

# Complex Analysis Notes

Before starting, I want to mention that the book “Basic Complex Analysis (3rd edition)” by Jerrold E. Marsden and Michael J. Hoffman. is used.

## §1 How Complex Is It?



### 1.1 Basic Operations

- $(a + bi) \pm (c + di) = (a \pm c) + (b \pm d)i$
- $(a + bi)(c + di) = (ac - bd) + (ad + bc)i$
- $\frac{a + bi}{c + di} = \frac{ac + bd}{c^2 + d^2} + \frac{bc - ad}{c^2 + d^2}i$

#### Problem

Fix a complex number  $z = x + iy$  and consider the linear mapping  $\phi_z : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  (that is, of  $\mathbb{C} \rightarrow \mathbb{C}$ ) defined by  $\phi_z(w) = z \cdot w$  (that is, multiplication by  $z$ ). Prove that the matrix of  $\phi_z$  in the standard basis  $(1, 0), (0, 1)$  of  $\mathbb{R}^2$  is given by

$$\begin{pmatrix} x & -y \\ y & x \end{pmatrix}.$$

Then show that  $\phi_{z_1 z_2} = \phi_{z_1} \circ \phi_{z_2}$ .

Let  $w = a + ib$ , then  $z \cdot w = (x + iy)(a + ib) = (xa - yb) + (xb + ya)i$ .

On the other hand,

$$\begin{pmatrix} x & -y \\ y & x \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} xa - yb \\ xb + ya \end{pmatrix}.$$

and we have

$$\phi_{z_1 z_2} = z_1 \cdot z_2 \cdot w = z_1 \cdot (z_2 \cdot w) = \phi_{z_1} \circ \phi_{z_2}.$$

## 1.2 What? There's More?

### Proposition (De Moivre's Formula)

If  $z = r(\cos \theta + i \sin \theta)$  then for some positive integer  $n$ ,

$$z^n = r^n(\cos n\theta + i \sin n\theta).$$

Some notable properties of **complex conjugation** and **norm**.

- $z\bar{z} = |z|^2$ .
- $\operatorname{Re}(z) = (z + \bar{z})/2$ ,  $\operatorname{Im}(z) = (z - \bar{z})/2i$
- $|\operatorname{Re}(z)| \leq |z|$ ,  $|\operatorname{Im}(z)| \leq |z|$
- Triangle Inequality:  $\left| \sum_{k=1}^n z_k \right| \leq \sum_{k=1}^n |z_k|$
- Cauchy-Schwarz Inequality:  $\left| \sum_{k=1}^n z_k w_k \right| \leq \sqrt{\sum_{k=1}^n |z_k|^2} \sqrt{\sum_{k=1}^n |w_k|^2}$

### Problem

If  $a, b \in \mathbb{C}$ , prove the **parallelogram identity**:  $|a-b|^2 + |a+b|^2 = 2(|a|^2 + |b|^2)$ .

Let  $a = p + iq$  and  $b = r + is$ , then

$$\begin{aligned} |a-b|^2 + |a+b|^2 &= (p-r)^2 + (q-s)^2 + (p+r)^2 + (q+s)^2 \\ &= 2(p^2 + q^2 + r^2 + s^2) \\ &= 2(|a|^2 + |b|^2) \end{aligned}$$

### Problem

Prove **Langrange's identity**:

$$\left| \sum_{k=1}^n z_k w_k \right|^2 = \left( \sum_{k=1}^n |z_k|^2 \right) \left( \sum_{k=1}^n |w_k|^2 \right) - \sum_{k < j} |z_k \bar{w}_j - z_j \bar{w}_k|.$$

We abuse the fact that  $z\bar{z} = |z|^2$ .

$$\begin{aligned}
\left| \sum_{k=1}^n z_k w_k \right|^2 &= \left( \sum_{k=1}^n z_k w_k \right) \overline{\left( \sum_{k=1}^n z_k w_k \right)} \\
&= \left( \sum_{k=1}^n z_k w_k \right) \left( \sum_{k=1}^n \overline{z_k w_k} \right) \\
&= \sum_{k=1}^n z_k w_k \overline{z_k w_k} + \sum_{j \neq k} z_j w_j \overline{z_k w_k} \\
&= \sum_{k=1}^n |z_k|^2 |w_k|^2 + \sum_{j \neq k} z_j w_j \overline{z_k w_k} - \sum_{j \neq k} z_k w_k \overline{z_j w_j} \\
&= \sum_{k=1}^n |z_k|^2 |w_k|^2 + \sum_{j \neq k} z_j w_j \overline{z_k w_k} - \sum_{j \neq k} z_k w_k \overline{z_j w_j} \\
&= \left( \sum_{k=1}^n |z_k|^2 \right) \left( \sum_{k=1}^n |w_k|^2 \right) + \sum_{j \neq k} z_j w_j \overline{z_k w_k} - \sum_{j \neq k} z_k w_k \overline{z_j w_j}
\end{aligned}$$

For some distinct indices  $j, k$  we have

$$\begin{aligned}
z_j w_j \overline{z_k w_k} + z_k w_k \overline{z_j w_j} - z_k w_k \overline{z_j w_j} - z_j w_j \overline{z_k w_k} &= z_j \overline{w_k} (w_j \overline{z_k} - w_k \overline{z_j}) + z_k \overline{w_j} (w_k \overline{z_j} - w_j \overline{z_k}) \\
&= (w_k \overline{z_j} - w_j \overline{z_k})(z_j \overline{w_k} - z_k \overline{w_j}) \\
&= -(w_k \overline{z_j} - w_j \overline{z_k}) \overline{(w_k \overline{z_j} - w_j \overline{z_k})} \\
&= -|w_k \overline{z_j} - w_j \overline{z_k}|^2
\end{aligned}$$

Summing up gives the desired result

### 1.3 Even Weirder Stuff

Using the fact that

$$re^{ix} = r(\cos x + i \sin x)$$

and thanks to Euler we generalize the complex numbers to even more functions.

