Two Dimensional Phase Retrieval from Local Measurements

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ABSTRACT

2D or not 2D, that is the tribe called question

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1. INTRODUCTION

In this paper we consider the problem of approximately recovering an unknown two dimensional sample transmission function $q : \mathbb{R}^2 \to \mathbb{C}$ with compact support, $\operatorname{supp}(q) \subset [0,1]^2$, from phaseless Fourier measurements of the form

$$|(\mathcal{F}[aS_{x_0,y_0}q])(u,v)|^2, \quad (u,v) \in \Omega \subset \mathbb{R}^2, \quad (x_0,y_0) \in \mathcal{L} \subset [0,1]^2 \tag{1}$$

where $\mathcal{F}$ denotes the 2 dimensional Fourier transform, $a : \mathbb{R}^2 \to \mathbb{C}$ is a known illumination function from an illuminating beam, S_{x_0,y_0} is a shift operator defined by $(S_{x_0,y_0}q)(x,y) := q(x-x_0, y-y_0)$, Ω is a finite set of sampled frequencies, and $\mathcal{L}$ is a finite set of shifts. When the illuminating beam is sharply focused, one can further assume that a is also (effectively) compactly supported within a smaller region $[0,\delta']^2$ for $\delta' \ll 1$. This is known as the *ptychographic imaging problem* and is of great interest in the physics community (see, e.g., Rodenburg[1]). Herein we will make the further assumption that all the utilized shifts of q also have their supports contained in $[0,1]^2$. That is, that

$$\bigcup_{(x_0,y_0)\in\mathcal{L}} \operatorname{supp}(S_{x_0,y_0}q) \subseteq [0,1]^2$$

holds. Note that an analogous assumption can always be achieved by dilating $[0,1]^2$.

Discretizing (1) using periodic boundary conditions we obtain a finite dimensional problem aimed at recovering an unknown matrix $Q \in \mathbb{C}^{d\times d}$ from phaseless measurements of the form

$$\left| \frac{1}{d^2} \sum_{j=1}^{d} \sum_{k=1}^{d} A_{j,k} \left(S_\ell Q S_{\ell'}^*\right)_{j,k} \mathbb{e}^{\frac{-2\pi \mathbb{i}}{d}(ju+kv)} \right|^2 \tag{2}$$

where $A \in \mathbb{C}^{d\times d}$ is a known measurement matrix representing our illuminating beam, and $S_\ell \in \mathbb{R}^{d\times d}$ is the discrete circular shift operator defined by $(S_\ell \mathbf{x})_j := x_{j-\ell \bmod d}$ for all $\mathbf{x} \in \mathbb{C}^d$ and $j, \ell \in [d] := \{1, \ldots, d\}$. Herein we will make the simplifying assumption that our original illuminating beam function a is not only sharply focused, but also separable. In particular, we assume that the weighted measurement matrix takes

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the form $\frac{1}{d^2}A := \mathbf{ab}^*$ where $\mathbf{a}, \mathbf{b} \in \mathbb{C}^d$ both have $a_j = b_j = 0$ for all $j \in [d] \setminus \{1, \ldots, \delta\}$. Here $\delta \in \mathbb{Z}^+$ is much smaller than d.

Using the small support and separability of $\frac{1}{d^2}A := \mathbf{ab}^*$ we can now rewrite the measurements (2) as

$$\left| \sum_{j=1}^{\delta} \sum_{k=1}^{\delta} a_j \overline{b_k} \left(S_\ell Q S_{\ell'}^* \right)_{j,k} \mathbb{e}^{\frac{-2\pi \mathrm{i}}{d}(ju+kv)} \right|^2 = \left| \sum_{j=1}^{\delta} \sum_{k=1}^{\delta} \overline{\overline{a_j} \mathbb{e}^{\frac{2\pi \mathrm{i} ju}{d}} b_k \mathbb{e}^{\frac{2\pi \mathrm{i} kv}{d}}} \left(S_\ell Q S_{\ell'}^* \right)_{j,k} \right|^2$$
$$= \left| \langle S_\ell Q S_{\ell'}^*, \mathbf{a}_u \mathbf{b}_v^* \rangle_{\mathrm{HS}} \right|^2 \tag{3}$$

