\frac{1}{4}$
(13)	$\int_0^\infty x^{1/2} \cos(bx)\, P_\nu^{1/4}(y)\, Q_\nu^{1/4}(y)\, dx = \dfrac{(\frac{1}{2}\pi)^{1/2} e^{\frac{1}{4}\pi i}\, \Gamma(\nu+\frac{3}{4})}{a b^{1/2}\, \Gamma(\nu+5/4)}$ $\times I_{\nu+1/2}\left(\dfrac{b}{2a}\right) K_{\nu+1/2}\left(\dfrac{b}{2a}\right)$ $\mathrm{Re}\, a > 0, \quad b > 0, \quad \mathrm{Re}\, \nu > -\frac{3}{4}$
(14)	$\int_0^\infty x^{1/2} y^{-1} \cos(bx)\, P_\nu^{-1/4}(y)\, P_\nu^{3/4}(y)\, dx = \dfrac{(2\pi)^{-1/2} a^{-2} b^{1/2}}{\Gamma(5/4+\nu)\,\Gamma(1/4-\nu)}$ $\times \left[K_{\nu+1/2}\left(\dfrac{b}{2a}\right)\right]^2$ $\mathrm{Re}\, a > 0, \quad b > 0, \quad -5/4 < \mathrm{Re}\, \nu < 1/4$

$y = (1 + a^2 x^2)^{1/2}$

Other variables (cont'd)

(15)	$\int_0^\infty x^{1/2}\, y^{-1} \cos(bx)\, P_\nu^{1/4}(y)\, P_{\nu-1}^{1/4}(y)\, dx$ $= \dfrac{(2\pi)^{-1/2}\, a^{-2}\, b^{1/2}}{\Gamma(3/4+\nu)\,\Gamma(3/4-\nu)}\, K_{\nu-1/2}\left(\dfrac{b}{2a}\right) K_{\nu+1/2}\left(\dfrac{b}{2a}\right)$ $\operatorname{Re} a > 0, \quad b > 0, \quad -3/4 < \operatorname{Re}\nu < 3/4$
(16)	$\int_0^a \left[\dfrac{\sin(a-x)}{\sin x}\right]^\kappa \mathrm{P}_\nu^{-\mu}(\cos x)\, \mathrm{P}_\nu^{-\kappa}[\cos(a-x)]\, \dfrac{dx}{\sin x}$ $= \dfrac{2^\kappa\, \Gamma(\mu-\kappa)\, \Gamma(\kappa+1/2)}{\pi^{1/2}\, \Gamma(\kappa+\mu+1)}\, (\sin a)^\kappa\, \mathrm{P}_\nu^{-\mu}(\cos a)$ $\operatorname{Re}\mu > \operatorname{Re}\kappa > -1/2$
(17)	$\int_0^a (\sin x)^\rho\, [\sin(a-x)]^\sigma\, \mathrm{P}_\nu^\mu(\cos x)\, \mathrm{P}_\kappa^\lambda[\cos(a-x)]\, dx$ For this integral and several particular cases see Bailey, W.N., 1931: *Proc. Cambridge Philos. Soc.* 27, 184-189 and 381-386.
(18)	$\int_0^\infty \cos(ax)\, P_\nu(\cosh x)\, dx$ $= -\dfrac{\sin(\nu\pi)}{4\pi^2}\, \Gamma\left(\dfrac{1+\nu+ia}{2}\right) \Gamma\left(\dfrac{1+\nu-ia}{2}\right) \Gamma\left(-\dfrac{\nu+ia}{2}\right) \Gamma\left(-\dfrac{\nu-ia}{2}\right)$ $a > 0, \quad -1 < \operatorname{Re}\nu < 0$
(19)	$\int_0^\infty \mathrm{P}_{-x-1/2}(\cos\theta)\, dx = 1/2 \csc(1/2\,\theta) \qquad 0 < \theta < \pi$
(20)	$\int_{-\infty}^\infty \mathrm{P}_x(\cos\theta)\, dx = \csc(1/2\,\theta) \qquad 0 < \theta < \pi$

$y = (1 + a^2 x^2)^{1/2}$

Other variables (cont'd)

(21)	$\int_0^\infty \cos(bx)\, P^{\mu}_{-\frac{1}{2}+ix}(\cosh a)\, dx$	
	$= 0$	$0 < a < b$
	$= \dfrac{(\frac{1}{2}\pi)^{\frac{1}{2}}\,(\sinh a)^{\mu}}{\Gamma(\frac{1}{2}-\mu)\,(\cosh a - \cosh b)^{\mu+\frac{1}{2}}}$	$0 < b < a$
(22)	$\int_0^\infty x^{-1} \tanh(\pi x)\, P_{-\frac{1}{2}+ix}(\cosh a)\, dx = 2e^{-\frac{1}{2}a} K(e^{-a})$	$a > 0$
(23)	$\displaystyle\int_0^\infty \frac{x \tanh(\pi x)}{a^2 + x^2}\, P_{-\frac{1}{2}+ix}(\cosh b)\, dx = Q_{a-\frac{1}{2}}(\cosh b)$	$\mathrm{Re}\, a > 0$
(24)	$\int_0^\infty \cos(bx)\, \Gamma(\mu + ix)\, \Gamma(\mu - ix)\, P^{\frac{1}{2}-\mu}_{-\frac{1}{2}+ix}(\cosh a)\, dx$	
	$= \dfrac{(\frac{1}{2}\pi)^{\frac{1}{2}}\, \Gamma(\mu)\, (\sinh a)^{\mu-\frac{1}{2}}}{(\cosh a + \cosh b)^{\mu}}$	$a, b > 0, \quad \mathrm{Re}\, \mu > 0$

CHAPTER XIX

BESSEL FUNCTIONS

For Bessel functions and related functions see H.T.F., vol. II, Chapter VII and the literature quoted there, especially the books by Watson (the standard treatise on the subject), Gray and Mathew, McLachlan (now in a second, revised, edition), and Weyrich. Integrals involving Bessel functions appear in almost every chapter of the present work, and the kernels of the integral transforms listed in Chapters VIII to XII are Bessel functions. The present chapter contains mainly integrals which have not already appeared in the earlier chapters, although some of the integrals already listed have been included for the sake of easy reference.

Bessel functions are particular confluent hypergeometric functions, and the expressions which follow may be used to reduce integrals involving Bessel functions to integrals involving hypergeometric functions.

$$
\begin{aligned}
J_\nu(z) &= \frac{(\tfrac{1}{2}z)^\nu}{\Gamma(\nu+1)}\, {}_0F_1(\nu+1; -\tfrac{1}{4}z^2) \\
&= \frac{(\tfrac{1}{2}z)^\nu}{\Gamma(\nu+1)}\, e^{-iz}\, {}_1F_1(\nu+\tfrac{1}{2}; 2\nu+1; 2iz) \\
&= \frac{z^{-1/2}\, e^{-\frac{1}{2}(\nu+\frac{1}{2})\pi i}}{2^{2\nu+1/2}\,\Gamma(\nu+1)} M_{0,\nu}(2iz)
\end{aligned}
$$

$$H_\nu^{(1)}(z) = (\tfrac{1}{2}\pi z)^{-1/2}\, e^{-\frac{1}{2}(\nu+\frac{1}{2})\pi i}\, W_{0,\nu}(-2iz)$$

$$H_\nu^{(2)}(z) = (\tfrac{1}{2}\pi z)^{-1/2}\, e^{\frac{1}{2}(\nu+\frac{1}{2})\pi i}\, W_{0,\nu}(2iz)$$

$$I_\nu(z) = \frac{(\frac{1}{2}z)^\nu}{\Gamma(\nu+1)} \, {}_0F_1(\nu+1; \tfrac{1}{4}z^2)$$

$$= \frac{(\frac{1}{2}z)^\nu}{\Gamma(\nu+1)} e^{-z} \, {}_1F_1(\nu+\tfrac{1}{2}; 2\nu+1; 2z)$$

$$= \frac{z^{-\frac{1}{2}} \, 2^{-2\nu-\frac{1}{2}}}{\Gamma(\nu+1)} M_{0,\nu}(2z)$$

$$K_\nu(z) = \left(\frac{\pi}{2z}\right)^{\frac{1}{2}} W_{0,\nu}(2z)$$

$$\mathbf{H}_\nu(z) = \frac{2(z/2)^{\nu+1}}{\pi^{\frac{1}{2}} \, \Gamma(\nu+3/2)} \, {}_1F_2(1; \nu+3/2, 3/2; -z^2/4)$$

$$\mathbf{L}_\nu(z) = \frac{2(z/2)^{\nu+1}}{\pi^{\frac{1}{2}} \, \Gamma(\nu+3/2)} \, {}_1F_2(1; \nu+3/2, 3/2; z^2/4)$$

$$s_{\mu,\nu}(z) = \frac{z^{\mu+1}}{(\mu-\nu+1)(\mu+\nu+1)} \, {}_1F_2\left(1; \frac{\mu+\nu+3}{2}, \frac{\mu-\nu+3}{2}; -\frac{z^2}{4}\right)$$

Expressions of various combinations of Bessel functions in terms of Meijer's G-function are given in the Appendix. Many integrals involving Bessel functions may be obtained by specializing parameters in the small number of known integrals involving the G-function. Likewise, in the tables which follow, many fairly general integrals involving Bessel functions have been evaluated in terms of confluent hypergeometric functions or G-functions. For special values of the parameters these expressions simplify considerably. Frequently these particular cases are not given separately, and the user of these tables is expected to perform the necessary operations, the requisite formulas being given in the Appendix.

19.1. Bessel functions of argument x. Finite intervals

(1)	$\int_0^a J_\nu(x)\,dx = 2 \sum_{n=0}^{\infty} J_{\nu+2n+1}(a)$	$\operatorname{Re}\nu > -1$
(2)	$\int_0^a x^\nu J_\nu(x)\,dx = 2^{\nu-1}\pi^{1/2}\Gamma(\nu+1/2)\,a\,[J_\nu(a)\,\mathbf{H}_{\nu-1}(a) - \mathbf{H}_\nu(a)\,J_{\nu-1}(a)]$	$\operatorname{Re}\nu > -1/2$
(3)	$\int_0^a x^{\nu+1} J_\nu(x)\,dx = a^{\nu+1} J_{\nu+1}(a)$	$\operatorname{Re}\nu > -1$
(4)	$\int_0^a x^{1-\nu} J_\nu(x)\,dx = \dfrac{1}{2^{\nu-1}\Gamma(\nu)} - a^{1-\nu} J_{\nu-1}(a)$	
(5)	$\int_0^a x^\mu J_\nu(x)\,dx = (\mu+\nu-1)\,a\,J_\nu(a)\,S_{\mu-1,\,\nu-1}(a) - a\,J_{\nu-1}(a)\,S_{\mu,\nu}(a) + 2^\mu \dfrac{\Gamma\left(\frac{1+\mu+\nu}{2}\right)}{\Gamma\left(\frac{1-\mu+\nu}{2}\right)}$	$\operatorname{Re}(\mu+\nu) > -1$
(6)	$\int_0^a (a-x)^{-1/2} J_\nu(x)\,dx = \pi(1/2a)^{1/2} J_{1/2\nu+1/4}(1/2a)\,J_{1/2\nu-1/4}(1/2a)$	$\operatorname{Re}\nu > -1$
(7)	$\int_0^a x^{-\nu}(a^2-x^2)^{-\nu-1/2} J_\nu(x)\,dx = \pi^{1/2}\,2^{-\nu-1}\,a^{-2\nu}\,\Gamma(1/2-\nu) \times J_\nu(1/2a)\,J_{-\nu}(1/2a)$	$\operatorname{Re}\nu < 1/2$

For other similar integrals see sections 8.5 and 13.1.

Bessel functions of x; finite intervals (cont'd)

(8)	$\int_0^a x^\rho (a^2 - 2abx + b^2)^{-\frac{1}{2}} J_\nu(x)\, dx$ See Bose, S.K., 1946: *Bull. Calcutta Math. Soc.* 38, 177-180.
(9)	$\int_0^a x^\nu \sin x\, J_\nu(x)\, dx = \dfrac{a^{\nu+1}}{2\nu+1} \sin a\, J_\nu(a) - \cos a\, J_{\nu+1}(a)$ $\operatorname{Re} \nu > -1$
(10)	$\int_0^a \sin(a-x)\, J_{2n}(x)\, dx = a\, J_{2n+1}(a) + (-1)^n\, 2n\, [\cos a$ $- J_0(a) - 2 \sum_{m=1}^{n} (-1)^m J_{2m}(a)]$ $n = 0, 1, 2, \ldots$
(11)	$\int_0^a \sin(a-x)\, J_{2n+1}(x)\, dx = a\, J_{2n+2}(a)$ $+ (-1)^n (2n+1)\, [\sin a - 2 \sum_{m=0}^{n} (-1)^m J_{2m+1}(a)]$ $n = 0, 1, 2, \ldots$
(12)	$\int_0^a \sin(a-x)\, J_\nu(x)\, dx = a\, J_{\nu+1}(a) - 2\nu \sum_{n=0}^{\infty} (-1)^n J_{\nu+2n+2}(a)$ $\operatorname{Re} \nu > -1$
(13)	$\int_0^a x^{-1} \sin(a-x)\, J_\nu(x)\, dx = 2\nu^{-1} \sum_{n=0}^{\infty} (-1)^n J_{\nu+2n+1}(a)$ $\operatorname{Re} \nu > 0$
(14)	$\int_0^a x^{-3/2} \sin(a-x)\, J_\nu(x)\, dx = (\nu^2 - \frac{1}{4})^{-1} a^{\frac{1}{2}} J_\nu(a)$ $\operatorname{Re} \nu > \frac{1}{2}$
(15)	$\int_0^a x^\nu \sin(a-x)\, J_\nu(x)\, dx = \dfrac{a^{\nu+1}}{2\nu+1} J_{\nu+1}(a)$ $\operatorname{Re} \nu > -\frac{1}{2}$

Bessel functions of x; finite interval (cont'd)

(16)	$\int_0^a x^\lambda \sin(a-x)\, J_\nu(x)\, dx$ $= 2a^{\lambda+1} \sum_{n=0}^{\infty} \frac{(-1)^n (\nu-\lambda)_{2n}}{(\nu+\lambda+1)_{2n+2}} (\nu+2n+1)\, J_{\nu+2n+1}(a)$ $\mathrm{Re}(\lambda+\nu) > -1$ See also Bailey, W. N., 1930: *Proc. London Math. Soc.* (2) 31, 200-208.
(17)	$\int_0^a (a^2-x^2)^{-1/2} \sin(\beta x)\, J_\nu(x)\, dx$ $= \pi \sum_{n=0}^{\infty} (-1)^n J_{2n+1}(a\beta)\, J_{\frac{1}{2}\nu+n+\frac{1}{2}}(\tfrac{1}{2}a)\, J_{\frac{1}{2}\nu-n-\frac{1}{2}}(a)$ $\mathrm{Re}\, \nu > -2$
(18)	$\int_0^a x^{\nu+1} \sin[\tfrac{1}{2}\beta(a^2-x^2)]\, J_\nu(x)\, dx = \beta^{-\nu-1}\, U_{\nu+2}(a^2\beta, a)$ $\mathrm{Re}\, \nu > -1$
(19)	$\int_0^a x^{\nu+1} \sin[b(a^2-x^2)^{1/2}]\, J_\nu(x)\, dx = (\tfrac{1}{2}\pi)^{1/2} a^{\nu+3/2}\, b(1+b^2)^{-\frac{1}{2}\nu-\frac{3}{4}}$ $\times J_{\nu+3/2}[a(1+b^2)^{1/2}]$ $\mathrm{Re}\, \mu > -1$
(20)	$\int_0^a x^\nu \cos x\, J_\nu(x)\, dx = \frac{a^{\nu+1}}{2\nu+1} [\cos a\, J_\nu(a) + \sin a\, J_{\nu+1}(a)]$ $\mathrm{Re}\, \nu > -\tfrac{1}{2}$
(21)	$\int_0^a \cos(a-x)\, J_{2n}(x)\, dx$ $= a\, J_{2n}(a) - (-1)^n 2n[\sin a - 2\sum_{m=0}^{n-1} (-1)^m J_{2m+1}(a)]$ $n = 0, 1, 2, \ldots$

Bessel functions of x; finite intervals (cont'd)

(22)	$\int_0^a \cos(a-x)\, J_{2n+1}(x)\, dx = a\, J_{2n+1}(a)$ $+ (-1)^n (2n+1)[\cos a - J_0(a) - 2 \sum_{m=1}^{n} (-1)^m J_{2m}(a)]$ $n = 0, 1, 2, \ldots$
(23)	$\int_0^a \cos(a-x)\, J_\nu(x)\, dx = a\, J_\nu(a) - 2\nu \sum_{n=0}^{\infty} (-1)^n J_{\nu+2n+1}(a)$ $\mathrm{Re}\,\nu > -1$
(24)	$\int_0^a x^{-1} \cos(a-x)\, J_\nu(x)\, dx = \nu^{-1} J_\nu(a) + 2\nu^{-1} \sum_{n=1}^{\infty} (-1)^n J_{\nu+2n}(a)$ $\mathrm{Re}\,\nu > 0$
(25)	$\int_0^a x^\nu \cos(a-x)\, J_\nu(x)\, dx = \dfrac{a^{\nu+1}}{2\nu+1} J_\nu(a)$ $\mathrm{Re}\,\nu > -\tfrac{1}{2}$
(26)	$\int_0^a x^\lambda \cos(a-x)\, J_\nu(x)\, dx = \dfrac{a^{\lambda+1} J_\nu(a)}{\lambda+\nu+1}$ $+ 2a^{\lambda+1} \sum_{n=1}^{\infty} \dfrac{(-1)^n (\nu-\lambda)_{2n-1}}{(\nu+\lambda+1)_{2n+1}} (\nu+2n)\, J_{\nu+2n}(a)$ $\mathrm{Re}(\lambda+\nu) > -1$ See also Bailey, W.N., 1930: *Proc. London Math. Soc.* (2) 31, 200-208.
(27)	$\int_0^a (a^2-x^2)^{-\frac{1}{2}} \cos(\beta x)\, J_\nu(x)\, dx = \tfrac{1}{2}\pi\, J_0(a\beta)\, [J_{\frac{1}{2}\nu}(\tfrac{1}{2}a)]^2$ $+ \pi \sum_{n=1}^{\infty} (-1)^n J_{2n}(a\beta)\, J_{\frac{1}{2}\nu+n}(\tfrac{1}{2}a)\, J_{\frac{1}{2}\nu-n}(\tfrac{1}{2}a)$ $\mathrm{Re}\,\nu > -1$
(28)	$\int_0^a x^{\nu+1} \cos[\tfrac{1}{2}\beta(a^2-x^2)]\, J_\nu(x)\, dx = \beta^{-\nu-1}\, U_{\nu+1}(a^2\beta, a)$ $\mathrm{Re}\,\nu > -1$

Bessel functions of x; finite intervals (cont'd)

(29)	$\int_0^a x(a^2-x^2)^{-1/2}\cos[\beta(a^2-x^2)^{1/2}]\,J_0(x)\,dx$ $=(\beta^2+1)^{-1/2}\sin[a(\beta^2+1)^{1/2}]$
(30)	$\int_0^a (a^2-x^2)^{-1/2}\cos[\beta(a^2-x^2)^{1/2}]\,J_\nu(x)\,dx$ $=\tfrac12\pi\,J_{\frac12\nu}(\tfrac12 ae^{u})\,J_{\frac12\nu}(\tfrac12 ae^{-u})$ $\qquad \beta=\sinh u,\quad \mathrm{Re}\,\nu>-1$
(31)	$\int_0^a x^{\mu-1}P_n(x/a)\,J_\nu(x)\,dx,\qquad \int_0^a x^{\mu-1}P_n(x/a)\,J_\mu(x)\,J_\nu(x)\,dx$ See Bose, S.K., 1946: *Bull. Calcutta Math. Soc.* 38, 177-180.
(32)	$\int_0^a x^{\mu-1}P_n(2x^2a^{-2}-1)\,J_\nu(x)\,dx$ $=\dfrac{2^{-\nu-1}a^{\mu+\nu}[\Gamma(\frac12\mu+\frac12\nu)]^2}{\Gamma(\nu+1)\Gamma(\frac12\mu+\frac12\nu+n+1)\Gamma(\frac12+\frac12\nu-n)}$ $\times\,{}_2F_3\left(\dfrac{\mu+\nu}{2},\dfrac{\mu+\nu}{2};\nu+1,\dfrac{\mu+\nu}{2}+n+1,\dfrac{\mu+\nu}{2}-n;-\dfrac{a^2}{4}\right)$ $\mathrm{Re}(\mu+\nu)>0$ For particular cases see Bose, B.N., 1944: *Bull. Calcutta Math. Soc.* 36, 125-132.
(33)	$\int_0^a x^{\frac12-\mu}(a^2-x^2)^{-\frac12\mu}\,\mathrm{P}_\nu^\mu(x/a)\,J_{\nu+\frac12}(x)\,dx$ $=(\tfrac12\pi)^{1/2}a^{1-\mu}J_{\frac12-\mu}(\tfrac12 a)\,J_{\nu+\frac12}(\tfrac12 a)$ $\qquad \mathrm{Re}\,\mu<1,\quad \mathrm{Re}(\mu-\nu)<2$
(34)	$\int_0^a x^{1/2}(a^2-x^2)^{-\frac12\nu-\frac14}\,\mathrm{P}_\mu^{\nu+\frac12}(2x^2a^{-2}-1)\,J_\nu(x)\,dx$ $=\pi^{1/2}\,2^{-\nu-1}a\,J_{\mu+\frac12}(\tfrac12 a)\,J_{-\mu-\frac12}(\tfrac12 a)$ $\qquad -1<\mathrm{Re}\,\nu<\tfrac12$

Bessel functions of x; finite intervals (cont'd)

(35)	$\int_0^a J_0(x)\, J_1(x)\, dx = \tfrac{1}{2} - \tfrac{1}{2}[J_0(a)]^2$
(36)	$\int_0^a J_n(x)\, J_{n+1}(x)\, dx = \tfrac{1}{2} - \tfrac{1}{2}[J_0(a)]^2 - \sum_{m=1}^{n} [J_m(a)]^2$ $n = 1, 2, 3, \ldots$
(37)	$\int_0^a J_\nu(x)\, J_{\nu+1}(x)\, dx = \sum_{n=0}^{\infty} [J_{\nu+n+1}(a)]^2$ $\quad$ $\mathrm{Re}\, \nu > -1$
(38)	$\int_0^a x^{\rho-1} (a^2 - x^2)^{\sigma-1} J_\mu(x)\, J_\nu(x)\, dx$ See Bailey, W.N., 1938: *Quart. J. Math. Oxford Ser.* 9, 141-147.
(39)	$\int_0^a x\, P_n(1 - 2x^2 a^{-2})\, [J_0(x)]^2\, dx = \dfrac{a^2}{2(2n+1)} \{[J_n(a)]^2 + [J_{n+1}(a)]^2\}$ $n = 0, 1, 2, \ldots$
(40)	$\int_0^a x^{2\nu+1} P_n(2x^2 a^{-2} - 1)\, [J_\nu(x)]^2\, dx$ See Bose, B.N., 1944: *Bull. Calcutta Math. Soc.* 36, 125-132.
(41)	$\displaystyle\int_a^b \frac{dx}{x\,[J_\nu(x)]^2} = \frac{\pi}{2}\left[\frac{Y_\nu(b)}{J_\nu(b)} - \frac{Y_\nu(a)}{J_\nu(a)}\right]$
(42)	$\displaystyle\int_a^b \frac{dx}{x\, J_\nu(x)\, J_{-\nu}(x)} = \frac{\pi}{2\sin(\nu\pi)} \log\left[\frac{J_{-\nu}(a)\, J_\nu(b)}{J_\nu(a)\, J_{-\nu}(b)}\right]$
(43)	$\int_0^a x^\nu\, Y_\nu(x)\, dx = 2^{\nu-1} \pi^{\frac{1}{2}} \Gamma(\nu + \tfrac{1}{2})\, a\, [Y_\nu(a)\, \mathbf{H}_{\nu-1}(a) - \mathbf{H}_\nu(a)\, Y_{\nu-1}(a)]$ $\mathrm{Re}\, \nu > -\tfrac{1}{2}$

Bessel functions of x; finite intervals (cont'd)

(44)	$\int_0^a x^{\nu+1} Y_\nu(x)\,dx = a^{\nu+1} Y_{\nu+1}(a) + 2^{\nu+1}\Gamma(\nu+1)$	$\operatorname{Re}\nu > -1$
(45)	$\int_0^a x^{1-\nu} Y_\nu(x)\,dx = \dfrac{\operatorname{ctn}(\nu\pi)}{2^{\nu-1}\Gamma(\nu)} - a^{1-\nu} Y_{\nu-1}(a)$	$\operatorname{Re}\nu < 1$
(46)	$\int_a^b Y_\nu(x)\,dx = 2\sum_{n=0}^{\infty}[Y_{\nu+2n+1}(b) - Y_{\nu+2n+1}(a)]$	
(47)	$\int_0^a (a^2-x^2)^{-1/2}\cos[\beta(a^2-x^2)^{1/2}]\,Y_{2\nu}(x)\,dx$ $= \tfrac14\pi[J_\nu(u)Y_\nu(v) + J_\nu(v)Y_\nu(u)] - \tfrac14\pi\tan(\nu\pi)[J_\nu(u)J_\nu(v) + Y_\nu(u)Y_\nu(v)]$ $u = \tfrac12 a[(\beta^2+1)^{1/2}+\beta],\quad v = \tfrac12 a[(\beta^2+1)^{1/2}-\beta],$	$-\tfrac12 < \operatorname{Re}\nu < \tfrac12$
(48)	$\int_0^a x P_n(1-2x^2a^{-2})\,J_0(x)\,Y_0(x)\,dx$ $= \dfrac{a^2}{2(2n+1)}[J_n(a)Y_n(a) + J_{n+1}(a)Y_{n+1}(a)]$	$n = 0, 1, 2, \ldots$
(49)	$\displaystyle\int_a^b \frac{dx}{x[Y_\nu(x)]^2} = \frac{\pi}{2}\left[\frac{J_\nu(a)}{Y_\nu(a)} - \frac{J_\nu(b)}{Y_\nu(b)}\right]$	
(50)	$\displaystyle\int_a^b \frac{dx}{x\,J_\nu(x)\,Y_\nu(x)} = \frac{\pi}{2}\log\left[\frac{J_\nu(a)\,Y_\nu(b)}{J_\nu(b)\,Y_\nu(a)}\right]$	

19.2. Bessel functions of argument x. Infinite intervals

(1)	$\int_0^\infty J_\nu(x)\,dx = 1$	$\operatorname{Re}\nu > -1$

Bessel functions of x; infinite intervals (cont'd)

(2)	$\int_0^\infty \frac{J_\nu(x)}{x^2+a^2}\,dx = \frac{\pi[\mathbf{J}_\nu(a) - J_\nu(a)]}{a\sin(\nu\pi)}$ $\qquad$ $\operatorname{Re} a > 0, \quad \operatorname{Re}\nu > -1$
(3)	$\int_0^\infty x^{-1}(a^2+x^2)^{-1/2}\,[J_0(x) - 1]\,dx = a^{-1}[\operatorname{Ei}(-a) - \log(\gamma a)]$ $\quad$ $\operatorname{Re} a > 0$
(4)	$\int_0^\infty x\,[(x^2+\alpha^2)(x^2+\beta^2)]^{-1/2}\,[(x^2+\alpha^2)^{1/2} + (x^2+\beta^2)^{1/2}]^{-2\nu}\,J_0(x)\,dx$ $= (\alpha^2-\beta^2)^{-\nu}\,I_\nu\left(\frac{\alpha-\beta}{2}\right)K_\nu\left(\frac{\alpha+\beta}{2}\right)$ $\qquad$ $\operatorname{Re}\alpha > 0, \quad \operatorname{Re}\beta > 0, \quad \operatorname{Re}\nu > -\tfrac{3}{4}$
(5)	$⨍_0^\infty x^3(x^4-a^4)^{-1}\,J_0(x)\,dx = \tfrac{1}{2}K_0(a) - \tfrac{1}{4}\pi\,Y_0(a)$ $\qquad$ $a > 0$
(6)	$\int_0^\infty x^\nu(e^{\alpha x}-1)^{-1}\,J_\nu(x)\,dx = 2^\nu\pi^{-1/2}\,\Gamma(\nu+\tfrac{1}{2})\sum_{n=1}^\infty (1+\alpha^2 n^2)^{-\nu-1/2}$ $\qquad$ $\operatorname{Re}\alpha > 0, \quad \operatorname{Re}\nu > 0$
(7)	$\int_0^\infty x^{-1}\exp\left(-\frac{2a}{x}\right)J_\nu(x)\,dx = 2\,J_\nu(2a^{1/2})\,K_\nu(2a^{1/2})$ $\qquad$ $\operatorname{Re} a > 0$
(8)	$\int_0^\infty \frac{x^\nu}{x+\beta}\sin(x+\beta)\,J_\nu(x)\,dx = \tfrac{1}{2}\pi\sec(\nu\pi)\,\beta^\nu J_{-\nu}(\beta)$ $\qquad$ $\lvert\arg\beta\rvert < \pi, \quad \lvert\operatorname{Re}\nu\rvert < \tfrac{1}{2}$
(9)	$\int_0^\infty \frac{x^\nu}{x+\beta}\cos(x+\beta)\,J_\nu(x)\,dx = -\tfrac{1}{2}\pi\sec(\nu\pi)\,\beta^\nu Y_{-\nu}(\beta)$ $\qquad$ $\lvert\arg\beta\rvert < \pi, \quad \lvert\operatorname{Re}\nu\rvert < \tfrac{1}{2}$

Bessel functions of x; infinite intervals (cont'd)

(10)	$\int_0^\infty x^{1/4} \sin(2ax^{1/2})\, J_{-1/4}(x)\, dx = \pi^{1/2} a^{3/2} J_{3/4}(a^2)$	$a > 0$
(11)	$\int_0^\infty x^{1/4} \sin(2ax^{1/2})\, J_{3/4}(x)\, dx = \pi^{1/2} a^{3/2} J_{-1/4}(a^2)$	$a > 0$
(12)	$\int_0^\infty x^{1/4} \cos(2ax^{1/2})\, J_{1/4}(x)\, dx = \pi^{1/2} a^{3/2} J_{-3/4}(a^2)$	$a > 0$
(13)	$\int_0^\infty x^{1/4} \cos(2ax^{1/2})\, J_{-3/4}(x)\, dx = \pi^{1/2} a^{3/2} J_{1/4}(a^2)$	$a > 0$
(14)	$\int_0^\infty x^{-1/4} e^{-\alpha x} \sin(2\beta x^{1/2})\, J_{-1/4}(x)\, dx$ $= \pi^{1/2} \left(\frac{\beta}{\alpha^2+1}\right)^{1/2} \exp\left(-\frac{\alpha\beta^2}{\alpha^2+1}\right) J_{-1/4}\left(\frac{\beta^2}{\alpha^2+1}\right)$	$\operatorname{Re} \alpha > 0$
(15)	$\int_0^\infty x^{-1/4} e^{-\alpha x} \sin(2\beta x^{1/2})\, J_{1/4}(x)\, dx$ $= \pi^{1/2} \left(\frac{\beta}{\alpha^2+1}\right)^{1/2} \exp\left(-\frac{\alpha\beta^2}{\alpha^2+1}\right) J_{1/4}\left(\frac{\beta^2}{\alpha^2+1}\right)$	$\operatorname{Re} \alpha > 0$
(16)	$\int_0^\infty x^{-1/2} \sin x \sin(4ax^{1/2})\, J_0(x)\, dx = (\tfrac{1}{2}\pi)^{1/2} \cos(a^2 + \tfrac{1}{4}\pi)\, J_0(a^2)$	$a > 0$
(17)	$\int_0^\infty x^{-1/3} \sin x \sin(4ax^{1/2})\, J_{1/3}(x)\, dx$ $= -2^{-5/2} \pi^{1/2} a^{1/3} [\sin(a^2 + \pi/12)\, J_{1/3}(a^2)$ $+ \cos(a^2 + \pi/12)\, Y_{1/3}(a^2)]$	$a > 0$
(18)	$\int_0^\infty x^{-1/2} \sin x \cos(4ax^{1/2})\, J_0(x)\, dx$ $= -2^{-3/2} \pi^{1/2} [\cos(a^2 - \tfrac{1}{4}\pi) J_0(a^2) - \sin(a^2 - \tfrac{1}{4}\pi)\, Y_0(a^2)]$	$a > 0$

Bessel functions of x; infinite intervals (cont'd)

(19)	$\int_0^\infty x^{-1/3} \sin x \cos(4ax^{1/2}) J_{-1/3}(x)\,dx$ $= -2^{-3/2}\pi^{1/2} a^{1/3} \sin(a^2 - \pi/12) J_{-1/3}(a^2)$	$a > 0$
(20)	$\int_0^\infty x^{-\frac{1}{2}} \cos x \sin(4ax^{\frac{1}{2}}) J_0(x)\,dx$ $= (\tfrac{1}{2}\pi)^{\frac{1}{2}} \cos(a^2 - \tfrac{1}{4}\pi) J_0(a^2)$	$a > 0$
(21)	$\int_0^\infty x^{-1/3} \cos x \sin(4ax^{1/2}) J_{1/3}(x)\,dx$ $= 2^{-5/2}\pi^{1/2} a^{1/3} [\cos(a^2 + \pi/12) J_{1/3}(a^2)$ $- \sin(a^2 + \pi/12) Y_{1/3}(a^2)]$	$a > 0$
(22)	$\int_0^\infty x^{-\frac{1}{2}} \cos x \cos(4ax^{\frac{1}{2}}) J_0(x)\,dx$ $= -2^{-3/2}\pi^{1/2}[\sin(a^2 - \tfrac{1}{4}\pi) J_0(a^2) + \cos(a^2 - \tfrac{1}{4}\pi) Y_0(a^2)]$	$a > 0$
(23)	$\int_0^\infty x^{-1/3} \cos x \cos(4ax^{1/2}) J_{-1/3}(x)\,dx$ $= 2^{-3/2}\pi^{1/2} a^{1/3} \cos(a^2 - \pi/12) J_{-1/3}(a^2)$	$a > 0$
(24)	$\int_0^\infty x^{-\rho} J_\mu(x) J_\nu(x)\,dx$ $= \dfrac{2^{-\rho}\Gamma(\rho)\Gamma[\tfrac{1}{2}(\mu+\nu-\rho+1)]}{\Gamma[\tfrac{1}{2}(\rho+\mu+\nu+1)]\Gamma[\tfrac{1}{2}(\rho-\mu+\nu+1)]\Gamma[\tfrac{1}{2}(\rho+\mu-\nu+1)]}$	$0 < \mathrm{Re}\,\rho < \mathrm{Re}(\mu+\nu)+1$
(25)	$\int_0^\infty x^{1-2\nu}[J_\nu(x)]^4\,dx = \dfrac{\Gamma(\nu)\Gamma(2\nu)}{2\pi[\Gamma(\nu+\tfrac{1}{2})]^2\Gamma(3\nu)}$	$\mathrm{Re}\,\nu > 0$
(26)	$\displaystyle\int_0^\infty \frac{x}{x^2+a^2}[J_\nu(x)]^2\,dx = I_\nu(a) K_\nu(a)$	$\mathrm{Re}\,a > 0,\quad \mathrm{Re}\,\nu > -1$

Bessel functions of x; infinite intervals (cont'd)

(27)	$\int_0^\infty (x^2+a^2)^{-\frac{1}{2}} J_\mu(x)\, J_\nu(x)\, dx$ See Bouwkamp, C.S., 1950: *Nederl. Akad. Wetensch., Proc.* 53, 654-661.
(28)	$\int_0^\infty x^{\rho-1}(x^2+a^2)^{-\sigma} J_\mu(x)\, J_\nu(x)\, dx$ See Watson, G.N., 1922: *A treatise on the theory of Bessel functions*, Cambridge, p. 436.
(29)	$\int_0^\infty x^{\mu+\nu} e^{-ax} J_\mu(x)\, J_\nu(x)\, dx$ See Watson, G.N., 1922: *A treatise on the theory of Bessel functions*, Cambridge, p. 390.
(30)	$\int_0^\infty \sin(2ax)\,[J_\nu(x)]^2\, dx$ $= \frac{1}{2} P_{\nu-\frac{1}{2}}(1-2a^2)$ $0<a<1$ $= \pi^{-1}\cos(\nu\pi)\, Q_{\nu-\frac{1}{2}}(2a^2-1)$ $a>1$ $\mathrm{Re}\,\nu>-1$
(31)	$\int_0^\infty \sin(2ax)\,[x^\nu J_\nu(x)]^2\, dx$ $= \dfrac{a^{-2\nu}\,\Gamma(\frac{1}{2}+\nu)}{2\pi^{\frac{1}{2}}\,\Gamma(1-\nu)}\; {}_2F_1(\tfrac{1}{2}+\nu, \tfrac{1}{2}; 1-\nu; a^2)$ $0<a<1$ $= \dfrac{a^{-4\nu-1}\,\Gamma(\frac{1}{2}+\nu)}{2\Gamma(1+\nu)\,\Gamma(\frac{1}{2}-2\nu)}\; {}_2F_1(\tfrac{1}{2}+\nu, \tfrac{1}{2}+2\nu; 1+\nu; a^{-2})$ $a>1$ $-\frac{1}{2}<\mathrm{Re}\,\nu<\frac{1}{2}$

Bessel functions of x; infinite intervals (cont'd)

(32)	$\int_0^\infty \cos(2ax)\,[J_\nu(x)]^2\,dx$ $= \pi^{-1}\, Q_{\nu-\frac{1}{2}}(1-2a^2)$ $= -\pi^{-1}\sin(\nu\pi)\, Q_{\nu-\frac{1}{2}}(2a^2-1)$	$0<a<1$ $a>1$ $\operatorname{Re}\nu > -\frac{1}{2}$
(33)	$\int_0^\infty \cos(2ax)\,[x^\nu J_\nu(x)]^2\,dx$ $= \dfrac{a^{-2\nu}\,\Gamma(\nu)}{2\pi^{\frac{1}{2}}\,\Gamma(\frac{1}{2}-\nu)}\; {}_2F_1(\nu+\frac{1}{2},\frac{1}{2};1-\nu;a^2)$ $+\dfrac{\Gamma(-\nu)\,\Gamma(\frac{1}{2}+2\nu)}{2\pi\,\Gamma(\frac{1}{2}-\nu)}\; {}_2F_1(\frac{1}{2}+\nu,\frac{1}{2}+2\nu;1+\nu;a^2)$ $= -\dfrac{\sin(\nu\pi)\,a^{-4\nu-1}\,\Gamma(\frac{1}{2}+2\nu)}{\Gamma(1+\nu)\,\Gamma(\frac{1}{2}-\nu)}\; {}_2F_1(\frac{1}{2}+\nu,\frac{1}{2}+2\nu;1+\nu;a^{-2})$	$0<a<1$ $a>1$ $-\frac{1}{4}<\operatorname{Re}\nu<\frac{1}{2}$
(34)	$\int_a^\infty (x^2-a^2)^{-\frac{1}{2}}\, J_\nu(x)\,dx = -\frac{1}{2}\pi\, J_{\frac{1}{2}\nu}(\frac{1}{2}a)\, Y_{\frac{1}{2}\nu}(\frac{1}{2}a)$	$\operatorname{Re}\nu > -1$
(35)	$\int_a^\infty x^{-1}(x^2-a^2)^{-\frac{1}{2}}\, J_0(x)\,dx = -\operatorname{si}(a)$	
(36)	$\int_a^\infty x^{\frac{1}{2}}\, P_{\nu-\frac{1}{2}}(x/a)\, J_\nu(x)\,dx = -(\frac{1}{2}a)^{-\frac{1}{2}}\,[\cos(\frac{1}{2}a)\, Y_\nu(\frac{1}{2}a)$ $+\sin(\frac{1}{2}a)\, J_\nu(\frac{1}{2}a)]$	$-\frac{1}{2}<\operatorname{Re}\nu<\frac{1}{2}$
(37)	$\int_a^\infty x^{\frac{1}{2}-\mu}(x^2-a^2)^{-\frac{1}{2}\mu}\, P^{\mu}_{\nu-\frac{1}{2}}(x/a)\, J_\nu(x)\,dx$ $= -2^{-3/2}\pi^{1/2}a^{1-\mu}[J_{\mu-\frac{1}{2}}(\frac{1}{2}a)\, Y_\nu(\frac{1}{2}a) + Y_{\mu-\frac{1}{2}}(\frac{1}{2}a)\, J_\nu(\frac{1}{2}a)]$	$-\frac{1}{4}<\operatorname{Re}\mu<1,\quad \lvert\operatorname{Re}\nu\rvert<\frac{1}{2}+2\operatorname{Re}\mu$

Bessel functions of x; infinite intervals (cont'd)

(38)	$\int_a^\infty x^{\nu}(x^2-a^2)^{\frac{1}{2}\lambda-\frac{1}{2}} P_\lambda^{\lambda-1}(x/a)\, J_\nu(x)\, dx$ $= \dfrac{2^{\lambda+\nu} a^{\nu}\,\Gamma(\frac{1}{2}+\nu)}{\pi^{\frac{1}{2}}\,\Gamma(1-\lambda)}\, S_{\lambda-\nu,\lambda+\nu}(a)$ $\operatorname{Re}\nu < 5/2, \quad \operatorname{Re}(2\lambda+\nu) < 3/2$
(39)	$\int_a^\infty x^{\frac{1}{2}}(x^2-a^2)^{\frac{1}{2}\nu-\frac{1}{4}} P_\mu^{\frac{1}{2}-\nu}(2x^2a^{-2}-1)\, J_\nu(x)\, dx$ $= -2^{\nu-2}\pi^{\frac{1}{2}}\, a \sec(\mu\pi)\,\{[J_{\mu+\frac{1}{2}}(\tfrac{1}{2}a)]^2 - [J_{-\mu-\frac{1}{2}}(\tfrac{1}{2}a)]^2\}$ $\operatorname{Re}\nu > -1/2, \quad \operatorname{Re}\nu - 3/2 < 2\operatorname{Re}\mu < 1/2 - \operatorname{Re}\nu$
(40)	$\int_a^\infty x^{1-2\nu}(x^2-a^2)^{\nu-3/2}\,[J_\nu(x)]^2\, dx = \dfrac{\Gamma(\nu-\frac{1}{2})}{2\pi^{\frac{1}{2}} a^{\nu+1}}\, \mathbf{H}_\nu(2a) \qquad \operatorname{Re}\nu > \tfrac{1}{2}$
(41)	$\int_a^\infty x^{2\nu+1}(a^2-x^2)^{-\nu-3/2}\,\{[J_\nu(x)]^2 + [J_{-\nu}(x)]^2\}\, dx$ $= \pi^{-\frac{1}{2}} a^{\nu-1}\,\Gamma(-\nu-\tfrac{1}{2})\sin(\nu\pi)\, J_\nu(2a) \qquad \operatorname{Re}\nu < -\tfrac{1}{2}$
(42)	$\displaystyle\int_{-\infty}^{\infty} \frac{\sin[a(x+\beta)]}{x+\beta}\, J_0(x)\, dx$ $= 2\int_0^a (1-u^2)^{-\frac{1}{2}} \cos(\beta u)\, du \qquad 0 \le a \le 1$ $= \pi J_0(\beta) \qquad 1 \le a < \infty$
(43)	$\displaystyle\int_{-\infty}^{\infty} \frac{\lvert x\rvert}{x+\beta} \sin[a(x+\beta)]\, J_0(x)\, dx = 0 \qquad 0 \le a < 1$
(44)	$\displaystyle\int_{-\infty}^{\infty} \frac{\sin[a(x+\beta)]}{x^{\nu}(x+\beta)}\, J_{\nu+2n}(x)\, dx = \pi\beta^{-\nu} J_{\nu+2n}(\beta) \qquad 1 \le a < \infty$ $n = 0, 1, 2, \ldots, \quad \operatorname{Re}\nu > -3/2$