- It's not hard to see that

$$\sin x = \frac{e^{ix} - e^{-ix}}{2i} \quad \text{and} \quad \cos x = \frac{e^{ix} + e^{-ix}}{2}$$

- Let  $z = re^{i\theta}$  then  $\ln z = \ln |r| + i \arg z$ .
- $z^w = e^{w \ln z}$  can be determined consequently.
- Moreover, we have

$$\sinh x = -i \sin(ix) \quad \text{and} \quad \cosh x = \cos(ix)$$

which can be deduced from

$$\sinh x = \frac{e^x - e^{-x}}{2} \quad \text{and} \quad \cosh x = \frac{e^x + e^{-x}}{2}$$

### Problem

Along which rays through the origin does  $\lim_{z \rightarrow \infty} |e^z|$  exist?

Let  $z = x + iy$ , then we have  $|e^z| = |e^x(\cos y + i \sin y)| = e^x$ . If  $x \rightarrow -\infty$  then  $e^x \rightarrow 0$ , but if  $x \rightarrow \infty$  then  $e^x \rightarrow \infty$  which the limit doesn't exist.

Hence the answers are all the rays passing through the nonnegative  $x$  plane.

### Problem

Prove the identity

$$z = \tan \left[ \frac{1}{i} \ln \left( \frac{1 + iz}{1 - iz} \right)^{1/2} \right]$$

for all real  $z$ .

$$\begin{aligned} \tan \left[ \frac{1}{i} \ln \left( \frac{1 + iz}{1 - iz} \right)^{1/2} \right] &= \tan \left[ \frac{1}{2i} (\ln(1 + iz) - \ln(1 - iz)) \right] \\ &= \tan \left[ \frac{1}{2i} (\ln |1 + iz| + i(\tan^{-1} z) - \ln |1 - iz| - i(\tan^{-1}(-z))) \right] \\ &= \tan \left[ \frac{1}{2i} (2i(\tan^{-1} z)) \right] \\ &= z \end{aligned}$$

### Problem

Use the equation  $\sin z = \sin x \cosh y + i \sinh y \cos x$  where  $z = x + iy$  to prove that  $|\sinh y| \leq |\sin z| \leq |\cosh y|$ .

Evaluating gives

$$|\sin z| = \sqrt{\sin^2 x \cosh^2 y + \sinh^2 y \cos^2 x}$$

Using the fact that  $\sinh x < \cosh x$ , we have

$$\sin^2 x \sinh^2 y + \sinh^2 y \cos^2 x < \sin^2 x \cosh^2 y + \sinh^2 y \cos^2 x < \sin^2 x \cosh^2 y + \cosh^2 y \cos^2 x$$

simplifying gives the desired result.

**Problem**

Using polar coordinates, show that  $z \mapsto z + 1/z$  maps the circle  $|z| = 1$  to the interval  $[-2, 2]$  on the  $x$  axis.

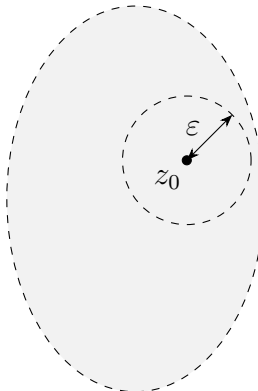
Let  $z = x + iy$ , then  $z + \frac{1}{z} = x + iy + \frac{x - iy}{x^2 + y^2}$  and since  $x^2 + y^2 = 1$ ,  $z + \frac{1}{z} = 2x$ . This means that for any complex number  $z = x + iy$  on the circle, it will be mapped to  $2x$ .

And since  $x$  is in the interval  $[-1, 1]$ , hence  $2x$  is in the interval  $[-2, 2]$ .

## 1.4 Topological Analysis of Complex Functions

### 1.4.1 Definitions

- **$r$  Disk**: The  $r$  disk is defined by  $D(z_0; r) = \{z \in \mathbb{C} \mid |z - z_0| < r\}$ . The **deleted  $r$  disk** is defined by  $D(z_0; r) \setminus \{z_0\}$ .
- **Open Sets**: The set  $A \subset \mathbb{C}$  is open when for any point  $z_0$  in  $A$ , there exists a real number  $\varepsilon$  such that if  $|z - z_0| < \varepsilon$  then  $z \in A$ .



- **Closed Sets**: A set  $F$  is closed if  $\mathbb{C} \setminus F$  is open.
  - The empty set and  $\mathbb{C}$  are both open and closed (known as **clopen sets**).
  - Let  $z_1, z_2, z_3, \dots$  are points in  $F$  and  $w = \lim_{n \rightarrow \infty} z_n$ , then  $w \in F$ .
    - \* **Sketch of proof**: Assume that  $w \notin F$ , then since  $\mathbb{C} \setminus F$  is open, we can always find a disk  $D(w; r)$  contained in  $\mathbb{C} \setminus F$ . This means that there exists some large enough  $n$  such that  $z_n \in D(w; r)$  by convergence, which implies  $z_n \notin F$ , a contradiction.
  - The **closure** of a set  $S$ , denoted by  $\overline{S}$  is the set  $S$  together with its limit points, or known as the **boundary**  $\partial(S)$ .
- **Limits**: The limit  $\lim_{z \rightarrow z_0} f(z) = L$  exists when for all  $\varepsilon > 0$ , there exists  $\delta > 0$  such that for all  $|z - z_0| < \delta$  ( $z \neq z_0$ ) we have  $|f(z) - L| < \varepsilon$ .