where $\mathbf{a}_u, \mathbf{b}_v \in \mathbb{C}^d$ are defined by $(a_u)_j := \overline{\mathbb{e}^{\frac{-2\pi \mathrm{i} ju}{d}} a_j}$ and $(b_v)_k := \overline{\mathbb{e}^{\frac{2\pi \mathrm{i} kv}{d}} b_k}$ for all $j, k \in [d]$. Continuing to rewrite (3) we can now see that our discretized measurements will all take the form of

$$\left| \langle S_\ell Q S_{\ell'}^*, \mathbf{a}_u \mathbf{b}_v^* \rangle_{\mathrm{HS}} \right|^2 = \left| \mathrm{Trace} \left(\mathbf{b}_v \mathbf{a}_u^* S_\ell Q S_{\ell'}^* \right) \right|^2 = \left| \mathrm{Trace} \left(S_{\ell'}^* \mathbf{b}_v \left(S_\ell^* \mathbf{a}_u \right)^* Q \right) \right|^2 = \left| \left\langle Q, S_\ell^* \mathbf{a}_u \left(S_{\ell'}^* \mathbf{b}_v \right)^* \right\rangle_{\mathrm{HS}} \right|^2 \tag{4}$$

for a finite set of frequencies $(u, v) \in \Omega \subset \mathbb{R}^2$ and shifts $(\ell, \ell') \in \mathcal{L} \subseteq [d] \times [d]$.

Motivated by ptychographic imaging we propose a new efficient numerical scheme for solving general discrete phase retrieval problems using measurements of type (4) herein. After a brief discussion of notation, we will outline our proposed method in §2 below. A preliminary numerical evaluation of the method is then presented in §3.

Notation and Preliminaries

For any $k \in \mathbb{N}$, we define $[k] := \{1, 2, \ldots, k\}$. For $i, j \in \mathbb{N}, e_i$ represents the standard basis vector and $E_{ij} = e_i e_j^*$; the dimensions of such an E_{ij} will always be clear from context. For a matrix $A \in \mathbb{C}^{m \times n}$,

$$\overrightarrow{A} := (a_{11}, a_{21}, \ldots, a_{m1}, \ldots, a_{mn})$$

denotes the column-major vectorization of A. $A \otimes B$ for arbitrary matrices denotes the standard Kronecker product. We remark that $\overrightarrow{\mathbf{ab}^*} = \overline{\mathbf{b}} \otimes \mathbf{a}$, and in particular

$$\overrightarrow{E_{jk}} \overrightarrow{E_{j'k'}}^* = \overrightarrow{e_j e_k^*} \overrightarrow{e_{j'} e_{k'}^*}^* = (e_k \otimes e_j)(e_{k'} \otimes e_{j'})^* = (e_k e_{k'}^*) \otimes (e_j e_{j'}^*) = E_{kk'} \otimes E_{jj'}. \tag{5}$$

We let $\langle A, B \rangle_{\mathrm{HS}} := \mathrm{Trace}(A^* B) = \langle \overrightarrow{A}, \overrightarrow{B} \rangle$ denote the Hilbert-Schmidt inner product on $\mathbb{C}^{n \times n}$ and remark that, for $\mathbf{x}, \mathbf{y} \in \mathbb{C}^n$,

$$|\langle \mathbf{x}, \mathbf{y} \rangle|^2 = \langle \mathbf{xx}^*, \mathbf{yy}^* \rangle_{\mathrm{HS}}. \tag{6}$$

2. AN EFFICIENT METHOD FOR SOLVING THE DISCRETE 2D PHASE RETRIEVAL PROBLEM

Our recovery method, outlined in Algorithm 1, aims to approximate an image $Q \in \mathbb{C}^{d \times d}$ from phaseless measurements of the form (4). Specifically, we consider the collection of measurements given by

$$y_{(\ell, \ell', u, v)} := \left| \left\langle Q, S_\ell^* \mathbf{a}_u \left(S_{\ell'}^* \mathbf{b}_v \right)^* \right\rangle_{\mathrm{HS}} \right|^2 \tag{7}$$

for all $(\ell, \ell', u, v) \in [d]^2 \times \Omega^2$ where $\Omega \subset [d]$ has $|\Omega| = 2\delta - 1$. Thus, we collect a total of $D := (2\delta - 1)^2 \cdot d^2$ measurements where each measurement is due to a vertical and horizontal shift of a rank one illumination pattern $\mathbf{a}_u \mathbf{b}_v^* \in \mathbb{C}^{d \times d}$. We assume that our measurements are *local* in the sense that $\mathrm{supp}(\mathbf{a}), \mathrm{supp}(\mathbf{b}) \subset [\delta]$. Recall that $\delta \ll d$,so the total number of measurements D is essentially linear in the problem size.