Bessel functions of x; infinite intervals (cont'd)

(45)	$\int_{-\infty}^{\infty} \frac{\sin[a(x+\beta)]}{x+\beta}[J_{n+\frac{1}{2}}(x)]^2\,dx = \pi[J_{n+\frac{1}{2}}(\beta)]^2$	$2 \le a < \infty$ $n = 0, 1, 2, \ldots$
(46)	$\int_{-\infty}^{\infty} \frac{\sin[a(x+\beta)]}{x+\beta} J_{n+\frac{1}{2}}(x)\,J_{-n-\frac{1}{2}}(x)\,dx = \pi J_{n+\frac{1}{2}}(\beta)J_{-n-\frac{1}{2}}(\beta)$	$n = 0, 1, 2, \ldots,\quad 2 \le a < \infty$
(47)	$\int_{-\infty}^{\infty} \frac{\sin[a(x+\beta)]}{x^{2\nu}(x+\beta)}[J_{\nu+n}(x)]^2\,dx = \pi\beta^{-2\nu}[J_{\nu+n}(\beta)]^2$	$2 \le a < \infty$ $n = 0, 1, 2, \ldots,\quad \mathrm{Re}\,\nu > -1$
(48)	$\int_0^{\infty} Y_\nu(x)\,dx = -\tan(\frac{1}{2}\nu\pi)$	$-1 < \mathrm{Re}\,\nu < 1$
(49)	$\int_0^{\infty} x^{\rho}(x^2+a^2)^{-\mu}\,Y_\nu(x)\,dx = \frac{a^{\rho-2\mu}}{\Gamma(\mu)}\,G_{24}^{31}\left(\frac{a^2}{4}\middle\vert \begin{matrix} 1-\frac{\rho}{2}, -\frac{\nu}{2} \\ \mu-\frac{\rho}{2}, \frac{1+\nu}{2}, \frac{1-\nu}{2}, -\frac{\nu}{2}\end{matrix}\right)$	$\vert\mathrm{Re}\,\nu\vert - 1 < \mathrm{Re}\,\rho < 2\,\mathrm{Re}\,\mu + \frac{1}{2}$
(50)	$\int_0^{\infty} x^{-1}\exp\left(-\frac{2a}{x}\right)Y_\nu(x)\,dx = 2Y_\nu(2a^{\frac{1}{2}})\,K_\nu(2a^{\frac{1}{2}})$	$\mathrm{Re}\,a > 0$
(51)	$\int_0^{\infty} x^{-1}\sin\left(\frac{a}{2x}\right)[\sin x\,J_0(x) + \cos x\,Y_0(x)]\,dx = \pi J_0(a^{\frac{1}{2}})\,Y_0(a^{\frac{1}{2}})$	$a > 0$

Bessel functions of x; infinite intervals (cont'd)

(52)	$\int_0^\infty x^{-1} \cos\left(\frac{a}{2x}\right) [\sin x \, Y_0(x) - \cos x \, J_0(x)] \, dx = \pi J_0(a^{1/2}) \, Y_0(a^{1/2})$ $\qquad a > 0$
(53)	$\int_0^\infty x^{1/4} \sin(2ax^{1/2}) \, Y_{3/4}(x) \, dx = -\pi^{1/2} a^{3/2} \mathbf{H}_{-1/4}(a^2)$ $\qquad a > 0$
(54)	$\int_0^\infty x^{1/4} \cos(2ax^{1/2}) \, Y_{1/4}(x) \, dx = -\pi^{1/2} a^{3/2} \mathbf{H}_{-3/4}(a^2)$ $\qquad a > 0$
(55)	$\int_0^\infty x^{-1/2} \sin x \cos(4ax^{1/2}) \, Y_0(x) \, dx = 2^{-3/2} \pi^{1/2} [3 \sin(a^2 - \frac{1}{4}\pi) \, J_0(a^2) - \cos(a^2 - \frac{1}{4}\pi) \, Y_0(a^2)]$ $\qquad a > 0$
(56)	$\int_0^\infty x^{-1/2} \cos x \cos(4ax^{1/2}) \, Y_0(x) \, dx = -2^{-3/2} \pi^{1/2} [3 \cos(a^2 - \frac{1}{4}\pi) \, J_0(a^2) + \sin(a^2 - \frac{1}{4}\pi) \, Y_0(a^2)]$ $\qquad a > 0$
(57)	$\int_0^\infty J_{\nu+n}(x) \, Y_{\nu-n}(x) \, dx = \frac{(-1)^{n+1}}{2}$ $\qquad \mathrm{Re}\, \nu > -\frac{1}{2}, \quad n = 0, 1, 2, \ldots$
(58)	$\int_0^\infty e^{-2ax} J_0(x) \, Y_0(x) \, dx = \frac{K[a(a^2+1)^{-1/2}]}{\pi (a^2+1)^{1/2}}$ $\qquad \mathrm{Re}\, a > 0$
(59)	$\int_0^\infty x^{2\nu+1} \exp(-ax^2) \, J_\nu(x) \, Y_\nu(x) \, dx = -\frac{1}{2}\pi^{-1/2} a^{-3\nu/2-1/2} \exp\left(-\frac{1}{2a}\right) W_{\frac{1}{2}\nu, \frac{1}{2}\nu}\left(\frac{1}{a}\right)$ $\qquad \mathrm{Re}\, a > 0, \quad \mathrm{Re}\, \nu > -\frac{1}{2}$

Bessel functions of x; infinite intervals (cont'd)

(60)	$\int_0^\infty \sin(2ax)\, J_0(x)\, Y_0(x)\, dx$ $= 0 \qquad 0 < a < 1$ $= -\dfrac{K[(1-a^{-2})^{1/2}]}{\pi a} \qquad a > 1$
(61)	$\int_0^\infty \cos(2ax)\, J_0(x)\, Y_0(x)\, dx$ $= -\pi^{-1} K(a) \qquad 0 < a < 1$ $= -(\pi a)^{-1} K(a^{-1}) \qquad a > 1$
(62)	$\int_0^\infty \cos(2ax)\, [Y_0(x)]^2\, dx$ $= \pi^{-1} K[(1-a^2)^{1/2}] \qquad 0 < a < 1$ $= 2(\pi a)^{-1} K[(1-a^{-2})^{1/2}] \qquad a > 1$
(63)	$\int_0^\infty x^{1-2\nu} \sin(2ax)\, J_\nu(x)\, Y_\nu(x)\, dx$ $= -\dfrac{\Gamma(3/2-\nu)\, a}{2\,\Gamma(2\nu-\tfrac{1}{2})\,\Gamma(2-\nu)}\, {}_2F_1\left(\dfrac{3}{2}-\nu, \dfrac{3}{2}-2\nu; 2-\nu; a^2\right)$ $0 < \operatorname{Re} \nu < 3/2, \quad 0 < a < 1$
(64)	$\int_0^\infty x^{1-2\nu} \sin(2ax)\, \{[J_\nu(x)]^2 - [Y_\nu(x)]^2\}\, dx$ $= \dfrac{\sin(2\nu\pi)\,\Gamma(3/2-\nu)\,\Gamma(3/2-2\nu)\, a}{\pi\,\Gamma(2-\nu)}\, {}_2F_1\left(\dfrac{3}{2}-\nu, \dfrac{3}{2}-2\nu; 2-\nu; a^2\right)$ $0 < \operatorname{Re} \nu < \tfrac{3}{4}, \quad 0 < a < 1$
(65)	$\int_0^\infty x^{2-2\nu} \sin(2ax)\, [J_\nu(x)\, J_{\nu-1}(x) - Y_\nu(x)\, Y_{\nu-1}(x)]\, dx$ $= -\dfrac{\sin(2\nu\pi)\Gamma(3/2-\nu)\Gamma(5/2-2\nu) a}{\pi\,\Gamma(2-\nu)}\, {}_2F_1\left(\dfrac{3}{2}-\nu, \dfrac{5}{2}-2\nu; 2-\nu; a^2\right)$ $1/2 < \operatorname{Re} \nu < 5/4, \quad 0 < a < 1$

Bessel functions of x; infinite intervals (cont'd)

(66)	$\int_0^\infty x^{2-2\nu} \sin(2ax)\,[J_\nu(x)\,Y_{\nu-1}(x) + Y_\nu(x)\,J_{\nu-1}(x)]\,dx$ $= -\dfrac{\Gamma(3/2-\nu)\,a}{\Gamma(2\nu-3/2)\,\Gamma(2-\nu)}\ {}_2F_1\left(\dfrac{3}{2}-\nu, \dfrac{5}{2}-2\nu; 2-\nu; a^2\right)$ $1/2 < \mathrm{Re}\,\nu < 5/2, \quad 0 < a < 1$
(67)	$\int_a^\infty x^{\frac{1}{2}-\mu}(a^2-x^2)^{-\frac{1}{2}\mu} P^\mu_{\nu-\frac{1}{2}}(x/a)\,Y_\nu(x)\,dx$ $= 2^{-3/2}\,\pi^{1/2}\,a^{1-\mu}[J_\nu(\tfrac{1}{2}a)\,J_{\mu-\frac{1}{2}}(\tfrac{1}{2}a) - Y_\nu(\tfrac{1}{2}a)\,Y_{\mu-\frac{1}{2}}(\tfrac{1}{2}a)]$ $-\tfrac{1}{4} < \mathrm{Re}\,\mu < 1, \quad \mathrm{Re}(2\mu-\nu) > -\tfrac{1}{2}$

19.3. Bessel functions of arguments $ax + \beta$, x^2, x^{-1}

(1)	$\int_0^\infty x^{\rho-1} J_\mu(ax)\,J_\nu(bx)\,dx = \dfrac{2^{\rho-1} a^\mu b^{-\mu-\rho}\Gamma(\frac{1}{2}\mu+\frac{1}{2}\nu+\frac{1}{2}\rho)}{\Gamma(\mu+1)\,\Gamma(1-\frac{1}{2}\mu+\frac{1}{2}\nu-\frac{1}{2}\rho)}$ $\times\ {}_2F_1\left(\dfrac{\mu+\nu+\rho}{2}, \dfrac{\mu-\nu+\rho}{2}; \mu+1; \dfrac{a^2}{b^2}\right)$ $\mathrm{Re}(\mu+\nu+\rho) > 0, \quad \mathrm{Re}\,\rho < 2, \quad 0 < a < b$
(2)	$\int_0^\infty x^{\frac{1}{2}}(x^2+\lambda^2)^{-\frac{1}{2}} J_\mu(ax)\,J_\nu(bx)\,dx$ See Bouwkamp, C.S., 1950: *Nederl. Akad. Wetensch., Proc.* 53, 654-661.
(3)	$\int_0^\infty x^{1+\nu}[J_\nu(ax)]^2\,J_\nu(2bx)\,dx$ $= -\dfrac{\Gamma(\frac{1}{2}+\nu)}{2\pi^{3/2}}\sin(\nu\pi)\,a^{2\nu}\,b^{-\nu-1}(b^2-a^2)^{-\nu-\frac{1}{2}} \qquad 0 < a < b$ $= \dfrac{\Gamma(\frac{1}{2}+\nu)}{2\pi^{3/2}}\,a^{2\nu}\,b^{-\nu-1}(a^2-b^2)^{-\nu-\frac{1}{2}} \qquad 0 < b < a$ $-\tfrac{1}{2} < \mathrm{Re}\,\nu < \tfrac{1}{2}$

Bessel functions of $\alpha x + \beta$, x^2, x^{-1} (cont'd)

(4)	$\int_0^\infty x^{1+\nu} J_\nu(ax)\, J_{-\nu}(ax)\, J_\nu(2bx)\,dx$	
	$= 0$	$0 < a < b$
	$= \dfrac{a^{2\nu} b^{-\nu-1}}{2\pi^{1/2}\,\Gamma(\frac{1}{2}-\nu)} (a^2 - b^2)^{-\nu-\frac{1}{2}}$	$0 < b < a$
		$-1 < \operatorname{Re} \nu < \frac{1}{2}$
(5)	$\int_0^\infty x\, J_{\frac{1}{2}\nu-\frac{1}{4}}(ax)\, J_{\frac{1}{2}\nu+\frac{1}{4}}(ax)\, J_\nu(2bx)\,dx$	
	$= 0$	$0 < a < b$
	$= 2^{-3/2}\,\pi^{-1}\,a^{-1/2}\,b^{-1}(a-b)^{-\frac{1}{2}}$	$0 < b < a$
		$\operatorname{Re} \nu > -1$
(6)	$\int_0^\infty x^{1+\nu} J_\mu(ax)\, J_{-\mu}(ax)\, J_\nu(2bx)\,dx$	
	$= 0$	$0 < a < b$
	$= \dfrac{(a^2 - b^2)^{-\frac{1}{2}\nu-\frac{1}{4}}}{2\pi^{\frac{1}{2}}\,ab^{\frac{1}{2}}} \mathrm{P}^{\nu+\frac{1}{2}}_{\mu-\frac{1}{2}}(2b^2 a^{-2} - 1)$	$0 < b < a$
		$-1 < \operatorname{Re} \nu < \frac{1}{2}$
(7)	$\int_0^\infty x^{\rho-1} J_\lambda(ax)\, J_\mu(ax)\, J_\nu(2bx)\,dx$	
	$= \dfrac{a^{\lambda+\mu}\, b^{-\lambda-\mu-\rho}\,\Gamma(\frac{1}{2}\lambda+\frac{1}{2}\mu+\frac{1}{2}\nu+\frac{1}{2}\rho)}{2^{\lambda+\mu}\,\Gamma(\lambda+1)\,\Gamma(\mu+1)\,\Gamma(1-\frac{1}{2}\lambda-\frac{1}{2}\mu+\frac{1}{2}\nu-\frac{1}{2}\rho)}$	
	$\times\, {}_4F_3\left(\dfrac{\lambda+\mu+1}{2}, \dfrac{\lambda+\mu}{2}+1, \dfrac{\lambda+\mu+\nu+\rho}{2}, \dfrac{\lambda+\mu-\nu+\rho}{2};\right.$	
	$\left.\lambda+1,\, \mu+1,\, \lambda+\mu+1;\, \dfrac{a^2}{b^2}\right)$	
		$\operatorname{Re}(\lambda+\mu+\nu+\rho) > 0, \quad 0 < a < b$

Bessel functions of $ax+\beta$, x^2, x^{-1}, (cont'd)

(8)	$\int_0^\infty x^{1-\nu} J_\nu(ax)\, J_\nu(bx)\, J_\nu(cx)\, dx$ $= \dfrac{2^{\nu-1}\Delta^{2\nu-1}}{\pi^{1/2}(abc)^\nu\,\Gamma(\nu+\frac{1}{2})}$ if a, b, c are sides of a triangle of area Δ $=0$ if a, b, c are not sides of a triangle $a, b, c>0, \quad \operatorname{Re}\nu>-\frac{1}{2}$
(9)	$\int_0^\infty x^{\rho-1} J_\lambda(ax)\, J_\mu(bx)\, J_\nu(cx)\, dx$ $= \dfrac{2^{\rho-1} a^\lambda b^\mu c^{-\lambda-\mu-\rho}\,\Gamma(\frac{1}{2}\lambda+\frac{1}{2}\mu+\frac{1}{2}\nu+\frac{1}{2}\rho)}{\Gamma(\lambda+1)\,\Gamma(\mu+1)\,\Gamma(1-\frac{1}{2}\lambda-\frac{1}{2}\mu+\frac{1}{2}\nu-\frac{1}{2}\rho)}$ $\times F_4\left(\dfrac{\lambda+\mu-\nu+\rho}{2}, \dfrac{\lambda+\mu+\nu+\rho}{2}; \lambda+1, \mu+1; \dfrac{a^2}{c^2}, \dfrac{b^2}{c^2}\right)$ $\operatorname{Re}(\lambda+\mu+\nu+\rho)>0, \quad \operatorname{Re}\rho<5/2, \quad a, b, c>0, \quad c>a+b$ For particular cases see Watson, G.N., 1922: *A treatise on the theory of Bessel functions*. Cambridge, Sec.13.46; Bailey, W.N., 1936: *Proc. London Math. Soc.* (2) 40, 37-48.
(10)	$\int_0^\infty x^{1-2\nu} [J_\nu(ax)\, J_\nu(bx)]^2\, dx$ $= \dfrac{a^{2\nu-1} b^{-1}\,\Gamma(\nu)}{2\pi\,\Gamma(\nu+\frac{1}{2})\,\Gamma(2\nu+\frac{1}{2})}\; {}_2F_1\left(\nu, \tfrac{1}{2}-\nu, 2\nu+\tfrac{1}{2}; \dfrac{a^2}{b^2}\right)$ $\operatorname{Re}\nu>0, \quad 0<a<b$
(11)	$\int_0^\infty x^{1-\nu} [J_\nu(x)\, Y_{-\nu}(x) + Y_\nu(x)\, J_{-\nu}(x)]\, J_\nu(2ax)\, dx$ $=0$ $\quad 0<a<1$ $= \dfrac{a^{\nu-1}(a^2-1)^{\nu-\frac{1}{2}}}{\pi^{1/2}\,\Gamma(\nu+\frac{1}{2})}$ $\quad a>1$ $-\frac{1}{2}<\operatorname{Re}\nu<1$

Bessel functions of $ax+\beta$, x^2, x^{-1} (cont'd)

(12)	$\int_0^\infty x^{\mu+1}[J_\nu(x)\,Y_\mu(x)+J_\mu(x)\,Y_\nu(x)]\,J_\nu(2ax)\,dx$ $=0 \qquad 0<a<1$ $=-\dfrac{a^{-\mu-1}(a^2-1)^{-\frac{1}{2}\mu-\frac{1}{4}}}{\pi^{\frac{1}{2}}\,2^{\mu+\frac{1}{2}}}\,P_{\nu-\frac{1}{2}}^{\mu+\frac{1}{2}}(a) \qquad a>1$ $-1<\operatorname{Re}\mu<\tfrac{1}{2},\quad \operatorname{Re}\nu>-1,\quad \operatorname{Re}(\mu+\nu)>-1$
(13)	$\int_0^\infty x^{1+\mu}\,Y_\mu(ax)\,J_\nu(bx)\,J_\nu(cx)\,dx=0$ $0<b<c,\quad 0<a<c-b$
(14)	$\int_0^\infty x\,J_\nu^2(ax)\,J_\nu(bx)\,Y_\nu(bx)\,dx$ $=0 \qquad 0<a<b$ $=-(2\pi ab)^{-1} \qquad 0<b<a$ $\operatorname{Re}\nu>-\tfrac{1}{2}$
(15)	$\int_0^\infty x^{2\nu+1}\,J_\nu(ax)\,Y_\nu(ax)\,J_\nu(bx)\,Y_\nu(bx)\,dx$ $=\dfrac{a^{2\nu}\,\Gamma(3\nu+1)}{2\pi\,b^{4\nu+2}\,\Gamma(\frac{1}{2}-\nu)\,\Gamma(2\nu+3/2)}\ {}_2F_1\left(\nu+\dfrac{1}{2},\ 3\nu+1;\ 2\nu+\dfrac{3}{2};\dfrac{a^2}{b^2}\right)$ $0<a<b,\quad -1/3<\operatorname{Re}\nu<1/2$
(16)	$\displaystyle\int_0^\infty \frac{J_\nu(ax)\,Y_\nu(bx)-J_\nu(bx)\,Y_\nu(ax)}{x\{[J_\nu(bx)]^2+[Y_\nu(bx)]^2\}}\,dx=-\frac{\pi}{2}\left(\frac{b}{a}\right)^\nu \qquad 0<b<a$
	For other similar integrals see sec. 6.8.

Bessel functions of $ax+\beta$, x^2, x^{-1} (cont'd)

(17)	$\displaystyle\int_0^\infty \frac{[J_\nu(ax)Y_\nu(bx)-J_\nu(bx)Y_\nu(ax)]}{[J_\nu(bx)]^2+[Y_\nu(bx)]^2} \times [J_{\nu+1}(cx)Y_\nu(bx)-J_\nu(bx)Y_{\nu+1}(cx)]\,dx$ $\displaystyle = -\frac{b^{2\nu}}{a^\nu c^{\nu+1}}$ $\displaystyle = \frac{a^\nu}{c^{\nu+1}} - \frac{b^{2\nu}}{a^\nu c^{\nu+1}}$	 $0<b<c<a$ $0<b<a<c$
(18)	$\displaystyle\int_0^\infty \frac{[J_\nu(ax)Y_\nu(bx)-J_\nu(bx)Y_\nu(ax)]}{[J_\nu(bx)]^2+[Y_\nu(bx)]^2} \times [J_{\nu+1}(ax)Y_\nu(bx)-J_\nu(bx)Y_{\nu+1}(ax)]\,dx$ $\displaystyle = \frac{1}{2a} - \frac{b^{2\nu}}{a^{2\nu+1}}$	 $0<b<a$
(19)	$\displaystyle\int_0^\infty \frac{J_0(ax)Y_0(bx)-J_0(bx)Y_0(ax)}{[J_0(bx)]^2+[Y_0(bx)]^2}\,\frac{x\,dx}{\lambda^2+x^2} = -\frac{\pi H_\nu^{(2)}(a\lambda)}{2H_\nu^{(2)}(b\lambda)}$	$\mathrm{Re}\,\lambda>0,\quad 0<b<a$
(20)	$\displaystyle\fint_0^\infty \frac{J_0(ax)Y_0(bx)-J_0(bx)Y_0(ax)}{[J_0(bx)]^2+[Y_0(bx)]^2}\,\frac{x\,dx}{c^2-x^2}$ $\displaystyle = \frac{\pi}{2}\,\frac{J_0(ac)J_0(bc)+Y_0(ac)Y_0(bc)}{[J_0(bc)]^2+[Y_0(bc)]^2}$	 $0<b<a,\quad c>0$
(21)	$\int_0^a J_\nu(x)J_{-\nu}(a-x)\,dx = \sin a$	$-1<\mathrm{Re}\,\nu<1$
(22)	$\int_0^a J_\nu(x)J_{1-\nu}(a-x)\,dx = J_0(a)-\cos a$	$-1<\mathrm{Re}\,\nu<2$

Bessel functions of $\alpha x + \beta$, x^2, x^{-2} (cont'd)

(23)	$\int_0^a J_\mu(x)\, J_\nu(a-x)\, dx = 2 \sum_{m=0}^{\infty} (-1)^m J_{\mu+\nu+2m+1}(a)$ $\operatorname{Re}\mu > -1, \quad \operatorname{Re}\nu > -1$
(24)	$\int_0^a x^{-1} J_\mu(x)\, J_\nu(a-x)\, dx = \mu^{-1} J_{\mu+\nu}(a) \qquad \operatorname{Re}\mu > 0, \quad \operatorname{Re}\nu > -1$
(25)	$\int_0^a x^{\lambda-1} J_\mu(x)\, J_\nu(a-x)\, dx = 2^\lambda \sum_{m=0}^{\infty} \frac{(-1)^m \Gamma(\lambda+\mu+m)\,(\lambda)_m}{m!\,\Gamma(\mu+m+1)}$ $\times J_{\lambda+\mu+\nu+2m}(a) \qquad \operatorname{Re}(\lambda+\mu) > 0, \quad \operatorname{Re}\nu > -1$
(26)	$\int_0^a x^{-1}(a-x)^{-1} J_\mu(x)\, J_\nu(a-x)\, dx = (\mu^{-1}+\nu^{-1})\, a^{-1} J_{\mu+\nu}(a)$ $\operatorname{Re}\mu > 0, \quad \operatorname{Re}\nu > 0$
(27)	$\int_0^a x^{\lambda-1}(a-x)^{-1} J_\mu(x)\, J_\nu(a-x)\, dx$ $= \frac{2^\lambda}{\nu a} \sum_{m=0}^{\infty} \frac{(-1)^m \Gamma(\lambda+\mu+m)\,(\lambda)_m}{m!\,\Gamma(\mu+m+1)} (\lambda+\mu+\nu+2m)$ $\times J_{\lambda+\mu+\nu+2m}(a) \qquad \operatorname{Re}(\lambda+\mu) > 0, \quad \operatorname{Re}\nu > 0$
(28)	$\int_0^a x^\mu (a-x)^\nu J_\mu(x)\, J_\nu(a-x)\, dx$ $= \frac{\Gamma(\mu+\frac{1}{2})\,\Gamma(\nu+\frac{1}{2})}{2^{1/2}\,\pi^{1/2}\,\Gamma(\mu+\nu+1)}\, a^{\mu+\nu+1/2} J_{\mu+\nu+1/2}(a)$ $\operatorname{Re}\mu > -\frac{1}{2}, \quad \operatorname{Re}\nu > -\frac{1}{2}$
(29)	$\int_0^a x^\mu (a-x)^{\nu+1} J_\mu(x)\, J_\nu(a-x)\, dx$ $= \frac{\Gamma(\mu+1/2)\,\Gamma(\nu+3/2)}{2^{1/2}\,\pi^{1/2}\,\Gamma(\mu+\nu+2)}\, a^{\mu+\nu+3/2} J_{\mu+\nu+1/2}(a)$ $\operatorname{Re}\nu > -1, \quad \operatorname{Re}\mu > -\frac{1}{2}$

Bessel functions of $ax+\beta$, x^2, x^{-1} (cont'd)

(30)	$\int_0^a x^{\mu}(a-x)^{-\mu-1} J_\mu(x)\, J_\nu(a-x)\, dx = \dfrac{2^\mu \Gamma(\mu+\frac{1}{2})\,\Gamma(\nu-\mu)}{\pi^{\frac{1}{2}}\,\Gamma(\mu+\nu+1)}\, a^\mu J_\nu(a)$ $\operatorname{Re}\nu > \operatorname{Re}\mu > -\frac{1}{2}$
(31)	$\int_0^a x^{\rho-1}(a-x)^{\sigma-1} J_\mu(x)\, J_\nu(a-x)\, dx$ $\int_0^a x^{\rho-1}(a-x)^{\sigma-1} J_\lambda(bx)\, J_\mu(cx)\, J_\nu(a-x)\, dx$ For these integrals and several particular cases see Bailey, W.N., 1930: *Proc. London Math. Soc.* (2) 30, 422-424 and 31, 200-208; Rutgers, J.G., 1931: *Nederl. Akad. Wetensch. Proc.* 44, 75-85.
(32)	$\int_{-\infty}^{\infty} \dfrac{J_\mu[a(x+y)]}{(x+y)^\mu}\, \dfrac{J_\nu[a(x+z)]}{(x+z)^\nu}\, dx$ $= \dfrac{(2\pi/a)^{\frac{1}{2}}\,\Gamma(\mu+\nu)}{\Gamma(\mu+\frac{1}{2})\,\Gamma(\nu+\frac{1}{2})}\, \dfrac{J_{\mu+\nu-\frac{1}{2}}[a(y-z)]}{(y-z)^{\mu+\nu-\frac{1}{2}}}$ $a>0,\quad \operatorname{Re}(\mu+\nu)>0$
(33)	$\int_0^\infty x^2 J_{2\nu}(2ax)\, J_{\nu-\frac{1}{2}}(x^2)\, dx = \frac{1}{2}a\, J_{\nu+\frac{1}{2}}(a^2) \qquad a>0, \quad \operatorname{Re}\nu > -\frac{1}{2}$
(34)	$\int_0^\infty x^3 J_{2\nu}(2ax)\, J_{\nu-\frac{1}{2}}(x^2)\, dx = \frac{1}{2}\csc(\nu\pi)\, \mathbf{H}_{-\nu-\frac{1}{2}}(a^2)$ $-\frac{1}{2}\operatorname{ctn}(\nu\pi)\, J_{\nu+\frac{1}{2}}(a^2) - \frac{1}{2}Y_{\nu+\frac{1}{2}}(a^2) \qquad a>0, \quad \operatorname{Re}\nu > -\frac{3}{4}$
(35)	$\int_0^\infty x^2 J_{2\nu}(2ax)\, J_{\nu+\frac{1}{2}}(x^2)\, dx = \frac{1}{2}a\, J_{\nu-\frac{1}{2}}(a^2) \qquad a>0, \quad \operatorname{Re}\nu > -2$
(36)	$\int_0^\infty x^2 J_{2\nu}(2ax)\, Y_{\nu+\frac{1}{2}}(x^2)\, dx = -\frac{1}{2}a\, \mathbf{H}_{\nu-\frac{1}{2}}(a^2) \qquad a>0, \quad \operatorname{Re}\nu > -2$

Bessel functions of $\alpha x + \beta$, x^2, x^{-1} (cont'd)

(37)	$\int_0^\infty x\, J_0(4ax)\, [J_0(x^2)]^2\, dx = -\tfrac{1}{4} J_0(a^2)\, Y_0(a^2)$	$a > 0$
(38)	$\int_0^\infty x\, J_0(4ax)\, J_0(x^2)\, Y_0(x^2)\, dx = -\tfrac{1}{4} [J_0(a^2)]^2$	$a > 0$
(39)	$\int_0^\infty x^{(\nu+2)/3} \sin(x^2)\, J_{(\nu+\frac{1}{2})/3}(x^2)\, J_\nu(4ax)\, dx$ $= -\dfrac{a^{(\nu-1)/3}}{8} \Big[J_{(\nu+\frac{1}{2})/3}(a^2) \sin\left(a^2 + \dfrac{\nu-1}{6}\pi\right)$ $+ Y_{(\nu+\frac{1}{2})/3}(a^2) \cos\left(a^2 + \dfrac{\nu-1}{4}\pi\right)\Big]$	$-5/2 < \operatorname{Re} \nu < -1/2$
(40)	$\int_0^\infty x^{(\nu+2)/3} \cos(x^2)\, J_{(\nu+\frac{1}{2})/3}(x^2)\, J_\nu(2ax)\, dx$ $= \dfrac{a^{(\nu-1)/3}}{8} \Big[J_{(\nu+\frac{1}{2})/3}(a^2) \cos\left(a^2 + \dfrac{\nu-1}{6}\pi\right)$ $+ Y_{(\nu+\frac{1}{2})/3}(a^2) \sin\left(a^2 + \dfrac{\nu-1}{6}\pi\right)\Big]$	$-1 < \operatorname{Re} \nu < 11/2$
(41)	$\int_0^\infty x \sin(ax^2)\, J_\nu(bx^2)\, J_{2\nu}(2cx)\, dx$ $= \tfrac{1}{2} h \sin(ac^2 h)\, J_\nu(bc^2 h)$ $0 < a < b$; $= \tfrac{1}{2} k \cos(ac^2 k)\, J_\nu(bc^2 k)$ $0 < b < a$	$\operatorname{Re} \nu > -1$, $h = (b^2 - a^2)^{-\frac{1}{2}}$, $k = (a^2 - b^2)^{-\frac{1}{2}}$
(42)	$\int_0^\infty x \cos(ax^2)\, J_\nu(bx^2)\, J_{2\nu}(2cx)\, dx$ $= \tfrac{1}{2} h \cos(ac^2 h)\, J_\nu(bc^2 h)$ $0 < a < b$; $= \tfrac{1}{2} k \sin(ac^2 k)\, J_\nu(bc^2 k)$ $0 < b < a$	$\operatorname{Re} \nu > -\tfrac{1}{2}$, $h = (b^2 - a^2)^{-\frac{1}{2}}$, $k = (a^2 - b^2)^{-\frac{1}{2}}$

Bessel functions of $ax + \beta$, x^2, x^{-1} (cont'd)

(43)	$\int_0^a \frac{x^2}{a^2 - x^2} J_{1/4}(x) J_{-1/4}(x) J_{2\nu}(2a^2 - 2x^2)\,dx = \frac{a}{4\nu} J_{\nu+1/4}(a^2) J_{\nu-1/4}(a^2)$ $\operatorname{Re}\nu > 0$
(44)	$\int_0^\infty J_\nu\left(\frac{x}{a}\right) J_{\nu+1}\left(\frac{b}{x}\right)\frac{dx}{x} = \left(\frac{a}{b}\right)^{1/2} J_{2\nu+1}\left(\frac{2b^{1/2}}{a^{1/2}}\right)$ $a, b > 0, \quad \operatorname{Re}\nu > -1$
(45)	$\int_0^\infty x^{\rho-1} J_\mu(ax)\, J_\nu(bx^{-1})\,dx = 2^{\rho-1} a^{-\rho} G^{20}_{04}\left(\frac{a^2b^2}{16}\,\middle\vert\, \frac{\nu}{2}, \frac{\rho+\mu}{2}, \frac{\rho-\mu}{2}, -\frac{\nu}{2}\right)$ $a, b > 0, \quad \operatorname{Re}(\rho - \nu) < 3/2, \quad \operatorname{Re}(\rho + \mu) > -3/2$
(46)	$\int_0^\infty J_\nu\left(\frac{a}{x}\right) Y_\nu\left(\frac{x}{b}\right) dx = b\left[\frac{2}{\pi} K_{2\nu}\left(\frac{2a^{1/2}}{b^{1/2}}\right) + Y_{2\nu}\left(\frac{2a^{1/2}}{b^{1/2}}\right)\right]$ $a, b > 0, \quad -\tfrac{1}{2} < \operatorname{Re}\nu < \tfrac{1}{2}$
(47)	$\int_0^\infty J_\nu\left(\frac{a}{x}\right) Y_\nu\left(\frac{x}{b}\right)\frac{dx}{x^2} = -\frac{1}{a}\left[\frac{2}{\pi} K_{2\nu}\left(\frac{2a^{1/2}}{b^{1/2}}\right) - Y_{2\nu}\left(\frac{2a^{1/2}}{b^{1/2}}\right)\right]$ $a, b > 0, \quad -\tfrac{1}{2} < \operatorname{Re}\nu < \tfrac{1}{2}$
(48)	$\int_0^\infty Y_\nu\left(\frac{a}{x}\right) Y_\nu\left(\frac{x}{b}\right) dx = -b\, J_{2\nu}\left(\frac{2a^{1/2}}{b^{1/2}}\right)$ $a, b > 0, \quad -\tfrac{1}{2} < \operatorname{Re}\nu < \tfrac{1}{2}$

19.4. Bessel functions of other arguments

(1)	$\int_0^\infty x^{\nu+1} y^{\mu} J_\nu(ax)\, J_\mu(by)\, dx$ $= a^\nu b^\mu (\beta/h)^{\mu+\nu+1} [\sin(\nu\pi)\, Y_{\mu+\nu+1}(\beta h) - \cos(\nu\pi) J_{\mu+\nu+1}(\beta h)]$ $\quad 0 < a < b$ $= -2\pi^{-1} a^\nu b^\mu (\beta/k)^{\mu+\nu+1} \sin(\mu\pi)\, K_{\mu+\nu+1}(\beta k)$ $\quad 0 < b < a$ $a, b > 0, \quad \mathrm{Re}\,\beta > 0, \quad \mathrm{Re}(\mu+\nu) < 0, \quad \mathrm{Re}\,\nu > -1$
(2)	$\int_0^\infty x^{\nu+1} y^{\mu} J_\nu(ax)\, Y_\mu(by)\, dx$ $= -a^\nu b^\mu (\beta/h)^{\mu+\nu+1} [\sin(\nu\pi)\, J_{\mu+\nu+1}(\beta h) + \cos(\nu\pi) Y_{\mu+\nu+1}(\beta h)]$ $\quad 0 < a < b$ $= -2\pi^{-1} a^\nu b^\mu (\beta/k)^{\mu+\nu+1} \cos(\mu\pi)\, K_{\mu+\nu+1}(\beta k)$ $\quad 0 < b < a$ $a, b > 0, \quad \mathrm{Re}\,\beta > 0, \quad \mathrm{Re}(\mu+\nu) < 0, \quad \mathrm{Re}\,\nu > -1$
(3)	$\int_0^\infty x^{\nu+1} y^{-\mu} J_\nu(ax)\, Y_\mu(by)\, dx$ $= a^\nu b^{-\mu} (\beta/h)^{\nu-\mu+1}\, Y_{\mu-\nu-1}(\beta h)$ $\quad 0 < a < b$ $= -2\pi^{-1} a^\nu b^{-\mu} (\beta/k)^{\nu-\mu+1}\, K_{\mu-\nu-1}(\beta k)$ $\quad 0 < b < a$ $a, b > 0, \quad \mathrm{Re}\,\beta > 0, \quad \mathrm{Re}\,\mu > \mathrm{Re}\,\nu > -1$
(4)	$\int_0^\infty x^{\rho-1} y^{-\mu} (x^2+\lambda^2)^{-1} \left[\cos\left(\frac{\rho-\nu}{2}\pi\right) J_\nu(ax)\right.$ $\left. + \sin\left(\frac{\rho-\nu}{2}\pi\right) Y_\nu(ax)\right] J_\mu(by)\, dx$ $= -\lambda^{\rho-2} \frac{J_\mu[b(\beta^2-\lambda^2)^{1/2}]}{(\beta^2-\lambda^2)^{1/2\mu}} K_\nu(a\lambda)$ $\quad 0 < b < a$ $\lvert\mathrm{Re}\,\nu\rvert < \mathrm{Re}\,\rho < \mathrm{Re}\,\mu + 4, \quad \mathrm{Re}\,\lambda > 0$

$y = (x^2+\beta^2)^{1/2}, \quad h = (b^2-a^2)^{1/2}, \quad k = (a^2-b^2)^{1/2}$

Bessel functions of other arguments (cont'd)

(5)	$\int_{-\infty}^{\infty} \frac{\sin[a(x+\lambda)]}{x+\lambda} \frac{J_\nu(by)}{y^\nu}\,dx = \pi \frac{J_\nu[b(\beta^2+\lambda^2)^{1/2}]}{(\beta^2+\lambda^2)^{\frac{1}{2}\nu}}$ $0 < b < a, \quad \operatorname{Re}\nu > -3/2$
(6)	$\int_{-\infty}^{\infty} \frac{\sin[a(x-z)]}{x-z} \frac{J_\nu[a(x^2-2bx\cos\theta+b^2)^{1/2}]}{(x^2-2bx\cos\theta+b^2)^{\frac{1}{2}\nu}}\,dx$ $= \pi \frac{J_\nu[a(z^2-2bz\cos\theta+b^2)^{1/2}]}{(z^2-2bz\cos\theta+b^2)^{\frac{1}{2}\nu}} \qquad \operatorname{Re}\nu > -\tfrac{1}{2}$
(7)	$\int_0^{\infty} x^{\rho-1}\left(\frac{a+bx}{ax+b}\right)^\nu J_{2\nu}[x^{-1/2}(a+bx)^{1/2}(ax+b)^{1/2}]\,dx$ $= -\pi[J_{\nu+\rho}(a)Y_{\nu-\rho}(b) + J_{\nu-\rho}(b)Y_{\nu+\rho}(a)]$ $a, b > 0, \quad -\tfrac{3}{4} < \operatorname{Re}\rho < \tfrac{3}{4}$
(8)	$\int_0^{\infty} x^{\rho-1}\left(\frac{a+bx}{ax+b}\right)^\nu Y_{2\nu}[x^{-1/2}(a+bx)^{1/2}(ax+b)^{1/2}]\,dx$ $= \pi[J_{\nu+\rho}(a)J_{\nu-\rho}(b) - Y_{\nu+\rho}(a)Y_{\nu-\rho}(b)]$ $a, b > 0, \quad -\tfrac{3}{4} < \operatorname{Re}\nu < \tfrac{3}{4}$
(9)	$\int_0^{\infty} x^{\rho-1}\left(\frac{a+bx}{ax+b}\right)^\nu H_\nu^{(2)}[x^{-1/2}(a+bx)^{1/2}(ax+b)^{1/2}]\,dx$ $= -i\pi H_{\nu+\rho}^{(2)}(a)H_{\nu-\rho}^{(2)}(b) \qquad a, b > 0, \quad -\tfrac{3}{4} < \operatorname{Re}\nu < \tfrac{3}{4}$
(10)	$\int_0^{\infty} \cosh x \cos(2a\sinh x) J_\nu(be^x) J_\nu(be^{-x})\,dx$ $= \tfrac{1}{2}(b^2-a^2)^{-1/2} J_{2\nu}[2(b^2-a^2)^{1/2}] \qquad 0 < a < b$ $= 0 \qquad 0 < b < a$ $\operatorname{Re}\nu > -1$

$y = (x^2+\beta^2)^{1/2}, \quad h = (b^2-a^2)^{1/2}, \quad k = (a^2-b^2)^{1/2}$

Bessel functions of other arguments (cont'd)

(11)	$\int_0^\infty \cosh x \cos(2a \sinh x)\, Y_\nu(be^x)\, Y_\nu(be^{-x})\, dx$ $= -\tfrac{1}{2}(b^2-a^2)^{-\frac{1}{2}} J_{2\nu}[2(b^2-a^2)^{\frac{1}{2}}]$ $\qquad 0<a<b$ $= 2\pi^{-1}\cos(\nu\pi)(a^2-b^2)^{-\frac{1}{2}} K_{2\nu}[2(a^2-b^2)^{\frac{1}{2}}]$ $\qquad 0<b<a$ $-1<\mathrm{Re}\,\nu<1$
(12)	$\int_0^\infty \cosh x \sin(2a \sinh x)\,[J_\nu(be^x)\, Y_\nu(be^{-x}) - Y_\nu(be^x) J_\nu(be^{-x})]dx$ $=0$ $\qquad 0<a<b$ $= -2\pi^{-1}\cos(\nu\pi)(a^2-b^2)^{-\frac{1}{2}} K_{2\nu}[2(a^2-b^2)^{\frac{1}{2}}]$ $\qquad 0<b<a$ $-\tfrac{1}{2}<\mathrm{Re}\,\nu<\tfrac{1}{2}$
(13)	$\int_0^\pi \sin(2\mu x)\, J_{2\nu}(2a\sin x)\, dx = \pi \sin(\mu\pi)\, J_{\nu-\mu}(a)\, J_{\nu+\mu}(a)$ $\mathrm{Re}\,\nu>-1$
(14)	$\int_0^\pi \cos(2\mu x)\, J_{2\nu}(2a\sin x)\, dx = \pi \cos(\mu\pi)\, J_{\nu-\mu}(a)\, J_{\nu+\mu}(a)$ $\mathrm{Re}\,\nu>-\tfrac{1}{2}$
(15)	$\int_0^{\frac{1}{2}\pi} \cos(2nx)\, J_0(2a\sin x)\, dx = \tfrac{1}{2}\pi\,[J_n(a)]^2$ $\qquad n=0,1,2,\ldots$
(16)	$\int_0^{\frac{1}{2}\pi} \cos(2nx)\, Y_0(2a\sin x)\, dx = \tfrac{1}{2}\pi\, J_n(a)\, Y_n(a)$ $n=0,1,2,\ldots$
(17)	$\int_0^\pi [\tan(\tfrac{1}{2}x)]^{-2\kappa}\, e^{-\beta\cos x}\, J_{2\nu}(a\sin x)\, dx$ $= \dfrac{\Gamma(\frac{1}{2}+\kappa+\nu)\Gamma(\frac{1}{2}-\kappa+\nu)}{a[\Gamma(2\nu+1)]^2} M_{\kappa,\nu}[\beta+(\beta^2-a^2)^{\frac{1}{2}}]\, M_{\kappa,\nu}[\beta-(\beta^2-a^2)^{\frac{1}{2}}]$ $\mathrm{Re}\,\nu+\tfrac{1}{2}>\lvert\mathrm{Re}\,\kappa\rvert$

Bessel functions of other arguments (cont'd)

