Homework 6 Solutions
STA4321 and STA5325, Spring 2022
Introduction to Probability
Department of Statistics, University of Florida
Due on Friday, March 18th, 2022 at 4:00pm

All work must be shown for complete credit. W.M.S. denotes the course textbook (*Mathematical Statistics with Applications*).

1. A circle of radius r has area $A = \pi r^2$. If a random circle has radius that is uniformly distributed on $[0, \theta]$:

   (a) What are the mean and variance of the area of the circle?

   $$\mathrm{E}(A) = \mathrm{E}(\pi r^2) = \pi \mathrm{E}(r^2) = \pi \left\{ \mathrm{Var}(r) + [\mathrm{E}(r)]^2 \right\} = \pi \left[\tfrac{\theta^2}{12} + \tfrac{\theta^2}{4} \right] = \tfrac{\pi\theta^2}{4} \left[\tfrac{1}{3} + 1 \right] = \tfrac{\pi\theta^2}{3}$$

   since we know the expected value and variance of a uniform random variable. Notice, from above, we have $\mathrm{E}(r^2) = \left[\frac{\theta^2}{12} + \frac{\theta^2}{4} \right] = \frac{\theta^2}{3}$ (which we could also get by integration). Then,

   $$\mathrm{Var}(A) = \mathrm{Var}(\pi r^2) = \pi^2 \mathrm{Var}(r^2) = \pi^2 \left\{ \mathrm{E}(r^4) - [\mathrm{E}(r^2)]^2 \right\} = \pi^2 \left\{ \mathrm{E}(r^4) - \frac{\theta^4}{9} \right\}.$$

   Now, it only remains to compute $\mathrm{E}(r^4)$.

   $$\mathrm{E}(r^4) = \int_0^\theta \frac{r^4}{\theta} dr = \left[\frac{r^5}{5\theta} \right]_0^\theta = \frac{\theta^4}{5}.$$

   Hence,

   $$\mathrm{Var}(A) = \pi^2 \left\{ \frac{\theta^4}{5} - \frac{\theta^4}{9} \right\} = \pi^2 \theta^4 \left[\frac{9}{45} - \frac{5}{45} \right] = \frac{4\pi^2\theta^4}{45}.$$

   (b) Suppose $\theta = 1$. What is the probability distribution function of A? That is, what is $F_A(a) = P(A \leq a)$?

   Notice

   $$\begin{aligned} F_A(a) = P(A \leq a) &= P(\pi r^2 \leq a) \\ &= P\left(r^2 \leq \frac{a}{\pi} \right) \end{aligned}$$

   then, taking the square root of both sides, we have:

   $$= P\left(-\sqrt{\frac{a}{\pi}} \leq r \leq \sqrt{\frac{a}{\pi}} \right)$$

so we have shown

$$\begin{aligned} F_A(a) &= P\left(-\sqrt{\frac{a}{\pi}} \leq r \leq \sqrt{\frac{a}{\pi}}\right) \\ &= P\left(0 \leq r \leq \sqrt{\frac{a}{\pi}}\right) \qquad\qquad \text{since } r \in [0,1] \end{aligned}$$

Hence we must consider three cases:
First consider the case that $\sqrt{\frac{a}{\pi}} < 0$. Then,

$$F_A(a) = P\left(0 \leq r \leq \sqrt{\frac{a}{\pi}}\right) = \int_0^{\sqrt{\frac{a}{\pi}}} \frac{1}{1} dr = \int_0^0 dr = 0$$

Next consider the case that $0 \leq \sqrt{\frac{a}{\pi}} \leq 1$. Then,

$$F_A(a) = P\left(0 \leq r \leq \sqrt{\frac{a}{\pi}}\right) = \int_0^{\sqrt{\frac{a}{\pi}}} \frac{1}{1} dr = [r]_0^{\sqrt{\frac{a}{\pi}}} = \sqrt{\frac{a}{\pi}}$$

And finally, consider $\sqrt{\frac{a}{\pi}} > 1$. Then, we have

$$F_A(a) = P\left(0 \leq r \leq \sqrt{\frac{a}{\pi}}\right) = \int_0^{\sqrt{\frac{a}{\pi}}} \frac{1}{1} dr = \int_0^1 \frac{1}{1} dr = 1.$$