Algorithm 1 consists of first rephrasing the system (7) as a linear system on the space of $d^2 \times d^2$ matrices (following Candes, et al.[2]), and then estimating a projection $\mathcal{P}(\overrightarrow{Q}\overrightarrow{Q}^*)$ of the rank one matrix $\overrightarrow{Q}\overrightarrow{Q}^*$ from this system. Next, we estimate the magnitudes of the entries of Q directly from $\mathcal{P}(\overrightarrow{Q}\overrightarrow{Q}^*)$ and their phases

Algorithm 1 Two Dimensional Phase Retrieval from Local Measurements

Input: Measurements $\mathbf{y} \in \mathbb{R}^D$ as per (7)
Output: $X \in \mathbb{C}^{d\times d}$ with $X \approx \mathbb{e}^{-\mathbb{i}\theta} Q$ for some $\theta \in [0, 2\pi]$

1. Compute the Hermitian matrix $P = \left(\left(\mathcal{M}\big|_{\mathcal{P}}\right)^{-1} \mathbf{y}\right)/2 + \left(\left(\mathcal{M}\big|_{\mathcal{P}}\right)^{-1} \mathbf{y}\right)^*/2 \in \mathcal{P}\left(\mathbb{C}^{d^2\times d^2}\right)$ as an estimate of $\mathcal{P}\left(\overrightarrow{Q}\overrightarrow{Q}^*\right)$. $\mathcal{M}$ and $\mathcal{P}$ are as defined in (8) and §2.1.
2. Form the matrix of phases, $\widetilde{P} \in \mathcal{P}\left(\mathbb{C}^{d^2\times d^2}\right)$, by normalizing the non-zero entries of P. We expect that $\widetilde{P} \approx \widetilde{Q}$.
3. Compute the principal eigenvector of $\widetilde{P}$ and use it to compute $U_{j,k} \approx \operatorname{sgn}(Q_{j,k})\ \forall j,k \in [d]$ as per §2.2.
4. Use the diagonal entries of P to compute $M_{j,k} \approx |Q_{j,k}|^2$ for all $j,k \in [d]$ as per §2.3.
5. Set $X_{j,k} = \sqrt{M_{j,k}} \cdot U_{j,k}$ for all $j,k \in [d]$ to form X

using an eigenvector approach. Together, the magnitude and phase estimates provide an approximation of Q.

Toward producing the linear system of step 1, we observe that

$$\begin{aligned} y_{(\ell,\ell',u,v)} &= \left|\left\langle Q, S_\ell^*\mathbf{a}_u \left(S_{\ell'}^*\mathbf{b}_v\right)^*\right\rangle_{\text{HS}}\right|^2 = |\langle \overrightarrow{Q}, S_{\ell'}^*\overline{\mathbf{b}_u} \otimes S_\ell^*\mathbf{a}_v\rangle|^2 \\ &= \left\langle \overrightarrow{Q}\overrightarrow{Q}^*,\ S_{\ell'}^*\overline{\mathbf{b}_u} \otimes S_\ell^*\mathbf{a}_v \left(S_{\ell'}^*\overline{\mathbf{b}_u} \otimes S_\ell^*\mathbf{a}_v\right)^*\right\rangle, \end{aligned}$$

which allows us to naturally define $\mathcal{M} : \mathbb{C}^{d^2\times d^2} \mapsto \mathbb{R}^D$ as the linear measurement operator given by

$$(\mathcal{M}(Z))_{(\ell,\ell',u,v)} := \left\langle Z,\ S_{\ell'}^*\overline{\mathbf{b}_u} \otimes S_\ell^*\mathbf{a}_v \left(S_{\ell'}^*\overline{\mathbf{b}_u} \otimes S_\ell^*\mathbf{a}_v\right)^*\right\rangle_{\text{HS}} = \left\langle Z,\ S_{\ell'}^*\overline{\mathbf{b}_u\mathbf{b}_u}^* S_{\ell'} \otimes S_\ell^*\mathbf{a}_v\mathbf{a}_v^* S_\ell\right\rangle_{\text{HS}}, \quad (8)$$

so that $\mathbf{y} = \mathcal{M}(\overrightarrow{Q}\overrightarrow{Q}^*)$. Our new objective is to solve for $\mathcal{P}(\overrightarrow{Q}\overrightarrow{Q}^*)$, the projection of $\overrightarrow{Q}\overrightarrow{Q}^*$ onto the rowspace of $\mathcal{M}$ using our measurements $\mathbf{y}$. Once we have solved for $\mathcal{P}(\overrightarrow{Q}\overrightarrow{Q}^*)$, we show that finding its principal eigenvector suffices to compute $\overrightarrow{Q}$ (and therefore Q) up to a global phase multiple.

