# Belief as Willingness to Bet

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# Probabilistic Reasoning

- We study modal logic for *qualitative* probabilistic reasoning:
  - $K\varphi$ means $P(\varphi) = 1$,
  - $B\varphi$ means $P(\varphi) > c$ for a fixed $c \in (0,1) \cap \mathbb{Q}$.
- Some previous work:
  - Lenzen (1980): probabilistically complete logic for $c = \frac{1}{2}$.
  - Halpern et al. (pre-2003, 2003 book): *quantitative* logics.
  - Herzig (2003): qualitative logics of action.

    $$BA \text{ means } P(A) > P(\neg A).$$

    - Equivalent to Lenzen's logic.
    - Soundness but no completeness.
  - Kyberg and Teng (2012): "acceptance" of $A$ iff $P(\neg A) \leq \epsilon$.
- Our contributions:
  - Modern reformulation of Lenzen's syntax, semantics, and results.
  - New epistemic neighborhood semantics:
    - Lenzen's logic is sound and complete for a sub-class of our models.
    - Truth-preserving maping: probabilistic $\rightarrow$ neighborhood semantics.

# Our Epistemic Probability Models

$$\mathcal{M} = (W, R, V, P)$$

- $(W, R, V)$ is a finite $\mathsf{S5}$ Kripke model, where

$$[w] := \{v \in W \mid wRv\} \ .$$

- $P$ is a probability measure on $\wp(W)$ satisfying *full support*:

$$P(w) > 0 \text{ for each } w \in W \ .$$

- For event $X \subseteq W$, agent assigns to $X$ at $w \in W$ the probability

$$P_w(X) := \frac{P(X \cap [w])}{P([w])} = P(X|[w]) \ .$$

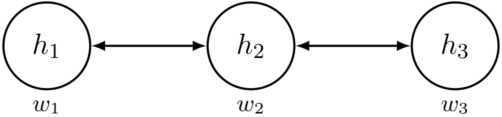
(Denominator $\neq 0$ by reflexivity & full support.)

**Notes.**

- Multi-agent version defined straightforwardly.
- Taking $\mathsf{S5}$ and full support is atypical but unproblematic.

# Our Epistemic Probability Models

## Example 1

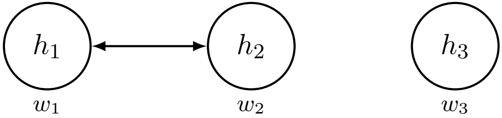



$$P \quad = \quad \{w_1 : \tfrac{3}{6}, w_2 : \tfrac{2}{6}, w_3 : \tfrac{1}{6}\}$$

- Agent considers each of worlds $w_1$, $w_2$, $w_3$ possible.
- Agent assigns odds 3:2:1 to these worlds.
- Letter $h_i$ true at $w_i$ (“Horse $h_i$ wins the race in world $w_i$).

# Our Epistemic Probability Models

## Example 2




$$P \quad = \quad \{w_1 : \tfrac{3}{6}, w_2 : \tfrac{2}{6}, w_3 : \tfrac{1}{6}\}$$

- Agent does not consider $w_3$ possible (relative to $w_1$ or $w_2$):

$$P_{w_1}(w_3) = P(\{w_3\}|[w_1]) = \frac{P(\{w_3\} \cap [w_1])}{P([w_1])} = \frac{P(\emptyset)}{P(\{w_1, w_2\})} = 0 \quad .$$

- Probability will always be evaluated w/r/t a world $w$ via $P_w(X)$.
- So even with full support, worlds can have (relative) probability 0.

