Belief as Willingness to Bet

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Abstract

We investigate modal logics of knowledge and belief in which knowledge is probabilistic certainty and belief is probability exceeding a fixed rational threshold $c \geq \frac{1}{2}$. Taking c as an agent's betting threshold leads to the motto "belief is willingness to bet." The logic $\mathsf{KB}^{0.5}$ for $c = \frac{1}{2}$ has $\mathsf{S5}$ knowledge modalities along with sub-normal belief modalities that extend the minimal modal logic $\mathsf{EMND45} + \neg B_a \bot$ by way of certain schemes relating knowledge and belief. A key property first articulated in the present form by Lenzen [Len03, Len80] originates from a result in measure theory due to Kraft, Pratt, and Seidenberg [KPS59] as presented and extended by Scott [Sco64]. Our approach here is a slight variation and extensive modernization of Lenzen's for $c = \frac{1}{2}$, but ours also permits for the possibility of thresholds exceeding $\frac{1}{2}$. Along with new results, we present Lenzen's proof of probability completeness for $\mathsf{KB}^{0.5}$. We also show that $\mathsf{KB}^{0.5}$ is sound and complete with respect to a new epistemic neighborhood semantics that bears an exact relationship with the probabilistic semantics for $c = \frac{1}{2}$. This yields a certain link between probabilistic and modal neighborhood semantics that may be of use in future cross-discipline work.

1 Introduction

A number of authors have studied connections between modal logic and probability; see, e.g., [Hal03, Her03] and citations therein. Our interest here is in identifying the modal logic of probabilistic certainty and belief exceeding a fixed $c \in [\frac{1}{2}, 1) \cap \mathbb{Q}$ and then relating this logic to a more familiar modal semantics.

Lenzen [Len03, Len80] is to our knowledge the first to consider this study for $c = \frac{1}{2}$. Actually, his perspective on the relation between knowledge and belief was slightly different from ours. He identified "the agent is convinced of A" with $P(A) = 1$ and "B is believed" by $P(B) > \frac{1}{2}$. Conviction (German: *Überzeugung*) does not imply truth. One of Lenzen's key contributions was the identification of a modal scheme that ensures the existence of a probability function realizing these

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notions of conviction and belief. Lenzen's scheme is closely related to a scheme used by Segerberg [Seg71] for a similar purpose in a modal logic of qualitative probabilistic comparisons (with non-nested binary modal operators for "A is at least as probable as B"). Lenzen's and Segerberg's logics are different because the link with probability is different. We therefore refer to the "Lenzen scheme," even though Lenzen himself credits Segerberg. Either way, both researchers' schemes exploit the Kraft-Pratt-Seidenberg theorem from measure theory [KPS59] or related results due to Scott [Sco64].

In more recent related work, Herzig [Her03] considers a logic of belief and action in which belief in A is identified with $P(A) > P(\neg A)$. This is equivalent to Lenzen's notion. Another more recent work by Kyberg and Teng [KT12] investigates a notion of "acceptance" in which A is accepted whenever the probability of $\neg A$ is at most some small ϵ. This gives rise to the minimal modal logic EMN, which is different than the logic under investigation in the present paper.

We herein consider belief à la Lenzen not only for the case $c = \frac{1}{2}$ but also for the case $c > \frac{1}{2}$. As it turns out, the logics for these cases are different, though our focus will be on the logic for $c = \frac{1}{2}$. Probability completeness for $c > \frac{1}{2}$ is still open. Thresholds $c < \frac{1}{2}$ permit simultaneous belief of A and $\neg A$ along with other unusual properties; these "low-threshold" beliefs are also left for future work (though we say a bit more on this later).

In Section 2 we identify a Kripke-style semantics for probability logic similar to [EoS, Hal03] (and no doubt to many others). We require that all worlds are probabalistically possible but not necessarily epistemically so, and we provide some examples of how this semantics works. In particular, we demonstrate that our requirement is not problematic.

In Section 3 we define our modal notions of certain knowledge and belief exceeding threshold c, explain the motto "belief is willingness to bet," and prove a number of properties of certain knowledge and this "betting" belief. For instance, we show that knowledge is $\mathsf{S5}$ and belief is not normal. We show a number of other threshold-specific properties of betting belief as well. In particular, we see that the belief modality extends the minimal modal logic $\mathsf{EMND45} + \neg B_a \bot$ by way of certain schemes relating knowledge and belief.[1]

We then introduce a formal modal language in Section 4 and relate this language to the probabilistic notions of belief and knowledge. We also introduce a neighborhood semantics for this language with a new epistemic twist. The relationship between the neighborhood and probabilistic semantics is provided in Section 5. There we prove that for each threshold c, there is a translation from epistemic probability models to epistemic neighborhood models such that certain knowledge, betting belief, and Boolean truth are all preserved. We also show that for $c = \frac{1}{2}$, certain desirable properties are preserved.

In Section 6, we introduce the modal theory $\mathsf{KB}^{0.5}$ of knowledge and belief. We prove this is theory sound and complete with respect to a certain class of our epistemic neighborhood models. To complete this proof, we extend Segerberg's [Seg71] ingenious concept of "logical finiteness" of neighborhood models in a way that guarantees definability of certain sets within a finite countermodel. Using our notation and concepts, we also present Lentzen's proof [Len80] that $\mathsf{KB}^{0.5}$ determines belief with threshold $c = \frac{1}{2}$. Given the link between our new epistemic neighborhood

[1] The definition of this logic is provided later. See [Che80, Ch. 8] for further details on naming of some minimal modal logics.

semantics and probability, our results may be viewed as a contribution to the study connecting two schools of rational decision making: the probabilist (e.g., [Kör08]) and the AI-based (e.g., [KT12]).

2 Epistemic Probability Models

Definition 2.1. We fix a finite nonempty set A of "agents" and a set $\mathbf{P}$ of propositional letters. An *epistemic probability model* is a structure

$$\mathcal{M} = (W, R, V, P)$$

satisfying the following.

- (W, R, V) is a finite multi-agent $\mathsf{S5}$ Kripke model:
  - W is a finite nonempty set of "worlds." An *event* is a set $X \subseteq W$ of worlds. When convenient, we identify a world w with the singleton event $\{w\}$.
  - $R : A \to \mathcal{P}(W \times W)$ assigns an equivalence relation R_a on W to each agent $a \in A$. We let
    $$[w]_a := \{v \in W \mid wR_a v\}$$
    denote the a-equivalence class of world w. This is the set of worlds that agent a cannot distinguish from w.
  - $V : W \to \mathcal{P}(\mathbf{P})$ assigns a set $V(w)$ of propositional letters to each world $w \in W$.
- $P : A \to W \to [0,1]$ assigns to each agent $a \in A$ and world $w \in W$ a real number $P_a(w) \in [0,1]$ subject to the following conditions:
  - *Unit total*: $\sum_{w \in W} P_a(w) = 1$ for each $a \in A$.
    This ensures that P_a is a probability function (on worlds).
  - *Full support*: $P_a(w) > 0$ for each $(a, w) \in A \times W$.
    This ensures that each world is probabilistically possible.

  We extend P_a to the set of events: for each event $X \subseteq W$, set
  $$P_a(X) := \sum_{w \in X} P_a(w) \quad .$$

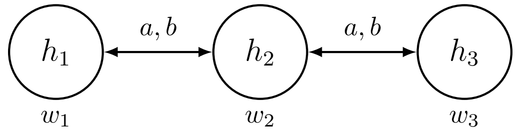
A *pointed epistemic probability model* is a pair $(\mathcal{M}, w)$ consisting of an epistemic probability model $\mathcal{M} = (W, R, V, P)$ and world $w \in W$ called the *point*.

Agent a's uncertainty as to which world is the actual world is given by the equivalence relation R_a. If w is the actual world, then the probability agent a assigns to an event X at w is given by

$$P_{a,w}(X) := \frac{P_a(X \cap [w]_a)}{P_a([w]_a)} \quad . \tag{1}$$

In words: the probability agent a assigns to event X at world w is the probability she assigns to X conditional on her knowledge at w. This reading makes sense because the right side of (1) is just $P_a(X|[w]_a)$, the probability of X conditional on $[w]_a$. Note that $P_{a,w}(X)$ is always well-defined: we have $w \in [w]_a$ by the reflexivity of R_a and hence $0 < P_a(w) \leq P_a([w]_a)$ by full support, so the denominator on the right side of (1) is nonzero.

Example 2.2 (Horse Racing)**.** Three horses compete in a race. For each $i \in \{1, 2, 3\}$, horse h_i wins the race in world w_i. Neither agent can distinguish between these three possibilities. Agent a assigns the horses winning chances of 3:2:1, while b assigns chances 1:2:1. We represent this situation in the form of an epistemic probability model $\mathcal{M}_{2.2}$ pictured as follows:



$$\begin{array}{rcl} P_a & = & \{w_1 : \frac{3}{6}, w_2 : \frac{2}{6}, w_3 : \frac{1}{6}\} \\ P_b & = & \{w_1 : \frac{1}{4}, w_2 : \frac{2}{4}, w_3 : \frac{1}{4}\} \end{array}$$

$\mathcal{M}_{2.2}$

When we picture epistemic probability models, the arrows of each individual agent are to be closed under reflexivity and transitivity. With this convention in place, it is not difficult to verify the following.

1. $P_{a,w_1}(\{w_1, w_3\}) = \frac{2}{3}$.

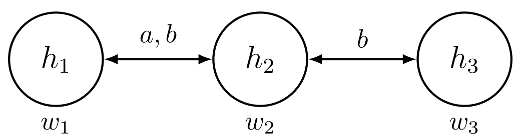
   At w_1, agent a assigns probability $\frac{2}{3}$ to the event that the winner is horse 1 or horse 3.

2. $P_{b,w_1}(\{w_1, w_3\}) = \frac{1}{2}$.

   At w_1, agent b assigns probability $\frac{1}{2}$ to the event that the winner is horse 1 or horse 3.

The property of full support says that each world is probabilistially possible. Therefore, in order to represent a situation in which agent a is certain that horse 3 can never win, we simply make the h_3-worlds inaccessible via R_a.

Example 2.3 (Certainty of impossibility)**.** We modify Example 2.2 by eliminating the a-arrow between worlds w_2 and w_3.



$$\begin{array}{rcl} P_a & = & \{w_1 : \frac{3}{6}, w_2 : \frac{2}{6}, w_3 : \frac{1}{6}\} \\ P_b & = & \{w_1 : \frac{1}{4}, w_2 : \frac{2}{4}, w_3 : \frac{1}{4}\} \end{array}$$

$\mathcal{M}_{2.3}$

At world w_1 in this picture, there is no a-accessible world at which horse 3 wins. Therefore, at world w_1, agent a assigns probability 0 to the event that horse 3 wins: $P_{a,w_1}(w_3) = 0$.

We define a language $\mathcal{L}$ for reasoning about epistemic probability models.

Definition 2.4. The language $\mathcal{L}$ of *multi-agent probability logic* is defined by the following grammar.

$$\begin{array}{rcl} \phi & ::= & \top \mid p \mid \neg\phi \mid \phi \wedge \phi \mid t_a \geq 0 \\ t_a & ::= & q \mid q \cdot P_a(\phi) \mid t_a + t_a \\ & & p \in \mathbf{P}, q \in \mathbb{Q}, a \in A \end{array}$$

We adopt the usual abbreviations for Boolean connectives. We define the relational symbols $\leq$, $>$, $<$, and $=$ in terms of $\geq$ as usual. For example, $t = s$ abbreviates $(t \geq s) \wedge (s \geq t)$. We also use the obvious abbreviations for writing general linear inequalities. For example, $P_a(p) \leq 1 - q$ abbreviates $1 + (-q) + (-1) \cdot P_a(p) \geq 0$.

Definition 2.5. Let $\mathcal{M} = (W, R, V, P)$ be an epistemic probability model. We define a binary truth relation $\models_{\mathsf{p}}$ between a pointed epistemic probability model $(\mathcal{M}, w)$ and $\mathcal{L}$-formulas as follows.

$$\begin{array}{lll} \mathcal{M}, w \models_{\mathsf{p}} \top & & \\ \mathcal{M}, w \models_{\mathsf{p}} p & \text{iff} & p \in V(w) \\ \mathcal{M}, w \models_{\mathsf{p}} \neg\phi & \text{iff} & \mathcal{M}, w \not\models_{\mathsf{p}} \phi \\ \mathcal{M}, w \models_{\mathsf{p}} \phi \wedge \psi & \text{iff} & \mathcal{M}, w \models_{\mathsf{p}} \phi \text{ and } \mathcal{M}, w \models_{\mathsf{p}} \psi \\ \mathcal{M}, w \models_{\mathsf{p}} t_a \geq 0 & \text{iff} & [\![t_a]\!]_w \geq 0 \end{array}$$

$$\begin{array}{rcl} [\![\phi]\!]_{\mathsf{p}} & := & \{u \in W \mid \mathcal{M}, u \models_{\mathsf{p}} \phi\} \\ P_{a,w}(X) & := & \dfrac{P_a(X \cap [w]_a)}{P_a([w]_a)} \\ [\![q]\!]_w & := & q \\ [\![q \cdot P_a(\phi)]\!]_w & := & q \cdot P_{a,w}([\![\phi]\!]_{\mathsf{p}}) \\ [\![t_a + t'_a]\!]_w & := & [\![t_a]\!]_w + [\![t'_a]\!]_w \end{array}$$

Validity of $\phi \in \mathcal{L}$ in epistemic probability model $\mathcal{M}$, written $\mathcal{M} \models_{\mathsf{p}} \phi$, means that $\mathcal{M}, w \models_{\mathsf{p}} \phi$ for each world $w \in W$. Validity of $\phi \in \mathcal{L}$, written $\models_{\mathsf{p}} \phi$, means that $\mathcal{M} \models_{\mathsf{p}} \phi$ for each epistemic probability model $\mathcal{M}$.

3 Certainty and Belief

[Eij13] formulates and proves a "certainty theorem" relating certainty in epistemic probability models to knowledge in a version of these models in which the probabilistic information is removed. This motivates the following definition.

Definition 3.1 (Knowledge as certainty)**.** We adopt the following abbreviations.

- $K_a\phi$ abbreviates $P_a(\phi) = 1$.

  We read $K_a\phi$ as "agent a knows ϕ."

- $\check{K}_a\phi$ abbreviates $\neg K_a\neg\phi$.

  We read $\check{K}_a\phi$ as "ϕ is consistent with agent a's knowledge."

Proposition 3.2 (Properties of knowledge as certainty; [Eij13])**.** K_a is an $\mathsf{S5}$ modal operator:

1. $\models_{\mathsf{p}} \phi$ for each $\mathcal{L}$-instance ϕ of a scheme of classical propositional logic.

   Axioms of classical propositional logic are valid.

2. $\models_{\mathsf{p}} K_a(\phi \to \psi) \to (K_a\phi \to K_a\psi)$

   Knowledge is closed under logical consequence.

3. $\models_{\mathsf{p}} K_a\phi \to \phi$

   Knowledge is veridical.

4. $\models_{\mathsf{p}} K_a\phi \to K_aK_a\phi$

   Knowledge is positive introspective: it is known what is known.

5. $\models_{\mathsf{p}} \neg K_a\phi \to K_a\neg K_a\phi$

   Knowledge is negative introspective: it is known what is not known.

6. $\models_{\mathsf{p}} \phi$ implies $\models_{\mathsf{p}} K_a\phi$

   All validities are known.

7. $\models_{\mathsf{p}} \phi \to \psi$ and $\models_{\mathsf{p}} \phi$ together imply $\models_{\mathsf{p}} \psi$.

   Validities are closed under the rule of Modus Ponens.

