# Two Dimensional Phase Retrieval from Local Measurements


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## ABSTRACT

2D or not 2D, that is the tribe called question



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## 1. INTRODUCTION

In this paper we consider the problem of approximately recovering an unknown two dimensional sample transmission function $q : \mathbb{R}^2 \to \mathbb{C}$ with compact support, $\text{supp}(q) \subset [0,1]^2$, from phaseless Fourier measurements of the form

$$\left|\left(\mathcal{F}\left[aS_{x_0,y_0}q\right]\right)(u,v)\right|^2, \quad (u,v) \in \Omega \subset \mathbb{R}^2, \quad (x_0,y_0) \in \mathcal{L} \subset [0,1]^2 \tag{1}$$

where $\mathcal{F}$ denotes the 2 dimensional Fourier transform, $a : \mathbb{R}^2 \to \mathbb{C}$ is a known illumination function from an illuminating beam, $S_{x_0,y_0}$ is a shift operator defined by $\left(S_{x_0,y_0}q\right)(x,y) := q(x-x_0, y-y_0)$, $\Omega$ is a finite set of sampled frequencies, and $\mathcal{L}$ is a finite set of shifts. When the illuminating beam is sharply focussed one can further assume that $a$ is also (effectively) compactly supported within a smaller region $[0,\delta']^2$ for $\delta' \ll 1$. This is known as the *ptychographic imaging problem* and is of great interest in the physics community (see, e.g., Rodenburg[?]). Herein we will make the further assumption that all the utilized shifts of $q$ also have their supports contained in $[0,1]^2$. That is, that

$$\bigcup_{(x_0,y_0)\in\mathcal{L}} \text{supp}\left(S_{x_0,y_0}q\right) \subseteq [0,1]^2$$

holds. Note that this assumption can always be achieved via rescaling.

Discretizing (**??**) using periodic boundary conditions we obtain a finite dimensional problem aimed at recovering an unknown matrix $Q \in \mathbb{C}^{d\times d}$ from phaseless measurements of the form

$$\left|\frac{1}{d^2}\sum_{j=1}^{d}\sum_{k=1}^{d} A_{j,k}\left(S_\ell Q S_{\ell'}^*\right)_{j,k} \mathbb{e}^{\frac{-2\pi \mathbb{i}}{d}(ju+kv)}\right|^2 \tag{2}$$

where $A \in \mathbb{C}^{d\times d}$ is a known measurement matrix representing our illuminating beam, and $S_\ell : \mathbb{C}^d \mapsto \mathbb{C}^d$ is the discrete circular shift operator defined by $(S_\ell \mathbf{x})_j := x_{j+\ell \text{ mod } d}$ for all $\mathbf{x} \in \mathbb{C}^d$ and $j,\ell \in [d] := \{1,\dots,d\}$. Herein we will make the simplifying assumption that our original illuminating beam funciton $a$ is not only sharply focused, but also separable. Using this assumption we let weighted measurement matrix be $\frac{1}{d^2}A := \mathbf{a}\mathbf{b}^*$ where $\mathbf{a},\mathbf{b} \in \mathbb{C}^d$ both have $a_j = b_j = 0$ for all $j \in [d] \setminus \{1,\dots,\delta\}$. Here $\delta \in \mathbb{Z}^+$ is much smaller than $d$.


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Using the small support and separability of $\frac{1}{d^2}A := \mathbf{a}\mathbf{b}^*$ we can now rewrite the measurements (**??**) as

$$\left|\sum_{j=1}^{\delta}\sum_{k=1}^{\delta} a_j\overline{b_k}\left(S_\ell Q S_{\ell'}^*\right)_{j,k} \mathbb{e}^{\frac{-2\pi\mathbb{i}}{d}(ju+kv)}\right|^2 = \left|\sum_{j=1}^{\delta}\sum_{k=1}^{\delta} \overline{\overline{a_j}\mathbb{e}^{\frac{2\pi\mathbb{i}ju}{d}} b_k \mathbb{e}^{\frac{2\pi\mathbb{i}kv}{d}}}\left(S_\ell Q S_{\ell'}^*\right)_{j,k}\right|^2$$
$$= \left|\langle S_\ell Q S_{\ell'}^*, \mathbf{a}_u\mathbf{b}_v^*\rangle_{\mathrm{HS}}\right|^2 \qquad (3)$$