# Probabilistic Language

$$
\begin{array}{rcl}
\phi & ::= & \top \mid p \mid \neg\phi \mid \phi \wedge \phi \mid t \geq 0 \\
t & ::= & q \mid q \cdot P(\phi) \mid t + t \\
 & & p \in \mathbf{P},\, q \in \mathbb{Q}
\end{array}
$$

- Abbreviations for: Booleans, $\leq$, $>$, $<$, $=$, and linear (in)equalities.
- Standard semantics for Booleans, to which we add:

$$\mathcal{M}, w \models_{\mathsf{p}} t \geq 0 \quad \text{iff} \quad [\![t]\!]_w \geq 0$$

$$
\begin{array}{rcl}
[\![\phi]\!]_{\mathsf{p}} & := & \{u \in W \mid \mathcal{M}, u \models_{\mathsf{p}} \phi\} \\
[\![q]\!]_w & := & q \\
[\![q \cdot P(\phi)]\!]_w & := & q \cdot P_w([\![\phi]\!]_{\mathsf{p}}) = q \cdot P([\![\phi]\!]_{\mathsf{p}} | [w]) \\
[\![t + t']\!]_w & := & [\![t]\!]_w + [\![t']\!]_w
\end{array}
$$

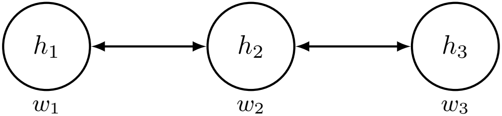
# Knowledge and Belief in the Probabilistic Language

$$\begin{aligned} K\varphi &:= P(\varphi) = 1 \\ B^c\varphi &:= P(\varphi) > c \quad \text{for } c \in (0,1) \cap \mathbb{Q} \end{aligned}$$

- Duals denoted by $\check{K}$ and $\check{B}^c$ (i.e., $\check{O}$ is $\neg O \neg$); $B\phi$ means $B^{0.5}\phi$.
- Easy results:
  - $K$ is S5.
  - Meaning of belief dual:

    $$\models_{\mathsf{p}} \check{B}^c\phi \Leftrightarrow (P(\phi) \geq 1 - c) \ ,$$

    so $\check{B}^{0.5}\phi$ says that $P(\phi) \geq 0.5$.
  - $B^c$ is not normal (i.e., does not satisfy $K$):




$$P = \{w_1 : \tfrac{1}{3}, w_2 : \tfrac{1}{3}, w_3 : \tfrac{1}{3}\}$$

$$\mathcal{M} \models_{\mathsf{p}} B\neg h_1 \wedge B\neg h_2 \text{ and yet } \mathcal{M} \not\models_{\mathsf{p}} B(\neg h_1 \wedge \neg h_2).$$

# Knowledge and Belief in the Probabilistic Language

$$\begin{aligned} K\varphi &:= P(\varphi) = 1 \\ B^c\varphi &:= P(\varphi) > c \quad \text{for } c \in (0,1) \cap \mathbb{Q} \end{aligned}$$

- Intuition for this notion of belief:
  - Agent believes $\phi$ iff she is "pretty sure" of truth (i.e., $P(\phi) > c$).
  - So "less sure" about conjunction $A \wedge B$ if $A \not\subseteq B$ and $B \not\subseteq A$.
- This permits the "lottery paradox": it is consistent to believe
  - there is some winning lottery ticket among the $n$ tickets, and
  - for each ticket $i = 1, \ldots, n$, ticket $i$ is not winning.
- This notion of belief comes from subjective probability:
  - Suppose agent believes $\varphi$ with threshold $p/q$; i.e., $B^{p/q}\varphi$.
  - Wagers $p$ dollars for chance to win $q - p$ dollars on bet $\varphi$ is true.
  - Expected win:
    $$(q-p) \cdot P(\varphi) - p \cdot (1 - P(\varphi)) = q \cdot P(\varphi) - p \ .$$
  - Positive iff $P(\varphi) > p/q$, which is guaranteed by $B^{p/q}\varphi$.
  - So bet is good iff has belief. "Belief as willingness to bet."
- Will set aside philsophical considerations, accept belief here as-is.