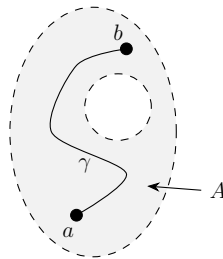
Limits are **unique** if they exist.

- **Continuity**:  $f$  is continuous at  $z_0 \in A$  if and only if

$$\lim_{z \rightarrow z_0} f(z) = f(z_0).$$

- **Cauchy Sequence**: A sequence is *Cauchy* if for every  $\varepsilon > 0$ , we can find some integer  $N$  such that whenever integers  $m, n$  are greater than  $N$ ,  $|z_m - z_n| < \varepsilon$ .
- **Path-Connected**: A set  $A \in \mathbb{C}$  is path-connected if for every  $a, b \in A$  there exists a *continuous map*  $\gamma : [0, 1] \rightarrow A$  such that  $\gamma(0) = a, \gamma(1) = b$ .

$\gamma$  is a **path** joining  $a$  and  $b$ .



Definition: A set  $C \in \mathbb{C}$  is **not connected** if there are open sets  $U, V$  such that

- $C \subset (U \cup V)$ ;
- $(C \cap U \neq \emptyset) \wedge (C \cap V \neq \emptyset)$ ;
- $C \cap U \cap V = \emptyset$ .

If a set is not “not connected”, then it is **connected**.

- **A path-connected set is connected, but a connected set may not be path-connected.**
- Example: **Topologist’s Sine Curve**

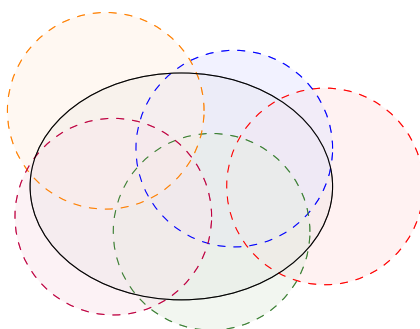
$$f(x) = \begin{cases} \sin \frac{1}{x} & x > 0 \\ 0 & x = 0 \end{cases}$$

**Sketch of proof:** Let the two sets be  $A, B$ . WLOG let  $(0, 0) \in A$ .

If some part of  $\sin(1/x)$  is in  $A$ , then  $B$  should be covering the other parts. But since both sets are open, there’s a point that is not covered.

If no part of  $\sin(1/x)$  is in  $A$ , then  $B$  must be covering the entire line of  $\sin(1/x)$ . But this is impossible since we cannot cover all points near  $x = 0^+$ .

- **Cover**: Let  $U$  be a collection of open sets.  $U$  is a cover of a set  $K$  if  $K$  is contained in the union of sets in  $U$ .



A **subcover** is a subset of  $U$  but can still cover  $K$ .

- **Compactness**: A set  $K$  is **compact** if every cover of  $K$  has a finite subcover.
  - **Heine-Borel Theorem**: A set  $K$  is compact if and only if  $K$  is closed and bounded.

**Sketch of proof:**

\* Sufficiency:

**Boundedness:** Assume that  $K$  is not bounded. Consider the set of open covers  $U = \{D(O; r)\}$ , (open) disks centered at the origin, then for all finite subcover  $U'$  of  $U$ , consider  $R = \max(r)$  and choose some point  $z \in K$  but  $|z| > R$ .

**Closedness:** Assume that  $K$  is not closed, then there exists some  $w \notin K$  such that the sequence  $\{z_i\}$  in  $K$  converges to  $w$ . So the set of open covers  $U = \{D(w, r)\}$  does not have a finite subcover.

\* Necessity: Assume that  $K$  is closed and bounded, then let  $z \in K$  such that  $|z|$  attains maximum value. Choose the open cover  $D(O; |z| + 1)$ .

### 1.4.2 On Functions

- If  $f$  is a continuous function defined on a connected set  $C$ , then  $f(C)$  is connected.

**Sketch of proof:** FTSOC, let  $A|B$  be a partition of  $f(C)$ . Then  $f^{-1}(A)$  and  $f^{-1}(B)$  are open and disjoint (since each value  $f^{-1}(x)$  can only belong to either one of  $f^{-1}(A)$  and  $f^{-1}(B)$ .)

- If  $f$  is a continuous function defined on a compact set  $C$ , then  $f(C)$  is compact.

**Sketch of proof:** Let  $U$  be an open cover of  $f(C)$ , then for each  $f(z) \in U$  and  $f(z) \in f(C)$ , we have  $z \in C$  and  $z \in f^{-1}(U)$ .

- **Extreme Value Theorem**: Let  $K$  be a compact set and  $f : K \rightarrow \mathbb{R}$  is a continuous function, then  $f$  attains **finite** maximum and minimum values.

**Sketch of proof:**  $K$  is compact implies  $f(K)$  is compact, or  $f(K)$  is bounded, therefore finite maximum and minimum exists.