where $\mathbf{a}_u, \mathbf{b}_v \in \mathbb{C}^d$ are defined by $(a_u)_j := \overline{\mathbb{e}^{\frac{-2\pi\mathbb{i}ju}{d}} a_j}$ and $(b_v)_k := \overline{\mathbb{e}^{\frac{2\pi\mathbb{i}kv}{d}} b_k}$ for all $j,k \in [d]$. Continuing to rewrite (**??**) we can now see that our discretized measurements will all take the form of

$$\left|\langle S_\ell Q S_{\ell'}^*, \mathbf{a}_u\mathbf{b}_v^*\rangle_{\mathrm{HS}}\right|^2 = \left|\mathrm{Trace}\left(\mathbf{b}_v\mathbf{a}_u^* S_\ell Q S_{\ell'}^*\right)\right|^2 = \left|\mathrm{Trace}\left(S_{\ell'}^*\mathbf{b}_v\left(S_\ell^*\mathbf{a}_u\right)^* Q\right)\right|^2 = \left|\left\langle Q, S_\ell^*\mathbf{a}_u\left(S_{\ell'}^*\mathbf{b}_v\right)^*\right\rangle_{\mathrm{HS}}\right|^2 \quad (4)$$

for a finite set of frequencies $(u,v) \in \Omega \subset \mathbb{R}^2$ and shifts $(\ell,\ell') \in \mathcal{L} \subseteq [d]\times[d]$.

Motivated by ptychographic imaging we propose a new efficient numerical scheme for solving discrete phase retrieval problems using measurements of type (**??**) herein.

OUTLINE OF REMAINING SECTIONS HERE!!!!

## 2. AN EFFICIENT METHOD FOR SOLVING THE DISCRETE 2D PHASE RETRIEVAL PROBLEM

In this section we present a lifted formulation[?] of the discrete 2D phase retrieval from local measurements of type (**??**). We then use this lifted formulation to rapidly solve for $Q \in \mathbb{C}^{d\times d}$ using a modified variant of the BlockPR agorithm.[?,?] More specifically, we will consider the collection of measurements given by

$$y_{(\ell,\ell',u,v)} := \left|\langle Q, S_\ell^*\mathbf{a}_u\mathbf{b}_v^* S_{\ell'}\rangle_{\mathrm{HS}}\right|^2$$

for all $(\ell,\ell',u,v) \in [d]^2 \times [2\delta-1]^2$ herein.* Thus, we collect a total of $D := (2\delta-1)^2 \cdot d^2$ measurements, where each measurement is due to a vertical and horizontal shift of a rank one 2D illumination pattern $\mathbf{a}_u\mathbf{b}_v^*$. As above, the inner product is the Hilbert-Schmidt inner product.

Let $X \in \mathbb{C}^{d\times d}$ and consider measurements of the form

$$y_I = \left|\langle X, S_\ell^*\mathbf{m}_j\mathbf{m}_{j'}^* S_{\ell'}\rangle_{HS}\right|^2$$

where the inner product is the Hilbert-Schmidt inner product. Here, $S_\ell : \mathbb{C}^d \mapsto \mathbb{C}^d$ and $\mathbf{m}_j$ are as usual, and $I = (j,j',\ell,\ell')$ is a multi-index with $j,j' \in [2\delta-1]$, and $\ell,\ell' \in [d]_0$. Define the multi-index set $\mathcal{I} = [2\delta-1]^2 \times [d]_0^2$ and note that, .

