Belief as Willingness to Bet

Bryan Renne

(with Jan van Eijck)

LIRa, ILLC Amsterdam

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Probabilistic Reasoning

- We study modal logic for *qualitative* probabilistic reasoning:
  - $K\varphi$ means $P(\varphi) = 1$,
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- Our contributions:
  - Modern reformulation of Lenzen's syntax, semantics, and results.
  - New epistemic neighborhood semantics:
    - Lenzen's logic is sound and complete for a sub-class of our models.
    - Truth-preserving maping: probabilistic $\to$ neighborhood semantics.

Our Epistemic Probability Models

$$\mathcal{M} = (W, R, V, P)$$

- (W, R, V) is a <u>finite</u> $\mathsf{S5}$ Kripke model, where

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(Denominator $\neq 0$ by reflexivity & full support.)

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Notes.

- Multi-agent version defined straightforwardly.
- Taking $\mathsf{S5}$ and full support is atypical but unproblematic.

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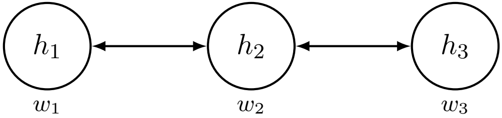
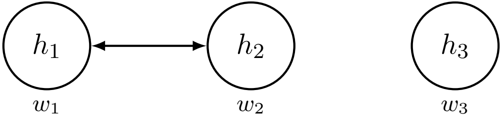


$$P \quad = \quad \{w_1 : \tfrac{3}{6}, w_2 : \tfrac{2}{6}, w_3 : \tfrac{1}{6}\}$$

- Agent considers each of worlds w_1, w_2, w_3 possible.
- Agent assigns odds 3:2:1 to these worlds.
- Letter h_i true at w_i (“Horse h_i wins the race in world w_i).

Our Epistemic Probability Models

Example 2



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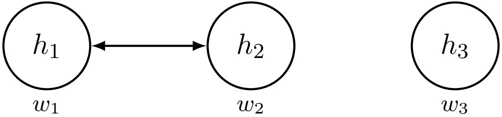

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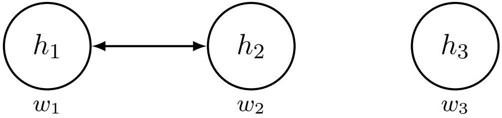

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- Probability will always be evaluated w/r/t a world w via $P_w(X)$.
- So even with full support, worlds can have (relative) probability 0.

Probabilistic Language

$$\begin{array}{rcl} \phi & ::= & \top \mid p \mid \neg\phi \mid \phi \wedge \phi \mid t \geq 0 \\ t & ::= & q \mid q \cdot P(\phi) \mid t + t \\ & & p \in \mathbf{P},\ q \in \mathbb{Q} \end{array}$$

- Abbreviations for: Booleans, $\leq$, $>$, $<$, $=$, and linear (in)equalities.
- Standard semantics for Booleans, to which we add:

$$\mathcal{M}, w \models_{\mathsf{p}} t \geq 0 \quad \text{iff} \quad [\![t]\!]_w \geq 0$$

$$\begin{array}{rcl} [\![\phi]\!]_{\mathsf{p}} & := & \{u \in W \mid \mathcal{M}, u \models_{\mathsf{p}} \phi\} \\ [\![q]\!]_w & := & q \\ [\![q \cdot P(\phi)]\!]_w & := & q \cdot P_w([\![\phi]\!]_{\mathsf{p}}) = q \cdot P([\![\phi]\!]_{\mathsf{p}} | [w]) \\ [\![t + t']\!]_w & := & [\![t]\!]_w + [\![t']\!]_w \end{array}$$

Knowledge and Belief in the Probabilistic Language

$$\begin{aligned} K\varphi &:= P(\varphi) = 1 \\ B^c\varphi &:= P(\varphi) > c \quad \text{for } c \in (0,1) \cap \mathbb{Q} \end{aligned}$$

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  - Meaning of belief dual:

$$\models_{\mathsf{p}} \check{B}^c\phi \Leftrightarrow (P(\phi) \geq 1 - c) \ ,$$

so $\check{B}^{0.5}\phi$ says that $P(\phi) \geq 0.5$.

