Belief as Willingness to Bet

Jan van Eijck

CWI & ILLC, Amsterdam

Bryan Renne[*]

ILLC, University of Amsterdam

December 17, 2014

Abstract

We investigate modal logics of high probability having two unary modal operators: an operator K expressing probabilistic certainty and an operator B expressing probability exceeding a fixed rational threshold $c \geq \frac{1}{2}$. Identifying knowledge with the former and belief with the latter, we may think of c as the agent's betting threshold, which leads to the motto "belief is willingness to bet." The logic KB.5 for $c = \frac{1}{2}$ has an S5 K modality along with a sub-normal B modality that extends the minimal modal logic EMND45 by way of four schemes relating K and B, one of which is a complex scheme arising out of a theorem due to Scott. Lenzen was the first to use Scott's theorem to show that a version of this logic is sound and complete for the probability interpretation. We reformulate Lenzen's results and present them here in a modern and accessible form. In addition, we introduce a new epistemic neighborhood semantics that will be more familiar to modern modal logicians. Using Scott's theorem, we provide the Lenzen-derivative properties that must be imposed on finite epistemic neighborhood models so as to guarantee the existence of a probability measure respecting the neighborhood function in the appropriate way for threshold $c = \frac{1}{2}$. This yields a link between probabilistic and modal neighborhood semantics that we hope will be of use in future work on modal logics of qualitative probability. We leave open the question of which properties must be imposed on finite epistemic neighborhood models so as to guarantee existence of an appropriate probability measure for thresholds $c \neq \frac{1}{2}$.

1   Introduction

De Finetti [dF51, dF37] proposed the following axiomatization of qualitative probabilistic comparison (presented here based on [Sco64]): for sets X, Y, and Z coming from the powerset $\wp(W)$ of a nonempty finite set W, we have

1. $W \not\preceq \emptyset$,

---

2. $\emptyset \preceq X$,

3. $X \preceq Y$ or $Y \preceq X$,

4. $X \preceq Y \preceq Z$ implies $X \preceq Z$, and

5. $X \preceq Y$ if and only if $X \cup Z \preceq Y \cup Z$ for Z disjoint from X and Y.

De Finetti conjectured that any binary relation $\preceq$ on $\wp(W)$ that satisfies these conditions is *realizable by a probability measure P on $\wp(W)$*, which means that we have $X \preceq Y$ if and only if $P(X) \leq P(Y)$. While every probability measure realizing a binary relation $\preceq$ on $\wp(W)$ satisfies de Finetti's conditions, these conditions do not in general guarantee the existence of a realizing probability measure: it was shown by Kraft, Pratt, and Seidenberg [KPS59] (presented here as in [Seg71]) that for $W = \{a, b, c, d, e\}$, the relations

$$\{c\} \prec \{a, b\}, \quad \{b, d\} \prec \{a, c\},$$
$$\{a, e\} \prec \{b, c\}, \quad \{a, b, c\} \prec \{d, e\}$$

may be extended to a binary relation over $\wp(W)$ that satisfies de Finetti's conditions and yet has no realizing probability measure. Kraft, Pratt, and Seidenberg ("KPS") also determined what was missing from de Finetti's axiomatization; Scott [Sco64] later presented the KPS conditions in a linear algebraic form.

Theorem 1.1 ([Sco64, Theorem 4.1] reformulated with $\wp(W)$ instead of a general Boolean algebra)**.** Let W be a nonempty finite set. Given $X \in \wp(W)$, let $\iota : W \to \{0, 1\}$ be the characteristic function of X (i.e., $\iota(X)(w) = 1$ if $w \in X$, and $\iota(X)(w) = 0$ if $w \notin X$). Construe functions $x : W \to \mathbb{R}$ as vectors: $x(w)$ indicates the real-number value of vector x at coordinate w. Addition and negation of these vectors is taken component-wise: $(x + y)(w) := x(w) + y(w)$ and $(-x)(w) := -(x(w))$. A binary relation $\preceq$ on $\wp(W)$ is realizable by a probability measure if and only if it satisfies each of the following: for each $m \in \mathbb{Z}^+$ and $X, Y, X_1, \ldots, X_m, Y_1, \ldots, Y_m \in \wp(W)$, we have

1. $\emptyset \prec W$;

2. $\emptyset \preceq X$;

3. $X \preceq Y$ or $Y \preceq X$; and

4. if $X_i \preceq Y_i$ for each $i \leq m$ and $\sum_{i=1}^{m} \iota(X_i) = \sum_{j=1}^{m} \iota(Y_j)$, then $Y_j \preceq X_j$ for each $j \leq m$.

Scott's fourth condition is the most difficult. The algebraic component

$$\sum_{i=1}^{m} \iota(X_i) = \sum_{j=1}^{m} \iota(Y_j) \tag{1}$$

of this condition says: for each coordinate $w \in W$, the number of X_i's that contain w is equal to the number of Y_j's that contain w. Intuitively, Scott's forth condition tells us that if two length-m sequences of coordinate sets are related component-wise by the relation "is no more probable than"

and the occurrence multiplicity of any given world is the same in each of the sequences, then the sets are also related component-wise by the relation "has the same probability."

Using Scott's theorem to prove completeness, Segerberg [Seg71] studied a modal logic of qualitative probability. Segerberg's logic has a binary operator $\preceq$ expressing qualitative probabilistic comparison and a unary operator $\Box$ expressing necessity. Gärdenfors [Gär75] considered a simplified version of Segerberg's logic that, among other differences, eliminated the necessity operator in lieu of the abbreviation $\Box\varphi := (1 \preceq \varphi)$, which has the semantic meaning that φ has probability 1 (and implies that φ is true at all outcomes with nonzero probability). Both Gärdenfors and Segerberg express the algebraic component (1) of Scott's fourth condition using Segerberg's notation

$$(\varphi_1, \ldots, \varphi_m \mathbb{E} \psi_1, \ldots, \psi_m) \ ,$$

which we sometimes shorten to $(\varphi_i \mathbb{E} \psi_i)_{i=1}^m$. This expression abbreviates the formula

$$\Box(F_0 \vee \cdots \vee F_m) \ ,$$

where each F_i is the disjunction of all conjunctions

$$d_1\varphi_1 \wedge \cdots \wedge d_m\varphi_m \wedge e_1\psi_1 \wedge \cdots \wedge e_m\psi_m \wedge$$

satisfying the property that exactly i of the d_k's are the empty string, exactly i of the e_k's are the empty string, and the rest of the d_k's and e_k's are the negation sign $\neg$. Intuitively, F_i says that i of the φ_k's are true and i of the ψ_k's are true; $F_0 \vee \cdots \vee F_m$ says that the number of true φ_k's is the same as the number of true ψ_k's; and $(\varphi_i \mathbb{E} \psi_i)_{i=1}^m := \Box(F_0 \vee \cdots \vee F_m)$ says that at every outcome with nonzero probability, the number of true φ_k's is the same as the number of true ψ_k's. Using this notation, it is possible to express the fourth condition of Scott's theorem and thereby obtain completeness for the probabilistic interpretation.

In the present paper, we follow this tradition of studying probability from a qualitative (i.e., non-numerical) point of view using modal logic. However, our focus shall not be on the binary relation $\preceq$ of qualitative probabilistic comparison but instead on the unary notions of certainty (i.e., having probability 1) and "high" probability (i.e., having a probability greater than some fixed rational-number threshold $c \geq \frac{1}{2}$). That is, our interest is in *unary modal logics of high probability*.

For convenience in this study, we shall identify epistemic notions with probabilistic assignment, which suggests a connection with subjective probability [Jef04]. In particular, we identify knowledge with probabilistic certainty (i.e., probability 1) and belief with probability greater than some fixed rational-number threshold $c \geq \frac{1}{2}$. Therefore, instead of the unary operator $\Box$, we shall use the unary operator K and assign this operator an epistemic reading: $K\varphi$ says that the agent knows φ, which means she assigns φ subjective probability 1. We shall use the unary modal operator B to express belief: $B\varphi$ says that the agent believes φ, which means she assigns φ a subjective probability exceeding the threshold c (which will always be a fixed value within a given context or theory). Though our readings of these formulas are epistemic and doxastic, we stress that our technical results are independent of this reading, so someone who disagrees with subjective probability or our epistemic/doxastic readings is encouraged to think of our work purely in terms of high

probability: $K\varphi$ says $P(\varphi) = 1$, and $B\varphi$ says $P(\varphi) > c$ for some fixed $c \in [\frac{1}{2}, 1) \cap \mathbb{Q}$. That is, the technical results of our work are in no way dependent on our use of epistemic/doxastic notions or on the philosophy of subjective probability.

Lenzen [Len03, Len80] is to our knowledge the first to consider a modal logic of high probability for the threshold $c = \frac{1}{2}$. Actually, his perspective is slightly different than the one we adopt here. First, his reading of formulas is different (though not in any deep way): he identifies "the agent is convinced of φ" with $P(\varphi) = 1$ and "ψ is believed" by $P(\psi) > \frac{1}{2}$. More substantially, Lenzen's conviction (German: *Überzeuging*) does not imply truth. Technically, this amounts to permitting the possibility that there are outcomes having probability zero. For reasons of personal preference, we forbid this in our study here, though this difference is non-essential, as it is completely trivial from the technical perspective to change our setting to allow zero-probability outcomes or to change Lenzen's setting to forbid them. Therefore, we credit Lenzen's work [Len80] as the first to provide a proof of probabilistic completeness for $c = \frac{1}{2}$. As with Segerberg's and Gärdenfors' probabilistic completeness results, Lenzen's proof made crucial use of Scott's work.

In more recent work, Herzig [Her03] considered a logic of belief and action in which belief in φ is identified with $P(\varphi) > P(\neg\varphi)$. This is equivalent to Lenzen's notion, though Herzig does not study completeness. Another recent work by Kyberg and Teng [KT12] investigated a notion of "acceptance" in which φ is accepted whenever the probability of $\neg\varphi$ is at most some small ϵ. This gives rise to the minimal modal logic EMN, which is different than Lenzen's logic.

We herein consider belief à la Lenzen not only for the case $c = \frac{1}{2}$ but also for the case $c > \frac{1}{2}$. As it turns out, the logics for these cases are different, though our focus will be on the logic for $c = \frac{1}{2}$ because this is the only threshold for which a probability completeness result is known. In particular, probability completeness for $c > \frac{1}{2}$ is still open. Thresholds $c < \frac{1}{2}$ permit simultaneous belief of φ and $\neg\varphi$ while avoiding belief of any self-contradictory sentence such as the propositional constant $\bot$ for falsehood. This might suggest some connection with paraconsistent logic. However, we leave these logics of low probability for future work, though we shall say a few words more about them later in this paper.

In Section 2, we identify a Kripke-style semantics for probability logic similar to [EoS14, Hal03] (and no doubt to many others). We require that all worlds are probabilistically possible but not necessarily epistemically so, and we provide some examples of how this semantics works. In particular, we demonstrate that our requirement is not problematic: world v can be made to have probability zero relative to world w if we cut the epistemic accessibility relation between these worlds.

In Section 3, we define our modal notions of certain knowledge and of belief exceeding threshold c, explain the motto "belief is willingness to bet," and prove a number of properties of certain knowledge and this "betting" belief. For instance, we show that knowledge is S5 and belief is not normal. We show a number of other threshold-specific properties of betting belief as well. In particular, we see that the belief modality extends the minimal modal logic EMND45 $+ \neg B\bot$ by way of certain schemes relating knowledge and belief.

We then introduce a formal modal language in Section 4, relate this language to the probabilistic notions of belief and knowledge, and introduce an epistemic neighborhood semantics for the language. We study the relationship between the neighborhood and probabilistic semantics.

4

In particular, we introduce a notion of "agreement" between epistemic probability models and epistemic neighborhood models, the key component of which is this: an event X is a neighborhood of a world w if and only if the probability measure P_w at w satisfies $P_w(X) > c$. We use one of Scott's theorems to prove that epistemic neighborhood models satisfying certain properties give rise to agreeing epistemic probability models for the threshold $c = \frac{1}{2}$. This result we credit to Lenzen; however, we prove this result anew in a modern, streamlined form that we hope will make it more accessible. The main remaining open problem is to prove the analogous result for thresholds $c \neq \frac{1}{2}$ (i.e., find the additional sufficient conditions on epistemic neighborhood models we need to impose so as to guarantee the existence of an agreeing epistemic probability model for threshold $c \neq \frac{1}{2}$). Finally, we prove that epistemic probability models always give rise to agreeing epistemic neighborhood models.

In Section 5, we introduce a basic modal theory KB that is probabilistically sound. We adapt an example due to Walley and Fine [WF79] that shows KB is probabilistically incomplete. This leads us to add additional principles to KB, thereby producing the modal theory KB.5, our name for our modern reformulation of Lenzen's modal theory of knowledge and belief (or, in Lenzen's terminology, his theory of "acceptance" and belief). Using the results from Section 4, we prove that this logic is sound and complete for epistemic probability models using threshold $c = \frac{1}{2}$. Regarding the semantics based on our epistemic neighborhood models, we prove that KB is sound and complete for the full class of these models and that KB.5 is sound and complete for the smaller class that satisfies the additional Lenzen-derivative properties needed to guarantee the existence of an agreeing probability measure for threshold $c = \frac{1}{2}$.

Stated in an analogy: KB is to de Finetti's axiomatization as KB.5 is to the KPS/Scott axiomatization. However, do not be misled: de Finetti, KPS, and Scott considered qualitative probabilistic comparison, which is a binary notion based on a binary operator $\preceq$. See also [HI13] for a revival of this tradition. We, on the other hand, consider high probability, which is a unary notion based on unary operators we denote as K and B.

Another version of our main open question can be restated in the following syntactic form: given a threshold $c \neq \frac{1}{2}$, find the additional principles that must be added to our probabilistically sound but incomplete base logic KB in order to obtain a probabilistically sound and complete logic for threshold c. In our conclusion, we present some additional sound principles that might come up in this work, but we have not been able to find the probabilistically sound and complete axiomatization for thresholds $c \neq \frac{1}{2}$.

Given the link between epistemic neighborhood models and epistemic probability models, our results may be viewed as a contribution to the study connecting two schools of rational decision making: the probabilist (e.g., [Kör08]) and the AI-based (e.g., [KT12]). We also hope that it will be of some use in future work on qualitative probability.

2 Epistemic Probability Models

Definition 2.1. We fix a set $\mathbf{P}$ of propositional letters. An *epistemic probability model* is a structure $\mathcal{M} = (W, R, V, P)$ satisfying the following.

- (W, R, V) is a finite single-agent S5 Kripke model:

- W is a finite nonempty set of "worlds" or "outcomes." An *event* is a set $X \subseteq W$ of worlds. When convenient, we identify a world w with the singleton event $\{w\}$.

- $R \subseteq W \times W$ is an equivalence relation R on W. We let

$$[w] := \{v \in W \mid wRv\}$$

denote the equivalence class of world w. This is the set of worlds that agent cannot distinguish from w.

- $V : W \to \wp(\mathbf{P})$ assigns a set $V(w)$ of propositional letters to each world $w \in W$.

- $P : \wp(W) \to [0,1]$ is a probability measure over the finite algebra $\wp(W)$ satisfying the property of *full support*: $P(w) \neq 0$ for each $w \in W$.

A *pointed epistemic probability model* is a pair $(\mathcal{M}, w)$ consisting of an epistemic probability model $\mathcal{M} = (W, R, V, P)$ and world $w \in W$ called the *point*.

The agent's uncertainty as to which world is the actual world is given by the equivalence relation R. If w is the actual world, then the probability the agent assigns to an event X at w is given by

$$P_w(X) := \frac{P(X \cap [w])}{P([w])} \quad . \tag{2}$$

In words: the probability the agent assigns to event X at world w is the probability she assigns to X conditional on her knowledge at w. Slogan: subjective probability is always conditioned, and the most general condition is given by the knowledge of the agent. This makes sense because the right side of (2) is just $P(X|[w])$, the probability of X conditional on $[w]$. Note that $P_w(X)$ is always well-defined: we have $w \in [w]$ by the reflexivity of R and hence $0 < P(w) \leq P([w])$ by full support, so the denominator on the right side of (2) is nonzero.

Example 2.2 (Horse racing). Three horses compete in a race. For each $i \in \{1, 2, 3\}$, horse h_i wins the race in world w_i. The agent can distinguish between these three possibilities, and she assigns the horses winning chances of $3:2:1$. We represent this situation in the form of an epistemic probability model $\mathcal{M}_{2.2}$ pictured as follows:
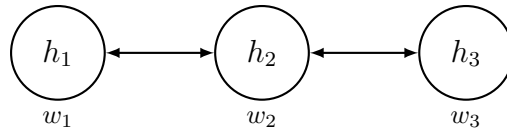

$$P = \{w_1 : \tfrac{3}{6}, w_2 : \tfrac{2}{6}, w_3 : \tfrac{1}{6}\}$$

$$\mathcal{M}_{2.2}$$

When we picture epistemic probability models, the arrows of the agent are to be closed under reflexivity and transitivity. With this convention in place, it is not difficult to verify that $P_{w_1}(\{w_1, w_3\}) = \frac{2}{3}$; that is, at w_1, the assigns probability $\frac{2}{3}$ to the event that the winner is horse 1 or horse 3.

The property of full support says that each world is probabilistically possible. Therefore, in order to represent a situation in which the agent is certain that horse 3 can never win, we simply make the h_3-worlds inaccessible via R.

Example 2.3 (Certainty of impossibility)**.** We modify Example 2.2 by eliminating the arrow between worlds w_2 and w_3.

$$P = \{w_1 : \tfrac{3}{6}, w_2 : \tfrac{2}{6}, w_3 : \tfrac{1}{6}\}$$

$$\mathcal{M}_{2.3}$$
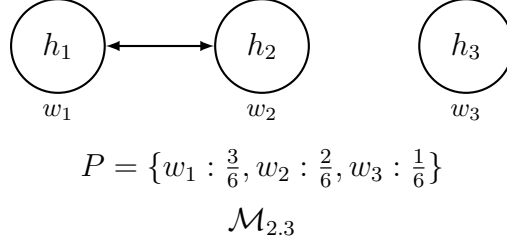
At world w_1 in this picture, there is no accessible world at which horse 3 wins. Therefore, at world w_1, the agent assigns probability 0 to the event that horse 3 wins: $P_{w_1}(w_3) = 0$.