(18)	$\int_0^{\frac{1}{2}\pi} \cos(2\beta \cos x)\, J_{2\nu}(2a \sin x)\, dx$ $= \frac{1}{2}\pi\, J_\nu[(\beta^2 + a^2)^{\frac{1}{2}} + \beta]\, J_\nu[(\beta^2 + a^2)^{\frac{1}{2}} - \beta]$ $\operatorname{Re} \nu > -\frac{1}{2}$
(19)	$\int_0^{\frac{1}{2}\pi} (\sin x)^{\nu+1} \cos(\beta \cos x)\, J_\nu(a \sin x)\, dx$ $= 2^{-\frac{1}{2}}\, \pi^{\frac{1}{2}}\, a^\nu (a^2 + \beta^2)^{-\frac{1}{2}\nu - \frac{1}{4}}\, J_{\nu+\frac{1}{2}}[(a^2 + \beta^2)^{\frac{1}{2}}]$ $\operatorname{Re} \nu > -1$
(20)	$\int_0^{\frac{1}{2}\pi} \sin(2x)\, P_n(\cos 2x)\, J_0(a \sin x)\, dx = a^{-1}\, J_{2n+1}(a)$ $n = 0, 1, 2, \ldots$
(21)	$\int_0^{\frac{1}{2}\pi} (\sin x)^{\nu+1} \cos(a \cos\theta \cos x)\, C_{2n}^{\nu+\frac{1}{2}}(\cos x)\, J_\nu(a \sin\theta \sin x)\, dx$ $= (-1)^n\, 2^{-\frac{1}{2}}\, \pi^{\frac{1}{2}}\, (\sin\theta)^\nu\, a^{-\frac{1}{2}}\, C_{2n}^{\nu+\frac{1}{2}}(\cos\theta)\, J_{\nu+2n+\frac{1}{2}}(a)$ $n = 0, 1, 2, \ldots,$ $\operatorname{Re} \nu > -1$
(22)	$\int_0^{\frac{1}{2}\pi} (\sin x)^{\nu+1} \sin(a \cos\theta \cos x)\, C_{2n+1}^{\nu+\frac{1}{2}}(\cos x)\, J_\nu(a \sin\theta \sin x)\, dx$ $= (-1)^n\, 2^{-\frac{1}{2}}\, \pi^{\frac{1}{2}}\, (\sin\theta)^\nu\, a^{-\frac{1}{2}}\, C_{2n+1}^{\nu+\frac{1}{2}}(\cos\theta)\, J_{\nu+2n+3/2}(a)$ $n = 0, 1, 2, \ldots,$ $\operatorname{Re} \nu > -1$
(23)	$\int_0^{\frac{1}{2}\pi} \cos(2\mu x)\, J_{2\nu}(2a \cos x)\, dx = \frac{1}{2}\pi\, J_{\nu+\mu}(a)\, J_{\nu-\mu}(a)$ $\operatorname{Re} \nu > -\frac{1}{2}$
(24)	$\int_0^{\frac{1}{2}\pi} \cos(2\mu x)\, Y_{2\nu}(2a \cos x)\, dx = \frac{1}{2}\pi \operatorname{ctn}(2\nu\pi)\, J_{\nu+\mu}(a)\, J_{\nu-\mu}(a)$ $- \frac{1}{2}\pi \csc(2\nu\pi)\, J_{\mu-\nu}(a)\, J_{-\mu-\nu}(a)$ $-\frac{1}{2} < \operatorname{Re} \nu < \frac{1}{2}$

Bessel functions of other arguments (cont'd)

(25)	$\int_0^{\frac{1}{2}\pi} (\sin x)^{\mu+1} (\cos x)^{\nu+1} J_\mu(a \sin x) J_\nu(\beta \cos x)\, dx$ $= a^\mu \beta^\nu (a^2+\beta^2)^{-\frac{1}{2}(\mu+\nu+1)} J_{\mu+\nu+1}[(a^2+\beta^2)^{\frac{1}{2}}]$ $\operatorname{Re}\mu > -1, \quad \operatorname{Re}\nu > -1$
(26)	$\int_0^{\frac{1}{2}\pi} (\sin x)^\rho (\cos x)^\sigma J_\mu(a \sin x) J_\nu(\beta \sin x)\, dx$ $\int_0^{\frac{1}{2}\pi} (\sin x)^\rho (\cos x)^\sigma J_\mu(a \sin x) J_\nu(\beta \cos x)\, dx$ See Bailey, W.N., 1938: *Quart. J. Math. Oxford Ser.* 9, 141-147.
(27)	$\int_0^\pi (\sin x)^{2\nu} \frac{J_\nu(\omega)}{\omega^\nu}\, dx = 2^\nu \pi^{\frac{1}{2}} \Gamma(\nu+\frac{1}{2}) \frac{J_\nu(a)}{a^\nu} \frac{J_\nu(\beta)}{\beta^\nu}$ $\omega = (a^2+\beta^2-2a\beta\cos x)^{\frac{1}{2}}, \quad \operatorname{Re}\nu > -\frac{1}{2}$
(28)	$\int_0^\pi (\sin x)^{2\nu} \frac{Y_\nu(\omega)}{\omega^\nu}\, dx = 2^\nu \pi^{\frac{1}{2}} \Gamma(\nu+\frac{1}{2}) \frac{J_\nu(a)}{a^\nu} \frac{Y_\nu(\beta)}{\beta^\nu}$ $\omega = (a^2+\beta^2-2a\beta\cos x)^{\frac{1}{2}}, \quad \lvert a\rvert < \lvert\beta\rvert, \quad \operatorname{Re}\nu > -\frac{1}{2}$
(29)	$\int_0^\pi (\sin x)^{2\nu} C_n^\nu(\cos x) \frac{J_\nu(\omega)}{\omega^\nu}\, dx = \frac{\pi\Gamma(2\nu+n)}{2^{\nu-1} n!\,\Gamma(\nu)} \frac{J_{\nu+n}(a)}{a^\nu} \frac{J_{\nu+n}(\beta)}{\beta^\nu}$ $n = 0, 1, 2, \ldots, \quad \omega = (a^2+\beta^2-2a\beta\cos x)^{\frac{1}{2}}, \quad \operatorname{Re}\nu > -\frac{1}{2}$
(30)	$\int_0^\pi (\sin x)^{2\nu} C_n^\nu(\cos x) \frac{Y_\nu(\omega)}{\omega^\nu}\, dx = \frac{\pi\Gamma(2\nu+n)}{2^{\nu-1} n!\,\Gamma(\nu)} \frac{J_{\nu+n}(a)}{a^\nu} \frac{Y_{\nu+n}(\beta)}{\beta^\nu}$ $n = 0, 1, 2, \ldots, \quad \omega = (a^2+\beta^2-2a\beta\cos x)^{\frac{1}{2}}, \quad \lvert a\rvert < \lvert\beta\rvert$ $\operatorname{Re}\nu > -\frac{1}{2}$

Bessel functions of other arguments (cont'd)

(31)	$\int_0^\infty e^{-2\mu x} Y_{2\nu}(2a \sinh x)\, dx = \operatorname{ctn}(2\nu\pi) I_{\mu+\nu}(a) K_{\mu-\nu}(a) - \csc(2\nu\pi) I_{\mu-\nu}(a) K_{\mu+\nu}(a)$ $a > 0, \quad \operatorname{Re}\mu > -3/2, \quad -1/2 < \operatorname{Re}\nu < 1/2$
(32)	$\int_0^\infty [\operatorname{ctnh}(\tfrac{1}{2}x)]^{-2\kappa} \{\sin[(\mu-\kappa)\pi] J_{2\mu}(a \sinh x) + \cos[(\mu-\kappa)\pi] Y_{2\mu}(a \sinh x)\}\, dx = -a^{-1} W_{\kappa,\mu}(a) W_{-\kappa,\mu}(a)$ $a > 0, \quad \operatorname{Re}\kappa > \lvert\operatorname{Re}\mu\rvert - \tfrac{1}{2}$
(33)	$\int_0^\infty \sinh x\, [\tanh(\tfrac{1}{2}x)]^{\nu} e^{-\beta\cosh x} J_\nu(a \sinh x)\, dx = (a^2+\beta^2)^{-\frac{1}{2}} \left[\dfrac{(a^2+\beta^2)^{\frac{1}{2}}+\beta}{(a^2+\beta^2)^{\frac{1}{2}}-\beta}\right]^{-\frac{1}{2}\nu} \exp[-(a^2+\beta^2)^{\frac{1}{2}}]$ $\operatorname{Re}\beta > \lvert\operatorname{Re}a\rvert, \quad \operatorname{Re}\nu > -1$
(34)	$\int_0^\infty [\operatorname{ctnh}(\tfrac{1}{2}x)]^{2\kappa} e^{-\beta\cosh x} J_{2\mu}(a \sinh x)\, dx = \dfrac{\Gamma(\tfrac{1}{2}-\kappa+\mu)}{a\,\Gamma(2\mu+1)} M_{-\kappa,\mu}[(a^2+\beta^2)^{\frac{1}{2}}-\beta]\, W_{\kappa,\mu}[(a^2+\beta^2)^{\frac{1}{2}}+\beta]$ $\operatorname{Re}\beta > \lvert\operatorname{Re}a\rvert, \quad \operatorname{Re}(\mu-\kappa) > -\tfrac{1}{2}$
(35)	$\int_0^\infty [\operatorname{ctnh}(\tfrac{1}{2}x)]^{2\kappa} e^{-\beta\cosh x} Y_{2\mu}(a \sinh x)\, dx = -a^{-1}\sec[(\mu+\kappa)\pi] W_{\kappa,\mu}(h) W_{-\kappa,\mu}(k) - \dfrac{\tan[(\mu+\kappa)\pi]\,\Gamma(\tfrac{1}{2}-\kappa+\mu)}{a\,\Gamma(2\mu+1)} W_{\kappa,\mu}(h) M_{-\kappa,\mu}(k)$ $h = (a^2+\beta^2)^{\frac{1}{2}}+\beta, \quad k = (a^2+\beta^2)^{\frac{1}{2}}-\beta$ $\operatorname{Re}\beta > \lvert\operatorname{Re}a\rvert, \quad \operatorname{Re}\kappa < \tfrac{1}{2} - \lvert\operatorname{Re}\mu\rvert$

Bessel functions of other arguments (cont'd)

(36)	$\int_0^\infty (\sinh x)^{\mu+1} (\cosh x)^{\nu+1} J_\mu(a \sinh x) H_\nu^{(2)}(b \cosh x)\,dx$ $= -e^{-\mu\pi i} a^\mu b^\nu h^{-\mu-\nu-1} H_{\mu+\nu+1}^{(2)}(h) \qquad 0 < a < b$ $= 2i\pi^{-1} e^{\nu\pi i} a^\mu b^\nu k^{-\mu-\nu-1} K_{\mu+\nu+1}(k) \qquad 0 < b < a$ $h = (b^2 - a^2)^{1/2}, \quad k = (a^2 - b^2)^{1/2}$ $\operatorname{Re}\mu > -1, \quad \operatorname{Re}(\mu+\nu) < 0$
(37)	$\int_0^\infty (\sinh x)^{\mu+1} (\cosh x)^{1-\nu} J_\mu(a \sinh x) H_\nu^{(2)}(b \cosh x)\,dx$ $= a^\mu b^{-\nu} h^{\nu-\mu-1} H_{\nu-\mu-1}^{(2)}(h) \qquad 0 < a < b$ $= 2i\pi^{-1} a^\mu b^{-\nu} k^{\nu-\mu-1} K_{\nu-\mu-1}(k) \qquad 0 < b < a$ $h = (b^2 - a^2)^{1/2}, \quad k = (a^2 - b^2)^{1/2}, \quad \operatorname{Re}\nu > \operatorname{Re}\mu > -1$
(38)	$\int_{-\infty}^\infty \operatorname{sech} x\, e^{2\kappa x - \beta\tanh x} J_{2\mu}(a \operatorname{sech} x)\,dx$ $= \dfrac{\Gamma(\frac{1}{2}+\kappa+\mu)\,\Gamma(\frac{1}{2}-\kappa+\mu)}{a[\Gamma(2\mu+1)]^2} M_{\kappa,\mu}(h)\, M_{\kappa,\mu}(k)$ $h + k = 2\beta, \quad hk = a^2, \quad \operatorname{Re}\mu > \lvert\operatorname{Re}\kappa\rvert - \frac{1}{2}$

19.5. Modified Bessel functions of argument x

For integrals involving $\operatorname{ber}_\nu x$, $\operatorname{bei}_\nu x$, $\operatorname{ker}_\nu x$, $\operatorname{kei}_\nu x$ and similar functions see McLachlan, N.W., 1954: *Bessel functions for engineers*. Oxford, Second edition.

(1)	$\int_0^a I_\nu(x)\,dx = 2 \sum_{n=0}^{\infty} (-1)^n I_{\nu+2n+1}(a) \qquad \operatorname{Re}\nu > -1$
(2)	$\int_0^a x^\nu I_\nu(x)\,dx = 2^{\nu-1}\pi^{1/2}\Gamma(\nu+\frac{1}{2})\,a\,[I_\nu(a)\,\mathbf{L}_{\nu-1}(a) - \mathbf{L}_\nu(a)\, I_{\nu-1}(a)]$ $\operatorname{Re}\nu > -\frac{1}{2}$

Modified functions of x (cont'd)

(3)	$\int_0^a x^{\nu+1} I_\nu(x)\,dx = a^{\nu+1} I_{\nu+1}(a)$ $\qquad \mathrm{Re}\,\nu > -1$
(4)	$\int_0^a x^{1-\nu} I_\nu(x)\,dx = a^{1-\nu} I_{\nu-1}(a) - \dfrac{2^{1-\nu}}{\Gamma(\nu)}$
(5)	$\int_0^a x^\nu (a^2-x^2)^{\nu-\frac{1}{2}} I_\nu(x)\,dx = 2^{-\nu-1}\pi^{\frac{1}{2}} a^{2\nu}\Gamma(\nu+\tfrac{1}{2})[I_\nu(\tfrac{1}{2}a)]^2$
(6)	$\int_0^a x^{\nu+1}(a^2-x^2)^{\sigma-1} I_\nu(x)\,dx = 2^{\sigma-1} a^{\nu+\sigma}\Gamma(\sigma) I_{\nu+\sigma}(a)$ $\mathrm{Re}\,\nu > -1, \quad \mathrm{Re}\,\sigma > 0$
(7)	$\int_0^a x^{\rho-1}(a^2-x^2)^{\sigma-1} I_\nu(x)\,dx$ $= \dfrac{\Gamma(\frac{\nu+\rho}{2})\;\Gamma(\sigma)\,a^{\nu+\rho+2\sigma-2}}{2^{\nu+1}\Gamma(\nu+1)\Gamma(\frac{\nu+\rho}{2}+\sigma)}\;{}_1F_2\left(\dfrac{\nu+\rho}{2};\,\nu+1,\dfrac{\nu+\rho}{2}+\sigma;\dfrac{a^2}{4}\right)$ $\mathrm{Re}\,(\rho+\nu) > 0, \quad \mathrm{Re}\,(\sigma) > 0$
(8)	$\int_0^a x^{n+1} e^{-x^2} I_n(2ax)\,dx = \tfrac{1}{4}a^n[e^{a^2} - e^{-a^2}\sum_{r=-n}^{n} I_r(2a^2)]$ $n = 0, 1, 2, \ldots$
(9)	$\int_0^a x^{\nu+1} y^{-1}\cos y\, I_\nu(x)\,dx = \dfrac{\pi^{\frac{1}{2}} a^{2\nu+1}}{2^{\nu+1}\Gamma(\nu+3/2)}$ $y = (a^2-x^2)^{\frac{1}{2}}, \quad \mathrm{Re}\,\nu > -1$
(10)	$\int_0^a y^{-1}\cosh(y\sinh t)\, I_{2\nu}(x)\,dx = \tfrac{1}{2}\pi\, I_\nu(ae^t)\, I_\nu(ae^{-t})$ $y = (a^2-x^2)^{\frac{1}{2}}, \quad \mathrm{Re}\,\nu > -\tfrac{1}{2}$

Modified functions of x (cont'd)

(11)	$\int_0^a e^{-x} P_n(1-2xa^{-1}) I_0(x)\,dx = \dfrac{ae^{-a}}{2n+1}[I_n(a)+I_{n+1}(a)]$
(12)	$\int_0^a x^{\mu} e^{-x} P_{\nu}(1-2x/a) I_{\mu}(x)\,dx$ See Bose, B.N., 1948: *Bull. Calcutta Math. Soc.* 40, 8-14.
(13)	$\int_0^{\infty} x^{-1/3} e^{-x} \sin(4ax^{1/2}) I_{1/3}(x)\,dx = (2\pi)^{-1/2} a^{1/3} e^{-a^2} K_{1/3}(a^2)$ $a>0$
(14)	$\int_0^{\infty} x^{-\nu} e^{-x} \sin(4ax^{1/2}) I_{\nu}(x)\,dx = (2^{3/2}a)^{\nu-1} e^{-a^2}$ $\times W_{1/2-3\nu/2,\,1/2-\nu/2}(2a^2)$ $\qquad a>0, \quad \mathrm{Re}\,\nu>0$
(15)	$\int_0^{\infty} x^{-1/2} e^{-x} \cos(4ax^{1/2}) I_0(x)\,dx = (2\pi)^{-1/2} e^{-a^2} K_0(a^2)$ $\qquad a>0$
(16)	$\int_0^{\infty} x^{-\nu-1/2} e^{-x} \cos(4ax^{1/2}) I_{\nu}(x)\,dx$ $= 2^{3\nu/2-1} a^{\nu-1} e^{-a^2} W_{-3\nu/2,\,\nu/2}(2a^2)$ $\qquad a>0, \quad \mathrm{Re}\,\nu>-1/2$
(17)	$\int_a^{\infty} (x^2-a^2)^{-1/2} T_n(ax^{-1}) K_{2\mu}(x)\,dx = \dfrac{\pi}{2a} W_{1/2 n,\,\mu}(a)\, W_{-1/2 n,\,\mu}(a)$ $n=0, 1, 2, \ldots$
(18)	$\int_0^{\infty} (1+x/a)^{\mu} e^{-x} P_{\nu}^{-2\mu}(1+2x/a) I_{\mu}(x)\,dx = 0$ $-1/2<\mathrm{Re}\,\mu<0, \quad -1/2+\mathrm{Re}\,\mu<\mathrm{Re}\,\nu<-1/2-\mathrm{Re}\,\mu$

Modified functions of x (cont'd)

(19)	$\int_0^\infty (x+a)^{-\mu} e^{-x} P_\nu^{-2\mu}(1+2x/a)\, I_\mu(x)\, dx$ $= \dfrac{2^{\mu-1}\,\Gamma(\mu+\nu+\tfrac{1}{2})\,\Gamma(\mu-\nu-\tfrac{1}{2})}{\pi^{1/2}\,\Gamma(2\mu+\nu+1)\,\Gamma(2\mu-\nu)}\, e^{a}\, W_{1/2-\mu,\,1/2+\nu}(2a)$ $\lvert\arg a\rvert < \pi, \quad \operatorname{Re}\mu > \lvert\operatorname{Re}\nu + \tfrac{1}{2}\rvert$
(20)	$\int_a^\infty x^{1-n} e^{-x^2} I_n(2ax)\, dx = \tfrac{1}{4} a^{-n}[e^{a^2} - e^{-a^2} \sum_{r=1-n}^{n-1} I_r(2a^2)]$ $n = 1, 2, \ldots$
(21)	$\int_0^a x^\nu K_\nu(x)\, dx = 2^{\nu-1}\pi^{1/2}\,\Gamma(\nu+\tfrac{1}{2})\, a\, [K_\nu(a)\,\mathbf{L}_{\nu-1}(a) + \mathbf{L}_\nu(a) K_{\nu-1}(a)]$ $\operatorname{Re}\nu > -\tfrac{1}{2}$
(22)	$\int_0^a x^{\nu+1} K_\nu(x)\, dx = 2^\nu\,\Gamma(\nu+1) - a^{\nu+1} K_{\nu+1}(a)$ $\operatorname{Re}\nu > -1$
(23)	$\int_0^a x^{1-\nu} K_\nu(x)\, dx = 2^{-\nu}\,\Gamma(1-\nu) - a^{1-\nu} K_{\nu-1}(a)$ $\operatorname{Re}\nu < 1$
(24)	$\int_0^a x^\mu (a^2-x^2)^{\mu-1/2} K_\mu(x)\, dx = 2^{\mu-1}\pi^{1/2} a^{2\mu}\,\Gamma(\mu+\tfrac{1}{2})$ $\times I_\mu(\tfrac{1}{2}a)\, K_\mu(\tfrac{1}{2}a)$ $\operatorname{Re}\mu > -\tfrac{1}{2}$
(25)	$\int_0^a y^{-1}\cosh(y\sinh t)\, K_{2\nu}(x)\, dx = \tfrac{1}{4}\pi^2\csc(\nu\pi)[I_{-\nu}(ae^{t})\, I_{-\nu}(ae^{-t})$ $- I_\nu(ae^{t})\, I_\nu(ae^{-t})]$ $y = (a^2-x^2)^{1/2}, \quad -\tfrac{1}{2} < \operatorname{Re}\nu < \tfrac{1}{2}$
(26)	$\int_0^a x\, J_\nu(\lambda x)\, K_\nu(\kappa x)\, dx = (\kappa^2+\lambda^2)^{-1}[(\lambda/\kappa)^\nu + \lambda a\, J_{\nu+1}(\lambda a) K_\nu(\kappa a)$ $- \kappa a\, J_\nu(\lambda a)\, K_{\nu+1}(\kappa a)]$ $\operatorname{Re}\nu > -1$

Modified functions of x (cont'd)

(27)	$\int_0^a x^{2\nu+1} P_\mu(1-2x^2 a^{-2})\, I_\nu(x)\, K_\nu(x)\, dx$ See Bose, B.N., 1948: *Bull. Calcutta Math. Soc.* 40, 8-14.
(28)	$\int_0^a x^{2\nu+1} P_n[(1-x^2 a^{-2})^{1/2}]\, I_\nu(x)\, K_\nu(x)\, dx$ See Bose, B.N., 1944: *Bull. Calcutta Math. Soc.* 36, 125-132.
(29)	$\int_0^\infty x^{-1/2}(x+a)^{-1} e^{-x} K_\nu(x)\, dx = \dfrac{\pi e^a}{a^{1/2}\cos(\nu\pi)} K_\nu(a)$ $\lvert\arg a\rvert < \pi, \quad \lvert\mathrm{Re}\,\nu\rvert < \tfrac{1}{2}$
(30)	$\int_0^\infty x^{-1/4} \exp(-2ax^{1/2})\, K_{1/4}(x)\, dx = \left(\dfrac{a\pi}{2}\right)^{1/2} K_{1/4}(a^2)$ $+ \tfrac{1}{2}\pi^{3/2} a^{1/2}[\mathbf{L}_{-1/4}(a^2) - \mathbf{L}_{1/4}(a^2)]$
(31)	$\int_0^\infty x^{-1/2} y^{-1} e^{-y} K_\nu(x)\, dx = \dfrac{\pi^{3/2} a^{-1/2} \sec(\nu\pi)}{\Gamma(\tfrac{3}{4}+\tfrac{1}{2}\nu)\,\Gamma(\tfrac{3}{4}-\tfrac{1}{2}\nu)} K_\nu(a)$ $y = (x^2+a^2)^{1/2}, \quad \mathrm{Re}\, a > 0, \quad -\tfrac{1}{2} < \mathrm{Re}\,\nu < \tfrac{1}{2}$
(32)	$\int_0^\infty x^{-1/2} y^{-1} e^{-y} K_\nu(x)\, dx = \pi a^{-1/2} \sec(\nu\pi)\, P_{\nu-1/2}(-\cos\phi)\, K_\nu(a)$ $y = (x^2+a^2-2ax\cos\phi)^{1/2}, \quad \lvert\arg a\rvert + \lvert\mathrm{Re}\,\phi\rvert < \pi, \quad \lvert\mathrm{Re}\,\nu\rvert < \tfrac{1}{2}$
(33)	$\int_0^\infty x^{-3/2}(1+a^2/x)^{-1/2} \exp[-(\beta+x)(1+a^2/x)^{1/2}]\, K_\nu(x)\, dx$ $= 4a^{-1} K_\nu(\beta)\, K_{2\nu}(2a\beta^{1/2}) \qquad \mathrm{Re}\, a > 0, \quad \mathrm{Re}(a\beta) > 0$
(34)	$\displaystyle\int_0^\infty x \sin\left(\frac{a}{2x}\right) K_0(x)\, dx = \tfrac{1}{2}\pi a\, J_1(a^{1/2})\, K_1(a^{1/2}) \qquad a > 0$

Modified functions of x (cont'd)

(35)	$\int_0^\infty x \cos\left(\frac{a}{2x}\right) K_0(x)\,dx = -\tfrac{1}{2}\pi\, a\, Y_1(a^{1/2})\, K_1(a^{1/2})$	$a > 0$
(36)	$\int_0^\infty x^{-1/3} e^{x} \sin(4ax^{1/2})\, K_{1/3}(x)\,dx = (\tfrac{1}{2}\pi)^{1/2} a^{1/3} e^{a^2} K_{1/3}(a^2)$	$a > 0$
(37)	$\int_0^\infty x^{-1/3} e^{-x} \sin(4ax^{1/2})\, K_{1/3}(x)\,dx = 2^{-1/2}\pi^{3/2} a^{1/3} e^{-a^2} I_{1/3}(a^2)$	
(38)	$\int_0^\infty x^{-\nu} e^{x} \sin(4ax^{1/2})\, K_\nu(x)\,dx = (2^{3/2} a)^{\nu-1}\pi\, \frac{\Gamma(3/2-2\nu)}{\Gamma(1/2+\nu)} \times e^{a^2} W_{3\nu/2-1/2,\, 1/2-\nu/2}(2a^2)$	$a > 0, \quad 0 < \operatorname{Re}\nu < \tfrac{3}{4}$
(39)	$\int_0^\infty x^{\rho-3/2} e^{-x} \sin(4ax^{1/2})\, K_\nu(x)\,dx = \frac{\pi^{1/2} a\, \Gamma(\rho+\nu)\,\Gamma(\rho-\nu)}{2^{\rho-2}\,\Gamma(\rho+\frac{1}{2})} \times {}_2F_2(\rho+\nu, \rho-\nu; 3/2, \rho+1/2; -2a^2)$	$\operatorname{Re}\rho > \lvert\operatorname{Re}\nu\rvert$
(40)	$\int_0^\infty x^{-1/2} e^{x} \cos(4ax^{1/2})\, K_0(x)\,dx = (\tfrac{1}{2}\pi)^{1/2} e^{a^2} K_0(a^2)$	$a > 0$
(41)	$\int_0^\infty x^{-1/2} e^{-x} \cos(4ax^{1/2})\, K_0(x)\,dx = 2^{-1/2}\pi^{3/2} e^{-a^2} I_0(a^2)$	
(42)	$\int_0^\infty x^{-\nu-1/2} e^{x} \cos(4ax^{1/2})\, K_\nu(x)\,dx = 2^{3\nu/2-1}\pi a^{\nu-1}\frac{\Gamma(\frac{1}{2}-2\nu)}{\Gamma(\frac{1}{2}+\nu)} e^{a^2} W_{3\nu/2,\,-\nu/2}(2a^2)$	$a > 0, \quad -\tfrac{1}{2} < \operatorname{Re}\nu < \tfrac{1}{4}$

Modified functions of x (cont'd)

(43)	$\int_0^\infty x^{\rho-1} e^{-x} \cos(4ax^{1/2}) K_\nu(x)\, dx = \dfrac{\pi^{1/2}\,\Gamma(\rho+\nu)\,\Gamma(\rho-\nu)}{2^\rho\,\Gamma(\rho+\frac{1}{2})}$ $\times {}_2F_2(\rho+\nu, \rho-\nu; \frac{1}{2}, \rho+\frac{1}{2}; -2a^2)$ $\qquad \operatorname{Re}\rho > \lvert\operatorname{Re}\nu\rvert$
(44)	$\int_0^\infty x^{n+2\nu-1/2} \exp[-(1+a)x]\, L_n^{2\nu}(ax)\, K_\nu(x)\, dx$ $= \dfrac{\pi^{1/2}\,\Gamma(n+\nu+\frac{1}{2})\,\Gamma(n+3\nu+\frac{1}{2})}{2^{n+2\nu+1/2}\, n!\;\Gamma(2\nu+1)}$ $\times {}_2F_1(n+\nu+\frac{1}{2}, n+3\nu+\frac{1}{2}; 2\nu+1; -\frac{1}{2}a)$ $\operatorname{Re} a > -2, \quad \operatorname{Re}(n+\nu) > -\frac{1}{2}, \quad \operatorname{Re}(n+3\nu) > -\frac{1}{2}$
(45)	$\int_a^\infty x^{-\nu}(x^2-a^2)^{1/4-1/2\nu}\, \mathrm{P}_\mu^{\nu-1/2}(2a^2x^{-2}-1)\, K_\nu(x)\, dx$ $= \pi^{1/2}\, 2^{-\nu} a^{-1/2-\nu}\, W_{\mu+1/2,\,\nu-1/2}(a)\, W_{-\mu-1/2,\,\nu-1/2}(a)$ $\qquad \operatorname{Re}\nu < 3/2$
(46)	$\displaystyle\int_0^\infty x^{-1} \exp\left(\frac{a^2}{2x} - x\right) \operatorname{Erfc}\left[\frac{a}{(2x)^{1/2}}\right] K_\nu(x)\, dx$ $= \frac{1}{4}\pi^{5/2} \sec(\nu\pi)\, \{[J_\nu(a)]^2 + [Y_\nu(a)]^2\}$ $\operatorname{Re} a > 0, \quad -\frac{1}{2} < \operatorname{Re}\nu < \frac{1}{2}$
(47)	$\int_0^\infty J_\mu(x)\, K_\nu(x)\, x^{\mu-\nu+1}\, dx = \frac{1}{2}\Gamma(\mu-\nu+1)$ $\operatorname{Re}\mu > -1, \quad \operatorname{Re}(\mu-\nu) > -1$
(48)	$\int_0^\infty e^{-2ax}\, I_0(x)\, K_0(x)\, dx$ $= \frac{1}{2}K[(1-a^2)^{1/2}]$ $\qquad 0 < a < 1$ $= (2a)^{-1}\, K[(1-a^{-2})^{1/2}]$ $\qquad 1 < a < \infty$

Modified functions of x (cont'd)

(49)	$\int_0^\infty x \exp\left(-\frac{x^2}{2a}\right)[I_\nu(x)+I_{-\nu}(x)]\,K_\nu(x)\,dx = a e^{a} K_\nu(a)$ Re $a>0$, $\quad -1<\text{Re}\,\nu<1$
(50)	$\int_0^\infty x^{\rho-1}\sin(2ax)\,K_\mu(x)\,K_\nu(x)\,dx = \frac{2^{\rho-1}a}{\Gamma(\rho+1)}\Gamma\left(\frac{\rho+\mu+\nu+1}{2}\right)$ $\times\Gamma\left(\frac{\rho+\mu-\nu+1}{2}\right)\Gamma\left(\frac{\rho-\mu+\nu+1}{2}\right)\Gamma\left(\frac{\rho-\mu-\nu+1}{2}\right)$ $\times\ {}_4F_3\left(\frac{\rho+\mu+\nu+1}{2},\frac{\rho+\mu-\nu+1}{2},\frac{\rho-\mu+\nu+1}{2},\frac{\rho-\mu-\nu+1}{2};\right.$ $\left.\frac{3}{2},\frac{\rho+1}{2},\frac{\rho}{2}+1;-a^2\right)$ $\lvert\text{Re}\,a\rvert<1$, $\quad \text{Re}\,\rho>\lvert\text{Re}\,\mu\rvert+\lvert\text{Re}\,\nu\rvert-1$
(51)	$\int_0^\infty x^{\rho-1}\cos(2ax)\,K_\mu(x)\,K_\nu(x)\,dx = \frac{2^{\rho-3}}{\Gamma(\rho)}\Gamma\left(\frac{\rho+\mu+\nu}{2}\right)$ $\times\Gamma\left(\frac{\rho+\mu-\nu}{2}\right)\Gamma\left(\frac{\rho-\mu+\nu}{2}\right)\Gamma\left(\frac{\rho-\mu-\nu}{2}\right)$ $\times\ {}_4F_3\left(\frac{\rho+\mu+\nu}{2},\frac{\rho+\mu-\nu}{2},\frac{\rho-\mu+\nu}{2},\frac{\rho-\mu-\nu}{2};\frac{1}{2},\frac{\rho}{2},\frac{\rho+1}{2};-a^2\right)$ $\lvert\text{Re}\,a\rvert<1$, $\quad \text{Re}\,\rho>\lvert\text{Re}\,\mu\rvert+\lvert\text{Re}\,\nu\rvert$
(52)	$\int_0^\infty x^3\cos\left(\frac{x^2}{2a}\right)Y_1(x)\,K_1(x)\,dx = -a^3\,K_0(a)$ $\qquad a>0$

19.6. Modified Bessel functions of other arguments

(1)	$\int_0^\infty \cos\left(\frac{x^2}{2a}\right) K_{2\nu}(xe^{i\pi/4})\, K_{2\nu}(xe^{-i\pi/4})\, dx$ $= \frac{\Gamma(\frac14+\nu)\,\Gamma(\frac14-\nu)}{8a^{\frac12}\,\pi^{-\frac12}}\, W_{\frac14,\nu}(ae^{i\pi/2})\, W_{\frac14,\nu}(ae^{-i\pi/2})$ $a>0, \quad -\frac14 < \operatorname{Re}\nu < \frac14$
(2)	$\int_0^\infty x^{-\frac12}\, I_\nu(x)\, K_\nu(x)\, K_\mu(2x)\, dx$ $= \frac{\Gamma(\frac14+\frac12\mu)\,\Gamma(\frac14-\frac12\mu)\,\Gamma(\frac14+\nu+\frac12\mu)\,\Gamma(\frac14+\nu-\frac12\mu)}{4\,\Gamma(\frac34+\nu+\frac12\mu)\,\Gamma(\frac34+\nu-\frac12\mu)}$ $\lvert\operatorname{Re}\mu\rvert < \frac12, \quad 2\operatorname{Re}\nu > \lvert\operatorname{Re}\mu\rvert - \frac12$
(3)	$\int_0^\infty [J_0(ax)\, Y_1(bx) + 2\pi^{-1} I_0(ax)\, K_1(bx)]\, dx = 0 \qquad 0<a<b$
(4)	$\int_0^\infty x^\rho [Y_\mu(ax) \pm 2\pi^{-1} K_\mu(ax)]\,[Y_\nu(bx) \pm 2\pi^{-1} K_\mu(bx)]\, dx$ See Dixon, A.L. and W.L. Ferrar, 1930: *Quart. J. Math.* Oxford Ser. 1, 122-145.
(5)	$\int_0^\infty \sinh(cx)\, K_1(ax)\, J_0(bx)\, dx, \qquad \int_0^\infty \cosh(cx)\, K_0(ax)\, J_0(bx)\, dx$ See Watson, G.N., 1928: *J. London Math. Soc.* 3, 22-27.
(6)	$\int_0^\infty x\, J_0(ax)\, I_0(\beta x)\, K_0(\gamma x)\, dx = [(a^2+\beta^2+\gamma^2)^2 - 4\beta^2\gamma^2]^{-\frac12}$ $\operatorname{Re}\gamma > \lvert\operatorname{Im} a\rvert + \lvert\operatorname{Re}\beta\rvert$
(7)	$\int_0^\infty x\, J_0(ax)\, I_1(\beta x)\, K_1(\gamma x)\, dx = \frac{1}{2\beta\gamma}\{(a^2+\beta^2+\gamma^2)$ $\times [(a^2+\beta^2+\gamma^2)^2 - 4\beta^2\gamma^2]^{\frac12} - 1\} \qquad \operatorname{Re}\gamma > \lvert\operatorname{Im} a\rvert + \lvert\operatorname{Re}\beta\rvert$

Modified functions of other arguments (cont'd)

(8)	$\int_0^\infty x^{\lambda-1} J_\mu(ax) J_\nu(\beta x) K_\rho(\gamma x)\,dx = \dfrac{2^{\lambda-2} a^\mu \beta^\nu \gamma^{-\lambda-\mu-\nu}}{\Gamma(\mu+1)\Gamma(\nu+1)}$ $\times \Gamma\left(\dfrac{\lambda+\mu+\nu-\rho}{2}\right)\Gamma\left(\dfrac{\lambda+\mu+\nu+\rho}{2}\right)$ $\times F_4\left(\dfrac{\lambda+\mu+\nu-\rho}{2}, \dfrac{\lambda+\mu+\nu+\rho}{2}; \mu+1, \nu+1; -\dfrac{a^2}{\gamma^2}, -\dfrac{\beta^2}{\gamma^2}\right)$ $\mathrm{Re}(\lambda+\mu+\nu) > \lvert\mathrm{Re}\,\rho\rvert, \quad \mathrm{Re}\,\gamma > \lvert\mathrm{Im}\,a\rvert + \lvert\mathrm{Im}\,\beta\rvert$
(9)	$\int_0^\infty x\,[J_0(ax) K_0(bx)]^2\,dx = \dfrac{\pi}{8ab} - \dfrac{1}{4ab}\sin^{-1}\left(\dfrac{b^2-a^2}{b^2+a^2}\right)$ $a, b > 0$
(10)	$\int_0^\infty J_\nu(ax) J_\nu(bx) K_\nu(ax) K_\nu(bx)\, x^{2\nu+1}\,dx$ $= \dfrac{2^{\nu-3} a^{2\nu}\Gamma(\frac{\nu+1}{2})\,\Gamma(\nu+\frac{1}{2})\,\Gamma(\frac{3\nu+1}{2})}{b^{4\nu+2}\pi^{1/2}\,\Gamma(\nu+1)}$ $\times {}_2F_1\left(\nu+\dfrac{1}{2}, \dfrac{3\nu+1}{2};\ 2\nu+1; 1-\dfrac{a^4}{b^4}\right)$ $0 < a < b, \quad \mathrm{Re}\,\nu > -1/3$
	For other similar integrals see sec. 6.8.
(11)	$\int_0^\infty x^{-\mu} e^x P_\nu^{2\mu}(1+2x/a) K_\mu(x+a)\,dx$ $= \pi^{-1/2} 2^{\mu-1}\cos(\mu\pi)\,\Gamma(\mu+\nu+\tfrac{1}{2})\Gamma(\mu-\nu+\tfrac{1}{2})\, W_{\frac{1}{2}-\mu,\frac{1}{2}+\nu}(2a)$ $\lvert\arg a\rvert < \pi, \quad \mathrm{Re}\,\mu > \lvert\mathrm{Re}\,\nu+\tfrac{1}{2}\rvert$

Modified functions of other arguments (cont'd)

(12)	$\int_0^\infty x^{-\frac{1}{2}\mu}(x+a)^{-\frac{1}{2}}\, e^{-x}\, \mathrm{P}^{\mu}_{\nu-\frac{1}{2}}\left(\frac{a-x}{a+x}\right) K_\nu(a+x)\,dx$ $= (\frac{1}{2}\pi)^{\frac{1}{2}}\, a^{-\frac{1}{2}\mu}\,\Gamma(\mu,\, 2a)$ $\qquad a>0, \quad \operatorname{Re}\mu<1$
(13)	$\int_0^\infty x^{\mu-1}(x+\beta)^{-\mu}\, I_\mu(x+\beta)\, K_\nu(x)\,dx$ See MacRobert, T.M., 1950: *Functions of a complex variable.* Macmillan, p. 379.
(14)	$\int_0^\infty x^{\mu-1}\, \|x-b\|^{-\mu}\, K_\mu(\|x-b\|)\, K_\nu(x)\,dx$ $= \pi^{-\frac{1}{2}}(2b)^{-\mu}\,\Gamma(\frac{1}{2}-\mu)\,\Gamma(\mu+\nu)\,\Gamma(\mu-\nu)\,K_\nu(b)$ $b>0, \quad \operatorname{Re}\mu<\frac{1}{2}, \quad \operatorname{Re}\mu>\|\operatorname{Re}\nu\|$
(15)	$\int_0^\infty x^{\mu-1}(x+\beta)^{-\mu}\, K_\mu(x+\beta)\, K_\nu(x)\,dx = \frac{\pi^{\frac{1}{2}}\,\Gamma(\mu+\nu)\,\Gamma(\mu-\nu)}{2^\mu\,\beta^\mu\,\Gamma(\mu+\frac{1}{2})}\, K_\nu(\beta)$ $\|\arg\beta\|<\pi, \quad \operatorname{Re}\mu>\|\operatorname{Re}\nu\|$
(16)	$\int_0^\infty x^{1+2\nu}\, J_{2\nu-1}(2ax)\, K_{2\nu-1}(2ax)\, J_\nu(x^2)\,dx$ $= \pi^{-1/2}\, 2^{\nu-2}\, a^{2\nu-1}\, K_{\nu-\frac{1}{2}}(2a^2)$ $\qquad \|\arg a\|<\frac{1}{4}\pi, \quad \operatorname{Re}\nu>0$
(17)	$\int_0^\infty x^{1-2\nu}\, J_{2\nu+1}(2ax)\, K_{2\nu+1}(2ax)\, J_\nu(x^2)\,dx$ $= \pi^{\frac{1}{2}}\, 2^{-\nu-3}\, a^{-2\nu-1}\csc(\nu\pi)\,[I_{\nu+\frac{1}{2}}(2a^2) - \mathbf{L}_{\nu+\frac{1}{2}}(2a^2)]$ $\|\arg a\|<\frac{1}{4}\pi, \quad \operatorname{Re}\nu>-1$
(18)	$\int_0^\infty x^{1-2\nu}\, Y_{2\nu+1}(2ax)\, K_{2\nu+1}(2ax)\, J_\nu(x^2)\,dx = \pi^{\frac{1}{2}}\, 2^{-\nu-3}\, a^{-2\nu-1}\operatorname{ctn}(\nu\pi)$ $\times\,[I_{\nu+\frac{1}{2}}(2a^2) - \mathbf{L}_{\nu+\frac{1}{2}}(2a^2) + 2\pi^{-1}\sec(\nu\pi)\,K_{\nu+\frac{1}{2}}(2a^2)]$ $\|\arg a\|<\frac{1}{4}\pi, \quad -1<\operatorname{Re}\nu<0$

Modified functions of other arguments (cont'd)