We define belief in a proposition ϕ as willingness to take bets on ϕ with the odds being better than some rational number $c \in [\frac{1}{2}, 1) \cap \mathbb{Q}$. This leads to a number of degrees of belief, one for each threshold c.[2]

Definition 3.3 (Belief as willingness to bet)**.** Fix a threshold $c \in [\frac{1}{2}, 1) \cap \mathbb{Q}$.

[2]Belief based on threshold $c = 0$ or $c = 1$ is trivial to express in terms of negation, K_a, and falsehood $\bot$. So we do not consider these thresholds here. Beliefs based on low-thresholds $c \in (0, \frac{1}{2}) \cap \mathbb{Q}$ have unintuitive and unusual features. First, low-threshold beliefs unintuitively permit inconsistency of the kind that an agent can believe both ϕ and $\neg\phi$ while avoiding inconsistency of the kind that the agent can believe a single contradictory formula such as $\bot$. Second, the dual of a low-threshold belief implies the belief at that threshold (i.e., $\check{B}^c_a\phi \to B^c_a\phi$), which is unusual if we assign the usual "consistency" reading to dual operators (i.e., "ϕ is consistent with the agent's beliefs implies ϕ is believed" is unusual). Since low-threshold $c \in (0, \frac{1}{2}) \cap \mathbb{Q}$ beliefs have these unintuitive and unusual features, we leave their study for future work.

- $B_a^c\phi$ abbreviates $P_a(\phi) > c$.

  We read $B_a^c\phi$ as "agent a believes ϕ with threshold c."

- $\check{B}_a^c\phi$ abbreviates $\neg B_a^c\neg\phi$.

  We read $\check{B}_a\phi$ as "ϕ is consistent with agent a's threshold-c beliefs."

If the threshold c is omitted (either in the notations $B_a^c\phi$ and $\check{B}_a^c\phi$ or in the informal readings of these notations), it is assumed that $c = \frac{1}{2}$.

This notion of belief comes from subjective probability [Jef04]. In particular, fix a threshold $c = p/q \in [\frac{1}{2}, 1) \cap \mathbb{Q}$. Suppose that agent a believes ϕ with threshold $c = p/q$; that is, $P_a(\phi) > p/q$. If the agent wagers p dollars for a chance to win $q - p$ dollars on a bet that ϕ is true, then she expects to win

$$(q - p) \cdot P_a(\phi) - p \cdot (1 - P_a(\phi)) = q \cdot P_a(\phi) - p$$

dollars on this bet. This is a positive number of dollars if and only if $q \cdot P_a(\phi) > p$. But notice that the latter is guaranteed by the assumption $P_a(\phi) > p/q$. Therefore, it is rational for agent a to take this bet. Said in the parlance of the subjective probability literature, "If agent a stakes p to win $q - p$ in a bet on ϕ, then her winning expectation is positive in case she believes ϕ with threshold c." Or in a short motto: "Belief is willingness to bet."

The following lemma provides a useful characterization of the dual $\check{B}_a^c\phi$.

Lemma 3.4. Let $\mathcal{M} = (W, R, V, P)$ be an epistemic probability model.

1. $\mathcal{M}, w \models_{\mathsf{p}} \check{B}_a^c\phi$ iff $\mathcal{M}, w \models_{\mathsf{p}} P_a(\phi) \geq 1 - c$.

2. $\mathcal{M}, w \models_{\mathsf{p}} \check{B}_a^{\frac{1}{2}}\phi$ iff $\mathcal{M}, w \models_{\mathsf{p}} P_a(\phi) \geq \frac{1}{2}$.

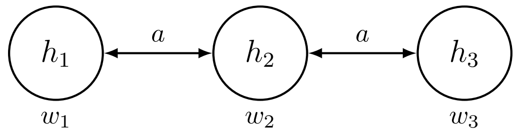
Proof. For Item 1, we have the following:

$$\begin{array}{lll}
 & \mathcal{M}, w \models_{\mathsf{p}} \check{B}_a^c\phi & \\
\text{iff} & \mathcal{M}, w \models_{\mathsf{p}} \neg B_a^c\neg\phi & \text{by definition of } \check{B}_a^c\phi \\
\text{iff} & P_{a,w}([\![\neg\phi]\!]_{\mathsf{p}}) \not> c & \text{by definition of } B_a^c\phi \text{ and } \models_{\mathsf{p}} \\
\text{iff} & P_{a,w}([\![\neg\phi]\!]_{\mathsf{p}}) \leq c & \text{since } \mathbb{Q} \text{ is totally ordered} \\
\text{iff} & P_{a,w}([\![\phi]\!]_{\mathsf{p}}) \geq 1 - c & \text{since } [\![\neg\phi]\!]_{\mathsf{p}} = W - [\![\phi]\!]_{\mathsf{p}}
\end{array}$$

For Item 2, we observe that $1 - \frac{1}{2} = \frac{1}{2}$ and apply Item 1. $\square$

We now consider a simple example.

Example 3.5 (Non-normality)**.** In this single-agent variation, all horses have equal chances of winning and agent a knows this.



$$P_a = \{w_1 : \tfrac{1}{3}, w_2 : \tfrac{1}{3}, w_3 : \tfrac{1}{3}\}$$

$\mathcal{M}_{3.5}$

Recalling that an omitted threshold c is implicitly assumed to be $\frac{1}{2}$, the following are readily verified.

1. $\mathcal{M}_{3.5} \models_{\mathsf{p}} B_a(h_1 \vee h_2 \vee h_3)$.

   Agent a believes the winning horse is among the three.

   (Agent a is willing to bet that the winning horse is among the three.)

2. $\mathcal{M}_{3.5} \models_{\mathsf{p}} B_a(h_1 \vee h_2) \wedge B_a(h_1 \vee h_3) \wedge B_a(h_2 \vee h_3)$.

   Agent a believes the winning horse is among any two.

   (Agent a is willing to bet that the winning horse is among any two.)

3. $\mathcal{M}_{3.5} \models_{\mathsf{p}} B_a\neg h_1 \wedge B_a\neg h_2 \wedge B_a\neg h_3$.

   Agent a believes the winning horse is not any particular one.

   (Agent a is willing to bet that the winning horse is not any particular one.)

4. $\mathcal{M}_{3.5} \models_{\mathsf{p}} \neg B_a(\neg h_1 \wedge \neg h_2)$.

   Agent a does not believe that both horses 1 and 2 do not win.

   (Agent a is not willing to bet that both horses 1 and 2 do not win.)

It follows from Items 3 and 4 of Example 3.5 that the present notion of belief is not closed under conjunction. This is discussed as part of the literature on the "Lottery Paradox" [Kyb61].[3] However, there is no reason in general that it is paradoxical to assign a conjunction $\phi \wedge \psi$ a lower probability than either of its conjunctions. Indeed, if ϕ and ψ are independent, then the probability of their conjunction equals the product of their probabilities, so unless one of ϕ or ψ is certain or impossible, the probability of $\phi \wedge \psi$ will be less than the probability of ϕ and less than the probability of ψ.

We set aside philosophical arguments for or against closure of belief under conjunction and instead turn our attention to the study of the properties of the present notion of belief. One of these is a complicated but useful property due to Lenzen [Len03] that makes use of notation due to Segerberg [Seg71].

[3] The usual formulation of the Lottery Paradox: it is paradoxical for an agent to believe that one of n lottery tickets will be a winner (i.e., "some ticket is a winner") without believing of any particular ticket that it is the winner (i.e., "for each $i \in \{1, \dots, n\}$, ticket i is not a winner").

Definition 3.6 (Segerberg notation; [Seg71]). Fix a positive integer $m \in \mathbb{Z}^+$ and formulas $\phi_1, \dots, \phi_m$ and $\psi_1, \dots, \psi_m$. The expression

$$(\phi_1, \dots, \phi_m \mathbb{I}_a \psi_1, \dots, \psi_m) \tag{2}$$

abbreviates the formula

$$K_a(C_0 \vee C_1 \vee C_2 \vee \cdots \vee C_m) \ ,$$

where C_i is the disjunction of all conjunctions

$$d_1\phi_1 \wedge \cdots \wedge d_m\phi_m \wedge e_1\psi_1 \wedge \cdots \wedge e_m\psi_m$$

satisfying the property that *exactly* i of the d_k's are the empty string, *at least* i of the e_k's are the empty string, and the rest of the d_k's and e_k's are the negation sign $\neg$. We may write $(\phi_i \mathbb{I}_a \psi_i)_{i=1}^m$ as an abbreviation for (2). Finally, let

$$(\phi_i \mathbb{E}_a \psi_i)_{i=1}^m \quad \text{abbreviate} \quad (\phi_i \mathbb{I}_a \psi_i)_{i=1}^m \wedge (\psi_i \mathbb{I}_a \phi_i)_{i=1}^m \ .$$

We also allow the use of $\mathbb{E}_a$ in a notation similar to (2).

The formula $(\phi_i \mathbb{I}_a \psi_i)_{i=1}^m$ says that agent a knows that the number of true ϕ_i's is less than or equal to the number of true ψ_i's. Put another way, $(\phi_i \mathbb{I}_a \psi_i)_{i=1}^m$ is true if and only if every one of a's epistemically accessible worlds satisfies at least as many ψ_i's as ϕ_i's. The formula $(\phi_i \mathbb{E}_a \psi_i)_{i=1}^m$ says that every one of a's epistemically accessible worlds satisfies exactly as many ψ_i's as ϕ_i's.

Definition 3.7 (Lenzen scheme; [Len03]). We define the following scheme:

$$[(\phi_i \mathbb{I}_a \psi_i)_{i=1}^m \wedge B_a^c\phi_1 \wedge \textstyle\bigwedge_{i=2}^m \check{B}_a^c\phi_i] \to \textstyle\bigvee_{i=1}^m B_a^c\psi_i \quad \text{(Len)}$$

If $m = 1$, then $\bigwedge_{i=2}^m \check{B}_a^c\phi_i$ is $\top$. Note that (Len) is meant to encompass the indicated scheme for each positive integer $m \in \mathbb{Z}^+$.

(Len) says that if agent a knows the number of true ϕ_i's is less than or equal to the number of true ψ_i's, agent a believes ϕ_1 with threshold c, and the remaining ϕ_i's are each consistent with agent a's threshold-c beliefs, then agent a believes one of the ψ_i's with threshold c. Adapting a proof of Segerberg [Seg71], we show that belief with threshold $c = \frac{1}{2}$ satisfies (Len).

We report this result along with a number of other properties in the following proposition.

Proposition 3.8 (Properties of belief as willingness to bet). For $c \in [\frac{1}{2}, 1) \cap \mathbb{Q}$, we have:

1. $\not\models_{\mathsf{p}} B_a^c(\phi \to \psi) \to (B_a^c\phi \to B_a^c\psi)$.

   Belief is not closed under logical consequence.

   (So B_a^c is not a normal modal operator.)

2. $\not\models_{\mathsf{p}} B_a^c\phi \to \phi$.

   Belief is not veridical.

3. $\models_{\mathsf{p}} K_a\phi \to B^c_a\phi$.

   What is known is believed.

4. $\models_{\mathsf{p}} \neg B^c_a\bot$.

   The propositional constant $\bot$ for falsehood is not believed.

5. $\models_{\mathsf{p}} B^c_a\top$.

   The propositional constant $\top$ for truth is believed.

6. $\models_{\mathsf{p}} B^c_a\phi \to K_a B^c_a\phi$.

   What is believed is known to be believed.

7. $\models_{\mathsf{p}} \neg B^c_a\phi \to K_a\neg B^c_a\phi$.

   What is not believed is known to be not believed.

8. $\models_{\mathsf{p}} K_a(\phi \to \psi) \to (B^c_a\phi \to B^c_a\psi)$.

   Belief is closed under known logical consequence.

9. $\models_{\mathsf{p}} B^c_a\phi \to \check{B}^c_a\phi$.

   Belief is consistent: belief in ϕ implies disbelief in $\neg\phi$.

10. $\models_{\mathsf{p}} \check{B}^{\frac{1}{2}}_a\phi \wedge \check{K}_a(\neg\phi \wedge \psi) \to B^{\frac{1}{2}}_a(\phi \vee \psi)$.

    For mid-threshold belief, if ϕ is consistent with agent a's beliefs and $\neg\phi \wedge \psi$ is consistent with agent a's knowledge, then agent a believes $\phi \vee \psi$.

11. $\models_{\mathsf{p}} [(\phi_i \mathbb{I}_a \psi_i)_{i=1}^m \wedge B^{\frac{1}{2}}_a\phi_1 \wedge \bigwedge_{i=2}^m \check{B}^{\frac{1}{2}}_a\phi_i] \to \bigvee_{i=1}^m B^{\frac{1}{2}}_a\psi_i$.

    Mid-threshold belief satisfies (Len).

Proof. We consider each item in turn.

1. Given $c \in (0,1)\cap\mathbb{Q}$ and integers p and q such that $p/q = c$, we define $\mathcal{M}$ as the modification of the model $\mathcal{M}_{3.5}$ of Example 3.5 obtained by changing P_a as follows:

   $$P_a := \left\{ w_1 : \frac{q-p}{2q},\ w_2 : \frac{p}{q},\ w_3 : \frac{q-p}{2q} \right\} \quad .$$

   Since $0 < p < q$, it follows that

   $$\begin{array}{rcl} P_{a,w_1}([\![\neg h_1 \to h_2]\!]_{\mathsf{p}}) & = & P_{a,w_1}(\{w_1, w_2\}) = \dfrac{q+p}{2q} > \dfrac{2p}{2q} = \dfrac{p}{q}, \\ P_{a,w_1}([\![\neg h_1]\!]_{\mathsf{p}}) & = & P_{a,w_1}(\{w_2, w_3\}) = \dfrac{q+p}{2q} > \dfrac{2p}{2q} = \dfrac{p}{q}, \text{ and} \\ P_{a,w_1}([\![h_2]\!]_{\mathsf{p}}) & = & P_{a,w_1}(w_2) = \dfrac{p}{q} \quad . \end{array}$$

Therefore, we have

$$\mathcal{M}, w_1 \models_{\mathsf{p}} B_a^c(\neg h_1 \to h_2) \wedge B_a^c \neg h_1 \wedge \neg B_a^c h_2 \ \ .$$

2. For $\mathcal{M}$ defined in the proof of Item 1, we have

$$\mathcal{M}, w_1 \models_{\mathsf{p}} h_1 \wedge B_a^c \neg h_1 \ \ .$$

3. $\mathcal{M}, w \models_{\mathsf{p}} K_a\phi$ implies $P_{a,w}([\![\phi]\!]_{\mathsf{p}}) = 1 > c$. Hence $\mathcal{M}, w \models_{\mathsf{p}} B_a^c\phi$.

4. $P_{a,w}([\![\bot]\!]_{\mathsf{p}}) = 0 < c$. Hence $\mathcal{M}, w \models \neg B_a^c \bot$.

5. $P_{a,w}([\![\top]\!]_{\mathsf{p}}) = 1 > c$. Hence $\mathcal{M}, w \models_{\mathsf{p}} B_a^c \top$.

6. $\mathcal{M}, w \models_{\mathsf{p}} B_a^c\phi$ implies $P_{a,w}([\![\phi]\!]_{\mathsf{p}}) > c$. To show that $\mathcal{M}, w \models_{\mathsf{p}} K_a B_a^c \phi$, we must argue that

$$P_{a,w}([\![B_a^c\phi]\!]_{\mathsf{p}}) = \frac{P_a([\![B_a^c\phi]\!]_{\mathsf{p}} \cap [w]_a)}{P_a([w]_a)} = 1 \ \ .$$

To show this, we prove that $[\![B_a^c\phi]\!]_{\mathsf{p}} \cap [w]_a = [w]_a$. So choose $u \in [w]_a$. Since R_a is an equivalence relation, we have

$$P_{a,u}([\![\phi]\!]_{\mathsf{p}}) = \frac{P_a([\![\phi]\!]_{\mathsf{p}} \cap [u]_a)}{P_a([u]_a)} = \frac{P_a([\![\phi]\!]_{\mathsf{p}} \cap [w]_a)}{P_a([w]_a)} = P_{a,w}([\![\phi]\!]_{\mathsf{p}}) > c \ \ ,$$

which implies $u \in [\![B_a^c\phi]\!]_{\mathsf{p}}$. The result follows.