We define a language $\mathcal{L}$ for reasoning about epistemic probability models.

Definition 2.4. The language $\mathcal{L}$ of *(single-agent) probability logic* is defined by the following grammar.

$$
\begin{aligned}
\varphi &::= \ \top \mid p \mid \neg\varphi \mid \varphi \wedge \varphi \mid t \geq 0 \\
t &::= \ q \mid q \cdot P(\varphi) \mid t + t \\
&\qquad p \in \mathbf{P}, q \in \mathbb{Q}
\end{aligned}
$$

We adopt the usual abbreviations for Boolean connectives. We define the relational symbols $\leq, >, <$, and $=$ in terms of $\geq$ as usual. For example, $t = s$ abbreviates $(t \geq s) \wedge (s \geq t)$. We also use the obvious abbreviations for writing linear inequalities. For example, $P(p) \leq 1 - q$ abbreviates $1 + (-q) + (-1) \cdot P(p) \geq 0$.

Definition 2.5. Let $\mathcal{M} = (W, R, V, P)$ be an epistemic probability model. We define a binary truth relation $\models_{\mathsf{p}}$ between a pointed epistemic probability model $(\mathcal{M}, w)$ and $\mathcal{L}$-formulas as follows.

$$
\begin{aligned}
\mathcal{M}, w &\models_{\mathsf{p}} \top \\
\mathcal{M}, w &\models_{\mathsf{p}} p && \text{iff} \quad p \in V(w) \\
\mathcal{M}, w &\models_{\mathsf{p}} \neg\varphi && \text{iff} \quad \mathcal{M}, w \not\models_{\mathsf{p}} \varphi \\
\mathcal{M}, w &\models_{\mathsf{p}} \varphi \wedge \psi && \text{iff} \quad \mathcal{M}, w \models_{\mathsf{p}} \varphi \text{ and } \mathcal{M}, w \models_{\mathsf{p}} \psi \\
\mathcal{M}, w &\models_{\mathsf{p}} t \geq 0 && \text{iff} \quad [\![t]\!]_w \geq 0
\end{aligned}
$$

$$
\begin{aligned}
[\![\varphi]\!]_{\mathsf{p}} &:= \{u \in W \mid \mathcal{M}, u \models_{\mathsf{p}} \varphi\} \\
P_w(X) &:= \frac{P(X \cap [w])}{P([w])} \\
[\![q]\!]_w &:= q \\
[\![q \cdot P(\varphi)]\!]_w &:= q \cdot P_w([\![\varphi]\!]_{\mathsf{p}}) \\
[\![t + t']\!]_w &:= [\![t]\!]_w + [\![t']\!]_w
\end{aligned}
$$

7

Validity of $\varphi \in \mathcal{L}$ in epistemic probability model $\mathcal{M}$, written $\mathcal{M} \models_\mathsf{p} \varphi$, means that $\mathcal{M}, w \models_\mathsf{p} \varphi$ for each world $w \in W$. Validity of $\varphi \in \mathcal{L}$, written $\models_\mathsf{p} \varphi$, means that $\mathcal{M} \models_\mathsf{p} \varphi$ for each epistemic probability model $\mathcal{M}$.

3 Certainty and Belief

[Eij13] formulates and proves a "certainty theorem" relating certainty in epistemic probability models to knowledge in a version of these models in which the probabilistic information is removed. This motivates the following definition.

Definition 3.1 (Knowledge as Certainty)**.** We adopt the following abbreviations.

- $K\varphi$ abbreviates $P(\varphi) = 1$.

  We read $K\varphi$ as "the agent knows φ."

- $\check{K}\varphi$ abbreviates $\neg K \neg \varphi$.

  We read $\check{K}\varphi$ as "φ is consistent with the agent's knowledge."

Theorem 3.2 ([Eij13])**.** K is an S5 modal operator:

1. $\models_\mathsf{p} \varphi$ for each $\mathcal{L}$-instance φ of a scheme of classical propositional logic.

   Axioms of classical propositional logic are valid.

2. $\models_\mathsf{p} K(\varphi \rightarrow \psi) \rightarrow (K\varphi \rightarrow K\psi)$

   Knowledge is closed under logical consequence.

3. $\models_\mathsf{p} K\varphi \rightarrow \varphi$

   Knowledge is veridical.

4. $\models_\mathsf{p} K\varphi \rightarrow KK\varphi$

   Knowledge is positive introspective: it is known what is known.

5. $\models_\mathsf{p} \neg K\varphi \rightarrow K\neg K\varphi$

   Knowledge is negative introspective: it is known what is not known.

6. $\models_\mathsf{p} \varphi$ implies $\models_\mathsf{p} K\varphi$

   All validities are known.

7. $\models_\mathsf{p} \varphi \rightarrow \psi$ and $\models_\mathsf{p} \varphi$ together imply $\models_\mathsf{p} \psi$.

   Validities are closed under the rule of Modus Ponens.

We define belief in a proposition φ as willingness to take bets on φ with the odds being better than some rational number $c \in (0, 1) \cap \mathbb{Q}$. This leads to a number of degrees of belief, one for each threshold c.

Definition 3.3 (Belief as Willingness to Bet). Fix a threshold $c \in (0,1) \cap \mathbb{Q}$.

- $B^c\varphi$ abbreviates $P(\varphi) > c$.

  We read $B^c\varphi$ as "the agent believes φ with threshold c."

- $\check{B}^c\varphi$ abbreviates $\neg B^c\neg\varphi$.

  We read $\check{B}\varphi$ as "φ is consistent with the agent's threshold-c beliefs."

If the threshold c is omitted (either in the notations $B^c\varphi$ and $\check{B}^c\varphi$ or in the informal readings of these notations), it is assumed that $c = \frac{1}{2}$.

This notion of belief comes from subjective probability [Jef04]. In particular, fix a threshold $c = p/q \in (0,1) \cap \mathbb{Q}$. Suppose that the agent believes φ with threshold $c = p/q$; that is, $P(\varphi) > p/q$. If the agent wagers p dollars for a chance to win q dollars on a bet that φ is true, then she expects her net winnings to be

$$[(q - p) \cdot P(\varphi)] - [p \cdot (1 - P(\varphi))] = q \cdot P(\varphi) - p$$

dollars on this bet. This is a positive number of dollars if and only if $q \cdot P(\varphi) > p$. But notice that the latter is guaranteed by the assumption $P(\varphi) > p/q$. Therefore, it is rational for the agent to take this bet. Said in the parlance of the subjective probability literature: "If the agent stakes p to win q in a bet on φ, then her winning expectation is positive in case she believes φ with threshold $c = p/q$." Or in a short motto: "Belief is willingness to bet."

Remark 3.4. Belief based on threshold $c = 0$ or $c = 1$ is trivial to express in terms of negation, K, and falsehood $\bot$. So we do not consider these thresholds here. Beliefs based on low-thresholds $c \in (0, \frac{1}{2}) \cap \mathbb{Q}$ have unintuitive and unusual features. First, low-threshold beliefs unintuitively permit inconsistency of the kind that an agent can believe both φ and $\neg\varphi$ while avoiding inconsistency of the kind that the agent can believe a self-contradictory formula such as $\bot$. (This suggests some connection with paraconsistent logic.) Second, the dual of a low-threshold belief implies the belief at that threshold (i.e., $\check{B}^c\varphi \to B^c\varphi$), which is unusual if we assign the usual "consistency" reading to dual operators (i.e., "φ is consistent with the agent's beliefs implies φ is believed" is unusual). Since low-threshold $c \in (0, \frac{1}{2}) \cap \mathbb{Q}$ beliefs have these unintuitive and unusual features, we leave their study for future work, focusing instead on thresholds $c \in [\frac{1}{2}, 1) \cap \mathbb{Q}$.

The following lemma provides a useful characterization of the dual $\check{B}^c\varphi$.

Lemma 3.5. Let $\mathcal{M} = (W, R, V, P)$ be an epistemic probability model.

1. $\mathcal{M}, w \models_\mathsf{p} \check{B}^c\varphi$ iff $\mathcal{M}, w \models_\mathsf{p} P(\varphi) \geq 1 - c$.

2. $\mathcal{M}, w \models_\mathsf{p} \check{B}^{\frac{1}{2}}\varphi$ iff $\mathcal{M}, w \models_\mathsf{p} P(\varphi) \geq \frac{1}{2}$.

Proof. For Item 1, we have the following:

$$\mathcal{M}, w \models_{\mathsf{p}} \check{B}^c \varphi$$

$$\begin{array}{lll} \text{iff} & \mathcal{M}, w \models_{\mathsf{p}} \neg B^c \neg \varphi & \text{by definition of } \check{B}^c \varphi \\ \text{iff} & P_w(\llbracket \neg \varphi \rrbracket_{\mathsf{p}}) \not> c & \text{by definition of } B^c \varphi \text{ and } \models_{\mathsf{p}} \\ \text{iff} & P_w(\llbracket \neg \varphi \rrbracket_{\mathsf{p}}) \leq c & \text{since } \mathbb{Q} \text{ is totally ordered} \\ \text{iff} & P_w(\llbracket \varphi \rrbracket_{\mathsf{p}}) \geq 1 - c & \text{since } \llbracket \neg \varphi \rrbracket_{\mathsf{p}} = W - \llbracket \varphi \rrbracket_{\mathsf{p}} \end{array}$$

For Item 2, apply Item 1 with $c = \frac{1}{2}$. □

We now consider a simple example.

Example 3.6 (Non-normality). In this variation, all horses have equal chances of winning and the agent knows this.

$$P = \{w_1 : \tfrac{1}{3}, w_2 : \tfrac{1}{3}, w_3 : \tfrac{1}{3}\}$$

$$\mathcal{M}_{3.6}$$
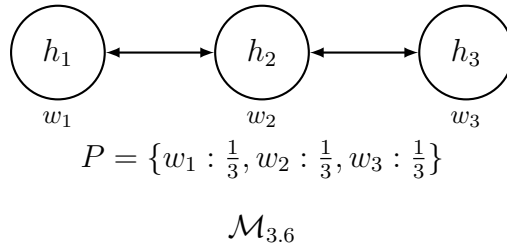
Recalling that an omitted threshold c is implicitly assumed to be $\frac{1}{2}$, the following are readily verified.

1. $\mathcal{M}_{3.6} \models_{\mathsf{p}} B(h_1 \vee h_2 \vee h_3)$.

   The agent believes the winning horse is among the three.

   (The agent is willing to bet that the winning horse is among the three.)

2. $\mathcal{M}_{3.6} \models_{\mathsf{p}} B(h_1 \vee h_2) \wedge B(h_1 \vee h_3) \wedge B(h_2 \vee h_3)$.

   The agent believes the winning horse is among any two.

   (The agent is willing to bet that the winning horse is among any two.)

3. $\mathcal{M}_{3.6} \models_{\mathsf{p}} B_a \neg h_1 \wedge B_a \neg h_2 \wedge B_a \neg h_3$.

   The agent believes the winning horse is not any particular one.

   (The agent is willing to bet that the winning horse is not any particular one.)

4. $\mathcal{M}_{3.6} \models_{\mathsf{p}} \neg B(\neg h_1 \wedge \neg h_2)$.

   The agent does not believe that both horses 1 and 2 do not win.

   (The agent is not willing to bet that both horses 1 and 2 do not win.)

It follows from Items 3 and 4 of Example 3.6 that the present notion of belief is not closed under conjunction. This is discussed as part of the literature on the "Lottery Paradox" [Kyb61].[1] However, there is no reason in general that it is paradoxical to assign a conjunction $\varphi \wedge \psi$ a lower probability than either of its conjunctions. Indeed, if φ and ψ are independent, then the probability of their conjunction equals the product of their probabilities, so unless one of φ or ψ is certain or impossible, the probability of $\varphi \wedge \psi$ will be less than the probability of φ and less than the probability of ψ.

We set aside philosophical arguments for or against closure of belief under conjunction and instead turn our attention to the study of the properties of the present notion of belief. One of these is a complicated but useful property due to Scott [Sco64] that makes use of notation due to Segerberg [Seg71].

Definition 3.7 (Segerberg notation; [Seg71]). Fix a positive integer $m \in \mathbb{Z}^+$ and formulas $\varphi_1, \ldots, \varphi_m$ and $\psi_1, \ldots, \psi_m$. The expression

$$(\varphi_1, \ldots, \varphi_m \mathbb{I} \psi_1, \ldots, \psi_m) \tag{3}$$

abbreviates the formula

$$K(F_0 \vee F_1 \vee F_2 \vee \cdots \vee F_m) \ ,$$

where F_i is the disjunction of all conjunctions

$$d_1 \varphi_1 \wedge \cdots \wedge d_m \varphi_m \wedge e_1 \psi_1 \wedge \cdots \wedge e_m \psi_m$$

satisfying the property that *exactly* i of the d_k's are the empty string, *at least* i of the e_k's are the empty string, and the rest of the d_k's and e_k's are the negation sign $\neg$. We may write $(\varphi_i \mathbb{I} \psi_i)_{i=1}^m$ as an abbreviation for (3). Finally, let

$$(\varphi_i \mathbb{E} \psi_i)_{i=1}^m \quad \text{abbreviate} \quad (\varphi_i \mathbb{I} \psi_i)_{i=1}^m \wedge (\psi_i \mathbb{I} \varphi_i)_{i=1}^m \ .$$

We also allow the use of $\mathbb{E}$ in a notation similar to (3).

The formula $(\varphi_i \mathbb{I} \psi_i)_{i=1}^m$ says that the agent knows that the number of true φ_i's is less than or equal to the number of true ψ_i's. Put another way, $(\varphi_i \mathbb{I} \psi_i)_{i=1}^m$ is true if and only if every one of the agent's epistemically accessible worlds satisfies at least as many ψ_i's as φ_i's. The formula $(\varphi_i \mathbb{E} \psi_i)_{i=1}^m$ says that every one of the agent's epistemically accessible worlds satisfies exactly as many ψ_i's as φ_i's.

Definition 3.8 (Scott scheme; [Sco64]). We define the following scheme:

$$[(\varphi_i \mathbb{I} \psi_i)_{i=1}^m \wedge B^c \varphi_1 \wedge \bigwedge_{i=2}^m \check{B}^c \varphi_i] \to \bigvee_{i=1}^m B^c \psi_i \tag{Scott}$$

If $m = 1$, then $\bigwedge_{i=2}^m \check{B}^c \varphi_i$ is $\top$. Note that (Scott) is meant to encompass the indicated scheme for each positive integer $m \in \mathbb{Z}^+$.

---

[1] The usual formulation of the Lottery Paradox: it is paradoxical for an agent to believe that one of n lottery tickets will be a winner (i.e., "some ticket is a winner") without believing of any particular ticket that it is the winner (i.e., "for each $i \in \{1, \ldots, n\}$, ticket i is not a winner").

(Scott) says that if the agent knows the number of true φ_i's is less than or equal to the number of true ψ_i's, she believes φ_1 with threshold c, and the remaining φ_i's are each consistent with her threshold-c beliefs, then she believes one of the ψ_i's with threshold c. Adapting a proof of Segerberg [Seg71], we show that belief with threshold $c = \frac{1}{2}$ satisfies (Scott).

We report this result along with a number of other properties in the following proposition.

Theorem 3.9 (Properties of Belief)**.** For $c \in (0, 1) \cap \mathbb{Q}$, we have:

1. $\not\models_\mathsf{p} B^c(\varphi \to \psi) \to (B^c\varphi \to B^c\psi)$.

   Belief is not closed under logical consequence.

   (So B^c is not a normal modal operator.)

2. $\not\models_\mathsf{p} B^c\varphi \to \varphi$.

   Belief is not veridical.

3. $\models_\mathsf{p} K\varphi \to B^c\varphi$.

   What is known is believed.

4. $\models_\mathsf{p} \neg B^c\bot$.

   The propositional constant $\bot$ for falsehood is not believed.

5. $\models_\mathsf{p} B^c\top$.

   The propositional constant $\top$ for truth is believed.

6. $\models_\mathsf{p} B^c\varphi \to KB^c\varphi$.

   What is believed is known to be believed.

7. $\models_\mathsf{p} \neg B^c\varphi \to K\neg B^c\varphi$.

   What is not believed is known to be not believed.

8. $\models_\mathsf{p} K(\varphi \to \psi) \to (B^c\varphi \to B^c\psi)$.

   Belief is closed under known logical consequence.

9. If $c \in [\frac{1}{2}, 1)$, then $\models_\mathsf{p} B^c\varphi \to \check{B}^c\varphi$.

   High-threshold belief is consistent: belief in φ implies disbelief in $\neg\varphi$.

10. $\models_\mathsf{p} \check{B}^{\frac{1}{2}}\varphi \wedge \check{K}(\neg\varphi \wedge \psi) \to B^{\frac{1}{2}}(\varphi \vee \psi)$.

    For mid-threshold belief, if φ is consistent with the agent's beliefs and $\neg\varphi \wedge \psi$ is consistent with her knowledge, then she believes $\varphi \vee \psi$.

11. $\models_\mathsf{p} [(\varphi_i \mathbb{I} \psi_i)_{i=1}^m \wedge B^{\frac{1}{2}}\varphi_1 \wedge \bigwedge_{i=2}^m \check{B}^{\frac{1}{2}}\varphi_i] \to \bigvee_{i=1}^m B^{\frac{1}{2}}\psi_i$.

    Mid-threshold belief satisfies (Scott).

Proof. We consider each item in turn.