Hence,

$$F_A(a) = \begin{cases} 0 & : a < 0 \\ \sqrt{\frac{a}{\pi}} & : 0 \leq a \leq \pi \\ 1 & : a > \pi \end{cases}$$

2. Suppose X is a random variable having the Uniform$[0,6]$ distribution. Compute the following conditional probabilities:

   (a) $P(X \in [0,1] \mid X < 3)$

   For each of the following, recall that $X < 3 \implies X \in [0,3)$, and conversely, $X \geq 3 \implies X \in [3,6]$. Then,

$$\begin{aligned} P(X \in [0,1] \mid X < 3) &= \frac{P(X \in [0,1] \cap X \in [0,3))}{P(X \in [0,3))} \\ &= \frac{P(X \in [0,1])}{P(X \in [0,3))} \end{aligned}$$

   since $[0,1]$ is entirely contained in $[0,3)$. Then,

$$P(X \in [0,1]) = \int_0^1 \frac{1}{6} dx = \frac{1}{6}, \quad \text{and} \quad P(X \in [0,3)) = P(X \in [0,3]) = \int_0^3 \frac{1}{6} dx = \frac{1}{2},$$

   so

$$P(X \in [0,1] \mid X < 3) = \frac{P(X \in [0,1])}{P(X \in [0,3))} = \frac{(\frac{1}{6})}{(\frac{1}{2})} = \frac{1}{3}.$$

(b) $P(X \in [0,4] \mid X < 3)$

$$P(X \in [0,4] \mid X < 3) = \frac{P(X \in [0,4] \cap X \in [0,3))}{P(X \in [0,3))}$$

and because $[0,3)$ is contained in $[0,4]$,

$$\begin{aligned} &= \frac{P(X \in [0,3])}{P(X \in [0,3))} \\ &= \frac{P(X \in [0,3])}{P(X \in [0,3])} \\ &= 1 \end{aligned}$$

(c) $P(X \in [0,4] \mid X \geq 3)$

$$P(X \in [0,4] \mid X \geq 3) = \frac{P(X \in [0,4] \cap X \in [3,6])}{P(X \in [3,6])}$$

and because the intersection of $[0,4]$ and $[3,6]$ is $[3,4]$,

$$\begin{aligned} &= \frac{P(X \in [3,4])}{P(X \in [3,6])} \\ &= \frac{(\frac{1}{6})}{(\frac{1}{2})} \\ &= \frac{1}{3} \end{aligned}$$

3. If a point is randomly located on an interval (a,b) and if Y denotes the location of the point, then Y is assumed to have a uniform distribution over (a,b). A plant efficiency expert randomly selects a location along a 500-foot assembly line from which to observe the work habits of the workers on the line. What is the probability that the point she selects is:

   (a) within 25 feet of the end of the line?

   $\int_{475}^{500} \frac{1}{500-0} dx = \frac{25}{500} = \frac{1}{20}$

   (b) within 25 feet of the beginning of the line?

   $\int_{0}^{25} \frac{1}{500-0} dx = \frac{25}{500} = \frac{1}{20}$

   (c) closer to the beginning of the line than to the end of the line?

   $P(X \in [250,500]) = \int_{250}^{500} \frac{1}{500-0} dr = \frac{250}{500} = \frac{1}{2}$

4. Let $Z = \log(Y)$ where Z is a random variable following the standard normal distribution. Compute $\mathrm{E}(Y)$. (Recall that log refers to the natural log whenever we use it in class.