(19)	$\int_0^\infty x^{3-2\nu} K_{2\nu}(2ax)\,[\cos(\nu\pi)\,J_{2\nu}(2ax) - \sin(\nu\pi)\,Y_{2\nu}(2ax)]\,J_\nu(x^2)\,dx$ $= \pi^{-1/2}\,2^{-\nu-1}\,a^{1-2\nu}\,K_{\nu+1/2}(2a^2)$ $\quad |\arg a| < \tfrac{1}{4}\pi, \quad -1 < \mathrm{Re}\,\nu < 1$
(20)	$\int_0^\infty x\,J_\nu(x^2)\,[\sin(\nu\pi)\,J_\nu(x^2) - \cos(\nu\pi)\,Y_\nu(x^2)]\,J_{4\nu}(4ax)\,dx$ $= \tfrac{1}{4}\,J_\nu(a^2)\,J_{-\nu}(a^2)$ $\quad a > 0, \quad \mathrm{Re}\,\nu > -1$
(21)	$\int_0^\infty x^{\rho-1} K_\mu(ax)\,J_\nu(bx^{-1})\,dx = 2^{\rho-2}\,a^{-\rho}\,G_{04}^{30}\left(\frac{a^2b^2}{16}\middle\vert \frac{\nu}{2}, \frac{\rho+\mu}{2}, \frac{\rho-\mu}{2}, -\frac{\nu}{2}\right)$ $\quad b > 0, \quad \mathrm{Re}\,a > 0, \quad \mathrm{Re}\,\rho > \lvert\mathrm{Re}\,\mu\rvert - 3/2$
(22)	$\int_0^\infty x^{\rho-1} K_\mu(ax)\,Y_\nu(bx^{-1})\,dx$ $= (-1)^{m+1}\,2^{\rho-2}\,a^{-\rho}\,G_{15}^{40}\left(\frac{a^2b^2}{16}\middle\vert \begin{matrix} \frac{1-\nu}{2} - m \\ \frac{\nu}{2}, -\frac{\nu}{2}, \frac{\rho+\nu}{2}, \frac{\rho-\mu}{2}, \frac{1-\nu}{2} - m \end{matrix}\right)$ $\quad m \text{ integer}, \quad b > 0, \quad \mathrm{Re}\,a > 0, \quad \mathrm{Re}\,\rho > \lvert\mathrm{Re}\,\mu\rvert - 3/2$
(23)	$\int_0^\infty K_\nu(ax)\,K_\nu(\beta x^{-1})\,dx = \pi a^{-1}\,K_{2\nu}(2a^{1/2}\,\beta^{1/2})$ $\quad \mathrm{Re}\;a > 0, \quad \mathrm{Re}\,\beta > 0$
(24)	$\int_0^\infty x^{2\nu-1/2}\,K_{1/2-\nu}(ax)\,K_\nu(\beta x^{-1})\,dx$ $= (2\pi)^{1/2}\,a^{-\nu-1/2}\,\beta^\nu\,K_{2\nu}[(2a\beta)^{1/2}\,e^{\frac{1}{4}\pi i}]\,K_{2\nu}[(2a\beta)^{1/2}\,e^{-\frac{1}{4}\pi i}]$ $\quad \mathrm{Re}\,a > 0, \quad \mathrm{Re}\,\beta > 0$
(25)	$\int_0^\infty x^{\rho-1}\,K_\mu(ax)\,K_\nu(\beta x^{-1})\,dx$ $= 2^{\rho-3}\,a^{-\rho}\,G_{04}^{40}\left(\frac{a^2\beta^2}{16}\middle\vert \frac{\nu}{2}, -\frac{\nu}{2}, \frac{\rho+\mu}{2}, \frac{\rho-\mu}{2}\right)$ $\quad \mathrm{Re}\,a > 0, \quad \mathrm{Re}\,\beta > 0$

Modified functions of other arguments (cont'd)

(26)	$\int_0^\infty x^{-2}[K_\nu(ax)]^2 J_0(bx^{-1})\,dx = -2\pi b^{-1} K_{2\nu}(2a^{1/2} b^{1/2})$ $\times[\sin(\nu\pi) J_{2\nu}(2a^{1/2} b^{1/2}) + \cos(\nu\pi) Y_{2\nu}(2a^{1/2} b^{1/2})]$ $b>0,\quad \mathrm{Re}\, a>0,\quad -\tfrac14<\mathrm{Re}\,\nu<\tfrac14$
(27)	$\int_0^a x^{\mu+1} y^{-\mu-2} J_\mu(x) I_\nu(y)\,dx = \dfrac{\Gamma(\tfrac12\nu-\tfrac12\mu)(\tfrac12 a)^\mu}{\Gamma(\tfrac12\nu+\tfrac12\mu+1)} J_\nu(a)$ $y=(a^2-x^2)^{1/2},\quad \mathrm{Re}\,\nu>\mathrm{Re}\,\mu>-1$
(28)	$\int_0^a x^{2\nu} y^{-2\nu-1} J_{2\nu-1}(2x) K_{2\mu}(2y)\,dx = -\tfrac14\Gamma(\tfrac12+\mu-\nu)\Gamma(\tfrac12-\mu-\nu)$ $\times a^{2\nu-1}\{\sin[(\mu-\nu)\pi] J_{2\mu}(2a) + \cos[(\mu-\nu)\pi] Y_{2\mu}(2a)\}$ $y=(a^2-x^2)^{1/2},\quad 0<\mathrm{Re}\,\nu<\tfrac12-\lvert\mathrm{Re}\,\mu\rvert$
(29)	$\int_0^a x^{1-3\nu} y^{2\nu-1} J_{-3\nu}(2x) I_\nu(y) I_{\nu-1}(y)\,dx = \dfrac{\Gamma(\nu+\tfrac12)}{2\pi^{1/2} a^{2\nu}} J_\nu(a) J_{-\nu}(a)$ $y=(a^2-x^2)^{1/2},\quad 0<\mathrm{Re}\,\nu<1/3$
(30)	$\int_0^a x^{1-2\nu} y^{2\nu-3/2} I_{-\nu}(x) K_\nu(x) J_{2\nu-3/2}(2y)\,dx$ $= -\tfrac14\Gamma(\tfrac12-\nu) a^{\nu-1} Y_\nu(2a) \qquad y=(a^2-x^2)^{1/2},\quad \tfrac14<\mathrm{Re}\,\nu<\tfrac12$
(31)	$\int_0^a x J_\lambda(2x) I_\lambda(2x) J_\mu(2y) I_\mu(2y)\,dx = \dfrac{a^{2\lambda+2\mu+2}}{2\Gamma(\lambda+1)\Gamma(\mu+1)\Gamma(\lambda+\mu+2)}$ $\times {}_1F_4\left(\dfrac{\lambda+\mu+1}{2};\lambda+1,\mu+1,\lambda+\mu+1,\dfrac{\lambda+\mu+3}{2};-a^4\right)$ $y=(a^2-x^2)^{1/2},\quad \mathrm{Re}\,\lambda,\ \mathrm{Re}\,\mu>-1$

Modified functions of other arguments (cont'd)

(32)	$\int_0^a x^{\kappa-\frac{1}{2}}(a-x)^{-\kappa-\frac{1}{2}} e^{-x\sinh t} I_{2\mu}[x^{\frac{1}{2}}(a-x)^{\frac{1}{2}}]\,dx$ $= \dfrac{2\Gamma(\frac{1}{2}+\kappa+\mu)\,\Gamma(\frac{1}{2}-\kappa+\mu)}{a\,[\Gamma(2\mu+1)]^2} M_{\kappa,\mu}(\frac{1}{2}ae^{t})\,M_{-\kappa,\mu}(\frac{1}{2}ae^{-t})$ $\operatorname{Re}\mu > \lvert\operatorname{Re}\kappa\rvert - \frac{1}{2}$
(33)	$\int_0^\infty x^{\nu-1}(a^2+x^2)^{-\frac{1}{2}\nu} K_\nu[(a^2+x^2)^{\frac{1}{2}}]\,K_\mu(x)\,dx$ $= 2^{\nu-2}a^{-\nu}\,\Gamma\left(\dfrac{\nu-\mu}{2}\right)\Gamma\left(\dfrac{\nu+\mu}{2}\right)K_\mu(a)$ $\operatorname{Re}\nu > \lvert\operatorname{Re}\mu\rvert, \quad \operatorname{Re}a > 0$
(34)	$\int_0^\infty x^{\nu-1} y^{-\nu} K_\nu(y)\,K_\mu(x)\,dx = 2^{-\frac{1}{2}}\pi^{\frac{1}{2}}\beta^{\frac{1}{2}-\nu}c^{-\frac{1}{2}}\,\Gamma(\nu+\mu)\,\Gamma(\nu-\mu)$ $\times P_{\mu-\frac{1}{2}}^{\frac{1}{2}-\nu}(a/c)\,K_\mu(c) \qquad y = [(x+a)^2+\beta^2]^{\frac{1}{2}}, \quad c = (a^2+\beta^2)^{\frac{1}{2}}$ $\operatorname{Re}\beta > \lvert\operatorname{Im}a\rvert, \quad \operatorname{Re}\nu > \lvert\operatorname{Re}\mu\rvert$
(35)	$\int_0^\infty x^{-\kappa-\frac{1}{2}}(a+x)^{\kappa-\frac{1}{2}} K_{2\mu}[x^{\frac{1}{2}}(a+x)^{\frac{1}{2}}]\,dx$ $= a^{-1}\,\Gamma(\frac{1}{2}-\kappa+\mu)\,\Gamma(\frac{1}{2}-\kappa-\mu)\,W_{\kappa,\mu}(\frac{1}{2}ae^{i\pi/2})\,W_{\kappa,\mu}(\frac{1}{2}ae^{-i\pi/2})$ $\lvert\arg a\rvert < \pi, \quad \operatorname{Re}\kappa + \lvert\operatorname{Re}\mu\rvert < \frac{1}{2}$
(36)	$\int_0^\infty x^{-\frac{1}{2}}(a+x)^{-\frac{1}{2}} e^{-x\cosh t} K_\nu[x^{\frac{1}{2}}(a+x)^{\frac{1}{2}}]\,dx$ $= \frac{1}{2}\sec(\frac{1}{2}\nu\pi)\,e^{\frac{1}{2}a\cosh t} K_{\frac{1}{2}\nu}(\frac{1}{4}ae^{t})\,K_{\frac{1}{2}\nu}(\frac{1}{4}ae^{-t}) \qquad -1 < \operatorname{Re}\nu < 1$
(37)	$\int_0^\infty x^{-\kappa-\frac{1}{2}}(a+x)^{\kappa-\frac{1}{2}} \exp(-\beta x)\,K_{2\mu}[x^{\frac{1}{2}}(a+x)^{\frac{1}{2}}]\,dx$ $= a^{-1}e^{\frac{1}{2}a\beta}\,\Gamma(\frac{1}{2}-\kappa+\mu)\,\Gamma(\frac{1}{2}-\kappa-\mu)\,W_{\kappa,\mu}(z_1)\,W_{\kappa,\mu}(z_2)$ $\lvert\arg a\rvert < \pi, \quad \operatorname{Re}\beta > -1, \quad \operatorname{Re}\kappa + \lvert\operatorname{Re}\mu\rvert < \frac{1}{2}$ $z_1, z_2 = \frac{1}{2}a[\beta \pm (\beta^2-1)^{\frac{1}{2}}]$

Modified functions of other arguments (cont'd)

(38)	$\int_0^\infty x^{\rho-1}\left(\frac{a+\beta x}{ax+\beta}\right)^\nu K_{2\nu}[x^{-1/2}(a+\beta x)^{1/2}(ax+\beta)^{1/2}]\,dx$ $= 2K_{\nu+\rho}(a)\,K_{\nu-\rho}(\beta)$ $\qquad \operatorname{Re} a>0, \quad \operatorname{Re}\beta>0$
(39)	$\int_0^{\frac{1}{2}\pi} \cos[(\mu-\nu)x]\, I_{\mu+\nu}(2a\cos x)\,dx = \tfrac{1}{2}\pi\, I_\mu(a)\, I_\nu(a)$ $\operatorname{Re}(\mu+\nu)>-1$
(40)	$\int_0^{\frac{1}{2}\pi} \cos[(\mu-\nu)x]\, K_{\mu+\nu}(2a\cos x)\,dx = \tfrac{1}{2}\pi\csc[(\mu+\nu)\pi]$ $\times[I_{-\mu}(a)\, I_{-\nu}(a) - I_\mu(a)\, I_\nu(a)]$ $\qquad -1<\operatorname{Re}(\mu+\nu)<1$
(41)	$\int_0^{\frac{1}{2}\pi} \sec x\cos(2\kappa x)\, K_{2\mu}(a\sec x)\,dx = \frac{\pi}{2a} W_{\kappa,\mu}(a)\, W_{-\kappa,\mu}(a)$ $\operatorname{Re} a>0$
(42)	$\int_0^\infty \cosh(2\mu x)\, K_{2\nu}(2a\cosh x)\,dx = \tfrac{1}{2}K_{\mu+\nu}(a)\, K_{\mu-\nu}(a)$ $\qquad \operatorname{Re} a>0$
(43)	$\int_0^\infty \operatorname{sech} x\cosh(2\kappa x)\, I_{2\mu}(a\operatorname{sech} x)$ $= \frac{\Gamma(\frac{1}{2}+\kappa+\mu)\,\Gamma(\frac{1}{2}-\kappa+\mu)}{2a\,[\Gamma(2\mu+1)]^2} M_{\kappa,\mu}(a)\, M_{-\kappa,\mu}(a)$ $\qquad \lvert\operatorname{Re}\kappa\rvert - \operatorname{Re}\mu < \tfrac{1}{2}$
(44)	$\int_0^\infty (\sinh x)^{\mu+1}(\cosh x)^{-2\mu-3/2}\, P_\nu^{-\mu}[\cosh(2x)]\, I_{\mu-\frac{1}{2}}(a\operatorname{sech} x)\,dx$ $= \frac{2^{\mu-\frac{1}{2}}\,\Gamma(\mu-\nu)\,\Gamma(\mu+\nu+1)}{\pi^{1/2}\, a^{\mu+3/2}\,[\Gamma(\mu+1)]^2} M_{\nu+1/2,\mu}(a)\, M_{-\nu-1/2,\mu}(a)$ $\operatorname{Re}\mu>\operatorname{Re}\nu, \quad \operatorname{Re}\mu>-\operatorname{Re}\nu-1$

Modified functions of other arguments (cont'd)

(45)	$\int_{-\infty}^{\infty} e^{\rho x} \left(\frac{\alpha+\beta e^{x}}{\alpha e^{x}+\beta}\right)^{\nu} K_{2\nu}[(\alpha^2+\beta^2+2\alpha\beta \cosh x)^{1/2}]\, dx$ $= 2K_{\nu+\rho}(\alpha)\, K_{\nu-\rho}(\beta)$ Re α, Re $\beta > 0$

19.7. Bessel functions and modified Bessel functions of variable order

(1)	$\int_{-\infty}^{\infty} J_{\nu-x}(a)\, J_{\mu+x}(a)\, dx = J_{\mu+\nu}(2a)$ $\mathrm{Re}(\mu+\nu) > 1$
(2)	$\int_{-\infty}^{\infty} a^{-\mu-x}\, b^{-\nu+x}\, e^{cxi}\, J_{\mu+x}(a)\, J_{\nu-x}(b)\, dx$ $= \left[\frac{2\cos(\frac{1}{2}c)}{a^2 e^{-\frac{1}{2}ci}+b^2 e^{\frac{1}{2}ci}}\right]^{\frac{1}{2}\mu+\frac{1}{2}\nu} \exp[\tfrac{1}{2}c(\nu-\mu)i]$ $\times J_{\mu+\nu}\{[2\cos(\tfrac{1}{2}c)\,(a^2 e^{-\frac{1}{2}ci}+b^2 e^{\frac{1}{2}ci})]^{\frac{1}{2}}\}$ $-\pi < c < \pi$ $= 0$ $c \geq \pi$ or $c \leq -\pi$ $\mathrm{Re}(\mu+\nu) > 1$
(3)	$\int_{-\infty}^{\infty} J_{\kappa+x}(a)\, J_{\lambda-x}(a)\, J_{\mu+x}(a)\, J_{\nu-x}(a)\, dx$ $= \frac{\Gamma(\kappa+\lambda+\mu+\nu+1)}{\Gamma(\kappa+\lambda+1)\,\Gamma(\lambda+\mu+1)\,\Gamma(\mu+\nu+1)\,\Gamma(\nu+\kappa+1)}$ $\times {}_4F_5\Big(\frac{\kappa+\lambda+\mu+\nu+1}{2}, \frac{\kappa+\lambda+\mu+\nu+1}{2}, \frac{\kappa+\lambda+\mu+\nu}{2}+1, \frac{\kappa+\lambda+\mu+\nu}{2}+1;$ $\kappa+\lambda+\mu+\nu+1, \kappa+\lambda+1, \lambda+\mu+1, \mu+\nu+1, \nu+\kappa+1; -4a^2\Big)$ $\mathrm{Re}(\kappa+\lambda+\mu+\nu) > -1$

For similar integrals see Vol. I, p. 59 ff. and p. 123 ff.

Variable order (cont'd)

(4)	$\int_0^\infty J_x(xz)\, J_{-x}(xz) \cos(\pi x)\, dx = \tfrac{1}{4}(1-z^2)^{-\frac{1}{2}}$	$\lvert z\rvert < 1$
(5)	$\int_0^\infty [J_x(xz)\, J_{-x}(xz) \cos(\pi x) - 1]\, x^{-2}\, dx = -\tfrac{1}{2}\pi^2$	
(6)	$\int_{-\infty}^\infty \operatorname{sech}(\tfrac{1}{2}\pi x)\, J_{ix}(a)\, dx = 2 \sin a$	$a > 0$
(7)	$-\!\!\!\!\!\!\int_{-\infty}^\infty \operatorname{csch}(\tfrac{1}{2}\pi x)\, J_{ix}(a)\, dx = -2i \cos a$	$a > 0$
(8)	$-\!\!\!\!\!\!\int_{-\infty}^\infty \dfrac{e^{\frac{1}{2}\pi x} \cos(bx)}{\sinh(\pi x)}\, J_{ix}(a)\, dx = -i \exp(ia \cosh b)$	$a,\ b > 0$
(9)	$\int_{-\infty}^\infty e^{-cxi}[J_{\nu-ix}(a)\, Y_{\nu+ix}(b) + Y_{\nu-ix}(a)\, J_{\nu+ix}(b)]\, dx$ $= -2(h/k)^{2\nu} J_{2\nu}(hk)$	$a,\ b > 0,\quad c$ real; $h = (ae^{\frac{1}{2}c} + be^{-\frac{1}{2}c})^{\frac{1}{2}},\quad k = (ae^{-\frac{1}{2}c} + be^{\frac{1}{2}c})^{\frac{1}{2}}$
(10)	$\int_{-\infty}^\infty e^{-cxi}[J_{\nu-ix}(a)\, J_{\nu+ix}(b) - Y_{\nu-ix}(a)\, Y_{\nu+ix}(b)]\, dx$ $= 2(h/k)^{2\nu}\, Y_{2\nu}(hk)$	$a,\ b > 0,\quad c$ real; $h = (ae^{\frac{1}{2}c} + be^{-\frac{1}{2}c}),\quad k = (ae^{-\frac{1}{2}c} + be^{\frac{1}{2}c})^{\frac{1}{2}}$
(11)	$\int_{-\infty}^\infty e^{-cxi}\, H^{(2)}_{\nu-ix}(a)\, H^{(2)}_{\nu+ix}(b)\, dx = 2i(h/k)^{2\nu} H^{(2)}_{2\nu}(hk)$	$a,\ b > 0,\quad c$ real, $\quad h = (ae^{\frac{1}{2}c} + be^{-\frac{1}{2}c})^{\frac{1}{2}},\quad k = (ae^{-\frac{1}{2}c} + be^{\frac{1}{2}c})^{\frac{1}{2}}$
(12)	$\int_0^\infty \operatorname{sech}(\pi x)\, \{[J_{ix}(a)]^2 + [Y_{ix}(a)]^2\}\, dx = -Y_0(2a) - \mathbf{E}_0(2a)$	$a > 0$

Variable order (cont'd)

(13)	$\int_0^\infty x e^{\pi x} \tanh(\pi x)\, H_{ix}^{(2)}(a)\, H_{ix}^{(2)}(b)\, dx = -\dfrac{2(ab)^{1/2}}{\pi(a+b)} \exp[-ik(a+b)]$ $a,\ b > 0$
(14)	$\int_0^\infty x e^{\pi x} \sinh(\pi x)\, \Gamma(\nu + ix)\, \Gamma(\nu - ix)\, H_{ix}^{(2)}(a)\, H_{ix}^{(2)}(b)\, dx$ $= i\, 2^\nu \pi^{1/2}\, \Gamma(\tfrac{1}{2} + \nu)\, (ab)^\nu (a+b)^{-\nu} K_\nu(a+b)$ $a,\ b > 0, \quad \operatorname{Re}\nu > 0$
(15)	$\int_0^\infty x e^{\pi x} \sinh(\pi x) \cosh(\pi x)\, \Gamma(\nu + ix)\, \Gamma(\nu - ix)\, H_{ix}^{(2)}(a)\, H_{ix}^{(2)}(b)\, dx$ $= \dfrac{i\pi^{3/2}\, 2^\nu}{\Gamma(\tfrac{1}{2} - \nu)} (b-a)^{-\nu} H_\nu^{(2)}(b-a) \qquad 0 < a < b, \quad 0 < \operatorname{Re}\nu < \tfrac{1}{2}$
(16)	$\int_0^\infty x e^{\pi x} \sinh(\pi x)\, \Gamma\left(\dfrac{\nu + ix}{2}\right) \Gamma\left(\dfrac{\nu - ix}{2}\right) H_{ix}^{(2)}(a)\, H_{ix}^{(2)}(b)\, dx$ $= i\pi\, 2^{2-\nu} (ab)^\nu (a^2 + b^2)^{-\frac{1}{2}\nu} H_\nu^{(2)}[(a^2 + b^2)^{1/2}]$ $a,\ b > 0, \quad \operatorname{Re}\nu > 0$
(17)	$\int_0^\infty x e^{\pi x} \tanh(\pi x)\, P_{-\frac{1}{2} + ix}(-\cos\phi)\, H_{ix}^{(2)}(a)\, H_{ix}^{(2)}(b)\, dx$ $= -\dfrac{2(ab)^{1/2}}{\pi R} e^{-iR}$ $a,\ b > 0, \quad 0 < \phi < \pi, \quad R = (a^2 + b^2 - 2ab\cos\phi)^{1/2}$
(18)	$\int_0^\infty x e^{\pi x} \sinh(\pi x)\, \Gamma(\nu + ix)\, \Gamma(\nu - ix)\, P_{-\frac{1}{2} + ix}^{\frac{1}{2} - \nu}(-\cos\phi)$ $\times H_{ix}^{(2)}(a)\, H_{ix}^{(2)}(b)\, dx = i(2\pi)^{1/2} (\sin\phi)^{\nu - \frac{1}{2}} (ab)^\nu R^{-\nu} H_\nu^{(2)}(R)$ $a,\ b > 0, \quad 0 < \phi < \pi, \quad R = (a^2 + b^2 - 2ab\cos\phi)^{1/2}, \quad \operatorname{Re}\nu > 0$

Variable order (cont'd)

(19)	$\int_0^\infty \cosh(\tfrac{1}{2}\pi x)\, K_{ix}(a)\, dx = \tfrac{1}{2}\pi$	$a > 0$
(20)	$\int_0^\infty x \sinh(\tfrac{1}{2}\pi x)\, K_{ix}(a)\, dx = \tfrac{1}{2}\pi a$	$a > 0$
(21)	$\int_{-\infty}^\infty K_{ix+iy}(\alpha)\, K_{ix+iz}(\beta)\, dx = \pi\, K_{iy-iz}(\alpha+\beta)$	$\lvert\arg\alpha\rvert + \lvert\arg\beta\rvert < \pi$
(22)	$\int_{-\infty}^\infty e^{\pi x} K_{ix+iy}(a)\, K_{ix+iz}(b)\, dx = \pi e^{-\pi z} K_{iy-iz}(a-b)$	$a > b > 0$
(23)	$\int_{-\infty}^\infty e^{i\rho x} K_{\nu+ix}(\alpha)\, K_{\nu-ix}(\beta)\, dx = \pi \left(\dfrac{\alpha+\beta e^{\rho}}{\alpha e^{\rho}+\beta}\right)^{\nu} K_{2\nu}(w)$ $\lvert\arg\alpha\rvert + \lvert\arg\beta\rvert + \lvert\operatorname{Im}\rho\rvert < \pi, \quad w = (\alpha^2+\beta^2+2\alpha\beta\cosh\rho)^{1/2}$	
(24)	$\int_{-\infty}^\infty \exp[(\pi-\gamma)x]\, K_{ix+iy}(a)\, K_{ix+iz}(b)\, dx = \pi e^{-\beta y-\alpha z} K_{iy-iz}(c)$ where $0 < \gamma < \pi$, $a, b, c > 0$, and α, β, γ are angles of the triangle with the sides a, b, c.	
(25)	$\int_{-\infty}^\infty (n+\nu+ix)^{-1} \sin[(\nu+ix)\pi]\, K_{\nu+ix}(a)\, K_{\nu-ix}(b)\, dx$ $= \pi^2 I_n(a)\, K_{n+2\nu}(b)$ $= \pi^2 K_{n+2\nu}(a)\, I_n(b)$	 $0 < a < b$ $0 < b < a$ $n = 0, 1, 2, \ldots$
(26)	$\int_0^\infty \sin(bx) \sinh(\pi x)\, [K_{ix}(a)]^2\, dx = \tfrac{1}{4}\pi^2 J_0\,[2a \sinh(\tfrac{1}{2}b)]$	$a, b > 0$

Variable order (cont'd)

(27)	$\int_0^\infty \cos(bx)\cosh(\pi x)\,[K_{ix}(a)]^2\,dx = -\tfrac{1}{4}\pi^2\,Y_0[2a\sinh(\tfrac{1}{2}b)]$ $a,\ b>0$
(28)	$\int_0^\infty \cosh(\rho x)\,K_{\nu+ix}(a)\,K_{\nu-ix}(a)\,dx = \tfrac{1}{2}\pi\,K_{2\nu}[2a\cos(\tfrac{1}{2}\rho)]$ $2\,\lvert\arg a\rvert + \lvert\mathrm{Re}\,\rho\rvert < \pi$
(29)	$\int_{-\infty}^\infty (\nu-\tfrac{1}{2}+ix)\Gamma(\tfrac{1}{2}-ix)\Gamma(2\nu-\tfrac{1}{2}+ix)\,P^{\frac{1}{2}-\nu}_{\nu+ix-1}(\cos\phi)$ $\times\, I_{\nu-\frac{1}{2}+ix}(a)\,K_{\nu-\frac{1}{2}+ix}(b)\,dx = (2\pi)^{\frac{1}{2}}(\sin\phi)^{\nu-\frac{1}{2}}(ab/\omega)^\nu\,K_\nu(\omega)$ $\omega = (a^2+b^2+2ab\cos\phi)^{\frac{1}{2}}$
	See also Chapter XII for similar integrals.

19.8. Functions related to Bessel functions

(1)	$\int_0^\infty x\,J_\nu(ax)\,[\mathbf{J}_\nu(x) - J_\nu(x)]\,dx = \dfrac{\sin(\nu\pi)}{\pi a(a+1)}$ $\quad a>0,\quad \mathrm{Re}\,\nu>-1$
(2)	$\int_0^\infty x^{-\nu-1}\,\mathbf{H}_\nu(x)\,dx = \dfrac{2^{-\nu-1}\pi}{\Gamma(\nu+1)}$ $\quad \mathrm{Re}\,\nu>-3/2$
(3)	$\int_{-\infty}^\infty x^{-\nu-1}\,\dfrac{\sin[a(x+\lambda)]}{x+\lambda}\,\mathbf{H}_\nu(x)\,dx = \pi\lambda^{-\nu-1}\,\mathbf{H}_\nu(\lambda)$ $a\geq 1,\quad \mathrm{Re}\,\nu>5/2$
(4)	$\int_0^\infty x^{1-\mu-\nu}\,J_\nu(x)\,\mathbf{H}_\mu(x)\,dx = \dfrac{(2\nu-1)\,2^{-\mu-\nu}}{(\mu+\nu-1)\,\Gamma(\mu+\tfrac{1}{2})\,\Gamma(\nu+\tfrac{1}{2})}$ $\mathrm{Re}\,\nu>\tfrac{1}{2},\quad \mathrm{Re}(\mu+\nu)>1$

Related functions (cont'd)

(5)	$\int_0^\infty [\cos(\tfrac{1}{2}\nu\pi) J_\nu(x) + \sin(\tfrac{1}{2}\nu\pi)\,\mathbf{H}_\nu(x)] \dfrac{dx}{x^2+a^2}$ $= \dfrac{\pi}{2a} [I_\nu(a) - \mathbf{L}_\nu(a)]$ $\quad$ $\operatorname{Re} a > 0, \quad -\tfrac{1}{2} < \operatorname{Re}\nu < 2$
(6)	$\int_a^\infty x^{\frac{1}{2}} (x^2 - a^2)^{-\frac{1}{4}-\frac{1}{2}\nu} P_\mu^{\nu+\frac{1}{2}}(2x^2a^{-2} - 1) [\mathbf{H}_\nu(x) - Y_\nu(x)]\, dx$ $= 2^{-\nu-2} \pi^{\frac{1}{2}} a \csc(\mu\pi) \cos(\nu\pi) \{[Y_\nu(\tfrac{1}{2}a)]^2 - [J_\nu(\tfrac{1}{2}a)]^2\}$ $-1 < \operatorname{Re}\mu < 0, \quad \operatorname{Re}\nu < \tfrac{1}{2}$
(7)	$\int_0^\infty x^{\rho-1} K_\mu(ax) K_\nu(ax)\,\mathbf{H}_\lambda(x)\, dx$ See Mohan, B., 1942: *Bull. Calcutta Math. Soc.* 34, 55-59.
	For other integrals involving Bessel functions and Struve functions see McLachlan, N.W. and A.L. Meyers, 1936: *Philos. Mag.* 21, 425-448.
(8)	$\int_0^\infty x^{-\mu-\nu}\,\mathbf{H}_\mu(x)\,\mathbf{H}_\nu(x)\, dx = \dfrac{2^{-\mu-\nu}\pi^{\frac{1}{2}}\Gamma(\mu+\nu)}{\Gamma(\mu+\frac{1}{2})\Gamma(\nu+\frac{1}{2})\Gamma(\mu+\nu+\frac{1}{2})}$ $\operatorname{Re}(\mu+\nu) > 0$
(9)	$\int_0^{\frac{1}{2}\pi} \cos[(\nu+1)x]\,\mathbf{H}_\nu(a\cos x)\, dx = \pi^{\frac{1}{2}} a^{-\frac{1}{2}} \sin(\tfrac{1}{2}a) J_{\nu+\frac{1}{2}}(\tfrac{1}{2}a)$ $\operatorname{Re}\nu > -2$
(10)	$\displaystyle\int_0^{\frac{1}{2}\pi} \frac{\cos(2\nu x)}{\cos x} [\mathbf{H}_{2\nu}(a\sec x) - Y_{2\nu}(a\sec x)]\, dx$ $= \dfrac{\pi^{\frac{1}{2}} a^{-1}}{\Gamma(2\nu+\frac{1}{2})} W_{\nu,\nu}(ae^{i\pi/2})\, W_{\nu,\nu}(ae^{-i\pi/2})$ $\quad$ $\operatorname{Re}\nu < \tfrac{1}{2}$

Related functions (cont'd)

(11)	$\int_0^\infty \exp[(\nu+1)x]\,\mathbf{H}_\nu(a\sinh x)\,dx = \pi^{1/2} a^{-1/2}\csc(\nu\pi)$ $\times[\sinh(\tfrac12 a)\,I_{\nu+1/2}(\tfrac12 a) - \cosh(\tfrac12 a)\,I_{-\nu-1/2}(\tfrac12 a)]$ $\qquad \mathrm{Re}\,a>0,\quad -2<\mathrm{Re}\,\nu<0$
(12)	$\int_0^\infty x^{-1}\cos(2ax^{-1})\,[I_0(x)-\mathbf{L}_0(x)]\,dx = 2\,J_0(2a^{1/2})\,K_0(2a^{1/2}) \qquad a>0$
(13)	$\int_0^\infty x^{\nu-1/2}\exp[-(1+a)x]\,K_0(ax)\,\mathbf{L}_\nu(x)\,dx$ $=\dfrac{\pi^{1/2}[\Gamma(\nu+\tfrac12)]^2}{(2a)^{\nu+1/2}\,\Gamma(\nu+1)}\,P_{\nu-1/2}(1+a^{-1}) \qquad \mathrm{Re}\,a>0,\quad \mathrm{Re}\,\nu>-\tfrac12$
(14)	$\int_0^a x\,P_n(1-2x^2a^{-2})\,[I_0(x)-\mathbf{L}_0(x)]\,dx$ $=(-1)^n a\,[I_{2n+1}(a)-\mathbf{L}_{2n+1}(a)] \qquad n=0,1,2,\ldots$
(15)	$\int_a^\infty x^{1/2}(x^2-a^2)^{-1/4-\frac12\nu}\,P_\mu^{\nu+1/2}(2x^2a^{-2}-1)\,[I_{-\nu}(x)-\mathbf{L}_\nu(x)]\,dx$ $=2^{-\nu-1}\pi^{1/2}a\csc(2\mu\pi)\cos(\nu\pi)\{[I_\nu(\tfrac12 a)]^2-[I_{-\nu}(\tfrac12 a)]^2\}$ $\qquad -1<\mathrm{Re}\,\mu<0,\quad \mathrm{Re}\,\nu<\tfrac12$
(16)	$\int_0^{\frac12\pi}\dfrac{\cos(2\nu x)}{\cos x}\,[I_{2\nu}(a\sec x)-\mathbf{L}_{2\nu}(a\sec x)]\,dx$ $=\dfrac{\pi^{1/2}a^{-1}}{\Gamma(2\nu+1)}\,W_{\nu,\nu}(a)\,M_{-\nu,\nu}(a) \qquad \mathrm{Re}\,\nu<\tfrac12$
(17)	$\int_0^\infty x^{\lambda-1}s_{\mu,\nu}(x)\,dx = \dfrac{\Gamma(\frac{1+\lambda+\mu}{2})\,\Gamma(\frac{1-\lambda-\mu}{2})\,\Gamma(\frac{1+\mu+\nu}{2})\,\Gamma(\frac{1+\mu-\nu}{2})}{2^{2-\lambda-\mu}\,\Gamma(1-\frac{\lambda+\nu}{2})\,\Gamma(1-\frac{\lambda-\nu}{2})}$ $\qquad -\mathrm{Re}\,\mu<\mathrm{Re}\,\lambda+1<5/2$

Related functions (cont'd)

(18)	$\int_0^\infty x^{-\mu-1}\cos(ax)\, s_{\mu,\nu}(x)\,dx$ $= 0 \qquad a > 1$ $= 2^{\mu-\frac{1}{2}}\pi^{\frac{1}{2}}\,\Gamma\left(\frac{\mu+\nu+1}{2}\right)\Gamma\left(\frac{\mu-\nu+1}{2}\right)(1-a^2)^{\frac{1}{2}\mu+\frac{1}{4}}\,\mathrm{P}_{\nu-\frac{1}{2}}^{-\mu-\frac{1}{2}}(a)$ $0 < a < 1$
(19)	$\int_0^\infty x^{-\mu} J_\nu(ax)\, s_{\mu+\nu,\,\mu-\nu+1}(x)\,dx$ $= 0 \qquad a > 1$ $= 2^{\nu-1}\Gamma(\nu)\,a^{-\nu}(1-a^2)^{\mu} \qquad 0 < a < 1$ $\mathrm{Re}\,\nu > -1$
(20)	$\int_0^\infty x^\nu K_{1-2\nu}(ax^{\frac{1}{2}}e^{i\pi/4})\,K_{1-2\nu}(ax^{\frac{1}{2}}e^{-i\pi/4})\,s_{\mu,\nu}(x)\,dx$ $= 2^{2\mu+2\nu-2}\pi^{-\frac{1}{2}}\,\Gamma\left(\frac{1+\mu+\nu}{2}\right)\Gamma\left(\frac{1+\mu-\nu}{2}\right)$ $\times\Gamma\left(\frac{\mu+3\nu+1}{2}\right)\Gamma\left(\frac{\mu+\nu}{2}+1\right)\, S_{-\mu-2\nu-\frac{1}{2},\,\frac{1}{2}-\nu}(a)$ $\lvert\arg a\rvert < \frac{1}{4}\pi, \quad \mathrm{Re}(\mu-\nu) > -3, \quad \mathrm{Re}(\mu+3\nu) > -1$
(21)	$\int_0^{\frac{1}{2}\pi}\cos[(\mu+1)x]\, s_{\mu,\nu}(a\cos x)\,dx = 2^{\mu-2}\pi\,\Gamma\left(\frac{\mu+\nu+1}{2}\right)\Gamma\left(\frac{\mu-\nu+1}{2}\right)$ $\times J_{\frac{1}{2}(\mu+\nu+1)}(\tfrac{1}{2}a)\,J_{\frac{1}{2}(\mu-\nu+1)}(\tfrac{1}{2}a) \qquad \mathrm{Re}\,\mu > -2$
(22)	$\int_0^\infty \exp[(\mu+1)x]\, s_{\mu,\nu}(a\sinh x)\,dx = 2^{\mu-2}\pi\csc(\mu\pi)\,\Gamma(\rho)\,\Gamma(\sigma)$ $\times[I_\rho(\tfrac{1}{2}a)\,I_\sigma(\tfrac{1}{2}a) - I_{-\rho}(\tfrac{1}{2}a)\,I_{-\sigma}(\tfrac{1}{2}a)]$ $2\rho = \mu+\nu+1, \quad 2\sigma = \mu-\nu+1, \quad a > 0, \quad -2 < \mathrm{Re}\,\mu < 0$

Related functions (cont'd)

(23)	$\int_0^\infty x^{-\mu} \sin(ax)\, S_{\mu,\nu}(x)\, dx = 2^{-\mu-1/2}\, \pi^{1/2}\, \Gamma\left(1-\frac{\mu+\nu}{2}\right) \Gamma\left(1-\frac{\mu-\nu}{2}\right)$ $\times (a^2-1)^{\frac{1}{2}\mu-\frac{1}{4}}\, P_{\nu-1/2}^{\mu-1/2}(a)$ $\qquad a > 1, \quad \mathrm{Re}\,\mu < 1 - \lvert\mathrm{Re}\,\nu\rvert$
(24)	$\int_0^a x^{\frac{1}{2}(\nu-\mu-1)} (a^2-x^2)^{\frac{1}{4}(\nu-\mu-2)}\, \mathrm{P}_{\nu-1/2}^{\frac{1}{2}(\mu-\nu+2)}(x/a)\, S_{\mu,\nu}(x)\, dx$ $= 2^{\mu-3/2}\, \pi^{1/2}\, a^{(\nu-\mu)/2}\, \Gamma\left(\frac{\mu+\nu+3}{4}\right) \Gamma\left(\frac{\mu-3\nu+3}{4}\right) \cos[\tfrac{1}{2}(\mu-\nu)\pi]$ $\times [J_\nu(\tfrac{1}{2}a)\, Y_{-\frac{1}{2}(\mu-\nu+1)}(\tfrac{1}{2}a) - Y_\nu(\tfrac{1}{2}a)\, J_{-\frac{1}{2}(\mu-\nu+1)}(\tfrac{1}{2}a)]$ $\mathrm{Re}(\mu-\nu) < 0, \quad -1 < \mathrm{Re}(\mu+\nu) < 1, \quad \mathrm{Re}(\mu-3\nu) < 1$
(25)	$\int_a^\infty x^{1/2} (x^2-a^2)^{-\frac{1}{2}\beta}\, P_\nu^\beta(x/a)\, S_{\mu,1/2}(x)\, dx$ $= \dfrac{a^{1/2}\, \Gamma(\frac{\beta-\mu+\nu}{2}+\frac{1}{4})\, \Gamma(\frac{\beta-\mu-\nu}{2}-\frac{1}{4})}{\pi^{1/2}\, 2^{3/2-\beta+\mu}\, \Gamma(\frac{1}{2}-\mu)}\, S_{\mu-\beta+1,\, \nu+1/2}(a)$ $\mathrm{Re}\,\beta < 1, \quad \mathrm{Re}(\mu+\nu-\beta) < -\tfrac{1}{2}, \quad \mathrm{Re}(\mu-\nu-\beta) < \tfrac{1}{2}$
(26)	$\int_a^\infty x (x^2-a^2)^{-\frac{1}{2}\nu}\, P_\lambda^\nu(2x^2a^{-2}-1)\, S_{\mu,\nu}(x)\, dx$ $= \dfrac{a\, \Gamma(\frac{\nu-\mu-1}{2}-\lambda)\, \Gamma(\frac{\nu-\mu+1}{2}+\lambda)}{2\, \Gamma(\frac{1-\mu-\nu}{2})\, \Gamma(\frac{1-\mu+\nu}{2})}\, S_{\mu-\nu+1,\, 2\lambda+1}(a)$ $\mathrm{Re}\,\nu < 1, \quad \mathrm{Re}(\mu-\nu+\lambda) < -1, \quad \mathrm{Re}(\mu-\nu-\lambda) < 0$
(27)	$\int_a^\infty x^{-\nu} (x^2-a^2)^{\frac{1}{4}-\frac{1}{2}\nu}\, \mathrm{P}_{\frac{1}{2}\mu-\frac{1}{2}\nu}^{\nu-1/2}(2a^2x^{-2}-1)\, S_{\mu,\nu}(x)\, dx$ $= \dfrac{\pi^{1/2}\, 2^{\mu-\nu}\, \Gamma(\frac{3\nu-\mu-1}{2})}{a^{\nu+1/2}\, \Gamma(\frac{1+\nu-\mu}{2})}\, W_{\rho,\sigma}(ae^{i\pi/2})\, W_{\rho,\sigma}(ae^{-i\pi/2})$ $\rho = \tfrac{1}{2}(\mu+1-\nu), \quad \sigma = \nu - \tfrac{1}{2}$ $\mathrm{Re}(\mu-\nu) < 0, \quad \mathrm{Re}\,\nu < 3/2, \quad \mathrm{Re}(3\nu-\mu) > 1$

Related functions (cont'd)

(28)	$\int_0^\infty x^{1-\mu-\nu} J_\nu(ax)\, S_{\mu,-\mu-2\nu}(x)\, dx = \dfrac{\pi^{1/2} a^{\nu-1}\Gamma(1-\mu-\nu)}{2^{\mu+2\nu}\Gamma(\nu+\frac{1}{2})}$ $\times (a^2-1)^{\frac{1}{2}(\mu+\nu-1)} P_{\mu+\nu}^{\mu+\nu-1}(a)$ $a > 1, \quad \operatorname{Re}\nu > -\tfrac{1}{2}, \quad \operatorname{Re}(\mu+\nu) < 1$
(29)	$\int_0^{\frac{1}{2}\pi} \cos(2\mu x)\, S_{2\mu-1,2\nu}(a\cos x)\, dx = \dfrac{\pi\, 2^{2\mu-3} a^{2\mu}\csc(2\nu\pi)}{\Gamma(1-\mu-\nu)\,\Gamma(1-\mu+\nu)}$ $\times [J_{\mu+\nu}(\tfrac{1}{2}a)\, Y_{\mu-\nu}(\tfrac{1}{2}a) - J_{\mu-\nu}(\tfrac{1}{2}a)\, Y_{\mu+\nu}(\tfrac{1}{2}a)]$ $\operatorname{Re}\mu > -2, \quad -1 < \operatorname{Re}\nu < 1$
(30)	$\displaystyle\int_0^{\frac{1}{2}\pi} \frac{\cos(2\mu x)}{\cos x}\, S_{2\mu,2\nu}(a\sec x)\, dx = \frac{\pi\, 2^{2\mu-1}}{a} W_{\mu,\nu}(ae^{i\pi/2})$ $\times W_{\mu,\nu}(ae^{-i\pi/2})$ $\lvert\arg a\rvert < \pi, \quad \operatorname{Re}\mu < 1$
(31)	$\int_0^\infty (\sinh x)^{1/2} \cosh(\nu x)\, S_{\mu,\frac{1}{2}}(a\cosh x)\, dx$ $= \dfrac{\Gamma(\frac{1}{4}-\frac{\mu+\nu}{2})\,\Gamma(\frac{1}{4}-\frac{\mu-\nu}{2})}{2^{\mu+3/2} a^{1/2}\,\Gamma(\frac{1}{2}-\mu)} S_{\mu+\frac{1}{2},\nu}(a)$ $\lvert\arg a\rvert < \pi, \quad \operatorname{Re}\mu + \lvert\operatorname{Re}\nu\rvert < \tfrac{1}{2}$
(32)	$\int_0^\infty x^{2\nu-1} U_\nu(w,x)\, dx = 2^{\nu-1}\Gamma(\nu)\, w^\nu \cos(\tfrac{1}{2}w)$ $\operatorname{Re}\nu > 0$
(33)	$\int_0^\infty x^{2\nu-3} U_\nu(w,x)\, dx = 2^{\nu-2}\Gamma(\nu-1)\, w^{\nu-1}\sin(\tfrac{1}{2}w)$ $\operatorname{Re}\nu > 1$
(34)	$\int_0^\infty x^{1-\nu}\sin(\tfrac{1}{2}ax)\, U_\nu(x,z)\, dx$ $= 0 \qquad a > 1$ $= \tfrac{1}{2}\pi(1-a)^{\frac{1}{2}\nu-1} z^{2-\nu} J_{\nu-2}[z(1-a)^{1/2}] \qquad 0 < a < 1$

Related functions (con t'd)

(35)	$\int_0^\infty x^{-\nu} \cos(\tfrac{1}{2}ax)\, U_\nu(x, z)\, dx$ $= 0 \qquad a > 1$ $= \tfrac{1}{2}\pi(1-a)^{\frac{1}{2}\nu - \frac{1}{2}} z^{1-\nu} J_{\nu-1}[z(1-a)^{\frac{1}{2}}] \qquad 0 < a < 1$
(36)	$\int_0^\infty x^{\nu-1} J_{\frac{1}{2}\nu-1}\left(\frac{x^2}{2w}\right) U_\nu(w, x)\, dx = \frac{w^\nu}{2(\nu-1)} J_{\frac{1}{2}\nu}\left(\frac{w}{2}\right)$ $\operatorname{Re} \nu > 1$
(37)	$\int_{-\infty}^\infty \frac{\sin[a(x+z)]}{x+z} U_\nu(w, x)\, dx = \pi\, U_\nu(w, z) \qquad a > 1$

CHAPTER XX

HYPERGEOMETRIC FUNCTIONS

Most of the higher transcendental functions of Chapters XVI to XIX are special hypergeometric functions. In the present chapter we list those special hypergeometric functions not included in the former chapters, and also generalized hypergeometric functions. The integrals listed in sections 20.4 and 20.5 are key formulas from which an enormous number of integrals involving special hypergeometric functions may be derived. For particular cases of the E-function and of the G-function see the Appendix. We do not list in this chapter integrals involving the generalized hypergeometric series ${}_pF_q$: since

$$\begin{aligned}
&{}_pF_q(a_1, \dots, a_p; b_1, \dots, b_q; x) \\
&= \frac{\Gamma(b_1) \cdots \Gamma(b_q)}{\Gamma(a_1) \cdots \Gamma(a_p)} E(a_1, \dots, a_p; b_1, \dots, b_q; -x^{-1}) \\
&= \frac{\Gamma(b_1) \cdots \Gamma(b_q)}{\Gamma(a_1) \cdots \Gamma(a_p)} G^{p,1}_{q+1,p}\left(-\frac{1}{x} \,\middle|\, \begin{matrix} 1, b_1, \dots, b_q \\ a_1, \dots, a_p \end{matrix}\right)
\end{aligned}$$

such integrals may be derived from those given in sections 20.4 and 20.5. Because of the great importance of integrals involving E-functions and G-functions, we have repeated integrals given in the earlier chapters, and in some cases have given more elaborate conditions of validity.