1. Given $c \in (0, 1) \cap \mathbb{Q}$ and integers p and q such that $p/q = c$, we define $\mathcal{M}$ as the modification of the model $\mathcal{M}_{3.6}$ of Example 3.6 obtained by changing P as follows:

$$P := \left\{ w_1 : \frac{q-p}{2q}, \ w_2 : \frac{p}{q}, \ w_3 : \frac{q-p}{2q} \right\} .$$

Since $0 < p < q$, it follows that

$$P_{w_1}(\llbracket \neg h_1 \to h_2 \rrbracket_{\mathsf{p}}) = P_{w_1}(\{w_1, w_2\}) = \frac{q+p}{2q} > \frac{2p}{2q} = \frac{p}{q},$$

$$P_{w_1}(\llbracket \neg h_1 \rrbracket_{\mathsf{p}}) = P_{w_1}(\{w_2, w_3\}) = \frac{q+p}{2q} > \frac{2p}{2q} = \frac{p}{q}, \text{ and}$$

$$P_{w_1}(\llbracket h_2 \rrbracket_{\mathsf{p}}) = P_{w_1}(w_2) = \frac{p}{q} .$$

Therefore, we have

$$\mathcal{M}, w_1 \models_{\mathsf{p}} B^c(\neg h_1 \to h_2) \wedge B^c \neg h_1 \wedge \neg B^c h_2 .$$

2. For $\mathcal{M}$ defined in the proof of Item 1, we have

$$\mathcal{M}, w_1 \models_{\mathsf{p}} h_1 \wedge B^c \neg h_1 .$$

3. $\mathcal{M}, w \models_{\mathsf{p}} K\varphi$ implies $P_w(\llbracket \varphi \rrbracket_{\mathsf{p}}) = 1 > c$. Hence $\mathcal{M}, w \models_{\mathsf{p}} B^c \varphi$.

4. $P_w(\llbracket \bot \rrbracket_{\mathsf{p}}) = 0 < c$. Hence $\mathcal{M}, w \models \neg B^c \bot$.

5. $P_w(\llbracket \top \rrbracket_{\mathsf{p}}) = 1 > c$. Hence $\mathcal{M}, w \models_{\mathsf{p}} B^c \top$.

6. $\mathcal{M}, w \models_{\mathsf{p}} B^c \varphi$ implies $P_w(\llbracket \varphi \rrbracket_{\mathsf{p}}) > c$. To show that $\mathcal{M}, w \models_{\mathsf{p}} KB^c \varphi$, we must prove that

$$P_w(\llbracket B^c \varphi \rrbracket_{\mathsf{p}}) = \frac{P(\llbracket B^c \varphi \rrbracket_{\mathsf{p}} \cap [w])}{P([w])} = 1 .$$

To show this, we prove that $\llbracket B^c \varphi \rrbracket_{\mathsf{p}} \cap [w] = [w]$. So choose $u \in [w]$. Since R is an equivalence relation, we have

$$P_u(\llbracket \varphi \rrbracket_{\mathsf{p}}) = \frac{P(\llbracket \varphi \rrbracket_{\mathsf{p}} \cap [u])}{P([u])} = \frac{P(\llbracket \varphi \rrbracket_{\mathsf{p}} \cap [w])}{P([w])} = P_w(\llbracket \varphi \rrbracket_{\mathsf{p}}) > c ,$$

which implies $u \in \llbracket B^c \varphi \rrbracket_{\mathsf{p}}$. The result follows.

7. The argument is similar to that for Item 6, though we note that $\mathcal{M}, w \models_{\mathsf{p}} \neg B^c \varphi$ implies $P_w(\llbracket \varphi \rrbracket_{\mathsf{p}}) \le c$.

8. We assume that $\mathcal{M}, w \models_\mathsf{p} K(\varphi \to \psi)$ and $\mathcal{M}, w \models_\mathsf{p} B^c\varphi$. This means that $P_w(\llbracket \varphi \to \psi \rrbracket_\mathsf{p}) = 1$ and $P_w(\llbracket \varphi \rrbracket_\mathsf{p}) > c$. But then it follows that $P_w(\llbracket \psi \rrbracket_\mathsf{p}) > c$ as well, which is what it means to have $\mathcal{M}, w \models_\mathsf{p} B^c\psi$.

9. Assume $c \in [\frac{1}{2}, 1) \cap \mathbb{Q}$ and $\mathcal{M}, w \models_\mathsf{p} B^c\varphi$. Then $P_w(\llbracket \varphi \rrbracket_\mathsf{p}) > c \geq 1-c$. So $P_w(\llbracket \varphi \rrbracket_\mathsf{p}) \geq 1-c$. The result therefore follows by Lemma 3.5.

10. We prove something more general. Assume $c \in (0, \frac{1}{2}] \cap \mathbb{Q}$ and $\mathcal{M}, w \models_\mathsf{p} \check{B}^c\varphi$. By Lemma 3.5, it follows that $P_w(\llbracket \varphi \rrbracket_\mathsf{p}) \geq c$. Let us assume further that $\mathcal{M}, w \models_\mathsf{p} \check{K}(\neg\varphi \wedge \psi)$. This means

$$1 \neq P_w(\llbracket \neg(\neg\varphi \wedge \psi) \rrbracket_\mathsf{p}) = \frac{P(\llbracket \neg(\neg\varphi \wedge \psi) \rrbracket_\mathsf{p} \cap [w])}{P([w])} \quad,$$

which implies there exists $v \in \llbracket \neg\varphi \wedge \psi \rrbracket_\mathsf{p} \cap [w]$. Since $P(v) > 0$ by full support, it follows that

$$
\begin{aligned}
P_w(\llbracket \varphi \vee \psi \rrbracket_\mathsf{p}) &= \frac{P(\llbracket \varphi \vee \psi \rrbracket_\mathsf{p} \cap [w])}{P([w])} \\
&= \frac{P(\llbracket \varphi \rrbracket_\mathsf{p} \cap [w])}{P([w])} + \frac{P(\llbracket \neg\varphi \wedge \psi \rrbracket_\mathsf{p} \cap [w])}{P([w])} \\
&\geq \frac{P(\llbracket \varphi \rrbracket_\mathsf{p} \cap [w])}{P([w])} + \frac{P(v)}{P([w])} \\
&= P_w(\llbracket \varphi \rrbracket_\mathsf{p}) + \frac{P(v)}{P([w])} \\
&\geq c + \frac{P(v)}{P([w])} > c \quad.
\end{aligned}
$$

That is, $\mathcal{M}, w \models_\mathsf{p} B^c(\varphi \vee \psi)$.

11. Again, we prove something more general. We assume $c \in (0, \frac{1}{2}] \cap \mathbb{Q}$ plus the following:

$$\mathcal{M}, w \models_\mathsf{p} (\varphi_i \mathbb{I} \psi_i)_{i=1}^m \tag{4}$$

$$\mathcal{M}, w \models_\mathsf{p} B^c\varphi_1 \tag{5}$$

$$\mathcal{M}, w \models_\mathsf{p} \bigwedge_{i=2}^m \check{B}^c\varphi_i \tag{6}$$

We recall the meaning of (4): for each $v \in [w]$, the number of φ_i's true at v is less than or equal to the number of ψ_k's true at v. It therefore follows from (4) that

$$P_w(\llbracket \varphi_1 \rrbracket_\mathsf{p}) + \cdots + P_w(\llbracket \varphi_m \rrbracket_\mathsf{p}) \leq P_w(\llbracket \psi_1 \rrbracket_\mathsf{p}) + \cdots + P_w(\llbracket \psi_m \rrbracket_\mathsf{p}) \quad. \tag{7}$$

Outlining an argument due to Segerberg [Seg71, pp. 344–346], the reason for this is as follows: we think of each world $v \in [w]$ as being assigned a "weight" $P_w(v)$. A member $P_w(\llbracket \varphi_i \rrbracket_\mathsf{p})$ of the sum on the left of (7) is just a total of the weight of every $v \in [w]$ that satisfies φ_i; that is,

$$P_w(\llbracket \varphi_i \rrbracket_\mathsf{p}) = \sum \{P_w(v) \mid v \in \llbracket \varphi_i \rrbracket_\mathsf{p} \cap [w]\} \quad.$$

Assumption (4) tells us that for each $v \in [w]$, the number of totals $P_w(\llbracket \varphi_i \rrbracket_\mathsf{p})$ on the left of (7) to which v contributes its weight is less than or equal to the number of totals $P_w(\llbracket \psi_k \rrbracket_\mathsf{p})$ on the right of (7) to which v contributes its weight. But then the sum of totals on the left must be less than or equal to the sum of totals on the right. Hence (7) follows.

Having established (7), we now proceed further with the overall proof. By (5), we have $P_w(\llbracket \varphi_1 \rrbracket_\mathsf{p}) > c$. Applying (6) and Lemma 3.5, we have $P_w(\varphi_i) \geq c$ for each $i \in \{2, \ldots, m\}$. Hence

$$P_w(\llbracket \psi_1 \rrbracket_\mathsf{p}) + \cdots + P_w(\llbracket \psi_m \rrbracket_\mathsf{p}) \geq P_w(\llbracket \varphi_1 \rrbracket_\mathsf{p}) + \cdots + P_w(\llbracket \varphi_m \rrbracket_\mathsf{p}) > mc \ .$$

That is, the sum of the $P_w(\llbracket \psi_k \rrbracket_\mathsf{p})$'s must exceed mc. Since each member of this m-member sum is non-negative, it follows that at least one member must exceed c. That is, there exists $j \in \{1, \ldots, m\}$ such that $P_w(\llbracket \psi_j \rrbracket_\mathsf{p}) > c$. Hence $\mathcal{M}, w \models_\mathsf{p} \bigvee_{j=1}^{m} B^c \psi_j$. $\qquad\square$

4 Epistemic Neighborhood Models

The modal formulas $K\varphi$ and $B^c\varphi$ were taken as abbreviations in the language $\mathcal{L}$ of probability logic. We wish to consider a propositional modal language that has knowledge and belief operators as primitives.

Definition 4.1. The language $\mathcal{L}_\mathsf{KB}$ of *(single-agent) knowledge and belief* is defined by the following grammar.

$$\varphi \quad ::= \quad \top \mid p \mid \neg\varphi \mid \varphi \wedge \varphi \mid K\varphi \mid B\varphi$$
$$p \in \mathbf{P}$$

We adopt the usual abbreviations for other Boolean connectives and define the dual operators $\check{K} := \neg K \neg$ and $\check{B} := \neg B \neg$. Finally, the $\mathcal{L}_\mathsf{KB}$-formula

$$(\varphi_1, \ldots, \varphi_m \mathbb{I} \psi_1, \ldots, \psi_m)$$

and its abbreviation $(\varphi_i \mathbb{I} \psi_i)_{i=1}^{m}$ are given as in Definition 3.7 except that all formulas are taken from the language $\mathcal{L}_\mathsf{KB}$.

Our goal will be to develop a possible worlds semantics for $\mathcal{L}_\mathsf{KB}$ that links with the probabilistic setting by making the following translation truth-preserving.

Definition 4.2 (Translation). For $c \in (0, 1) \cap \mathbb{Q}$, we define $c : \mathcal{L}_\mathsf{KB} \to \mathcal{L}$ as follows.

$$\begin{aligned}
\top^c &:= \top \\
p^c &:= p \\
(\neg\varphi)^c &:= \neg\varphi^c \\
(\varphi \wedge \psi)^c &:= \varphi^c \wedge \psi^c \\
(K\varphi)^c &:= P(\varphi^c) = 1 \quad && (= K\varphi^c \text{ in } \mathcal{L}) \\
(B\varphi)^c &:= P(\varphi^c) > c \quad && (= B^c\varphi^c \text{ in } \mathcal{L})
\end{aligned}$$

Since we have seen that the probabilistic belief operator B^c is not a normal modal operator (Theorem 3.9(1)), we opt for a neighborhood semantics for $\mathcal{L}_{\mathsf{KB}}$ [Che80, Ch. 7] with an epistemic twist.

Definition 4.3. An *epistemic neighborhood model* is a structure

$$\mathcal{M} = (W, R, V, N)$$

satisfying the following.

- (W, R, V) is a finite single-agent S5 Kripke model (as in Definition 2.1). As before, we let

  $$[w] := \{v \in W \mid wRv\}$$

  denote the equivalence class of world w. This is the set of worlds the agent cannot distinguish from w.

- $N : W \to \wp(\wp(W))$ is a *neighborhood function* that assigns to each world $w \in W$ a collection $N(w)$ of sets of worlds—each such set called a *neighborhood* of w—subject to the following conditions.

  (kbc) $\forall X \in N(w) : X \subseteq [w]$.

  (kbf) $\emptyset \notin N(w)$.

  (n) $[w] \in N(w)$.

  (a) $\forall v \in [w] : N(v) = N(w)$.

  (kbm) $\forall X \subseteq Y \subseteq [w] :$ if $X \in N(w)$, then $Y \in N(w)$.

A *pointed epistemic neighborhood model* is a pair $(\mathcal{M}, w)$ consisting of an epistemic neighborhood model $\mathcal{M}$ and a world w in M.

An epistemic neighborhood model is a variation of a neighborhood model that includes an epistemic component R. Intuitively, $[w]$ is the set of worlds the agent knows to be possible at w and each $X \in N(w)$ represents a proposition that the agent believes at w. The condition that R be an equivalence relation ensures that knowledge is closed under logical consequence, veridical (i.e., only true things can be known), positive introspective (i.e., the agent knows what she knows), and negative introspective (i.e., the agent knows what she does not know).

Property (kbc) ensures that the agent does not believe a proposition $X \subseteq W$ that she knows to be false: if X contains a world in $w' \in (W - [w])$ that the agent knows is not possible with respect to the actual world w, then she knows that X cannot be the case and hence she does not believe X. Property (kbf) ensures that no logical falsehood is believed, while Property (n) ensures that every logical truth is believed. Property (a) ensures that X is believed if and only if it is known that X is believed. Property (kbm) says that belief is monotonic: if an agent believes X, then she believes all propositions $Y \supseteq X$ that follow from X.

We now turn to the definition of truth for the language $\mathcal{L}_{\mathsf{KB}}$.

16

Definition 4.4. Let $\mathcal{M} = (W, R, V, N)$ be an epistemic neighborhood model. We define a binary truth relation $\models_n$ between a pointed epistemic neighborhood model $(\mathcal{M}, w)$ and $\mathcal{L}_{KB}$-formulas and a function $[\![\cdot]\!]_n^{\mathcal{M}} : \mathcal{L}_{KB} \to \wp(W)$ as follows.

$$
\begin{aligned}
[\![\varphi]\!]_n^{\mathcal{M}} &:= \{v \in W \mid \mathcal{M}, v \models_n \varphi\} \\
\mathcal{M}, w \models_n p \quad &\text{iff} \quad p \in V(w) \\
\mathcal{M}, w \models_n \neg\varphi \quad &\text{iff} \quad \mathcal{M}, w \not\models_n \varphi \\
\mathcal{M}, w \models_n \varphi \wedge \psi \quad &\text{iff} \quad \mathcal{M}, w \models_n \varphi \text{ and } \mathcal{M}, w \models_n \psi \\
\mathcal{M}, w \models_n K\varphi \quad &\text{iff} \quad [w] \subseteq [\![\varphi]\!]_n^{\mathcal{M}} \\
\mathcal{M}, w \models_n B\varphi \quad &\text{iff} \quad [w] \cap [\![\varphi]\!]_n^{\mathcal{M}} \in N(w)
\end{aligned}
$$

Validity of $\varphi \in \mathcal{L}_{KB}$ in an epistemic neighborhood model $\mathcal{M}$, written $\mathcal{M} \models_n \varphi$, means that $\mathcal{M}, w \models_n \varphi$ for each world $w \in W$. Validity of $\varphi \in \mathcal{L}_{KB}$, written $\models_n \varphi$, means that $\mathcal{M} \models_n \varphi$ for each epistemic neighborhood model $\mathcal{M}$. For a class $\mathcal{C}$ of epistemic neighborhood models, we write $\mathcal{C} \models_n \varphi$ to mean that $\mathcal{M} \models_n \varphi$ for each $\mathcal{M} \in \mathcal{C}$.

Intuitively, $K\varphi$ is true at w iff φ holds at all worlds epistemically possible with respect to w, and $B\varphi$ holds at w iff the epistemically possible φ-worlds make up a neighborhood of w. Note that it follows from this definition that the dual for belief $\check{B}\varphi$ is true at w iff $[w] \cap [\![\neg\varphi]\!]_n^{\mathcal{M}} \notin N(w)$. The latter says that the epistemically possible $\neg\varphi$-worlds do not make up a neighborhood of w.

4.1 Neighborhood and Probability Model Agreement

Epistemic neighborhood models describe agent knowledge and belief. Epistemic probability models can be used for the same purpose along the lines we have discussed above once we establish a belief threshold $c \in (0, 1) \cap \mathbb{Q}$. This gives rise to a natural question: is there some sense in which these two models for knowledge and belief can be seen to agree?

Definition 4.5 (Model Agreement). Let $\mathcal{M} = (W, R, V, N)$ be an epistemic neighborhood model. For a threshold $c \in (0, 1) \cap \mathbb{Q}$, to say that a probability measure $P : \wp(W) \to [0, 1]$ *agrees with* $\mathcal{M}$ *for threshold* c means we have the following:

- P satisfies full support (i.e., $P(w) \neq 0$ for each $w \in W$); and

- for each $w \in W$ and $X \subseteq [w]$, we have

$$
X \in N(w) \quad \text{iff} \quad P_w(X) := P(X|[w]) > c \ .
$$

To say that an epistemic probability model $\mathcal{M}' = (W', R', V', P')$ *agrees with* $\mathcal{M}$ *for threshold* c means that $(W', R', V') = (W, R, V)$ and P' agrees with $\mathcal{M}$ for threshold c. If the threshold c is not mentioned, it is assumed that $c = \frac{1}{2}$.

Agreement for threshold c between an epistemic neighborhood model and an epistemic probability model makes the translation $c : \mathcal{L}_{KB} \to \mathcal{L}$ (Definition 4.2) truth-preserving.