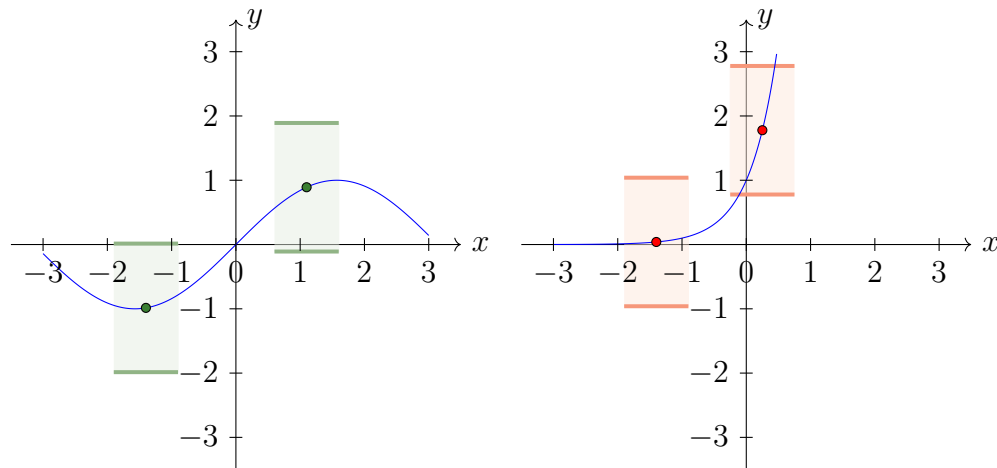
- **Distance Lemma:** Let  $K$  be a compact set and  $C$  be a closed set and  $K \cap C = \emptyset$ . Then there exists a number  $\rho > 0$ , such that whenever  $z \in K$  and  $w \in C$  then  $|z - w| > \rho$ .

**Sketch of proof:**  $K$  is closed and bounded. Assume that  $\rho$  doesn't exist,  $\rho \rightarrow 0$  since we can always find some  $|z - w| < \rho_0$  if  $\rho_0$  is fixed. Consider the sequences  $\{z_k\}$  and  $\{w_k\}$ . Thus  $\lim_{k \rightarrow \infty} |z_k - w_k| = 0$  which means  $\lim_{k \rightarrow \infty} z_k = \lim_{k \rightarrow \infty} w_k$ .

But since both sets are closed, we must have  $\lim_{k \rightarrow \infty} z_k \in K$  and  $\lim_{k \rightarrow \infty} w_k \in C$ , hence a contradiction.

- **Uniform Continuity:** A function  $f : A \rightarrow \mathbb{C}$  is **uniformly continuous** on  $A$  if for any  $\varepsilon > 0$ , there exists a  $\delta > 0$  such that  $|f(s) - f(t)| < \varepsilon$  whenever  $s, t \in A$  and  $|s - t| < \delta$ .

An example  $f(x) = \sin x$  and a counterexample  $f(x) = 2^x$  are shown below. Choose  $\delta = \varepsilon/2 = 0.5$ .



- **Heine-Cantor Theorem:** Let  $f : A \rightarrow \mathbb{C}$  be a continuous function. If  $A$  is compact then  $f(A)$  is uniformly continuous.

**Sketch of proof:**

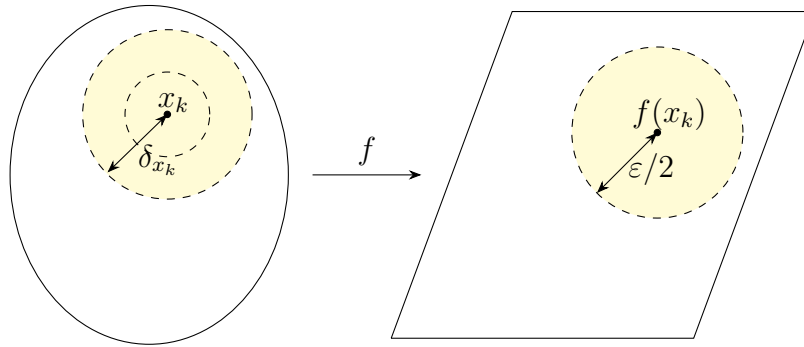
- Let  $x$  be some point in  $A$ . By continuity, there is a number  $\delta_x$  such that whenever for some point  $y$  satisfying  $|x - y| < \delta_x$  then  $|f(x) - f(y)| < \varepsilon/2$ .
- For a sequence of points  $x$ , say  $\{x_i\}$ , consider disks  $D_k = D(x_k; \delta_{x_k}/2)$ . These disks cover  $A$  by compactness. Let the minimal radius over all disks be  $\delta$ .
- For points  $s, t$  satisfying  $|s - t| < \delta$ , we must have  $t$  contained in some disk  $D_k$ . Thus  $|t - x_k| < \delta_{x_k}/2$ , implies that  $|f(t) - f(x_k)| < \varepsilon/2$ . We have

$$|s - x_k| \leq |s - t| + |t - x_k| < \delta + \delta_{x_k}/2 \leq \delta_{x_k}$$