7. The argument is similar to that for Item 6, though we note that $\mathcal{M}, w \models_{\mathsf{p}} \neg B_a^c\phi$ implies $P_{a,w}([\![\phi]\!]_{\mathsf{p}}) \leq c$.

8. We assume that $\mathcal{M}, w \models_{\mathsf{p}} K_a(\phi \to \psi)$ and $\mathcal{M}, w \models_{\mathsf{p}} B_a^c\phi$. This means that $P_{a,w}([\![\phi \to \psi]\!]_{\mathsf{p}}) = 1$ and $P_{a,w}([\![\phi]\!]_{\mathsf{p}}) > c$. But then it follows that $P_{a,w}([\![\psi]\!]_{\mathsf{p}}) > c$ as well, which is what it means to have $\mathcal{M}, w \models_{\mathsf{p}} B_a^c\psi$.

9. Assume $c \in [\frac{1}{2}, 1) \cap \mathbb{Q}$ and $\mathcal{M}, w \models_{\mathsf{p}} B_a^c\phi$. Then $P_{a,w}([\![\phi]\!]_{\mathsf{p}}) > c \geq 1 - c$. So $P_{a,w}([\![\phi]\!]_{\mathsf{p}}) \geq 1 - c$. The result therefore follows by Lemma 3.4.

10. Assume $c \in (0, \frac{1}{2}] \cap \mathbb{Q}$ and $\mathcal{M}, w \models_{\mathsf{p}} \check{B}_a^c\phi$. By Lemma 3.4, it follows that $P_{a,w}([\![\phi]\!]_{\mathsf{p}}) \geq c$. Let us assume further that $\mathcal{M}, w \models_{\mathsf{p}} \check{K}_a(\neg\phi \wedge \psi)$. This means

$$1 \neq P_{a,w}([\![\neg(\neg\phi \wedge \psi)]\!]_{\mathsf{p}}) = \frac{P_a([\![\neg(\neg\phi \wedge \psi)]\!]_{\mathsf{p}} \cap [w]_a)}{P_a([w]_a)} \ \ ,$$

which implies there exists $v \in [\![\neg\phi \wedge \psi]\!]_{\mathsf{p}} \cap [w]_a$. Since $P_a(v) > 0$ by full support, it follows that

$$\begin{aligned}
P_{a,w}([\![\phi \vee \psi]\!]_{\mathsf{p}}) &= \frac{P_a([\![\phi \vee \psi]\!]_{\mathsf{p}} \cap [w]_a)}{P_a([w]_a)} \\
&= \frac{P_a([\![\phi]\!]_{\mathsf{p}} \cap [w]_a)}{P_a([w]_a)} + \frac{P_a([\![\neg\phi \wedge \psi]\!]_{\mathsf{p}} \cap [w]_a)}{P_a([w]_a)} \\
&\geq \frac{P_a([\![\phi]\!]_{\mathsf{p}} \cap [w]_a)}{P_a([w]_a)} + \frac{P_a(v)}{P_a([w]_a)} \\
&= P_{a,w}([\![\phi]\!]_{\mathsf{p}}) + \frac{P_a(v)}{P_a([w]_a)} \\
&\geq c + \frac{P_a(v)}{P_a([w]_a)} > c \ .
\end{aligned}$$

That is, $\mathcal{M}, w \models_{\mathsf{p}} B_a^c(\phi \vee \psi)$.

11. We assume $c \in (0, \frac{1}{2}] \cap \mathbb{Q}$ plus the following:

$$\mathcal{M}, w \models_{\mathsf{p}} (\phi_i \mathbb{I}_a \psi_i)_{i=1}^m \tag{3}$$

$$\mathcal{M}, w \models_{\mathsf{p}} B_a^c \phi_1 \tag{4}$$

$$\mathcal{M}, w \models_{\mathsf{p}} \textstyle\bigwedge_{i=2}^m \check{B}_a^c \phi_i \tag{5}$$

We recall the meaning of (3): for each $v \in [w]_a$, the number of ϕ_i's true at v is less than or equal to the number of ψ_k's true at v. It therefore follows from (3) that

$$P_{a,w}([\![\phi_1]\!]_{\mathsf{p}}) + \cdots + P_{a,w}([\![\phi_m]\!]_{\mathsf{p}}) \leq P_{a,w}([\![\psi_1]\!]_{\mathsf{p}}) + \cdots + P_{a,w}([\![\psi_m]\!]_{\mathsf{p}}) \ . \tag{6}$$

Outlining an argument due to Segerberg [Seg71, pp. 344–346], the reason for this is as follows: we think of each world $v \in [w]_a$ as being assigned a "weight" $P_{a,w}(v)$. A member $P_{a,w}([\![\phi_i]\!]_{\mathsf{p}})$ of the sum on the left of (6) is just a total of the weight of every $v \in [w]_a$ that satisfies ϕ_i; that is,

$$P_{a,w}([\![\phi_i]\!]_{\mathsf{p}}) = \sum \{P_{a,w}(v) \mid v \in [\![\phi_i]\!]_{\mathsf{p}} \cap [w]_a\} \ .$$

Assumption (3) tells us that for each $v \in [w]_a$, the number of totals $P_{a,w}([\![\phi_i]\!]_{\mathsf{p}})$ on the left of (6) to which v contributes its weight is less than or equal to the number of totals $P_{a,w}([\![\psi_k]\!]_{\mathsf{p}})$ on the right of (6) to which v contributes its weight. But then the sum of totals on the left must be less than or equal to the sum of totals on the right. Hence (6) follows.

Having established (6), we now proceed further with the overall proof. By (4), we have $P_{a,w}([\![\phi_1]\!]_{\mathsf{p}}) > c$. Applying (5) and Lemma 3.4, we have $P_{a,w}(\phi_i) \geq c$ for each $i \in \{2, \ldots, m\}$. Hence

$$P_{a,w}([\![\psi_1]\!]_{\mathsf{p}}) + \cdots + P_{a,w}([\![\psi_m]\!]_{\mathsf{p}}) \geq P_{a,w}([\![\phi_1]\!]_{\mathsf{p}}) + \cdots + P_{a,w}([\![\phi_m]\!]_{\mathsf{p}}) > mc \ .$$

That is, the sum of the $P_{a,w}([\![\psi_k]\!]_{\mathsf{p}})$'s must exceed mc. Since each member of this m-member sum is non-negative, it follows that at least one member must exceed c. That is, there exists $j \in \{0, \ldots, m\}$ such that $P_{a,w}([\![\psi_j]\!]_{\mathsf{p}}) > c$. Hence $\mathcal{M}, w \models_{\mathsf{p}} \bigvee_{j=1}^m B_a^c \psi_j$. □

4 Epistemic Neighborhood Models

The modal formulas $K_a\phi$ and $B_a^c\phi$ were taken as abbreviations in the language $\mathcal{L}$ of multi-agent probability logic. We wish to consider a propositional modal language that has knowledge and belief operators as primitives.

Definition 4.1. The language $\mathcal{L}_{\mathsf{KB}}$ of *multi-agent knowledge and belief* is defined by the following grammar.

$$\begin{array}{rcl}\phi & ::= & \top \mid p \mid \neg\phi \mid \phi\wedge\phi \mid K_a\phi \mid B_a\phi \\ & & p\in\mathbf{P},\, a\in A\end{array}$$

We adopt the usual abbreviations for other Boolean connectives and define the dual operators $\check{K}_a := \neg K_a \neg$ and $\check{B}_a := \neg B_a\neg$. Finally, the $\mathcal{L}_{\mathsf{KB}}$-formula

$$(\phi_1,\dots,\phi_m \mathbb{I}_a \psi_1,\dots,\psi_m)$$

and its abbreviation $(\phi_i \mathbb{I}_a \psi_i)_{i=1}^m$ are given as in Definition 3.6 except that all formulas are taken from the language $\mathcal{L}_{\mathsf{KB}}$.

Our goal will be to develop a possible worlds semantics for $\mathcal{L}_{\mathsf{KB}}$ that links with the probabilistic setting by making the following translation truth-preserving.

Definition 4.2 (Translation)**.** For each $c\in[\frac{1}{2},1)\cap\mathbb{Q}$, we define a translation $c:\mathcal{L}_{\mathsf{KB}}\to\mathcal{L}$ as follows.

$$\begin{array}{rcll}\top^c & := & \top & \\ p^c & := & p & \\ (\neg\phi)^c & := & \neg\phi^c & \\ (\phi\wedge\psi)^c & := & \phi^c\wedge\psi^c & \\ (K_a\phi)^c & := & P_a(\phi^c)=1 & (= K_a\phi^c \text{ in } \mathcal{L}) \\ (B_a\phi)^c & := & P_a(\phi^c)>c & (= B_a^c\phi^c \text{ in } \mathcal{L})\end{array}$$

Since we have seen that the probabilistic belief operator B_a^c is not a normal modal operator (Proposition 3.8(1)), we opt for a neighborhood semantics for $\mathcal{L}_{\mathsf{KB}}$ [Che80, Ch. 7] with a new epistemic twist.

Definition 4.3. An *epistemic neighborhood model* is a structure

$$\mathcal{M} = (W,R,V,N)$$

satisfying the following.

- (W,R,V) is a finite multi-agent S5 Kripke model (as in Definition 2.1). As before, we let

  $$[w]_a := \{v\in W \mid wR_av\}$$

  denote the a-equivalence class of world w. This is the set of worlds a cannot distinguish from w.

- $N : A \times W \to \mathcal{P}(\mathcal{P}(W))$ is a *neighborhood function* that assigns to each agent $a \in A$ and world $w \in W$ a collection $N_a(w)$ of sets of worlds—each such set called a *neighborhood* of w—subject to the following conditions.

  (kbc) $\forall X \in N_a(w) : X \subseteq [w]_a$.

  (kbf) $\emptyset \notin N_a(w)$.

  (n) $[w]_a \in N_a(w)$.

  (a) $\forall v \in [w]_a : N_a(v) = N_a(w)$.

  (kbm) $\forall X \subseteq Y \subseteq [w]_a :$ if $X \in N_a(w)$, then $Y \in N_a(w)$.

A *pointed epistemic neighborhood model* is a pair $(\mathcal{M}, w)$ consisting of an epistemic neighborhood model $\mathcal{M}$ and a world w in M.

An epistemic neighborhood model is a variation of a neighborhood model that includes an epistemic component R_a for each agent a. Intuitively, $[w]_a$ is the set of worlds agent a knows to be possible at w and each $X \in N_a(w)$ represents a proposition that the agent believes at w. The condition that R_a be an equivalence relation ensures that knowledge is closed under logical consequence, veridical (i.e., only true things can be known), positive introspective (i.e., the agent knows what she knows), and negative introspective (i.e., the agent knows what she does not know).

Property (kbc) ensures that the agent does not believe a proposition $X \subseteq W$ that she knows to be false: if X contains a world in $w' \in (W - [w]_a)$ that the agent knows is not possible with respect to the actual world w, then she knows that X cannot be the case and hence she does not believe X. Property (kbf) ensures that no logical falsehood is believed, while Property (n) ensures that every logical truth is believed. Property (a) ensures that X is believed if and only if it is known that X is believed. Property (kbm) says that belief is monotonic: if an agent believes X, then she believes all propositions $Y \supseteq X$ that follow from X.

We now turn to the definition of truth for the language $\mathcal{L}_{\mathsf{KB}}$.

Definition 4.4. Let $\mathcal{M} = (W, R, V, N)$ be an epistemic neighborhood model. We define a binary truth relation $\models_{\mathsf{n}}$ between a pointed epistemic neighborhood model $(\mathcal{M}, w)$ and $\mathcal{L}_{\mathsf{KB}}$-formulas and a function $[\![\cdot]\!]^{\mathcal{M}}_{\mathsf{n}} : \mathcal{L}_{\mathsf{KB}} \to \mathcal{P}(W)$ as follows.

$$
\begin{array}{rcl}
[\![\phi]\!]^{\mathcal{M}}_{\mathsf{n}} & := & \{v \in W \mid \mathcal{M}, v \models_{\mathsf{n}} \phi\} \\
\mathcal{M}, w \models_{\mathsf{n}} p & \text{iff} & p \in V(w) \\
\mathcal{M}, w \models_{\mathsf{n}} \neg\phi & \text{iff} & \mathcal{M}, w \not\models_{\mathsf{n}} \phi \\
\mathcal{M}, w \models_{\mathsf{n}} \phi \wedge \psi & \text{iff} & \mathcal{M}, w \models_{\mathsf{n}} \phi \text{ and } \mathcal{M}, w \models_{\mathsf{n}} \psi \\
\mathcal{M}, w \models_{\mathsf{n}} K_a\phi & \text{iff} & [w]_a \subseteq [\![\phi]\!]^{\mathcal{M}}_{\mathsf{n}} \\
\mathcal{M}, w \models_{\mathsf{n}} B_a\phi & \text{iff} & [w]_a \cap [\![\phi]\!]^{\mathcal{M}}_{\mathsf{n}} \in N_a(w)
\end{array}
$$

Validity of $\phi \in \mathcal{L}_{\mathsf{KB}}$ in an epistemic neighborhood model $\mathcal{M}$, written $\mathcal{M} \models_{\mathsf{n}} \phi$, means that $\mathcal{M}, w \models_{\mathsf{n}} \phi$ for each world $w \in W$. Validity of $\phi \in \mathcal{L}_{\mathsf{KB}}$, written $\models_{\mathsf{n}} \phi$, means that $\mathcal{M} \models_{\mathsf{n}} \phi$ for each epistemic neighborhood model $\mathcal{M}$. For a class $\mathcal{C}$ of epistemic neighborhood models, we write $\mathcal{C} \models_{\mathsf{n}} \phi$ to mean that $\mathcal{M} \models_{\mathsf{n}} \phi$ for each $\mathcal{M} \in \mathcal{C}$.

Intuitively, $K_a\phi$ is true at w iff ϕ holds at all worlds epistemically possible with respect to w, and $B_a\phi$ holds at w iff the epistemically possible ϕ-worlds make up a neighborhood of w. Note that it follows from this definition that the dual for belief $\check{B}_a\phi$ is true at w iff $[w]_a \cap [\![\neg\phi]\!]^{\mathcal{M}}_{\mathsf{n}} \notin N_a(w)$. The latter says that the epistemically possible $\neg\phi$-worlds do not make up a neighborhood of w.

Here are a few additional properties of epistemic neighborhood models that will arise later on.

Definition 4.5 (Extra Properties)**.** Let $\mathcal{M} = (W, R, V, N)$ be an epistemic neighborhood model. For $m \in \mathbb{Z}^+$ and sets of worlds $X_1, \dots, X_m$ and $Y_1, \dots, Y_m$, we write

$$X_1, \dots, X_m \mathbb{I}_a Y_1, \dots, Y_m \tag{7}$$

to mean that for each $v \in W$, the number of X_i's containing v is less than or equal to the number of Y_i's containing v. This is the semantic counterpart of the formula from Definition 3.6. We may write $(X_i \mathbb{I}_a Y_i)_{i=1}^m$ as an abbreviation for (7). Also, we write $(X_i \mathbb{E}_a Y_i)_{i=1}^m$ to mean that both $(X_i \mathbb{I}_a Y_i)_{i=1}^m$ and $(Y_i \mathbb{I}_a X_i)_{i=1}^m$ hold, and we allow the notation with $\mathbb{E}_a$ to be used in a form as in (7). The following is a list of properties that $\mathcal{M}$ may satisfy.