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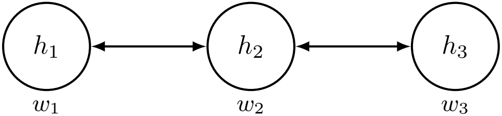
    so $\check{B}^{0.5}\phi$ says that $P(\phi) \geq 0.5$.
  - B^c is not normal (i.e., does not satisfy K):



$$P = \{w_1 : \tfrac{1}{3}, w_2 : \tfrac{1}{3}, w_3 : \tfrac{1}{3}\}$$

$$\mathcal{M} \models_{\mathsf{p}} B\neg h_1 \wedge B\neg h_2 \text{ and yet } \mathcal{M} \not\models_{\mathsf{p}} B(\neg h_1 \wedge \neg h_2).$$

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- Intuition for this notion of belief:
  - Agent believes ϕ iff she is "pretty sure" of truth (i.e., $P(\phi) > c$).
  - So "less sure" about conjunction $A \wedge B$ if $A \not\subseteq B$ and $B \not\subseteq A$.

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  - there is some winning lottery ticket among the n tickets, and
  - for each ticket $i = 1, \ldots, n$, ticket i is not winning.

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- This notion of belief comes from subjective probability:
  - Suppose agent believes φ with threshold p/q; i.e., $B^{p/q}\varphi$.
  - Wagers p dollars for chance to win $q - p$ dollars on bet φ is true.
  - Expected win:

$$(q-p) \cdot P(\varphi) - p \cdot (1 - P(\varphi)) = q \cdot P(\varphi) - p \ .$$

  - Positive iff $P(\varphi) > p/q$, which is guaranteed by $B^{p/q}\varphi$.
  - So bet is good iff has belief. "Belief as willingness to bet."

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  - So bet is good iff has belief. "Belief as willingness to bet."
- Will set aside philsophical considerations, accept belief here as-is.

Segerberg Notation

$$(\phi_1, \ldots, \phi_m \mathbb{I} \psi_1, \ldots, \psi_m) \ ,$$

also written $(\phi_i \mathbb{I} \psi_i)_{i=1}^m$, abbreviates the formula

$$K(C_0 \vee C_1 \vee C_2 \vee \cdots \vee C_m) \ ,$$

where C_i is the disjunction of all conjunctions

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satisfying:

- exactly i of d_k's are the empty string,
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Intuitive meaning: the agent knows that the number of true ψ_k's is is no less than the number of true ϕ_k's.

Lenzen Scheme

$$[(\phi_i \mathbb{I} \psi_i)_{i=1}^m \wedge B^c\phi_1 \wedge \bigwedge_{i=2}^m \check{B}^c\phi_i] \rightarrow \bigvee_{i=1}^m B^c\psi_i \tag{Len}$$

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Lenzen showed:

(Len) is key to modal-language completeness for fixed $c = \frac{1}{2}$.

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  The propositional constant $\bot$ for falsehood is not believed.
- $\models_{\mathsf{p}} B^c\top$. (Minimal Logic's N)
  The propositional constant $\top$ for truth is believed.
- $\models_{\mathsf{p}} B^c\phi \to KB^c\phi$.
  What is believed is known to be believed.
- $\models_{\mathsf{p}} \neg B^c\phi \to K\neg B^c\phi$.
  What is not believed is known to be not believed.

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- $\models_{\mathsf{p}} \check{B}^c\phi \wedge \check{K}(\neg\phi \wedge \psi) \rightarrow B^c(\phi \vee \psi)$ for $c \in \left(0, \frac{1}{2}\right] \cap \mathbb{Q}$.
  If ϕ is consistent with agent a's beliefs and $\neg\phi \wedge \psi$ is consistent with agent a's knowledge, then agent a believes $\phi \vee \psi$.