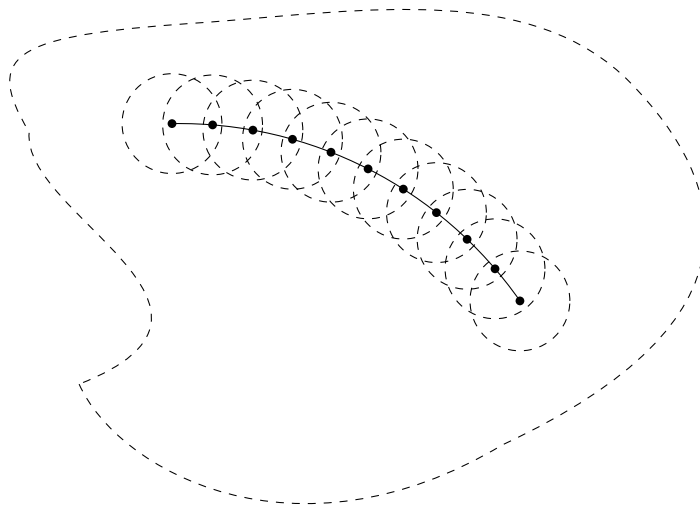
- On the other hand,

$$|f(s) - f(t)| \leq |f(s) - f(x_k)| + |f(x_k) - f(t)| < \varepsilon$$





- **Path-Covering Lemma:** Suppose  $\gamma : [0, 1] \rightarrow K$  is a continuous path into an open subset  $K$  of  $\mathbb{C}$ . We can find a number  $\rho > 0$  and a subdivision of  $[0, 1]$ , namely  $0 = t_0 < t_1 < t_2 < \cdots < t_{N-1} < t_N = 1$  such that
  - (a)  $D(\gamma(t_k); \rho) \subset G$  for all  $k$
  - (b)  $\gamma(t) \in D(\gamma(t_0); \rho)$  for  $t_0 \leq t \leq t_1$
  - (c)  $\gamma(t) \in D(\gamma(t_k); \rho)$  for  $t_{k-1} \leq t \leq t_{k+1}$
  - (d)  $\gamma(t) \in D(\gamma(t_N); \rho)$  for  $t_{N-1} \leq t \leq t_N$



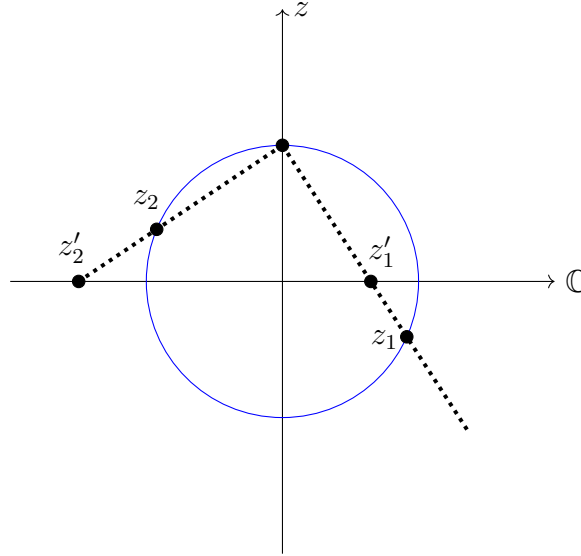
### Sketch of proof:

- By the Distance Lemma, we can find some  $\rho > 0$  such that the distance from  $\gamma([0, 1])$  to  $\mathbb{C} \setminus K$  is at least  $\rho$  since both sets are closed.
- By Heine-Cantor Theorem,  $\gamma([0, 1])$  is uniformly continuous. So for any two points  $s, t$ , if  $|s - t| < \delta$  then  $|f(s) - f(t)| < \rho$ .
- Choose  $t_k$  to be fine enough such that  $t_{k+1} - t_k < \delta$ .
- **Riemann Sphere:** We may want to define the value  $\infty$  in the complex plane.
  - $\lim_{z \rightarrow \infty} f(z) = L$  means for any  $\varepsilon > 0$ , there exists  $Z > 0$  such that whenever  $|z| > Z$  implies  $|f(z) - L| < \varepsilon$ .
  - $\lim_{z \rightarrow z_0} f(z) = \infty$  means for any  $R > 0$ , there exists  $\delta > 0$  such that whenever  $|z - z_0| < \delta$  implies  $|f(z)| > R$ .

- $\lim_{z \rightarrow \infty} f(z) = \infty$  means for any  $Z > 0$ , there exists  $R > 0$  such that whenever  $|z| > Z$  implies  $|f(z)| > R$ .

Consider the sphere  $x^2 + y^2 + z^2 = 1$  in  $\mathbb{R}^3$ . A point  $z'$  on the plane  $\mathbb{C}$  is the **stereographic projection** of some point  $z$  on the sphere through  $(0, 0, 1)$ .

A 2-D illustration as an example:



### Problem

Show that  $\lim_{z \rightarrow \infty} \frac{1}{z} = 0$ .

By our definition, we must have  $|z| \rightarrow \infty$ . Let  $z = x + iy$  so that  $|z| = \sqrt{x^2 + y^2}$ . Clearly, at least one of  $|x|, |y|$  must tend to  $\infty$ .

So

$$\lim_{z \rightarrow \infty} \frac{1}{z} = \lim_{|x| \rightarrow \infty \text{ or } |y| \rightarrow \infty} \left( \frac{x}{x^2 + y^2} - i \frac{y}{x^2 + y^2} \right)$$

which, implies that  $\lim_{z \rightarrow \infty} \frac{1}{z} = 0$ .

**Problem**

(a) Show that

$$|\operatorname{Re}(z_1) - \operatorname{Re}(z_2)| \leq |z_1 - z_2| \leq |\operatorname{Re}(z_1) - \operatorname{Re}(z_2)| + |\operatorname{Im}(z_1) - \operatorname{Im}(z_2)|$$

for any two complex numbers  $z_1$  and  $z_2$ .

(b) If  $f(z) = u(x, y) + iv(x, y)$ , show that

$$\lim_{z \rightarrow z_0} f(z) = \lim_{\substack{x \rightarrow x_0 \\ y \rightarrow y_0}} u(x, y) + \lim_{\substack{x \rightarrow x_0 \\ y \rightarrow y_0}} iv(x, y)$$

exists if both limits on the right of the equation exist. Conversely, if the limit on the left exists, show that both limits on the right exist as well and equality holds.

(a) Let  $z_1 = x_1 + iy_1$  and  $z_2 = x_2 + iy_2$  for real numbers  $x_i, y_i$ . Let  $\mathcal{X} = |x_1 - x_2|$  and  $\mathcal{Y} = |y_1 - y_2|$ . The inequality above can be expressed as

$$\mathcal{X} \leq \sqrt{\mathcal{X}^2 + \mathcal{Y}^2} \leq \mathcal{X} + \mathcal{Y}$$

which is obvious by squaring each part in the inequality.