# Segerberg Notation

$$(\phi_1, \ldots, \phi_m \mathbb{I} \psi_1, \ldots, \psi_m) \ ,$$

also written $(\phi_i \mathbb{I} \psi_i)_{i=1}^m$, abbreviates the formula

$$K(C_0 \vee C_1 \vee C_2 \vee \cdots \vee C_m) \ ,$$

where $C_i$ is the disjunction of all conjunctions

$$d_1\phi_1 \wedge \cdots \wedge d_m\phi_m \wedge e_1\psi_1 \wedge \cdots \wedge e_m\psi_m$$

satisfying:

- exactly $i$ of $d_k$'s are the empty string,
- at least $i$ of the $e_k$'s are the empty string,
- and the remaining $d_k$'s and $e_k$'s are the negation sign $\neg$.

**Intuitive meaning:** the agent knows that the number of true $\psi_k$'s is is no less than the number of true $\phi_k$'s.

# Lenzen Scheme

$$[(\phi_i \mathbb{I} \psi_i)_{i=1}^m \wedge B^c\phi_1 \wedge \bigwedge_{i=2}^m \check{B}^c\phi_i] \to \bigvee_{i=1}^m B^c\psi_i \qquad \text{(Len)}$$

If we have that:

- the agent knows the true $\psi_k$'s are at least as many as the true $\phi_k$'s,
- the agent believes $\phi_1$ (with threshold $c$), and
- the remaining $\phi_k$'s are consistent with the agent's (thresh.-$c$) beliefs

then the agent believes one of the $\psi_k$'s (with threshold $c$).

**Theorem (Lenzen for $c = \frac{1}{2}$).** (Len) is valid for $c \in \left(0, \frac{1}{2}\right] \cap \mathbb{Q}$:

$$\models_{\mathsf{p}} [(\phi_i \mathbb{I} \psi_i)_{i=1}^m \wedge B^c\phi_1 \wedge \bigwedge_{i=2}^m \check{B}^c\phi_i] \to \bigvee_{i=1}^m B^c\psi_i \ .$$

Lenzen showed:

(Len) is key to modal-language completeness for fixed $c = \frac{1}{2}$.

# Other Principles

- $\not\models_{\mathsf{p}} B^c(\phi \to \psi) \to (B^c\phi \to B^c\psi)$.
  Belief is not closed under logical consequence (i.e., not normal).
- $\not\models_{\mathsf{p}} B^c\phi \to \phi$.
  Belief is not veridical.
- $\models_{\mathsf{p}} K\phi \to B^c\phi$.
  What is known is believed.
- $\models_{\mathsf{p}} \neg B^c\bot$.
  The propositional constant $\bot$ for falsehood is not believed.
- $\models_{\mathsf{p}} B^c\top$. (Minimal Logic's N)
  The propositional constant $\top$ for truth is believed.
- $\models_{\mathsf{p}} B^c\phi \to KB^c\phi$.
  What is believed is known to be believed.
- $\models_{\mathsf{p}} \neg B^c\phi \to K\neg B^c\phi$.
  What is not believed is known to be not believed.

# Other Principles

- $\models_{\mathsf{p}} B^c\phi \to B^cB^c\phi.$ (4)
  Belief is positive introspective.
- $\models_{\mathsf{p}} \neg B^c\phi \to B^c\neg B^c\phi.$ (5)
  Belief is negative introspective.
- $\models_{\mathsf{p}} K(\phi \to \psi) \to (B^c\phi \to B^c\psi).$
  Belief is closed under known logical consequence.
- $\models_{\mathsf{p}} \phi \to \psi$ implies $\models_{\mathsf{p}} B^c\phi \to B^c\psi.$ (Minimal Logic's RM)
  Belief distributes over provable implication.
- $\models_{\mathsf{p}} B^c(\phi \wedge \psi) \to (B^c\phi \wedge B^c\psi).$ (Minimal Logic's M)
  Belief of a conjunction implies conjunction of beliefs.
- $\models_{\mathsf{p}} B^c\phi \to \check{B}^c\phi.$
  Belief is consistent: belief in $\phi$ implies disbelief in $\neg\phi$.
- $\models_{\mathsf{p}} \check{B}^c\phi \wedge \check{K}(\neg\phi \wedge \psi) \to B^c(\phi \vee \psi)$ for $c \in \left(0, \frac{1}{2}\right] \cap \mathbb{Q}$.
  If $\phi$ is consistent with agent $a$'s beliefs and $\neg\phi \wedge \psi$ is consistent with agent $a$'s knowledge, then agent $a$ believes $\phi \vee \psi$.