(d) $\forall X \in N_a(w) : [w]_a - X \notin N_a(w)$.

(sc) $\forall X, Y \subseteq [w]_a$: if $[w]_a - X \notin N_a(w)$ and $X \subsetneq Y$, then $Y \in N_a(w)$.

(l) $\forall m \in \mathbb{Z}^+, \forall X_1, \dots, X_m, Y_1, \dots, Y_m \subseteq [w]_a$:

$$\begin{array}{ll} \text{if} & X_1, \dots, X_m \mathbb{I}_a Y_1, \dots, Y_m \quad \text{and} \\ & X_1 \in N_a(w) \quad \text{and} \\ & \forall i \in \{2, \dots, m\} : [w]_a - X_i \notin N_a(w) \ , \\ \text{then} & \exists j \in \{1, \dots, m\} : Y_j \in N_a(w) \ . \end{array}$$

To say an epistemic neighborhood model is *mid-threshold* means it satisfies (d), (sc), and (l). We may drop the word "epistemic" in referring to mid-threshold epistemic neighborhood models. Pointed versions of mid-threshold neighborhood models are defined in the obvious way.

Property (d) ensures that beliefs are consistent in the sense that the agent does not believe both X and its complement $\overline{X} := [w]_a - X$. Property (sc) is a form of "strong commitment": if the agent does not believe the complement $\overline{X}$, then she must believe any strictly weaker Y implied by X. Property (l) is a version of the Lenzen scheme (Len) from Definition 3.7. We will come back to this property later.

5 Relating Belief and Willingness to Bet

We now relate the definition of belief in neighborhood models to the notion of belief as willingness to bet.

Definition 5.1. Given an epistemic probability model $\mathcal{M} = (W, R, V, P)$ and a threshold $c \in [\frac{1}{2}, 1) \cap \mathbb{Q}$, we define the structure

$$\mathcal{M}^c := (W, R, V, N^c)$$

by setting

$$N_a^c(w) := \{X \subseteq [w]_a \mid P_{a,w}(X) > c\} \ .$$

Intuitively, agent a believes a proposition X at world w (i.e., $X \in N_a^c(w)$) if and only if X is epistemically possible (i.e., $X \subseteq [w]_a$) and the probability a assigns to X at world w is greater than c (i.e., $P_{a,w}(X) > c$).

Lemma 5.2 (Correctness)**.** If $\mathcal{M}$ is an epistemic probability model and $c \in [\frac{1}{2}, 1) \cap \mathbb{Q}$, then $\mathcal{M}^c$ is an epistemic neighborhood model. Furthermore, $\mathcal{M}^{\frac{1}{2}}$ is a mid-threshold neighborhood model.

Proof. We verify that N_a^c satisfies the required properties.

- For (kbc), $X \in N_a^c(w)$ implies $X \subseteq [w]_a$ by definition.
- For (kbf), $P_{a,w}(\emptyset) = 0 < c$, so $\emptyset \notin N_a^c(w)$.
- For (n), $P_{a,w}([w]_a) = 1 > c$, so $[w]_a \in N_a^c(w)$.
- For (a), suppose $X \in N_a^c(w)$ and $v \in [w]_a$. Then $P_{a,w}(X) > c$. Since $v \in [w]_a$ implies $[w]_a = [v]_a$, we have

  $$P_{a,w}(X) = \frac{P_a(X \cap [w]_a)}{P_a([w]_a)} = \frac{P_a(X \cap [v]_a)}{P_a([v]_a)} = P_{a,v}(X) \ .$$

  Hence $P_{a,v}(X) > c$, so $X \in N_a^c(v)$.
- For (kbm), suppose $X \in N_a^c(w)$. Then $P_{a,w}(X) > c$. Hence if Y satisfies $X \subseteq Y \subseteq [w]_a$, we have $P_{a,w}(Y) > c$ and so $Y \in N_a^c(w)$.

So $\mathcal{M}^c$ is an epistemic neighborhood model. We now show that $\mathcal{M}^{\frac{1}{2}}$ satisfies the additional required properties.

- For (d), assume $c \in [\frac{1}{2}, 1) \cap \mathbb{Q}$ and $X \in N_a^c(w)$. Then $P_{a,w}(X) > c$, and therefore $P_{a,w}([w]_a - X) \leq 1 - c \leq c$. Hence $[w]_a - X \notin N_a^c(w)$.
- For (sc), assume $X' := [\Gamma]_a - X \notin N_a^{\frac{1}{2}}(w)$ and $X \subsetneq Y \subseteq [\Gamma]_a$. From the first assumption, we have $P_{a,w}(X') \leq \frac{1}{2}$, and therefore that $P_{a,w}(X) \geq \frac{1}{2}$. Applying the second assumption, $P_{a,w}(Y) > P_{a,w}(X) \geq \frac{1}{2}$, and hence $X \in N_a^{\frac{1}{2}}(w)$.

- For (l), we assume $c \in (0, \frac{1}{2}] \cap \mathbb{Q}$ along with the following:

$$(X_i \mathbb{I}_a Y_i)_{i=1}^m \tag{8}$$
$$X_1^{w,a} \in N_a^c(w, i) \tag{9}$$
$$\forall i \in \{2, \dots, m\} : [\Gamma]_a - X \notin N_a^c(w) \tag{10}$$

From (8) it follows that

$$P_{a,w}(X_1) + \cdots + P_{a,w}(X_m) \leq P_{a,w}(Y_1) + \cdots + P_{a,w}(Y_m) \tag{11}$$

The argument for this is similar to an argument for (6) in proof of Proposition 3.8(11). From (9), we have $P_{a,w}(X_1) > c$. From (10), we have for each $i \in \{2, \dots, m\}$ that $P_{a,w}([w]_a - X_i) \leq c$ and therefore that $P_{a,w}(X_i) \geq 1 - c \geq c$ since $c \in (0, \frac{1}{2}] \cap \mathbb{Q}$. Hence the left side of (11) exceeds mc. Since every summand on the right side of the inequality is positive and $mc > 0$, it follows that at least one member of the right side of (11) must exceed c. That is, there exists $j \in \{1, \dots, m\}$ such that $P_{a,w}(Y_j) > c$ and hence $Y_j \in N_a^c(w)$. □

Theorem 5.3 (Betting and Certainty)**.** For each epistemic probability model $\mathcal{M} = (W, R, V, P)$, world $w \in W$, threshold $c \in (0, 1) \cap \mathbb{Q}$, and formula $\phi \in \mathcal{L}_{\mathsf{KB}}$, we have:

$$\mathcal{M}^c, w \models_{\mathsf{n}} \phi \quad \text{iff} \quad \mathcal{M}, w \models_{\mathsf{p}} \phi^c \ .$$

Proof. Induction on the structure of ϕ. The non-modal cases are obvious.

We first consider knowledge formulas. Assume $\mathcal{M}^c, w \models_{\mathsf{n}} K_a\psi$. This means $[w]_a \subseteq [\![\psi]\!]_{\mathsf{n}}^{\mathcal{M}^c}$. Applying the induction hypothesis, this is equivalent to $[w]_a \subseteq [\![\psi^c]\!]_{\mathsf{p}}^{\mathcal{M}}$. But the latter holds if and only if

$$P_{a,w}([\![\psi^c]\!]_{\mathsf{p}}^{\mathcal{M}}) = \frac{P_a([\![\psi^c]\!]_{\mathsf{p}}^{\mathcal{M}} \cap [w]_a)}{P_a([w]_a)} = 1 \ ,$$

which is what it means to have $\mathcal{M}, w \models_{\mathsf{p}} P_a(\psi^c) = 1$. Since $P_a(\psi^c) = 1$ is what is abbreviated by $(K_a\psi)^c$, the result follows.

Now we move to belief formulas. Assume $\mathcal{M}^c, w \models_{\mathsf{n}} B_a\psi$. This means that $[w]_a \cap [\![\psi]\!]_{\mathsf{n}}^{\mathcal{M}^c} \in N_a^c(w)$. Applying induction hypothesis, this is equivalent to $[w]_a \cap [\![\psi^c]\!]_{\mathsf{p}}^{\mathcal{M}} \in N_a^c(w)$. By the definition of N_a^c, the latter holds iff $P_{a,w}([w]_a \cap [\![\psi^c]\!]_{\mathsf{p}}^{\mathcal{M}}) > c$. But this is equivalent to $P_{a,w}([\![\psi^c]\!]_{\mathsf{p}}^{\mathcal{M}}) > c$, which is what it means to have $\mathcal{M}, w \models_{\mathsf{p}} P_a(\psi^c) > c$. Since $P_a(\psi^c) > c$ is what is abbreviated by $(B_a\psi)^c$, the result follows. □

6 Calculi for Belief as Willingness to Bet

Definition 6.1. We define the following theories in the language $\mathcal{L}_{\mathsf{KB}}$.

- KB is defined in Table 1.
- $\mathsf{KB}^{0.5}$ is obtained from KB by adding (D), (SC), and (L) from Table 2.
- $\mathsf{KB}_{-}^{0.5}$ is obtained from $\mathsf{KB}^{0.5}$ by omitting (KBM).

	AXIOM SCHEMES
(CL)	Schemes of Classical Propositional Logic
(KS5)	S5 axiom schemes for each K_a
(KBC)	$K_a\phi \to B_a\phi$
(BF)	$\neg B_a\bot$
(N)	$B_a\top$
(Ap)	$B_a\phi \to K_aB_a\phi$
(An)	$\neg B_a\phi \to K_a\neg B_a\phi$
(KBM)	$K_a(\phi\to\psi)\to(B_a\phi\to B_a\psi)$

RULES

$$\frac{\phi\to\psi \quad \phi}{\psi}\ \text{(MP)} \qquad \frac{\phi}{K_a\phi}\ \text{(MN)}$$

Table 1: The theory KB

(D)	$B_a\phi \to \check{B}_a\phi$
(SC)	$\check{B}_a\phi \wedge \check{K}_a(\neg\phi\wedge\psi)\to B_a(\phi\vee\psi)$
(L)	$[(\phi_i \mathbb{I}_a \psi_i)_{i=1}^m \wedge B_a\phi_1 \wedge \bigwedge_{i=2}^m \check{B}_a\phi_i] \to \bigvee_{i=1}^m B_a\psi_i$

Table 2: Additional axiom schemes for the theory $\mathsf{KB}^{0.5}$

6.1 Results for the Basic Calculus KB

The following result shows that if we restrict attention to provable statements whose only modality is single-agent belief $B_a\phi$, then KB is an extension of the minimal modal logic $\mathsf{EMN45}+\neg B_a\bot = \mathsf{EMN45}+\text{(BF)}$ obtained by adding S5-knowledge and the knowledge-belief connection principles (KBC), (Ap), (An), and (KBM).[4] The modal theory $\mathsf{KB}^{0.5}$ is therefore a similar knowledge-inclusive extension of $\mathsf{EMND45}+\text{(BF)}+\text{(L)}$ that adds the additional connection principle (SC).[5] In Section 6.2, we will show that $\mathsf{KB}^{0.5}$ is the modal logic for probabilistic belief with threshold $c=\frac{1}{2}$.

Proposition 6.2 (KB Derivables)**.** We have each of the following.

1. $\mathsf{KB}\vdash B_a(\phi\wedge\psi)\to(B_a\phi\wedge B_a\psi)$.

[4] $\mathsf{EMN45}+\text{(BF)}$ is the logic of single-agent belief (without knowledge) having Schemes (CL) (Table 1), M (Prop. 6.2(1)), (N) (Table 1), 4 (Prop. 6.2(4)), 5 (Prop. 6.2(5)), and (BF) (Table 1) along with Rules (MP) (Table 1) and RE (Prop. 6.2(11)). This is a "monotonic" system of modal logic satisfying positive and negative belief introspection (4 and 5) and the property (BF) that falsehood $\bot$ is not believed. See [Che80, Ch. 8] for details on naming minimal modal logics.

[5] $\mathsf{EMND45}+\text{(BF)}+\text{(L)}$ is $\mathsf{EMN45}+\text{(BF)}$ plus Schemes (D) and (L) from Table 2.

This is “Scheme M” [Che80, Ch. 8].

2. $\mathsf{KB} \vdash K_a\phi \wedge B_a\psi \to B_a(\phi \wedge \psi)$.

   If the antecedent $K_a\phi$ were replaced by $B_a\phi$, then we would obtain “Scheme C” [Che80, Ch. 8]. So we do not have Scheme C outright but instead a knowledge-weakened version: in order to conclude belief of a conjunction from belief of one of the conjuncts, the other conjunct must be known (and not merely believed, as is required by the stronger, non-KB-provable Scheme C).

3. $\mathsf{KB} \vdash K_a(\phi \to \psi) \to (\check{B}_a\phi \to \check{B}_a\psi)$.

   This is the dual version of our (KBM).

4. $\mathsf{KB} \vdash B_a\phi \to B_aB_a\phi$.

   This is “Scheme 4” [Che80, Ch. 8].

5. $\mathsf{KB} \vdash \neg B_a\phi \to B_a\neg B_a\phi$.

   This is “Scheme 5” [Che80, Ch. 8].

6. $\mathsf{KB} \vdash B_a\phi \leftrightarrow K_aB_a\phi$.

   This says that belief and knowledge of belief are equivalent.

7. $\mathsf{KB} \vdash \neg B_a\phi \leftrightarrow K_a\neg B_a\phi$.

   This says that non-belief and knowledge of non-belief are equivalent.

8. $\mathsf{KB} \vdash \phi$ implies $\mathsf{KB} \vdash B_a\phi$.

   This is the rule of Modus Ponens (or Modal Necessitation), sometimes called “Rule RN” [Che80, Ch. 8].

9. $\mathsf{KB} \vdash \phi \to \psi$ implies $\mathsf{KB} \vdash B_a\phi \to B_a\psi$.

   This is “Rule RM” [Che80, Ch. 8].

10. $\mathsf{KB} \vdash \phi \to \psi$ implies $\mathsf{KB} \vdash \check{B}_a\phi \to \check{B}_a\psi$.

    This is the dual version of RM.

11. $\mathsf{KB} \vdash \phi \leftrightarrow \psi$ implies $\mathsf{KB} \vdash B_a\phi \leftrightarrow B_a\psi$.

    This is “Rule RE” [Che80, Ch. 8].

12. $\mathsf{KB} \vdash \phi \to \bot$ implies $\mathsf{KB} \vdash \neg B_a\phi$.

    This says that no self-contradictory sentence is believed. This may be viewed as a certain generalization of (BF) (Table 1).

Proof. We reason in KB. For 1, we derive

$$K_a((\phi \wedge \psi) \to \phi) \to (B_a(\phi \wedge \psi) \to B_a\phi) \tag{12}$$

by (KBM), and the antecedent of (12) by (CL) and (MN). Therefore, the consequent of (12) is derivable by (MN). By a similar argument, $B_a(\phi \wedge \psi) \to B_a\psi$ is derivable. By classical reasoning, 1 is derivable.

For 2, we derive

$$K_a\phi \to K_a(\psi \to (\phi \wedge \psi)) \text{ and} \tag{13}$$

$$K_a(\psi \to (\phi \wedge \psi)) \to (B_a\psi \to B_a(\phi \wedge \psi)) \ . \tag{14}$$

(13) follows by $\mathsf{S5}$ reasoning. (14) follows by (KBM). Applying classical reasoning to (13) and (14), we obtain

$$K_a\phi \to (B_a\psi \to B_a(\phi \wedge \psi)) \ ,$$

from which 2 follows by classical reasoning.