(b) **Necessity:** Assume that both  $\lim_{\substack{x \rightarrow x_0 \\ y \rightarrow y_0}} u(x, y) = U$  and  $\lim_{\substack{x \rightarrow x_0 \\ y \rightarrow y_0}} v(x, y) = V$  exists, then for some  $\varepsilon$ , there exists  $\delta_u$  and  $\delta_v$  such that:

- $|u(x, y) - U| < \varepsilon/2$  whenever  $|(x, y) - (x_0, y_0)| < \delta_u$ .
- $|v(x, y) - V| < \varepsilon/2$  whenever  $|(x, y) - (x_0, y_0)| < \delta_v$ .

By the limit laws,  $\lim_{z \rightarrow z_0} f(z) = U + iV$ , then there exists  $\delta = \min\{\delta_u, \delta_v\}$  such that whenever  $|z - z_0| < \delta$ ,

$$\begin{aligned} |f(z) - Z| &= |u(x, y) + iv(x, y) - U - iV| \\ &\leq |u(x, y) - U| + |i||v(x, y) - V| \\ &< \varepsilon \end{aligned}$$

**Sufficiency:** Assume that  $\lim_{z \rightarrow z_0} f(z) = U + iV$  exists. Then for all  $\varepsilon > 0$  there exists  $\delta_u > 0$  such that whenever  $0 < |z - z_0| < \delta_u$  then  $|u(x, y) - U| < \varepsilon$  and  $\delta_v > 0$  such that whenever  $0 < |z - z_0| < \delta_v$  then  $|v(x, y) - V| < \varepsilon$ . Choose  $\delta = \min\{\delta_u, \delta_v\}$ .

**Problem**

Introduce the **chordal metric**  $\rho$  on  $\bar{\mathbb{C}}$  by setting  $\rho(z_1, z_2) = d(z'_1, z'_2)$  where  $z'_1$  and  $z'_2$  are the corresponding points on the Riemann sphere and  $d$  is the usual distance between points in  $\mathbb{R}^3$ .

- (a) Show that  $z_n \rightarrow z$  in  $\mathbb{C}$  if and only if  $\rho(z_n, z) \rightarrow 0$ .
- (b) Show that  $z_n \rightarrow \infty$  if and only if  $\rho(z_n, \infty) \rightarrow 0$ .
- (c) If  $f(z) = (az + b)/(cz + d)$  and  $ad - bc \neq 0$ , show that  $f$  is continuous at  $\infty$ .

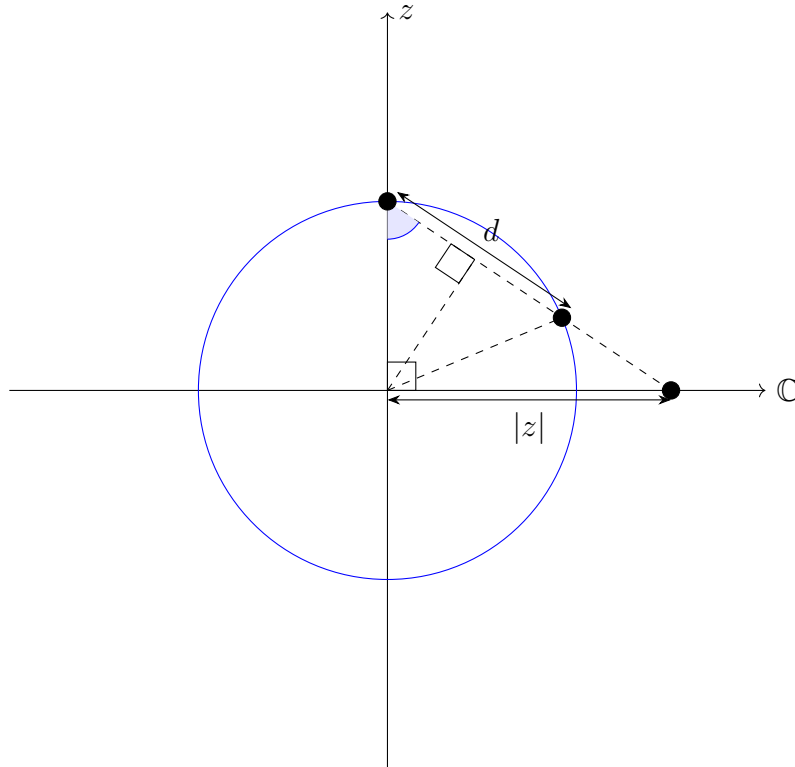
- (a) If  $z_n \rightarrow z$ , then  $\rho(z_n, z) \rightarrow \rho(z, z) = d(z', z') = 0$ .

On the other hand, let  $z'_n = (x_n, y_n, t_n)$  and  $z' = (x, y, t)$ . Then if  $\rho(z_n, z) = d(z'_n, z') \rightarrow 0$ , we have

$$\begin{aligned}\sqrt{(x_n - x)^2 + (y_n - y)^2 + (t_n - t)^2} &\rightarrow 0 \\ (x_n - x)^2 + (y_n - y)^2 + (t_n - t)^2 &\rightarrow 0\end{aligned}$$

FTSOC, WLOG assume  $x_n$  does not converge to  $x$ , then since  $(x_n - x)^2 + (y_n - y)^2 + (t_n - t)^2 \geq 0$ , we have  $(x_n - x)^2 + (y_n - y)^2 + (t_n - t)^2$  converges to at least  $(x_n - x)^2$ , contradiction.

- (b) If  $z_n \rightarrow \infty$ ,  $|z_n| \rightarrow \infty$ . By drawing the Riemann sphere again,



We see that  $\tan \theta = |z|$  and  $d = 2 \cos \theta$  ( $\theta$  is the measure of the angle marked

in blue). Substituting gives

$$d = \frac{2}{\sqrt{|z|^2 + 1}}$$

which is obvious that  $d \rightarrow 0$ .