Parabolic cylinder functions

For the theory of these functions see H.T.F. vol. II, Chapter VIII and the literature quoted there, also Buchholz, Herbert, 1953: *Die konfluente hypergeometrische Funktion*. Springer Verlag.

$$D_\nu(z) = 2^{\frac{1}{2}\nu+\frac{1}{4}} z^{-\frac{1}{2}} W_{\frac{1}{2}\nu+\frac{1}{4}, \pm\frac{1}{4}}\left(\frac{z^2}{2}\right)$$
$$= \frac{2^{-\nu-1} z^\nu}{\pi^{\frac{1}{2}} \Gamma(-\nu)} \exp\left(-\frac{z^2}{4}\right) E\left(-\frac{\nu}{2}, \frac{1-\nu}{2} :: \frac{z^2}{2}\right)$$
$$= 2^{\frac{1}{2}\nu} \exp\left(\frac{z^2}{4}\right) G^{20}_{12}\left(\frac{z^2}{2}\middle| \begin{matrix} \frac{1}{2} - \frac{1}{2}\nu \\ 0, \frac{1}{2} \end{matrix}\right)$$
$$= \frac{2^{-\frac{1}{2}\nu-1}}{\pi^{\frac{1}{2}} \Gamma(-\nu)} \exp\left(-\frac{z^2}{4}\right) G^{21}_{12}\left(\frac{z^2}{2}\middle| \begin{matrix} 1 + \frac{1}{2}\nu \\ 0, \frac{1}{2} \end{matrix}\right)$$

Other expressions in terms of the G-function, and expressions for products of parabolic cylinder functions may be derived by means of the formulas given in the Appendix.

Gauss' hypergeometric series

For the theory of these series see H.T.F. vol. I, Chapter II and the literature quoted there, especially the monographs by Goursat, Kampé de Fériet, Klein, Snow (now available in a second edition), and Chapter XIV of Whittaker and Watson.

$$_2F_1(a, b; c; x) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} E(a, b: c: -x^{-1})$$
$$= \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} G^{21}_{22}\left(-\frac{1}{x}\middle| \begin{matrix} 1, c \\ a, b \end{matrix}\right)$$

The evaluation of integrals involving Gauss' series is often facilitated by the use of the transformation formulas: for these see H.T.F. vol. I, sections 2.9 and 2.11.

Confluent hypergeometric functions

For the theory of these functions see H.T.F. vol. I, Chapter VI and the literature quoted there, especially Chapter XVI of Whittaker and Watson, and also Tricomi, F.G., 1952: *Lezioni sulle funzioni ipergeometriche confluenti.* Torino, Gheroni and Buchholz, Herbert, 1953: *Die konfluente hypergeometrische Funktion.* Springer Verlag.

$$M_{\kappa,\mu}(z) = z^{\mu+\frac12} e^{-\frac12 z} \; {}_1F_1(\tfrac12 - \kappa + \mu;\, 2\mu + 1;\, z)$$

$$= z^{\mu+\frac12} e^{\frac12 z} \; {}_1F_1(\tfrac12 + \kappa + \mu;\, 2\mu + 1;\, -z)$$

$$= \frac{\Gamma(2\mu+1)}{\Gamma(\frac12 - \kappa + \mu)} z^{\mu+\frac12} e^{-\frac12 z} E(\tfrac12 - \kappa + \mu : 2\mu + 1 : -z^{-1})$$

$$= \frac{\Gamma(2\mu+1)}{\Gamma(\frac12 + \kappa + \mu)} z^{\mu+\frac12} e^{\frac12 z} E(\tfrac12 + \kappa + \mu : 2\mu + 1 : z^{-1})$$

$$= \frac{\Gamma(2\mu+1)}{\Gamma(\frac12 + \kappa + \mu)} e^{\frac12 z} G^{11}_{12}\left(z \,\middle|\, \begin{matrix} 1-\kappa \\ \frac12 + \mu,\ \frac12 - \mu \end{matrix}\right)$$

$$W_{\kappa,\mu}(z) = \frac{z^{\kappa} e^{-\frac12 z}}{\Gamma(\frac12 - \kappa + \mu)\,\Gamma(\frac12 - \kappa - \mu)} E(\tfrac12 - \kappa + \mu,\ \tfrac12 - \kappa - \mu :: z)$$

$$= \frac{e^{-\frac12 z}}{\Gamma(\frac12 - \kappa + \mu)\,\Gamma(\frac12 - \kappa - \mu)} G^{21}_{12}\left(z \,\middle|\, \begin{matrix} 1+\kappa \\ \frac12 + \mu,\ \frac12 - \mu \end{matrix}\right)$$

$$= e^{\frac12 z} G^{20}_{12}\left(z \,\middle|\, \begin{matrix} 1-\kappa \\ \frac12 + \mu,\ \frac12 - \mu \end{matrix}\right)$$

For other expressions in terms of the G-function, and for expressions of products of confluent hypergeometric functions see the Appendix.

MacRobert's E-function

A brief introduction to this function is given in H.T.F. vol. I, sections 5.2-5.2.2, and a more detailed presentation of its theory may be found in MacRobert, T.M., 1950: *Functions of a complex variable*. Macmillan, Appendix V and Miscellaneous Examples III. See also the papers by MacRobert listed on p. 246 ff. of H.T.F. vol. I, and further papers by Professor MacRobert and his pupils in *Proc. Glasgow Math. Ass.* vol. I, 1953.

$$E(a_1, \dots, a_p : b_1, \dots, b_q : x) = G^{p,1}_{q+1,p}\left(z \,\middle|\, \begin{matrix} 1, b_1, \dots, b_q \\ a_1, \dots, a_p \end{matrix}\right)$$

Numerous higher transcendental functions and some of their combinations are special instances of the E-function: a selection of these is given in the Appendix.

Meijer's G - function

For the theory of this function see H.T.F. vol. I, sections 5.3-5.6 and the papers by Meijer listed on p. 247 of H.T.F. vol. I, and also further papers by Professor Meijer in recent volumes of *Proc. Nederl. Akad. Wetensch.* It has already been mentioned that a very large number of integrals involving special functions may be reduced to integrals involving the G-function. Examples of this process, and the necessary formulas, are given in Professor Meijer's papers: a selection of reduction formulas is also given in the Appendix.

20.1. Parabolic cylinder functions

See also under confluent hypergeometric functions, E-function, G-function.

(1)	$\int_0^\infty x^{\nu-1} \exp\left(-\frac{3x^2}{4}\right) D_\nu(x)\, dx$ $= 2^{-\frac{1}{2}\nu} \Gamma(\nu) \cos(\frac{1}{4}\nu\pi)$ Re $\nu > 0$
(2)	$\int_0^\infty x^\nu \exp\left(-\frac{3x^2}{4}\right) D_{\nu-1}(x)\, dx$ $= 2^{-\frac{1}{2}\nu-1} \Gamma(\nu) \sin(\frac{1}{4}\nu\pi)$ Re $\nu > -1$
(3)	$\int_0^a x^{2\nu-1} (a^2 - x^2)^{\lambda-1} \exp\left(\frac{x^2}{4}\right) D_{-2\lambda-2\nu}(x)\, dx$ $= \frac{\Gamma(\lambda)\Gamma(2\nu)}{\Gamma(2\lambda+2\nu)} 2^{\lambda-1} a^{2\lambda+2\nu-2} \exp\left(\frac{a^2}{4}\right) D_{-2\nu}(a)$ Re $\lambda > 0$, Re $\nu > 0$
(4)	$\int_a^\infty x^\nu (x-a)^{\frac{1}{2}\mu-\frac{1}{2}\nu-1} \exp[-\frac{1}{4}(x-a)^2] D_\mu(x)\, dx$ $= 2^{\mu-\nu-2} a^{\mu-1} \Gamma\left(\frac{\mu-\nu}{2}\right) D_\nu(a)$ Re $(\mu-\nu) > 0$
(5)	$\int_{-\infty}^\infty (x - ia)^{-1} \exp\left(-\frac{x^2}{4}\right) D_n(x) dx = -(2\pi)^{\frac{1}{2}} (-i)^n n!$ $\times \exp\left(-\frac{a^2}{4}\right) D_{-n-1}(a)$ $n = 0, 1, 2, \ldots,$ Re $a > 0$

Parabolic cylinder functions (cont'd)

(6)	$\int_0^\infty x^\nu (x^2+a^2)^{-1} \exp\left(-\frac{x^2}{4}\right) D_\nu(x)\,dx$ $= (\frac{1}{2}\pi)^{\frac{1}{2}} a^{\nu-1}\Gamma(\nu+1)\exp\left(\frac{a^2}{4}\right) D_{-\nu-1}(a)$ $\operatorname{Re} a > 0, \quad 0 < \operatorname{Re}\nu < 1$
(7)	$\int_0^\infty x^{\nu-1}(x^2+a^2)^{-\frac{1}{2}} \exp\left(-\frac{x^2}{4}\right) D_\nu(x)\,dx$ $= a^{\nu-1}\Gamma(\nu)\exp\left(\frac{a^2}{4}\right) D_{-\nu}(a)$ $\operatorname{Re} a > 0, \quad \operatorname{Re}\nu > 0$
(8)	$\int_0^\infty x^{2\rho-1} \sin(ax) \exp\left(-\frac{x^2}{4}\right) D_{2\nu}(x)\,dx$ $= 2^{\nu-\rho-\frac{1}{2}}\pi^{\frac{1}{2}} a \frac{\Gamma(2\rho+1)}{\Gamma(\rho-\nu+1)}\, {}_2F_2\left(\rho+\frac{1}{2}, \rho+1; \frac{3}{2}, \rho-\nu+1; -\frac{a^2}{2}\right)$ $\operatorname{Re}\rho > -\frac{1}{2}$
(9)	$\int_0^\infty x^{2\rho-1} \sin(ax) \exp\left(\frac{x^2}{4}\right) D_{2\nu}(x)\,dx$ $= \frac{2^{\rho-\nu-2}}{\Gamma(-2\nu)} G^{22}_{23}\left(\frac{a^2}{2}\middle\vert \begin{matrix} \frac{1}{2}-\rho, 1-\rho \\ -\rho-\nu, \frac{1}{2}, 0\end{matrix}\right)$ $a > 0, \quad \operatorname{Re}\rho > -\frac{1}{2}, \quad \operatorname{Re}(\rho+\nu) < \frac{1}{2}$
(10)	$\int_0^\infty x^{2\rho-1} \cos(ax) \exp\left(-\frac{x^2}{4}\right) D_{2\nu}(x)\,dx$ $= \frac{2^{\nu-\rho}\Gamma(2\rho)\pi^{\frac{1}{2}}}{\Gamma(\rho-\nu+\frac{1}{2})}\, {}_2F_2\left(\rho, \rho+\frac{1}{2}; \frac{1}{2}, \rho-\nu+\frac{1}{2}; -\frac{a^2}{2}\right)$ $\operatorname{Re}\nu > 0$
(11)	$\int_0^\infty x^{2\rho-1} \cos(ax) \exp\left(\frac{x^2}{4}\right) D_{2\nu}(x)\,dx$ $= \frac{2^{\rho-\nu-2}}{\Gamma(-2\nu)} G^{22}_{23}\left(\frac{a^2}{2}\middle\vert \begin{matrix} \frac{1}{2}-\rho, 1-\rho \\ -\rho-\nu, 0, \frac{1}{2}\end{matrix}\right)$ $a > 0, \quad \operatorname{Re}\rho > 0, \quad \operatorname{Re}(\rho+\nu) < \frac{1}{2}$

Parabolic cylinder functions (cont'd)

(12)	$\int_0^\infty x^{\nu-\frac{1}{2}} \exp[-(x+a)^2]\, I_{\nu-\frac{1}{2}}(2ax)\, D_\nu(2x)\, dx$ $= \frac{1}{2}\pi^{-\frac{1}{2}}\, \Gamma(\nu)\, a^{\nu-\frac{1}{2}}\, D_{-\nu}(2a)$ Re $a > 0$, Re $\nu > 0$
(13)	$\int_0^\infty x^{\nu-3/2} \exp[-(x+a)^2]\, I_{\nu-3/2}(2ax)\, D_\nu(2x)\, dx$ $= \frac{1}{2}\pi^{-1/2}\, \Gamma(\nu)\, a^{\nu-3/2}\, D_{-\nu}(2a)$ Re $a > 0$, Re $\nu > 1$
(14)	$\int_0^\infty [D_\nu(x)]^2\, dx = (\frac{1}{2}\pi)^{\frac{1}{2}}\, \Gamma(\nu+1) + \dfrac{\pi^{1/2}}{2^{3/2}\,\Gamma(-\nu)}$ $\times \left[\psi\left(\dfrac{\nu+1}{2}\right) - \psi\left(\dfrac{\nu}{2}+1\right)\right]$
(15)	$\int_0^\infty D_\nu(x)\, D_\mu(x)\, dx = \dfrac{\pi\, 2^{\frac{1}{2}(\mu+\nu+1)}}{\mu-\nu} \left[\dfrac{1}{\Gamma(-\frac{\nu}{2})\,\Gamma(\frac{1-\mu}{2})} - \dfrac{1}{\Gamma(-\frac{\mu}{2})\,\Gamma(\frac{1-\nu}{2})}\right]$
(16)	$\int_0^\infty J_0(xy)\, D_\nu(x)\, D_{\nu-1}(-x)\, dx = y^{-1}[D_\nu(y)\, D_{\nu-1}(y)$ $+ \frac{1}{2}D_\nu(y)\, D_{\nu-1}(-y) + \frac{1}{2}D_\nu(-y)\, D_{\nu-1}(y)]$ $y > 0$
(17)	$\int_0^\infty J_0(xy)\, D_\nu(x)\, D_{\nu-1}(x)\, dx$ $= \dfrac{1}{2y}[D_\nu(y)\, D_{\nu-1}(-y) - D_\nu(-y)\, D_{\nu-1}(y)]$
(18)	$\int_0^\infty J_0(xy)\, D_\nu(-x)\, D_{\nu-1}(x)\, dx = y^{-1}[\frac{1}{2}D_\nu(y)\, D_{\nu-1}(-y)$ $+ \frac{1}{2}D_\nu(-y)\, D_{\nu-1}(y) - D_\nu(y)\, D_{\nu-1}(y)]$
(19)	$\int_0^{\frac{1}{2}\pi} (\sin x)^{-\nu}\, (\cos x)^{-\mu-2}\, D_\nu(a\sin x)\, D_\mu(a\cos x)\, dx$ $= -(\frac{1}{2}\pi)^{\frac{1}{2}}\, (\mu+1)^{-1}\, D_{\mu+\nu+1}(a)$ Re $\nu < 1$, Re $\mu < -1$

Parabolic cylinder functions (cont'd)

(20)	$\int_0^\infty \cosh(2\mu x)\exp[-(a\sinh x)^2]\,D_{2\kappa}(2a\cosh x)\,dx$ $= 2^{\kappa-3/2}\pi^{1/2}a^{-1}W_{\kappa,\mu}(2a^2)$ $\qquad \operatorname{Re} a^2>0$
(21)	$\int_0^\infty \cosh(2\mu x)\exp[(a\sinh x)^2]\,D_{2\kappa}(2a\cosh x)\,dx$ $=\dfrac{\Gamma(\mu-\kappa)\Gamma(-\mu-\kappa)}{2^{\kappa+5/2}a\Gamma(-2\kappa)}W_{\kappa+\frac{1}{2},\mu}(2a^2)$ $\qquad \lvert\arg a\rvert<3\pi/4,\quad \operatorname{Re}\kappa+\lvert\operatorname{Re}\mu\rvert<0$
(22)	$\int_0^\infty \cos(ax)\,D_{x-\frac{1}{2}}(\beta)\,D_{-x-\frac{1}{2}}(\beta)\,dx$ $=\dfrac{1}{2}\left(\dfrac{\pi}{\cos a}\right)^{\frac{1}{2}}\exp\left(-\dfrac{\beta^2}{2\sec a}\right)$ $\qquad -\frac{1}{2}\pi<a<\frac{1}{2}\pi$ $=0$ $\qquad a<-\frac{1}{2}\pi$ or $a>\frac{1}{2}\pi$

20.2. Gauss' hypergeometric series

See also under Legendre functions, E-function, and G-function.

(1)	$\int_0^1 x^{a-\gamma}(1-x)^{\gamma-\beta-1}\,{}_2F_1(a,\beta;\gamma;x)\,dx$ $=\dfrac{\Gamma(1+\frac{1}{2}a)\Gamma(\gamma)\Gamma(a-\gamma+1)\Gamma(\gamma-\frac{1}{2}a-\beta)}{\Gamma(1+a)\Gamma(1+\frac{1}{2}a-\beta)\Gamma(\gamma-\frac{1}{2}a)}$ $\qquad \operatorname{Re} a+1>\operatorname{Re}\gamma>\operatorname{Re}\beta,\quad \operatorname{Re}(\gamma-\frac{1}{2}a-\beta)>0$
(2)	$\int_0^1 x^{\rho-1}(1-x)^{\beta-\gamma-n}\,{}_2F_1(-n,\beta;\gamma;x)\,dx$ $=\dfrac{\Gamma(\gamma)\Gamma(\rho)\Gamma(\beta-\gamma+1)\Gamma(\gamma-\rho+n)}{\Gamma(\gamma+n)\Gamma(\gamma-\rho)\Gamma(\beta-\gamma+\rho+1)}$ $\qquad n=0,1,2,\ldots,\quad \operatorname{Re}\rho>0,\quad \operatorname{Re}(\beta-\gamma)>n-1$

Gauss' series (cont'd)

(3)	$\int_0^1 x^{\rho-1}(1-x)^{\beta-\rho-1}\,{}_2F_1(\alpha,\beta;\gamma;x)\,dx = \dfrac{\Gamma(\gamma)\Gamma(\rho)\Gamma(\beta-\rho)\Gamma(\gamma-\alpha-\rho)}{\Gamma(\beta)\Gamma(\gamma-\alpha)\Gamma(\gamma-\rho)}$ $\operatorname{Re}\rho>0,\quad \operatorname{Re}(\beta-\rho)>0,\quad \operatorname{Re}(\gamma-\alpha-\rho)>0$
(4)	$\int_0^1 x^{\gamma-1}(1-x)^{\rho-1}\,{}_2F_1(\alpha,\beta;\gamma;x)\,dx = \dfrac{\Gamma(\gamma)\Gamma(\rho)\Gamma(\gamma+\rho-\alpha-\beta)}{\Gamma(\gamma+\rho-\alpha)\Gamma(\gamma+\rho-\beta)}$ $\operatorname{Re}\gamma>0,\quad \operatorname{Re}\rho>0,\quad \operatorname{Re}(\gamma+\rho-\alpha-\beta)>0$
(5)	$\int_0^1 x^{\rho-1}(1-x)^{\sigma-1}\,{}_2F_1(\alpha,\beta;\gamma;x)\,dx = \dfrac{\Gamma(\rho)\Gamma(\sigma)}{\Gamma(\rho+\sigma)}\,{}_3F_2(\alpha,\beta,\rho;\gamma,\rho+\sigma;1)$ $\operatorname{Re}\rho>0,\quad \operatorname{Re}\sigma>0,\quad \operatorname{Re}(\gamma+\sigma-\alpha-\beta)>0$
(6)	$\int_0^1 x^{\gamma-1}(1-x)^{\rho-1}(1-zx)^{-\sigma}\,{}_2F_1(\alpha,\beta;\gamma;x)\,dx = \dfrac{\Gamma(\gamma)\Gamma(\rho)\Gamma(\gamma+\rho-\alpha-\beta)}{\Gamma(\gamma+\rho-\alpha)\Gamma(\gamma+\rho-\beta)}(1-z)^{\sigma}$ $\times\,{}_3F_2\left(\rho,\sigma,\gamma+\rho-\alpha-\beta;\gamma+\rho-\alpha,\gamma+\rho-\beta;\dfrac{z}{z-1}\right)$ $\operatorname{Re}\gamma>0,\quad \operatorname{Re}\rho>0,\quad \operatorname{Re}(\gamma+\rho-\alpha-\beta)>0,\quad \lvert\arg(1-z)\rvert<\pi$
(7)	$\int_0^1 x^{\rho-1}(1-x)^{\sigma-1}\,{}_2F_1(\alpha,\beta;\gamma;xz)\,dx = \dfrac{\Gamma(\rho)\Gamma(\sigma)}{\Gamma(\rho+\sigma)}\,{}_3F_2(\alpha,\beta,\rho;\gamma,\sigma;z)$ $\operatorname{Re}\rho>0,\quad \operatorname{Re}\sigma>0,\quad \lvert\arg(1-z)\rvert<\pi$

Gauss' series (cont'd)

(8)	$\int_0^1 x^{\gamma-1}(1-x)^{\rho-1} e^{-xz}\,{}_2F_1(\alpha,\beta;\gamma;x)\,dx$ $= \dfrac{\Gamma(\gamma)\Gamma(\rho)\Gamma(\gamma+\rho-\alpha-\beta)}{\Gamma(\gamma+\rho-\alpha)\Gamma(\gamma+\rho-\beta)} e^{-z}$ $\times\,{}_2F_2(\rho,\gamma+\rho-\alpha-\beta;\gamma+\rho-\alpha,\gamma+\rho-\beta;z)$ $\operatorname{Re}\gamma>0,\quad \operatorname{Re}\rho>0,\quad \operatorname{Re}(\gamma+\rho-\alpha-\beta)>0$
(9)	$\int_0^\infty x^{\gamma-1}(1+x)^{-\sigma}\,{}_2F_1(\alpha,\beta;\gamma;-x)\,dx$ $= \dfrac{\Gamma(\gamma)\Gamma(\alpha-\gamma+\sigma)\Gamma(\beta-\gamma+\sigma)}{\Gamma(\sigma)\Gamma(\alpha+\beta-\gamma+\sigma)}$ $\operatorname{Re}\gamma>0,\quad \operatorname{Re}(\alpha-\gamma+\sigma)>0,\quad \operatorname{Re}(\beta-\gamma+\sigma)>0$
(10)	$\int_0^\infty x^{\gamma-1}(x+z)^{-\sigma}\,{}_2F_1(\alpha,\beta;\gamma;-x)\,dx$ $= \dfrac{\Gamma(\alpha-\gamma+\sigma)\Gamma(\beta-\gamma+\sigma)\Gamma(\gamma)}{\Gamma(\alpha+\beta-\gamma+\sigma)\Gamma(\sigma)}$ $\times\,{}_2F_1(\alpha-\gamma+\sigma,\beta-\gamma+\sigma;\alpha+\beta-\gamma+\sigma;1-z)$ $\operatorname{Re}\gamma>0,\quad \operatorname{Re}(\alpha-\gamma+\sigma)>0,\quad \operatorname{Re}(\beta-\gamma+\sigma)>0,\quad \lvert\arg z\rvert<\pi$
(11)	$\int_0^1 x^{\gamma-1}(1-x)^{\delta-\gamma-1}\,{}_2F_1(\alpha,\beta;\gamma;xz)$ $\times\,{}_2F_1[\delta-\alpha,\delta-\beta;\delta-\gamma;(1-x)\zeta]\,dx$ $= \dfrac{\Gamma(\gamma)\Gamma(\delta-\gamma)}{\Gamma(\delta)}(1-\zeta)^{2\alpha-\delta}\,{}_2F_1(\alpha,\beta;\delta;z+\zeta-z\zeta)$ $0<\operatorname{Re}\gamma<\operatorname{Re}\delta,\quad \lvert\arg(1-z)\rvert<\pi,\quad \lvert\arg(1-\zeta)\rvert<\pi$
(12)	$\int_0^1 x^{\gamma-1}(1-x)^{\epsilon-1}(1-xz)^{-\delta}\,{}_2F_1(\alpha,\beta;\gamma;xz)$ $\times\,{}_2F_1\left[\delta,\beta-\gamma;\epsilon;\dfrac{(1-x)z}{1-xz}\right]dx = \dfrac{\Gamma(\gamma)\Gamma(\epsilon)}{\Gamma(\gamma+\epsilon)}\,{}_2F_1(\alpha+\delta,\beta;\gamma+\epsilon;z)$ $\operatorname{Re}\gamma>0,\quad \operatorname{Re}\epsilon>0,\quad \lvert\arg(z-1)\rvert<\pi$

Gauss' series (cont'd)

(13)	$\int_0^\infty e^{-\lambda x}\, {}_2F_1(\alpha, \beta; \tfrac{1}{2}; -x^2)\, dx = \lambda^{\alpha+\beta-1} S_{1-\alpha-\beta,\, \alpha-\beta}(\lambda)$ $\operatorname{Re}\lambda > 0$
(14)	$\int_0^\infty x e^{-\lambda x}\, {}_2F_1(\alpha, \beta; 3/2; -x^2)\, dx = \lambda^{\alpha+\beta-2} S_{1-\alpha-\beta,\, \alpha-\beta}(\lambda)$ $\operatorname{Re}\lambda > 0$
(15)	$\int_0^\infty x^{\gamma-1}(x+y)^{-\alpha}(x+z)^{-\beta} e^{-x}\, {}_2F_1\left[\alpha, \beta; \gamma; \dfrac{x(x+y+z)}{(x+y)(x+z)}\right] dx$ $= \Gamma(\gamma)\,(xy)^{-\frac{1}{2}-\mu} e^{\frac{1}{2}y+\frac{1}{2}z} W_{\kappa,\mu}(y)\, W_{\lambda,\mu}(z)$ $2\kappa = 1-\alpha+\beta-\gamma, \quad 2\lambda = 1+\alpha-\beta-\gamma, \quad 2\mu = \alpha+\beta-\gamma$ $\operatorname{Re}\gamma > 0, \quad \lvert\arg y\rvert < \pi, \quad \lvert\arg z\rvert < \pi$
(16)	$\int_0^\infty x^{\alpha+\beta-2\nu-1}(x+1)^{-\nu} e^{xz} K_\nu[(x+1)z]\, {}_2F_1(\alpha, \beta; \alpha+\beta-2\nu; -x)\, dx$ $= \pi^{-\frac{1}{2}} \cos(\nu\pi)\,\Gamma(\tfrac{1}{2}-\alpha+\nu)\,\Gamma(\tfrac{1}{2}-\beta+\nu)\,\Gamma(\gamma)$ $\times (2z)^{-\frac{1}{2}-\frac{1}{2}\gamma} W_{\frac{1}{2}\gamma,\, \frac{1}{2}(\beta-\alpha)}(2z)$ $\operatorname{Re}(\alpha+\beta-2\nu) > 0, \quad \operatorname{Re}(\tfrac{1}{2}-\alpha+\nu) > 0, \quad \operatorname{Re}(\tfrac{1}{2}-\beta+\nu) > 0$ $\lvert\arg z\rvert < 3\pi/2, \quad \gamma = \alpha+\beta-2\nu$

20.3. Confluent hypergeometric functions

See also under E-function, G-function.

For special confluent hypergeometric functions see also sections 16.5, 16.6, 17.3, 20.1, and Chapter XIX.

(1)	$\int_0^a x^{\beta-1}(a-x)^{\gamma-1}\, {}_1F_1(\alpha; \beta; x)\, dx = \dfrac{\Gamma(\beta)\,\Gamma(\gamma)}{\Gamma(\beta+\gamma)}\, a^{\beta+\gamma-1}$ $\times\, {}_1F_1(\alpha; \beta+\gamma; a)$ $\qquad \operatorname{Re}\beta > 0, \quad \operatorname{Re}\gamma > 0$

Confluent hypergeometric functions (cont'd)

(2)	$\int_0^a x^{\beta-1}(a-x)^{\delta-1}\ {}_1F_1(\alpha;\beta;x)\ {}_1F_1(\gamma;\delta;a-x)\,dx$ $=\dfrac{\Gamma(\beta)\,\Gamma(\delta)}{\Gamma(\beta+\delta)}\,a^{\beta+\delta-1}\ {}_1F_1(\alpha+\gamma;\beta+\delta;a)$ $\quad$ $\operatorname{Re}\beta>0,\quad \operatorname{Re}\delta>0$
(3)	$\int_0^1 x^{\beta-1}(1-x)^{\sigma-\beta-1}\ {}_1F_1(\alpha;\beta;\lambda x)\ {}_1F_1[\sigma-\alpha;\sigma-\beta;\mu(1-x)]\,dx$ $=\dfrac{\Gamma(\beta)\,\Gamma(\sigma-\beta)}{\Gamma(\sigma)}\,e^{\lambda}\ {}_1F_1(\alpha;\sigma;\mu-\lambda)$ $\quad$ $0<\operatorname{Re}\beta<\operatorname{Re}\sigma$
(4)	$\int_0^\infty \cos(ax)\ {}_1F_1(\nu+1;1;ix)\ {}_1F_1(\nu+1;1;-ix)\,dx$ $=-a^{-1}\sin(\nu\pi)\,P_\nu(2a^{-2}-1)$ $\quad$ $0<a<1$ $=0$ $\quad$ $1<a<\infty$ $-1<\operatorname{Re}\nu<0$
(5)	$\int_0^a x^{-1}(a-x)^{\kappa-1}\,e^{\frac{1}{2}(a-x)}\,M_{\kappa,\mu}(x)\,dx=\dfrac{\Gamma(\kappa)\,\Gamma(2\mu+1)}{\Gamma(\kappa+\mu+\frac{1}{2})}\,\pi^{\frac{1}{2}}\,a^{\kappa-\frac{1}{2}}\,I_\mu(\tfrac{1}{2}a)$ $\operatorname{Re}\kappa>0,\quad \operatorname{Re}\mu>-\tfrac{1}{2}$
(6)	$\int_0^a x^{\kappa-1}(a-x)^{\lambda-1}\,e^{\frac{1}{2}(a-x)}\,M_{\kappa+\lambda,\mu}(x)\,dx$ $=\dfrac{\Gamma(\lambda)\,\Gamma(\kappa+\mu+\frac{1}{2})}{\Gamma(\kappa+\lambda+\mu+\frac{1}{2})}\,a^{\kappa+\lambda-1}\,M_{\kappa,\mu}(a)$ $\operatorname{Re}(\kappa+\mu)>-\tfrac{1}{2},\quad \operatorname{Re}\lambda>0$
(7)	$\int_0^a x^{\mu-\frac{1}{2}}(a-x)^{\nu-\frac{1}{2}}\,M_{\kappa,\mu}(x)\,M_{\lambda,\nu}(a-x)\,dx$ $=\dfrac{\Gamma(2\mu+1)\,\Gamma(2\nu+1)}{\Gamma(2\mu+2\nu+2)}\,a^{\mu+\nu}\,M_{\kappa+\lambda,\mu+\nu+\frac{1}{2}}(a)$ $\operatorname{Re}\mu>-\tfrac{1}{2},\quad \operatorname{Re}\nu>-\tfrac{1}{2}$

Confluent hypergeometric functions (cont'd)

(8)	$\int_0^\infty x^{\rho-1}[x^{1/2}+(a+x)^{1/2}]^{2\sigma}e^{-1/2x}M_{\kappa,\mu}(x)\,dx$ $= -\frac{\sigma\Gamma(2\mu+1)a^\sigma}{\pi^{1/2}\Gamma(1/2+\kappa+\mu)}G^{23}_{34}\left(a\,\middle\vert\,\begin{matrix}1/2,\ 1,\ 1-\kappa+\rho\\ 1/2+\mu+\rho,\ -\sigma,\ \sigma,\ 1/2-\mu+\rho\end{matrix}\right)$ $\lvert\arg a\rvert<\pi,\quad \mathrm{Re}(\mu+\rho)>-1/2,\quad \mathrm{Re}(\kappa-\rho-\sigma)>0$
(9)	$\int_0^\infty x^{\rho-1}(a+x)^{-1/2}[x^{1/2}+(a+x)^{1/2}]^{2\sigma}e^{-1/2x}M_{\kappa,\mu}(x)\,dx$ $= \frac{\Gamma(2\mu+1)a^\sigma}{\pi^{1/2}\Gamma(1/2+\kappa+\mu)}G^{23}_{34}\left(a\,\middle\vert\,\begin{matrix}0,\ 1/2,\ 1/2-\kappa+\rho\\ -\sigma,\ \rho+\mu,\ \rho-\mu,\ \sigma\end{matrix}\right)$ $\lvert\arg a\rvert<\pi,\quad \mathrm{Re}(\rho+\mu)>-1/2,\quad \mathrm{Re}(\kappa-\rho-\sigma)>-1/2$
(10)	$\int_0^\infty x^{-\mu-1/2}e^{-1/2x}\sin(2ax^{1/2})M_{\kappa,\mu}(x)\,dx$ $= \pi^{1/2}a^{\kappa+\mu-1}\frac{\Gamma(3-2\mu)}{\Gamma(1/2+\kappa+\mu)}\exp\left(-\frac{a^2}{2}\right)W_{\rho,\sigma}(a^2)$ $a>0,\quad \mathrm{Re}(\kappa+\mu)>0,\quad 2\rho=\kappa-3\mu+1,\quad 2\sigma=\kappa+\mu-1$
(11)	$\int_0^\infty x^{-1/2}(a+x)^{\mu}e^{-1/2x}P_\nu^{-2\mu}(1+2x/a)M_{\kappa,\mu}(x)\,dx$ $= -\frac{\sin(\nu\pi)}{\pi\Gamma(\kappa)}\Gamma(2\mu+1)\Gamma(\kappa-\mu+\nu+1/2)\Gamma(\kappa-\mu-\nu-1/2)$ $\times e^{1/2a}W_{\rho,\sigma}(a)\qquad \rho=1/2-\kappa+\mu,\quad \sigma=1/2+\nu$ $\lvert\arg a\rvert<\pi,\quad \mathrm{Re}\,\mu>-1/2,\quad \mathrm{Re}(\kappa-\mu)>\lvert\mathrm{Re}\,\nu+1/2\rvert$
(12)	$\int_0^\infty x^{-1/2}(a+x)^{-\mu}e^{-1/2x}P_\nu^{-2\mu}(1+2x/a)M_{\kappa,\mu}(x)\,dx$ $= \frac{\Gamma(2\mu+1)\Gamma(\kappa+\mu+\nu+1/2)\Gamma(\kappa+\mu-\nu-1/2)}{\Gamma(\kappa+\mu+1/2)\Gamma(2\mu+\nu+1)\Gamma(2\mu-\nu)}\cdot$ $\times e^{1/2a}W_{1/2-\kappa-\mu,\,1/2+\nu}(a)$ $\lvert\arg a\rvert<\pi,\quad \mathrm{Re}\,\mu>-1/2,\quad \mathrm{Re}(\kappa+\mu)>\lvert\mathrm{Re}\,\nu+1/2\rvert$

Confluent hypergeometric functions (cont'd)

(13)	$\int_0^\infty x^{-\frac{1}{2}} e^{-\frac{1}{2}x} P_\nu^{-2\mu}[(1+x/a)^{\frac{1}{2}}] M_{\kappa,\mu}(x)\,dx$ $= \dfrac{\Gamma(2\mu+1)\Gamma(\kappa+\frac{1}{2}\nu)\Gamma(\kappa-\frac{1}{2}\nu-\frac{1}{2})}{2^{2\mu}a^{\frac{1}{4}}\Gamma(\kappa+\mu+\frac{1}{2})\Gamma(\mu+\frac{1}{2}\nu+\frac{1}{2})\Gamma(\mu-\frac{1}{2}\nu)}$ $\times e^{\frac{1}{2}a} W_{\frac{3}{4}-\kappa,\frac{1}{4}+\frac{1}{2}\nu}(a)$ $\lvert\arg a\rvert<\pi,\quad \mathrm{Re}\,\kappa>\frac{1}{2}\mathrm{Re}\,\nu-\frac{1}{2},\quad \mathrm{Re}\,\kappa>-\frac{1}{2}\mathrm{Re}\,\nu$
(14)	$\int_0^\infty x^{\mu-\frac{1}{4}\kappa-\frac{1}{2}\nu-\frac{1}{2}}(a+x)^{\frac{1}{2}\nu} e^{-\frac{1}{2}x} Q^\nu_{\mu-\kappa+3/2}(1+2x/a) M_{\kappa,\mu}(x)\,dx$ $= \dfrac{e^{\nu\pi i}\Gamma(1+2\mu-\nu)\Gamma(1+2\mu)\Gamma(5/2-\kappa+\mu+\nu)}{2\Gamma(\frac{1}{2}+\kappa+\mu)}$ $\times a^{\frac{1}{4}(\kappa+2\mu-2\nu+5)} e^{\frac{1}{2}a} W_{\rho,\sigma}(a)$ $\lvert\arg a\rvert<\pi,\quad \mathrm{Re}\,\mu>-\frac{1}{2},\quad \mathrm{Re}\,(2\mu-\nu)>-1$ $2\rho=\frac{1}{2}-\kappa-\mu+2\nu,\quad 2\sigma=\kappa-3\mu-3/2$
(15)	$\int_0^\infty x^{\frac{1}{2}(\kappa+\mu+\nu)-1}(a+x)^{-\frac{1}{2}} e^{-\frac{1}{2}x} Q^{1-\kappa+\mu-\nu}_{\kappa-\mu-\nu-1}[(1+x/a)^{\frac{1}{2}}] M_{\kappa,\mu}(x)\,dx$ $= e^{(1-\kappa+\mu-\nu)\pi i}\, 2^{\mu-\kappa-\nu}\, a^{\frac{1}{2}(\kappa+\mu-1)} \dfrac{\Gamma(\frac{1}{2}-\nu)\Gamma(1+2\mu)\Gamma(\kappa+\mu+\nu)}{\Gamma(\kappa+\mu+\frac{1}{2})}$ $\times e^{\frac{1}{2}a} W_{\rho,\sigma}(a)$ $\mathrm{Re}\,\mu>-\frac{1}{2},\quad \mathrm{Re}(\kappa+\mu+\nu)>0$ $\lvert\arg a\rvert<\pi,\quad \rho=\frac{1}{2}-\kappa-\frac{1}{2}\nu,\quad \sigma=\mu+\frac{1}{2}\nu$
(16)	$\int_0^\infty x^{\nu-\frac{1}{2}} e^{-\frac{1}{2}x} Q^{2\mu-2\nu}_{2\kappa-2\nu-3}[(1+x/a)^{\frac{1}{2}}] M_{\kappa,\mu}(x)\,dx$ $= e^{2(\mu-\nu)\pi i}\, 2^{2\mu-2\nu-1}\, a^{\frac{1}{2}(\kappa+\mu-1)} e^{\frac{1}{2}a}$ $\times \dfrac{\Gamma(2\mu+1)\Gamma(\nu+1)\Gamma(\kappa+\mu-2\nu-\frac{1}{2})}{\Gamma(\kappa+\mu+\frac{1}{2})} W_{\rho,\sigma}(a)$ $\lvert\arg a\rvert<\pi,\quad \mathrm{Re}\,\mu>-\frac{1}{2},\quad \mathrm{Re}\,\nu>-1,\quad \mathrm{Re}(\kappa+\mu-2\nu)>\frac{1}{2}$ $2\rho=1-\kappa+\mu-2\nu,\quad 2\sigma=\kappa-\mu-2\nu-2$

Confluent hypergeometric functions (cont'd)

