# Tutorial 4 - Markov Decision Processes and Tic Tac toe

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This problem set will consists of two parts

1. We will discuss Markov Decision Processes (MDP) .
2. We will discuss an example of an MDP, the game tic tac toe, and write an algorithm that can perform the game.

## Markov Decision Processes

**Setting**: we consider a setting where there is an agent that at moment $t$ choose an action $A_t$ from a set of actions $a \in \mathcal{A}$. The goal of the agent is to maximize the long-term reward, with the reward at $t$ denoted as $r_t$. This is the setting that we have also considered in previous tutorials, but now we add one additional piece of information; the agent is in a certain state $S_t$ from all possible states $s \in \mathcal{S}$.

We also assume that this is decision process is *Markov*. This means each probability for a subsequent state $S_{t+1}$ and its accompanying reward $r_{t+1}$ only depends on preceding state and action, $S_t$ and $A_t$. This probability is denoted as $p(S_{t+1}|s, a)$.

**Goal**: In order to know what action to take, we try to learn a *policy*; this is simply which action the agent should take, given the state $s$. A policy denoted as $\pi(a|s) = p(A_t = a|S_t = s)$, e.g. the probability that at $t$ an action is $a$ given the state $s$. When an agent in state $S_t$ takes an action $A_t$ as described by policy $\pi$, it transitions to a new state $S_{t+1}$ and a reward $r_{t+1}$. This create a trajectory, where the agent observes:

$$S_0, A_0, R_1, S_1, A_1, R_2, ...$$

Our goal is to maximize our long-term reward. Each reward from the future is discounted by a factor $\gamma$. This long-term reward can be described as:

$$\begin{aligned} G_t &= r_{t+1} + \sum_{k=0}^{\text{inf}} \gamma^k r_{t+k+1} \\ &= r_{t+1} + G_{t+1} \end{aligned}$$

**What is the best policy?** we determine this with help of two functions. First, the state-value function gives us the expected return from a policy $\pi$ when starting in state $s$:

$$\begin{aligned} v_\pi(s) &= \mathrm{E}[G_t|s] \\ &= \mathrm{E}[r_{t+1} + G_{t+1}|s] \end{aligned}$$

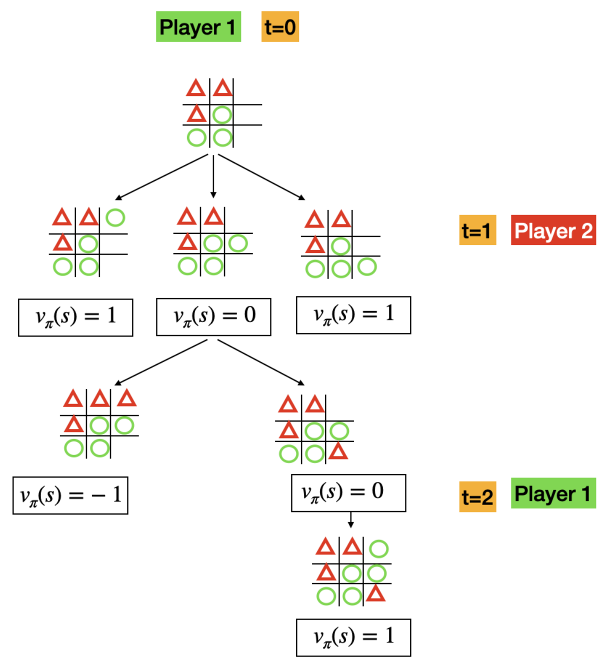
The second function is the action-value function, which gives us the value of taking action $a$ in state $s$ under policy $\pi$ and following that policy afterwards:

$$
\begin{aligned}
q_\pi(s,a) &= \mathrm{E}[G_t|a,s] \\
&= \mathrm{E}[r_{t+1} + G_{t+1}|a,s]
\end{aligned}
$$

## Tic Tac Toe

The game tic tac toe can be modelled as a markov decision process, just like the example we just discussed - see the example below. The state $s$ in this case is the board, and the action $a$ is putting either a circle or triangle on an open position of the board. If you win the game, the reward $r_t = 1$, a loss means $r_t = -1$, and when there is no winner the reward is $r_t = 0$.

The idea is visualized below:




In order to write an algorithm that can play this game, we need several functions:

1. A function that based on a state $s$, calculates all the possible states for the next moment $t$.
2. A function that evaluates the state $s$: has the game ended, and if yes who won?
3. A maximizer and minimzer function, which can be used to decide the best move given a state $s$.