Conversely, if  $d \rightarrow 0$ , it can be shown that  $\sqrt{|z|^2 + 1} \rightarrow \infty$  which gives  $z_n \rightarrow \infty$ .

(c) A function  $f(z)$  is continuous at infinity if the limits when  $z \rightarrow +\infty$  and  $z \rightarrow -\infty$  are equal.

- $\lim_{z \rightarrow \infty} \frac{az + b}{cz + d} = \frac{a}{c}.$
- $\lim_{z \rightarrow -\infty} \frac{az + b}{cz + d} = \frac{a}{c}.$

Hence the function is continuous at  $\infty$ .

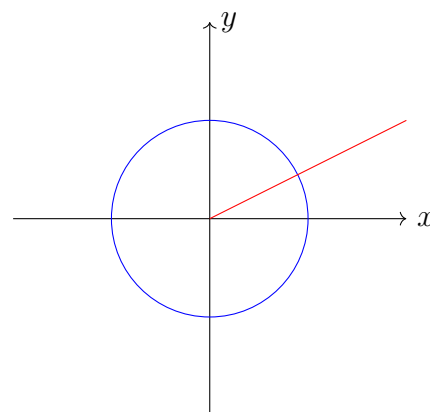
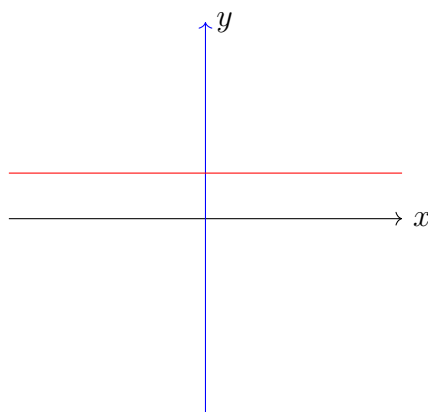
## 1.5 Analytic Functions: What and Why

- functions that are differentiable in complex.
- “regular”, “holomorphic”, “analytic”
- a function  $f(z)$  is differentiable at  $z_0$  if  $\lim_{z \rightarrow z_0} \frac{f(z) - f(z_0)}{z - z_0}$  exists.

### 1.5.1 Differentiation

Differentiation laws apply here, including product rule, quotient rule, and chain rule.

- **Conformal Maps:** A function  $f : A \rightarrow \mathbb{C}$  is conformal if it preserves **angles** between intersecting curves.
  - “conformal transformation”, “angle-preserving transformation”, “biholomorphic map”
  - An example of  $e^z = e^{x+iy}$ .

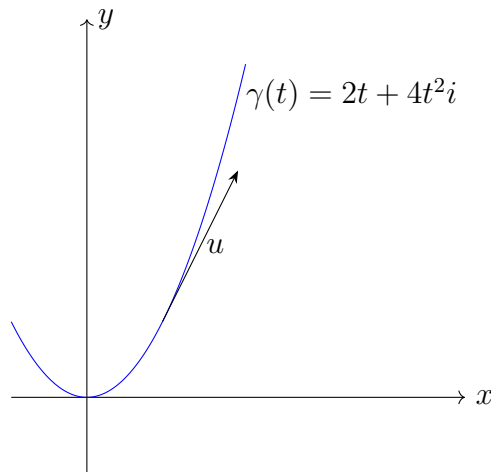


- What does it mean to differentiate complex numbers?
  - Clearly  $f'(z_0) = x + iy$  is a vector  $\begin{pmatrix} x \\ y \end{pmatrix}$  in the plane. Assume for some curve  $\gamma(t) : \mathbb{R} \rightarrow \mathbb{C}$ .

**Claim —** Let  $\gamma(t) = x(t) + iy(t)$ , define  $\gamma'(t) = x'(t) + iy'(t)$ , then the vector  $u = \begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix}$  is tangent to the curve  $\gamma$  at  $(x(t), y(t))$ .

This is trivial by noticing

$$\frac{y'(t)}{x'(t)} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{dy}{dx}$$



- **Conformal Mapping Theorem:** If  $A \rightarrow \mathbb{C}$  is analytic and  $f'(z_0) \neq 0$ , then  $f$  is conformal at  $z_0$ .

**Sketch of Proof:** Define  $\sigma(t) = f(\gamma(t))$ , then obviously  $\sigma(t)$  is also a curve. Taking

$$\frac{df(\gamma(t))}{dt} = \frac{df(\gamma(t))}{dz} \cdot \frac{dz}{dt}$$

Assume  $\gamma(t_0) = z_0$ , letting  $t = t_0$  we have

$$\sigma'(t_0) = f'(z_0)\gamma'(t_0)$$

Since by our definition,  $f'(z_0)$  is independent of  $\gamma$ , choose  $t_1$  and  $t_2$  so that

$$\frac{\sigma'(t_1)}{\gamma'(t_1)} = \frac{\sigma'(t_2)}{\gamma'(t_2)}$$

Taking the argument we have

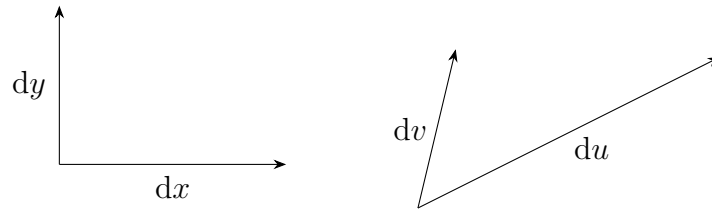
$$\arg \sigma'(t_1) - \arg \sigma'(t_2) \equiv \arg \gamma'(t_1) - \arg \gamma'(t_2) \pmod{2\pi}$$

hence it is clear that angles are preserved.