# Modal Language

$$\begin{array}{rcl} \phi & ::= & \top \mid p \mid \neg\phi \mid \phi \wedge \phi \mid K\phi \mid B\phi \\ & & p \in \mathbf{P} \end{array}$$

- Threshold $c \in (0,1) \cap \mathbb{Q}$ omitted; will come from the semantics.
- Define the Segerberg formula $(\phi_i \mathbb{I} \psi_i)_{i=1}^m$ as before.
- Each $c \in (0,1) \cap \mathbb{Q}$ maps modal formulas to probability formulas:

$$\begin{array}{rcll} p^c & := & p \text{ for } p \in \mathbf{P} \cup \{\top\} & \\ (\neg\phi)^c & := & \neg\phi^c & \\ (\phi \wedge \psi)^c & := & \phi^c \wedge \psi^c & \\ (K\phi)^c & := & P(\phi^c) = 1 & (= K\phi^c \text{ in probability lang.}) \\ (B\phi)^c & := & P(\phi^c) > c & (= B^c\phi^c \text{ in probability lang.}) \end{array}$$

- We define a neighborhood semantics; maps are truth-preserving.

# Our Epistemic Neighborhood Models

$$\mathcal{M} = (W, R, V, N)$$

- $(W, R, V)$ is a <u>finite</u> $\mathsf{S5}$ Kripke model; notation $[w]$ as before.
- $N : W \to \wp(\wp(W))$ is a neighborhood function satisfying:
  - $X \in N(w)$ implies $X \subseteq [w]$,
    Agent does not believe something known to be false;
  - $\emptyset \notin N(w)$,
    Agent does not believe logical falsehoods;
  - $[w] \in N(w)$,
    Agent believes logical truths;
  - $v \in [w]$ implies $N(v) = N(w)$,
    Agent believes $X$ iff she knows she believes $X$; and
  - $X \subseteq Y \subseteq [w]$ and $X \in N(w)$ together imply $Y \in N(w)$,
    Agent believes all logcal consequences of a given belief.
- Semantics $(\models_{\mathsf{n}}, [\![\cdot]\!]_{\mathsf{n}})$: standard Boolean semantics plus

$$\begin{array}{lcl} \mathcal{M}, w \models_{\mathsf{n}} K\phi & \text{iff} & [w] \subseteq [\![\phi]\!]_{\mathsf{n}}^{\mathcal{M}} \\ \mathcal{M}, w \models_{\mathsf{n}} B\phi & \text{iff} & [w] \cap [\![\phi]\!]_{\mathsf{n}}^{\mathcal{M}} \in N(w) \end{array}$$

# Our Epistemic Neighborhood Models

## Additional Properties on $N$ for "Mid-Threshold" Models

$$\mathcal{M} = (W, R, V, N)$$

- $X \in N(w)$ implies $[w] - X \notin N(w)$,
  Agent does not believe $X$ and $[w] - X$ (i.e., belief consistency);
- $[w] - X \notin N(w)$ and $X \subsetneq Y \subseteq [w]$ together imply $Y \in N(w)$,
  Agent believes strict implications of a non-belief's negation; and
- if $(X_i \mathbb{I} Y_i)_{i=1}^m$, $X_1 \in N(w)$, and $[w] - X_2, \ldots, [w] - X_m \notin N(w)$,
  then $Y_j \in N(w)$ for some $j$,
  a Lenzen-like property for neighborhoods.
  (Note: $(X_i \mathbb{I} Y_i)_{i=1}^m$ is defined semantically.)