(17)	$\int_0^\infty x^{\kappa-3/2} \exp[-\tfrac{1}{2}(a+1)x]\, K_\nu(\tfrac{1}{2}ax)\, M_{\kappa,\nu}(x)\, dx$ $= \dfrac{\pi^{1/2}\,\Gamma(\kappa)\,\Gamma(\kappa+2\nu)}{a^{\kappa+\nu}\,\Gamma(\kappa+\nu+\frac{1}{2})}\, {}_2F_1(\kappa,\ \kappa+2\nu;\ 2\nu+1;\ -a^{-1})$ $\operatorname{Re} a > 0, \quad \operatorname{Re}\kappa > 0, \quad \operatorname{Re}(\kappa+2\nu) > 0$
(18)	$\int_0^\infty x^{-1/2}\, J_\nu(ax^{1/2})\, K_{\frac{1}{2}\nu-\mu}(\tfrac{1}{2}x)\, M_{\kappa,\mu}(x)\, dx$ $= \dfrac{\Gamma(2\mu+1)}{a\,\Gamma(\kappa+\frac{1}{2}\nu+1)}\, W_{\frac{1}{2}(\kappa-\mu),\,\frac{1}{2}\kappa-\frac{1}{4}\nu}\left(\dfrac{a^2}{2}\right) M_{\frac{1}{2}(\kappa+\mu),\,\frac{1}{2}\kappa+\frac{1}{4}\nu}\left(\dfrac{a^2}{2}\right)$ $a > 0, \quad \operatorname{Re}\kappa > -\tfrac{1}{4}, \quad \operatorname{Re}\mu > -\tfrac{1}{2}, \quad \operatorname{Re}\nu > -1$
(19)	$\int_0^\infty x^{\rho-1}\, e^{-\frac{1}{2}x}\, M_{\gamma+\rho,\,\beta+\rho+\frac{1}{2}}(x)\, {}_2F_1(a,\ \beta;\ \gamma;\ -\lambda/x)\, dx$ $= \dfrac{\Gamma(a+\beta+2\rho)\,\Gamma(2\beta+2\rho)\,\Gamma(\gamma)}{\Gamma(\beta)\,\Gamma(\beta+\gamma+2\rho)}\, \lambda^{\frac{1}{2}\beta+\rho-\frac{1}{2}}\, e^{\frac{1}{2}\lambda}\, W_{\kappa,\mu}(\lambda)$ $\lvert\arg\lambda\rvert < \pi, \quad \operatorname{Re}(\beta+\rho) > 0, \quad \operatorname{Re}(a+\beta+2\rho) > 0, \quad \operatorname{Re}\gamma > 0$ $\kappa = \tfrac{1}{2} - a - \tfrac{1}{2}\beta - \rho, \quad \mu = \tfrac{1}{2}\beta + \rho$
(20)	$\int_0^\infty x^{2\lambda-1}\, (a+x)^{-\mu-\frac{1}{2}}\, e^{-\frac{1}{2}x}\, M_{\kappa,\mu}(a+x)\, dx$ $= \dfrac{a^{\lambda-\mu-\frac{1}{2}}\,\Gamma(2\lambda)\,\Gamma(2\mu+1)\,\Gamma(\kappa+\mu-2\lambda+\frac{1}{2})}{\Gamma(\kappa+\mu+\frac{1}{2})\,\Gamma(1-2\lambda+2\mu)}\, M_{\kappa-\lambda,\,\mu-\lambda}(a)$ $\operatorname{Re}\lambda > 0, \quad \operatorname{Re}(\kappa+\mu-2\lambda) > -\tfrac{1}{2}$
(21)	$\int_0^a x^{-\kappa-\lambda-1}\, (a-x)^{\lambda-1}\, e^{\frac{1}{2}x}\, W_{\kappa,\mu}(x)\, dx$ $= \dfrac{\Gamma(\lambda)\,\Gamma(\frac{1}{2}-\kappa-\lambda+\mu)\,\Gamma(\frac{1}{2}-\kappa-\lambda-\mu)}{a^{\kappa+1}\,\Gamma(\frac{1}{2}-\kappa+\mu)\,\Gamma(\frac{1}{2}-\kappa-\mu)}\, W_{\kappa+\lambda,\mu}(a)$ $\operatorname{Re}\lambda > 0, \quad \operatorname{Re}(\kappa+\lambda) < \tfrac{1}{2} - \lvert\operatorname{Re}\mu\rvert$

Confluent hypergeometric functions (cont'd)

(22)	$\int_0^\infty W_{\kappa,\mu}(x)\frac{dx}{x} = \frac{\pi^{3/2}\, 2^{\kappa} \sec(\mu\pi)}{\Gamma(\frac{3}{4}-\frac{1}{2}\kappa+\frac{1}{2}\mu)\,\Gamma(\frac{3}{4}-\frac{1}{2}\kappa-\frac{1}{2}\mu)}$ $-\frac{1}{2} < \mathrm{Re}\,\mu < \frac{1}{2}$
(23)	$\int_0^\infty x^{\kappa+2\mu-1} e^{-3x/2} W_{\kappa,\mu}(x)\,dx = \frac{\Gamma(\kappa+\mu+\frac{1}{2})\Gamma[\frac{1}{4}(2\kappa+6\mu+5)]}{(\kappa+3\mu+\frac{1}{2})\,\Gamma[\frac{1}{4}(2\mu-2\kappa+3)]}$ $\mathrm{Re}(\kappa+\mu) > -\frac{1}{2}, \quad \mathrm{Re}(\kappa+3\mu) > -\frac{1}{2}$
(24)	$\int_0^\infty x^{\rho-1}[x^{1/2}+(a+x)^{1/2}]^{2\sigma} e^{-\frac{1}{2}x} W_{\kappa,\mu}(x)\,dx$ $= -\pi^{-1/2}\,\sigma a^{\sigma} G_{34}^{32}\left(a \,\middle\vert\, \begin{matrix} \frac{1}{2},\, 1,\, 1-\kappa+\rho \\ \frac{1}{2}+\mu+\rho,\, \frac{1}{2}-\mu+\rho,\, -\sigma,\, \sigma \end{matrix}\right)$ $\lvert\arg a\rvert < \pi, \quad \mathrm{Re}\,\rho > \lvert\mathrm{Re}\,\mu\rvert - \frac{1}{2}$
(25)	$\int_0^\infty x^{\rho-1}[x^{1/2}+(a+x)^{1/2}]^{2\sigma} e^{\frac{1}{2}x} W_{\kappa,\mu}(x)\,dx$ $= -\frac{\sigma\pi^{-1/2} a^{\sigma}}{\Gamma(\frac{1}{2}-\kappa+\mu)\,\Gamma(\frac{1}{2}-\kappa-\mu)} G_{34}^{33}\left(a \,\middle\vert\, \begin{matrix} \frac{1}{2},\, 1,\, 1+\kappa+\rho \\ \frac{1}{2}+\mu+\rho,\, \frac{1}{2}-\mu+\rho,\, -\sigma,\, \sigma \end{matrix}\right)$ $\lvert\arg a\rvert < \pi, \quad \mathrm{Re}\,\rho > \lvert\mathrm{Re}\,\mu\rvert - \frac{1}{2}, \quad \mathrm{Re}(\kappa+\rho+\sigma) < 0$
(26)	$\int_0^\infty x^{\rho-1}(a+x)^{-1/2}[x^{1/2}+(a+x)^{1/2}]^{2\sigma} e^{\frac{1}{2}x} W_{\kappa,\mu}(x)\,dx$ $= \frac{\pi^{-1/2} a^{\sigma}}{\Gamma(\frac{1}{2}-\kappa+\mu)\,\Gamma(\frac{1}{2}-\kappa-\mu)} G_{34}^{33}\left(a \,\middle\vert\, \begin{matrix} 0,\, \frac{1}{2},\, \frac{1}{2}+\kappa+\rho \\ -\sigma,\, \rho+\mu,\, \rho-\mu,\, \sigma \end{matrix}\right)$ $\lvert\arg a\rvert < \pi, \quad \mathrm{Re}\,\rho > \lvert\mathrm{Re}\,\mu\rvert - \frac{1}{2}, \quad \mathrm{Re}(\kappa+\rho+\sigma) < \frac{1}{2}$
(27)	$\int_0^\infty x^{\rho-1}(a+x)^{-1/2}[x^{1/2}+(a+x)^{1/2}]^{2\sigma} e^{-\frac{1}{2}x} W_{\kappa,\mu}(x)\,dx$ $= \pi^{-1/2} a^{\sigma} G_{34}^{32}\left(a \,\middle\vert\, \begin{matrix} 0,\, \frac{1}{2},\, \frac{1}{2}-\kappa+\rho \\ -\sigma,\, \rho+\mu,\, \rho-\mu,\, \sigma \end{matrix}\right) \quad \lvert\arg a\rvert < \pi, \mathrm{Re}\,\rho > \lvert\mathrm{Re}\,\mu\rvert - \frac{1}{2}$

Confluent hypergeometric functions (cont'd)

(28)	$\int_0^\infty x^{\rho-1} \sin(cx^{1/2})\, e^{-1/2 x}\, W_{\kappa,\mu}(x)\, dx = \frac{\Gamma(1+\mu+\rho)\,\Gamma(1-\mu+\rho)}{\Gamma(3/2-\kappa+\rho)}$ $\times\, c_2F_2\left(1+\mu+\rho,\ 1-\mu+\rho;\ \frac{3}{2},\ \frac{3}{2}-\kappa+\rho;\ -\frac{c^2}{4}\right)$ $\mathrm{Re}\,\rho > \lvert\mathrm{Re}\,\mu\rvert - 1$
(29)	$\int_0^\infty x^{\rho-1} \sin(cx^{1/2})\, e^{1/2 x}\, W_{\kappa,\mu}(x)\, dx$ $= \frac{\pi^{1/2}}{\Gamma(1/2-\kappa+\mu)\,\Gamma(1/2-\kappa-\mu)}\, G_{23}^{22}\left(\frac{c^2}{4}\middle\vert \begin{matrix} 1/2+\mu-\rho,\ 1/2-\mu-\rho \\ 1/2,\ -\kappa-\rho,\ 0 \end{matrix}\right)$ $c > 0, \quad \mathrm{Re}\,\rho > \lvert\mathrm{Re}\,\mu\rvert - 1, \quad \mathrm{Re}(\kappa+\rho) < 1/2$
(30)	$\int_0^\infty x^{\rho-1} \cos(cx^{1/2})\, e^{-1/2 x}\, W_{\kappa,\mu}(x)\, dx = \frac{\Gamma(1/2+\mu+\rho)\,\Gamma(1/2-\mu+\rho)}{\Gamma(1-\kappa+\rho)}$ $\times\, {}_2F_2\left(\frac{1}{2}+\mu+\rho,\ \frac{1}{2}-\mu+\rho;\ \frac{1}{2},\ 1-\kappa+\rho;\ -\frac{c^2}{4}\right)$ $\mathrm{Re}\,\rho > \lvert\mathrm{Re}\,\mu\rvert - 1/2$
(31)	$\int_0^\infty x^{\rho-1} \cos(cx^{1/2})\, e^{1/2 x}\, W_{\kappa,\mu}(x)\, dx = \frac{\pi^{1/2}}{\Gamma(1/2-\kappa+\mu)\,\Gamma(1/2-\kappa-\mu)}$ $\times\, G_{23}^{22}\left(\frac{c^2}{4}\middle\vert \begin{matrix} 1/2+\mu-\rho,\ 1/2-\mu-\rho \\ 0,\ -\kappa-\rho,\ 1/2 \end{matrix}\right)$ $c > 0, \quad \mathrm{Re}\,\rho > \lvert\mathrm{Re}\,\mu\rvert - 1/2, \quad \mathrm{Re}(\kappa+\rho) < 1/2$
(32)	$\int_0^\infty x^{-1/2-1/2\mu-\nu}\,(a+x)^{1/2\mu}\, e^{-1/2 x}\, P^{\mu}_{\kappa+\nu-3/2}(1+2x\,a^{-1})\, W_{\kappa,\nu}(x)\, dx$ $= \frac{\Gamma(1-\mu-2\nu)}{\Gamma(3/2-\kappa-\mu-\nu)}\, a^{-1/4+1/2\kappa-1/2\nu}\, e^{1/2 a}\, W_{\rho,\sigma}(a)$ $2\rho = 1/2 + 2\mu + \nu - \kappa, \quad 2\sigma = \kappa + 3\nu - 3/2$ $\lvert\arg a\rvert < \pi, \quad \mathrm{Re}\,\mu < 1, \quad \mathrm{Re}(\mu+2\nu) < 1$

Confluent hypergeometric functions (cont'd)

(33)	$\int_0^\infty x^{-\frac{1}{2}-\frac{1}{2}\mu-\nu}(a+x)^{-\frac{1}{2}\mu} e^{-\frac{1}{2}x} P^{\mu}_{\kappa+\mu+\nu-3/2}(1+2x\,a^{-1})\, W_{\kappa,\nu}(x)\,dx$ $= \dfrac{\Gamma(1-\mu-2\nu)}{\Gamma(3/2-\kappa-\mu-\nu)}\, a^{-\frac{1}{4}+\frac{1}{2}\kappa-\frac{1}{2}\nu} e^{\frac{1}{2}a} W_{\rho,\sigma}(a)$ $2\rho=\frac{1}{2}-\kappa+\nu, \quad 2\sigma=\kappa+2\mu+3\nu-3/2$ $\lvert\arg a\rvert<\pi, \quad \operatorname{Re}\mu<1, \quad \operatorname{Re}(\mu+2\nu)<1$
(34)	$\int_0^\infty x^{-\frac{1}{2}-\frac{1}{2}\mu-\nu} e^{-\frac{1}{2}x} P^{\mu}_{2\kappa+\mu+2\nu-3}[(1+x/a)^{\frac{1}{2}}]\, W_{\kappa,\nu}(x)\,dx$ $= \dfrac{2^{\mu}\Gamma(1-\mu-2\nu)}{\Gamma(3/2-\kappa-\mu-\nu)}\, a^{-\frac{1}{2}+\frac{1}{2}\kappa-\frac{1}{2}\nu} e^{\frac{1}{2}a} W_{\rho,\sigma}(a)$ $2\rho=1-\kappa+\mu+\nu, \quad 2\sigma=\kappa+\mu+3\nu-2$ $\lvert\arg a\rvert<\pi, \quad \operatorname{Re}\mu<1, \quad \operatorname{Re}(\mu+2\nu)<1$
(35)	$\int_0^\infty x^{-\frac{1}{2}-\frac{1}{2}\mu-\nu}(a+x)^{-\frac{1}{2}} e^{-\frac{1}{2}x} P^{\mu}_{2\kappa+\mu+2\nu-2}[(1+x/a)^{\frac{1}{2}}]\, W_{\kappa,\nu}(x)\,dx$ $= \dfrac{2^{\mu}\Gamma(1-\mu-2\nu)}{\Gamma(3/2-\kappa-\mu-\nu)}\, a^{-\frac{1}{2}+\frac{1}{2}\kappa-\frac{1}{2}\nu} e^{\frac{1}{2}a} W_{\rho,\sigma}(a)$ $2\rho=\mu+\nu-\kappa, \quad 2\sigma=\kappa+\mu+3\nu-1$ $\lvert\arg a\rvert<\pi, \quad \operatorname{Re}\mu>0, \quad \operatorname{Re}\nu>0$
(36)	$\int_0^\infty x^{-\kappa-3/2}\exp[-\frac{1}{2}(a-1)x]\, K_{\mu}(\frac{1}{2}ax)\, W_{\kappa,\mu}(x)\,dx$ $= \dfrac{\pi\,\Gamma(-\kappa)\Gamma(2\mu-\kappa)\Gamma(-2\mu-\kappa)}{\Gamma(\frac{1}{2}-\kappa)\Gamma(\frac{1}{2}+\mu-\kappa)\Gamma(\frac{1}{2}-\mu-\kappa)}\, 2^{2\kappa+1} a^{\kappa-\nu}$ $\times\, {}_2F_1(-\kappa, 2\mu-\kappa; -2\kappa; 1-a^{-1})$ $\operatorname{Re} a>0, \quad \operatorname{Re}\kappa<2\operatorname{Re}\mu<-\operatorname{Re}\kappa$

Confluent hypergeometric functions (cont'd)

(37)	$\int_0^\infty x^{\rho-1} e^{-\frac{1}{2}x} J_{\lambda+\nu}(ax^{\frac{1}{2}}) J_{\lambda-\nu}(ax^{\frac{1}{2}}) W_{\kappa,\mu}(x)\,dx$ $= \dfrac{(\frac{1}{2}a)^{2\lambda}\Gamma(\frac{1}{2}+\lambda+\mu+\rho)\,\Gamma(\frac{1}{2}+\lambda-\mu+\rho)}{\Gamma(1+\lambda+\nu)\,\Gamma(1+\lambda-\nu)\,\Gamma(1+\lambda-\kappa+\rho)}$ $\times\, {}_4F_4(1+\lambda,\ \frac{1}{2}+\lambda,\ \frac{1}{2}+\lambda+\mu+\rho,\ \frac{1}{2}+\lambda-\mu+\rho;\ 1+\lambda+\nu,\ 1+\lambda-\nu,\ 1+2\lambda,\ 1+\lambda-\kappa+\rho;\ -a^2)$ $\lvert\operatorname{Re}\mu\rvert < \operatorname{Re}(\lambda+\rho)+\frac{1}{2}$
(38)	$\int_0^\infty x^{\rho-1} e^{-\frac{1}{2}x} I_{\lambda+\nu}(ax^{\frac{1}{2}}) K_{\lambda-\nu}(ax^{\frac{1}{2}}) W_{\kappa,\mu}(x)\,dx$ $= \dfrac{1}{2\pi^{\frac{1}{2}}} G^{24}_{45}\left(a^2 \middle\vert \begin{matrix} 0,\ \frac{1}{2},\ \frac{1}{2}+\mu-\rho,\ \frac{1}{2}-\mu-\rho \\ \lambda,\ \nu,\ -\lambda,\ -\nu,\ \kappa-\rho \end{matrix}\right)$ $\lvert\operatorname{Re}\mu\rvert < \operatorname{Re}(\lambda+\rho)+\frac{1}{2},\quad \lvert\operatorname{Re}\mu\rvert < \operatorname{Re}(\nu+\rho)+\frac{1}{2}$
(39)	$\int_0^\infty x^{-1} M_{\kappa,\mu}(x)\, W_{\lambda,\mu}(x)\,dx = \dfrac{\Gamma(2\mu+1)}{(\kappa-\lambda)\,\Gamma(\frac{1}{2}+\mu-\lambda)}$ $\operatorname{Re}\mu > -\frac{1}{2},\quad \operatorname{Re}(\kappa-\lambda) > 0$
(40)	$\int_0^\infty x^{-1} W_{\kappa,\mu}(x)\, W_{\lambda,\mu}(x)\,dx = \dfrac{1}{(\kappa-\lambda)\sin(2\mu\pi)}\left[\dfrac{1}{\Gamma(\frac{1}{2}-\kappa+\mu)\,\Gamma(\frac{1}{2}-\lambda-\mu)} - \dfrac{1}{\Gamma(\frac{1}{2}-\kappa-\mu)\,\Gamma(\frac{1}{2}-\lambda+\mu)}\right]$ $-\frac{1}{2} < \operatorname{Re}\mu < \frac{1}{2}$
(41)	$\int_0^\infty x^{\rho-1} W_{\kappa,\mu}(x)\, W_{-\kappa,\mu}(x)\,dx = \dfrac{\Gamma(\rho+1)\,\Gamma(\frac{\rho+1}{2}+\mu)\,\Gamma(\frac{\rho+1}{2}-\mu)}{2\Gamma(1+\frac{1}{2}\rho+\kappa)\,\Gamma(1+\frac{1}{2}\rho-\kappa)}$ $\operatorname{Re}\rho > 2\lvert\operatorname{Re}\mu\rvert - 1$

Confluent hypergeometric functions (cont'd)

(42)	$\int_0^\infty x^{\rho-1} W_{\kappa,\mu}(x) W_{\lambda,\nu}(x)\, dx$ $= \dfrac{\Gamma(1+\mu+\nu+\rho)\,\Gamma(1-\mu+\nu+\rho)\,\Gamma(-2\nu)}{\Gamma(\tfrac{1}{2}-\lambda-\nu)\,\Gamma(3/2-\kappa+\nu+\rho)}$ $\times {}_3F_2(1+\mu+\nu+\rho,\ 1-\mu+\nu+\rho,\ 1/2-\lambda+\nu;\ 1+2\nu,\ 3/2-\kappa+\nu+\rho;\ 1)$ $+\dfrac{\Gamma(1+\mu-\nu+\rho)\,\Gamma(1-\mu-\nu+\rho)\,\Gamma(2\nu)}{\Gamma(1/2-\lambda+\nu)\,\Gamma(3/2-\kappa-\nu+\rho)}$ $\times\ {}_3F_2(1+\mu-\nu+\rho,\ 1-\mu-\nu+\rho,\ 1/2-\lambda-\nu;$ $1-2\nu,\ 3/2-\kappa-\nu+\rho;\ 1)$ $\lvert\mathrm{Re}\,\mu\rvert+\lvert\mathrm{Re}\,\nu\rvert<\mathrm{Re}\,\rho+1$
(43)	$\int_0^\infty x^{\rho-1} \exp[-\tfrac{1}{2}(\alpha+\beta)x]\, M_{\kappa,\mu}(\alpha x)\, W_{\lambda,\nu}(\beta x)\, dx$ $= \dfrac{\Gamma(1+\mu+\nu+\rho)\,\Gamma(1+\mu-\nu+\rho)}{\Gamma(3/2-\lambda+\mu+\rho)}\ \alpha^{\mu+\frac{1}{2}}\ \beta^{-\mu-\rho-\frac{1}{2}}$ $\times\ {}_3F_2(1/2+\kappa+\mu,\ 1+\mu+\nu+\rho,\ 1+\mu-\nu+\rho;$ $2\mu+1,\ 3/2-\lambda+\mu+\rho;\ -\alpha/\beta)$ $\mathrm{Re}\,\alpha>0,\quad \mathrm{Re}\,\beta>0,\quad \mathrm{Re}(\rho+\mu)>\lvert\mathrm{Re}\,\nu\rvert-1$
(44)	$\int_0^\infty x^{\rho-1} \exp[\tfrac{1}{2}(\alpha+\beta)]\, W_{\kappa,\mu}(\alpha x)\, W_{\lambda,\nu}(\beta x)\, dx$ $= \beta^{-\rho}\,[\Gamma(\tfrac{1}{2}-\kappa+\mu)\,\Gamma(\tfrac{1}{2}-\kappa-\mu)\,\Gamma(\tfrac{1}{2}-\lambda+\nu)\,\Gamma(\tfrac{1}{2}-\lambda-\nu)]^{-1}$ $\times G^{33}_{33}\left(\dfrac{\beta}{\alpha}\,\middle\vert\, \begin{matrix}\tfrac{1}{2}+\mu,\ \tfrac{1}{2}-\mu,\ 1+\lambda+\rho\\ \tfrac{1}{2}+\nu+\rho,\ \tfrac{1}{2}-\nu+\rho,\ -\kappa\end{matrix}\right)$ $\lvert\mathrm{Re}\,\mu\rvert+\lvert\mathrm{Re}\,\nu\rvert<\mathrm{Re}\,\rho+1,\qquad \mathrm{Re}(\kappa+\lambda+\rho)<0$

Confluent hypergeometric functions (cont'd)

(45)	$\int_0^\infty x^{\rho-1} \exp[-\tfrac{1}{2}(\alpha-\beta)x]\, W_{\kappa,\mu}(\alpha x)\, W_{\lambda,\nu}(\beta x)\, dx$ $= \dfrac{\beta^{-\rho}}{\Gamma(\tfrac{1}{2}-\lambda+\nu)\,\Gamma(\tfrac{1}{2}-\lambda-\nu)} G^{23}_{33}\left(\dfrac{\beta}{\alpha}\middle\vert \begin{matrix} \tfrac{1}{2}+\mu,\ \tfrac{1}{2}-\mu,\ 1+\lambda+\rho \\ \tfrac{1}{2}+\nu+\rho,\ \tfrac{1}{2}-\nu+\rho,\ \kappa \end{matrix}\right)$ $\operatorname{Re}\alpha > 0, \quad \lvert\operatorname{Re}\mu\rvert + \lvert\operatorname{Re}\nu\rvert < \operatorname{Re}\rho + 1$
(46)	$\int_0^\infty x^{\rho-1} \exp[-\tfrac{1}{2}(\alpha+\beta)x]\, W_{\kappa,\mu}(\alpha x)\, W_{\lambda,\nu}(\beta x)\, dx$ $= \beta^{-\rho} G^{22}_{33}\left(\dfrac{\beta}{\alpha}\middle\vert \begin{matrix} \tfrac{1}{2}+\mu,\ \tfrac{1}{2}-\nu,\ 1-\lambda+\rho \\ \tfrac{1}{2}+\nu+\rho,\ \tfrac{1}{2}-\nu+\rho,\ \kappa \end{matrix}\right)$ $\operatorname{Re}(\alpha+\beta) > 0, \quad \lvert\operatorname{Re}\mu\rvert + \lvert\operatorname{Re}\nu\rvert < \operatorname{Re}\rho + 1$
(47)	$\int_0^\infty x^{2\lambda-1}(\alpha+x)^{-\mu-\frac{1}{2}} e^{-\frac{1}{2}x} W_{\kappa,\mu}(\alpha+x)\, dx$ $= \Gamma(2\lambda)\, \alpha^{\lambda-\mu-\frac{1}{2}} W_{\kappa-\lambda,\mu-\lambda}(\alpha) \qquad \lvert\arg\alpha\rvert < \pi, \quad \operatorname{Re}\lambda > 0$
(48)	$\int_0^\infty x^{\lambda-1}(\alpha+x)^{\kappa-\lambda-1} e^{-\frac{1}{2}x} W_{\kappa,\mu}(\alpha+x)\, dx$ $= \Gamma(\lambda)\, \alpha^{\kappa-1} W_{\kappa-\lambda,\mu}(\alpha) \qquad \lvert\arg\alpha\rvert < \pi, \quad \operatorname{Re}\lambda > 0$
(49)	$\int_0^\infty x^{\rho-1}(\alpha+x)^{-\sigma} e^{-\frac{1}{2}x} W_{\kappa,\mu}(\alpha+x)\, dx$ $= \Gamma(\rho)\, \alpha^{\rho} e^{\frac{1}{2}\alpha} G^{30}_{23}\left(\alpha\middle\vert \begin{matrix} 0,\ 1-\kappa-\sigma \\ -\rho,\ \tfrac{1}{2}+\mu-\sigma,\ \tfrac{1}{2}-\mu-\sigma \end{matrix}\right)$ $\lvert\arg\alpha\rvert < \pi, \quad \operatorname{Re}\rho > 0$
(50)	$\int_0^\infty x^{2\lambda-1}(\alpha+x)^{-\mu-\frac{1}{2}} e^{\frac{1}{2}x} W_{\kappa,\mu}(\alpha+x)\, dx$ $= \dfrac{\Gamma(2\lambda)\,\Gamma(\tfrac{1}{2}-\kappa+\mu-2\lambda)}{\Gamma(\tfrac{1}{2}-\kappa+\mu)} \alpha^{\lambda-\mu-\frac{1}{2}} W_{\kappa+\lambda,\mu-\lambda}(\alpha)$ $\lvert\arg\alpha\rvert < \pi, \quad 0 < 2\operatorname{Re}\lambda < \tfrac{1}{2} - \operatorname{Re}(\kappa+\mu)$

Confluent hypergeometric functions (cont'd)

(51)	$\int_0^\infty x^{\rho-1}(a+x)^{-\sigma}e^{\frac{1}{2}x}W_{\kappa,\mu}(a+x)\,dx = \dfrac{\Gamma(\rho)\,a^\rho\,e^{-\frac{1}{2}a}}{\Gamma(\frac{1}{2}-\kappa+\mu)\,\Gamma(\frac{1}{2}-\kappa-\mu)}$ $\times G_{23}^{31}\left(a\,\middle\vert\,\begin{matrix}\kappa-\sigma+1,\ 0\\ -\rho,\ \frac{1}{2}+\mu-\sigma,\ \frac{1}{2}-\mu-\sigma\end{matrix}\right)$ $\lvert\arg a\rvert<\pi, \quad 0<\mathrm{Re}\,\rho<\mathrm{Re}(\sigma-\kappa)$
(52)	$\int_0^\infty x^{\tau-1}(a+x)^{-\lambda}\,{}_2F_1(\rho,\sigma;\tau;-x/a)\,e^{\pm\frac{1}{2}x}W_{\kappa,\mu}(a+x)\,dx$ Express $W_{\kappa,\mu}$ in terms of G and then see under Meijer's G-function.
	Integrals involving products of $e^{\pm\frac{1}{2}x}W_{\kappa,\mu}(a+x)$ with Legendre functions. Express $W_{\kappa,\mu}$ in terms of G, and the Legendre function as a hypergeometric series, then see under Meijer's G-function.
(53)	$\int_0^\infty x^{-\frac{1}{2}}\exp\left[-\frac{1}{2}\left(\frac{x}{a}+\frac{\beta}{x}\right)\right]W_{\kappa,\mu}\left(\frac{x}{a}\right)W_{\kappa,\mu}\left(\frac{\beta}{x}\right)dx$ $=\pi^{\frac{1}{2}}\,2^{\frac{1}{2}-2\kappa}(a\beta)^{\frac{1}{4}}\exp\left[-\left(\frac{\beta}{a}\right)^{\frac{1}{2}}\right]W_{2\kappa-\frac{1}{2},2\mu}\left[2\left(\frac{\beta}{a}\right)^{\frac{1}{2}}\right]$ $\mathrm{Re}\,a>0, \quad \mathrm{Re}\,\beta>0$
(54)	$\int_0^\infty x^{\rho-1}\exp\left[-\frac{1}{2}\left(\frac{x}{a}+\frac{\beta}{x}\right)\right]W_{\kappa,\mu}\left(\frac{x}{a}\right)W_{\lambda,\nu}\left(\frac{\beta}{x}\right)dx$ $=\beta^\rho\,G_{24}^{40}\left(\frac{\beta}{a}\,\middle\vert\,\begin{matrix}1-\kappa,\ 1-\lambda-\rho\\ \frac{1}{2}+\mu,\ \frac{1}{2}-\mu,\ \frac{1}{2}+\nu-\rho,\ \frac{1}{2}-\nu-\rho\end{matrix}\right)$ $\mathrm{Re}\,a>0, \quad \mathrm{Re}\,\beta>0$
(55)	$\int_0^\infty x^{\rho-1}\exp\left[\frac{1}{2}\left(\frac{x}{a}-\frac{\beta}{x}\right)\right]W_{\kappa,\mu}\left(\frac{x}{a}\right)W_{\lambda,\nu}\left(\frac{\beta}{x}\right)dx$ $=\dfrac{\beta^\rho}{\Gamma(\frac{1}{2}-\kappa+\mu)\,\Gamma(\frac{1}{2}-\kappa-\mu)}\,G_{24}^{41}\left(\frac{\beta}{a}\,\middle\vert\,\begin{matrix}1+\kappa,\ 1-\lambda-\rho\\ \frac{1}{2}+\mu,\ \frac{1}{2}-\mu,\ \frac{1}{2}+\nu-\rho,\ \frac{1}{2}-\nu-\rho\end{matrix}\right)$ $\lvert\arg a\rvert<3\pi/2, \quad \mathrm{Re}\,\beta>0, \quad \mathrm{Re}(\kappa+\rho)<-\lvert\mathrm{Re}\,\nu\rvert-\frac{1}{2}$

Confluent hypergeometric functions (cont'd)

(56)	$\int_0^\infty x^{-\frac{1}{2}} \exp\left[\frac{1}{2}\left(\frac{x}{\alpha}+\frac{\beta}{x}\right)\right] W_{\kappa,\mu}\left(\frac{x}{\alpha}\right) W_{\kappa,\mu}\left(\frac{\beta}{x}\right) dx$ $= \frac{\pi^{\frac{1}{2}}\,\Gamma(-\kappa-\mu)\,\Gamma(-\kappa+\mu)\,(\alpha\beta)^{\frac{1}{4}}}{2^{\frac{1}{2}+2\kappa}\,\Gamma(\frac{1}{2}-\kappa+\mu)\,\Gamma(\frac{1}{2}-\kappa-\mu)}$ $\times \exp\left[\left(\frac{\beta}{\alpha}\right)^{\frac{1}{2}}\right] W_{2\kappa+\frac{1}{2},2\mu}\left[2\left(\frac{\beta}{\alpha}\right)^{\frac{1}{2}}\right]$ $\lvert\arg\alpha\rvert < 3\pi/2, \quad \lvert\arg\beta\rvert < 3\pi/2, \quad \operatorname{Re}\kappa < -\lvert\operatorname{Re}\mu\rvert$
(57)	$\int_0^\infty x^{\rho-1} \exp\left[\frac{1}{2}\left(\frac{x}{\alpha}+\frac{\beta}{x}\right)\right] W_{\kappa,\mu}\left(\frac{x}{\alpha}\right) W_{\lambda,\nu}\left(\frac{\beta}{x}\right) dx$ $= \frac{\beta^{\rho}}{\Gamma(\frac{1}{2}-\kappa+\mu)\,\Gamma(\frac{1}{2}-\kappa-\mu)\,\Gamma(\frac{1}{2}-\lambda+\nu)\,\Gamma(\frac{1}{2}-\lambda-\nu)}$ $\times G_{24}^{42}\left(\frac{\beta}{\alpha}\middle\vert \begin{matrix} 1+\kappa,\ 1+\lambda-\rho \\ \frac{1}{2}+\mu,\ \frac{1}{2}-\mu,\ \frac{1}{2}+\nu-\rho,\ \frac{1}{2}-\nu-\rho \end{matrix}\right)$ $\lvert\arg\alpha\rvert < 3\pi/2, \quad \lvert\arg\beta\rvert < 3\pi/2$ $\operatorname{Re}(\lambda-\rho) < \frac{1}{2}-\lvert\operatorname{Re}\mu\rvert, \quad \operatorname{Re}(\kappa+\rho) < \frac{1}{2}-\lvert\operatorname{Re}\nu\rvert$
(58)	$\int_{-\infty}^{\infty} e^{-2\rho x i}\,\Gamma(\frac{1}{2}+\nu+ix)\,\Gamma(\frac{1}{2}+\nu-ix)\,M_{ix,\nu}(2\alpha)\,dx$ $= 2^{\frac{1}{2}-\nu}\,\pi\,\beta^{\nu+\frac{1}{2}}\,(\cosh\rho)^{-2\nu-1}\exp(-\alpha\tanh\rho)\,\Gamma(2\nu+1)$ $\lvert\operatorname{Im}\rho\rvert < \frac{1}{2}\pi, \quad \operatorname{Re}\nu > -\frac{1}{2}$
(59)	$\int_{-i\infty}^{i\infty} \Gamma(\frac{1}{2}+\nu+\mu+x)\,\Gamma(\frac{1}{2}+\nu+\mu-x)\,\Gamma(\frac{1}{2}+\nu-\mu+x)\,\Gamma(\frac{1}{2}+\nu-\mu-x)$ $\times M_{\mu+ix,\nu}(\alpha)\,M_{\mu-ix,\nu}(\beta)\,dx$ $= \frac{2\pi(\alpha\beta)^{\nu+\frac{1}{2}}\,[\Gamma(2\nu+1)]^2\,\Gamma(2\nu+2\mu+1)\,\Gamma(2\nu-2\mu+1)}{(\alpha+\beta)^{2\nu+1}\,\Gamma(4\nu+2)}$ $\times M_{2\mu,\,2\nu+\frac{1}{2}}(\alpha+\beta) \qquad \operatorname{Re}\nu > \lvert\operatorname{Re}\mu\rvert - \frac{1}{2}$

Confluent hypergeometric functions (cont'd)

(60)	$\int_{-\infty}^{\infty} e^{-2\rho x i}\Gamma(\tfrac{1}{2}+\nu+ix)\,\Gamma(\tfrac{1}{2}+\nu-ix)\,M_{ix,\nu}(\alpha)\,M_{ix,\nu}(\beta)\,dx$ $= \dfrac{2\pi(\alpha\beta)^{\frac{1}{2}}}{\cosh\rho}\exp[-(\alpha+\beta)\tanh\rho]\,J_{2\nu}\left(\dfrac{2\alpha^{\frac{1}{2}}\beta^{\frac{1}{2}}}{\cosh\rho}\right)$ $\lvert\operatorname{Im}\rho\rvert < \tfrac{1}{2}\pi, \quad \operatorname{Re}\nu > -\tfrac{1}{2}$
(61)	$\int_{-\infty}^{\infty} \operatorname{sech}(\pi x)\,W_{ix,0}(\alpha)\,W_{-ix,0}(\beta)\,dx = 2\,\dfrac{(\alpha\beta)^{\frac{1}{2}}}{\alpha+\beta}\exp[-\tfrac{1}{2}(\alpha+\beta)]$
(62)	$\int_{-\infty}^{\infty} \Gamma(ix)\,\Gamma(2\kappa+ix)\,W_{\kappa+ix,\kappa-\frac{1}{2}}(\alpha)\,W_{-\kappa-ix,\kappa-\frac{1}{2}}(\beta)\,dx$ $= 2\pi^{\frac{1}{2}}\,\Gamma(2\kappa)\,(\alpha\beta)^{\kappa}\,(\alpha+\beta)^{\frac{1}{2}-2\kappa}\,K_{2\kappa-\frac{1}{2}}\left(\dfrac{\alpha+\beta}{2}\right)$
	For numerous integrals with respect to parameters see Buchholz, Herbert, 1953: *Die konfluente hypergeometrische Funktion*. Springer Verlag. Chapter VI.

20.4. MacRobert's E-function

See also under G-function.

(1)	$\int_0^1 x^{\beta-1}(1-x)^{\gamma-\beta-1}E(\alpha_1,\ldots,\alpha_p:\rho_1,\ldots,\rho_q:xz)\,dx$ See MacRobert, T.M., 1953: *Proc. Glasgow Math. Assoc.* 1, 118.
(2)	$\int_0^1 x^{\beta-1}(1-x)^{\gamma-\beta-1}E(\alpha_1,\ldots,\alpha_p:\rho_1,\ldots,\rho_q:x^{-m}z)\,dx$ $=\Gamma(\gamma-\beta)\,m^{\beta-\gamma}\,E(\alpha_1,\ldots,\alpha_{p+m}:\rho_1,\ldots,\rho_{q+m}:z)$ $\alpha_{p+k}=\dfrac{\beta+k-1}{m},\ \rho_{p+k}=\dfrac{\gamma+k-1}{m},\quad k=1,\ldots,m$ $\operatorname{Re}\gamma>\operatorname{Re}\beta>0,\quad m=1,2,\ldots,$

MacRobert's E-function (cont'd)

(3)	$\int_0^\infty x^{\rho-1}(1+x)^{-\sigma} E[a_1,\dots,a_p:\rho_1,\dots,\rho_q:(1+x)z]\,dx$ $=\Gamma(\rho)E(a_1,\dots,a_p,\sigma-\rho:\rho_1,\dots,\rho_q,\sigma:z)\qquad \mathrm{Re}\,\sigma>\mathrm{Re}\,\rho>0$
(4)	$\int_0^\infty x^{\beta-1}e^{-x}E(a_1,\dots,a_p:\rho_1,\dots,\rho_q:xz)\,dx$ $=\pi\csc(\beta\pi)[E(a_1,\dots,a_p:1-\beta,\rho_1,\dots,\rho_q:e^{\pm i\pi}z)$ $-z^{-\beta}E(a_1+\beta,\dots,a_p+\beta:1+\beta,\rho_1+\beta,\dots,\rho_q+\beta:e^{\pm i\pi}z)]$ $p\geq q+1,\quad \mathrm{Re}(a_r+\beta)>0,\quad r=1,\dots,p,\quad \lvert\arg z\rvert<\pi$ For $p\leq q$ the result holds if the integral is convergent.
(5)	$\int_0^\infty x^{\beta-1}e^{-x}E(a_1,\dots,a_p:\rho_1,\dots,\rho_q:x^{-m}z)\,dx$ $=(2\pi)^{\frac{1}{2}-\frac{1}{2}m}m^{\beta-\frac{1}{2}}E(a_1,\dots,a_{p+m}:\rho_1,\dots,\rho_q:m^{-m}z)$ $\mathrm{Re}\,\beta>0,\quad m=1,2,\dots,\quad a_{p+k}=(\beta+k-1)/m,\quad k=1,\dots,m$
(6)	$\int_{-1}^1(1-x)^{-a_p}(1-x^2)^{-\frac{1}{2}\mu}\mathrm{P}^\mu_{n-\mu}(x)E[a_1,\dots,a_p:\rho_1,\dots,\rho_q:(1-x)z]dx$ See MacRobert, T.M., 1953: *Proc. Glasgow Math. Assoc.* 1, 111-114.
(7)	$\int_0^\infty x^{\beta-1}J_\nu(x)E(a_1,\dots,a_p:\rho_1,\dots,\rho_q:x^{-2m}z)\,dx$ $=(2\pi)^{-m}(2m)^{\beta-1}\{\exp[\frac{1}{2}\pi(\beta-\nu-1)i]$ $\times E[a_1,\dots,a_{p+2m}:\rho_1,\dots,\rho_q:(2m)^{-2m}ze^{-m\pi i}]$ $+\exp[-\frac{1}{2}\pi(\beta-\nu-1)i]E[a_1,\dots,a_{p+2m}:\rho_1,\dots,\rho_q:(2m)^{-2m}ze^{m\pi i}]\}$ $\mathrm{Re}(\beta+\nu)>0,\quad \mathrm{Re}(2a_rm-\beta)>-3/2,\quad r=1,\dots,p$ $a_{p+k}=\dfrac{\beta+\nu+2k-2}{2m},\quad a_{p+m+k}=\dfrac{\beta-\nu+2k-2}{2m}$ $m=1,2,\dots,\quad k=1,\dots,m$

MacRobert's E-function (cont'd)

(8)	$\int_0^\infty x^{\beta-1} K_\nu(x) E(\alpha_1,\dots,\alpha_p:\rho_1,\dots,\rho_q: x^{-2m} z)\,dx$ $= (2\pi)^{1-m}\, 2^{\beta-2}\, m^{\beta-1} E[\alpha_1,\dots,\alpha_{p+2m}:\rho_1,\dots,\rho_q:(2m)^{-2m} z]$ $\alpha_{p+k} = \dfrac{\beta+\nu+2k-2}{2m}, \quad \alpha_{p+m+k} = \dfrac{\beta-\nu+2k-2}{2m}$ $\operatorname{Re}\beta > \lvert\operatorname{Re}\nu\rvert, \quad m = 1, 2, \dots, \quad k = 1, \dots, m$
(9)	$\int_0^\infty x^{\beta-1} e^x K_\nu(x) E(\alpha_1,\dots,\alpha_p:\rho_1,\dots,\rho_q: z/x)\,dx$ See Ragab, F.M., 1953: *Proc. Glasgow Math. Assoc.* 1, 192-195.
(10)	$\int_0^\infty x^{\beta-1} e^{-\frac{1}{2}x} W_{\kappa,\mu}(x) E(\alpha_1,\dots,\alpha_p:\rho_1,\dots,\rho_q: x^{-m} z)\,dx$ $= (2\pi)^{\frac{1}{2}-\frac{1}{2}m}\, m^{\beta+\kappa-\frac{1}{2}} E(\alpha_1,\dots,\alpha_{p+2m}:\rho_1,\dots,\rho_{q+m}: m^{-m} z)$ $\operatorname{Re}\beta > \lvert\operatorname{Re}\mu\rvert - \tfrac{1}{2}, \quad m = 1, 2, \dots$ $\alpha_{p+k} = (\beta+k+\mu-\tfrac{1}{2})/m, \quad \alpha_{p+m+k} = (\beta-\mu+k-\tfrac{1}{2})/m$ $\rho_{q+k} = (\beta-\kappa+k)/m, \quad k = 1, \dots, m$
(11)	$\int_0^\infty x^{\lambda-1} E(\alpha_1,\dots,\alpha_p:\rho_1,\dots,\rho_q: xy)\, E(\beta_1,\dots,\beta_r:\sigma_1,\dots,\sigma_s: xz)\,dx$ $\int_0^\infty x^{\lambda-1} E(\alpha_1,\dots,\alpha_p:\rho_1,\dots,\rho_q: xy)\, E(\beta_1,\dots,\beta_r:\sigma_1,\dots,\sigma_s: z/x)\,dx$ See Ragab, F.M., 1953: *Proc. Glasgow Math. Assoc.* 1, 192-195.