Let's first introduce how the states are recorded. Each state is represented as a vector of 9 elements. The 1 indicates a tile for player 1, the -1 for player 2. 0 indicates no tile has been played

```r
# each state is represented as a vector of 9 elements.
state <-c(1, -1, 0, -1, 1, -1, 1, 1, -1)


state_to_board <- function(state){
  # return matrix version
  board <- t(matrix(state, nrow = 3))

  return(board)
```

```r
}

state_to_board_viz <- function(state){
  board <- state_to_board(state)
  # 1 is for 0
  board[board==-1] = 'o'

  # -1 is for x
  board[board==1] = 'x'

  # 0 means open
  board[board==0] = '-'
  return(board)
}

# visualize the board
state_to_board_viz(state)
```

```
##      [,1] [,2] [,3]
## [1,] "x"  "o"  "-"
## [2,] "o"  "x"  "o"
## [3,] "x"  "x"  "o"
```

**Step 1:** write function that based on a state $s$, calculates all the possible states for the next moment $t$. Your function should fulfill the following requirements:

- It should take the state as an argument
- If the number of tiles played is uneven, then player 1 will put the next tile. Otherwise, player 2 will put down the next tile.
- It should return a matrix with each row representing possible future states. For an example of the output, see below.

```
}

state <-c(1, -1, 0, -1, 1, -1, 1, 1, 0)
child_states = create_states(state)
child_states
```

```
     [,1] [,2] [,3] [,4] [,5] [,6] [,7] [,8] [,9]
[1,]    1   -1    1   -1    1   -1    1    1    0
[2,]    1   -1    0   -1    1   -1    1    1    1
```

```r
# function that returns the states as a matrix,
create_states<- function(state){

    #TODO: write function in line with description of step 1



}

# to check if correct
state <-c(1, -1, 0, -1, 1, -1, 1, 1, 0)
create_states(state)
```

**Step 2:** write a function that evaluates the state $s$: has the game ended, and if yes who won? There are a couple of requirements:

- It should take the state as an argument
- It should return a list with two items; **terminal** is a boolean that tells if the game has ended, and **score** is an integer that contains a 1, 0 or -1 value based on the result of the game. A value of 1 means that the maximizer has won, -1 that the minimizer has won, and 0 means that there is no result yet. For an example, see below.

```
state <-c(1, -1, 0, -1, 1, -1, 1, 1, 0)
result = is_terminal(state)
result
```

```
$terminal
[1] FALSE

$score
[1] 0
```

```
is_terminal <- function(state, players=c(1, -1), scores = c(1, -1)){

  #TODO: write function in line with description of step 2
}


# to check if correct
state <-c(1, -1, 0, -1, 1, -1, 1, 1, 0)
is_terminal(state)
```

**Step 3**: write a maximizer and a minimizer function. These work recursively. In general, recursion means: when something is defined in terms of itself. In this case, recursion means that the function *mazimize* calls the *minimize* function, and that the *minimize* also calls the *mazimize* function. Your goal is to code:

- For the *maximize* function; which action it should take?
- For the *minimize* function; which action it should take?

```
# given a state, and a minimizer, what is the maximizing option?
maximize <- function(state){

  # check based on state if it is terminal
  result <- is_terminal(state)

  # if terminal, return result
  if(result$terminal){

    return(result)

  }
  max_state= NA
  max_score=-Inf

  # fill in all the possible states
```

```r
  max_states  # TODO: add here the states that are maximizing (hint; can be multiple)

  # create new states
  children = create_states(state)

  # go over each potential state
  for(i in seq(nrow(children))){

    # select state
    child = children[i, ]

    # what would the minimizer do?
    # TODO: add the result from the minimizer,

    # TODO: get the score based on choice of minimizer

    # TODO: if the state is maxiziming, add to max_states
  }
  # if multiple max_states, sample random
  sampled_state <- sample(nrow(max_states), 1)
  max_state = max_states[sampled_state, ]

  # return the state and score
  result <- list(state = max_state, score = max_score)


  return(result)


}

# given a state, and a maximizer, what is the minimizing option?
minimize <- function(state){

  # check based on state if it is terminal
  result <- is_terminal(state)

  # if terminal, return result
  if(result$terminal){
    return(result)

  }
  min_state= NA
  min_score=Inf

  # fill in all the possible states
  min_states # TODO: add here the states that are minimizing (hint; can be multiple)

  # create new states
  children = create_states(state)

  # go over each potential state
  for(i in seq(nrow(children))){
```

```r
    # select state
    child = children[i, ]

    # what would the maximizer do?
    # TODO: add the result from the maximizer,

    # TODO: get the score based on choice of maximizer

    # TODO: if the state is maxiziming, add to min_states
  }

  # if multiple, sample random
  sampled_state <- sample(nrow(min_states), 1)
  min_state = min_states[sampled_state, ]

  result <- list(state = min_state, score = min_score)


  return(result)
}
```

Now let's check if you have written the functions correctly. If all works well, the function should tell you to put a cross in the top right corner.

```r
# what is the optimal decision, given a state?
optimal_decision <- function(state,player = 'Maximizer'){
  if(player == 'Maximizer'){
    result = maximize(state)
  }else if (player=='Minimizer'){
    result = minimize(state)
  }
  return(result)
}


state_start <- c(1, 1, 0, 1, -1, 0, -1, -1, 0)
print('State at start')
state_to_board_viz(state_start)

decision <- optimal_decision(state_start, player='Maximizer')
print('Optimal decision:')

state_to_board_viz(decision$state)
```