For 3, we derive

$$K_a(\phi \to \psi) \to K_a(\neg\psi \to \neg\phi) \text{ and} \tag{15}$$

$$K_a(\neg\psi \to \neg\phi) \to (B_a\neg\psi \to B_a\neg\phi) \ . \tag{16}$$

(15) follows by $\mathsf{S5}$ reasoning. (16) follows by (KBM). Applying classical reasoning to (15) and (16), we obtain

$$K_a(\phi \to \psi) \to (B_a\neg\psi \to B_a\neg\phi) \ ,$$

from which 3 follows by classical reasoning (just contrapose the consequent).

4 follows by (Ap) and (KBC). 5 follows by (An) and (KBC). 6 follows by by (Ap) for the right-to-left and (KS5) for the left-to-right. 7 follows by (An) for the right-to-left and (KS5) for the left-to-right. 8 follows by (MN) and (KBC). 9 follows by (MN) and (KBM). 10 follows by contraposition, (MN), (KBM), and contraposition. 11 follows from 9 by classical reasoning.

For 12, we have

$$K_a(\phi \to \bot) \to (B_a\phi \to B_a\bot) \tag{17}$$

by (KBM). Therefore, if $\phi \to \bot$ is provable, it follows by (MN) that the antecedent of (17) is as well. By (MP), the consequent $B_a\phi \to B_a\bot$ is provable. Applying (BF) and classical reasoning, it follows by contraposition that $\neg B_a\phi$ is provable. □

Theorem 6.3 (KB Neighborhood Soundness)**.** KB is sound with respect to the class $\mathcal{C}$ of epistemic neighborhood models:

$$\forall \phi \in \mathcal{L}_{\mathsf{KB}} : \quad \mathsf{KB} \vdash \phi \quad \Rightarrow \quad \mathcal{C} \models_{\mathsf{n}} \phi \ .$$

Proof. By induction on the length of derivation. We first verify soundness of the axioms.

- Validity of (CL) immediate. Validity of (KS5) follows because the R_a's are equivalence relations [BdRV01].

- Scheme (KBC) is valid: $\models_{\mathsf{n}} K_a\phi \to B_a\phi$.

  $\mathcal{M}, w \models_{\mathsf{n}} K_a\phi$ means $[w]_a \subseteq [\![\phi]\!]_{\mathsf{n}}$. Since $[w]_a \in N_a(w)$ by (n), we have $[w]_a \cap [\![\phi]\!]_{\mathsf{n}} = [w]_a \in N_a(w)$. That is, $\mathcal{M}, w \models_{\mathsf{n}} B_a\phi$.

- Scheme (BF) is valid: $\models_{\mathsf{n}} \neg B_a\bot$.

  $[\![\bot]\!]_{\mathsf{n}} = \emptyset \notin N_a(w)$ by (kbf). Hence $\mathcal{M}, w \not\models_{\mathsf{n}} B_a\bot$.

- Scheme (N) is valid: $\models_{\mathsf{n}} B_a\top$.

  $[\![\top]\!]_{\mathsf{n}} \cap [w]_a = [w]_a \in N_a(w)$ by (n). Hence $\mathcal{M}, w \models_{\mathsf{n}} B_a\top$.

- Scheme (Ap) is valid: $\models_{\mathsf{n}} B_a\phi \to K_aB_a\phi$.

  Suppose $\mathcal{M}, w \models_{\mathsf{n}} B_a\phi$. Then $[w]_a \cap [\![\phi]\!]_{\mathsf{n}} \in N_a(w)$. Take $v \in [w]_a$. We have $[v]_a = [w]_a$ because R_a is an equivalence relation, and we have $N_a(v) = N_a(w)$ by (a). Hence $[v]_a \cap [\![\phi]\!]_{\mathsf{n}} \in N_a(v)$; that is, $\mathcal{M}, v \models_{\mathsf{n}} B_a\phi$. Since $v \in [w]_a$ was chosen arbitrarily, we have shown that $[w]_a \subseteq [\![B_a\phi]\!]_{\mathsf{n}}$. Hence $\mathcal{M}, w \models_{\mathsf{n}} K_aB_a\phi$.

- Scheme (An) is valid: $\models_{\mathsf{n}} \neg B_a\phi \to K_a\neg B_a\phi$.

  Replace $B_a\phi$ by $\neg B_a\phi$ and $\in$ by $\notin$ in the argument for the previous item.

- Scheme (KBM) is valid: $\models_{\mathsf{n}} K_a(\phi \to \psi) \to (B_a\phi \to B_a\psi)$.

  Suppose $\mathcal{M}, w \models_{\mathsf{n}} K_a(\phi \to \psi)$ and $\mathcal{M}, w \models_{\mathsf{n}} B_a\phi$. This means $[w]_a \subseteq [\![\phi \to \psi]\!]_{\mathsf{n}}$ and $[w]_a \cap [\![\phi]\!]_{\mathsf{n}} \in N_a(w)$. But then

  $$[w]_a \cap [\![\phi]\!]_{\mathsf{n}} \subseteq [w]_a \cap [\![\phi]\!]_{\mathsf{n}} \cap [\![\phi \to \psi]\!]_{\mathsf{n}} \subseteq [w]_a \cap [\![\psi]\!]_{\mathsf{n}} \ .$$

  Hence $[w]_a \cap [\![\psi]\!]_{\mathsf{n}} \in N_a(w)$ by (kbm). That is, $\mathcal{M}, w \models_{\mathsf{n}} B_a\psi$.

That validity is closed under applications of the rules MP and MN follows by the standard arguments [BdRV01]. □

In preparation for the proof of completeness of KB with respect to the class of finite epistemic neighborhood models, we require a few preliminary definitions and a key lemma. The first definition concerns maximal consistent sets and definability of collections of these sets.

Definition 6.4. Let L be a set of $\mathcal{L}_{\mathsf{KB}}$-formulas. An L-*world* is a maximal KB-consistent subset of L.[6] We write W^L for the set of L-worlds. To say that L-worlds Γ and Δ are a-*compatible*, written $\Gamma R_a \Delta$, means

$$\forall K_a\phi \in L : (K_a\phi \in \Gamma \Leftrightarrow K_a\phi \in \Delta) \ .$$

a-compatability is an equivalence relation and we write

$$[\Gamma]_a := \{\Delta \in W^L \mid \Gamma R_a \Delta\}$$

[6] To say that a set of $\mathcal{L}_{\mathsf{KB}}$-formulas is KB-*consistent* means that KB cannot be used to derive $\bot$ from a finite subset; the set is *maximal* KB-*consistent* iff adding a formula not already present will violate KB-consistency.

for the equivalence class of an L-world Γ under this relation. An *L-a-region* is a set $X \subseteq [\Gamma]_a$ of a-compatible L-worlds. To say L-a-regions X and Y are *a-compatible* means that they are each subsets of a single equivalence class $[\Gamma]_a$. For $L' \subseteq L$, to say that an L-a-region $X \subseteq [\Gamma]_a$ is *L'-definable* means

$$\exists \phi \in L' : X = \{\Delta \in [\Gamma]_a \mid \phi \in \Delta\} \ ;$$

i.e., fixing the equivalence class $[\Gamma]_a$ containing X, some $\phi \in L'$ satisfies the property that X is the set of all a-compatible worlds containing ϕ. The formula ϕ is said to *define* X. To say an L-world Γ is *L'-definable* means that the L-a-region $\{\Gamma\}$ is L'-definable.

Definition 6.5 ([Seg71])**.** Let L be a set of $\mathcal{L}_{\mathsf{KB}}$-formulas. A *basis* for L is a subset $L_0 \subseteq L$ satisfying the property

$$\forall \phi \in L, \exists \phi_0 \in L_0 : \mathsf{KB} \vdash \phi \leftrightarrow \phi_0 \ .$$

To say L is *logically finite* means L has a finite basis.

Logical finiteness is a useful property: a logically finite set L may be infinite and yet there will be only finitely many L-worlds (i.e., W^L is finite). Segerberg introduced logical finiteness in order to exploit this property in a filtration-completeness argument for a modal theory of qualitative probabilistic comparison [Seg71]. We will also exploit this property here. However, we will have an additional task, which takes the form of the following lemma.

Lemma 6.6. Let L be a finite, subformula-closed set of $\mathcal{L}_{\mathsf{KB}}$-formulas. Let L' be the Boolean closure of L.[7] For each unary symbol $M \in \{\neg, B_a, K_a\}$ and set of $\mathcal{L}_{\mathsf{KB}}$-formulas S, define $\bar{M}S := S \cup \{M\phi \mid \phi \in S\}$ and let $\bar{\vee}S$ denote the closure of S under finite nonempty disjunctions. Define

$$L^+ := \bigcup_{a \in A} \bar{\neg}\bar{K}_a\bar{\neg}\bar{\vee}\bar{B}_a L' \ .$$

L' and L^+ are logically finite. If L'_0 is a finite basis for L', then every L^+-a-region is L'_0-definable.

Proof. Logical finiteness of L' follows from the finiteness of L by a normal form argument: treat modal formulas as letters and then consider disjunctive normal forms of L'-formulas using this extended but still finite set of letters appearing in L. So let L'_0 be a finite basis for L'. Let $\bar{\vee}_{<\infty}S$ denote the closure of a set S of formulas under finite nonempty disjunctions without repeated disjuncts. Observe that $\bar{\vee}_{<\infty}S$ is finite if S is. We prove that

$$L_0^+ := \bigcup_{a \in A} \bar{\neg}\bar{K}_a\bar{\neg}\bar{\vee}_{<\infty}\bar{B}_a L'_0$$

is a finite basis for L^+. Since L'_0 is a finite basis for L' and L_0^+ is obviously finite, it suffices for us to show that for each $\phi \in L^+ - L'$, there exists $\phi_0 \in L_0^+$ such that $\phi \leftrightarrow \phi_0$ is KB-derivable. There are only a few cases to consider.

- Case: $\phi = B_a\psi$ with $\psi \in L'$.

  Choose $\psi \in L'_0$ such that $\psi \leftrightarrow \psi_0$ is provable. This is possible because L'_0 is a basis for L'. By Proposition 6.2(11), it follows that $B_a\psi \leftrightarrow B_a\psi_0$ is provable if $\psi \leftrightarrow \psi_0$ is. Since $B_a\psi_0 \in L_0^+$, the result follows.

[7] That is, L' is the smallest extension of L closed under negation, conjunction, disjunction, implication, and the addition of the propositonal constants $\bot$ (falsehood) and $\top$ (truth).

- Case: $\phi = \bigvee_{i=1}^{m} B_a\psi^i$ with each $\psi_i \in L'$.

  By the previous case we have provably equivalent $B_a\psi_0^i \in L_0^+$ for each i. Renumber the ψ^i's and choose the largest $n \leq m$ such that $\bigvee_{i=1}^{n} B_a\psi^i$ contains no repeated disjuncts. By classical reasoning, we have

  $$\vdash \bigvee_{i=1}^{m} B_a\psi^i \leftrightarrow \bigvee_{i=1}^{n} B_a\psi_0^i \ .$$

  Since $\bigvee_{i=1}^{n} B_a\psi_0^i$ contains no repetitions, this formula is a member of L_0^+ and the result follows.

- Case: $\phi = \neg\bigvee_{i=1}^{m} B_a\psi^i$ or $\phi = \neg\neg\bigvee_{i=1}^{m} B_a\psi^i$.

  By the previous case and the definition of L_0^+.

- Case: $\phi = K_a\chi$ and χ is $\neg\bigvee_{i=1}^{m} B_a\psi^i$, $\bigvee_{i=1}^{m} B_a\psi^i$, or $\psi \in L'$.

  By the previous cases or the fact that L_0' is a basis for L', there exists $\chi_0 \in L_0^+$ such that $\chi \leftrightarrow \chi_0$ is derivable. Since K_a is $\mathsf{S5}$, it follows that $K_a\chi \leftrightarrow K_a\chi_0$ is derivable as well. Since $K_a\chi_0 \in L_0^+$, the result follows.

- Case: $\phi = \neg K_a\chi$ and χ is as in the previous case.

  By the previous case and the definition of L_0^+.

Note that we need not consider the cases $\phi = \neg\psi$ or $\phi = \neg\neg\psi$ with $\psi \in L'$ because L' is closed under Boolean operations and hence such ϕ are already in L'. Conclusion: L_0^+ is a finite basis for L^+, so L^+ is logically finite.

Now let L_0' be an arbitrary finite basis for L'. What remains is to show that every L^+-a-region is L_0'-definable. But this follows if every L^+-world is L_0'-definable. Indeed, each L^+-a-region contains finitely many L^+-worlds because the logical finiteness of L^+ implies W^{L^+} is finite. Therefore, if the L^+-worlds $\Gamma_1, \ldots, \Gamma_n$ are respectively defined by the L_0'-formulas $\phi_1, \ldots, \phi_n$, then the disjunction of these formulas, which is a member of L' and is therefore provabily equivalent to another L_0'-formula, defines the L^+-a-region consisting of the worlds.

So all we must show is that every L^+-world is L_0'-definable. Proceeding, let Γ be an L^+-world. Let γ be the conjunction consisting the members of $\Gamma \cap L_0'$ along with the negation of each member of $L_0' - \Gamma$. Since L_0' is finite, γ is a formula in L' and is therefore provably equivalent to some $\gamma_0 \in L_0'$. Clearly $\gamma_0 \in \Gamma$ by the maximal KB-consistency of Γ. So suppose that $\Delta \in [\Gamma]_a$ is an arbitrary a-compatible L^+-world and $\gamma_0 \in \Delta$. If we can show that $\Delta = \Gamma$, then the proof is complete. Proceeding, it follows by the maximal KB-consistency of Δ that $\gamma \in \Delta$ and therefore $\Gamma \cap L_0' = \Delta \cap L_0'$. Since L_0' is a basis for L', it follows by maximal KB-consistency that $\Gamma \cap L' = \Delta \cap L'$. So to prove that $\Delta = \Gamma$, all that remains is to prove that for each $\phi \in L^+ - L'$, we have $\phi \in \Delta$ iff $\phi \in \Gamma$. For this there are only a few cases to consider.

- Case: $\phi = B_a\psi$ with $\psi \in L'$.

By the definition of L^+, maximal consistency, (Ap), the definition of a-compatibility, and the $\mathsf{S5}$ scheme $K_a\chi \to \chi$, we have

$$\begin{aligned} B_a\psi \in \Gamma &\Rightarrow K_a B_a\psi \in \Gamma \\ &\Rightarrow K_a B_a\psi \in \Delta \\ &\Rightarrow B_a\psi \in \Delta \ . \end{aligned}$$

By the definition of L^+, maximal consistency, (An), the definition of a-compatibility, and the same $\mathsf{S5}$ scheme, we also have

$$\begin{aligned} B_a\psi \in L^+ - \Gamma &\Rightarrow \neg B_a\psi \in \Gamma \\ &\Rightarrow K_a\neg B_a\psi \in \Gamma \\ &\Rightarrow K_a\neg B_a\psi \in \Delta \\ &\Rightarrow \neg B_a\psi \in \Delta \\ &\Rightarrow B_a\psi \in L^+ - \Delta \ . \end{aligned}$$

It follows that we have $B_a\psi \in \Gamma$ iff $B_a\psi \in \Delta$.

- Case: $\phi = \bigvee_{i=1}^m B_a\psi^i$, $\phi = \neg \bigvee_{i=1}^m B_a\psi^i$, or $\phi = \neg\neg \bigvee_{i=1}^m B_a\psi^i$.

  By the previous case and maximal consistency.

- Case: $\phi = K_a\chi$ or $\phi = \neg K_a\chi$ and χ has the form of a previous case.