(Will see: Lenzen's logic for $c = \frac{1}{2}$ is complete for mid-threshold models.)

# Connecting Probability and Neighborhood Models

Each $c \in (0,1) \cap \mathbb{Q}$ induces a map

$$\mathcal{M} = (W, R, V, P) \qquad \stackrel{c}{\mapsto} \qquad \mathcal{M}^c = (W, R, V, N^c)$$

given by setting

$$N^c(w) := \{X \subseteq [w] \mid P_w(X) > c\} \ .$$

**Theorem.** We have:

- $N^c$ satisfies the epistemic neighborhood model properties.
- $N^{\frac{1}{2}}$ satisfies the additional "mid-threshold" properties.
- For each modal formula $\phi$:

$$\mathcal{M}^c, w \models_{\mathsf{n}} \phi \quad \text{iff} \quad \mathcal{M}, w \models_{\mathsf{p}} \phi^c \ .$$

  $\Rightarrow$ The qualitative modal language "talks correctly" about probability.

  $\Rightarrow$ Epistemic neighborhood models are connected with probability models.

# The Basic Qualitative Theory $\mathsf{KB}$

**Axiom Schemes**

| | |
|---|---|
| (CL) | Schemes of Classical Propositional Logic |
| (KS5) | $\mathsf{S5}$ axiom schemes for each $K$ |
| (KBC) | $K\phi \to B\phi$ |
| (BF) | $\neg B\bot$ |
| (N) | $B\top$ |
| (Ap) | $B\phi \to KB\phi$ |
| (An) | $\neg B\phi \to K\neg B\phi$ |
| (KBM) | $K(\phi \to \psi) \to (B\phi \to B\psi)$ |

**Rules**

$$\frac{\phi \to \psi \quad \phi}{\psi}\ (\mathrm{MP}) \qquad \frac{\phi}{K\phi}\ (\mathrm{MN})$$

**Theorem.** $\mathsf{KB}$ is sound and complete for epist. neighborhood models.

Proof is tricky (extends Segerberg's "logical finiteness" notion).

# KB Probability Incompleteness

There exists a modal formula $\phi$ for which $\models_{\mathsf{p}} \phi^c$ and $\mathsf{KB} \nvdash \phi$.

Sketch of argument (adapted from Walley and Fine, 1979):

- Propositional letters are $a, b, c, \ldots, g$.
- $X := \{efg, abg, adf, bde, ace, cdg, bcf\}$ with, e.g., $efg = \{e, f, g\}$.
- $Y := \{fga, bca, beg, cef, bdf, dea, cdg\}$ with similar abbreviations.
- $\mathcal{M} := (W, R, V, N)$ defied by: $W$ is letter set, $R$ is total, $V$ satisfies $p$ at $p$, and $N(w)$ is the superset closure of $X$. This is an epist. neighborhood model.
- $\sigma$ is the modal formula describing $(\mathcal{M}, a)$: informally (easily formalized),

$$\sigma \quad := \quad a\bar{b}\cdots\bar{g} \wedge KW \wedge (\textstyle\bigwedge_{Z \in N(a)} BZ) \wedge (\textstyle\bigwedge_{Y \in \wp(W) - N(a)} \neg BY) \ .$$

- $\mathcal{M}, w \models_{\mathsf{n}} \sigma$, so $\not\models_{\mathsf{n}} \neg\sigma$ and hence $\mathsf{KB} \nvdash \neg\sigma$ by our completness result.
- Argue that $\models_{\mathsf{p}} \neg\sigma^c$ by *reductio*: assume $\mathcal{N}, w \models_{\mathsf{p}} \sigma^c$.
  - $\mathcal{N}, w \models_{\mathsf{p}} P(W) = 1 \wedge (\bigwedge_{Z \in N(a)} P(Z) > c) \wedge (\bigwedge_{Y \in \wp(W) - N(a)} P(Z) \leq c)$.
  - Each letter $p$ occurs in exactly three $X$-sets:

$$7c < \textstyle\sum_{x \in X} P_w(x) = \sum_{p \in W} 3 \cdot P\left(p \wedge \bigwedge_{q \in W - \{p\}} \neg q\right) \ .$$