2.1 Inverting the Linear Measurement Operator $\mathcal{M}$

Recall that $D = (2\delta-1)^2 \cdot d^2 \ll d^4$ from above so that our number of measurements (8) is severely underdetermined for arbitrary $Z \in \mathbb{C}^{d^2\times d^2}$. Toward circumventing the generally underdetermined nature of our measurements we observe that the local supports of both $\mathbf{a}_u$ and $\mathbf{b}_v$ ensure that $\mathcal{M}\left(\overrightarrow{E_{j,k}}\left(\overrightarrow{E_{j',k'}}\right)^*\right) = \mathbf{0}$ whenever either $|j-j'| \geq \delta$ or $|k-k'| \geq \delta$ holds (this is clear from (8) and (5)). As a result we can see that $\mathcal{M}(\mathcal{P}(Z)) = \mathcal{M}(Z)$ holds for all $Z \in \mathbb{C}^{d^2\times d^2}$ where $\mathcal{P} : \mathbb{C}^{d^2\times d^2} \mapsto \mathbb{C}^{d^2\times d^2}$ is the orthogonal projector onto the span of $\mathcal{B} := \left\{\overrightarrow{E_{j,k}}\left(\overrightarrow{E_{j',k'}}\right)^* \,\middle|\, |j-j'| < \delta, |k-k'| < \delta\right\}$.* Furthermore, the dimension of $\mathcal{P}\left(\mathbb{C}^{d^2\times d^2}\right) = \operatorname{span}(\mathcal{B})$ is D by construction. As a result, it is conceivable that the composition of $\mathcal{M}$ and $\mathcal{P}$ restricted to $\mathcal{P}\left(\mathbb{C}^{d^2\times d^2}\right)$, $\mathcal{M}\big|_{\mathcal{P}} : \operatorname{span}(\mathcal{B}) \mapsto \mathbb{R}^D$, is invertible on $\operatorname{span}(\mathcal{B})$. Indeed, numerical experiments indicate this turns out to be the case for many different choices of local pairs $\left\{(\mathbf{a}_u, \mathbf{b}_v) \,\middle|\, (u,v) \in \Omega^2\right\} \subset \mathbb{C}^d \times \mathbb{C}^d$ as long as $|\Omega| \geq 2\delta - 1$. Moreover, in this paper we prove the following proposition, a corollary of which identifies pairs $\mathbf{a}, \mathbf{b}$ which produce an invertible linear system:

Proposition 1. *Let $T_\delta : \mathbb{C}^{d\times d} \to \mathbb{C}^{d\times d}$ be the operator given by*

$$T_\delta(X)_{ij} = \begin{cases} X_{ij}, & |i-j| < \delta \mod d \\ 0, & \textit{otherwise} \end{cases}.$$

*Note that $\mathcal{P}$ can also be described as a restriction operator onto the indices associated with the elements of $\mathcal{B}$. Our periodic boundary conditions also imply that, e.g., $|j-j'| < \delta \Leftrightarrow \exists h \in \mathbb{Z}$ with $|h| < \delta$ s.t. $j' + h \equiv j \mod d$.

If the space $T_\delta(\mathbb{C}^{d\times d})$ is spanned by the collection $\{a_j a_j^*\}_{j=1}^K$, then $\mathcal{P}(\mathbb{C}^{d^2\times d^2})$ is spanned by

$$\{(a_j\otimes a_{j'})(a_j\otimes a_{j'})^*\}_{(j,j')\in[K]^2}=\{(a_j a_j^*)\otimes(a_{j'}a_{j'}^*)\}_{(j,j')\in[K]^2}.$$