  By the definition of a-compatibility and maximal consistency. □

Theorem 6.7 (KB Neighborhood Completeness)**.** KB is complete with respect to the class $\mathcal{C}$ of epistemic neighborhood models:

$$\forall\phi \in \mathcal{L}_{\mathsf{KB}} : \quad \mathcal{C} \models_{\mathsf{n}} \phi \quad \Rightarrow \quad \mathsf{KB} \vdash \phi \ .$$

Further, every $\phi \in \mathcal{L}_{\mathsf{KB}}$ that is satisfiable at a pointed epistemic neighborhood model is satisfiable at a finite pointed epistemic neighborhood model.

Proof. Fix $\theta \in \mathcal{L}_{\mathsf{KB}}$ satisfying $\mathsf{KB} \nvdash \neg\theta$. Let $\mathsf{sub}(\phi)$ denote the set of subformulas of ϕ (including ϕ itself). Define the finite, subformula-closed set $L := \mathsf{sub}(\theta)$ and let L' and L^+ be given as in Lemma 6.6. Fix a finite basis L_0' for L'. For each L^+-a-region X, we will conflate X with a fixed L_0'-formula defining X whose existence is guaranteed by Lemma 6.6.

We define the structure $\mathcal{M} = (W, R, V, N)$ as follows.

$$\begin{aligned} W &:= W^{L^+} \\ \Gamma R_a \Delta &\text{ iff } \forall K_a\phi \in L^+ : (K_a\phi \in \Gamma \Leftrightarrow K_a\phi \in \Delta) \\ [\Gamma]_a &:= \{\Delta \in W \mid \Gamma R_a \Delta\} \\ V(\Gamma) &:= \mathbf{P} \cap \Gamma \\ N_a(\Gamma) &:= \{X \subseteq [\Gamma]_a \mid B_a X \in \Gamma\} \end{aligned}$$

$\mathcal{M}$ is finite: the logical finiteness of L^+ implies W^{L^+} is finite, so each component of $\mathcal{M}$ is finite.

We verify that $\mathcal{M}$ is an epistemic neighborhood model. W is nonempty because $\mathsf{KB}^{0.5} \nvdash \neg\theta$ and therefore $\{\theta\}$ may be extended to a maximal consistent set $\Gamma_\theta \in W$. R_a is obviously an equivalence relation. What remains is to check that N_a satisfies (kbc), (kbf), (n), (a), and (kbm).

- For (kbc), we must show $X \in N_a(\Gamma)$ implies $X \subseteq [\Gamma]_a$. But this follows by the definition of $N_a(\Gamma)$.

- For (kbf), we must show that $\emptyset \notin N_a(\Gamma)$. Assume toward a contradiction that $\emptyset \in N_a(\Gamma)$. It follows that $B_a\emptyset \in \Gamma$. We have $\mathsf{KB} \vdash \emptyset \leftrightarrow \bot$ by the fact that $\bot$ defines the L^+-a-region $\emptyset$. By Proposition 6.2(9), we have $\mathsf{KB} \vdash B_a\emptyset \to B_a\bot$. Since $B_a\bot \in L^+$, we have $B_a\bot \in \Gamma$ by maximal consistency. But this contradicts the consistency of Γ because we have $\neg B_a\bot \in \Gamma$ by (BF), maximal consistency, and the fact that $\neg B_a\bot \in L^+$. Conclusion: $\emptyset \notin N_a(\Gamma)$.

- For (n), we must show that $[\Gamma]_a \in N_a(\Gamma)$. Proceeding, we have $B_a\top \in \Gamma$ by (N), maximal consistency, and the fact that $B_a\top \in L^+$. We have $\mathsf{KB} \vdash \top \leftrightarrow [\Gamma]_a$ by the fact that $\top$ defines the L^+-a-region $[\Gamma]_a$, and therefore $\mathsf{KB} \vdash B_a\top \to B_a[\Gamma]_a$ by Proposition 6.2(9). Hence $B_a[\Gamma]_a \in \Gamma$ by maximal consistency. But then $[\Gamma]_a \in N_a(\Gamma)$.

- For (a), we must show that $\Delta \in [\Gamma]_a$ implies $N_a(\Delta) = N_a(\Gamma)$. Choosing $\Delta \in [\Gamma]_a$, we have by our argument in the proof of Lemma 6.6 that $B_a\phi \in \Gamma$ iff $B_a\phi \in \Delta$ for each $B_a\phi \in L^+$. Applying the definition of N_a, we have $N_a(\Delta) = N_a(\Gamma)$.

- For (kbm), we must show $X \in N_a(\Gamma)$ and $X \subseteq Y \subseteq [\Gamma]_a$ together imply that $Y \in N_a(\Gamma)$. So suppose $X \in N_a(\Gamma)$ and $X \subseteq Y \subseteq [\Gamma]_a$. It follows that $B_aX \in \Gamma$. Since $X, Y \in L'_0$, it follows that $X \to Y \in L'$ and therefore that $K_a(X \to Y) \in L^+$. Further, it follows from $X \subseteq Y \subseteq [\Gamma]_a$ by maximal consistency that $K_a(X \to Y) \in \Gamma$. Applying (KBM), maximal consistency, and the fact that $B_aY \in L^+$, we obtain $B_aY \in \Gamma$. Hence $Y \in N_a(\Gamma)$.

So $\mathcal{M}$ is indeed an epistemic neighborhood model.

We prove the following *Truth Lemma*: for each $\phi \in L^+$ and $\Gamma \in W$, we have $\phi \in \Gamma$ iff $\mathcal{M}, \Gamma \models_{\mathsf{n}} \phi$. The proof is by induction on the construction of ϕ. Most cases are straightforward; we only consider the modal cases $\phi = K_a\psi$ and $\phi = B_a\psi$.

- For $K_a\psi \in L^+$: $K_a\psi \in \Gamma$ iff $\mathcal{M}, \Gamma \models_{\mathsf{n}} K_a\psi$.

  Suppose $K_a\psi \in \Gamma$. Take $\Delta \in [\Gamma]_a$. It follows that $K_a\psi \in \Delta$ by the definition of R_a. Hence $\psi \in \Delta$ by the definition of L, the $\mathsf{S5}$ scheme $K_a\chi \to \chi$, and maximal consistency. Applying the induction hypothesis, we have $\mathcal{M}, \Delta \models_{\mathsf{n}} \psi$. Since $\Delta \in [\Gamma]_a$ was chosen arbitrarily, we have shown that $[\Gamma]_a \subseteq [\![\psi]\!]_{\mathsf{n}}$. That is, $\mathcal{M}, \Gamma \models_{\mathsf{n}} K_a\psi$.

  Conversely, suppose $K_a\psi \in L^+ - \Gamma$. It follows that $\neg K_a\psi \in \Gamma$ by the definition of L^+ and maximal consistency. We claim that the set

$$S := \{\neg\psi\} \cup \{K_a\chi \in L^+ \mid K_a\chi \in \Gamma\}$$

is consistent. Toward a contradiction, suppose S is not consistent. It follows that there are $K_a\chi_1, \dots, K_a\chi_n \in \Gamma$ such that

$$\mathsf{KB} \vdash K_a\chi_1 \wedge \cdots \wedge K_a\chi_n \to \psi \ .$$

It follows by modal reasoning using the $\mathsf{S5}$ operator K_a that

$$\mathsf{KB} \vdash K_a\chi_1 \wedge \cdots \wedge K_a\chi_n \to K_a\psi \ .$$

Hence $K_a\psi \in \Gamma$ by maximal consistency, which contradicts the consistency of Γ because $\neg K_a\psi \in \Gamma$. So S is indeed consistent and can therefore be extended to a maximal consistent $\Delta \in W^{L^+}$. By construction, $\Delta \in [\Gamma]_a$ and, since $K_a\psi \in L^+$ implies $\neg\psi \in L^+$, we also have $\neg\psi \in \Delta$. But then $\psi \notin \Delta$ by consistency. By the definition of L^+, it follows from $K_a\psi \in L^+$ that $\psi \in L^+$, so we may apply the induction hypothesis: from $\psi \in L^+ - \Delta$, we conclude that $\mathcal{M}, \Delta \not\models_{\mathsf{n}} \psi$ and therefore $\mathcal{M}, \Gamma \not\models_{\mathsf{n}} K_a\psi$.

- For $B_a\psi \in L^+$: $B_a\psi \in \Gamma$ iff $\mathcal{M}, \Gamma \models_{\mathsf{n}} B_a\psi$.

  Let $X_\chi \subseteq [\Gamma]_a$ be the L^+-a-region defined by $\chi \in L^+$. Since $B_a\psi \in L^+$, it follows that $\psi \in L'$ and therefore we may choose $\psi_0 \in L'_0$ such that $\mathsf{KB} \vdash \psi \leftrightarrow \psi_0$.

  Now suppose $B_a\psi \in \Gamma$. By Proposition 6.2(9) and maximal consistency, this is equivalent to $B_a\psi_0 \in \Gamma$. This means $X_{\psi_0} \in N_a(\Gamma)$. But $X_{\psi_0} = X_\psi$. Applying the induction hypothesis, we have $X_\psi \in N_a(\Gamma)$ iff $[\![\psi]\!]_{\mathsf{n}} \cap [w]_a \in N_a(\Gamma)$. But the latter is the meaning of $\mathcal{M}, \Gamma \models_{\mathsf{n}} B_a\psi$.

This completes the proof of the Truth Lemma.

Since $\theta \in \Gamma_\theta$, it follows by the Truth Lemma that $\mathcal{M}, \Gamma_\theta \models_{\mathsf{n}} \theta$. Therefore, the satisfiable formula θ is satisfiable at the finite pointed epistemic neighborhood model $(\mathcal{M}, \Gamma_\theta)$. Completeness follows. □

Theorem 6.8 (KB Probability Soundness)**.** KB is sound for any threshold $c \in [\frac{1}{2}, 1) \cap \mathbb{Q}$ with respect to the class of epistemic probability models:

$$\forall c \in [\tfrac{1}{2}, 1) \cap \mathbb{Q}, \forall \phi \in \mathcal{L}_{\mathsf{KB}} : \quad \mathsf{KB} \vdash \phi \quad \Rightarrow \quad \models_{\mathsf{p}} \phi^c \ .$$

Proof. By Propositions 3.2 and 3.8. Note that by the proofs of these propositions, we can extend the result to all thresholds $c \in (0, 1) \cap \mathbb{Q}$. □

Theorem 6.9 ([WF79]; KB Probability Incompleteness)**.** $\mathsf{KB}^{>c}$ is incomplete for all thresholds $c \in [\frac{1}{2}, 1) \cap \mathbb{Q}$ with respect to the class of epistemic probability models:

$$\exists \phi \in \mathcal{L}_{\mathsf{KB}}, \forall c \in [\tfrac{1}{2}, 1) \cap \mathbb{Q} : \quad \models_{\mathsf{p}} \phi^c \quad \text{and} \quad \mathsf{KB} \nvdash \phi \ .$$

Proof. We adapt Example 2 from [WF79, pp. 344-345] to the present setting. Fix $c \in (0, 1) \cap \mathbb{Q}$. Let $\mathbf{P} := \{a, b, c, d, e, f, g\}$ and $A = \{1\}$. Define the set

$$\mathcal{X} := \{efg, abg, adf, bde, ace, cdg, bcf\} \ ,$$

where we use the notation xyz for $\{x, y, z\}$. Let $X_1, \ldots, X_7$ denote the members of $\mathcal{X}$ (named here as in the order above.) Define the single-agent structure $\mathcal{M}$ by

$$\begin{aligned}
\mathcal{M} &:= (W, R, V, N) \\
W &:= \mathbf{P} \\
R_1 &:= W \times W \\
V(p) &:= \{p\} \\
N_1(x) &:= \{Z \subseteq W \mid \exists X \in \mathcal{X} : X \subseteq Z\}
\end{aligned}$$

It is easy to verify that $\mathcal{M}$ satisfies (kbc), (kbf), (n), (a), and (kbm). So $\mathcal{M}$ is a finite epistemic neighborhood model. For each $x \in W$, let $\hat{x}$ denote the $\mathcal{L}_{\mathsf{KB}}$-formula

$$x \wedge (\textstyle\bigwedge_{y \in W - \{x\}} \neg y) \enspace .$$

Extend this to sets of worlds: for each $X \subseteq W$, let $\hat{X}$ denote the formula $\bigvee_{x \in X} \hat{x}$. Note that $\bigvee_{x \in \emptyset} \hat{x} = \bot$. Now let σ be the $\mathcal{L}_{\mathsf{KB}}$-formula

$$\begin{aligned}
\hat{a} \;\wedge\; K_1 \hat{W} \;\wedge\; & (\textstyle\bigwedge_{Z \in N_1(a)} B_1 \hat{Z}) \;\wedge \\
& (\textstyle\bigwedge_{Z \in \mathcal{P}(W) - N_1(a)} \neg B_1 \hat{Z}) \enspace .
\end{aligned}$$

It is clear that $\mathcal{M}, a \models_{\mathsf{n}} \sigma$ and therefore that $\not\models_{\mathsf{n}} \neg\sigma$. By neighborhood soundness (Theorem 6.3), we have $\mathsf{KB} \nvdash \neg\sigma$.

Define $\mathcal{Y}$ by a cyclic permutation of the letters making up the sets in $\mathcal{X}$: each letter goes to the next in the alphabet, except that g goes to a. That is,

$$\mathcal{Y} := \{fga, bca, beg, cef, bdf, dea, cdg\} \enspace .$$

Let $Y_1, \ldots, Y_7$ denote the members of $\mathcal{Y}$ (named here as in the order above.) Note that $\mathcal{Y} \cap \mathcal{X} = \emptyset$. Therefore, if $Y \in \mathcal{Y}$, then $Y \notin N_1(x)$ for any $x \in W$. In words: no member of $\mathcal{Y}$ is a neighborhood in $\mathcal{M}$. Further, we observe that each world in W is a member of exactly three sets in $\mathcal{Y}$ and a member of exactly three sets in $\mathcal{X}$.

We wish to show that $\models_{\mathsf{p}} \neg\sigma^c$. Note that the $\mathcal{L}$-formula σ^c is

$$\begin{aligned}
\hat{a} \;\wedge\; [P_1(\hat{W}) = 1] \;\wedge\; & (\textstyle\bigwedge_{Z \in N_1(a)} [P_1(\hat{Z}) > c]) \;\wedge \\
& (\textstyle\bigwedge_{Z \in \mathcal{P}(W) - N_1(a)} \neg [P_1(\hat{Z}) > c]) \enspace .
\end{aligned}$$

Toward a contradiction, assume $\mathcal{N}, w \models_{\mathsf{p}} \sigma^c$. It follows that

$$P_{1,w}([\![\hat{W}]\!]^{\mathcal{N}}_{\mathsf{p}}) = 1 \tag{18}$$

$$\forall Z \in N_1(a) : P_{1,w}([\![\hat{Z}]\!]^{\mathcal{N}}_{\mathsf{p}}) > c \tag{19}$$

$$\forall Z \in \mathcal{P}(W) - N_1(a) : P_{1,w}([\![\hat{Z}]\!]^{\mathcal{N}}_{\mathsf{p}}) \leq c \tag{20}$$

By (19) and the fact that each $\hat{x}$ occurs in exactly three $\hat{X}_i$'s, we have

$$7c < \sum_{i=1}^{7} P_{1,w}(\llbracket \hat{X}_i \rrbracket_{\mathsf{p}}^{\mathcal{N}}) = 3 \sum_{x \in W} P_{1,w}(\llbracket \hat{x} \rrbracket_{\mathsf{p}}^{\mathcal{N}}) \ . \tag{21}$$

By (20) and the fact that each $\hat{x}$ occurs in exactly three $\hat{Y}_i$'s, we have

$$7c \geq \sum_{i=1}^{7} P_{1,w}(\llbracket \hat{Y}_i \rrbracket_{\mathsf{p}}^{N}) = 3 \sum_{x \in W} P_{1,w}(\llbracket \hat{x} \rrbracket_{\mathsf{p}}^{\mathcal{N}}) \ . \tag{22}$$

But (21) contradicts (22). Therefore that we cannot have $\mathcal{N}, w \models_{\mathsf{p}} \sigma^c$ for any pointed epistemic probability model $(\mathcal{N}, w)$. Hence $\models_{\mathsf{p}} \neg\sigma^c$. And, as we have seen, $\mathsf{KB} \nvdash \neg\sigma$. $\square$

Remark 6.10. We use the notation of the proof of Theorem 6.9. It follows from this proof that there can be no epistemic probability model $\mathcal{N} = (W, R, V, P)$ based on the same underlying Kripke model (W, R, V) as $\mathcal{M}$ such that $\mathcal{N}^c = \mathcal{M}$. For if there were such an $\mathcal{N}$, then it would follow from $\mathcal{M}, a \models_{\mathsf{n}} \sigma$ by Theorem 5.3 that $\mathcal{N}, a \models_{\mathsf{p}} \sigma^c$. But as we have seen, no pointed epistemic probability model can satisfy σ^c.