  - Each letter $p$ occurs in exactly three $Y$-sets, none of which is in $N(a)$:

$$7c \geq \textstyle\sum_{y \in Y} P_w(y) = \sum_{p \in W} 3 \cdot P\left(p \wedge \bigwedge_{q \in W - \{p\}} \neg q\right) \ .$$

## $\mathsf{KB}^{0.5}$: (our name for) Lenzen's Theory for $c = \frac{1}{2}$

| | |
|---|---|
| (KB) | Schemes and rules of $\mathsf{KB}$ |
| (D) | $B\phi \to \check{B}\phi$ |
| (SC) | $\check{B}\phi \wedge \check{K}(\neg\phi \wedge \psi) \to B(\phi \vee \psi)$ |
| (L) | $[(\phi_i \mathbb{I} \psi_i)_{i=1}^{m} \wedge B\phi_1 \wedge \bigwedge_{i=2}^{m} \check{B}\phi_i] \to \bigvee_{i=1}^{m} B\psi_i$ |

In minimal modal logic terms: $\mathsf{KB}^{0.5}$ is

- $\mathsf{EMND45} + \neg B\bot +$ (L) for belief
- plus $\mathsf{S5}$ knowledge and knowledge-belief connection principles.

**Theorem.** $\mathsf{KB}^{0.5}$ is sound and complete for mid-threshold models.

**Theorem (Lenzen).** $\mathsf{KB}^{0.5}$ is probabilistically complete for $c = \frac{1}{2}$:

$$\mathsf{KB}^{0.5} \vdash \phi \qquad \text{iff} \qquad \models_{\mathsf{p}} \phi^{\frac{1}{2}} \ .$$

# Conclusions

- Contributions:
  - Formulate Lenzen's result in modern modal language.
  - Modal completeness for epistemic neighborhood semantics.
  - Truth-preserving connection: neighborhoods and probabilities.
- Open questions:
  - Probabilistic comleteness for high-thresholds $c \in \left(\frac{1}{2}, 1\right) \cap \mathbb{Q}$:

    $$\mathsf{KB}^c \vdash \phi \text{ iff } \models_{\mathsf{p}} \phi^c \text{ for which } \mathsf{KB}^c?$$

    Principles: for $s' := c/(1-c)$ and $s := \mathsf{ceiling}(s')$,

    $$\begin{array}{ll} (\mathrm{SC}_0^s) & (\check{K}\phi_0 \wedge \bigwedge_{i=1}^{s} \check{B}\phi_i \wedge \bigwedge_{i \neq j = 0}^{s} K(\phi_i \to \neg\phi_j)) \to B(\bigvee_{i=0}^{s} \phi_i) \\ (\mathrm{SC}_1^s) & (\bigwedge_{i=1}^{s} \check{B}\phi_i \wedge \bigwedge_{i \neq j = 1}^{s} K(\phi_i \to \neg\phi_j)) \to B(\bigvee_{i=1}^{s} \phi_i) \\ (\mathrm{WL}) & [(\phi_i \mathbb{I} \psi_i)_{i=1}^{m} \wedge \bigwedge_{i=1}^{m} B\phi_i] \to \bigvee_{i=1}^{m} B\psi_i \end{array}$$

    (WL) sound. $\mathrm{SC}_1^s$ sound if $s \neq s'$, $\mathrm{SC}_0^s$ sound if $s = s'$.
  - Segerberg's/Gardenfors' probability comparison $\phi \geq \psi$ connection?
  - De Jongh–Ghosh belief strength comparison $\phi \geq \psi$ connection?
  - Neighborhoods for public announcements (i.e., Bayesian updates)?

# The End

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