Proof. By (5), it suffices to show that

$$(e_k e_{k'}^*)\otimes(e_j e_{j'}^*)\in \operatorname{span}\{(a_n a_n^*)\otimes(a_{n'}a_{n'}^*)\}_{(n,n')\in[K]^2}$$

for any $|j-j'|,|k-k'|<\delta$. Indeed, we have that $\{E_{jj'}:|j-j'|<\delta \mod d\}$ forms a basis for $\mathcal{B}$, so $E_{jj'},E_{kk'}\in\operatorname{span}\{a_n a_n^*\}_{n\in[K]}$ and

$$(e_k e_{k'}^*)\otimes(e_j e_{j'}^*)\in \operatorname{span}\{(a_n a_n^*)\otimes(a_{n'}a_{n'}^*)\}_{(n,n')\in[K]^2}.$$

□

In theorem 4 of,[3] an illumination function $\mathbf{a}\in\mathbb{C}^d$ with $\operatorname{supp}(\mathbf{a})\subset[\delta]$ is offered such that $\{S_\ell\mathbf{a}_u\mathbf{a}_u^*S_\ell^*\}_{(\ell,u)\in[d]^2}$ spans $T_\delta(\mathbb{C}^d)$; by proposition 1, we have the following corollary.

Corollary 1. *Choose a constant $a\in[4,\infty)$ and let the vectors $\mathbf{a}_\ell$ be defined by $(\mathbf{a}_\ell)_k=\dfrac{e^{-k/a}}{\sqrt[4]{2\delta-1}}\cdot\mathbb{1}_{k\le\delta}$. Then if $2\delta-1$ divides d (with $d=k(2\delta-1)$), we have that*

$$\{S_\ell^*\overline{\mathbf{a}_u\mathbf{a}_u}^*S_\ell\otimes S_{\ell'}^*\mathbf{a}_v\mathbf{a}_v^*S_{\ell'}\}_{(u,v,\ell,\ell')\in[d]^2\times k[2\delta-1]^2}$$

spans $\mathcal{P}$.

We remark that the condition $2\delta-1|d$ may be met by zero padding the matrix Q.

2.2 Computing the Phases of the Entries of Q after Inverting $\mathcal{M}\big|_{\mathcal{P}}$

Assuming that $\mathcal{M}\big|_{\mathcal{P}}$ is invertible so that we can recover $\mathcal{P}(\overrightarrow{Q}\overrightarrow{Q}^*)$ from our measurements $\mathbf{y}$, we are still left with the problem of how to recover $\overrightarrow{Q}$ from $\mathcal{P}(\overrightarrow{Q}\overrightarrow{Q}^*)$. Our first step in solving for $\overrightarrow{Q}$ will be to compute all the phases of the entries of $\overrightarrow{Q}$ from $\mathcal{P}(\overrightarrow{Q}\overrightarrow{Q}^*)$. Thankfully, this can be solved as an angular synchronization problem[5] using the variant utilized by BlockPR.[4,6] Let $\mathbb{1}\in\mathbb{C}^{d^2\times d^2}$ be the vector of all ones, and $\operatorname{sgn}:\mathbb{C}\mapsto\mathbb{C}$ be

$$\operatorname{sgn}(z)=\begin{cases}\dfrac{z}{|z|}, & z\neq 0\\ 1, & \text{otherwise}\end{cases}.$$

We now define $\widetilde{Q}\in\mathbb{C}^{d^2\times d^2}$ by $\widetilde{Q}=\mathcal{P}\left(\operatorname{sgn}\left(\overrightarrow{Q}\overrightarrow{Q}^*\right)\right)$; as we shall see, the principal eigenvector of $\widetilde{Q}$ will provide us with all of the phases of the entries of $\overrightarrow{Q}$.

Working toward that goal we may note that

$$\widetilde{Q}=\operatorname{diag}\left(\operatorname{sgn}\left(\overrightarrow{Q}\right)\right)\mathcal{P}\left(\mathbb{1}\mathbb{1}^*\right)\operatorname{diag}\left(\overline{\operatorname{sgn}\left(\overrightarrow{Q}\right)}\right) \tag{9}$$

where sgn is applied component-wise to vectors, and where $\operatorname{diag}(\mathbf{x})\in\mathbb{C}^{d^2\times d^2}$ is diagonal with $(\operatorname{diag}(\mathbf{x}))_{j,j}:=x_j$ for all $\mathbf{x}\in\mathbb{C}^{d^2}$ and $j\in[d^2]$. After noting that $\operatorname{diag}(\operatorname{sgn}(\cdot))$ always produces a unitary diagonal matrix, we can further see that the spectral structure of $\widetilde{Q}$ is primarily determined by $\mathcal{P}(\mathbb{1}\mathbb{1}^*)$. The following theorem completely characterizes the eigenvalues and eigenvectors of $\mathcal{P}(\mathbb{1}\mathbb{1}^*)$.