Examining the model $\mathcal{M}$ further, one may verify that that no $X_j \in \mathcal{X}$ is contained in the complement $W - X_i$ of some $X_i \in \mathcal{X}$. It follows that $\mathcal{M}$ satisfies

$$(X \mathbb{I}_1 Y)_{i=1}^{7}, \ \ X_1 \in N_1(a), \text{and } \ \forall i \in \{2, \ldots, 7\} : W - X_i \notin N_1(a) \ ,$$

which is the antecedent of property (l), also from Definition 4.5. However, $\mathcal{M}$ does not satisfy

$$\exists j \in \{1, \ldots, 7\} : Y_j \in N_1(a) \ ,$$

which is the corresponding consequent of the indicated instance of (l). So we see that if we were to restrict ourselves to the class of finite epistemic neighborhood models satisfying this property, we would no longer able to use $\mathcal{M}$ as a counterexample to the claim "every epistemic neighborhood model $\mathcal{M}$ (in the class in question) gives rise to an epistemic probability model $\mathcal{N}$ on the same underlying Kripke model such that $\mathcal{N}^{\frac{1}{2}} = \mathcal{M}$." Of course ruling out $\mathcal{M}$ as a counterexample to this claim does not prove the claim. Nevertheless, it is suggestive of the problem with models that do not satisfy (l). And, as we will see in the proof of Theorem 6.14, this property is key among the sufficient (and necessary) conditions needed to guarantee that the claim holds.

6.2 Results for the Mid-Threshold Calculus $\mathsf{KB}^{0.5}$

We first show that the KB scheme (KBM) is redundant in the theory $\mathsf{KB}^{0.5}$.

Proposition 6.11. $\mathsf{KB}_{-}^{0.5}$ and $\mathsf{KB}^{0.5}$ derive the same theorems:

$$\forall \phi \in \mathcal{L}_{\mathsf{KB}} : \quad \mathsf{KB}_{-}^{0.5} \vdash \varphi \quad \Leftrightarrow \quad \mathsf{KB}^{0.5} \vdash \varphi \ .$$

Proof. It suffices to prove that the scheme (KBM) is derivable in $\mathsf{KB}_{-}^{0.5}$. By Definition 3.6, the formula $\phi \mathbb{I}_a \psi$ is just

$$K_a\big(\underbrace{(\neg\phi \wedge \neg\psi) \vee (\neg\phi \wedge \psi)}_{C_0} \vee \underbrace{(\phi \wedge \psi)}_{C_1} \big) \ , \tag{23}$$

where we have explicitly indicated the subformulas C_0 and C_1 used in the notation of Definition 3.6. Semantically, (23) says that in each of a's accessible worlds, ψ is true whenever ϕ is true. Now reasoning within $\mathsf{KB}^{0.5}_{-}$, it follows that $K_a(\phi \to \psi)$ is provably equivalent to $\phi \mathbb{I}_a \psi$. But then from $K_a(\phi \to \psi)$ and $B_a\phi$, we may derive $\phi \mathbb{I}_a \psi$ and $B_a\phi$, from which we may derive $B_a\psi$ by (L). Hence (KBM) is derivable. □

Theorem 6.12 ($\mathsf{KB}^{0.5}$ Neighborhood Soundness and Completeness)**.** $\mathsf{KB}^{0.5}$ is sound and complete with respect to the class $\mathcal{C}^{0.5}$ of mid-threshold neighborhood models:

$$\forall \phi \in \mathcal{L}_{\mathsf{KB}} : \quad \mathsf{KB}^{0.5} \vdash \phi \quad \Leftrightarrow \quad \mathcal{C}^{0.5} \models_{\mathsf{n}} \phi \ .$$

Further, every $\phi \in \mathcal{L}_{\mathsf{KB}}$ that is satisfiable at a pointed mid-threshold neighborhood model is satisfiable at a finite pointed mid-threshold neighborhood model.

Proof. Soundness is by induction on the length of derivation. Most cases are as in the proof of Theorem 6.7. We only need consider the remaining axiom schemes.

- Scheme (D) is valid: $\models_{\mathsf{n}} B_a\phi \to \check{B}_a\phi$.

  Suppose $\mathcal{M}, w \models_{\mathsf{n}} B_a\phi$. This means $[w]_a \cap [\![\phi]\!]_{\mathsf{n}} \in N_a(w)$. By (d),

  $$[w]_a \cap [\![\neg\psi]\!]_{\mathsf{n}} = [w]_a - [\![\phi]\!]_{\mathsf{n}} = [w]_a - ([w]_a \cap [\![\phi]\!]_{\mathsf{n}}) \notin N_a(w) \ .$$

  But this is what it means to have $\mathcal{M}, w \models_{\mathsf{n}} \check{B}_a\phi$.

- Scheme (SC) is valid: $\models_{\mathsf{n}} \check{B}_a\phi \wedge \check{K}_a(\neg\phi \wedge \psi) \to B_a(\phi \vee \psi)$.

  Suppose $\mathcal{M}, w \models_{\mathsf{n}} \check{B}_a\phi$ and $\mathcal{M}, w \models_{\mathsf{n}} \check{K}_a(\neg\phi \wedge \psi)$. It follows that

  $$[w]_a - ([w]_a \cap [\![\phi]\!]_{\mathsf{n}}) = [w]_a \cap [\![\neg\phi]\!]_{\mathsf{n}} \notin N_a(w)$$

  and that there exists $v \in [w]_a$ satisfying $\mathcal{M}, v \models \neg\phi \wedge \psi$. But then $[w]_a \cap [\![\phi \vee \psi]\!]_{\mathsf{n}} \supsetneq [w]_a \cap [\![\phi]\!]_{\mathsf{n}}$ and therefore $[w]_a \cap [\![\phi \vee \psi]\!]_{\mathsf{n}} \in N_a(w)$ by (sc). Hence $\mathcal{M}, w \models B_a(\phi \vee \psi)$.

- Scheme (L) is valid:

  $$\models_{\mathsf{n}} [(\phi_i \mathbb{I}_a \psi_i)_{i=1}^m \wedge B_a\phi_1 \wedge \textstyle\bigwedge_{i=2}^m \check{B}_a\phi_i] \to \bigvee_{i=1}^m B_a\psi_i \ .$$

  Suppose $(\mathcal{M}, w)$ satisfies the antecedent of scheme (L). It follows that each $v \in [w]_a$ satisfies just as many ϕ_i's as ψ_i's, that $[w]_a \cap [\![\psi_1]\!]_{\mathsf{n}} \in N_a(w)$, and that $[w]_a - [\![\phi_k]\!]_{\mathsf{n}} \notin N_a(w)$ for each $k \in \{2, \ldots, m\}$. Hence

  $$[w]_a \cap [\![\phi_1]\!]_{\mathsf{n}}, \ldots, [w]_a \cap [\![\phi_m]\!]_{\mathsf{n}} \mathbb{I}_a [w]_a \cap [\![\psi_1]\!]_{\mathsf{n}}, \ldots, [w]_a \cap [\![\psi_m]\!]_{\mathsf{n}} \ ,$$

  from which it follows by (l) that $[w]_a \cap [\![\psi_j]\!]_{\mathsf{n}} \in N_a(w)$ for some $j \in \{1, \ldots, m\}$. Hence $\mathcal{M}, w \models_{\mathsf{n}} B_a\psi_j$, and thus $\mathcal{M}, w \models_{\mathsf{n}} \bigvee_{i=1}^m B_a\psi_i$.

Soundness has been proved.

Completeness is as in the proof of Theorem 6.7, except that all relevant definitions and results that make use of derivability are changed so as to take derivability with respect to $\mathsf{KB}^{0.5}$. All that needs to be shown is that the model $\mathcal{M}$ defined as in the proof of Theorem 6.7 is a mid-threshold neighborhood model; the rest of the argument is as in that proof, *mutatis mutandis*. Most of the properties of $\mathcal{M}$ are shown in that proof. What remains is for us to show that $\mathcal{M}$ also satisfies (d), (sc), and (l).

- For (d), we must show $X \in N_a(\Gamma)$ implies $X' \notin N_a(\Gamma)$, where $X \subseteq [\Gamma]_a$ and $X' := [\Gamma]_a - X$. Toward a contradiction, assume $X, X' \in N_a(\Gamma)$. This means $B_aX, B_aX' \in N_a(\Gamma)$. Since $X, X' \in L'_0$, we have $X \to \neg X' \in L'$ and therefore $K_a(X \to \neg X') \in L^+$. But $X \cap X' = \emptyset$, so it follows by maximal consistency that $K_a(X \to \neg X') \in \Gamma$. By (M), maximal consistency, and the fact that $\neg X' \in L'$ and hence $B_a\neg X' \in L^+$, we obtain $B_a\neg X' \in \Gamma$. But since $B_aX' \in \Gamma$ and $X' \in L'$ implies $\neg B_a \neg X' \in L^+$, it follows by (D) and maximal consistency that $\neg B_a\neg X' \in \Gamma$, contradicting the consistency of Γ.

- For (sc), we must show $X' := [\Gamma]_a - X \notin N_a(\Gamma)$ and $X \subsetneq Y \subseteq [\Gamma]_a$ together imply $Y \in N_a(\Gamma)$. So suppose we have the antecedent of this implication. It follows that $B_aX' \notin \Gamma$ and therefore $\neg B_aX' \in \Gamma$ by the definition of L^+ and maximal consistency. Notice that $\mathsf{KB}^{0.5} \vdash X' \leftrightarrow \neg X$ because $X' = [\Gamma]_a - X$ and therefore we have by Proposition 6.2(9) (and the fact that $\mathsf{KB}^{0.5}$ extends $\mathsf{KB}^{>.5}$) that $\check{B}_aX \in \Gamma$. Further, $\check{K}_a(\neg X \wedge Y) \in L^+$, so since $X \subsetneq Y \subseteq [\Gamma]_a$, we have $\check{K}_a(\neg X \wedge Y) \in \Gamma$ by maximal consistency. Further, since $B_a(X \vee Y) \in L^+$, it follows by (SC) and maximal consistency that $B_a(X \vee Y) \in \Gamma$. Since $K_a(X \vee Y \to Y) \in L^+$, we have by $X \subsetneq Y \subseteq [\Gamma]_a$ and maximal consistency that $K_a(X \vee Y \to Y) \in \Gamma$. Hence by maximal consistency and (KBM), we have $B_aY \in \Gamma$ and therefore $Y \in N_a(\Gamma)$.

- For (l), we must show $(X_i \mathbb{I}_a Y_i)_{i=1}^m$, $X_1 \in N_a(\Gamma)$, and $[\Gamma]_a - X_i \notin N_a(\Gamma)$ for all $i \neq 1$ together imply that $Y_j \in N_a(\Gamma)$ for some j. Suppose we have the antecedent of this implication. Then $B_aX_1 \in \Gamma$. Further, as in the argument for (sc), our assumptions imply $\check{B}_aX_i \in \Gamma$ for all $i \neq 1$. By our argument for (wl), we have $(X_i \mathbb{I}_a Y_i)_{i=1}^m \in \Gamma$. Since $\bigvee_{i=1}^m Y_i \in L^+$, it follows by (L) and maximal consistency that $\bigvee_{i=1}^m Y_i \in \Gamma$ and therefore by maximal consistency that $Y_j \in \Gamma$ for some j. But then $Y_j \in N_a(\Gamma)$. □

We make use of the following theorem due to Scott [Sco64] in formulating Lenzen's proof of probability completeness for $\mathsf{KB}^{0.5}$ (Theorem 6.14).

Theorem 6.13 ([Sco64, Theorem 1.2])**.** Let X be a finite, rational, symmetric subset of the vector space L of all real-valued functions defined on a finite set S.[8] For each $N \subseteq X$, there exists a linear functional φ on L such that

$$N = \{x \in X \mid \varphi(x) \geq 0\}$$

[8]So L is the S-dimensional vector space. To say $x : S \to \mathbb{R}$ is *rational* means that all of its coordinates (i.e., values) are rational numbers. To say $X \subseteq L$ is *rational* means every $x \in X$ is rational. To say X is *symmetric* means that $X = -X := \{-x \mid x \in X\}$.

if and only if the following conditions are satisfied:

- for each $x \in X$, we have $x \in N$ or $-x \in N$; and
- for each integer $n \geq 0$ and $x_0, \ldots, x_n \in N$, we have

$$\sum_{i=0}^{n} x_i = 0 \quad \Rightarrow \quad -x_0 \in N \ .$$

That $\mathsf{KB}^{0.5}$ is sound and complete for the probabilistic semantics with threshold $c = \frac{1}{2}$ was originally proved by Lenzen [Len80].

Theorem 6.14 (Due to [Len80]; $\mathsf{KB}^{0.5}$ Probability Soundness and Completeness)**.** $\mathsf{KB}^{0.5}$ is sound and complete for threshold $\frac{1}{2}$ with respect to the class of epistemic probability models:

$$\forall \phi \in \mathcal{L}_{\mathsf{KB}}: \quad \mathsf{KB}^{0.5} \vdash \phi \quad \Leftrightarrow \quad \models_{\mathsf{p}} \phi^{\frac{1}{2}} \ .$$

Proof. Soundness is by Propositions 3.2 and 3.8.

For completeness, we will show that every finite mid-threshold neighborhood model can be extended to an epistemic probability model.

Let $\mathcal{M} = (W, R, V, N)$ be a mid-threshold neighborhood model for agent set A and proposition set $\mathbf{P}$. Assume that W is finite. Let $a \in A$ be some agent and let $w \in W$. We will construct a probability measure $P_{a,w}$ on $\mathcal{P}([w]_a)$ satisfying full support. From these $P_{a,w}$ a probability function P_a on W can be defined as follows. Let $n = |\{[w]_a \mid w \in W\}|$, i.e., let n be the size of the partition generated by R_a. Then P_a defined by $P_a(w) = \frac{P_{a,w}(w)}{n}$ is a probability measure on W satisfying full support.

