Agenda:

1. Basic set theory
2. Sample spaces and events
3. Probability

Basic set theory

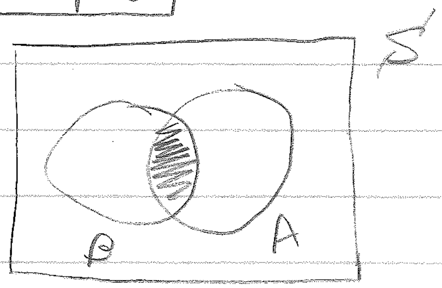
Def A set is a collection of distinct objects, which are called elements or points of the set.

- Sets are denoted by capital letters, e.g., $A, B, \ldots$

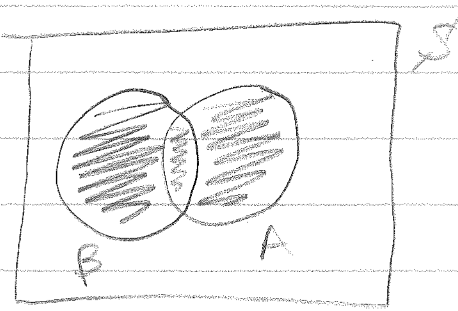
Notation

- Subset: "$A \subset B$" means A is a subset of B, i.e., all the elements of A are also in B.
- Null set/empty set "$\emptyset$" denotes the set which contains no points.
- Union: "$A \cup B$" denotes the collection of points which are in A or B (or both).
- Intersection: "$A \cap B$" denotes the collection of points which are in both A and B.
- Sample space/Universal set: "S" denotes the universal set, i.e., the set containing all points of interest in the current situation.
- Complement: "A^c" denotes the set of all points in S not contained in A.
- mutually exclusive: A and B are called mutually exclusive if $A \cap B = \emptyset$ (empty set).

Examples



$A \cap B$ is shaded

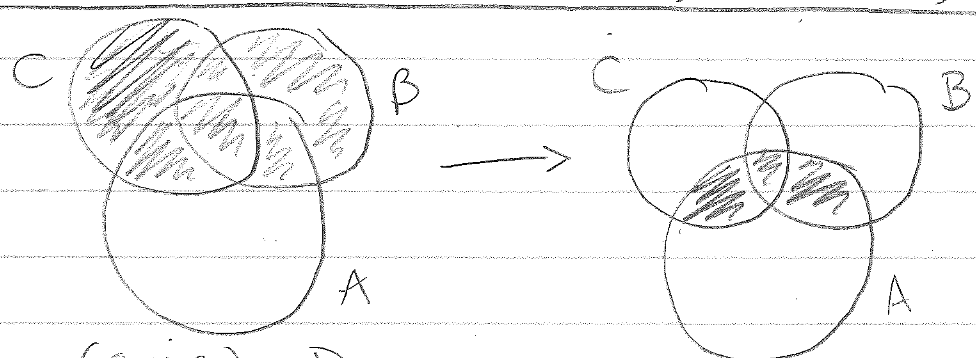


$A \cup B$ is shaded

$\boxed{Q}$ What area is $(A^c) \cap B$?

Distributive law

- $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
- $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$



$(B \cup C) = D$ $\qquad$ $D \cap A$

DeMorgan's law:

- $(A \cup B)^c = A^c \cap B^c$
- $(A \cap B)^c = A^c \cup B^c$

$\boxed{Q}$ Convince yourself using Venn diagram.

* Both laws will be useful for probability computations going forward.

Example Let $S = \{1,2,3,4\}$.

Let $A = \{1,2,3\}$, $B = \{2,3,4\}$, $C = \{4\}$

- $A \cup B = \{1,2,3,4\}$
- $A \cap B = \{2,3\}$
- $A \cap C = \emptyset$
- $C \subset B \Rightarrow C \cup B = B$, $C \cap B = C$
- $A^c = \{4\}$

Sample spaces and events

- Perform a random experiment/observe a random phenomenon. For example, consider the tossing of a coin 4 times or the proportion of people in a population affected by the flu.

Def The <u>sample space</u> S of a random experiment is the set of all possible outcomes of the experiment listed in a mutually exclusive and <u>exhaustive</u> way.

Examples of S

- Toss a coin 4 times?

  $\longrightarrow S = \{HHHH, THHH, \ldots, TTTT\}$

  $\therefore$ $2^4 = 16$ possible outcomes.

- Percentage of people in a population affected by the flu?

  $\longrightarrow S = [0, 100]$, i.e., all possible real numbers from 0 to 100. This is an example of a continuous or "uncountable" sample space.

Def An *event* is a collection of sample points. In other words, any subset of the sample space S is called an event.

Example

Toss a coin three times:

$$S = \{HHH, THH, \ldots\ldots, TTT\}$$

Let A = event that there is at least one heads
$= \{HHH, THH, HTH, HHT, TTH, HTT, THT\}$

B = event that there is at most one heads
$= \{TTH, HTT, THT, TTT\}$

Probability

Formal def of probability Intuitively, the "probability" of an event is a number between 0 and 1 expressing our belief in the occurence of the event in a single performance of an experiment.

(*) $\mathcal{A}$ = collection of all possible events

A probability assignment P for a random experiment is a numerically valued function that assigns a value $P(A)$ to every event A so that the following axioms hold:

(1) $P(A) \geq 0$ for every event $A \in \mathcal{A}$ (2) $P(S) = 1$

(3) If $A_1, A_2, A_3, \ldots$ is a sequence of mutually exclusive events (i.e. $A_i \cap A_j = \emptyset$ for $i \neq j$) then

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i)$$

Consequences of axioms

(i) $P(\emptyset) = 0$: Formal proof in a moment...

(ii) $A \cap B = \emptyset \Longrightarrow P(A \cup B) = P(A) + P(B)$

Hint: Set $A_1 = A$, $A_2 = B$, all others to $\emptyset$, apply 3.

(iii) $A \subset B \Longrightarrow P(A) \leq P(B)$ $\boxed{Q}$ Prove this.

$\boxed{\text{Proof of } P(\emptyset) = 0}$ For axiom 3, choose $A_1 = S$, $A_i = \emptyset$ for all $i \geq 2$. It is immediate then that these events are mutually exclusive. Hence, we get that

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i)$$

$$\Longrightarrow P(S) = P(S) + \sum_{i=2}^{\infty} P(A_i) \quad \text{since } \bigcup_{i=1}^{\infty} A_i = S$$

$$\Longrightarrow \sum_{i=2}^{\infty} P(\emptyset) = \sum_{i=2}^{\infty} P(A_i) = 0.$$

Since $P(\emptyset) \geq 0$ by axiom 1, it follows that $P(\emptyset) = 0$.

$\boxed{Q}$ Prove (ii) in a similar way using the hint.

To recap:

Defining and calculating the probability of an event by the sample point method (* for discrete sample space)

1. Define the experiment
2. Construct the sample space
3. Assign probabilities to each of the sample points (ensuring they sum to one)
4. Express event of interest as a collection of sample points.
5. Find P(A) by summing the probabilities of all the sample points in A.