Assessment 1: Coupled Harmonic Oscillator

Benji Tigg

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1 Introduction

In a system of multiple particles of mass M, connected by springs with spring constant K, the system can be written as a system of second order differential equations. If we wanted to find the frequencies that the system oscillates at, if we assume the solution $x_i = A_i \exp(-i\omega t)$ where $i \epsilon \{1, 2, ...\}$ we can form the matrix equation, it's possible to write this system of equations as the matrix equation $M\mathbf{A} = -\omega^2\mathbf{A}$; This is also a eigenvalue equation and so by finding the eigenvalues of $\mathbf{M}$, it's possible to find the frequency ω. For small systems this can be done analytically however for larger systems e.g. 3+ particles it's easier to solve the eigenvalue equation iteratively, this is most commonly done through a method called QR (or QU) factorization.

2 Code description

This code solves the eigenvalue equation for the coupled harmonic oscillator, using QU factorisation. The code can read input data from a file and output results to a .csv file for it to be plotted in python. The user can specify in the input file, how the system is set up e.g. the mass and the spring constant. The user can also specify and minimum and maximum value for m and k and the steps between them. The user also has the ability to set the system tolerance in the input file.

3 Analytical solving and verification

3.1 Analytical solving

In a system with 2 particles the system of differential equations can be written as

$$m\ddot{x}_1 = -2kx_1 + kx_2 \tag{1}$$

$$m\ddot{x}_2 = +kx_1 - 2kx_2 \tag{2}$$

We assume the form on the solution for x_1 and x_2 to be

$$A_1 = \exp(-i\omega_1 t) \tag{3}$$

$$A_2 = \exp(-i\omega_2 t) \tag{4}$$

Subbing this in gives

$$-m\omega^2 A_1 = -2kA_1 + kA_2 \tag{5}$$

$$-m\omega^2 A_2 = kA_1 - 2kA_2 \tag{6}$$

This can be rewritten as the matrix eigenvalue equation

$$M\mathbf{A} = -\omega^2 \mathbf{A} \tag{7}$$

To find the eigenvalues we have to solve the characteristic equation given by

$$det(M - \lambda I) = 0 \tag{8}$$

$$det\begin{bmatrix} -2k/m - \lambda & k/m \\ k/m & -2k/m - \lambda \end{bmatrix} = 0 \tag{9}$$

$$(-2k/m - \lambda)(-2k/m - \lambda) - (k/m)^2 = 0 \tag{10}$$

$$3(k/m)^2 + 4(k/m)\lambda + \lambda^2 = 0 \tag{11}$$

$$(\lambda + 3(k/m))(\lambda + (k/m)) = 0 \tag{12}$$

This gives us eigenvalues of

$$\lambda_1 = -\frac{3k}{m} \text{ and } \lambda_2 = -\frac{k}{m} \tag{13}$$

this gives us frequencies of

$$\omega_1 = \sqrt{\frac{3k}{m}} \text{ and } \omega_2 = \sqrt{\frac{k}{m}} \tag{14}$$

3.2 Verification

If we consider a system where the spring constant K = 2, based of equation 14 we should get a graph that looks a bit like this

