Agenda

1. Some examples to understand the formal definition of probability.
2. Fundamental principle of counting
3. Evaluating probabilities using permutations

Examples

Random experiment : choose a person from 4 persons at random with no preference to any person.

- What is the sample space?

  ↳ Since there are 4 possible outcomes, the sample space is $\mathcal{S} = \{1, 2, 3, 4\}$.

- What is the set of all possible events?

  ↳ Recall that $\mathcal{A}$, which denotes the set of all possible events, is essentially the collection of all possible subsets of $\mathcal{S}$. Hence

$$\mathcal{A} = \left\{ \begin{array}{l} \emptyset, \{1\}, \{2\}, \{3\}, \{4\}, \{1,2\} \\ \{1,3\}, \{1,4\}, \{2,3\}, \{2,4\}, \{3,4\} \\ \{1,2,3\}, \{1,2,4\}, \{1,3,4\}, \{2,3,4\} \\ \{1,2,3,4\} \end{array} \right\}$$

- How do we provide a probability assignment P, which reflects our belief in how the random experiment is conducted and also satisfies the three axioms?

  ↳ Based on your belief of how the random experiment is conducted, assign a chance to each point of the sample space $\mathcal{S}$ such that these numbers add up to one. For this experiment

Since there is no preference for any person,

$$P(\{1\}) = \frac{1}{4}, \quad P(\{2\}) = \frac{1}{4}, \quad P(\{3\}) = \frac{1}{4}$$

$$P(\{4\}) = \frac{1}{4}.$$

Then, define the probability of any event as the sum of the probabilities of the sample points in the given event. Hence, for A = event that person 1 or 2 is chosen,

$$P(A) = P(\{1,2\}) = P(\{1\}) + P(\{2\})$$

$$= \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$$

* This procedure ensures that the probability assignment satisfies the three axioms for discrete sample spaces.

<u>Random experiment</u>: There are 3 mailboxes. Three people put letters in at random (with no preference to any of the three mailboxes).

- Compute the probability of the event that each of the three mailboxes is chosen once.

→ Each outcome is a sequence of three numbers, each denoting the mailbox chosen by the respective person. Hence,

$$\mathcal{S} = \{123, 122, 131, \ldots\}$$ 27 points in total

Since the mailboxes have no special preference by any person, all outcomes are equally likely. Hence, we assign

$$P(\{s\}) = \frac{1}{27} \text{ for all } s \in \mathcal{S}$$

Finally, let A = event that each mailbox is chosen once,

$$A = \{123, 132, 213, 231, 312, 321\}.$$

Hence, $P(A) = P(\{123\}) + P(\{132\}) + P(\{213\}) + P(\{231\}) + P(\{312\}) + P(\{321\})$

$$= \frac{1}{27} + \cdots + \frac{1}{27} = \frac{6}{27}.$$

Do we always need to go through this procedure for calculating probabilities of events? No. We can often use <u>counting rules</u> to get around the situation.

Counting rule #1 Suppose we are performing an experiment where all outcomes are equally likely, hence we assign the same probability to every sample point (say there are N sample points). Suppose the event of interest A consists of n_A sample points. Then,

$$P(A) = \frac{n_a}{N} = \frac{\#\text{ of sample points in } A}{\#\text{ of total sample points}}$$

The fundamental principle of counting

Suppose that an experiment consists of two successive tasks. The first task can result in n_1 outcomes, the second task can result in n_2 outcomes. Then, the total number of outcomes in the experiment is $n_1 n_2$.

Example: "Birthday problem"

Random experiment: Choose 25 people.

- Compute the probability of the event, A, that there is at least one match in the 25 birthdays.
  → Sample space is the collection of all possible 25-sequences of birthdays. A typical sample point looks like:

$$\underbrace{x_1 \quad x_2 \quad x_3 \quad x_4 \quad \ldots \quad x_{25}}$$

where x_i is an integer from 1 to 365 for $i = 1, \ldots, 25$. Hence, the number of sample points is $N = (365)^{25}$ (ignoring leap years).

→ Assuming all birthdays are equally likely, each sample point has probability $\frac{1}{(365)^{25}}$.

- How many different ways are there for 25 people to have no birthday in common?
  → $365 \cdot 364 \cdot 363 \cdot \ldots \cdot 341 = \frac{365!}{340!}$, so that $n_A = (365)^{25} - \frac{365!}{340!}$. Hence, by counting rule #1,

→ $P(A) = \frac{n_A}{N} = \left[\frac{\left((365)^{25} - \frac{365!}{340!}\right)}{(365)^{25}}\right] = 0.5687.$

Permutations

Here are some identities we have been using informally until now, and which will be quite helpful in counting principles for determining probabilities of events.

- What is the number of ways to choose q objects <u>with replacement</u> from n objects (order is important)?

  → There are n different ways to choose each object. Hence, the answer is

$$n \cdot n \cdot n \cdot \ldots \cdot n = n^q.$$

- What is the number of ways to choose q objects <u>without replacement</u> from n objects (order is important)?

  → The first slot can be filled in n ways, the second in $(n-1)$ ways, the third in $(n-2)$ ways, and so on.

  Hence

$$n(n-1)(n-2)\ldots(n-q+1) = \frac{n!}{(n-q)!}$$