Theorem 4.6 (Agreement). Fix $c \in (0, 1) \cap \mathbb{Q}$, an epistemic neighborhood model $\mathcal{M}$, and an epistemic probability model $\mathcal{M}'$. If $\mathcal{M}$ and $\mathcal{M}'$ agree for threshold c, then we have for each $\varphi \in \mathcal{L}_{\mathsf{KB}}$ that

$$\mathcal{M}, w \models_\mathsf{n} \varphi \quad \text{iff} \quad \mathcal{M}', w \models_\mathsf{p} \varphi^c \ .$$

Proof. Induction on the structure of $\varphi \in \mathcal{L}_{\mathsf{KB}}$. The non-modal cases are obvious.

We first consider knowledge formulas. Assume $\mathcal{M}, w \models_\mathsf{n} K\psi$. This means $[w] \subseteq [\![\psi]\!]_\mathsf{n}^\mathcal{M}$. Applying the induction hypothesis, this is equivalent to $[w] \subseteq [\![\psi^c]\!]_\mathsf{p}^{\mathcal{M}'}$. By full support, the latter holds if and only if

$$P_w([\![\psi^c]\!]_\mathsf{p}^{\mathcal{M}'}) = \frac{P([\![\psi^c]\!]_\mathsf{p}^{\mathcal{M}'} \cap [w])}{P([w])} = 1 \ ,$$

which is what it means to have $\mathcal{M}', w \models_\mathsf{p} P(\psi^c) = 1$. Since $P(\psi^c) = 1$ is what is abbreviated by $(K\psi)^c$, the result follows.

Now we move to belief formulas. Assume $\mathcal{M}, w \models_\mathsf{n} B\psi$. This means that $[w] \cap [\![\psi]\!]_\mathsf{n}^\mathcal{M} \in N(w)$. Since $\mathcal{M}'$ agrees with $\mathcal{M}$, the latter holds iff $P_w([w] \cap [\![\psi^c]\!]_\mathsf{p}^{\mathcal{M}'}) > c$. But this is equivalent to $P_w([\![\psi^c]\!]_\mathsf{p}^{\mathcal{M}'}) > c$, which is what it means to have $\mathcal{M}', w \models_\mathsf{p} P(\psi^c) > c$. Since $P(\psi^c) > c$ is what is abbreviated by $(B\psi)^c$, the result follows. $\qquad\square$

4.2 Probability Measures on Epistemic Neighborhood Models

In this subsection, we take up the question of agreement between epistemic probability models and epistemic neighborhood models from the point of view of the latter: given an epistemic neighborhood model and a threshold c, can we find an agreeing epistemic probability model for this threshold? As we will see, we have a full answer only for the case $c = \frac{1}{2}$. The case for $c \neq \frac{1}{2}$ is open, though we will have some comments on this in the conclusion of the paper.

To begin, we adapt an example due to Walley and Fine [WF79] to show that not every epistemic neighborhood model gives rise to an agreeing probability measure.

Theorem 4.7 ([WF79]). There exists an epistemic neighborhood model $\mathcal{M}$ that has no agreeing probability measure for any threshold $c \in (0, 1) \cap \mathbb{Q}$.

Proof. We adapt Example 2 from [WF79, pp. 344-345] to the present setting. Fix $c \in (0, 1) \cap \mathbb{Q}$. Let $\mathbf{P} := \{a, b, c, d, e, f, g\}$. Define:

$$\mathcal{X} := \{efg, abg, adf, bde, ace, cdg, bcf\} \ ,$$
$$\mathcal{Y} := \{abcd, cdef, bceg, acfg, bdfg, abef, adeg\} \ .$$

Notation: in the above sets, xyz denotes $\{x, y, z\}$, and $wxyz$ denotes $\{w, x, y, z\}$. Now define

$$\mathcal{N} := \{X' \mid \exists X \in \mathcal{X} : X \subseteq X' \subseteq \mathbf{P}\} \ .$$

Let $\mathcal{M} := (W, R, V, N)$ be defined by $W := \mathbf{P}$, $R := W \times W$, $V(w) := \{w\}$ for each $w \in \mathbf{P}$, and $N(w) := \mathcal{N}$ for each $w \in W$. It is straightforward to verify that $\mathcal{M}$ is an epistemic neighborhood model and that $\mathcal{Y} \cap \mathcal{N} = \emptyset$.

Toward a contradiction, suppose there exists a probability measure P that agrees with $\mathcal{M}$. Since each letter $p \in W$ occurs in exactly three of the seven members of $\mathcal{X}$, we have:

$$\sum_{X \in \mathcal{X}} P(X) = \sum_{p \in W} 3 \cdot P(\{p\}) \ .$$

Since each letter $p \in W$ occurs in exactly four of the seven members of $\mathcal{Y}$, we have:

$$\sum_{Y \in \mathcal{Y}} P(Y) = \sum_{p \in W} 4 \cdot P(\{p\}) > \sum_{X \in \mathcal{X}} P(X) \ .$$

On the other hand, since $\mathcal{Y} \cap \mathcal{N} = \emptyset$, no member of $\mathcal{Y}$ is a neighborhood of $\mathcal{M}$ and therefore it follows by the agreement of P with $\mathcal{M}$ that we have $P(Y) \leq c < P(X)$ for each $Y \in \mathcal{Y}$ and $X \in \mathcal{X}$. But then

$$\sum_{Y \in \mathcal{Y}} P(Y) < \sum_{X \in \mathcal{X}} P(X) \ ,$$

and we have reached a contradiction. Conclusion: no such P exists. $\qquad\square$

Question: what are the additional restrictions on the neighborhood function that one must impose in order to guarantee the existence of an agreeing probability measure for a given threshold $c \in (0,1) \cap \mathbb{Q}$? For $c = \frac{1}{2}$, the restrictions are known. For thresholds $c \neq \frac{1}{2}$, the question is open.

The restrictions needed for $c = \frac{1}{2}$ were studied first in the form of a purely probabilistic semantics (i.e., something like epistemic probability models and not something like our epistemic neighborhood models). To our knowledge, Lenzen's [Len80] is the first complete study of the restrictions needed in such a purely probabilistic framework over a unary modal language similar to $\mathcal{L}_{\mathsf{KB}}$. The conditions Lenzen proposed are targeted to satisfy the conditions of a theorem due to Scott, which is the key result that gives rise to a probability measure in the completeness proof for Lenzen's logic. Here we state the required restrictions in the language of our epistemic neighborhood models. Later we will make the link with Lenzen's axiomatic system when we consider axiomatic theories in the language $\mathcal{L}_{\mathsf{KB}}$ targeted to our epistemic neighborhood models.

Definition 4.8 (Extra Properties for "Mid-Threshold" Models). Let $\mathcal{M} = (W, R, V, N)$ be an epistemic neighborhood model. For $m \in \mathbb{Z}^+$ and sets of worlds $X_1, \ldots, X_m$ and $Y_1, \ldots, Y_m$, we write

$$X_1, \ldots, X_m \mathbb{I} Y_1, \ldots, Y_m \tag{8}$$

to mean that for each $v \in W$, the number of X_i's containing v is less than or equal to the number of Y_i's containing v. This is the semantic counterpart of the formula from Definition 3.7. We may write $(X_i \mathbb{I} Y_i)_{i=1}^m$ as an abbreviation for (8). Also, we write $(X_i \mathbb{E} Y_i)_{i=1}^m$ to mean that both $(X_i \mathbb{I} Y_i)_{i=1}^m$ and $(Y_i \mathbb{I} X_i)_{i=1}^m$ hold, and we allow the notation with $\mathbb{E}$ to be used in a form as in (8). The following is a list of properties that $\mathcal{M}$ may satisfy.

(d) $\forall X \in N(w) : [w] - X \notin N(w)$.

(sc) $\forall X, Y \subseteq [w]$: if $[w] - X \notin N(w)$ and $X \subsetneq Y$, then $Y \in N(w)$.

(scott) $\forall m \in \mathbb{Z}^+, \forall X_1, \ldots, X_m, Y_1, \ldots, Y_m \subseteq [w] :$

$$
\begin{aligned}
\text{if} \quad & X_1, \ldots, X_m \mathbb{I} Y_1, \ldots, Y_m \quad \text{and} \\
& X_1 \in N(w) \quad \text{and} \\
& \forall i \in \{2, \ldots, m\} : [w] - X_i \notin N(w) \ , \\
\text{then} \quad & \exists j \in \{1, \ldots, m\} : Y_j \in N(w) \ .
\end{aligned}
$$

To say an epistemic neighborhood model is *mid-threshold* means it satisfies (d), (sc), and (scott). We may drop the word "epistemic" in referring to mid-threshold epistemic neighborhood models. Pointed versions of mid-threshold neighborhood models are defined in the obvious way.

Property (d) ensures that beliefs are consistent in the sense that the agent does not believe both X and its complement $[w] - X$. Property (sc) is a form of "strong commitment": if the agent does not believe the complement $[w] - X$, then she must believe any strictly weaker Y implied by X. Property (scott) is a version of the syntactic scheme (Scott) from Definition 3.8.

Let us return to the model $\mathcal{M}$ from the proof of Theorem 5.5. It is easy to see that for no $X_i \in \mathcal{X}$ do we have $W - X_i \in N(a) = \mathcal{N}$. So, numbering the members of $\mathcal{X}$ as $X_1, \ldots, X_7$ and the members of $\mathcal{Y}$ as $Y_1, \ldots, Y_7$, we see that $\mathcal{M}$ satisfies

$$(X_i \mathbb{I} Y_i)_{i=1}^7, \quad X_1 \in N(a), \text{ and } \forall i \in \{2, \ldots, 7\} : W - X_i \notin N(a) \ ,$$

which is the antecedent of property (scott) from Definition 4.8. However, $\mathcal{M}$ does not satisfy

$$\exists j \in \{1, \ldots, 7\} : Y_j \in N(a) \ ,$$

which is the corresponding consequent of the indicated instance of (scott). So we see that if we were to restrict ourselves to the class of epistemic neighborhood models satisfying this property, we would no longer be able to use $\mathcal{M}$ as a counterexample to the claim that not every epistemic neighborhood model gives rise to an agreeing probability measure. Of course ruling out $\mathcal{M}$ as a counterexample to this claim does not prove the claim. However, utilizing (scott) in conjunction with (d) and (sc), we are able to prove the claim. This proof makes crucial use of a theorem due to Scott that is closely related to [Sco64, Theorem 4.1].

In preparation for the statement of Scott's theorem, we recall some well-known notions from linear algebra. For a nonempty set S, let $L(S)$ denote the S-dimensional real vector space whose vectors consist of functions $x : S \to \mathbb{R}$ and whose operations of vector addition and scalar multiplication are defined coordinate-wise: given vectors $x, y : S \to \mathbb{R}$ and a scalar real $r \in \mathbb{R}$, the vector $(x + y) : S \to \mathbb{R}$ is defined by $(x + y)(s) := x(s) + y(s)$ for each coordinate $s \in S$ and the vector $(r \cdot x) : S \to \mathbb{R}$ is defined by $(r \cdot x)(s) := r \cdot x(s)$ for each coordinate $s \in S$. Note that we have just used the usual notational overloading wherein the $+$ or $\cdot$ symbol on one side of an equation refers to the vector operation, and yet the same symbol on the other side of the same equation refers to the operation in $\mathbb{R}$. Other common notational abbreviations such as omission of $\cdot$'s and writing $-x$ for $(-1) \cdot x$ will be used. To say that a vector $x : S \to \mathbb{R}$ is *rational* means that all of its coordinates (i.e., values) are rational numbers. To say a set $X \subseteq L(S)$ of vectors is rational means that every

vector in X is rational, and to say that X is *symmetric* means that $X = -X := \{-x \mid x \in X\}$. A *linear functional on* $L(S)$ is a function $f : L(S) \to \mathbb{R}$ satisfying the following property of *linearity*: for each $r_1, r_2 \in \mathbb{R}$ and $x, y \in L(S)$, we have $f(r_1 x + r_2 y) = r_1 \cdot f(x) + r_2 \cdot f(y)$.

Theorem 4.9 ([Sco64, Theorem 1.2]). Let S be a finite nonempty set and X be a finite, rational, symmetric subset of $L(S)$. For each $N \subseteq X$, there exists a linear functional f on $L(S)$ that *realizes* N, meaning

$$N = \{x \in X \mid f(x) \geq 0\} \ ,$$

if and only if the following conditions are satisfied:

- for each $x \in X$, we have $x \in N$ or $-x \in N$; and

- for each integer $n \geq 0$ and $x_0, \ldots, x_n \in N$, we have

$$\sum_{i=0}^{n} x_i = 0 \quad \Rightarrow \quad -x_0 \in N \ .$$

We use this theorem to show that mid-threshold models always give rise to an agreeing probability measure. That is, the neighborhood function of mid-threshold models picks out exactly those neighborhoods that may be assigned a probability exceeding $\frac{1}{2}$. Many of the key ideas of the proof of the following result are due to Lenzen [Len80]. However, the argument we present here has been rewritten in a streamlined, modern form and in the language of our epistemic neighborhood models. Despite this difference (and the necessary work we had to undertake to translate these results into this modern form), we are happy to credit Professor Lenzen for the following result.

Theorem 4.10 ([Len80]). Let $\mathcal{M} = (W, R, V, N)$ be a mid-threshold epistemic neighborhood model. There exists a probability measure $P : \wp(W) \to [0, 1]$ agreeing with $\mathcal{M}$ for threshold $\frac{1}{2}$; that is,

- P satisfies full support (i.e., $P(w) \neq 0$ for each $w \in W$); and

- for each $w \in W$ and $X \subseteq [w]$, we have

$$X \in N(w) \quad \text{iff} \quad P_w(X) := P(X|[w]) > \tfrac{1}{2} \ .$$

Proof. We credit Lenzen [Len80] for this proof, though we herein provide an original reformulation of his work within the setting of the epistemic neighborhood models introduced in this paper. Proceeding, for $w \in W$, define $S_w := [w]$. For each $X \subseteq S_w$, define the relative complement $X' := S_w - X$ and let $\iota(X) : S_w \to \{0, 1\}$ be the characteristic function of X:

$$\iota(X)(s) := \begin{cases} 1 & \text{if } s \in X, \\ 0 & \text{otherwise.} \end{cases}$$

We consider the following finite subsets of $L(S_w)$:

$$\begin{aligned}
\mathcal{A}_w &:= \{\iota(X) \mid X \subseteq S_w\} \ , \\
\mathcal{B}_w &:= \{\iota(X) - \iota(X') \mid X \subseteq S_w \ \& \ X' \notin N(w)\} \ , \\
\mathcal{N}_w &:= \mathcal{A}_w \cup \mathcal{B}_w \ , \\
\mathcal{X}_w &:= \mathcal{N}_w \cup (-\mathcal{N}_w) \ .
\end{aligned}$$

It is easy to see that $\mathcal{N}_w \subseteq \mathcal{X}_w$ and that $\mathcal{X}_w$ is a finite, rational, and symmetric subset of $L(S_w)$. We wish to show that $\mathcal{N}_w$ and $\mathcal{X}_w$ satisfy the conditions of Theorem 4.9. First, we note that $x \in \mathcal{X}_w$ implies $x \in \mathcal{N}_w$ or $-x \in \mathcal{N}_w$ by the definition of $\mathcal{X}_w$.

For the second condition of Theorem 4.9, suppose we are given an integer $n \geq 0$ such that $x_0, \ldots, x_n \in \mathcal{N}_w$ and $\sum_{i=0}^{n} x_i = 0$. We wish to show that $-x_0 \in \mathcal{N}_w$. Proceeding, there exists an integer ℓ satisfying:

$$\begin{aligned}
0 \leq i \leq \ell \quad &\text{implies} \quad x_i = \iota(X_i) - \iota(X_i') \in \mathcal{B}_w \quad \text{, and} \\
\ell < i \leq n \quad &\text{implies} \quad x_i = \iota(X_i) \in \mathcal{A}_w \ .
\end{aligned}$$

Toward a contradiction, assume there exists $i > \ell$ with $x_i \neq 0$. Then for $x^* := \sum_{i=\ell+1}^{n} x_i$, we have $x^*(s) \geq 0$ for all $s \in S_w$, and there exists $s^* \in S_w$ with $x^*(s^*) > 0$. Hence

$$\sum_{i=0}^{\ell} x_i = \sum_{i=0}^{\ell} \big(\iota(X_i) - \iota(X_i')\big) = -x^* \ ,$$

where $-x^*(s^*) < 0$ and $-x^*(s) \leq 0$ for all $s \in S_w$. So for each $s \in S_w$, the number of the sets in the list $X_0', \ldots, X_\ell'$ containing s is greater than or equal to the number of the sets in the list $X_0, \ldots, X_\ell$ containing s. Further, s^* is a member of strictly more sets in the former list than those in the latter. By renumbering, we may assume that $s^* \in X_0' - X_0$. Then we have

$$X_0 \cup \{s^*\}, X_1, \ldots, X_\ell \mathbb{I} X_0', X_1', \ldots, X_\ell' \ .$$

Since $X_0', \ldots, X_\ell' \notin N(w)$, it follows by (scott) that $X_0 \cup \{s^*\} \notin N(w)$. But $X_0 \subsetneq X_0 \cup \{s^*\} \notin N(w)$ and $X_0' \notin N(w)$, which violates (sc). Conclusion: $i > \ell$ implies $x_i = 0$. But then we have $\sum_{i=0}^{n} x_i = \sum_{i=0}^{\ell} x_i$. Since $x_i = \iota(X_i) - \iota(X_i')$ for $i \leq \ell$, it follows that $\sum_{i=0}^{\ell} \iota(X_i) = \sum_{i=0}^{\ell} \iota(X_i')$. But the latter is what it means to have $(X_i \mathbb{E} X_i')_{i=0}^{\ell}$. Since $X_i' \notin N(w)$ for $i \leq \ell$ by the definition of $\mathcal{B}_w$, it follows by (scott) that $X_0 \notin N(w)$. But then $\iota(X_0') - \iota(X_0) = -x_0 \in \mathcal{B}_w \subseteq \mathcal{N}_w$, as desired.