20.5. Meijer's G-function

(1)	$$\int_0^1 x^{\rho-1}(1-x)^{\sigma-1}\, G_{pq}^{mn}\left(\alpha x \,\middle\vert\, \begin{matrix} a_1,\dots,a_p \\ b_1,\dots,b_q \end{matrix}\right) dx$$ $$= \Gamma(\sigma)\, G_{p+1,q+1}^{m,n+1}\left(\alpha \,\middle\vert\, \begin{matrix} 1-\rho, a_1,\dots,a_p \\ b_1,\dots,b_q, 1-\rho-\sigma \end{matrix}\right)$$ First set of conditions of validity: $p+q<2(m+n), \quad \lvert\arg\alpha\rvert<(m+n-\tfrac{1}{2}p-\tfrac{1}{2}q)\pi$ $\operatorname{Re}(\rho+b_j)>0, \quad j=1,\dots,m, \quad \operatorname{Re}\sigma>0$ Second set of conditions of validity: $p+q\le 2(m+n), \quad \lvert\arg\alpha\rvert\le(m+n-\tfrac{1}{2}p-\tfrac{1}{2}q)\pi$ $\operatorname{Re}(\rho+b_j)>0, \quad j=1,\dots,m, \quad \operatorname{Re}\sigma>0$ $\operatorname{Re}\left[\sum_{j=1}^{p} a_j-\sum_{j=1}^{q} b_j+(p-q)(\rho-\tfrac{1}{2})\right]>-\tfrac{1}{2}$ Third set of conditions of validity: $p<q \quad (\text{or } p\le q \quad \text{and} \quad \lvert\alpha\rvert<1)$ $\operatorname{Re}(\rho+b_j)>0, \quad j=1,\dots,m, \quad \operatorname{Re}\sigma>0$
(2)	$$\int_1^\infty x^{-\rho}(x-1)^{\sigma-1}\, G_{pq}^{mn}\left(\alpha x \,\middle\vert\, \begin{matrix} a_1,\dots,a_p \\ b_1,\dots,b_q \end{matrix}\right) dx$$ $$= \Gamma(\sigma)\, G_{p+1,q+1}^{m+1,n}\left(\alpha \,\middle\vert\, \begin{matrix} a_1,\dots,a_p, \rho \\ \rho-\sigma, b_1,\dots,b_q \end{matrix}\right)$$ First set of conditions of validity: $p+q<2(m+n), \quad \lvert\arg\alpha\rvert<(m+n-\tfrac{1}{2}p-\tfrac{1}{2}q)\pi$ $\operatorname{Re}(\rho-\sigma-a_j)>-1, \quad j=1,\dots,n, \quad \operatorname{Re}\sigma>0$ Second set of conditions of validity: $p+q\le 2(m+n), \quad \lvert\arg\alpha\rvert\le(m+n-\tfrac{1}{2}p-\tfrac{1}{2}q)\pi$ $\operatorname{Re}(\rho-\sigma-a_j)>-1, \quad j=1,\dots,n, \quad \operatorname{Re}\sigma>0$ $\operatorname{Re}\left[\sum_{j=1}^{p} a_j-\sum_{j=1}^{q} b_j+(q-p)(\rho-\sigma+\tfrac{1}{2})\right]>-\tfrac{1}{2}$

Meijer's G-function (cont'd)

(2)	$\int_1^\infty x^{-\rho}(x-1)^{\sigma-1}\, G^{mn}_{pq}\left(ax\,\middle\vert\,\begin{matrix}a_1,\dots,a_p\\ b_1,\dots,b_q\end{matrix}\right)dx$ $=\Gamma(\sigma)\, G^{m+1,n}_{p+1,q+1}\left(a\,\middle\vert\,\begin{matrix}a_1,\dots,a_p,\rho\\ \rho-\sigma, b_1,\dots,b_q\end{matrix}\right)$ Third set of conditions of validity: $q<p$ (or $q\le p$ and $\lvert a\rvert>1$) $\operatorname{Re}(\rho-\sigma-a_j)>-1,\quad j=1,\dots,n,\quad \operatorname{Re}\sigma>0$
(3)	$\int_0^\infty x^{\rho-1}\, G^{mn}_{pq}\left(ax\,\middle\vert\,\begin{matrix}a_1,\dots,a_p\\ b_1,\dots,b_q\end{matrix}\right)dx$ $=\dfrac{\prod_{j=1}^{m}\Gamma(b_j+\rho)\prod_{j=1}^{n}\Gamma(1-a_j-\rho)}{\prod_{j=m+1}^{q}\Gamma(1-b_j-\rho)\prod_{j=n+1}^{p}\Gamma(a_j+\rho)}\,a^{\rho}$ $p+q<2(m+n),\quad \lvert\arg a\rvert<(m+n-\tfrac12 p-\tfrac12 q)\pi$ $-\min_{1\le j\le m}\operatorname{Re}b_j<\operatorname{Re}\rho<1-\max_{1\le j\le n}\operatorname{Re}a_j$
(4)	$\int_0^\infty x^{\rho-1}(x+\beta)^{-\sigma}\, G^{mn}_{pq}\left(ax\,\middle\vert\,\begin{matrix}a_1,\dots,a_p\\ b_1,\dots,b_q\end{matrix}\right)dx$ $=\dfrac{\beta^{\rho-\sigma}}{\Gamma(\sigma)}\, G^{m+1,n+1}_{p+1,q+1}\left(a\beta\,\middle\vert\,\begin{matrix}1-\rho, a_1,\dots,a_p\\ \sigma-\rho, b_1,\dots,b_q\end{matrix}\right)$ First set of conditions of validity: $p+q<2(m+n),\quad \lvert\arg a\rvert<(m+n-\tfrac12 p-\tfrac12 q)\pi,\quad \lvert\arg\beta\rvert<\pi$ $\operatorname{Re}(\rho+b_j)>0,\quad j=1,\dots,m,\quad \operatorname{Re}(\rho-\sigma+a_j)<1,\quad j=1,\dots,n$

Meijer's G-function (cont'd)

(4)	$\int_0^\infty x^{\rho-1}(x+\beta)^{-\sigma} G_{pq}^{mn}\left(\alpha x \middle\vert \begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix}\right) dx$ $= \dfrac{\beta^{\rho-\sigma}}{\Gamma(\sigma)} G_{p+1,q+1}^{m+1,n+1}\left(\alpha\beta \middle\vert \begin{matrix} 1-\rho, a_1, \dots, a_p \\ \sigma-\rho, b_1, \dots, b_q \end{matrix}\right)$ Second set of conditions of validity: $p \le q,\ p+q \le 2(m+n),\ \lvert\arg\alpha\rvert \le (m+n-\tfrac{1}{2}p-\tfrac{1}{2}q)\pi,\ \lvert\arg\beta\rvert < \pi$ $\mathrm{Re}(\rho+b_j) > 0,\ j=1, \dots, m,\ \mathrm{Re}(\rho-\sigma+a_j) < 1,\ j=1, \dots, n$ $\mathrm{Re}[\sum_{j=1}^{p} a_j - \sum_{j=1}^{q} b_j - (q-p)(\rho-\sigma-\tfrac{1}{2})] > 1$ Third set of conditions of validity: $p \ge q,\ p+q \le 2(m+n),\ \lvert\arg\alpha\rvert \le (m+n-\tfrac{1}{2}p-\tfrac{1}{2}q)\pi,\ \lvert\arg\beta\rvert < \pi$ $\mathrm{Re}(\rho+b_j) > 0,\ j=1, \dots, m,\ \mathrm{Re}(\rho-\sigma+a_j) < 1,\ j=1, \dots, n$ $\mathrm{Re}[\sum_{j=1}^{p} a_j - \sum_{j=1}^{q} b_j + (p-q)(\rho-\tfrac{1}{2})] > 1$
(5)	$\int_0^\infty x^{-\rho} e^{-\beta x} G_{pq}^{mn}\left(\alpha x \middle\vert \begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix}\right) dx$ $= \beta^{\rho-1} G_{p+1,q}^{m,n+1}\left(\dfrac{\alpha}{\beta} \middle\vert \begin{matrix} \rho, a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix}\right)$ $p+q < 2(m+n),\ \lvert\arg\alpha\rvert < (m+n-\tfrac{1}{2}p-\tfrac{1}{2}q)\pi,\ \lvert\arg\beta\rvert < \tfrac{1}{2}\pi$ $\mathrm{Re}(b_j-\rho) > -1,\quad j=1, \dots, m$
(6)	$\int_0^\infty e^{-\beta x} G_{pq}^{mn}\left(\alpha x^2 \middle\vert \begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix}\right) dx$ $= \pi^{-\frac{1}{2}} \beta^{-1} G_{p+2,q}^{m,n+2}\left(\dfrac{4\alpha}{\beta^2} \middle\vert \begin{matrix} 0, \tfrac{1}{2}, a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix}\right)$ $p+q < 2(m+n),\ \lvert\arg\alpha\rvert < (m+n-\tfrac{1}{2}p-\tfrac{1}{2}q)\pi,\ \lvert\arg\beta\rvert < \tfrac{1}{2}\pi$ $\mathrm{Re}\, b_j > -\tfrac{1}{2},\quad j=1, \dots, m$

Meijer's G-functions (cont'd)

(7)	$\int_0^\infty \sin(cx)\, G_{pq}^{mn}\left(ax^2 \,\middle\vert\, \begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix}\right) dx$ $= \pi^{1/2} c^{-1} G_{p+2,q}^{m,n+1}\left(\frac{4a}{c^2} \,\middle\vert\, \begin{matrix} 0, a_1, \dots, a_p, \frac{1}{2} \\ b_1, \dots, b_q \end{matrix}\right)$ $p + q < 2(m+n), \quad \lvert\arg a\rvert < (m + n - \tfrac{1}{2}p - \tfrac{1}{2}q)\pi, \quad c > 0$ $\operatorname{Re} b_j > -1, \quad j = 1, \dots, m, \quad \operatorname{Re} a_j < \tfrac{1}{2}, \quad j = 1, \dots, n$
(8)	$\int_0^\infty \cos(cx)\, G_{pq}^{mn}\left(ax^2 \,\middle\vert\, \begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix}\right) dx$ $= \pi^{1/2} c^{-1} G_{p+2,q}^{m,n+1}\left(\frac{4a}{c^2} \,\middle\vert\, \begin{matrix} \frac{1}{2}, a_1, \dots, a_p, 0 \\ b_1, \dots, b_q \end{matrix}\right)$ $p + q < 2(m+n), \quad \lvert\arg a\rvert < (m + n - \tfrac{1}{2}p - \tfrac{1}{2}q)\pi, \quad c > 0$ $\operatorname{Re} b_j > -\tfrac{1}{2}, \quad j = 1, \dots, m, \quad \operatorname{Re} a_j < \tfrac{1}{2}, \quad j = 1, \dots, n$
(9)	$\int_0^\infty x^{-\rho} J_\nu(2x^{1/2})\, G_{pq}^{mn}\left(ax \,\middle\vert\, \begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix}\right) dx$ $= G_{p+2,q}^{m,n+1}\left(a \,\middle\vert\, \begin{matrix} \rho - \frac{1}{2}\nu, a_1, \dots, a_p, \rho + \frac{1}{2}\nu \\ b_1, \dots, b_q \end{matrix}\right).$ $p + q < 2(m+n), \quad \lvert\arg a\rvert < (m + n - \tfrac{1}{2}p - \tfrac{1}{2}q)\pi$ $-\tfrac{3}{4} + \max\limits_{1\le j\le n} \operatorname{Re} a_j < \operatorname{Re}\rho < 1 + \tfrac{1}{2}\operatorname{Re}\nu + \min\limits_{1\le j\le m} \operatorname{Re} b_j$
(10)	$\int_0^\infty x^{-\rho} Y_\nu(2x^{1/2})\, G_{pq}^{mn}\left(ax \,\middle\vert\, \begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix}\right) dx$ $= G_{p+3,q+1}^{m,n+2}\left(a \,\middle\vert\, \begin{matrix} \rho - \frac{1}{2}\nu, \rho + \frac{1}{2}\nu, a_1, \dots, a_p, \rho + \frac{1}{2} + \frac{1}{2}\nu \\ b_1, \dots, b_q, \rho + \frac{1}{2} + \frac{1}{2}\nu \end{matrix}\right)$ $p + q < 2(m+n), \quad \lvert\arg a\rvert < (m + n - \tfrac{1}{2}p - \tfrac{1}{2}q)\pi$ $-\tfrac{3}{4} + \max\limits_{1\le j\le n} \operatorname{Re} a_j < \operatorname{Re}\rho < \min\limits_{1\le j\le m} \operatorname{Re} b_j + \tfrac{1}{2}\lvert\operatorname{Re}\nu\rvert + 1$

Meijer's G-function (cont'd)

(11)	$\int_0^\infty x^{-\rho} K_\nu(2x^{1/2})\, G_{pq}^{mn}\left(ax \middle\vert \begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix}\right) dx$ $= \tfrac{1}{2} G_{p+2,\, q}^{m,\, n+2}\left(a \middle\vert \begin{matrix} \rho - \tfrac{1}{2}\nu,\ \rho + \tfrac{1}{2}\nu,\ a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix}\right)$ $p + q < 2(m+n), \quad \lvert\arg a\rvert < (m + n - \tfrac{1}{2}p - \tfrac{1}{2}q)\pi$ $\operatorname{Re}\rho < 1 - \tfrac{1}{2}\lvert\operatorname{Re}\nu\rvert + \min_{1\le j\le m} \operatorname{Re} b_j$
(12)	$\int_0^\infty x^{-\rho}\, \mathbf{H}_\nu(2x^{1/2})\, G_{pq}^{mn}\left(ax \middle\vert \begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix}\right) dx$ $= G_{p+3,\, q+1}^{m+1,\, n+1}\left(a \middle\vert \begin{matrix} \rho - \tfrac{1}{2} - \tfrac{1}{2}\nu,\ a_1, \dots, a_p,\ \rho + \tfrac{1}{2}\nu,\ \rho - \tfrac{1}{2}\nu \\ \rho - \tfrac{1}{2} - \tfrac{1}{2}\nu,\ b_1, \dots, b_q \end{matrix}\right)$ $p + q < 2(m+n), \quad \lvert\arg a\rvert < (m + n - \tfrac{1}{2}p - \tfrac{1}{2}q)\pi$ $\max\left(-\frac{3}{4}, \operatorname{Re}\frac{\nu-1}{2}\right) + \max_{1\le j\le n} \operatorname{Re} a_j < \operatorname{Re}\rho < \min_{1\le j\le m} \operatorname{Re} b_j + \frac{1}{2}\operatorname{Re}\nu + \frac{3}{2}$
(13)	$\int_1^\infty x^{-\rho}(x-1)^{\sigma-1}\, {}_2F_1(\kappa+\sigma-\rho,\, \lambda+\sigma-\rho;\, \sigma;\, 1-x)\, G_{pq}^{mn}\left(ax \middle\vert \begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix}\right) dx$ $= \Gamma(\sigma)\, G_{p+2,\, q+2}^{m+2,\, n}\left(a \middle\vert \begin{matrix} a_1, \dots, a_p,\ \kappa + \lambda + \sigma - \rho,\ \rho \\ \kappa,\ \lambda,\ b_1, \dots, b_q \end{matrix}\right)$ First set of conditions of validity: $p + q < 2(m+n), \quad \lvert\arg a\rvert < (m + n - \tfrac{1}{2}p - \tfrac{1}{2}q)\pi$ $\operatorname{Re}\sigma > 0, \quad \operatorname{Re}\kappa \ge \operatorname{Re}\lambda > \operatorname{Re} a_j - 1, \quad j = 1, \dots, n$ Second set of conditions of validity: $p + q \le 2(m+n), \quad \lvert\arg a\rvert \le (m + n - \tfrac{1}{2}p - \tfrac{1}{2}q)\pi$ $\operatorname{Re}\sigma > 0, \quad \operatorname{Re}\kappa \ge \operatorname{Re}\lambda > \operatorname{Re} a_j - 1, \quad j = 1, \dots, n$ $\operatorname{Re}\left[\sum_{j=1}^{p} a_j - \sum_{j=1}^{q} b_j + (q-p)(\kappa + \tfrac{1}{2})\right] > -\tfrac{1}{2}$ $\operatorname{Re}\left[\sum_{j=1}^{p} a_j - \sum_{j=1}^{q} b_j + (q-p)(\lambda + \tfrac{1}{2})\right] > -\tfrac{1}{2}$

Meijer's G-function (cont'd)

(14)	$$\int_0^\infty G_{pq}^{mn}\left(\alpha x \,\middle|\, \begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix}\right) G_{rs}^{kl}\left(\beta x \,\middle|\, \begin{matrix} c_1, \dots, c_r \\ d_1, \dots, d_s \end{matrix}\right) dx$$ $$= \alpha^{-1} G_{q+r,\, p+s}^{k+n,\, l+m}\left(\frac{\beta}{\alpha} \,\middle|\, \begin{matrix} -b_1, \dots, -b_m,\, c_1, \dots, c_r,\, -b_{m+1}, \dots, -b_q \\ -a_1, \dots, -a_n,\, d_1, \dots, d_s,\, -a_{n+1}, \dots, -a_p \end{matrix}\right)$$ For (five sets of) conditions of validity see Meijer, C.S., 1941: *Nederl. Akad. Wetensch., Proc.* 44, 82-92.

APPENDIX

NOTATIONS AND DEFINITIONS OF HIGHER TRANSCENDENTAL FUNCTIONS

H.T.F. I refers to volume I, and H.T.F. II to volume II, of *Higher transcendental functions* by the same authors as the present work.

Miscellaneous notations

Ad hoc notations are explained where they occur. Notations occurring several times on a page are explained at the bottom of the page.

In general, real variables and parameters are denoted by Latin letters, and complex variables and parameters by Greek letters. Exceptions are made to preserve traditional notations (such as y in chapter XIV). The letters m, n denote integers mostly.

Re z, Im z. Real and imaginary parts of a complex quantity z.

$|z|$, arg z. Modulus and argument (phase) of a complex quantity.

Cauchy Principal Value. If the integrand has a singularity at c, $a < c < b$, the Cauchy Principal Value of

$$\int_a^b f(x)\,dx$$

is

$$⨍_a^b f(x)\,dx = \lim\left[\int_a^{c-\epsilon} f(x)\,dx + \int_{c+\epsilon}^b f(x)\,dx\right] \qquad \epsilon > 0, \quad \epsilon \to 0.$$

Empty sums are to be interpreted as zero, and empty products as unity. $\sum\limits_{n=a}^{b}$, $\prod\limits_{n=a}^{b}$ are empty if $b < a$.

$[x]$ largest integer $\leq x$.

$$(a)_\nu = \Gamma(a+\nu)/\Gamma(a)$$

$$(a)_0 = 1$$

$$(a)_n = a(a+1)\cdots(a+n-1) \qquad n = 1, 2, \ldots$$

$$(a)_n = (-1)^n (1-a-n)_n \qquad n \text{ integer}$$

$$(a)_{-n} = (-1)^n/(1-a)_n \qquad n \text{ integer}$$

Binomial coefficient

$$\binom{a}{\beta} = \frac{\Gamma(a+1)}{\Gamma(\beta+1)\,\Gamma(a-\beta+1)}.$$

$$\operatorname{sgn} x = \begin{cases} -1 & x<0 \\ 0 & x=0 \\ 1 & x>0 \end{cases}$$

Euler-Mascheroni constant.

$$C = \lim_{m\to\infty} \left(\sum_{n=1}^{m} 1/n - \log m \right) = 0.5772156649\ldots$$

$$\gamma = e^C.$$

Note that in H.T.F. and many other books C is denoted by γ.

Orthogonal polynomials

See also H.T.F. II Chapter X and pp. 265-269 of the present volume.

Legendre polynomial

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2-1)^n.$$

Gegenbauer polynomial

$$C_n^\nu(x) = \frac{(-2)^n (\nu)_n}{n!(n+2\nu)_n} (1-x^2)^{\frac{1}{2}-\nu} \frac{d^n}{dx^n} (1-x^2)^{n+\nu-\frac{1}{2}}.$$

Tchebichef polynomials

$$T_n(x) = \cos(n \cos^{-1} x)$$

$$U_n(x) = \frac{\sin[(n+1)\cos^{-1} x]}{\sin(\cos^{-1} x)}.$$

Jacobi polynomial

$$P_n^{(\alpha,\beta)}(x) = \frac{(-1)^n}{2^n n!} (1-x)^{-\alpha} (1+x)^{-\beta} \frac{d^n}{dx^n} [(1-x)^{n+\alpha} (1+x)^{n+\beta}].$$

Laguerre polynomial

$$L_n^{\alpha}(z) = \frac{e^z z^{-\alpha}}{n!} \frac{d^n}{dz^n}(e^{-z} z^{n+\alpha})$$

$$L_n^0(z) = L_n(z).$$

Hermite polynomials

$$\mathrm{He}_n(x) = (-1)^n e^{\frac{1}{2}x^2} \frac{d^n}{dx^n}(e^{-\frac{1}{2}x^2})$$

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}.$$

Charlier polynomial

$$p_n(x; a) = n!\, a^{-n} L_n^{x-n}(a).$$

The gamma function and related functions

See also H.T.F. I Chapter I.

Gamma function

$$\Gamma(z) = \int_0^{\infty} e^{-t} t^{z-1}\, dt \qquad \mathrm{Re}\, z > 0.$$

Logarithmic derivative of the gamma function

$$\psi(z) = \frac{\Gamma'(z)}{\Gamma(z)}, \quad \psi'(z) = \frac{d\psi}{dz}, \quad \text{etc.}$$

Beta function

$$\mathrm{B}(x, y) = \frac{\Gamma(x)\,\Gamma(y)}{\Gamma(x+y)}.$$

Euler's dilogarithm

$$L_2(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^2} = -\int_0^z \frac{\log(1-z)}{z}\, dz.$$

Incomplete gamma functions. See under Confluent hypergeometric functions.

Incomplete beta function. See under Hypergeometric functions.

Riemann's zeta function and related functions

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$$

$$\xi(t) = -\tfrac{1}{2}(t^2 + \tfrac{1}{4})\,\pi^{-\frac{1}{2}it-\frac{1}{4}}\,\Gamma(\tfrac{1}{2}it + \tfrac{1}{4})\,\zeta(it + \tfrac{1}{2})$$

$$\zeta(z, a) = \sum_{n=0}^{\infty} (n + a)^{-z}, \qquad \Phi(z, s, v) = \sum_{n=0}^{\infty} \frac{z^n}{(v + n)^s}$$

Legendre functions

See also H.T.F. I Chapter III. For expressions of products of Legendre functions as hypergeometric series see Meijer, C.S., 1936: *Math. Ann.* 112, 469-489 and *Proc. Nederl. Akad. Wetensch.* 39, 394-403 and 519-527; 1938: *Nieuw Arch. Wiskunde* (2) 19, 207-234.

$$P_\nu^\mu(z) = \frac{1}{\Gamma(1-\mu)}\left(\frac{z+1}{z-1}\right)^{\frac{1}{2}\mu} {}_2F_1(-\nu,\ \nu+1;\ 1-\mu;\ \tfrac{1}{2} - \tfrac{1}{2}z)$$

$$Q_\nu^\mu(z) = \frac{e^{\mu\pi i}\,\pi^{\frac{1}{2}}\,\Gamma(\mu+\nu+1)}{2^{\nu+1}\,\Gamma(\nu + 3/2)}\; z^{-\mu-\nu-1}(z^2 - 1)^{\frac{1}{2}\mu}$$

$$\times\ {}_2F_1\left(\frac{\mu+\nu+1}{2}, \frac{\mu+\nu+2}{2};\ \nu + \frac{3}{2};\ \frac{1}{z^2}\right)$$

z in the complex plane cut along the real axis from -1 to 1.

$$\mathrm{P}_\nu^\mu(x) = \frac{1}{\Gamma(1-\mu)}\left(\frac{1+x}{1-x}\right)^{\frac{1}{2}\mu} {}_2F_1(-\nu, \nu+1; 1-\mu; \tfrac{1}{2}-\tfrac{1}{2}x) \qquad -1 < x < 1$$

$$\mathrm{Q}_\nu^\mu(x) = \tfrac{1}{2}e^{-i\mu\pi}[e^{-\frac{1}{2}\mu\pi i}Q_\nu^\mu(x+i0) + e^{\frac{1}{2}\mu\pi i}Q_\nu^\mu(x-i0)] \qquad -1 < x < 1$$

$$P_\nu(z) = P_\nu^0(z), \qquad Q_\nu(z) = Q_\nu^0(z).$$

Bessel functions and related functions

See also H.T.F. II Chapter VII, and pp. 331-332 of the present volume.

Bessel functions

$$J_\nu(z) = \sum_{m=0}^{\infty} \frac{(-1)^m\,(\tfrac{1}{2}z)^{\nu+2m}}{m!\,\Gamma(\nu + m + 1)}$$

$$Y_\nu(z) = \operatorname{cosec}\nu\pi\,[J_\nu(z)\cos\nu\pi - J_{-\nu}(z)]$$

$$H_\nu^{(1)}(z) = J_\nu(z) + i\,Y_\nu(z)$$

$$H_\nu^{(2)}(z) = J_\nu(z) - i\,Y_\nu(z)$$

$$Ji_\nu(x) = \int_\infty^x J_\nu(t)\,\frac{dt}{t}\,.$$

Modified Bessel functions

$$I_\nu(z) = \sum_{m=0}^{\infty} \frac{(\tfrac{1}{2}z)^{\nu+2m}}{m!\,\Gamma(\nu+m+1)}$$

$$K_\nu(z) = \frac{\pi}{2}\,\frac{I_{-\nu}(z) - I_\nu(z)}{\sin \nu\pi}\,.$$

Kelvin's and related functions

$$\mathrm{ber}_\nu(z) + i\,\mathrm{bei}_\nu(z) = J_\nu(ze^{\frac{3}{4}\pi i})$$

$$\mathrm{ber}_\nu(z) - i\,\mathrm{bei}_\nu(z) = J_\nu(ze^{-\frac{3}{4}\pi i})$$

$$\mathrm{ker}_\nu(z) + i\,\mathrm{kei}_\nu(z) = K_\nu(ze^{\frac{1}{4}\pi i})$$

$$\mathrm{ker}_\nu(z) - i\,\mathrm{kei}_\nu(z) = K_\nu(ze^{-\frac{1}{4}\pi i})$$

$$\mathrm{ber}(z) = \mathrm{ber}_0(z), \qquad \mathrm{bei}(z) = \mathrm{bei}_0(z),$$

$$\mathrm{ker}(z) = \mathrm{ker}_0(z), \qquad \mathrm{kei}(z) = \mathrm{kei}_0(z).$$

Note that the definition of $\mathrm{ker}_\nu(z)$ and $\mathrm{kei}_\nu(z)$ differs from that given in H.T.F. II sec. 7.2.3.

$$X_\nu^{(b)}(z) = \mathrm{ber}_\nu^2(z) + \mathrm{bei}_\nu^2(z)$$

$$V_\nu^{(b)}(z) = [\mathrm{ber}_\nu'(z)]^2 + [\mathrm{bei}_\nu'(z)]^2$$

$$W_\nu^{(b)}(z) = \mathrm{ber}_\nu(z)\,\mathrm{bei}_\nu'(z) - \mathrm{bei}_\nu(z)\,\mathrm{ber}_\nu'(z)$$

$$\tfrac{1}{2}Z_\nu^{(b)}(z) = \mathrm{ber}_\nu(z)\,\mathrm{bei}_\nu'(z) + \mathrm{bei}_\nu(z)\,\mathrm{ber}_\nu'(z).$$

Neumann polynomials

$$O_0(x) = \frac{1}{x}; \qquad O_n(x) = \frac{1}{4}\sum_{m=0}^{\leq \frac{1}{2}n} \frac{n(n-m-1)!}{m!\,(\tfrac{1}{2}x)^{n-2m+1}} \qquad n = 1, 2, \ldots$$

$$O_{-n}(x) = (-1)^n\,O_n(x). \qquad n = 1, 2, \ldots$$

Anger-Weber functions

$$\mathbf{J}_\nu(z) = \pi^{-1} \int_0^\pi \cos(\nu\theta - z \sin\theta)\, d\theta$$

$$\mathbf{E}_\nu(z) = \pi^{-1} \int_0^\pi \sin(\nu\theta - z \sin\theta)\, d\theta.$$

Struve's functions

$$\mathbf{H}_\nu(z) = \sum_{m=0}^{\infty} \frac{(-1)^m (z/2)^{\nu+2m+1}}{\Gamma(m+3/2)\,\Gamma(\nu+m+3/2)}$$

$$= \frac{(z/2)^{\nu+1}}{\Gamma(3/2)\,\Gamma(\nu+3/2)}\, {}_1F_2(1;\, 3/2,\, \nu+3/2;\, -z^2/4)$$

$$= 2^{1-\nu}\, \pi^{-1/2}\, [\Gamma(\nu+1/2)]^{-1}\, s_{\nu,\nu}(z).$$

$$\mathbf{L}_\nu(z) = e^{-\frac{1}{2}(\nu+1)\pi i}\, \mathbf{H}_\nu(z e^{\frac{1}{2} i\pi}).$$

Lommel's functions

$$s_{\mu,\nu}(z) = \frac{z^{\mu+1}}{(\mu-\nu+1)(\mu+\nu+1)}\, {}_1F_2\left(1;\, \frac{\mu-\nu+3}{2},\, \frac{\mu+\nu+3}{2};\, -\frac{z^2}{4}\right)$$

$$S_{\mu,\nu}(z) = s_{\mu,\nu}(z) + 2^{\mu-1}\, \Gamma\left(\frac{\mu-\nu+1}{2}\right) \Gamma\left(\frac{\mu+\nu+1}{2}\right)$$
$$\times \left[\sin\left(\frac{\mu-\nu}{2}\pi\right) J_\nu(z) - \cos\left(\frac{\mu-\nu}{2}\pi\right) Y_\nu(z)\right].$$

Lommel's functions of two variables

$$U_\nu(w, z) = \sum_{m=0}^{\infty} (-1)^m \left(\frac{w}{z}\right)^{\nu+2m} J_{\nu+2m}(z)$$

$$V_\nu(w, z) = \cos\left(\frac{w}{2} + \frac{z^2}{2w} + \frac{\nu\pi}{2}\right) + U_{2-\nu}(w, z).$$

Hypergeometric functions

See also H.T.F. I Chapters II, IV.

Generalized hypergeometric series

$$ {}_mF_n(a_1, \dots, a_m; \gamma_1, \dots, \gamma_n; z) = \sum_{k=0}^{\infty} \frac{(a_1)_k \cdots (a_m)_k}{(\gamma_1)_k \cdots (\gamma_n)_k} \frac{z^k}{k!}. $$

${}_2F_1(a, b; c; z)$ is Gauss' hypergeometric series and is often (for instance in H.T.F. I Chapter II) denoted by $F(a, b; c; z)$.

${}_1F_1(a; c; z)$ is Kummer's confluent hypergeometric series and is sometimes (for instance in H.T.F. I Chapter VI) denoted by $\Phi(a; c; z)$.

${}_mF_n(a_1, \dots, a_m; \gamma_1, \dots, \gamma_n; z)$ is sometimes written as

$$ {}_mF_n\begin{bmatrix} a_1, \dots, a_m; z \\ \gamma_1, \dots, \gamma_n \end{bmatrix}. $$

Incomplete beta function

$$ B_x(p, q) = \int_0^x t^{p-1}(1-t)^{q-1}\,dt = p^{-1} x^p \; {}_2F_1(p, 1-q; p+1; x). $$

$$ I_x(p, q) = \frac{B_x(p, q)}{B_1(p, q)} $$

$$ S_n(b_1, b_2, b_3, b_4; z) = \sum_{h=1}^{n} \frac{\prod_{j=1}^{n}{}' \, \Gamma(b_j - b_h)}{\prod_{j=n+1}^{4} \Gamma(1 + b_h - b_j)} z^{1+2b_h} $$

$$ \times \; {}_0F_3\left(1 + b_h - b_1, \dots, *, \dots, 1 + b_h - b_4; (-1)^n z^2\right) $$

The prime in Π' and the asterisk in ${}_0F_3$ mean that the term containing $b_h - b_h$ is to be omitted. For $n = 1$ the product Π in the numerator, for $n = 4$ that in the denominator is to be replaced by unity.

Confluent hypergeometric functions

See also H.T.F. I Chapter VI and H.T.F. II Chapters VIII and IX. See also under Hypergeometric functions, Orthogonal polynomials, E-function, G-function.

Whittaker's functions

$$M_{\kappa,\mu}(z) = z^{\frac{1}{2}+\mu} e^{-\frac{1}{2}z} {}_1F_1(\tfrac{1}{2}+\mu-\kappa;\, 2\mu+1;\, z)$$

$$W_{\kappa,\mu}(z) = \frac{\Gamma(-2\mu)\, M_{\kappa,\mu}(z)}{\Gamma(\frac{1}{2}-\mu-\kappa)} + \frac{\Gamma(2\mu)\, M_{\kappa,-\mu}(z)}{\Gamma(\frac{1}{2}+\mu-\kappa)}.$$

Parabolic cylinder functions

$$D_\nu(z) = 2^{\frac{1}{2}\nu+\frac{1}{4}} z^{-\frac{1}{2}} W_{\frac{1}{2}\nu+\frac{1}{4},\frac{1}{4}}(\tfrac{1}{2}z^2)$$

$$D_n(z) = (-1)^n e^{\frac{1}{4}z^2} \frac{d^n}{dz^n}(e^{-\frac{1}{2}z^2}).$$

Bateman's function

$$k_{2\nu}(z) = \frac{1}{\Gamma(\nu+1)} W_{\nu,\frac{1}{2}}(2z).$$

The exponential integral and related functions

$$-\mathrm{Ei}(-x) = E_1(x) = \int_x^\infty e^{-t}\frac{dt}{t} = \Gamma(0,x) \qquad -\pi < \arg x < \pi$$

$$\mathrm{Ei}^+(x) = \mathrm{Ei}(x+i0), \quad \mathrm{Ei}^-(x) = \mathrm{Ei}(x-i0) \qquad x > 0$$

$$\overline{\mathrm{Ei}}(x) = \tfrac{1}{2}[\mathrm{Ei}^+(x) + \mathrm{Ei}^-(x)] \qquad x > 0.$$

The last function is denoted by $E^*(x)$ in H.T.F. II sec. 9.7.

$$\mathrm{li}(z) = \int_0^z \frac{dt}{\log t} = \mathrm{Ei}(\log z)$$

$$\mathrm{si}(x) = -\int_x^\infty \frac{\sin t}{t}\, dt = \frac{1}{2i}[\mathrm{Ei}(ix) - \mathrm{Ei}(-ix)]$$

$$\mathrm{Si}(x) = \int_0^x \frac{\sin t}{t}\, dt = \tfrac{1}{2}\pi + \mathrm{si}(x)$$

$$\mathrm{Ci}(x) = -\int_x^\infty \frac{\cos t}{t}\, dt = -\,\mathrm{ci}(x) = \tfrac{1}{2}[\mathrm{Ei}(ix) + \mathrm{Ei}(-ix)]$$

Error functions and related functions

$$\operatorname{Erf}(x) = 2\pi^{-\frac{1}{2}} \int_0^x e^{-t^2}\, dt = \frac{2x}{\sqrt{\pi}}\ {}_1F_1\left(\frac{1}{2}, \frac{3}{2}; -x^2\right)$$

$$\operatorname{Erfc}(x) = 2\pi^{-\frac{1}{2}} \int_x^{\infty} e^{-t^2}\, dt = 1 - \operatorname{Erf}(x) = (\pi x)^{-\frac{1}{2}}\, e^{-\frac{1}{2}x^2}\, W_{-\frac{1}{4},\frac{1}{4}}(x^2).$$

These functions differ by the factor $2\pi^{-\frac{1}{2}}$ from the functions introduced in H.T.F. II sec. 9.9.

$$C(x) = 2^{-\frac{1}{2}}\, \pi^{-\frac{1}{2}} \int_0^x t^{-\frac{1}{2}} \cos t\, dt$$

$$S(x) = 2^{-\frac{1}{2}}\, \pi^{-\frac{1}{2}} \int_0^x t^{-\frac{1}{2}} \sin t\, dt.$$

Incomplete gamma functions

$$\gamma(a, x) = \int_0^x e^{-t}\, t^{a-1}\, dt = a^{-1}\, x^{a}\ {}_1F_1(a; a+1; -x)$$

$$\Gamma(a, x) = \int_x^{\infty} e^{-t}\, t^{a-1}\, dt = \Gamma(a) - \gamma(a, x)$$

$$= x^{\frac{1}{2}(a-1)}\, e^{-\frac{1}{2}x}\, W_{\frac{1}{2}(a-1),\, \frac{1}{2}a}(x).$$

Particular cases of Whittaker's functions

$$M_{-\frac{1}{4},\frac{1}{4}}(x) = \tfrac{1}{2}\pi^{-\frac{1}{2}}\, x^{\frac{1}{4}}\, e^{\frac{1}{2}x} \operatorname{Erf}(x^{\frac{1}{2}})$$

$$M_{n+\frac{1}{4},-\frac{1}{4}}(x) = \frac{(-2)^{-n}}{(\frac{1}{2})_n} x^{\frac{1}{4}}\, e^{-\frac{1}{2}x}\, \mathrm{He}_{2n}[(2x)^{\frac{1}{2}}]$$

$$M_{n+\frac{3}{4},\frac{1}{4}}(x) = \frac{(-1)^n . 2^{-n-\frac{1}{2}}}{(3/2)_n}\, x^{\frac{1}{4}}\, e^{-\frac{1}{2}x}\, \mathrm{He}_{2n+1}[(2x)^{\frac{1}{2}}]$$

$$M_{0,\mu}(x) = 2^{2\mu}\, \Gamma(\mu+1)\, x^{\frac{1}{2}}\, I_{\mu}(\tfrac{1}{2}x)$$

$$M_{0,\mu}(\pm ix) = 2^{2\mu}\Gamma(\mu+1)\, e^{\pm\frac{1}{2}\mu\pi i} x^{\frac{1}{2}}\, J_\mu(\tfrac{1}{2}x)$$

$$M_{\mu+\frac{1}{2},\mu}(x) = x^{\mu+\frac{1}{2}}\, e^{-\frac{1}{2}x}$$

$$M_{-\mu-\frac{1}{2},\mu}(x) = x^{\mu+\frac{1}{2}}\, e^{\frac{1}{2}x}$$

$$M_{\mu-\frac{1}{2},\mu}(x) = 2\mu x^{\frac{1}{2}-\mu} e^{\frac{1}{2}x}\, \gamma(2\mu, x)$$

$$M_{\mu+n+\frac{1}{2},\mu}(x) = \frac{n!}{(2\mu+1)_n}\, x^{\mu+\frac{1}{2}}\, e^{-\frac{1}{2}x}\, L_n^{2\mu}(x)$$

$$W_{-\frac{1}{2},0}(x) = -x^{\frac{1}{2}}\, e^{\frac{1}{2}x}\, \mathrm{Ei}(-x)$$

$$W_{-\frac{1}{4},\pm\frac{1}{4}}(x) = \pi^{\frac{1}{2}}\, x^{\frac{1}{4}}\, e^{\frac{1}{2}x}\, \mathrm{Erfc}\,(x^{\frac{1}{2}})$$

$$W_{\frac{1}{2}n+\frac{1}{4},\pm\frac{1}{4}}(x) = 2^{-\frac{1}{2}n}\, x^{\frac{1}{4}}\, e^{-\frac{1}{2}x}\, \mathrm{He}_n[(2x)^{\frac{1}{2}}]$$

$$W_{\mu,\pm\frac{1}{4}}(x) = 2^{-\mu}(2x)^{\frac{1}{4}}\, D_{2\mu-\frac{1}{2}}[(2x)^{\frac{1}{2}}]$$

$$W_{0,\mu}(x) = \left(\frac{x}{\pi}\right)^{\frac{1}{2}} K_\mu\left(\frac{x}{2}\right)$$

$$W_{0,\mu}(ix) = \frac{(\pi x)^{\frac{1}{2}}}{2}\, \exp[-(\tfrac{1}{2}\nu+\tfrac{1}{4})\pi i]\, H_\mu^{(2)}\left(\frac{x}{2}\right)$$

$$W_{0,\mu}(-ix) = \frac{(\pi x)^{\frac{1}{2}}}{2}\, \exp[(\tfrac{1}{2}\nu+\tfrac{1}{4})\pi i]\, H_\mu^{(1)}\left(\frac{x}{2}\right)$$

$$W_{\mu+\frac{1}{2},\pm\mu}(x) = x^{\mu+\frac{1}{2}}\, e^{-\frac{1}{2}x}$$

$$W_{\mu-\frac{1}{2},\pm\mu}(x) = x^{\frac{1}{2}-\mu}\, e^{\frac{1}{2}x}\, \Gamma(2\mu, x)$$

$$W_{\mu+n+\frac{1}{2},\pm\mu}(x) = (-1)^n\, n!\, x^{\mu+\frac{1}{2}}\, e^{-\frac{1}{2}x}\, L_n^{2\mu}(x)$$

MacRobert's E - function

See also H.T.F. I Chapter V.

If $p \geq q+1$,

$$E(p\,;\,a_r : q\,;\,\rho_s : x) = \sum_{r=1}^{p} \frac{\prod_{s=1}^{p}{}' \,\Gamma(a_s - a_r)}{\prod_{t=1}^{q} \Gamma(\rho_t - a_r)} \Gamma(a_r)\, x^{a_r}$$

$$\times {}_{q+1}F_{p-1}(a_r, a_r - \rho_1 + 1, \ldots, a_r - \rho_q + 1;\ a_r - a_1 + 1, \ldots, *, \ldots, a_r - a_p + 1;$$
$$(-1)^{p-q} x).$$

where $|x| < 1$ when $p = q+1$.

If $p \leq q+1$,

$$E(p\,;\,a_r : q\,;\,\rho_s : x) = \frac{\prod_{r=1}^{p} \Gamma(a_r)}{\prod_{s=1}^{q} \Gamma(\rho_s)}\, {}_pF_q(a_1, \ldots, a_p; \rho_1, \ldots, \rho_q; -1/x)$$

where $x \neq 0$ and $|x| > 1$ if $p = q+1$. If $p > q+1$, the last relation gives the asymptotic expansion of the E-function for large x.

$$E(a::x) = \Gamma(a)\,(1 + x^{-1})^{-a}$$

$$E(:\nu + 1 : x) = x^{\frac{1}{2}\nu}\, J_\nu(2x^{-\frac{1}{2}})$$

$$E(\tfrac{1}{2} + \nu, \tfrac{1}{2} - \nu :: 2x) = \sec(\nu\pi)\,(2\pi x)^{\frac{1}{2}}\, e^x\, K_\nu(x)$$

$$E(a, \beta :: x) = \Gamma(a)\,\Gamma(\beta)\, x^{-k}\, e^{\frac{1}{2}x}\, W_{k,m}(x)$$

$$k = \tfrac{1}{2}(1 - a - \beta), \qquad m = \tfrac{1}{2}(a - \beta)$$

$$E\left(\alpha, \beta, \frac{\alpha+\beta}{2}, \frac{\alpha+\beta+1}{2} : \alpha+\beta : \frac{x^2}{4}\right)$$

$$= \pi^{1/2}\,\Gamma(\alpha)\,\Gamma(\beta)\left(\frac{x}{2}\right)^{-2k} W_{k,m}(ix)\,W_{k,m}(-ix)$$

$$k = \tfrac{1}{2}(1-\alpha-\beta), \qquad m = \tfrac{1}{2}(\alpha-\beta)$$

Meijer's G-function

See also H.T.F. I Chapter V.

$$G^{m,n}_{p,q}\left(x \,\middle|\, \begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix}\right)$$

$$= \frac{1}{2\pi i}\int_L \frac{\prod_{j=1}^{m}\Gamma(b_j - s)\prod_{j=1}^{n}\Gamma(1-a_j+s)}{\prod_{j=m+1}^{q}\Gamma(1-b_j+s)\prod_{j=n+1}^{p}\Gamma(a_j - s)}\, x^s ds$$

where L is a path separating the poles of $\Gamma(b_1 - s) \cdots \Gamma(b_m - s)$ from those of $\Gamma(1-a_1+s)\cdots\Gamma(1-a_n+s)$. For a more detailed definition see H.T.F. I sec. 5.3.