Modal Language

$$\begin{array}{rcl}\phi & ::= & \top \mid p \mid \neg\phi \mid \phi \wedge \phi \mid K\phi \mid B\phi \\ & & p \in \mathbf{P}\end{array}$$

- Threshold $c \in (0,1) \cap \mathbb{Q}$ omitted; will come from the semantics.

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- We define a neighborhood semantics; maps are truth-preserving.

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- Semantics ($\models_{\mathsf{n}}$, $[\![\cdot]\!]_{\mathsf{n}}$): standard Boolean semantics plus

$$\begin{array}{lll} \mathcal{M}, w \models_{\mathsf{n}} K\phi & \text{iff} & [w] \subseteq [\![\phi]\!]_{\mathsf{n}}^{\mathcal{M}} \\ \mathcal{M}, w \models_{\mathsf{n}} B\phi & \text{iff} & [w] \cap [\![\phi]\!]_{\mathsf{n}}^{\mathcal{M}} \in N(w) \end{array}$$

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Additional Properties on N for "Mid-Threshold" Models

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- if $(X_i \mathbb{I} Y_i)_{i=1}^m$, $X_1 \in N(w)$, and $[w] - X_2, \ldots, [w] - X_m \notin N(w)$,
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(Will see: Lenzen's logic for $c = \frac{1}{2}$ is complete for mid-threshold models.)

Connecting Probability and Neighborhood Models

Each $c \in (0,1) \cap \mathbb{Q}$ induces a map

$$\mathcal{M} = (W, R, V, P) \qquad \overset{c}{\mapsto} \qquad \mathcal{M}^c = (W, R, V, N^c)$$

given by setting

$$N^c(w) := \{X \subseteq [w] \mid P_w(X) > c\} \ .$$

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- For each modal formula ϕ:

$$\mathcal{M}^c, w \models_{\mathsf{n}} \phi \quad \text{iff} \quad \mathcal{M}, w \models_{\mathsf{p}} \phi^c \ .$$

$\Rightarrow$ The qualitative modal language "talks correctly" about probability.

$\Rightarrow$ Epistemic neighborhood models are connected with probability models.

The Basic Qualitative Theory KB

Axiom Schemes

(CL)	Schemes of Classical Propositional Logic
(KS5)	$\mathsf{S5}$ axiom schemes for each K
(KBC)	$K\phi \to B\phi$
(BF)	$\neg B\bot$
(N)	$B\top$
(Ap)	$B\phi \to KB\phi$
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$$\frac{\phi \to \psi \quad \phi}{\psi}\ (\mathrm{MP}) \qquad \frac{\phi}{K\phi}\ (\mathrm{MN})$$

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Theorem. KB is sound and complete for epist. neighborhood models.

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Proof is tricky (extends Segerberg's "logical finiteness" notion).

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Sketch of argument (adapted from Walley and Fine, 1979):

- Propositional letters are $a, b, c, \ldots, g$.
- $X := \{efg, abg, adf, bde, ace, cdg, bcf\}$ with, e.g., $efg = \{e, f, g\}$.
- $Y := \{fga, bca, beg, cef, bdf, dea, cdg\}$ with similar abbreviations.
- $\mathcal{M} := (W, R, V, N)$ defied by: W is letter set, R is total, V satisfies p at p, and $N(w)$ is the superset closure of X. This is an epist. neighborhood model.
- σ is the modal formula describing $(\mathcal{M}, a)$: informally (easily formalized),

$$\sigma \quad := \quad a\bar{b}\cdots\bar{g} \wedge KW \wedge (\textstyle\bigwedge_{Z\in N(a)} BZ) \wedge (\textstyle\bigwedge_{Y\in\wp(W)-N(a)} \neg BY) \ .$$