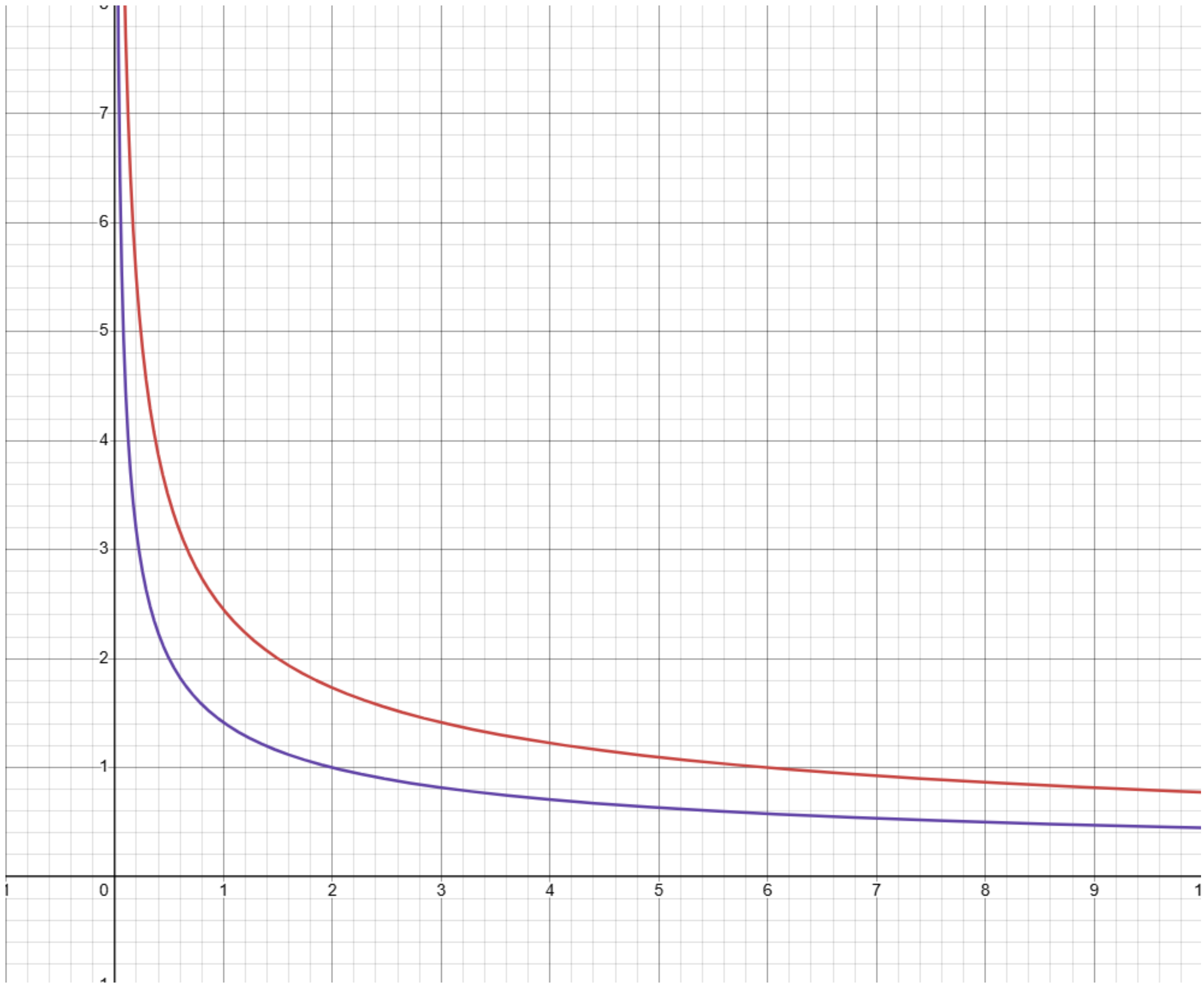


Figure 1: A graph showing how ω (y axis) should change with increasing mass (x axis)

When looking at the code generated by the QR factorization code for the same setup, accounting for some difference due to the aspect ratio we get a similar looking graph.