So we may apply Theorem 4.9: there exists a linear functional f_w on $L(S_w)$ that realizes $\mathcal{N}_w$. That is,

$$\mathcal{N}_w = \{x \in \mathcal{X}_w \mid f_w(x) \geq 0\} \ .$$

Define $g_w : \wp(S_w) \to \mathbb{R}$ by the composition $g_w(X) := f_w(\iota(X))$. This function satisfies a few important properties.

1. $X \in N(w)$ iff $g_w(X) > g_w(X')$.

   Suppose $X \in N(w)$. Then $X' \notin N(w)$ by (d). Hence $\iota(X) - \iota(X') \in \mathcal{B}_w$ and $\iota(X') - \iota(X) \notin \mathcal{B}_w$. Since $S_w \in N(w)$ by (n), it follows that $X \neq \emptyset = S_w'$. But then the coordinates of

22

$\iota(X') - \iota(X)$ contain at least one 1 and at least one -1. Since every $x \in \mathcal{A}_w$ has coordinates that are 1's or 0's only, it follows that $\iota(X') - \iota(X) \notin \mathcal{N}_w$. As $\iota(X) - \iota(X') \in \mathcal{B}_w \subseteq \mathcal{N}_w$ and f_w is linear and realizes $\mathcal{N}_w$, it follows that $g_w(X) \geq g_w(X')$ and $g_w(X') \not\geq g_w(X)$. That is, $g_w(X) > g_w(X')$.

Conversely, suppose $g_w(X) > g_w(X')$. Since f_w is linear and realizes $\mathcal{N}_w$, it follows that $\iota(X') - \iota(X) \notin \mathcal{N}_w \supseteq \mathcal{B}_w$. Applying the definition of $\mathcal{B}_w$, we have $X \in N(w)$.

2. $g_w(S_w) > g_w(\emptyset) = 0$.

   We have $g_w(\emptyset) = f_w(0) = 0$ by the linearity of f_w. Since $S_w \in N(w)$ by (n), it follows that $g_w(S_w) > g_w(\emptyset)$ by property 1.

3. If $0 \leq g_w(X) \leq g_w(S_w)$.

   Since $\iota(X) \in \mathcal{A}_w \subseteq \mathcal{N}_w$ and f_w realizes $\mathcal{N}_w$, we have $g_w(X) \geq 0$. So each $X \subseteq S_w$ satisfies $g_w(X) \geq 0$. From this it follows by the linearity of f_w that for each $X \subseteq S_w$, we have

$$g_w(X) = \sum_{v \in X} g_w(\{v\}) \leq \sum_{v \in S_w} g_w(\{v\}) = g_w(S_w) \ .$$

4. If $X, Y \subseteq S_w$ and $X \cap Y = \emptyset$, then $g_w(X \cup Y) = g_w(X) + g_w(Y)$.

   By the linearity of f_w.

5. $\emptyset \neq X \subseteq S_w$ implies $g_w(X) > 0$.

   Suppose $\emptyset \neq X \subseteq S_w$. By property 2, it suffices to prove the result for $X \neq S_w$. Toward a contradiction, assume $g_w(X) = 0$ for $\emptyset \subsetneq X \subsetneq S_w$. By property 4, we have $g_w(S_w) = g_w(X) + g_w(X') = g_w(X')$. Since f_w is linear and realizes $\mathcal{N}_w$ and

$$\iota(X') - \iota(S_w) = -(\iota(S_w) - \iota(X')) = -\iota(X) \in \mathcal{X}_w \ ,$$

   we obtain $-\iota(X) \in \mathcal{N}_w$. But $\emptyset \subsetneq X \subsetneq S_w$ implies that $-\iota(X)$ has coordinates containing at least one -1 and at least one 0. Since members of $\mathcal{A}_w$ have coordinates made up of 0's and 1's, members of $\mathcal{B}_w$ have coordinates made up of -1's and 1's, and $\mathcal{N}_w = \mathcal{A}_w \cup \mathcal{B}_w$, it cannot be the case that $-\iota(X) \in \mathcal{N}_w$. Contradiction. Conclusion: $g_w(X) > 0$.

Now take $v \in [w]$. Since $N(v) = N(w)$ by (a), it follows that g_w also realizes $\mathcal{N}_v$. So, letting $[W]$ be the set $\{[w] \mid w \in W\}$ of equivalence classes, let $h : [W] \to W$ be a choice function that selects for each class $[w] \in [W]$ a representative $h([w]) \in [w]$. Using a notational overloading that ought to be harmless, we define a new function $h_w : \wp([w]) \to \mathbb{R}$ by setting $h_w(X) := g_{h([w])}(X)$. Obviously, $v \in [w]$ implies $h_v = h_w$. Finally, we define $P : \wp(W) \to [0, 1]$ by

$$P(X) := \sum_{[w] \in [W]} \frac{h_w(X \cap [w])}{h_w([w])} \ .$$

Note that by property 2, the denominator $h_w([w])$ is always nonzero.

23

We prove that P is a probability measure on $\wp(W)$ satisfying full support. First, P satisfies the Kolmogorov axioms over the finite algebra $\wp(W)$: we have $P(X) \geq 0$ by property 3, $P(W) = 1$ by property 2 and the definition of P, and $P(X \cup Y) = P(X) + P(Y)$ for disjoint X and Y by property 4 and the definition of P. Second, full support follows by property 5.

Finally, for $X \subseteq [w]$, we have by property 1 that $X \in N(w)$ iff $h_w(X) > h_w(X')$. But the latter holds iff we have (making use of property 4) that

$$2 \cdot h_w(X) > h_w(X) + h_w(X') = h_w([w]) \ .$$

By property 2, the definition of P, and the fact that $X \subseteq [w]$, the above inequality holds iff

$$P(X) = \frac{h_w(X)}{h_w([w])} > \tfrac{1}{2} \ .$$

$\square$

Corollary 4.11. Let $\mathcal{M} = (W, R, V, N)$ be a mid-threshold epistemic neighborhood model. There exists an epistemic probability model $\mathcal{N} = (W, R, V, P)$ that agrees with $\mathcal{M}$ for threshold $\tfrac{1}{2}$.

Proof. Let P be the measure given by Theorem 4.10. $\square$

4.3 Epistemic Neighborhood Models from Probability Measures

In the last subsection, we investigated the question of whether an epistemic neighborhood model gives rise to an agreeing epistemic probability model. In this section, we look at this question the other way around: given an epistemic probability model and a threshold c, is there an agreeing epistemic neighborhood model? As we will see, the answer is always "yes."

Definition 4.12. Given an epistemic probability model $\mathcal{M} = (W, R, V, P)$ and a threshold $c \in [\tfrac{1}{2}, 1) \cap \mathbb{Q}$, we define the structure $\mathcal{M}^c := (W, R, V, N^c)$ by setting

$$N^c(w) := \{X \subseteq [w] \mid P_w(X) > c\} \ .$$

Intuitively, the agent believes a proposition X at world w (i.e., $X \in N^c(w)$) if and only if X is epistemically possible (i.e., $X \subseteq [w]$) and the probability she assigns to X at world w exceeds the threshold (i.e., $P_w(X) > c$).

Lemma 4.13 (Correctness). Fix $c \in (0, 1) \cap \mathbb{Q}$. If $\mathcal{M}$ is an epistemic probability model, then $\mathcal{M}^c$ is an epistemic neighborhood model. Furthermore, $\mathcal{M}^{\frac{1}{2}}$ is a mid-threshold neighborhood model.

Proof. We verify that N^c satisfies the required properties.

- For (kbc), $X \in N^c(w)$ implies $X \subseteq [w]$ by definition.

- For (kbf), $P_w(\emptyset) = 0 < c$, so $\emptyset \notin N^c(w)$.

- For (n), $P_w([w]) = 1 > c$, so $[w] \in N^c(w)$.

- For (a), suppose $X \in N^c(w)$ and $v \in [w]$. Then $P_w(X) > c$. Since $v \in [w]$ implies $[w] = [v]$, we have

$$P_w(X) = \frac{P(X \cap [w])}{P([w])} = \frac{P(X \cap [v])}{P([v])} = P_v(X) \ .$$

  Hence $P_v(X) > c$, so $X \in N^c(v)$.

- For (kbm), suppose $X \in N^c(w)$. Then $P_w(X) > c$. Hence if Y satisfies $X \subseteq Y \subseteq [w]$, we have $P_w(Y) > c$ and so $Y \in N^c(w)$.

So $\mathcal{M}^c$ is an epistemic neighborhood model. We now show that $\mathcal{M}^{\frac{1}{2}}$ satisfies the additional required properties.

- For (d), assume $c \in [\frac{1}{2}, 1) \cap \mathbb{Q}$ and $X \in N^c(w)$. Then $P_w(X) > c$, and therefore $P_w([w] - X) \leq 1 - c \leq c$. Hence $[w] - X \notin N^c(w)$.

- For (sc), assume $X' := [\Gamma] - X \notin N^{\frac{1}{2}}(w)$ and $X \subsetneq Y \subseteq [\Gamma]$. From the first assumption, we have $P_w(X') \leq \frac{1}{2}$, and therefore that $P_w(X) \geq \frac{1}{2}$. Applying the second assumption, $P_w(Y) > P_w(X) \geq \frac{1}{2}$, and hence $X \in N^{\frac{1}{2}}(w)$.

- For (scott), we assume $c \in (0, \frac{1}{2}] \cap \mathbb{Q}$ along with the following:

$$(X_i \mathbb{I} Y_i)_{i=1}^m \tag{9}$$
$$X_1 \in N^c(w) \tag{10}$$
$$\forall i \in \{2, \ldots, m\} : [\Gamma] - X_i \notin N^c(w) \tag{11}$$

  From (9) it follows that

$$P_w(X_1) + \cdots + P_w(X_m) \leq P_w(Y_1) + \cdots + P_w(Y_m) \tag{12}$$

  The argument for this is similar to an argument for (7) in proof of Theorem 3.9(11). From (10), we have $P_w(X_1) > c$. From (11), we have for each $i \in \{2, \ldots, m\}$ that $P_w([w] - X_i) \leq c$ and therefore that $P_w(X_i) \geq 1 - c \geq c$ since $c \in (0, \frac{1}{2}] \cap \mathbb{Q}$. Hence the left side of (12) exceeds mc. Since every summand on the right side of the inequality is positive and $mc > 0$, it follows that at least one member of the right side of (12) must exceed c. That is, there exists $j \in \{1, \ldots, m\}$ such that $P_w(Y_j) > c$ and hence $Y_j \in N^c(w)$. $\qquad\square$

Theorem 4.14. Let $c \in (0, 1) \cap \mathbb{Q}$ and $\mathcal{M} = (W, R, V, P)$ be an epistemic probability model. The epistemic neighborhood model $\mathcal{M}^c = (W, R, V, N^c)$ agrees with $\mathcal{M}$ for threshold c.

Proof. By definition of N^c. $\qquad\square$

5   Calculi for Belief as Willingness to Bet

We now consider an axiomatic link both with epistemic neighborhood models and with epistemic probability models. We study two calculi: the calculus KB of epistemic neighborhood models, and the calculus KB.5 of mid-threshold neighborhood models. Regarding the probability interpretation, KB is sound for every threshold but not complete for any threshold. KB.5 is both sound and complete for the probability interpretation with threshold $c = \frac{1}{2}$.

KB.5 is our modern reformulation of Lenzen's [Len80] calculus for the logic of knowledge (i.e., Lenzen's "acceptance") as probabilistic certainty and belief as probability exceeding threshold $\frac{1}{2}$. Lenzen's intended semantic structures are something like epistemic probability models. Our intended semantic structures are our mid-threshold neighborhood models, though there is a natural link with epistemic probability models via Theorem 4.10. In fact, many of the main ideas of our proof of Theorem 4.10 are not doubt translations of Lenzen's ideas into the language of our epistemic neighborhood models. Since we have rewritten all proofs using our own approach and modern modal notions, it is difficult to determine whether we have introduced novel mathematical results on top of Lenzen's existing work, though we suspect that anything new we may have added along these lines (excluding of course epistemic neighborhood models themselves and all related results except Theorem 4.10) may be slight at best. Therefore, we are happy to credit Professor Lenzen for the probabilistic soundness and completeness of KB.5 and for Theorem 4.10. Nevertheless, we do think that it is worth our effort to provide this modern reformulation of his results. In particular, we believe that in using semantic structures more familiar to the modern modal logician, our modern reformulation of Lenzen's results will make the mathematical details of Lenzen's work more accessible to a modern English-language audience. We also hope that our use of the modal neighborhood structures will suggest directions for further study of qualitative probability via tools from modal logic.

Definition 5.1. We define the following theories in the language $\mathcal{L}_{\mathsf{KB}}$.

- KB is defined in Table 1.

- KB.5 is obtained from KB by adding (D), (SC), and (Scott) from Table 2.

- KB.5$^-$ is obtained from KB.5 by omitting (BF) and (KBM).

We will see later in Theorem 5.6 that KB.5 and KB.5$^-$ derive the same theorems.

5.1   Results for the Basic Calculus KB

The following result shows that if we restrict attention to provable statements whose only modality is single-agent belief $B\varphi$, then KB is an extension of the minimal modal logic EMN45 $+ \neg B\bot =$ EMN45 $+$ (BF) obtained by adding S5-knowledge and the knowledge-belief connection principles (Ap), (An), and (KBM).[2] The modal theory KB.5, which we will see is equivalent to KB.5$^-$, is a

---

[2]EMN45 $+$ (BF) is the logic of single-agent belief (without knowledge) having Schemes (CL) (Table 1), M (Theorem 5.2(2)), (N) (Table 1), 4 (Theorem 5.2(5)), 5 (Theorem 5.2(6)), and (BF) (Table 1) along with Rules (MP)

<div style="border:1px solid">

AXIOM SCHEMES

(CL)	Schemes of Classical Propositional Logic
(KS5)	S5 axiom schemes for each K
(BF)	$\neg B\bot$
(N)	$B\top$
(Ap)	$B\varphi \to KB\varphi$
(An)	$\neg B\varphi \to K\neg B\varphi$
(KBM)	$K(\varphi \to \psi) \to (B\varphi \to B\psi)$

RULES

$$\frac{\varphi \to \psi \quad \varphi}{\psi} \ \text{(MP)} \qquad \frac{\varphi}{K\varphi} \ \text{(MN)}$$

</div>

Table 1: The theory KB

<div style="border:1px solid">

(D)	$B\varphi \to \check{B}\varphi$
(SC)	$\check{B}\varphi \wedge \check{K}(\neg\varphi \wedge \psi) \to B(\varphi \vee \psi)$
(Scott)	$[(\varphi_i \mathbb{I} \psi_i)_{i=1}^m \wedge B\varphi_1 \wedge \bigwedge_{i=2}^m \check{B}\varphi_i] \to \bigvee_{i=1}^m B\psi_i$

</div>

Table 2: Additional axiom schemes for the theory KB.5

knowledge-inclusive extension of $\mathsf{EMND45} + (\text{Scott})$ that adds the additional connection principle (SC).[3] In Section 5.2, we will show that KB.5 is the modal logic for probabilistic belief with threshold $c = \frac{1}{2}$.

Theorem 5.2 (KB Derivables). We have each of the following.

1. $\mathsf{KB} \vdash K\varphi \to B\varphi$.

   "Knowledge implies belief."

2. $\mathsf{KB} \vdash B(\varphi \wedge \psi) \to (B\varphi \wedge B\psi)$.

   This is "Scheme M" [Che80, Ch. 8].

3. $\mathsf{KB} \vdash K\varphi \wedge B\psi \to B(\varphi \wedge \psi)$.

   If the antecedent $K\varphi$ were replaced by $B\varphi$, then we would obtain "Scheme C" [Che80, Ch. 8]. So we do not have Scheme C outright but instead a knowledge-weakened version:

(Table 1) and RE (Theorem 5.2(12)). This is a "monotonic" system of modal logic satisfying positive and negative belief introspection (4 and 5) and the property (BF) that falsehood $\bot$ is not believed. See [Che80, Ch. 8] for details on naming minimal modal logics.

[3]$\mathsf{EMND45} + (\text{Scott})$ is $\mathsf{EMN45} + (\text{BF})$ minus Scheme (BF) plus Schemes (D) and (Scott) from Table 2.

in order to conclude belief of a conjunction from belief of one of the conjuncts, the other conjunct must be known (and not merely believed, as is required by the stronger, non-KB-provable Scheme C).

4. $\mathsf{KB} \vdash K(\varphi \to \psi) \to (\check{B}\varphi \to \check{B}\psi)$.

   This is the dual version of our (KBM).

5. $\mathsf{KB} \vdash B\varphi \to BB\varphi$.

   This is "Scheme 4" for belief [Che80, Ch. 8].

6. $\mathsf{KB} \vdash \neg B\varphi \to B\neg B\varphi$.

   This is "Scheme 5" for belief [Che80, Ch. 8].

7. $\mathsf{KB} \vdash B\varphi \leftrightarrow KB\varphi$.

   This says that belief and knowledge of belief are equivalent.

8. $\mathsf{KB} \vdash \neg B\varphi \leftrightarrow K\neg B\varphi$.

   This says that non-belief and knowledge of non-belief are equivalent.

9. $\mathsf{KB} \vdash \varphi$ implies $\mathsf{KB} \vdash B\varphi$.

   This is the rule of Modus Ponens (or Modal Necessitation), sometimes called "Rule RN" [Che80, Ch. 8].

10. $\mathsf{KB} \vdash \varphi \to \psi$ implies $\mathsf{KB} \vdash B\varphi \to B\psi$.

    This is "Rule RM" [Che80, Ch. 8].

11. $\mathsf{KB} \vdash \varphi \to \psi$ implies $\mathsf{KB} \vdash \check{B}\varphi \to \check{B}\psi$.

    This is the dual version of RM.

12. $\mathsf{KB} \vdash \varphi \leftrightarrow \psi$ implies $\mathsf{KB} \vdash B\varphi \leftrightarrow B\psi$.

    This is "Rule RE" [Che80, Ch. 8].