Based on our definition, $Y = \exp(Z)$. Then, by our definition of $\mathrm{E}[g(X)]$ for all functions g, letting $g(x) = \exp(x)$, we have

$$\begin{aligned}
\mathrm{E}(Y) &= \mathrm{E}[\exp(Z)] \\
&= \int_{-\infty}^{\infty} \frac{\exp(z)}{\sqrt{2\pi}} \exp\left(-\frac{z^2}{2}\right) dz \\
&= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{z^2}{2} + z\right) dz
\end{aligned}$$

and then, because $-\frac{(z-1)^2}{2} = -\left(\frac{z^2-2z+1}{2}\right)$, we can add and subtract $1/2$ inside the exp term to complete the square

$$\begin{aligned}
&= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{z^2}{2} + z \underbrace{-\frac{1}{2} + \frac{1}{2}}_{\text{adding zero}}\right) dz \\
&= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{(z-1)^2}{2} + \frac{1}{2}\right) dz \\
&= \exp\left(\frac{1}{2}\right) \underbrace{\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{(z-1)^2}{2}\right) dz}_{A}
\end{aligned}$$

and A is the integral from $-\infty$ to ∞ of the density of normal random variable with $\mu = 1$ and $\sigma = 1$. Hence, $A = 1$, and it follows that

$$= \exp\left(\frac{1}{2}\right).$$

Thus, we have shown $\mathrm{E}(Y) = \exp\left(\frac{1}{2}\right)$.

5. Let Z be a standard normal random variable. Then, using statistical software, we know $P(Z \leq 1) = 0.841$ and $P(Z \leq 2) = 0.977$. Using this information, answer the following: Suppose that the measured voltage in a certain electric circuit has the normal distribution with mean 120 and standard deviation 2. If three independent measurements of the voltage are made, what is the probability that all three measurements will lie between 116 and 118?

$$
\begin{aligned}
P(116 \leq X \leq 118) &= P\left(\frac{116-120}{2} \leq \frac{X-120}{2} \leq \frac{118-120}{2}\right) \\
&= P\left(-2 \leq Z \leq -1\right) \\
&= P(Z \leq -1) - P(Z < -2) \\
&= P(Z \leq -1) - P(Z \leq -2) \qquad\qquad \text{since } P(Z=-2)=0
\end{aligned}
$$

and by the symmetric of the normal distribution:

$0.977 = P(Z \leq 2) = 1 - P(Z > 2) = 1 - P(Z \leq -2) \implies P(Z \leq -2) = 1 - 0.977 = 0.023,$

and by the same logic, $P(Z \leq -1) = 0.159$ so

$$
\begin{aligned}
&= 0.159 - 0.023 \\
&= 0.136
\end{aligned}
$$

Thus, the probability that all three measurements lie in this interval is

$$(0.136)^3 \approx 0.0025$$

6. Prove that if Y is a random variable following the Gamma distribution with parameters $\alpha > 0$ and $\beta > 0$

$$\mathrm{E}(Y^3) = \frac{\beta^3 \Gamma(3+\alpha)}{\Gamma(\alpha)}.$$

$$
\begin{aligned}
\mathrm{E}(Y^3) &= \int_{-\infty}^{\infty} y^3 \frac{1}{\Gamma(\alpha)\beta^\alpha} y^{\alpha-1} \exp\left(-\frac{y}{\beta}\right) dy \\
&= \int_{-\infty}^{\infty} \frac{1}{\Gamma(\alpha)\beta^\alpha} y^{\alpha+2} \exp\left(-\frac{y}{\beta}\right) dy \\
&= \frac{1}{\Gamma(\alpha)\beta^\alpha} \int_{-\infty}^{\infty} y^{\alpha+2} \exp\left(-\frac{y}{\beta}\right) dy
\end{aligned}
$$

and with a change of variables $y = \beta u$ (and hence, $dy = \beta du$)

$$
\begin{aligned}
&= \frac{1}{\Gamma(\alpha)\beta^\alpha} \int_{-\infty}^{\infty} (\beta u)^{\alpha+2} \exp\left(-u\right) \beta du \\
&= \frac{\beta^{\alpha+3}}{\Gamma(\alpha)\beta^\alpha} \underbrace{\int_{-\infty}^{\infty} u^{\alpha+2} \exp\left(-u\right) du}_{\Gamma(\alpha+3)} \\
&= \frac{\beta^3 \Gamma(\alpha+3)}{\Gamma(\alpha)}
\end{aligned}
$$

7. (Extra credit) Suppose X is a random variable whose distribution depends on parameters $\alpha > 0$ and $\beta > 0$ with density

$$f_X(x) = \begin{cases} \frac{\beta\alpha^\beta}{x^{\beta+1}} & \alpha < x < \infty \\ 0 & x \leq \alpha \end{cases}$$