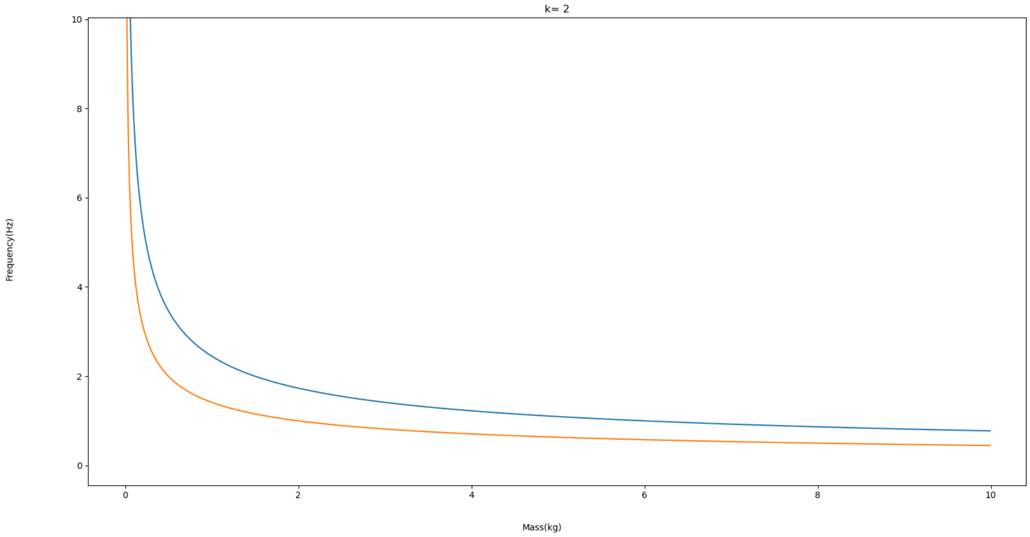


Figure 2: A graph showing how ω (y axis) should change with increasing mass (x axis)

Just comparing the shape of the graph, shows us that our method is reproducing the right physics however to check how accurate they are we need to compare the values produced by the analytical solution to that of the QR method.

k/m	analytical	QR	error
1	1 , $\sqrt{3}$	1 , 1.73205	0 , 8.075e-7
2	$\sqrt{2}$, $\sqrt{6}$	1.41421 , 2.44949	3.56237e-6 , 2.5722e-7
4	2 , 2 $\sqrt{3}$	2 , 3.4641	0 , 1.61514e-6
5	$\sqrt{5}$, $\sqrt{15}$	2.23607 , 3.87298	2.0225e-6 , 3.34621e-06

Table 1: shows the values produced by the analytical solution compared to that of the QR solution and compares there errors

For this setup I had the tolerance set to 1e-6 and so having a error on the order of 1e-6 or less is to be expected.

4 Investigation of how frequency changes with mass

From the analytical solution we know that we have a line that is proportional to $\frac{1}{\sqrt{m}}$, as shown by Figure 3.

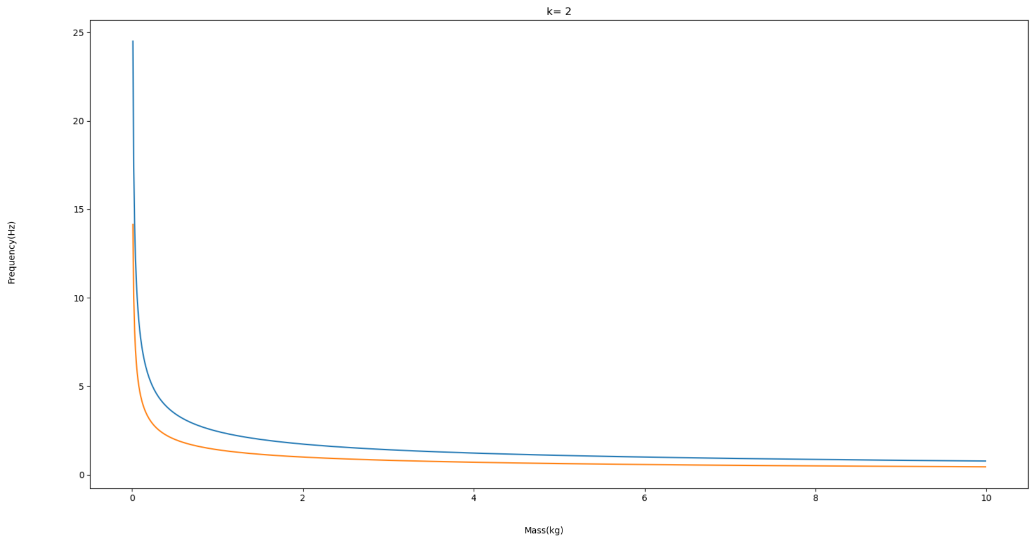
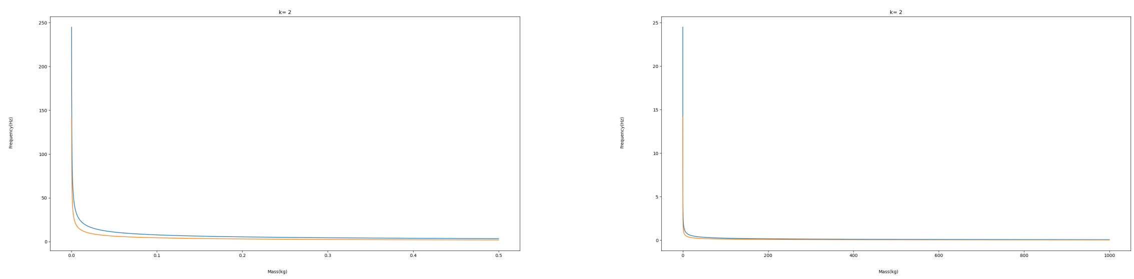