13. $\mathsf{KB} \vdash \varphi \to \bot$ implies $\mathsf{KB} \vdash \neg B\varphi$.

    This says that no self-contradictory sentence is believed. This may be viewed as a certain generalization of (BF) (Table 1).

Proof. We reason in KB. For 1, we have $K\varphi \to K(\top \to \varphi)$ by elementary modal reasoning. But then from this, $B\top$ by (N), and $K(\top \to \varphi) \to (B\top \to B\varphi)$ by (KBM), it follows by classical reasoning that we have $K\varphi \to B\varphi$.

For 2, we derive

$$K((\varphi \wedge \psi) \to \varphi) \to (B(\varphi \wedge \psi) \to B\varphi) \tag{13}$$

by (KBM), and the antecedent of (13) by (CL) and (MN). Therefore, the consequent of (13) is derivable by (MN). By a similar argument, $B(\varphi \wedge \psi) \to B\psi$ is derivable. By classical reasoning, 2 is derivable.

For 3, we derive

$$K\varphi \to K(\psi \to (\varphi \wedge \psi)) \quad \text{and} \tag{14}$$

$$K(\psi \to (\varphi \wedge \psi)) \to (B\psi \to B(\varphi \wedge \psi)) \ . \tag{15}$$

(14) follows by S5 reasoning. (15) follows by (KBM). Applying classical reasoning to (14) and (15), we obtain

$$K\varphi \to (B\psi \to B(\varphi \wedge \psi)) \ ,$$

from which 3 follows by classical reasoning.

For 4, we derive

$$K(\varphi \to \psi) \to K(\neg\psi \to \neg\varphi) \quad \text{and} \tag{16}$$

$$K(\neg\psi \to \neg\varphi) \to (B\neg\psi \to B\neg\varphi) \ . \tag{17}$$

(16) follows by S5 reasoning. (17) follows by (KBM). Applying classical reasoning to (16) and (17), we obtain

$$K(\varphi \to \psi) \to (B\neg\psi \to B\neg\varphi) \ ,$$

from which 4 follows by classical reasoning (just contrapose the consequent).

5 follows by (Ap) and 1. 6 follows by (An) and 1. 7 follows by by (Ap) for the right-to-left and (KS5) for the left-to-right. 8 follows by (An) for the right-to-left and (KS5) for the left-to-right. 9 follows by (MN) and 1. 10 follows by (MN) and (KBM). 11 follows by contraposition, (MN), (KBM), and contraposition. 12 follows from 10 by classical reasoning.

For 13, we have

$$K(\varphi \to \bot) \to (B\varphi \to B\bot) \tag{18}$$

by (KBM). Therefore, if $\varphi \to \bot$ is provable, it follows by (MN) that the antecedent of (18) is as well. By (MP), the consequent $B\varphi \to B\bot$ is provable. Applying (BF) and classical reasoning, it follows by contraposition that $\neg B\varphi$ is provable. $\qquad \square$

Theorem 5.3 (KB Neighborhood Soundness and Completeness)**.** KB is sound and complete with respect to the class $\mathcal{C}$ of epistemic neighborhood models:

$$\forall \varphi \in \mathcal{L}_{\mathsf{KB}}: \quad \mathsf{KB} \vdash \varphi \quad \Leftrightarrow \quad \mathcal{C} \models_{\mathsf{n}} \varphi \ .$$

Proof. By induction on the length of derivation. We first verify soundness of the axioms.

- Validity of (CL) immediate. Validity of (KS5) follows because the R's are equivalence relations [BdRV01].

- Scheme (BF) is valid: $\models_{\mathsf{n}} \neg B\bot$.
  $[\![\bot]\!]_{\mathsf{n}} = \emptyset \notin N(w)$ by (kbf). Hence $\mathcal{M}, w \not\models_{\mathsf{n}} B\bot$.

29

- Scheme (N) is valid: $\models_n B\top$.

  $[\![\top]\!]_n \cap [w] = [w] \in N(w)$ by (n). Hence $\mathcal{M}, w \models_n B\top$.

- Scheme (Ap) is valid: $\models_n B\varphi \rightarrow KB\varphi$.

  Suppose $\mathcal{M}, w \models_n B\varphi$. Then $[w] \cap [\![\varphi]\!]_n \in N(w)$. Take $v \in [w]$. We have $[v] = [w]$ because R is an equivalence relation, and we have $N(v) = N(w)$ by (a). Hence $[v] \cap [\![\varphi]\!]_n \in N(v)$; that is, $\mathcal{M}, v \models_n B\varphi$. Since $v \in [w]$ was chosen arbitrarily, we have shown that $[w] \subseteq [\![B\varphi]\!]_n$. Hence $\mathcal{M}, w \models_n KB\varphi$.

- Scheme (An) is valid: $\models_n \neg B\varphi \rightarrow K\neg B\varphi$.

  Replace $B\varphi$ by $\neg B\varphi$ and $\in$ by $\notin$ in the argument for the previous item.

- Scheme (KBM) is valid: $\models_n K(\varphi \rightarrow \psi) \rightarrow (B\varphi \rightarrow B\psi)$.

  Suppose $\mathcal{M}, w \models_n K(\varphi \rightarrow \psi)$ and $\mathcal{M}, w \models_n B\varphi$. This means $[w] \subseteq [\![\varphi \rightarrow \psi]\!]_n$ and $[w] \cap [\![\varphi]\!]_n \in N(w)$. But then

  $$[w] \cap [\![\varphi]\!]_n \subseteq [w] \cap [\![\varphi]\!]_n \cap [\![\varphi \rightarrow \psi]\!]_n \subseteq [w] \cap [\![\psi]\!]_n \ .$$

  Hence $[w] \cap [\![\psi]\!]_n \in N(w)$ by (kbm). That is, $\mathcal{M}, w \models_n B\psi$.

That validity is closed under applications of the rules MP and MN follows by the standard arguments [BdRV01]. This completes the proof of soundness.

Before we prove completeness, we first prove an important result that we will use tacitly throughout the completeness proof proper. Let M be the set of all $\mathcal{L}_{KB}$-formulas having one of the forms $K\varphi$, $\neg K\varphi$, $B\varphi$, or $\neg B\varphi$. We prove the following *Modal-Assumption Deduction Theorem*: for each finite $F \subseteq M$, we have

$$F \vdash_{KB} \varphi \quad \text{iff} \quad \vdash_{KB} (\bigwedge F) \rightarrow \varphi \ .$$

The right-to-left direction straightforward. The proof of the left-to-right direction is by induction on the length of derivation. All cases are standard except for the induction step in which (MN) is applied, so we focus on this case. Suppose $F \vdash_{KB} K\varphi$ is derived by (MP) from φ such that $F \vdash_{KB} \varphi$. By the induction hypothesis, we have $\vdash_{KB} (\bigwedge_{\chi \in F} \chi) \rightarrow \varphi$. By (MN) and K reasoning, we have $\vdash_{KB} (\bigwedge_{\chi \in F} K\chi) \rightarrow K\varphi$. However, it also follows by S5 reasoning (using schemes 4 and 5), scheme (Ap), scheme (An), and the fact that $F \subseteq M$ that we have $\vdash_{KB} \chi \rightarrow K\chi$ for each $\chi \in F$. Hence $\vdash_{KB} (\bigwedge_{\chi \in F} \chi) \rightarrow (\bigwedge_{\chi \in F} K\chi)$, where $(\bigwedge_{\chi \in F} \chi) = \bigwedge F$. Conclusion: $\vdash_{KB} (\bigwedge F) \rightarrow K\varphi$.

To prove completeness, it suffices to show that $KB \nvdash \neg\theta$ implies θ is satisfiable at a pointed epistemic neighborhood model. For two sets F and F' of $\mathcal{L}_{KB}$-formulas, to say that F is *maxcons in F'* means that $F \subseteq F'$, the set F is KB-consistent (i.e., for no finite $G \subseteq F$ do we have $\vdash_{KB} (\bigwedge G) \rightarrow \bot$), and adding any formula $\psi \in F'$ not already in F will produce a KB-inconsistent (i.e., not KB-consistent) set.

For a set F of $\mathcal{L}_{\mathsf{KB}}$-formulas, we define the *single-negation closure* $\pm F$ of F and the *modal closure* $\mathsf{MCl}(F)$ of F to be the sets

$$
\begin{aligned}
\pm F &:= F \cup \{\neg\varphi \mid \varphi \in F\} \cup \{\bot, \top\} \ , \\
\mathsf{MCl}(F) &:= F \cup \{X\varphi \mid \varphi \in F \text{ and } X \in \{K, \neg K, B, \neg B\}\} \ .
\end{aligned}
$$

In particular, $\mathsf{MCl}(F)$ is obtained from F by adding for each formula $\varphi \in F$ the additional formulas $K\varphi$, $\neg K\varphi$, $B\varphi$, and $\neg B\varphi$. We say the each of the latter four formulas is a *modalization* of φ.

Let S be the set of subformulas of θ, including θ itself. Let C_0 be the Boolean closure of S; that is, C_0 is the smallest extension of S that contains the propositional constants $\top$ (truth) and $\bot$ (falsehood), or their abbreviations in $\mathcal{L}_{\mathsf{KB}}$ if they are not primitive, and is closed under the Boolean connectives (e.g., negation, conjunction, implication, and disjunction) definable in the language. Finally, define $C := \mathsf{MCl}(C_0)$.

We define the structure $\mathcal{M} = (W, R, V, N)$ as follows:

$$
\begin{aligned}
W &:= \{w \subseteq C \mid w \text{ is maxcons in } C\}, \\
[\varphi] &:= \{w \in W \mid \varphi \in w\} \text{ for } \varphi \in C, \\
R &:= \{(w, v) \in W^2 \mid w \cap M = v \cap M\}, \\
V(w) &:= \mathbf{P} \cap w, \\
N(w) &:= \{X \subseteq [w] \mid \exists \varphi \in C : (X = [\varphi] \cap [w] \text{ and } w \cap M \vdash_{\mathsf{KB}} B\varphi)\}.
\end{aligned}
$$

We make use (often tacitly) of the following *In-class Identity Lemma*: for each $u, v \in W$, if $[u] = [v]$ and $u \cap \pm S = v \cap \pm S$, then $u = v$. So suppose $[u] = [v]$ and $u \cap S' = v \cap S'$. Given $\varphi \in u$, we wish to show that $\varphi \in u$. There are two cases to consider.

- Case: $\varphi \in u \cap C_0$.

  φ is a Boolean combination of members of S and is therefore KB-provably equivalent to a formula φ' that is a disjunction of conjunctions of maxcons subsets of $\pm S$. It follows by the maximal KB-consistency of u that $\varphi' \in u$ and hence one of the disjuncts φ'' of φ' is a member of u. Applying the maimal KB-consistency of u, it follows from $\varphi'' \in u$ that every conjunct of φ'' is a member of u. But each conjunct of φ'' is a member of $\pm S \subseteq S'$ and hence each conjunct of φ'' is a member of v by our assumption $u \cap S' = v \cap S'$. Applying the maximal KB-consistency of v, it follows that $\varphi'' \in v$, hence $\varphi' \in v$, and hence $\varphi \in v$.

- Case: $\varphi \in u \cap (C - C_0)$.

  φ is a modalization $X\psi$ of a Boolean combination of members of S. But $X\psi \in M$ and our assumption $[u] = [v]$ implies $u \cap M = v \cap M$. So $X\psi \in v$.

The converse is proved similarly.

The In-class Identity Lemma gives rise to the following *Identity Lemma*: for each $u, v \in W$, if $u \cap \mathsf{MCl}(\pm S) = v \cap \mathsf{MCl}(\pm S)$, then $u = v$. Indeed, suppose $u \cap \mathsf{MCl}(\pm S) = v \cap \mathsf{MCl}(\pm S)$. If $[u] = [v]$, then it follows from $\pm S \subseteq \mathsf{MCl}(\pm S)$ that we have $u \cap \pm S = v \cap \pm S$, and therefore $u = v$ by the In-class Identity Lemma. So it suffices to prove that $[u] = [v]$; that is, we prove that

$u \cap M = v \cap M$. Proceeding, take $X\varphi \in u \cap M$. If $X\varphi \in C_0$, then we have $X\varphi \in v \cap M$ by the argument in the first case of the In-class Identity Lemma. So suppose $X\varphi \in C - C_0$ so that $X\varphi$ is a modalization of $\varphi \in C_0$. We have $\vdash_{\mathsf{KB}} \varphi \leftrightarrow \varphi'$, where φ' is a disjunction of conjunctions of maxcons subsets of $\pm S$. Applying (MN) and K reasoning, we obtain

$$\vdash_{\mathsf{KB}} K\varphi \leftrightarrow K\varphi' \quad \text{and} \quad \vdash_{\mathsf{KB}} \neg K\varphi \leftrightarrow \neg K\varphi' \ . \tag{19}$$

Applying (KBM) and classical reasoning to (19), it follows that

$$\vdash_{\mathsf{KB}} B\varphi \leftrightarrow B\varphi' \quad \text{and} \quad \vdash_{\mathsf{KB}} \neg B\varphi \leftrightarrow \neg B\varphi' \ . \tag{20}$$

Since $X\varphi \in u$, we have by (19), (20), and the maximal KB-consistency of u that $X\varphi' \in u \cap$ $\mathsf{MCl}(\pm S)$. Since $u \cap \mathsf{MCl}(\pm S) = v \cap \mathsf{MCl}(\pm S)$, it follows that $X\varphi' \in v$, and hence $X\varphi \in v$ by the maximal KB-consistency of v. The converse is proved similarly.

We may make use (often tacitly) of the following *Definability Lemma*: for each $w \in W$ and each $X \subseteq [w]$, defining

$$X^d := \bigvee_{v \in X} \bigwedge (v \cap \pm S) \ ,$$

it follows that $X^d \in C_0 \subseteq C$ and $[X^d] \cap [w] = X$. For the proof, first note that $X^d \in C_0$ because C_0 is closed under Boolean operations and $\pm S \subseteq C_0 \subseteq C$. So assume $u \in [X^d] \cap [w]$, which implies $X^d \in u$ and $[u] = [w]$. Since u is maxcons in $C \supseteq \pm S$, we have by the above definition of X^d as a disjunction over $v \in X$ that there exists $v \in X$ such that $\bigwedge (v \cap \pm S) \in u$ and hence $v \cap \pm S \subseteq u$. Since u is maxcons in C and hence maxcons in $\pm S$ and since $\pm S$ is closed under the operation $\sim : \mathcal{L}_{\mathsf{KB}} \to \mathcal{L}_{\mathsf{KB}}$ defined by

$$\sim \varphi := \begin{cases} \psi & \text{if } \varphi = \neg \psi \\ \neg \varphi & \text{otherwise,} \end{cases}$$

it follows that $u \cap \pm S = v \cap \pm S$. So since $[u] = [v]$ and $u \cap \pm S = v \cap \pm S$, we have $u = v \in X$ by the Identity Lemma. Conversely, suppose $u \in X \subseteq [w]$. By the definition of X^d, we have $\mathsf{KB} \vdash \bigwedge (u \cap \pm S) \to X^d$ and therefore $X^d \in u$ because u is maxcons in C and $\bigwedge (u \cap \pm S) \in u$. Hence $u \in [X^d] \cap [w]$ because $u \in X \subseteq [w]$.

Our definitions above specify the structure $\mathcal{M} = (W, R, V, N)$. W is nonempty because θ is consistent and so may be extended to a maxcons $w_\theta \in W$. Since $\mathsf{MCl}(\pm S)$ is finite, it follows by the Identity Lemma that W is finite. Further, R is an equivalence relation. So to conclude that M is an epistemic neighborhood model, all that remains is for us to show that N satisfies the neighborhood function properties.

(kbc) $X \in N(w)$ implies $X \subseteq [w]$.

By definition.

(bf) $\emptyset \notin N(w)$.

Choose $\varphi \in C$ satisfying $[\varphi] \cap [w] = \emptyset$. It follows that $w \cap M \vdash_{\mathsf{KB}} \varphi \to \bot$, since otherwise we could extend $(w \cap M) \cup \{\varphi\}$ to some $v \in [\varphi] \cap [w]$, which would contradict $[\varphi] \cap [w] = \emptyset$.

So by (MN), we have $w \cap M \vdash_{\mathsf{KB}} K(\varphi \to \bot)$ and hence $w \cap M \vdash_{\mathsf{KB}} B\varphi \to B\bot$ by (KBM). Since $w \cap M \vdash \neg B\bot$ by (BF), it follows that $w \cap M \vdash_{\mathsf{KB}} \neg B\varphi$. So we have shown that $w \cap M \vdash_{\mathsf{KB}} \neg B\varphi$ for each $\varphi \in C$ satisfying $[\varphi] \cap [w] = \emptyset$. KB is consistent (just apply soundness to any epistemic neighborhood model), and therefore we have $w \cap M \nvdash_{\mathsf{KB}} B\varphi$ for each $\varphi \in C$ satisfying $[\varphi] \cap [w] = \emptyset$. Conclusion: $\emptyset \notin N(w)$.

(n) $[w] \in N(w)$.

$w \cap M \vdash_{\mathsf{KB}} B\top$ by (N). Hence $[\top] \cap [w] = [w] \in N(w)$.

(a) $v \in [w]$ implies $N(v) = N(w)$.

$v \in [w]$ implies $[v] = [w]$ and $v \cap M = w \cap M$. Therefore for each $X \subseteq [v] = [w]$, we have $\varphi \in C$ satisfying $X = [\varphi] \cap [v]$ and $v \cap M \vdash_{\mathsf{KB}} B\varphi$ iff $X = [\varphi] \cap [w]$ and $w \cap M \vdash_{\mathsf{KB}} B\varphi$. Hence $N(v) = N(w)$.

(kbm) If $X \subseteq Y \subseteq [w]$ and $X \in N(w)$, then $Y \in N(w)$.