Theorem 1. *Let $F\in\mathbb{C}^{d\times d}$ be the unitary discrete Fourier transform matrix with $F_{j,k}:=\frac{1}{\sqrt{d}}\mathbb{e}^{2\pi\mathbb{i}\frac{(j-1)(k-1)}{d}}$ $\forall j,k\in[d]$, and let $D\in\mathbb{C}^{d\times d}$ be the diagonal matrix with $D_{j,j}=1+2\sum_{k=1}^{\delta-1}\cos\left(\frac{2\pi(j-1)k}{d}\right)$ $\forall j\in[d]$. Then,*

$$\mathcal{P}\left(\mathbb{1}\mathbb{1}^*\right)=\left(F\otimes F\right)\left(D\otimes D\right)\left(F\otimes F\right)^*.$$

In particular, the principal eigenvector of $\mathcal{P}(\mathbb{1}\mathbb{1}^*)$ *is* $\mathbb{1}$ *and its associated eigenvector is* $(2\delta-1)^2$.

Proof. From the definition of $\mathcal{P}$ we have that

$$\begin{aligned}\mathcal{P}(\mathbb{1}\mathbb{1}^*) &= \sum_{j=1}^{d}\sum_{|j-j'|<\delta}\sum_{k=1}^{d}\sum_{|k-k'|<\delta}\overrightarrow{E_{j,k}}\left(\overrightarrow{E_{j',k'}}\right)^* \\ &= \sum_{j=1}^{d}\sum_{|j-j'|<\delta}\sum_{k=1}^{d}\sum_{|k-k'|<\delta}E_{kk'}\otimes E_{jj'} \\ &= \left(\sum_{k=1}^{d}\sum_{|k-k'|<\delta}E_{kk'}\right)\otimes\left(\sum_{j=1}^{d}\sum_{|j-j'|<\delta}E_{jj'}\right) \\ &= T_\delta(\mathbb{1}\mathbb{1}^*)\otimes T_\delta(\mathbb{1}\mathbb{1}^*)\end{aligned}$$

Thankfully the eigenvectors and eigenvalues of T_δ are known (see Lemma 1 of Iwen, Preskitt, Saab, and Viswanathan[4]). In particular, $T_\delta = FDF^*$ which then yields the desired result by Theorem 4.2.12 of Horn and Johnson.[7] □

Theorem 1 in combination with (9) makes it clear that $\operatorname{sgn}\left(\overrightarrow{Q}\right)$ will be the principal eigenvector of $\widetilde{Q}$. As a result, we can rapidly compute the phases of all the entries of $\overrightarrow{Q}$ by using, e.g., a shifted inverse power method[8] in order compute the eigenvector of $\widetilde{Q}$ corresponding to the eigenvalue $(2\delta-1)^2$.

2.3 Computing the Magnitudes of the Entries of Q after Inverting $\mathcal{M}\big|_{\mathcal{P}}$

Having found the phases of each entry of $\overrightarrow{Q}$ using $\mathcal{P}(\overrightarrow{Q}\overrightarrow{Q}^*)$ it only remains to find each entry's magnitude as well. This is comparably easy to achieve. Note that the set $\mathcal{B}$ above always contains $\overrightarrow{E_{j,k}}(\overrightarrow{E_{j,k}})^*$ for all $j,k\in[d]$. As a result, $\mathcal{P}(\overrightarrow{Q}\overrightarrow{Q}^*)$ is guaranteed to always provide the diagonal entries of $\overrightarrow{Q}\overrightarrow{Q}^*$ for all $\delta\geq 1$, and the diagonal entries of $\overrightarrow{Q}\overrightarrow{Q}^*$ are exactly the squared magnitudes of each entry in $\overrightarrow{Q}$. Combined with the phase information recovered above in §2.2 we are finally able to reconstruct every entry of $\overrightarrow{Q}$ up to a global phase. See Algorithm 1 for complete pseudocode.

3. NUMERICAL EVALUATION

We will now demonstrate the efficiency and robustness of Algorithm 1.

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