Formulas involving the G-function may be used as key formulas from which many integrals with Bessel functions, Legendre functions, and other higher transcendental functions follow by specializing parameters. The following two lists give expressions of certain special G-functions in terms of well-known higher transcendental functions, and, conversely, expressions for higher transcendental functions in terms of G-functions. The list is not complete. See also H.T.F. I sec. 5.6.

Particular cases of the G-function

$$G^{10}_{02}(x|a, b) = x^{\frac{1}{2}(a+b)} J_{a-b}(2x^{\frac{1}{2}})$$

$$G^{20}_{02}(x|a, b) = 2x^{\frac{1}{2}(a+b)} K_{a-b}(2x^{\frac{1}{2}})$$

$$G^{11}_{12}\left(x \,\middle|\, \begin{matrix} \tfrac{1}{2} \\ b, -b \end{matrix}\right) = \pi^{\frac{1}{2}} e^{-\frac{1}{2}x} I_b(\tfrac{1}{2}x)$$

$$G_{12}^{11}\left(x\,\middle|\,\begin{matrix}a\\ b,\ c\end{matrix}\right)=\frac{\Gamma(1-a+b)}{\Gamma(1+b-c)}\,x^{b}\,{}_1F_1(1-a+b;\,1+b-c;\,-x)$$

$$G_{12}^{20}\left(x\,\middle|\,\begin{matrix}\tfrac12\\ b,\ -b\end{matrix}\right)=\pi^{-1/2}\,e^{-\frac12 x}\,K_b(\tfrac12 x)$$

$$G_{12}^{20}\left(x\,\middle|\,\begin{matrix}a\\ b,\ c\end{matrix}\right)=x^{\frac12(b+c-1)}\,e^{-\frac12 x}\,W_{k,m}(x)$$

$$k=\tfrac12(1+b+c)-a,\qquad m=\tfrac12 b-\tfrac12 c$$

$$G_{12}^{21}\left(x\,\middle|\,\begin{matrix}\tfrac12\\ b,\ -b\end{matrix}\right)=\frac{\pi^{1/2}}{\cos b\pi}\,e^{\frac12 x}\,K_b(\tfrac12 x)$$

$$G_{12}^{21}\left(x\,\middle|\,\begin{matrix}a\\ b,\ c\end{matrix}\right)=\Gamma(b-a+1)\,\Gamma(c-a+1)\,x^{\frac12(b+c-1)}\,e^{\frac12 x}\,W_{k,m}(x)$$

$$k=a-\tfrac12(b+c+1),\qquad m=\tfrac12 b-\tfrac12 c$$

$$G_{04}^{10}(x\,|\,a,\ b,\ 2b-a,\ b+\tfrac12)=\pi^{-1/2}x^{b}\,I_{2(a-b)}(2^{3/2}x^{1/4})\,J_{2(a-b)}(2^{3/2}x^{1/4})$$

$$G_{04}^{10}(x\,|\,a,\ a+\tfrac12,\ b,\ 2a-b)=\tfrac12\pi^{-1/2}\sec(b-a)\pi$$
$$\times\, x^{a}[J_{2(a-b)}(2^{3/2}x^{1/4})\,I_{2(b-a)}(2^{3/2}x^{1/4})$$
$$+\,I_{2(a-b)}(2^{3/2}x^{1/4})\,J_{2(b-a)}(2^{3/2}x^{1/4})]$$

$$G_{04}^{10}(x\,|\,a+\tfrac12,\ a,\ b,\ 2a-b)=\tfrac12\pi^{-1/2}[\sin(a-b)\pi]^{-1}$$
$$\times\, x^{a}[J_{2(a-b)}(2^{3/2}x^{1/4})\,I_{2(b-a)}(2^{3/2}x^{1/4})$$
$$-\,I_{2(a-b)}(2^{3/2}x^{1/4})\,J_{2(b-a)}(2^{3/2}x^{1/4})]$$

$$G_{04}^{20}(x\,|\,a,\ a+\tfrac12,\ b,\ b+\tfrac12)=x^{\frac12(a+b)}\,J_{2(a-b)}(4x^{1/4})$$

$$G_{04}^{20}(x\,|\,a,\ -a,\ 0,\ \tfrac12)=-\pi^{1/2}(\sin 2a\pi)^{-1}[J_{2a}(ze^{\pi i/4})\,J_{2a}(ze^{-\pi i/4})$$
$$-\,J_{-2a}(ze^{\pi i/4})\,J_{-2a}(ze^{-\pi i/4})]\qquad z=2^{3/2}x^{1/4}$$

$$G^{20}_{04}(x|0,\ \tfrac{1}{2},\ a,\ -a) = \pi^{\frac{1}{2}}\, i^{-1} (\sin 2a\pi)^{-1}$$
$$\times [e^{2a\pi i} J_{2a}(ze^{-\pi i/4})\, J_{-2a}(ze^{\pi i/4}) - e^{-2a\pi i} J_{2a}(ze^{\pi i/4})$$
$$\times J_{-2a}(ze^{-\pi i/4})] \qquad z = 2^{3/2} x^{1/4}$$

$$G^{30}_{04}(x|\,3a-\tfrac{1}{2},\ a,\ -a-\tfrac{1}{2},\ a-\tfrac{1}{2}) = 2\pi^{\frac{1}{2}} (\cos 2a\pi)^{-1}$$
$$\times x^{a-1/2} K_{4a}(2^{3/2} x^{1/4})\, [J_{4a}(2^{3/2} x^{1/4}) + J_{-4a}(2^{3/2} x^{1/4})]$$

$$G^{30}_{04}(x|\,0,\ a-\tfrac{1}{2},\ -a-\tfrac{1}{2},\ -\tfrac{1}{2}) = 4\pi^{\frac{1}{2}} x^{-\frac{1}{2}}$$
$$\times K_{2a}(2^{3/2} x^{1/4})\, [J_{2a}(2^{3/2} x^{1/4}) \cos a\pi - Y_{2a}(2^{3/2} x^{1/4}) \sin a\pi]$$

$$G^{30}_{04}(x|-\tfrac{1}{2},\ a-\tfrac{1}{2},\ -a-\tfrac{1}{2},\ 0) = -4\pi^{\frac{1}{2}} x^{-\frac{1}{2}}$$
$$\times K_{2a}(2^{3/2} x^{1/4})\, [J_{2a}(2^{3/2} x^{1/4}) \sin a\pi + Y_{2a}(2^{3/2} x^{1/4}) \cos a\pi]$$

$$G^{30}_{04}(x|\,a,\ b+\tfrac{1}{2},\ b,\ 2b-a) = \pi^{\frac{1}{2}}\, 2^{\frac{1}{2}}\, x^{b} K_{2(a-b)}(2^{3/2} x^{1/4})$$
$$\times J_{2(a-b)}(2^{3/2} x^{1/4})$$

$$G^{40}_{04}(x|\,a,\ a+\tfrac{1}{2},\ b,\ b+\tfrac{1}{2}) = 4\pi x^{\frac{1}{2}(a+b)} K_{2(a-b)}(4x^{\frac{1}{4}})$$

$$G^{40}_{04}(x|\,a,\ a+\tfrac{1}{2},\ b,\ 2a-b) = 2^{3}\, \pi^{\frac{1}{2}}\, x^{a}$$
$$\times K_{2(b-a)}(2^{3/2} x^{1/4} e^{\pi i/4})\, K_{2(b-a)}(2^{3/2} x^{1/4} e^{-\pi i/4})$$

$$G^{n0}_{04}(x|a,\ b,\ c,\ d) = x^{-\frac{1}{2}} S_{n}(a,\ b,\ c,\ d;\ x^{\frac{1}{2}}) \qquad n = 1,\ 2,\ 3,\ 4$$

$$G^{11}_{13}\left(x\,\middle|\,\begin{matrix}\tfrac{1}{2}\\ a,\ 0,\ -a\end{matrix}\right) = \pi^{\frac{1}{2}} J^{2}_{a}(x^{\frac{1}{2}})$$

$$G^{11}_{13}\left(x\,\middle|\,\begin{matrix}\tfrac{1}{2}\\ 0,\ a,\ -a\end{matrix}\right) = \pi^{\frac{1}{2}} J_{a}(x^{\frac{1}{2}})\, J_{-a}(x^{\frac{1}{2}})$$

$$G^{11}_{13}\left(x\,\middle|\,\begin{matrix}a\\ a,\ b,\ a-\tfrac{1}{2}\end{matrix}\right) = x^{\frac{1}{2}a+\frac{1}{2}b-\frac{1}{4}}\, \mathbf{H}_{a-b-\frac{1}{2}}(2x^{\frac{1}{2}})$$

$$G^{20}_{13}\left(x\,\middle|\,\begin{matrix}a-\tfrac{1}{2}\\ a,\ b,\ a-\tfrac{1}{2}\end{matrix}\right) = x^{\frac{1}{2}(a+b)}\, Y_{b-a}(2x^{\frac{1}{2}})$$

$$G_{13}^{20}\left(x\,\middle|\,\begin{matrix} a+\tfrac{1}{2} \\ b,\ a,\ 2a-b\end{matrix}\right) = -\pi^{1/2}\, x^{a}\, J_{b-a}(x^{1/2})\, Y_{b-a}(x^{1/2})$$

$$G_{13}^{20}\left(x\,\middle|\,\begin{matrix} \tfrac{1}{2} \\ a,\ -a,\ 0\end{matrix}\right) = \pi^{1/2}\, 2^{-1}\,(\sin a\pi)^{-1}\,[J_{-a}^{2}(x^{1/2}) - J_{a}^{2}(x^{1/2})]$$

$$G_{13}^{21}\left(x\,\middle|\,\begin{matrix} \tfrac{1}{2} \\ a,\ 0,\ -a\end{matrix}\right) = 2\pi^{1/2}\, I_{a}(x^{1/2})\, K_{a}(x^{1/2})$$

$$G_{13}^{21}\left(x\,\middle|\,\begin{matrix} \tfrac{1}{2} \\ a,\ -a,\ 0\end{matrix}\right) = \pi^{3/2}\,(\sin 2a\pi)^{-1}\,[I_{-a}^{2}(x^{1/2}) - I_{a}^{2}(x^{1/2})]$$

$$G_{13}^{21}\left(x\,\middle|\,\begin{matrix} a+\tfrac{1}{2} \\ a+\tfrac{1}{2},\ b,\ a\end{matrix}\right) = \frac{\pi x^{\frac{1}{2}(a+b)}}{\cos(a-b)\pi}\,[I_{b-a}(2x^{1/2}) - \mathbf{L}_{a-b}(2x^{1/2})]$$

$$G_{13}^{21}\left(x\,\middle|\,\begin{matrix} a+\tfrac{1}{2} \\ a,\ a+\tfrac{1}{2},\ b\end{matrix}\right) = \pi x^{\frac{1}{2}(a+b)}\,[I_{a-b}(2x^{1/2}) - \mathbf{L}_{a-b}(2x^{1/2})]$$

$$G_{13}^{30}\left(x\,\middle|\,\begin{matrix} a+\tfrac{1}{2} \\ a+b,\ a-b,\ a\end{matrix}\right) = 2\pi^{-1/2}\, x^{a}\, K_{b}^{2}(x^{1/2})$$

$$G_{13}^{31}\left(x\,\middle|\,\begin{matrix} a+\tfrac{1}{2} \\ a+\tfrac{1}{2},\ -a,\ a\end{matrix}\right) = \frac{\pi^{2}}{\cos 2a\pi}\,[\mathbf{H}_{2a}(2x^{1/2}) - Y_{2a}(2x^{1/2})]$$

$$G_{13}^{31}\left(x\,\middle|\,\begin{matrix} a \\ a,\ b,\ -b\end{matrix}\right) = 2^{-2a+2}\,\Gamma(1-a-b)\,\Gamma(1-a+b)\,S_{2a-1,\,2b}(2x^{1/2})$$

$$G_{13}^{31}\left(x\,\middle|\,\begin{matrix} a+\tfrac{1}{2} \\ b,\ 2a-b,\ a\end{matrix}\right) = \pi^{5/2}\, 2^{-1}\,[\cos(b-a)\pi]^{-1}$$
$$\times x^{a}\, H_{b-a}^{(1)}(x^{1/2})\, H_{b-a}^{(2)}(x^{1/2})$$

$$G_{22}^{12}\left(x\,\middle|\,\begin{matrix} -c_{1},\ -c_{2} \\ a-1,\ -b\end{matrix}\right) = \frac{\Gamma(a+c_{1})\,\Gamma(a+c_{2})}{\Gamma(a+b)}$$
$$\times x^{a-1}\,{}_{2}F_{1}(a+c_{1},\ a+c_{2};\ a+b;\ -x)$$

$$G_{24}^{12}\left(x\,\middle|\,\begin{matrix} a+\tfrac{1}{2},\ a \\ b+a,\ a-c,\ a+c,\ a-b\end{matrix}\right) = \pi^{1/2}\, x^{a}\, J_{b+c}(x^{1/2})\, J_{b-c}(x^{1/2})$$

$$G_{24}^{22}\left(x\,\middle|\,\begin{matrix} a,\ a+\tfrac{1}{2} \\ b,\ c,\ 2a-c,\ 2a-b \end{matrix}\right) = 2\pi^{1/2}\, x^{a}\, I_{b+c-2a}(x^{1/2})\, K_{b-c}(x^{1/2})$$

$$G_{24}^{30}\left(x\,\middle|\,\begin{matrix} 0,\ \tfrac{1}{2} \\ a,\ b,\ -b,\ -a \end{matrix}\right) = i\,2^{-2}\,\pi^{1/2}$$

$$\times\left[H_{a-b}^{(1)}(x^{1/2})\,H_{a+b}^{(1)}(x^{1/2}) - H_{a-b}^{(2)}(x^{1/2})\,H_{a+b}^{(2)}(x^{1/2})\right]$$

$$G_{24}^{31}\left(x\,\middle|\,\begin{matrix} \tfrac{1}{2}+a,\ \tfrac{1}{2}-a \\ 0,\ \tfrac{1}{2},\ b,\ -b \end{matrix}\right) = \frac{\pi^{1/2}\,\Gamma(\tfrac{1}{2}-a+b)\,x^{-1/2}}{\Gamma(1+2a)}\,W_{a,b}(2x^{1/2})\,M_{-a,b}(2x^{1/2})$$

$$G_{24}^{40}\left(x\,\middle|\,\begin{matrix} \tfrac{1}{2}+a,\ \tfrac{1}{2}-a \\ 0,\ \tfrac{1}{2},\ b,\ -b \end{matrix}\right) = \pi^{1/2}\,x^{-1/2}\,W_{a,b}(2x^{1/2})\,W_{-a,b}(2x^{1/2})$$

$$G_{24}^{40}\left(x\,\middle|\,\begin{matrix} a,\ a+\tfrac{1}{2} \\ b+c,\ b-c,\ b+\tfrac{1}{2}+c,\ b+\tfrac{1}{2}-c \end{matrix}\right)$$

$$= \pi^{1/2}\,2^{-k}\,x^{b-1/4}\,e^{-x^{1/2}}\,W_{k,2c}(2x^{1/2}) \qquad k=\tfrac{1}{2}+2b-2c$$

$$G_{24}^{40}\left(x\,\middle|\,\begin{matrix} a,\ a+\tfrac{1}{2} \\ a+b,\ a+c,\ a-c,\ a-b \end{matrix}\right) = 2\pi^{-1/2}\,x^{a}\,K_{b+c}(x^{1/2})\,K_{b-c}(x^{1/2})$$

$$G_{24}^{41}\left(x\,\middle|\,\begin{matrix} 0,\ \tfrac{1}{2} \\ a,\ b,\ -b,\ -a \end{matrix}\right) = \frac{-2^{-2}\,\pi^{5/2}}{i\,\sin a\pi\,\sin b\pi}$$

$$\times\left[e^{-b\pi i}H_{a-b}^{(1)}(x^{1/2})\,H_{a+b}^{(2)}(x^{1/2}) - e^{b\pi i}H_{a+b}^{(1)}(x^{1/2})\,H_{a-b}^{(2)}(x^{1/2})\right]$$

$$G_{24}^{41}\left(x\,\middle|\,\begin{matrix} \tfrac{1}{2},\ 0 \\ a,\ b,\ -b,\ -a \end{matrix}\right) = \frac{2^{-2}\,\pi^{5/2}}{\cos a\pi\,\cos b\pi}$$

$$\times\left[e^{-b\pi i}H_{a-b}^{(1)}(x^{1/2})\,H_{a+b}^{(2)}(x^{1/2}) + e^{b\pi i}H_{a+b}^{(1)}(x^{1/2})\,H_{a-b}^{(2)}(x^{1/2})\right]$$

$$G_{24}^{41}\left(x\,\middle|\,\begin{matrix} \tfrac{1}{2}+a,\ \tfrac{1}{2}-a \\ 0,\ \tfrac{1}{2},\ b,\ -b \end{matrix}\right) = x^{-1/2}\,\pi^{1/2}\,\Gamma(\tfrac{1}{2}+b-a)\,\Gamma(\tfrac{1}{2}-b-a)$$

$$\times\,W_{a,b}(2ix^{1/2})\,W_{a,b}(-2ix^{1/2})$$

$$G_{24}^{42}\left(x\left|\begin{matrix}a,\ a+\tfrac{1}{2}\\ b+c,\ b-c,\ b+\tfrac{1}{2}+c,\ b+\tfrac{1}{2}-c\end{matrix}\right.\right)$$
$$=2^{k+1}\pi^{3/2}\,\Gamma(1-2a+2b+2c)\,\Gamma(1-2a+2b-2c)$$
$$\times x^{b-\frac{1}{4}}e^{x^{\frac{1}{2}}}W_{k,2c}(2x^{\frac{1}{2}})\qquad k=2a-2b-\tfrac{1}{2}$$

$$G_{44}^{14}\left(x\left|\begin{matrix}a-1,-c_1,-c_2,-c_3\\ -b_1,-b_2,-b_3,-b_4\end{matrix}\right.\right)=\frac{\prod_{h=1}^{4}\Gamma(a+b_h)}{\prod_{h=1}^{3}\Gamma(a+c_h)}x^{a-1}$$
$$\times{}_4F_3(a+b_1,a+b_2,a+b_3,a+b_4;a+c_1,a+c_2,a+c_3;-x)$$

$$G_{pq}^{1p}\left(x\left|\begin{matrix}a_1,\dots,a_p\\ b_1,\dots,b_q\end{matrix}\right.\right)=\frac{\prod_{j=1}^{p}\Gamma(1+b_1-a_j)}{\prod_{j=2}^{q}\Gamma(1+b_1-b_j)}x^{b_1}$$
$$\times{}_pF_{q-1}(1+b_1-a_1,\dots,1+b_1-a_p;$$
$$1+b_1-b_2,\dots,1+b_1-b_q;-x)\qquad p\le q$$

$$G_{pq}^{1n}\left(x\left|\begin{matrix}a_1,\dots,a_p\\ b_1,\dots,b_q\end{matrix}\right.\right)=\frac{\prod_{j=1}^{n}\Gamma(1+b_1-a_j)\,x^{b_1}}{\prod_{j=2}^{q}\Gamma(1+b_1-b_j)\prod_{j=n+1}^{p}\Gamma(a_j-b_1)}$$
$$\times{}_pF_{q-1}(1+b_1-a_1,\dots,1+b_1-a_p;$$
$$1+b_1-b_2,\dots,1+b_1-b_q;-x)\qquad p\le q$$

$$G_{pq}^{q1}\left(x\left|\begin{matrix}a_1,\dots,a_p\\ b_1,\dots,b_q\end{matrix}\right.\right)$$
$$\times x^{a_1-1}E(1-a_1+b_1,\dots,1-a_1+b_q:1-a_1+a_2,\dots,1-a_1+a_p:x)$$

Functions expressible in terms of the G-function

$$x^{\mu} J_{\nu}(x) = 2^{\mu} G_{02}^{10}(\tfrac14 x^2 | \tfrac12\nu + \tfrac12\mu, \tfrac12\mu - \tfrac12\nu)$$

$$x^{\mu} J_{\nu}(x) = 4^{\mu} G_{04}^{20}(4^{-4} x^4 | \tfrac14\nu + \tfrac14\mu, \tfrac14\nu + \tfrac14\mu + \tfrac12, \tfrac14\mu - \tfrac14\nu, \tfrac12 + \tfrac14\mu - \tfrac14\nu)$$

$$x^{\mu} Y_{\nu}(x) = 2^{\mu} G_{13}^{20}\left(\tfrac14 x^2 \middle| \begin{matrix} \tfrac12\mu - \tfrac12\nu - \tfrac12 \\ \tfrac12\mu - \tfrac12\nu, \tfrac12\mu + \tfrac12\nu, \tfrac12\mu - \tfrac12\nu - \tfrac12 \end{matrix}\right)$$

$$x^{\mu} K_{\nu}(x) = 2^{\mu-1} G_{02}^{20}(\tfrac14 x^2 | \tfrac12\mu + \tfrac12\nu, \tfrac12\mu - \tfrac12\nu)$$

$$x^{\mu} K_{\nu}(x) = 4^{\mu-1}\pi^{-1} \times G_{04}^{40}(4^{-4} x^4 | \tfrac14\nu + \tfrac14\mu, \tfrac12 + \tfrac14\nu + \tfrac14\mu, -\tfrac14\nu + \tfrac14\mu, \tfrac12 - \tfrac14\nu + \tfrac14\mu)$$

$$e^{-x} I_{\nu}(x) = \pi^{-\frac12} G_{12}^{11}\left(2x \middle| \begin{matrix} \tfrac12 \\ \nu, -\nu \end{matrix}\right)$$

$$e^{-x} K_{\nu}(x) = \pi^{\frac12} G_{12}^{20}\left(2x \middle| \begin{matrix} \tfrac12 \\ \nu, -\nu \end{matrix}\right)$$

$$e^{x} K_{\nu}(x) = \pi^{-\frac12} \cos\nu\pi\, G_{12}^{21}\left(2x \middle| \begin{matrix} \tfrac12 \\ \nu, -\nu \end{matrix}\right)$$

$$x^{\mu} \mathbf{H}_{\nu}(x) = 2^{\mu} G_{13}^{11}\left(\tfrac14 x^2 \middle| \begin{matrix} \tfrac12 + \tfrac12\nu + \tfrac12\mu \\ \tfrac12 + \tfrac12\nu + \tfrac12\mu, \tfrac12\mu - \tfrac12\nu, \tfrac12\mu + \tfrac12\nu \end{matrix}\right)$$

$$\mathbf{H}_{\nu}(x) - Y_{\nu}(x) = \pi^{-2} \cos\nu\pi\, G_{13}^{31}\left(\tfrac14 x^2 \middle| \begin{matrix} \tfrac12 + \tfrac12\nu \\ \tfrac12 + \tfrac12\nu, -\tfrac12\nu, \tfrac12\nu \end{matrix}\right)$$

$$x^{\mu}[I_{\nu}(x) - \mathbf{L}_{\nu}(x)] = \pi^{-1} 2^{\mu} G_{13}^{21}\left(\tfrac14 x^2 \middle| \begin{matrix} \tfrac12\mu + \tfrac12\nu + \tfrac12 \\ \tfrac12\mu + \tfrac12\nu, \tfrac12\mu + \tfrac12\nu + \tfrac12, \tfrac12\mu - \tfrac12\nu \end{matrix}\right)$$

$$x^{\mu}[I_{-\nu}(x) - \mathbf{L}_{\nu}(x)]$$
$$= \pi^{-1}\, 2^{\mu} \cos \nu\pi\, G^{21}_{13}\left(\tfrac{1}{4}x^2 \,\middle|\, \begin{matrix} \tfrac{1}{2} + \tfrac{1}{2}\nu + \tfrac{1}{2}\mu \\ \tfrac{1}{2} + \tfrac{1}{2}\nu + \tfrac{1}{2}\mu,\ \tfrac{1}{2}\mu - \tfrac{1}{2}\nu,\ \tfrac{1}{2}\mu + \tfrac{1}{2}\nu \end{matrix}\right)$$

$$S_{\mu,\nu}(x) = 2^{\mu-1} \frac{1}{\Gamma(\tfrac{1}{2} - \tfrac{1}{2}\mu - \tfrac{1}{2}\nu)\,\Gamma(\tfrac{1}{2} - \tfrac{1}{2}\mu + \tfrac{1}{2}\nu)}$$
$$\times G^{31}_{13}\left(\tfrac{1}{4}x^2 \,\middle|\, \begin{matrix} \tfrac{1}{2} + \tfrac{1}{2}\mu \\ \tfrac{1}{2} + \tfrac{1}{2}\mu,\ \tfrac{1}{2}\nu,\ -\tfrac{1}{2}\nu \end{matrix}\right)$$

$$J^2_{\nu}(x) = \pi^{-\frac{1}{2}}\, G^{11}_{13}\left(x^2 \,\middle|\, \begin{matrix} \tfrac{1}{2} \\ \nu,\ 0,\ -\nu \end{matrix}\right)$$

$$J_{\nu}(x)\, J_{-\nu}(x) = \pi^{-\frac{1}{2}}\, G^{11}_{13}\left(x^2 \,\middle|\, \begin{matrix} \tfrac{1}{2} \\ 0,\ \nu,\ -\nu \end{matrix}\right)$$

$$x^{\sigma} J_{\mu}(x)\, J_{\nu}(x)$$
$$= \pi^{-\frac{1}{2}} G^{12}_{24}\left[x^2 \,\middle|\, \begin{matrix} \tfrac{1}{2} + \tfrac{1}{2}\sigma,\ \tfrac{1}{2}\sigma \\ \tfrac{1}{2}(\mu + \nu + \sigma),\ \tfrac{1}{2}(\nu + \sigma - \mu),\ \tfrac{1}{2}(\mu + \sigma - \nu),\ \tfrac{1}{2}(\sigma - \mu - \nu) \end{matrix}\right]$$

$$x^{\mu} I_{\nu}(x)\, J_{\nu}(x) = \pi^{\frac{1}{2}}\, 2^{3\mu/2}\, G^{10}_{04}\left(\frac{x^4}{64} \,\middle|\, \tfrac{1}{4}\mu + \tfrac{1}{2}\nu,\ \tfrac{1}{4}\mu - \tfrac{1}{2}\nu,\ \tfrac{1}{4}\mu,\ \tfrac{1}{4}\mu + \tfrac{1}{2}\right)$$

$$I_{\nu}(x)\, J_{-\nu}(x) = \pi^{\frac{1}{2}} \cos(\tfrac{1}{2}\nu\pi)\, G^{10}_{04}\left(\frac{x^4}{64} \,\middle|\, 0,\ \tfrac{1}{2},\ \tfrac{1}{2}\nu,\ -\tfrac{1}{2}\nu\right)$$
$$- \pi^{\frac{1}{2}} \sin(\tfrac{1}{2}\nu\pi)\, G^{10}_{04}\left(\frac{x^4}{64} \,\middle|\, \tfrac{1}{2},\ 0,\ \tfrac{1}{2}\nu,\ -\tfrac{1}{2}\nu\right)$$

$$x^{\mu} J_{\nu}(x)\, Y_{\nu}(x) = -\pi^{-\frac{1}{2}}\, G^{20}_{13}\left(x^2 \,\middle|\, \begin{matrix} \tfrac{1}{2} + \tfrac{1}{2}\mu \\ \nu + \tfrac{1}{2}\mu,\ \tfrac{1}{2}\mu,\ \tfrac{1}{2}\mu - \nu \end{matrix}\right)$$

$$I_{\nu}(x)\, K_{\nu}(x) = 2^{-1}\, \pi^{-\frac{1}{2}}\, G^{21}_{13}\left(x^2 \,\middle|\, \begin{matrix} \tfrac{1}{2} \\ \nu,\ 0,\ -\nu \end{matrix}\right)$$

$$x^{\mu} K_{\nu}(x)\, J_{\nu}(x) = \pi^{-1/2}\, 2^{3\mu/2-1/2}$$

$$\times\, G^{30}_{04}\left(\frac{1}{64}x^{4}\,\middle|\, \tfrac{1}{4}\mu+\tfrac{1}{2}\nu,\ \tfrac{1}{4}\mu+\tfrac{1}{2},\ \tfrac{1}{4}\mu,\ \tfrac{1}{4}\mu-\tfrac{1}{2}\nu\right)$$

$$x^{\sigma} I_{\nu}(x)\, K_{\mu}(x) = 2^{-1}\, \pi^{-1/2}$$

$$\times\, G^{22}_{24}\left[x^{2}\,\middle|\, \begin{matrix} \tfrac{1}{2}\sigma,\ \tfrac{1}{2}\sigma+\tfrac{1}{2} \\ \tfrac{1}{2}(\nu+\mu+\sigma),\ \tfrac{1}{2}(\nu+\sigma-\mu),\ \tfrac{1}{2}(\mu+\sigma-\nu),\ \tfrac{1}{2}(\sigma-\nu-\mu)\end{matrix}\right]$$

$$x^{\mu} H^{(1)}_{\nu}(x)\, H^{(2)}_{\nu}(x) = \pi^{-5/2}\, 2\cos\nu\pi$$

$$\times\, G^{31}_{13}\left(x^{2}\,\middle|\, \begin{matrix} \tfrac{1}{2}+\tfrac{1}{2}\mu \\ \tfrac{1}{2}\mu+\nu,\ \tfrac{1}{2}\mu-\nu,\ \tfrac{1}{2}\mu\end{matrix}\right)$$

$$x^{\mu} K^{2}_{\nu}(x) = 2^{-1}\, \pi^{1/2}\, G^{30}_{13}\left(x^{2}\,\middle|\, \begin{matrix} \tfrac{1}{2}+\tfrac{1}{2}\mu \\ \nu+\tfrac{1}{2}\mu,\ -\nu+\tfrac{1}{2}\mu,\ \tfrac{1}{2}\mu\end{matrix}\right)$$

$$x^{\sigma} K_{\nu}(x)\, K_{\mu}(x) = 2^{-1}\, \pi^{1/2}$$

$$\times\, G^{40}_{24}\left[x^{2}\,\middle|\, \begin{matrix} \tfrac{1}{2}\sigma,\ \tfrac{1}{2}\sigma+\tfrac{1}{2} \\ \tfrac{1}{2}(\nu+\mu+\sigma),\ \tfrac{1}{2}(\nu+\sigma-\mu),\ \tfrac{1}{2}(\mu+\sigma-\nu),\ \tfrac{1}{2}(\sigma-\nu-\mu)\end{matrix}\right]$$

$$x^{2\mu} K_{2\nu}(xe^{\pi i/4})\, K_{2\nu}(xe^{-\pi i/4}) = 2^{3\mu-3}\, \pi^{-1/2}$$

$$\times\, G^{40}_{04}\left(\frac{1}{64}x^{4}\,\middle|\, \tfrac{1}{2}\mu,\ \tfrac{1}{2}\mu+\tfrac{1}{2},\ \tfrac{1}{2}\mu+\nu,\ \tfrac{1}{2}\mu-\nu\right)$$

$$x^{l} e^{-\frac{1}{2}x} M_{k,m}(x) = \frac{\Gamma(2m+1)}{\Gamma(\tfrac{1}{2}+k+m)}\, G^{11}_{12}\left(x\,\middle|\, \begin{matrix} 1-k+l \\ \tfrac{1}{2}+l+m,\ \tfrac{1}{2}+l-m\end{matrix}\right)$$

$$x^{l} e^{-\frac{1}{2}x} W_{k,m}(x) = G^{20}_{12}\left(x\,\middle|\, \begin{matrix} l-k+1 \\ m+l+\tfrac{1}{2},\ l-m+\tfrac{1}{2}\end{matrix}\right)$$

$$x^{l} e^{\frac{1}{2}x} W_{k,m}(x) = \frac{1}{\Gamma(\tfrac{1}{2}+m-k)\,\Gamma(\tfrac{1}{2}-m-k)}\, G^{21}_{12}\left(x\,\middle|\, \begin{matrix} k+l+1 \\ l-m+\tfrac{1}{2},\ m+l+\tfrac{1}{2}\end{matrix}\right)$$

$$e^{-\frac{1}{2}x} W_{k,m}(x) = \pi^{-\frac{1}{2}} x^{\frac{1}{2}} 2^{k-\frac{1}{2}}$$

$$\times G^{40}_{24}\left(2^{-2} x^2 \middle| \begin{matrix} \frac{1}{4} - \frac{1}{2}k, \frac{3}{4} - \frac{1}{2}k \\ \frac{1}{2} + \frac{1}{2}m, \frac{1}{2} - \frac{1}{2}m, \frac{1}{2}m, -\frac{1}{2}m \end{matrix}\right)$$

$$e^{x} W_{k,m}(2x) = \frac{x^{\frac{1}{2}} 2^{-(k+1)} \pi^{-3/2}}{\Gamma(\frac{1}{2} + m - k)\Gamma(\frac{1}{2} - m - k)}$$

$$\times G^{42}_{24}\left(x^2 \middle| \begin{matrix} \frac{1}{4} + \frac{1}{2}k, \frac{3}{4} + \frac{1}{2}k \\ \frac{1}{2}m, \frac{1}{2} + \frac{1}{2}m, -\frac{1}{2}m, \frac{1}{2} - \frac{1}{2}m \end{matrix}\right)$$

$$W_{k,m}(x) M_{-k,m}(x) = \frac{\pi^{-\frac{1}{2}}\Gamma(1+2m)}{\Gamma(\frac{1}{2} - k + m)} G^{31}_{24}\left(\tfrac{1}{4}x^2 \middle| \begin{matrix} 1+k, 1-k \\ \frac{1}{2}, 1, \frac{1}{2} + m, \frac{1}{2} - m \end{matrix}\right)$$

$$x^{l} W_{k,m}(2ix) W_{k,m}(-2ix) = \frac{x\pi^{-\frac{1}{2}}}{\Gamma(\frac{1}{2} + m - k)\Gamma(\frac{1}{2} - m - k)}$$

$$\times G^{41}_{24}\left(x^2 \middle| \begin{matrix} \frac{1}{2} + \frac{1}{2}l + k, \frac{1}{2} + \frac{1}{2}l - k \\ \frac{1}{2}l, \frac{1}{2} + \frac{1}{2}l, \frac{1}{2}l + m, \frac{1}{2}l - m \end{matrix}\right)$$

$$W_{k,m}(x) W_{-k,m}(x)$$

$$= \pi^{-\frac{1}{2}} G^{40}_{24}\left(\tfrac{1}{4}x^2 \middle| \begin{matrix} k+1, -k+1 \\ \frac{1}{2}, 1, m + \frac{1}{2}, -m + \frac{1}{2} \end{matrix}\right)$$

$${}_2F_1(a, b; c; -x) = \frac{\Gamma(c)\, x}{\Gamma(a)\Gamma(b)} G^{12}_{22}\left(x \middle| \begin{matrix} -a, -b \\ -1, -c \end{matrix}\right)$$

$${}_4F_3(a, b, c, d; e, f, l; -x) = \frac{\Gamma(e)\Gamma(f)\Gamma(l)}{\Gamma(a)\Gamma(b)\Gamma(c)\Gamma(d)} x$$

$$\times G^{14}_{44}\left(x \middle| \begin{matrix} -a, -b, -c, -d \\ -1, -e, -f, -l \end{matrix}\right)$$

$$ {}_pF_q(a_1, \dots, a_p; b_1, \dots, b_q; -x) = \frac{\prod_{j=1}^{q} \Gamma(b_j)}{\prod_{j=1}^{p} \Gamma(a_j)} $$

$$ \times x\, G^{1,p}_{p,q+1}\left(x \,\middle|\, \begin{matrix} -a_1, \dots, -a_p \\ -1, -b_1, \dots, -b_q \end{matrix}\right) \qquad p \le q+1 $$

$$ E(p; a_r : q; \beta_s : x) = G^{p,1}_{q+1,p}\left(x \,\middle|\, \begin{matrix} 1, \beta_1, \dots, \beta_q \\ a_1, \dots, a_p \end{matrix}\right) $$

For further special functions expressible in terms of the G-function, in particular for combinations of Legendre functions, and also combinations of generalized hypergeometric series, see C.S. Meijer, *Nederl. Akad. Wetensch., Proc.* 43 (1940), 198-210 and 366-378; 44 (1941), 82-92, 186-194, 298-307, 435-451, 590-605, 1062-1070; 49 (1946), 227-235, 344-356, 457-469, 632-641, 765-772, 936-943, 1063-1072, 1164-1175; 55 (1952), 369-379, 483-487; 56 (1953), 43-49, 187-193.

Hypergeometric series of several variables

See also H.T.F. I Chapter V.

Hypergeometric series of two variables. In all double sums m and n run for 0 to ∞.

$$ F_1(a; \beta, \beta'; \gamma; x, y) = \sum \frac{(a)_{m+n} (\beta)_m (\beta')_n}{(\gamma)_{m+n}\, m!\, n!} x^m y^n $$

$$ F_2(a; \beta, \beta'; \gamma, \gamma'; x, y) = \sum \frac{(a)_{m+n} (\beta)_m (\beta')_n}{(\gamma)_m (\gamma')_n\, m!\, n!} x^m y^n $$

$$ F_3(a, a', \beta, \beta'; \gamma; x, y) = \sum \frac{(a)_m (a')_n (\beta)_m (\beta')_n}{(\gamma)_{m+n}\, m!\, n!} x^m y^n $$

$$ F_4(a, \beta; \gamma, \gamma'; x, y) = \sum \frac{(a)_{m+n} (\beta)_{m+n}}{(\gamma)_m (\gamma')_n\, m!\, n!} x^m y^n $$

$$ \Phi_1(a, \beta, \gamma; x, y) = \sum \frac{(a)_{m+n} (\beta)_m}{(\gamma)_{m+n}\, m!n!} x^m y^n $$

$$\Phi_2(\beta, \beta'; \gamma; x, y) = \sum \frac{(\beta)_m (\beta')_n}{(\gamma)_{m+n}\, m!\, n!} x^m y^n$$

$$\Phi_3(\beta, \gamma; x, y) = \sum \frac{(\beta)_m}{(\gamma)_{m+n}\, m!\, n!} x^m y^n$$

$$\Psi_1(\alpha, \beta, \gamma, \gamma'; x, y) = \sum \frac{(\alpha)_{m+n} (\beta)_m}{(\gamma)_m (\gamma')_n\, m!\, n!} x^m y^n$$

$$\Psi_2(\alpha, \gamma, \gamma'; x, y) = \sum \frac{(\alpha)_{m+n}}{(\gamma)_m (\gamma')_n\, m!\, n!} x^m y^n$$

$$\Xi_1(\alpha, \alpha'; \beta, \gamma; x, y) = \sum \frac{(\alpha)_m (\alpha')_n (\beta)_m}{(\gamma)_{m+n}\, m!\, n!} x^m y^n$$

$$\Xi_2(\alpha, \beta, \gamma; x, y) = \sum \frac{(\alpha)_m (\beta)_m}{(\gamma)_{m+n}\, m!\, n!} x^m y^n$$

For other hypergeometric series of two variables see H.T.F. I sec. 5.7.1.

Hypergeometric series of several variables. All summations run from 0 to ∞.

$$F_A(\alpha; \beta_1, \dots, \beta_n; \gamma_1, \dots, \gamma_n; z_1, \dots, z_n)$$
$$= \sum \frac{(\alpha)_{m_1 + \cdots + m_n} (\beta_1)_{m_1} \cdots (\beta_n)_{m_n}}{(\gamma_1)_{m_1} \cdots (\gamma_n)_{m_n}\, m_1! \cdots m_n!} z_1^{m_1} \cdots z_n^{m_n}$$

$$\Phi_2(\beta_1, \dots, \beta_n; \gamma; z_1, \dots, z_n) = \sum \frac{(\beta_1)_{m_1} \cdots (\beta_n)_{m_n}}{(\gamma)_{m_1 + \cdots + m_n}\, m_1! \cdots m_n!} z_1^{m_1} \cdots z_n^{m_n}$$

$$\Psi_2(\alpha; \gamma_1, \dots, \gamma_n; z_1, \dots, z_n) = \sum \frac{(\alpha)_{m_1 + \cdots + m_n}}{(\gamma_1)_{m_1} \cdots (\gamma_n)_{m_n}\, m_1! \cdots m_n!} z_1^{m_1} \cdots z_n^{m_n}$$

Elliptic functions and integrals

See also H.T.F. II Chapter XIII.

Complete elliptic integrals

$$K(k) = \int_0^{\frac{1}{2}\pi} (1 - k^2 \sin^2\phi)^{-\frac{1}{2}}\, d\phi = \tfrac{1}{2}\pi \;{}_2F_1(\tfrac{1}{2}, \tfrac{1}{2}; 1; k^2)$$

$$E(k) = \int_0^{\frac{1}{2}\pi} (1 - k^2 \sin^2\phi)^{\frac{1}{2}}\, d\phi = \tfrac{1}{2}\pi \;{}_2F_1(-\tfrac{1}{2}, \tfrac{1}{2}; 1; k^2)$$

Theta functions

$$\theta_0(v|\tau) = (-i\tau)^{-\frac{1}{2}} \sum_{n=-\infty}^{\infty} e^{-i\pi(v-\frac{1}{2}+n)^2/\tau}$$

$$\theta_1(v|\tau) = (-i\tau)^{-\frac{1}{2}} \sum_{n=-\infty}^{\infty} (-1)^n e^{-i\pi(v-\frac{1}{2}+n)^2/\tau}$$

$$\theta_2(v|\tau) = (-i\tau)^{-\frac{1}{2}} \sum_{n=-\infty}^{\infty} (-1)^n e^{-i\pi(v+n)^2/\tau}$$

$$\theta_3(v|\tau) = (-i\tau)^{-\frac{1}{2}} \sum_{n=-\infty}^{\infty} e^{-i\pi(v+n)^2/\tau}$$

$$\theta_4(v|\tau) = \theta_0(v|\tau).$$

The series given here are connected with the definitions given in H.T.F. II equations 13.19(10) to (13) by means of Jacobi's imaginary transformation, see H.T.F. II equations 13.22(8).

Modified theta functions

$$\hat{\theta}_0(v|\tau) = (-i\tau)^{-\frac{1}{2}} \Big[\sum_{n=0}^{\infty} e^{-i\pi(v+\frac{1}{2}+n)^2/\tau} - \sum_{n=-1}^{-\infty} e^{-i\pi(v+\frac{1}{2}+n)^2/\tau}\Big]$$

$$\hat{\theta}_2(v|\tau) = (-i\tau)^{-\frac{1}{2}} \Big[\sum_{n=0}^{\infty} (-1)^n e^{-i\pi(v+n)^2/\tau} - \sum_{n=-1}^{-\infty} (-1)^n e^{-i\pi(v+n)^2/\tau}\Big]$$

$$\hat{\theta}_3(v|\tau) = (-i\tau)^{-\frac{1}{2}} \Big[\sum_{n=0}^{\infty} e^{-i\pi(v+n)^2/\tau} - \sum_{n=-1}^{-\infty} e^{-i\pi(v+n)^2/\tau}\Big].$$

Miscellaneous functions

See also H.T.F. III Chapter XVIII.

$$\mu(x, a) = \int_0^\infty \frac{x^s s^a}{\Gamma(s+1)} ds$$

$$\nu(x) = \int_0^\infty \frac{x^s}{\Gamma(s+1)} ds$$

$$\nu(x, a) = \int_0^\infty \frac{x^{s+a}}{\Gamma(s+a+1)} ds = \int_a^\infty \frac{x^s}{\Gamma(s+1)} ds$$

INDEX OF NOTATIONS

Y

Z

GREEK LETTERS

MISCELLANEOUS NOTATIONS