Let $\mathcal{P}$ be the projection onto the span of

$$\left\{\overrightarrow{\left(S_\ell^*\mathbf{m}_j\right)^T \otimes \left(S_{\ell'}^*\mathbf{m}_{j'}\right)^*}\ \overrightarrow{\left(\left(S_\ell^*\mathbf{m}_j\right)^T \otimes \left(S_{\ell'}^*\mathbf{m}_{j'}\right)^*\right.}\right\}_{(j,j',\ell,\ell')},$$

i.e., $\mathcal{P}$ is analogous to $T_\delta$ from our last paper (as will become apparent shortly). Note that each of our measurements $y_I$ is of the form $|\langle X, \mathbf{a}\mathbf{b}^*\rangle|^2$ where $\mathbf{a},\mathbf{b} \in \mathbb{C}^d$. So, denoting by $\overrightarrow{X} \in \mathbb{C}^{d^2}$ the (matlab-style)

*For any $n \in \mathbb{Z}^+$ we will let $[n] := \{1,2,3,\ldots,n\} \subset \mathbb{Z}^+$.

vectorization of $X$, and by $A \otimes B$ the Kronecker product of $A$ and $B$, we can write

$$\begin{aligned}
|\langle X, \mathbf{a}\mathbf{b}^* \rangle_{HS}|^2 &= |\operatorname{Tr}(\mathbf{b}\mathbf{a}^* X)|^2 \\
&= |\mathbf{a}^* X \mathbf{b}|^2 = |\langle \overrightarrow{\mathbf{b}^T \otimes \mathbf{a}^*}, \overrightarrow{X} \rangle_{\ell_2^{d^2}}|^2 \\
&= \left( \langle \overrightarrow{\mathbf{b}^T \otimes \mathbf{a}^*}, \overrightarrow{X} \rangle_{\ell_2^{d^2}} \right)^* \cdot \langle \overrightarrow{\mathbf{b}^T \otimes \mathbf{a}^*}, \overrightarrow{X} \rangle_{\ell_2^{d^2}} \\
&= \overrightarrow{X}^* \ \overrightarrow{\mathbf{b}^T \otimes \mathbf{a}^*} \ (\overrightarrow{\mathbf{b}^T \otimes \mathbf{a}^*})^* \ \overrightarrow{X} \\
&= \operatorname{Tr}\left( \overrightarrow{X}\overrightarrow{X}^* \ \overrightarrow{\mathbf{b}^T \otimes \mathbf{a}^*} \ (\overrightarrow{\mathbf{b}^T \otimes \mathbf{a}^*})^* \right) \\
&= \left\langle \overrightarrow{X}\overrightarrow{X}^*, \overrightarrow{\mathbf{b}^T \otimes \mathbf{a}^*} \ (\overrightarrow{\mathbf{b}^T \otimes \mathbf{a}^*})^* \right\rangle_{HS}
\end{aligned} \tag{5}$$

Accordingly, let $\mathcal{A}$ be the linear operator given by

$$\begin{aligned}
\mathcal{A} : (\mathbb{C}^{d^2 \times d^2}) &\to \mathbb{R}^D \\
Z &\mapsto \left( \left\langle Z, \overrightarrow{(S_\ell^* \mathbf{m}_j)^T \otimes (S_{\ell'}^* \mathbf{m}_{j'})^*} \ (\overrightarrow{(S_\ell^* \mathbf{m}_j)^T \otimes (S_{\ell'}^* \mathbf{m}_{j'})^*})^* \right\rangle_{HS} \right)_{(j,j',\ell,\ell') \in \mathcal{I}}
\end{aligned} \tag{6}$$

To generalize our theorem from the ACHA submission, we need (at least) two ingredients: The spectral gap of the adjacency matrix $\mathcal{P}(\mathbb{1}\mathbb{1}^*)$ and the condition number of the linear operator $\mathcal{A}$.

## 2.1 Spectral gap of the adjacency matrix $\mathcal{P}(\mathbb{1}\mathbb{1}^*)$

Here note that the doubly indexed vertices $(i,j), (i',j') \in [d] \times [d]$ are connected by an edge if and only if

$$|i - i'| \mod d < \delta \quad \text{and} \quad |j - j'| \mod d < \delta.$$

That is, the graph is the tensor-product of two identical graphs, each with adjacency matrix $T_\delta(\mathbb{e}_d \mathbb{e}_d^*)$. So that now

$$\mathcal{P}(\mathbb{e}_{d^2} \mathbb{e}_{d^2}^*) = T_\delta(\mathbb{e}_d \mathbb{e}_d^*) \otimes T_\delta(\mathbb{e}_d \mathbb{e}_d^*).$$

Using the fact that the eigenvalues of the Kronecker product are the pairwise products of the eigenvalues of the individual matrices, the spectral gap is now $\mathcal{O}(\frac{\delta^4}{d^2})$.