(a) Verify that f_X is a valid probability density function (Hint: be mindful of the Range(X)).

$$\begin{aligned} \int_\alpha^\infty \frac{\beta\alpha^\beta}{x^{\beta+1}} dx &= \beta\alpha^\beta \int_\alpha^\infty \frac{1}{x^{\beta+1}} dx \\ &= \beta\alpha^\beta \left[-\frac{x^{-\beta}}{\beta}\right]_\alpha^\infty \\ &= \beta\alpha^\beta \left[0 + \frac{\alpha^{-\beta}}{\beta}\right] \\ &= \beta\alpha^\beta \left(\frac{\alpha^{-\beta}}{\beta}\right) \\ &= 1 \end{aligned}$$

(b) Derive the expected value and variance of X. Does the variance exist (i.e., is it finite) for all possible values of α and β?

$$\begin{aligned} \mathrm{E}(X) &= \int_\alpha^\infty x \frac{\beta\alpha^\beta}{x^{\beta+1}} dx \\ &= \beta\alpha^\beta \int_\alpha^\infty \frac{1}{x^\beta} dx \end{aligned}$$

then, multiplying by $1 = \frac{(\beta-1)\alpha^{\beta-1}}{(\beta-1)\alpha^{\beta-1}}$ (if $\beta \neq 1$)

$$= \frac{\beta\alpha^\beta}{(\beta-1)\alpha^{\beta-1}} \underbrace{\int_\alpha^\infty (\beta-1)\alpha^{\beta-1} \frac{1}{x^\beta} dx}_{A}$$

where A is the integral of the density of a random variable with the same density as X with β replaced with $\beta - 1$ as long as $\beta > 1$. Note that when $\beta \leq 1$, the integral A is infinite. Thus

$$\mathrm{E}(X) = \begin{cases} \texttt{DNE} & : \beta \leq 1 \\ \frac{\beta\alpha}{(\beta-1)} & : \beta > 1 \end{cases}$$

since by our definition, we say $\mathrm{E}(X)$ does not exist when $\beta \leq 1$. Similary, assuming $\beta > 1$ (so the expected value exists),

$$\begin{aligned} \mathrm{Var}(X) + [\mathrm{E}(X)]^2 = \mathrm{E}(X^2) &= \int_\alpha^\infty x^2 \frac{\beta\alpha^\beta}{x^{\beta+1}} dx \\ &= \beta\alpha^\beta \int_\alpha^\infty \frac{1}{x^{\beta-1}} dx \end{aligned}$$

and again multiplying by $1 = \frac{(\beta-2)\alpha^{\beta-2}}{(\beta-2)\alpha^{\beta-2}}$ (if $\beta \neq 2$)

$$= \frac{\beta\alpha^\beta}{(\beta-2)\alpha^{\beta-2}} \underbrace{\int_\alpha^\infty \frac{(\beta-2)\alpha^{\beta-2}}{x^{\beta-1}} dx}_{B}$$

where B is the integral of the density of random variable with the same density as X with β replaced with $\beta - 2$ as long as $\beta > 2$, so we have

$$\mathrm{E}(X^2) = \frac{\beta\alpha^\beta}{(\beta-2)\alpha^{\beta-2}}, \quad \beta > 2$$

Hence, when $\beta > 2$,

$$\begin{aligned} \mathrm{Var}(X) &= \frac{\beta\alpha^\beta}{(\beta-2)\alpha^{\beta-2}} - [\mathrm{E}(X)]^2 \\ &= \frac{\beta\alpha^2}{(\beta-2)} - \left[\frac{\beta\alpha}{(\beta-1)}\right]^2 \\ &= \frac{\beta\alpha^2}{(\beta-1)^2(\beta-2)} \end{aligned}$$

and thus,

$$\mathrm{Var}(X) = \begin{cases} \texttt{DNE} & \beta \leq 2 \\ \frac{\beta\alpha^2}{(\beta-1)^2(\beta-2)} & \beta > 2 \end{cases}$$