Let $[w]_a = \{w_1, \ldots, w_n\}$. For convenience in what follows, we will often conflate the function N with its value $N_a(w)$ for agent a at world w; this ought not cause confusion because every mention of the function N in this proof will only be to mention its value on (a, w). Consider the n-dimensional vector space $\mathbb{R}^n$, with the usual operations for vector addition and scalar multiplication. Let $X \subseteq [w]_a$. Then X is mapped to a vector by

$$V(X) = (\Psi_X(w_1), \ldots, \Psi_X(w_n)) \ ,$$

where Ψ_X is the decision function for X given by $\Psi_X(w) = 1$ if $w \in X$, $\Psi_X(w) = 0$ otherwise. Note that $V(\emptyset) = 0^n$ and $V([w]_a) = 1^n$. Let us write $[w]_a - X$ as $\overline{X}$. Let

$$\begin{aligned} M_1 &= \{V(X) \in \mathbb{R}^n \mid X \subseteq [w]_a\}, \\ M_2 &= \{(V(X) - V(\overline{X})) \in \mathbb{R}^n \mid X \subseteq [w]_a, \overline{X} \notin N\} \\ M &= M_1 \cup M_2 \\ K &= M \cup M^-, \end{aligned}$$

where $M^- = \{-r \mid r \in M\} = \{-1 \cdot r \mid r \in M\}$.

Note that K is finite, K is symmetric by definition (meaning: $K = K^-$), and K is rational (all vector values are rational numbers). We can apply a theorem of [Sco64] (Theorem 1.2): there is a real-valued linear function ϕ on $\mathbb{R}^n$ with ($\phi(v) \geq 0$ iff $v \in M$) iff the following two conditions hold:

1. $\forall r \in K : r \in M \vee -r \in M$.

2. $\forall m \geq 1\, \forall r_1, \dots, r_m \in K$:
   if $\sum_{i=1}^{m} r_i = 0^n$ then $r_1, \dots, r_{m-1} \in M$ implies $-r_m \in M$.

It follows immediately from the definition of M that condition (1) holds. For condition (2), assume $r_m \in K - M$. Then $r_m = -r$ for some $r \in M$, and therefore $-r \in M$, and done.

Now assume that $r_m \in M$. We have to show that $\sum_{i=1}^{m} r_i = 0^n$ and $r_1, \dots, r_m \in M$ together imply $-r_m \in M$.

Split the r_j as to whether they are from M_1 or M_2. Without loss of generality we may assume there is i with $\{r_1, \dots, r_i\} \subseteq M_1$ and $\{r_{i+1}, \dots, r_m\} \subseteq M_2$. Each r_j corresponds to a set $X_i \subseteq [w]_a$ via $r_j = V(X_j)$ for $r_j \in M_1$, and $r_j = V(X_j) - V(\overline{X_j})$ for $r_j \in M_2$.

Suppose for some vector r in $\{r_1, \dots, r_i\}$ we have $r \neq 0^n$. Then

$$\sum \{r_1, \dots, r_i\} = r^* = (r_1^*, \dots, r_n^*),$$

with all components ≥ 0, and some $r_k^* > 0$.

From $\sum\{r_1, \dots, r_i\} + \sum\{r_{i+1}, \dots, r_m\} = 0^n$ we get that $\sum\{r_{i+1}, \dots, r_m\} = -r^*$. By the definition of M_2 this means that

$$\sum_{j=i+1}^{m} (V(X_j) - V(\overline{X_j}) = -r^*,$$

so all components are ≤ 0, and $-r_k^* < 0$. So any $v \in [w]_a$ occurs at most as often in one of the $X_{i+1}, \dots, X_m$ as in one of the complements $\overline{X_{i+1}}, \dots, \overline{X_m}$. Moreover, w_k occurs less often in one of the $X_{i+1}, \dots, X_m$ than in one of the complements $\overline{X_{i+1}}, \dots, \overline{X_m}$. Without loss of generality, assume $w_k \notin X_{i+1}$. Then:

$$X_{i+1} \cup \{w_k\}, \dots, X_m \mathbb{I}_a \overline{X_{i+1}}, \dots, \overline{X_m}.$$

From (l) and the fact that $\overline{X_{i+1}} \notin N, \dots, \overline{X_m} \notin N$ it follows that $X_{i+1} \cup \{w_k\} \notin N$. But now we have $X_{i+1} \subsetneq X_{i+1} \cup \{w_k\} \notin N$ and $\overline{X_{i+1}} \notin N$, and contradiction with (sc).

So we can assume $r_i \in M_2$ for all i with $1 \leq i \leq m$. Therefore, we have, by definition of M_2:

$$\sum_{i=1}^{m} V(X_i) - V(\overline{X_i}) = 0^n.$$

This means, by definition of V, that for each $v \in [a]_w$ the following holds:

$$\sum_{i=1}^{m} \Psi_{X_i}(v) - \Psi_{\overline{X_i}}(v) = 0.$$

In other words, each $v \in [a]_w$ occurs exactly as often in one of the X_i as in one of the complements:

$$X_1, \dots, X_m \mathbb{E}_a \overline{X_1}, \dots, \overline{X_m}.$$

Since none of the complements is in N, it follows from this and (l) that $X_m \notin N$. Therefore $-r_m = V(\overline{X}_m - V(X_m) \in M_2$. This proves (2).

The conditions of Scott's theorem are fulfilled, so there is a real-valued linear function ϕ on $\mathbb{R}^n$ with $\phi(v) \geq 0$ iff $v \in M$. Suppose $\phi(V([w]_a) = \phi(1^n) = 0$. Since ϕ is linear, $\phi(0^n) = 0$. This gives, by linearity of ϕ again: $\phi(0^n - 1^n) = \phi(0^n) - \phi(1^n) = 0$, and therefore, by the conditions on ϕ: $0^n - 1^n = -1^n \in M$. Since $-1^n \notin M_1$ this gives $-1^n \in M_2$, and it follows by the definition of M_2 that $[w]_a \notin N$. Contradiction with (n).

We may conclude that $\phi(1^n) > 0$, and we can define our probability measure $P_{a,w}$ on $\mathcal{P}([w]_a)$ by means of

$$P_{a,w}(X) = \frac{\phi(V(X))}{\phi(1^n)}.$$

We still have to show that the value of this is $P_{a,w}(X) > \frac{1}{2}$ iff $X \in N$.

Suppose $\overline{X} \notin N$. Then, by definition of M_n, $V(X) - V(\overline{X}) \in M_2$, and by the properties of ϕ, $\phi(V(X) - V(\overline{X})) \geq 0$. Since $V(X) - V(\overline{X}) = 2V(X) - V([w]_a)$, we have by linearity of ϕ:

$$2\phi(V(X)) - \phi(V([w]_a)) \geq 0,$$

and therefore:

$$2\phi(V(X)) \geq \phi(1^n).$$

It follows that $P_{a,w}(X) \geq \frac{1}{2}$.

Suppose $\overline{X} \in N$. Then $V(X) - V(\overline{X}) \notin M_2$, and by definition of M, also $V(X) - V(\overline{X}) \notin M$. This means $\phi(V(X) - V(\overline{X}) < 0$, and by reasoning similar to the above we derive that $2\phi(V(X)) < \phi(1^n)$, from which it follows that $P_{a,w}(X) < \frac{1}{2}$.

Putting these two implications together, we get: $\overline{X} \notin N$ iff $P_{a,w}(X) \geq \frac{1}{2}$. From this, by substitution: $X \notin N$ iff $P_{a,w}(\overline{X}) \geq \frac{1}{2}$ iff $P_{a,w}(X) \leq \frac{1}{2}$. From this, by contraposition:

$$X \in N \text{ iff } P_{a,w}(X) > \frac{1}{2}.$$

The full support property of $P_{a,w}$ now follows easily from the above plus (sc). $\square$

7 Conclusion

Summary We have provided a study of the modal logic of certain knowledge and "betting" belief (i.e., belief of events greater than a rational probability $c \geq \frac{1}{2}$). Our study included both a probabilistic semantics and a neighborhood semantics with a new epistemic twist. We formulated Lenzen's proof that $\mathsf{KB}^{0.5}$ is the logic of threshold $c = \frac{1}{2}$. We also proved completeness with respect to the new neighborhood semantics. This proof made use of an ingenious neighborhood-completeness trick due to Segerberg ("logical finiteness"), which we extended in a way that ensures all epistemically possible propositions are definable using finitely many formulas that are all candidates for agent belief in the canonical model. Moving to a broder take on our results, we believe that our work provides connections between probabilistic and neighborhood semantics that may present interesting opportunities for future cross-disciplinary work.

Open Questions for Future Work

1. The main open question is the following: given a "high-threshold" $c \in (\frac{1}{2}, 1) \cap \mathbb{Q}$, find the exact extension KB^c of KB that is probabilistically sound and complete for threshold c with respect to the class of epistemic probability models, in the sense that we would have:

$$\forall\phi \in \mathcal{L}_{\mathsf{KB}} : \quad \mathsf{KB}^c \vdash \phi \quad \Leftrightarrow \quad \models_{\mathsf{p}} \phi^c \ .$$

Observing that (SC) and (L) are not valid for high-thresholds $c > \frac{1}{2}$, we conjecture that what is required are threshold-specific variants of (SC) and (L) that will together guarantee probability soundness and completeness. Toward this end, we suggest the following schemes as a starting point:

(SC$_0^s$) $(\check{K}_a\phi_0 \wedge \bigwedge_{i=1}^{s} \check{B}_a\phi_i \wedge \bigwedge_{i\neq j=0}^{s} K_a(\phi_i \to \neg\phi_j)) \to B_a(\bigvee_{i=0}^{s} \phi_i)$

(SC$_1^s$) $(\bigwedge_{i=1}^{s} \check{B}_a\phi_i \wedge \bigwedge_{i\neq j=1}^{s} K_a(\phi_i \to \neg\phi_j)) \to B_a(\bigvee_{i=1}^{s} \phi_i)$

(WL) $[(\phi_i \mathbb{I}_a \psi_i)_{i=1}^{m} \wedge \bigwedge_{i=1}^{m} B_a\phi_i] \to \bigvee_{i=1}^{m} B_a\psi_i$

Observe that (SC) is just (SC$_0^1$). Further, if we define $s' := c/(1-c)$ and $s := \text{ceiling}(s')$, then scheme (SC$_0^s$) is probabilistically sound if $s = s'$ and scheme (SC$_1^s$) is probabilistically sound if $s \neq s'$. The reasoning for this is as follows: s' tells us the number of $(1-c)$'s that divide c. In particular, recall from Lemma 3.4 that the probabilistic interpretation of $\check{B}_a\phi$ is that ϕ is assigned probability at least $1-c$. Therefore, if we have s disjoint propositions that each have probability at least $1-c$, then the probability of their disjunction will have probability $s\cdot(1-c) \geq c$. This inequality is strict if $s \neq s'$ and is in fact an equality if $s = s'$. Therefore, in the case $s \neq s'$, scheme (SC$_1^s$) is sound: s disjoint propositions each having probability $1-c$ together sum to a probability exceeding the threshold c. And in case $s = s'$, scheme (SC$_0^s$) is sound: s disjoint propositions each having probability $1-c$ together sum to a probability that equals c, so adding some additional probability from another disjoint proposition ϕ_0 will yield a disjunction whose probability again exceeds c. In either case, exceeding probability c is what we equate with belief, so soundness is proved. We note that scheme (WL) can be shown to be sound by adapting the proof Proposition 3.8(11). The epistemic neighborhood model versions of (SC$_0^s$), (SC$_1^s$), and (WL) are:

(sc$_0^s$) $\forall X_1, \dots, X_s, Y \subseteq [w]_a$: if $[w]_a - X_1, \dots, [w]_a - X_s \notin N_a(w)$, the X_i's are pairwise disjoint, and $Y \supsetneq \bigcup_{i=1}^{s} X_i$, then $Y \in N_a(w)$.

(sc$_1^s$) $\forall X_1, \dots, X_s \subseteq [w]_a$: if $[w]_a - X_1, \dots, [w]_a - X_s \notin N_a(w)$ and the X_i's are pairwise disjoint, then $\bigcup_{i=1}^{s} X_i \in N_a(w)$.

(wl) $\forall m \in \mathbb{Z}^+, \forall X_1, \dots, X_m, Y_1, \dots, Y_m \subseteq [w]_a$:

$$\begin{array}{ll} \text{if} & X_1, \dots, X_m \mathbb{I}_a Y_1, \dots, Y_m \quad \text{and} \\ & \forall i \in \{1, \dots, m\} : X_i \in N_a(w) \ , \\ \text{then} & \exists j \in \{1, \dots, m\} : Y_j \in N_a(w) \ . \end{array}$$

If M is an epistemic neighborhood model, then a slight modification of the proof of property (wl) in Lemma 5.2 shows that M^c satisfies (wl). We presume that an adaptation of the proof for the proof of property (sc) in the same lemma will show that M^c satisfies (sc_0^s) if $s = s'$ and (sc_1^s) if $s \neq s'$.

We remark that (WL) is not threshold-specific, though it is sound for all high-thresholds $c > \frac{1}{2}$. We suspect that a threshold-specific variant may be required in order to adapt Lenzen's proof of $\mathsf{KB}^{0.5}$ probability soundness and completeness for threshold $c = \frac{1}{2}$ (Theorem 6.14). In particular, using terminology and notation from that proof: take $c = p/q$ and redefine set M_2 by setting

$$M_2 := \{((q-p)V(X) - pV(\overline{X}) \in \mathbb{R}^n \mid X \subseteq [w]_a, \overline{X} \notin N\} \ .$$

Observe that we have

$$(q-p)V(X) - pV(\overline{X})) = qV(X) - pV([w]_a) \ .$$

So if we have an appropriate linear functional ϕ and $\overline{X} \notin N$, then $q \cdot \phi(V(X)) - p \cdot \phi(V([w]_a)) \geq 0$. Since $\phi(V([w]_a)) = 1$, it follows that $P_{a,w}(X) \geq p/q = c$. The argument for the case $\overline{X} \in N$ is similar. However, we note that the existence of ϕ seems to require a threshold-specific version of (L) and it is not clear how this might come about so as to follow the logic of the proof of Theorem 6.14. Perhaps some variant of (WL) that takes into account the specific values of p and q is required.

2. Another open question is the exact relationship between Segerberg's comparitive operator $\phi \geq \psi$ ("ϕ is at least as probable as ψ") [Seg71] and betting belief. $B_a\phi$ is equivalent to $\phi > \neg\phi$, where the strict inequality $>$ is defined in the natural way. However, it is not clear how the logics of these operators are related.

3. Yet another direction is the extension of our work to Bayesian updating. Given a pointed epistemic probability model $(\mathcal{M}, w)$ satisfying ϕ, let

$$\mathcal{M}[\phi] = (W[\phi], R[\phi], V[\phi], P[\phi])$$

be defined by

$$\begin{aligned} W[\phi] &:= [\![\phi]\!]^{\mathcal{M}}_{\mathsf{p}} \\ R[\phi]_a &:= R_a \cap (W[\phi] \times W[\phi]) \\ V[\phi](w) &:= V(w) \text{ for } w \in W[\phi] \\ P[\phi]_a(w) &:= \frac{P_a(w)}{P_a([\![\phi]\!]^{\mathcal{M}}_{\mathsf{p}})} \end{aligned}$$

It is not difficult to see that $\mathcal{M}[\phi]$ is an epistemic probability model and

$$P[\phi]_a(X) = \frac{P_a(X \cap [\![\phi]\!]^{\mathcal{M}}_{\mathsf{p}})}{P_a([\![\phi]\!]^{\mathcal{M}}_{\mathsf{p}})} = P[\phi]_a(X|[\![\phi]\!]^{\mathcal{M}}_{\mathsf{p}}) \ ,$$

where the value on the right is the probability of X conditional on $[\![\phi]\!]^{\mathcal{M}}_{\mathsf{p}}$. It would be interesting to investigate the analog of this operation in epistemic neighborhood models. The operation may also have a close relationship with the study of updates in Probabilistic Dynamic Epistemic Logic [vBGK09, BS08].

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