Suppose $X \subseteq Y \subseteq [w]$ and $X \in N(w)$. Then there is $\varphi \in C$ satisfying $X = [\varphi] \cap [w]$ and $w \cap M \vdash_{\mathsf{KB}} B\varphi$. Since $X \subseteq Y$, it follows that $[\varphi] \cap [w] \subseteq [Y^d] \cap [w]$. From this we obtain that $w \cap M \vdash_{\mathsf{KB}} \varphi \to Y^d$, since otherwise we could extend $(w \cap M) \cup \{\varphi, \neg Y^d\}$ to some $v \in [\varphi] \cap [\neg Y^d] \cap [w]$, which would contradict $[\varphi] \cap [w] \subseteq [Y^d] \cap [w]$. Hence $w \cap M \vdash_{\mathsf{KB}} K(\varphi \to Y^d)$ by (MN) and so $w \cap M \vdash_{\mathsf{KB}} B\varphi \to BY^d$ by (KBM). Since $w \cap M \vdash_{\mathsf{KB}} B\varphi$, we have $w \cap M \vdash_{\mathsf{KB}} BY^d$. Hence $Y \in N(w)$.

So M is indeed and epistemic neighborhood model. To complete our overall argument, it suffices to prove the *Truth Lemma*: for each $\varphi \in C$ and $w \in W$, we have $\varphi \in w$ iff $\mathcal{M}, w \models_{\mathsf{n}} \varphi$. The argument is by induction on the construction of $\varphi \in C$. Boolean cases are straightforward, so we restrict our attention to the modal cases: formulas $B\varphi$ and $K\varphi$ in C. Note that by the definition of C as the Boolean closure of the set S of subformulas of θ, either of $B\varphi \in C$ or $K\varphi \in C$ implies $\varphi \in C$.

Suppose $B\varphi \in w$. Then $w \cap M \vdash_{\mathsf{KB}} B\varphi$ and hence $[\varphi] \cap [w] \in N(w)$ by the definition of N and the fact that $\varphi \in C$. Applying the induction hypothesis, $[\varphi] = [\![\varphi]\!]_{\mathsf{n}}^{\mathcal{M}}$, so $[\![\varphi]\!]_{\mathsf{n}}^{\mathcal{M}} \cap [w] \in N(w)$. But this is what it means to have $\mathcal{M}, w \models_{\mathsf{n}} B\varphi$.

Conversely, assume $\mathcal{M}, w \models_{\mathsf{n}} B\varphi$ for $B\varphi \in C$. This means $[\![\varphi]\!]_{\mathsf{n}}^{\mathcal{M}} \cap [w] \in N(w)$. By the induction hypothesis and the fact that $\varphi \in C$, we have $[\varphi] = [\![\varphi]\!]_{\mathsf{n}}^{\mathcal{M}}$, so $[\varphi] \cap [w] \in N(w)$. By the definition of N, there exists $\psi \in C$ such that $w \cap M \vdash_{\mathsf{KB}} B\psi$ and $[\varphi] \cap [w] = [\psi] \cap [w]$. But then $w \cap M \vdash_{\mathsf{KB}} \psi \to \varphi$, for otherwise we could extend $(w \cap M) \cup \{\psi, \neg \varphi\}$ to some $v \in [w]$ such that $v \in [\psi] \cap [w]$ and $v \notin [\varphi] \cap [w]$, contradicting $[\varphi] \cap [w] = [\psi] \cap [w]$. Applying (MN), we have $w \cap M \vdash_{\mathsf{KB}} K(\psi \to \varphi)$ and hence $w \cap M \vdash_{\mathsf{KB}} B\psi \to B\varphi$ by (KBM). Since $w \cap M \vdash_{\mathsf{KB}} B\psi$, it follows that $w \cap M \vdash_{\mathsf{KB}} B\varphi$. And since w is maxcons in C, we conclude that $B\varphi \in w$.

Now suppose $K\varphi \in w$. Then for each $v \in [w]$, we have that $K\varphi \in v$ and therefore $\varphi \in v$ by S5 reasoning (using scheme T) and the fact that v is maxcons in C. But then we have shown that $[w] \subseteq [\varphi]$. Since $\varphi \in C$, it follows by the induction hypothesis that $[w] \subseteq [\![\varphi]\!]_{\mathsf{n}}^{\mathcal{M}}$, which is what it means to have $\mathcal{M}, w \models_{\mathsf{n}} K\varphi$.

Conversely, assume $\mathcal{M}, w \models_n K\varphi$ for $K\varphi \in C$. It follows that $[w] \subseteq [\![\varphi]\!]_n^{\mathcal{M}}$. By the induction hypothesis, $[w] \subseteq [\varphi]$. But then $w \cap M \vdash_{\mathsf{KB}} \varphi$, for otherwise we could extend $(w \cap M) \cup \{\neg\varphi\}$ to some $v \in [w]$ satisfying $v \notin [\varphi]$, contradicting $[w] \subseteq [\varphi]$. By (MN), we have $w \cap M \vdash_{\mathsf{KB}} K\varphi$. Since $K\varphi \in C$ and w is maxcons in C, it follows that $K\varphi \in w$. $\qquad\square$

Since KB is sound and complete with respect to the class of epistemic neighborhood models, we would expect that in light of Theorem 4.7 that KB is at most sound for the probability interpretation.

Theorem 5.4 (KB Probability Soundness)**.** KB is sound for any threshold $c \in (0,1) \cap \mathbb{Q}$ with respect to the class of epistemic probability models:

$$\forall c \in (0,1) \cap \mathbb{Q}, \forall \varphi \in \mathcal{L}_{\mathsf{KB}}: \quad \mathsf{KB} \vdash \varphi \quad \Rightarrow \quad \models_{\mathsf{p}} \varphi^c .$$

Proof. Theorems 3.2 and 3.9. $\qquad\square$

Theorem 5.5 (KB Probability Incompleteness)**.** KB is incomplete for all thresholds $c \in (0,1) \cap \mathbb{Q}$ with respect to the class of epistemic probability models:

$$\exists \varphi \in \mathcal{L}_{\mathsf{KB}}, \forall c \in (0,1) \cap \mathbb{Q}: \quad \models_{\mathsf{p}} \varphi^c \quad \text{and} \quad \mathsf{KB} \nvdash \varphi .$$

Proof. Take $\mathcal{M}$ as in the proof of Theorem 4.7. Let σ be the modal formula describing $(\mathcal{M}, a)$: informally (and easily formalizable),

$$\sigma \quad := \quad a\bar{b}\cdots\bar{g} \wedge KW \wedge \left(\bigwedge_{Z \in N(a)} BZ\right) \wedge \left(\bigwedge_{Z' \in \wp(W) - N(a)} \neg BZ'\right) .$$

We have $\mathcal{M}, w \models_n \sigma$ so that $\not\models_n \neg\sigma$ and therefore $\mathsf{KB} \nvdash \neg\sigma$ by Theorem 5.3. By the proof of Theorem 4.7, there is no probability measure agreeing with $\mathcal{M}$ for any threshold. Hence $\models_{\mathsf{p}} \neg\sigma^c$. So $\varphi := \neg\sigma$ gives us the desired formula. $\qquad\square$

5.2 Results for the Mid-Threshold Calculus KB.5

We first show that the KB schemes (BF) and (KBM) are redundant in the theory KB.5.

Theorem 5.6. KB.5$^-$ and KB.5 derive the same theorems:

$$\forall \varphi \in \mathcal{L}_{\mathsf{KB}}: \quad \mathsf{KB.5}^- \vdash \varphi \quad \Leftrightarrow \quad \mathsf{KB.5} \vdash \varphi .$$

Proof. It suffices to prove that the schemes (BF) and (KBM) are derivable in KB.5$^-$. For (KBM), we have by Definition 3.7 that the formula $\varphi \mathbb{I} \psi$ is just

$$K\left(\underbrace{(\neg\varphi \wedge \neg\psi) \vee (\neg\varphi \wedge \psi)}_{F_0} \vee \underbrace{(\varphi \wedge \psi)}_{F_1} \right) , \tag{21}$$

where we have explicitly indicated the subformulas F_0 and F_1 used in the notation of Definition 3.7. Semantically, (21) says that in each of the agent's accessible worlds, ψ is true whenever φ is true. Now reasoning within KB.5$^-$, it follows that $K(\varphi \rightarrow \psi)$ is provably equivalent to $\varphi \mathbb{I} \psi$. But then

from $K(\varphi \to \psi)$ and $B\varphi$, we may derive $\varphi \mathbb{I} \psi$ and $B\varphi$, from which we may derive $B\psi$ by (Scott). Hence (KBM) is derivable.

We now consider (BF). The formula $\bot \to \neg\top$ is a classical tautology and hence $K(\bot \to \neg\top)$ follows by (MN). Hence by an instance of (KBM), which can be defined away in terms of axioms other than (BF) as above, it follows that $B\bot \to B\neg\top$ and therefore that $\neg B\neg\top \to \neg B\bot$. Also by (N), (D), and (MP), we may derive $\neg B\neg\top$. That is, (BF) is derivable. $\qquad\square$

Theorem 5.7 (KB.5 Neighborhood Soundness and Completeness). KB.5 is sound and complete with respect to the class $\mathcal{C}^{.5}$ of mid-threshold neighborhood models:

$$\forall \varphi \in \mathcal{L}_{\mathsf{KB}} : \quad \mathsf{KB.5} \vdash \varphi \quad \Leftrightarrow \quad \mathcal{C}^{.5} \models_{\mathsf{n}} \varphi \ .$$

Proof. Soundness is by induction on the length of derivation. Most cases are as in the proof of Theorem 5.3. We only need consider the remaining axiom schemes.

- Scheme (D) is valid: $\models_{\mathsf{n}} B\varphi \to \check{B}\varphi$.

  Suppose $\mathcal{M}, w \models_{\mathsf{n}} B\varphi$. This means $[w] \cap [\![\varphi]\!]_{\mathsf{n}} \in N(w)$. By (d),

  $$[w] \cap [\![\neg\psi]\!]_{\mathsf{n}} = [w] - [\![\varphi]\!]_{\mathsf{n}} = [w] - ([w] \cap [\![\varphi]\!]_{\mathsf{n}}) \notin N(w) \ .$$

  But this is what it means to have $\mathcal{M}, w \models_{\mathsf{n}} \check{B}\varphi$.

- Scheme (SC) is valid: $\models_{\mathsf{n}} \check{B}\varphi \wedge \check{K}(\neg\varphi \wedge \psi) \to B(\varphi \vee \psi)$.

  Suppose $\mathcal{M}, w \models_{\mathsf{n}} \check{B}\varphi$ and $\mathcal{M}, w \models_{\mathsf{n}} \check{K}(\neg\varphi \wedge \psi)$. It follows that

  $$[w] - ([w] \cap [\![\varphi]\!]_{\mathsf{n}}) = [w] \cap [\![\neg\varphi]\!]_{\mathsf{n}} \notin N(w)$$

  and that there exists $v \in [w]$ satisfying $\mathcal{M}, v \models \neg\varphi \wedge \psi$. But then $[w] \cap [\![\varphi \vee \psi]\!]_{\mathsf{n}} \supsetneq [w] \cap [\![\varphi]\!]_{\mathsf{n}}$ and therefore $[w] \cap [\![\varphi \vee \psi]\!]_{\mathsf{n}} \in N(w)$ by (sc). Hence $\mathcal{M}, w \models B(\varphi \vee \psi)$.

- Scheme (Scott) is valid:

  $$\models_{\mathsf{n}} [(\varphi_i \mathbb{I} \psi_i)_{i=1}^m \wedge B\varphi_1 \wedge \bigwedge_{i=2}^m \check{B}\varphi_i] \to \bigvee_{i=1}^m B\psi_i \ .$$

  Suppose $(\mathcal{M}, w)$ satisfies the antecedent of scheme (Scott). It follows that each $v \in [w]$ satisfies at least as many φ_i's as ψ_i's, that $[w] \cap [\![\psi_1]\!]_{\mathsf{n}} \in N(w)$, and that $[w] - [\![\varphi_k]\!]_{\mathsf{n}} \notin N(w)$ for each $k \in \{2, \ldots, m\}$. Hence

  $$[w] \cap [\![\varphi_1]\!]_{\mathsf{n}}, \ldots, [w] \cap [\![\varphi_m]\!]_{\mathsf{n}} \mathbb{I} [w] \cap [\![\psi_1]\!]_{\mathsf{n}}, \ldots, [w] \cap [\![\psi_m]\!]_{\mathsf{n}} \ ,$$

  from which it follows by (scott) that $[w] \cap [\![\psi_j]\!]_{\mathsf{n}} \in N(w)$ for some $j \in \{1, \ldots, m\}$. Hence $\mathcal{M}, w \models_{\mathsf{n}} B\psi_j$, and thus $\mathcal{M}, w \models_{\mathsf{n}} \bigvee_{i=1}^m B\psi_i$.

Soundness has been proved.

For completeness, it suffices to show that the model $\mathcal{M}$ defined as in the proof of Theorem 5.3—except that now derivability is always taken with respect to KB.5—is a mid-threshold neighborhood model; the rest of the argument is as in that proof, *mutatis mutandis*. Most of the properties of $\mathcal{M}$ are shown in that proof. What remains is for us to show that $\mathcal{M}$ also satisfies (d), (sc), and (scott).

(d) $X \in N(w)$ implies $X' \notin N(w)$, where $X' := [w] - X$.

Suppose $X \in N(w)$. Then we have $\varphi \in C$ such that $X = [\varphi] \cap [w]$ and $w \cap M \vdash_{\mathsf{KB.5}} B\varphi$. By (D), it follows that $w \cap M \vdash_{\mathsf{KB.5}} \check{B}\varphi$. Choose $\psi \in C$ satisfying $X' = [\psi] \cap [w]$. We have $w \cap M \vdash_{\mathsf{KB.5}} \psi \to \neg\varphi$, since otherwise we could extend $(w \cap M) \cup \{\psi, \varphi\}$ to a $v \in [w]$ such that $v \in [\psi] \cap [w] = X'$ and $v \in [\varphi] \cap [w] = X$, contradicting $X' \cap X = \emptyset$. By (MP), we have $w \cap M \vdash_{\mathsf{KB.5}} K(\psi \to \neg\varphi)$ and therefore $w \cap M \vdash_{\mathsf{KB.5}} B\psi \to B\neg\varphi$ by (KBM). So since $\check{B}\varphi = \neg B\neg\varphi$, it follows by classical reasoning that $w \cap M \vdash_{\mathsf{KB.5}} \check{B}\varphi \to \neg B\psi$. Since $w \cap M \vdash_{\mathsf{KB.5}} \check{B}\varphi$, it follows that $w \cap M \vdash_{\mathsf{KB.5}} \neg B\psi$. By the consistency of KB.5 (which follows by applying soundness to any mid-threshold epistemic neighborhood model), we have $w \cap M \nvdash_{\mathsf{KB}} B\psi$. So we have shown that $w \cap M \nvdash_{\mathsf{KB.5}} B\psi$ for each $\psi \in C$ satisfying $X' = [\psi] \cap [w]$. Conclusion: $X' \notin N(w)$.

(sc) If $X' \notin N(w)$ and $X \subsetneq Y \subseteq [w]$, then $Y \in N(w)$.

Assume $X' \notin N(w)$ and $X \subsetneq Y \subseteq [w]$. It follows from $X' \notin N(w)$ that we have $w \cap M \nvdash_{\mathsf{KB.5}} B(X')^d$. Since $(X')^d \in C_0$, it follows that $B(X')^d \in C = \mathsf{MCl}(C_0)$ and therefore $\neg B(X')^d \in w \cap M$ because w is maxcons in C. Hence $w \cap M \vdash_{\mathsf{KB.5}} \neg B(X')^d$. Now we have $w \cap M \vdash_{\mathsf{KB.5}} \neg X^d \to (X')^d$, for otherwise we could extend $(w \cap M) \cup \{\neg X^d, \neg(X')^d\}$ to some $v \in [w]$ such that $v \in [\neg X^d] \cap [w] = X'$ and $v \in [\neg(X')^d] \cap [w] = X$, contradicting $X' \cap X = \emptyset$. By (MP), we have $w \cap M \vdash_{\mathsf{KB.5}} K(\neg X^d \to (X')^d)$ and therefore $w \cap M \vdash_{\mathsf{KB.5}} B\neg X^d \to B(X')^d$ by (KBM). Since $w \cap M \vdash_{\mathsf{KB.5}} \neg B(X')^d$, it follows by classical reasoning that $w \cap M \vdash_{\mathsf{KB.5}} \neg B\neg X^d$. That is, $w \cap M \vdash_{\mathsf{KB.5}} \check{B}X^d$.

Further, since $X \subsetneq Y \subseteq [w]$, it follows that there exists $y \in Y - X$ satisfying $y \in [Y^d] - [X^d] = [Y^d \wedge \neg X^d]$. Since $\neg(Y^d \wedge \neg X^d) \in C_0$, it follows that $\neg K\neg(Y^d \wedge \neg X^d) \in C = \mathsf{MCl}(C_0)$. But then $K\neg(Y^d \wedge \neg X^d) \notin w$, for otherwise it would follow from $y \in [w]$ that $w \cap M = y \cap M$ and hence $K\neg(Y^d \wedge \neg X^d) \in y$, from which it would follow by T and the fact that y is maxcons in C that $\neg(Y^d \wedge \neg X^d) \in y$, contradicting $y \in [Y^d \wedge \neg X^d]$. So since $K\neg(Y^d \wedge \neg X^d) \notin w$, we have by the fact that $\neg K\neg(Y^d \wedge \neg X^d) \in C$ and the maximal KB.5-consistency of w that $\neg K\neg(Y^d \wedge \neg X^d) = \check{K}(Y^d \wedge \neg X^d) \in w$. Hence $w \cap M \vdash_{\mathsf{KB.5}} \check{K}(Y^d \wedge \neg X^d)$. As $w \cap M \vdash_{\mathsf{KB.5}} \check{B}X^d$ as well, it follows by (SC) that $w \cap M \vdash_{\mathsf{KB.5}} B(Y^d \vee X^d)$. But $[Y^d \vee X^d] = Y$ by our assumption $X \subsetneq Y$ and therefore we have shown that $Y \in N(w)$.

