3] Markov Random Fields

Energy Function

a) Considered the use of iterated conditional modes (ICM) to minimize the energy function given by Eq (1). Write down an expression for the change in the values of the energy associated with the two states of a particular variable x_K, with all other variables held fixed, and show it depends only on quantities that are local to x_K in the graph.

- When we use the variable x_K for the states $\{-1, +1\}$, the energy function changes as follows:

$$E(x,y) = h\sum_{i \neq K} x_i + h x_K - \beta \sum_{i,j \neq K} x_i x_j -$$

$$\beta \sum_{m} x_K x_m - \eta \sum_{i \neq K} x_i y_i - \eta x_K y_K$$

- In this energy function equation, the terms hx_K, $-\beta\sum_m x_K x_m$ and $-\eta x_K y_K$ show how $E(x,y)$ is only dependent on x_K.
- In the term $-\beta\sum_m x_K x_m$, x_m is the neighbor of x_K.
- Since the terms $h\sum_{i\neq K} x_i$, $-\beta\sum_{i,j\neq K} x_i x_j$, $-\eta\sum_{i\neq K} x_i y_i$ are not affecting x_K we can rewrite $E(x,y)$ as:

$$E(x,y) = hx_K - \beta\sum_m x_K x_m - \eta x_K y_K$$

- Substituting the states $(-1,+1)$ in $E(x,y)$ as $x_K=-1$ and $x_K=+1$ and subtracting them to get the change in the energy we get,

$$E(x,y)\big|_{x_K=1} - E(x,y)\big|_{x_K=-1}$$

$$= \left(h(1) - \beta\sum_m (1)x_m - \eta(1)y_K\right) - \left(h(-1) - \beta\sum_m (-1)x_m - \eta(-1)y_K\right)$$

$$\therefore E(x,y) = \left(h - \beta \sum_m x_m - \eta y_K\right) - \left(-h + \beta \sum_m x_m + \eta y_K\right)$$

$$\therefore \boxed{E(x,y) = 2h - 2\beta \sum_m x_m - 2\eta y_K},$$

where the difference is implicted by h and depends on the target variable x_K, x_m is the represented neighber of x_K and its observed value is y_K.

b) Consider a particular case of the energy function given by Eq (1) in which the co-efficients $\beta = h = 0$. Show that the most probable configuration of the latent variable is given by $x_i = y_i$ for all i.

- If the co-efficients $\beta = 0$ and $h = 0$ then the energy function can be written as:

$$E(x,y) = h \sum_i x_i - \beta \sum_{\{i,j\}} x_i x_j - \eta \sum_i x_i y_i$$

$$\therefore E(x,y) = 0 \sum_i x_i - 0 \sum_{\{i,j\}} x_i x_j - \eta \sum_i x_i y_i$$

$$\therefore E(x,y) = -\eta \sum_i x_i y_i$$

- If $x_K, y_K \in \{-1, +1\}$ where k is any index and fulfills the relation $x_K \neq y_K$, then $x_K y_K$ will be equal to -1. When we change the sign of x_K, we can increase the value of $x_K y_K$ from -1 to $+1$ just like we can also decrease the value of $x_K y_K$ from $+1$ to -1.

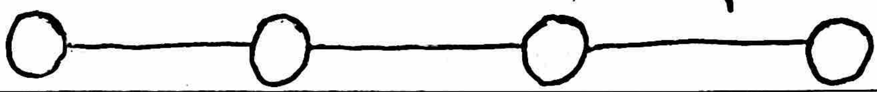
- When we want to decrease the energy function $E(x, y) = -\eta \sum_i x_i y_i$ we will simply increase the value of $x_i y_i$ by changing the sign of it and increasing it from -1 to $+1$.

- Therefore, given an array of binary pixels values $y_i \in \{-1, +1\}$, where the index $i = 1, 2, 3, \ldots D$ and $x_i \in \{-1, +1\}$, where the index $i = 1, 2, 3 \ldots D$ inorder for us to maintain the minimum energy in the energy function, the most probable configuration of latent variables x_i and y_i is to set $x_i = y_i$ for all i.

Undirected Graphs

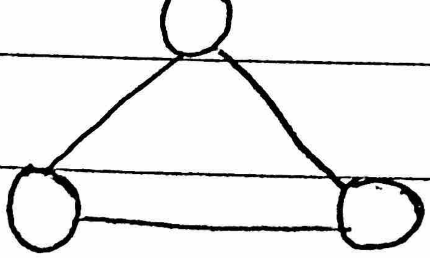
Show that there are $2^{M(M-1)/2}$ distinct undirected graphs over a set of M distinct random variables. Draw the 8 possibilities for the case of $M = 3$.