Figure 3: A graph showing how ω (y axis) should change with increasing mass (x axis)

From figure 3, we can see that in the limit of m as it tends to zero, the frequency will tend towards infinity and in the limit of m as it tends to infinity the frequency will tend towards 0, this can be shown in more detail in figure 4.



(a) **(b)**

Figure 4: (a) shows how the frequency changes as m tends towards 0, (b) shows how the frequency changes as m tends towards infinity

4.1 How the rate at which m decreases, changes with k

$$\omega \propto \sqrt{\frac{k}{m}} \tag{15}$$

$$\frac{d\omega}{dm} \propto -\frac{1}{2}\frac{\sqrt{k}}{m^{3/2}} \tag{16}$$

Equation 16 shows that, assuming identical springs, as the spring constant k increases the rate at which the frequency will tend to zero is decreased, as m has to become bigger to become the dominant term ij the equation, this can be seen in figure 5.

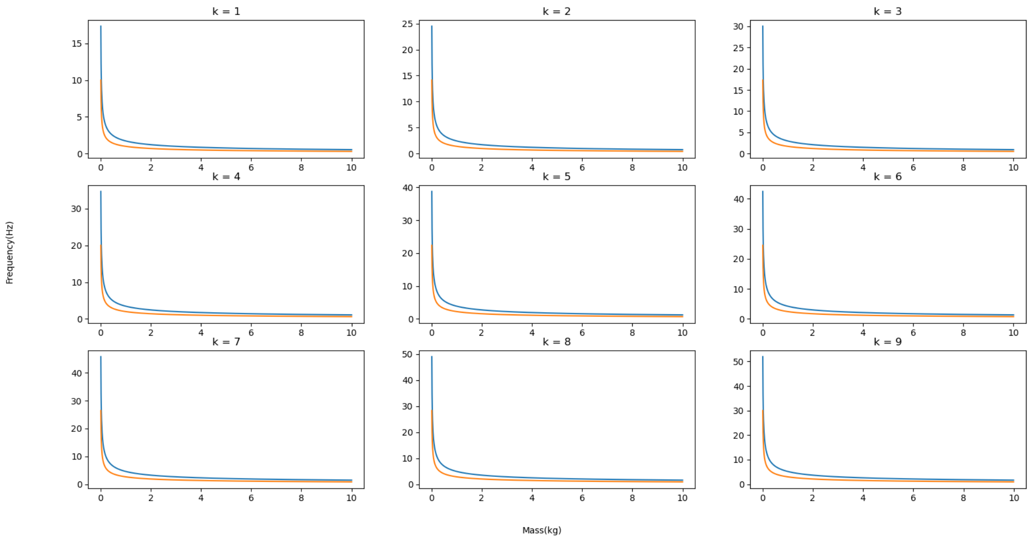


Figure 5: Plot showing how ω changes with m for various values of k