## 2.2 Condition number of the linear operator $\mathcal{A}$

We will show that $\mathcal{A} = \mathcal{M} \otimes \mathcal{M}$ for an appropriate linear operator $\mathcal{M} : T_\delta(\mathbb{C}^{d \times d}) \to \mathbb{R}^{d(2\delta - 1)}$.

**Observation 1:**

Recall that each row of the matrix representation of $\mathcal{A}$ from (**??**) is a vectorized version of a rank-1 matrix of the form $\overrightarrow{\mathbf{b}^T \otimes \mathbf{a}^*} \ (\overrightarrow{\mathbf{b}^T \otimes \mathbf{a}^*})^*$ as can be seen in (**??**), appropriately restricted (but let's worry about that later). Now, observe that

$$\overrightarrow{\mathbf{b}^T \otimes \mathbf{a}^*} \ (\overrightarrow{\mathbf{b}^T \otimes \mathbf{a}^*})^* = (b_1 \mathbf{a}^*, ..., b_d \mathbf{a}^*)^T (\bar{b}_1 \mathbf{a}, ..., \bar{b}_d \mathbf{a})$$

so that the entries of the rank-1 matrix $\overrightarrow{\mathbf{b}^T \otimes \mathbf{a}^*} \ (\overrightarrow{\mathbf{b}^T \otimes \mathbf{a}^*})^*$ can be multi-indexed via

$$(\overrightarrow{\mathbf{b}^T \otimes \mathbf{a}^*} \ (\overrightarrow{\mathbf{b}^T \otimes \mathbf{a}^*})^*)_{m,n,m',n'} = b_m \bar{a}_n \bar{b}_{m'} a_{n'} = b_m \bar{b}_{m'} \bar{a}_n a_{n'}. \tag{7}$$

On the other hand, note that

$$\overrightarrow{\mathbf{a}^T \otimes \mathbf{a}^*}\ (\overrightarrow{\mathbf{b}^T \otimes \mathbf{b}^*})^* = (a_1\mathbf{a}^*, ..., a_d\mathbf{a}^*)^T(\bar{b}_1\mathbf{b}, ..., \bar{b}_d\mathbf{b})$$

so that

$$(\overrightarrow{\mathbf{a}^T \otimes \mathbf{a}^*}\ (\overrightarrow{\mathbf{b}^T \otimes \mathbf{b}^*})^*)_{n',n,m',m} = b_m\bar{b}_{m'}\bar{a}_n a_{n'}. \tag{8}$$

Comparing (**??**) and (**??**) and noting that $m, n$ and $m', n'$ all range over the same set we see that (**??**) and (**??**) are the same up-to a permutation.

**Observation 2:**

In our case, each row of $\mathcal{A}$ corresponds to some $\mathbf{a} = S_{\ell'}^*\mathbf{m}_{j'}$ and $\mathbf{b} = S_\ell^*\mathbf{m}_j$, where every combination of $(j, j', \ell, \ell') \in \mathcal{I}$ is taken. That is, every row of $\mathcal{A}$ is a permutation of (a vectorized version) of

$$\overrightarrow{(S_\ell^*\mathbf{m}_j)^T \otimes (S_\ell^*\mathbf{m}_j)^*}\ (\overrightarrow{(S_{\ell'}^*\mathbf{m}_{j'})^T \otimes (S_{\ell'}^*\mathbf{m}_{j'})^*})^*$$

for some $(j, j', \ell, \ell') \in \mathcal{I}$. In other words we have a permutation of the Kronecker product of

$$\mathcal{M} : X \in \mathbb{C}^{d\times d} \mapsto (\langle X, (S_\ell^*\mathbf{m}_j)(S_\ell^*\mathbf{m}_j)^*\rangle)_{j,\ell}$$

with itself.

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