- $\mathcal{M}, w \models_{\mathsf{n}} \sigma$, so $\not\models_{\mathsf{n}} \neg\sigma$ and hence $\mathsf{KB} \nvdash \neg\sigma$ by our completness result.
- Argue that $\models_{\mathsf{p}} \neg\sigma^c$ by *reductio*: assume $\mathcal{N}, w \models_{\mathsf{p}} \sigma^c$.
  - $\mathcal{N}, w \models_{\mathsf{p}} P(W) = 1 \wedge (\bigwedge_{Z\in N(a)} P(Z) > c) \wedge (\bigwedge_{Y\in\wp(W)-N(a)} P(Z) \leq c)$.
  - Each letter p occurs in exactly three X-sets:
  $$7c < \textstyle\sum_{x\in X} P_w(x) = \sum_{p\in W} 3 \cdot P\left(p \wedge \bigwedge_{q\in W-\{p\}} \neg q\right) \ .$$
  - Each letter p occurs in exactly three Y-sets, none of which is in $N(a)$:
  $$7c \geq \textstyle\sum_{y\in Y} P_w(y) = \sum_{p\in W} 3 \cdot P\left(p \wedge \bigwedge_{q\in W-\{p\}} \neg q\right) \ .$$

$\mathsf{KB}^{0.5}$: (our name for) Lenzen's Theory for $c = \frac{1}{2}$

(KB)	Schemes and rules of KB
(D)	$B\phi \to \check{B}\phi$
(SC)	$\check{B}\phi \wedge \check{K}(\neg\phi \wedge \psi) \to B(\phi \vee \psi)$
(L)	$[(\phi_i \mathbb{I} \psi_i)_{i=1}^m \wedge B\phi_1 \wedge \bigwedge_{i=2}^m \check{B}\phi_i] \to \bigvee_{i=1}^m B\psi_i$

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Theorem (Lenzen). $\mathsf{KB}^{0.5}$ is probabilistically complete for $c = \frac{1}{2}$:

$$\mathsf{KB}^{0.5} \vdash \phi \qquad \text{iff} \qquad \models_{\mathsf{p}} \phi^{\frac{1}{2}} \ .$$

Conclusions

- Contributions:
  - Formulate Lenzen's result in modern modal language.
  - Modal completeness for epistemic neighborhood semantics.
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- Contributions:
  - Formulate Lenzen's result in modern modal language.
  - Modal completeness for epistemic neighborhood semantics.
  - Truth-preserving connection: neighborhoods and probabilities.
- Open questions:
  - Probabilistic comleteness for high-thresholds $c \in \left(\frac{1}{2}, 1\right) \cap \mathbb{Q}$:
    $$\mathsf{KB}^c \vdash \phi \text{ iff } \models_{\mathsf{p}} \phi^c \text{ for which } \mathsf{KB}^c?$$
    Principles: for $s' := c/(1-c)$ and $s := \mathsf{ceiling}(s')$,
    $$\begin{array}{ll} (\mathrm{SC}_0^s) & (\check{K}\phi_0 \wedge \bigwedge_{i=1}^{s} \check{B}\phi_i \wedge \bigwedge_{i \neq j=0}^{s} K(\phi_i \to \neg\phi_j)) \to B(\bigvee_{i=0}^{s} \phi_i) \\ (\mathrm{SC}_1^s) & (\bigwedge_{i=1}^{s} \check{B}\phi_i \wedge \bigwedge_{i \neq j=1}^{s} K(\phi_i \to \neg\phi_j)) \to B(\bigvee_{i=1}^{s} \phi_i) \\ (\mathrm{WL}) & [(\phi_i \mathbb{I} \psi_i)_{i=1}^{m} \wedge \bigwedge_{i=1}^{m} B\phi_i] \to \bigvee_{i=1}^{m} B\psi_i \end{array}$$
    (WL) sound. SC_1^s sound if $s \neq s'$, SC_0^s sound if $s = s'$.
  - Segerberg's/Gardenfors' probability comparison $\phi \geq \psi$ connection?
  - De Jongh–Ghosh belief strength comparison $\phi \geq \psi$ connection?
  - Neighborhoods for public announcements (i.e., Bayesian updates)?

The End

Bryan Renne

http://bryan.renne.org/