(scott) If $X_1, \ldots, X_m, Y_1, \ldots, Y_m \subseteq [w]$, $(X_i \mathbb{I} Y_i)_{i=1}^m$, $X_1 \in N(w)$, and $X_i' := [w] - X_i \notin N(w)$ for all $i \in \{2, \ldots, m\}$, then there exists $j \in \{1, \ldots, m\}$ such that $Y_j \in N(w)$.

Assume we have the above-stated antecedent of the (scott) property. It follows from $X_1 \in N(w)$ that $w \cap M \vdash_{\mathsf{KB.5}} BX_1^d$. For $i \in \{2, \ldots, m\}$, it follows from $X_i' \notin N(w)$ by an argument as in the above proof for (sc) that $w \cap M \vdash \check{B}X_i^d$. If we can prove that $w \cap M \vdash_{\mathsf{KB.5}} (X_i^d \mathbb{I} Y_i^d)_{i=1}^m$ as well, then we would have by (Scott) that $w \cap M \vdash_{\mathsf{KB.5}} \bigvee_{j=1}^m BY_j^d$. But then since $BY_j^d \in C$ for each $j \in \{1, \ldots, m\}$, we would have $BY_k^d \in w$ for some $k \in \{1, \ldots, m\}$ by the maximal KB.5-consistency of w, hence $w \cap M \vdash_{\mathsf{KB.5}} BY_k^d$, and hence $Y_k = [Y_k^d] \in N(w)$.

So it suffices for us to prove that $w \cap M \vdash_{\mathsf{KB.5}} (X_i^d \mathbb{I} Y_i^d)_{i=1}^m$. Proceeding, we recall that $(X_i^d \mathbb{I} Y_i^d)_{i=1}^m$ abbreviates the formula $K(F_0 \vee \cdots \vee F_m)$, where F_k is the disjunction of all conjunctions

$$d_1 X_1^d \wedge \cdots \wedge d_m X_m^d \wedge e_1 Y_1^d \wedge \cdots \wedge e_m Y_m^d \ , \tag{22}$$

satisfying the property that exactly k of the d_i's are the empty string, at least k of the e_i's are the empty string, and the rest of the d_i's and e_i's are the negation sign $\neg$. Since each of the X_i's and Y_i's is a member of C_0 and C_0 is closed under Boolean operations, each conjunction (22) is a member of C_0, and hence so is the disjunction $F_0 \vee \cdots \vee F_m$. But then $K(F_0 \vee \cdots \vee F_m) \in C = \mathsf{MCl}(C_0)$. We make use of these facts tacitly in what follows. Now we have by our assumption $(X_i \mathbb{I} Y_i)_{i=1}^m$ and the fact that the X_i's and Y_i's are subsets of $[w]$ that every world in $[w]$ is contained in at least as many of the X_i's as in the Y_i's. Every world in $[w]$ therefore contains at least one of the F_i's, for otherwise it would follow by maximal KB.5-consistency that we could find a world $v \in [w]$ that is not contained in at least as many of the X_i's as in the Y_i's, a contradiction. By maximal KB.5-consistency, every world in $[w]$ thereby contains the disjunction $F_0 \vee \cdots \vee F_m$. But then it follows by maximal KB.5-consistency and T-reasoning that $K(F_0 \vee \cdots \vee F_m) \in w$, and hence $w \cap M \vdash_{\mathsf{KB.5}} (X_i^d \mathbb{I} Y_i^d)_{i=1}^m$. $\qquad\square$

Since KB.5 is sound and complete with respect to mid-threshold neighborhood models, we would expect from Corollary 4.11 that KB.5 is sound and complete with respect to the probability interpretation for threshold $c = \frac{1}{2}$.

Theorem 5.8 (Due to [Len80]; KB.5 Probability Soundness and Completeness). KB.5 is sound and complete for threshold $\frac{1}{2}$ with respect to the class of epistemic probability models:

$$\forall \varphi \in \mathcal{L}_{\mathsf{KB}} : \quad \mathsf{KB.5} \vdash \varphi \quad \Leftrightarrow \quad \models_{\mathsf{p}} \varphi^{\frac{1}{2}} \ .$$

Proof. Soundness is by Theorems 3.2 and 3.9. Completeness is by Theorem 5.7 and Corollary 4.11. $\qquad\square$

6 Conclusion

Summary We have provided a study of unary modal logics of high probability. We introduced epistemic neighborhood models and studied their connection to traditional epistemic probability models by way of a natural notion of "agreement." We listed the Lenzen-derivative properties of epistemic neighborhood models that guarantee the existence of an agreeing probability measure for threshold $c = \frac{1}{2}$. The list of properties required to guarantee the existence of an agreeing probability measure for other thresholds is unknown. We also presented our study from a proof theoretic point of view by introducing a probabilistically sound but incomplete logic KB and our version of Lenzen's probabilistically sound and complete logic KB.5 for threshold $c = \frac{1}{2}$. It is open as to the principles one must add to KB in order to obtain probabilistic completeness for other thresholds. We also proved soundness and completeness of KB and of KB.5 with respect to a corresponding class of epistemic neighborhood models. The result for KB.5 along with our

Theorem 4.10, a theorem we credit to Lenzen, shows that KB.5 is the logic of probabilistic certainty and of probability exceeding $c = \frac{1}{2}$. It is our hope that our repackaging of Professor Lenzen's result will make his work more accessible to a broad audience of modern modal logicians. We also hope that the connection we have made with neighborhood semantics will prove useful in future work on modal logics of qualitative probability.

Open Questions for Future Work

1. The main open question is the following: given a "high-threshold" $c \in (\frac{1}{2}, 1) \cap \mathbb{Q}$, find the exact extension KB^c of KB that is probabilistically sound and complete for threshold c with respect to the class of epistemic probability models, in the sense that we would have:

$$\forall \varphi \in \mathcal{L}_{\mathsf{KB}} : \quad \mathsf{KB}^c \vdash \varphi \quad \Leftrightarrow \quad \models_{\mathsf{p}} \varphi^c .$$

Observing that (SC) and (Scott) are not valid for high-thresholds $c > \frac{1}{2}$, we conjecture that what is required are threshold-specific variants of (SC) and (Scott) that will together guarantee probability soundness and completeness. Toward this end, we suggest the following schemes as a starting point:

(SC$_0^s$) $(\check{K}\varphi_0 \wedge \bigwedge_{i=1}^s \check{B}\varphi_i \wedge \bigwedge_{i \neq j=0}^s K(\varphi_i \to \neg\varphi_j)) \to B(\bigvee_{i=0}^s \varphi_i)$

(SC$_1^s$) $(\bigwedge_{i=1}^s \check{B}\varphi_i \wedge \bigwedge_{i \neq j=1}^s K(\varphi_i \to \neg\varphi_j)) \to B(\bigvee_{i=1}^s \varphi_i)$

(WS) $[(\varphi_i \mathbb{I} \psi_i)_{i=1}^m \wedge \bigwedge_{i=1}^m B\varphi_i] \to \bigvee_{i=1}^m B\psi_i$

Observe that (SC) is just (SC$_0^1$). Further, if we define $s' := c/(1-c)$ and $s := \text{ceiling}(s')$, then scheme (SC$_0^s$) is probabilistically sound if $s = s'$ and scheme (SC$_1^s$) is probabilistically sound if $s \neq s'$. The reasoning for this is as follows: s' tells us the number of $(1-c)$'s that divide c. In particular, recall from Lemma 3.5 that the probabilistic interpretation of $\check{B}\varphi$ is that φ is assigned probability at least $1 - c$. Therefore, if we have s disjoint propositions that each have probability at least $1 - c$, then the probability of their disjunction will have probability $s \cdot (1-c) \geq c$. This inequality is strict if $s \neq s'$ and is in fact an equality if $s = s'$. Therefore, in the case $s \neq s'$, scheme (SC$_1^s$) is sound: s disjoint propositions each having probability $1 - c$ together sum to a probability exceeding the threshold c. And in case $s = s'$, scheme (SC$_0^s$) is sound: s disjoint propositions each having probability $1 - c$ together sum to a probability that equals c, so adding some additional probability from another disjoint proposition φ_0 will yield a disjunction whose probability again exceeds c. In either case, exceeding probability c is what we equate with belief, so soundness is proved. We note that scheme (WS) can be shown to be sound by adapting the proof Theorem 3.9(11). The epistemic neighborhood model versions of (SC$_0^s$), (SC$_1^s$), and (WS) are:

(sc$_0^s$) $\forall X_1, \ldots, X_s, Y \subseteq [w]$: if $[w] - X_1, \ldots, [w] - X_s \notin N(w)$, the X_i's are pairwise disjoint, and $Y \supsetneq \bigcup_{i=1}^s X_i$, then $Y \in N(w)$.

(sc$_1^s$) $\forall X_1, \ldots, X_s \subseteq [w]$: if $[w] - X_1, \ldots, [w] - X_s \notin N(w)$ and the X_i's are pairwise disjoint, then $\bigcup_{i=1}^s X_i \in N(w)$.

38

(ws) $\forall m \in \mathbb{Z}^+, \forall X_1, \ldots, X_m, Y_1, \ldots, Y_m \subseteq [w]$:

$$
\begin{aligned}
\text{if} \quad & X_1, \ldots, X_m \mathbb{I} Y_1, \ldots, Y_m \quad \text{and} \\
& \forall i \in \{1, \ldots, m\} : X_i \in N(w) \ , \\
\text{then} \quad & \exists j \in \{1, \ldots, m\} : Y_j \in N(w) \ .
\end{aligned}
$$

If M is an epistemic neighborhood model, then a slight modification of the proof of property (scott) in Lemma 4.13 shows that M^c satisfies (ws). We presume that an adaptation of the proof for the proof of property (sc) in the same lemma will show that M^c satisfies (sc_0^s) if $s = s'$ and (sc_1^s) if $s \neq s'$.

We remark that (WS) is not threshold-specific, though it is sound for all high-thresholds $c > \frac{1}{2}$. We suspect that a threshold-specific variant may be required in order to adapt Lenzen's proof of KB.5 probability soundness and completeness for threshold $c = \frac{1}{2}$ (Theorem 4.10).

2. Another open question is the exact relationship between Segerberg's comparitive operator $\varphi \preceq \psi$ ("φ is no more probable than ψ") [Gär75, Seg71] and our unary operators K and B. The formula $B\varphi$ is equivalent to $\neg\varphi \prec \varphi$. However, it is not clear how the logics of these operators are related. Also, we suspect that a language with $\preceq$ is strictly more expressive.

3. Yet another direction is the extension of our work to Bayesian updating. Given a pointed epistemic probability model $(\mathcal{M}, w)$ satisfying φ, let

$$
\mathcal{M}[\varphi] = (W[\varphi], R[\varphi], V[\varphi], P[\varphi])
$$

be defined by

$$
\begin{aligned}
W[\varphi] &:= \llbracket \varphi \rrbracket_{\mathsf{p}}^{\mathcal{M}} \\
R[\varphi] &:= R \cap (W[\varphi] \times W[\varphi]) \\
V[\varphi](w) &:= V(w) \text{ for } w \in W[\varphi] \\
P[\varphi](w) &:= \frac{P(w)}{P(\llbracket \varphi \rrbracket_{\mathsf{p}}^{\mathcal{M}})}
\end{aligned}
$$

It is not difficult to see that $\mathcal{M}[\varphi]$ is an epistemic probability model and

$$
P[\varphi](X) = \frac{P(X \cap \llbracket \varphi \rrbracket_{\mathsf{p}}^{\mathcal{M}})}{P(\llbracket \varphi \rrbracket_{\mathsf{p}}^{\mathcal{M}})} = P[\varphi](X | \llbracket \varphi \rrbracket_{\mathsf{p}}^{\mathcal{M}}) \ ,
$$

where the value on the right is the probability of X conditional on $\llbracket \varphi \rrbracket_{\mathsf{p}}^{\mathcal{M}}$. It would be interesting to investigate the analog of this operation in epistemic neighborhood models. The operation may also have a close relationship with the study of updates in Probabilistic Dynamic Epistemic Logic [vBGK09, BS08].

4. Finally, we have only considered a single-agent version of our logics KB and KB.5. The reason for this is that obtaining completeness for KB.5 with respect to the class of finite mid-threshold neighborhood models requires us to construct a finite countermodel satisfying (sc), as we did in the completeness portion of the proof of Theorem 5.7. However, this property has an antecedent that includes the negative condition $X' \notin N(w)$ and from this and $X \subsetneq Y \subseteq [w]$, we are to conclude the positive condition $Y \in N(w)$. Referring the reader to the completeness portions of the proofs of Theorems 5.3 and 5.7 for definitions and terminology, the trick to making things work in the single-agent case is to prove the Definability Lemma using a particular closure construction that ensures every potential neighborhood $X \subseteq [w]$ is definable by a formula X^d such that BX^d is a member of the closure set C. This makes crucial use of the In-class Identity Lemma. However, our proof of this lemma depends on the assumption that maxcons sets $u, v \in [w]$ differ only in non-modal formulas. In the straightforward multi-agent version of our setting, we would have an equivalence class $[w]_a$ consisting of all maxcons sets that agree on modal formulas $K_a\varphi$ and $B_a\psi$ for a given agent a. But then two worlds $u, v \in [w]_a$ could disagree on modal formulas $K_b\varphi$ or $B_b\psi$ for some agent $b \neq a$, which leads to a breakdown in the current proof of the In-class Identity Lemma and therefore presents problems for guaranteeing definability of potential neighborhoods satisfying the desired membership property. Remedying this in a multi-agent version of a finite mid-threshold neighborhood model is not straightforward because it is difficult to simultaneously satisfy (sc), all other properties of finite mid-threshold neighborhood models, and the definability-with-membership property. We therefore leave for future work the matter of proving completeness of multi-agent KB.5 with respect to the class of finite multi-agent mid-threshold neighborhood models. We note that multi-agent KB.5 is obtained from our existing axiomatization by simply adding a subscript to all occurrences of a modal operator K or B in our present axiomatization. Multi-agent KB is obtained similarly, though completeness for multi-agent KB with respect to the full class of finite multi-agent epistemic neighborhood models can be shown without much difficulty because the problematic property (sc) need not be satisfied.

References

[BdRV01] P. Blackburn, M. de Rijke, and Y. Venema. *Modal Logic*. Cambridge Tracts in Theoretical Computer Science. Cambridge University Press, 2001.

[BS08] Alexandru Baltag and Sonja Smets. Probabilistic dynamic belief revision. *Synthese*, 165(2):179–202, 2008.

[Che80] B.F. Chellas. *Modal Logic: An Introduction*. Cambridge University Press, 1980.

[dF37]   Bruno de Finetti. La prévision: ses lois logiques, ses sources subjectives. *Annals de l'Institut Henri Poincaré*, 7:1–68, 1937. Translated into English in H.E. Kyburg, Jr. and H.E. Smokler (eds.), *Studies in Subjective Probability*, 1964, pp. 93–158.

[dF51]   Bruno de Finetti. La "logica del plausibile" secondo la concezione di polya. In *Atti della XLII Riunione, Societa Italiana per il Progresso delle Scienze*, pages 227–236, 1951.

[Eij13]   Jan van Eijck. Learning about probability. available from `homepages.cwi.nl:~/jve/software/prodemo`, 2013.

[EoS14]   Jan van Eijck and Franois Schwarzentruber. Epistemic probability logic simplified. In Rajeev Goré, Barteld Kooi, and Agi Kurucz, editors, *Advances in Modal Logic*, volume 10, pages 158–177, 2014.

[Gär75]   Peter Gärdenfors. Qualitative probability as an intensional logic. *Journal of Philosophical Logic*, 4(2):171–185, 1975.

[Hal03]   J. Halpern. *Reasoning About Uncertainty*. MIT Press, 2003.

[Her03]   A. Herzig. Modal probability, belief, and actions. *Fundamenta Informaticae*, 57:323–344, 2003.

[HI13]   Wesley H. Holliday and Thomas F. Icard. Measure semantics and qualitative semantics for epistemic modals. In *Proceedings of SALT*, volume 23, pages 514–534, 2013.

[Jef04]   Richard Jeffrey. *Subjective Probability — The Real Thing*. Cambridge University Press, 2004.

[Kör08]   T.W. Körner. *Naive Decision Making: Mathematics Applied to the Social World*. Cambridge University Press, 2008.

[KPS59]   Charles H. Kraft, John W. Pratt, and A. Seidenberg. Intuitive probability on finite sets. *The Annals of Mathematical Statistics*, 30(2):408–419, 1959.

[KT12]   Henry E. Kyburg and Choh Man Teng. The logic of risky knowledge, reprised. *International Journal of Approximate Reasoning*, 53:274–285, 2012.

[Kyb61]   H.E. Kyburg. *Probability and the Logic of Rational Belief*. Wesleyan University Press, Middletown, CT, 1961.

[Len80]   Wolfgang Lenzen. *Glauben, Wissen und Wahrscheinlichkeit — Systene der epistemischen Logik*. Springer Verlag, Wien & New York, 1980.

[Len03]   Wolfgang Lenzen. Knowledge, belief, and subjective probability — outlines of a unified theory of epistemic/doxastic logic. In V.F. Hendricks, K.F. Jorgensen, and S.A. Pedersen, editors, *Knowledge Contributors*, number 322 in Synthese Library, pages 17–31. Kluwer, 2003.

[Sco64]   Dana Scott. Measurement structures and linear equalities. *Journal of Mathematical Psychology*, 1:233–247, 1964.

[Seg71]   K. Segerberg. Qualitative probability in a modal setting. In J. Fenstad, editor, *Proceedings of the 2nd Scandinavian Logic Symposium*, pages 341–352, Amsterdam, 1971. North Holland.

[vBGK09]   Johan van Benthem, Jelle Gerbrandy, and Barteld Kooi. Dynamic update with probabilities. *Studia Logica*, 93(1):67–96, 2009.

[WF79]   P. Walley and T.L. Fine. Varieties of modal (classificatory) and comparative probability. *Synthese*, 41:321–374, 1979.