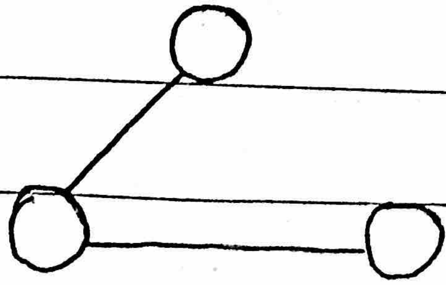
- When there is an undirected graph of M nodes, there is a possibility that there is a link between each pair of nodes.

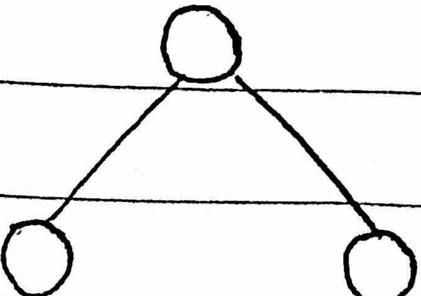


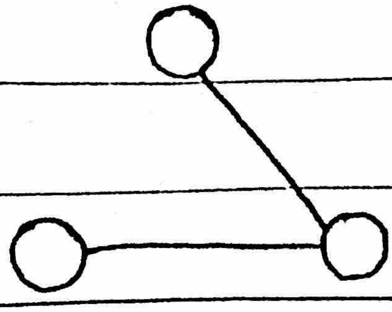
- Due to these links, the number of distinct graphs is then 2 raised to the power of the number of potential links.
- To calculate the number of distinct links, we have to take into consideration that there are M nodes and it can have a link to any of the other $M-1$ nodes, which gives a total of $M(M-1)$ distinct links.

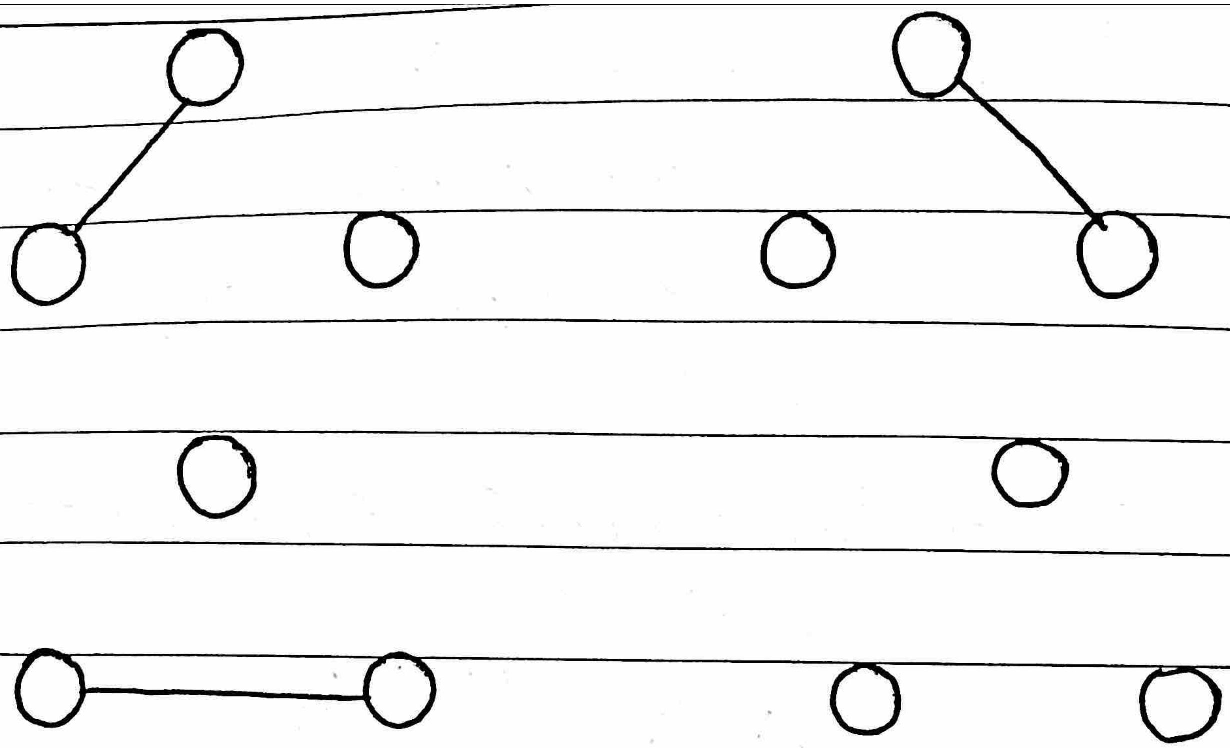
- Since, it is an undirected graph, the links are connected in both directions. For example, a link from node x to node y is equal to a link from node y to node x. Hence, the number of potential number of distinct links changes from $m(m-1)$ to $[m(m-1)]/2$.
- We need to calculate distinct graphs from the number of potential distinct links which is $[m(m-1)]/2$, so the number of distinct graphs $2^{m(m-1)/2}$.
- Following are the undirected graphs for $m=3$:











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