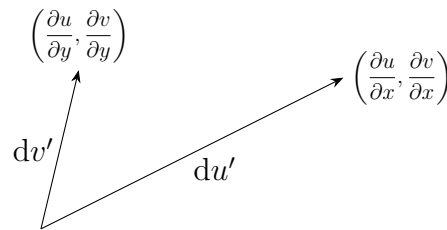
- **Cauchy-Riemann Equations:** Let  $f(x, y) = u(x, y) + iv(x, y)$ ,
  - The **Jacobian matrix** of  $f$  is

$$\mathbf{J}_f = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix}$$

What does the Jacobian matrix tell us? Recall that the transformation matrix  $\mathbf{T} = [\mathbf{v}_1 \ \mathbf{v}_2 \ \mathbf{v}_3 \ \dots \ \mathbf{v}_n]$ .



Taking ratios gives



- **Cauchy-Riemann Theorem:**

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

**Sketch of proof:** This is due to the derivative of  $f(z)$  can be approached from multiple directions.

Fix  $y$ , then when  $x^* \rightarrow x$ , we have

$$\begin{aligned} \lim_{x^* \rightarrow x} \frac{f(z^*) - f(z)}{z^* - z} &= \lim_{x^* \rightarrow x} \frac{u(x^*, y) + iv(x^*, y) - u(x, y) - iv(x, y)}{x^* - x} \\ &= \lim_{x^* \rightarrow x} \left( \frac{u(x^*, y) - u(x, y)}{x^* - x} + i \frac{v(x^*, y) - v(x, y)}{x^* - x} \right) \\ &= \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} \end{aligned}$$

Similarly, fix  $x$ , then when  $iy^* \rightarrow iy$ , we have

$$\lim_{y^* \rightarrow y} \frac{f(z^*) - f(z)}{z^* - z} = \frac{\partial v}{\partial y} - i \frac{\partial u}{\partial y}$$

Comparing both equations gives the desired result.

- Applying Cauchy-Riemann equations to the Jacobian matrix, we have

$$\mathbf{J}_f = \begin{pmatrix} \frac{\partial u}{\partial x} & -\frac{\partial v}{\partial x} \\ \frac{\partial v}{\partial x} & \frac{\partial u}{\partial x} \end{pmatrix}$$

- Recall that the multiplication of complex values is the product of matrices.

For example,  $(a + bi)(x + yi) = (ax - by) + i(ay + bx)$ ,

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix} \begin{pmatrix} x & -y \\ y & x \end{pmatrix} = \begin{pmatrix} ax - by & -(bx + ay) \\ bx + ay & ax - by \end{pmatrix}$$

- Inverse functions

- It is trivial that  $\frac{d}{dz}f^{-1}(z) = \frac{1}{f'(f^{-1}(z))}$
- By  $z = x + iy$ ,  $f(z) = u(x, y) + iv(x, y)$ ,

$$f'(z) = \frac{df}{dz} = \frac{\partial f}{\partial x} \frac{1}{\frac{\partial z}{\partial x}} = \frac{\partial f}{\partial x} = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x}$$

Thus the matrix representing  $f'(z)$  is surprisingly  $\mathbf{J}_f$ ! Moreover,

$$|f'(z)|^2 = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 = \det \mathbf{J}_f$$

- **Inverse Function Theorem:** Let  $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  be a (continuous, differentiable) function and let  $\mathbf{J}_f(\mathbf{p})$  denote the Jacobian matrix of  $f$  evaluated at point  $\mathbf{p}$  (here, we may assume that  $\mathbf{p}$  is a complex value on the plane), then

$$\mathbf{J}_{f^{-1}}(f(\mathbf{p})) = (\mathbf{J}_f(\mathbf{p}))^{-1}$$

**Sketch of proof:** Using the two facts above, we have

$$\mathbf{J}_{f^{-1}}(z)\mathbf{J}_f(f^{-1}(z)) = \mathbf{I}$$

Letting  $z = f(\mathbf{p})$  gives the desired result.

- **Recall on linear algebra** (taking 3-D system as example)

First, we define the **del/nabla** operator.

$$\nabla = \left( \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right)$$



1. **Gradient**: Denoted by  $\nabla f$  – Scalar multiplication of  $\nabla$  and  $f$ .

$$\nabla f(x, y, z) = \begin{pmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \\ \frac{\partial f}{\partial z} \end{pmatrix} = \frac{\partial f}{\partial x} \mathbf{i} + \frac{\partial f}{\partial y} \mathbf{j} + \frac{\partial f}{\partial z} \mathbf{k}$$

2. **Divergence**: Denoted by  $\nabla \cdot \mathbf{f}$  – Dot product of  $\nabla$  and  $\mathbf{f}$ .

$$\nabla \cdot \mathbf{f} = \nabla \cdot (F_x, F_y, F_z) = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$$

3. **Curl**: Denoted by  $\nabla \times \mathbf{f}$  – Cross product of  $\nabla$  and  $\mathbf{f}$ .

$$\nabla \times \mathbf{f} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix}$$

4. **Laplacian**: Denoted by  $\nabla \cdot \nabla \mathbf{f}$  or  $\nabla^2 \mathbf{f}$  – Divergence of gradient.

$$\nabla \cdot \nabla \mathbf{f} = \frac{\partial^2 F_x}{\partial x^2} + \frac{\partial^2 F_y}{\partial y^2} + \frac{\partial^2 F_z}{\partial z^2}$$

Now let's get back to the 2-D plane.

- **Harmonic Functions**: A function  $f : A \subseteq \mathbb{R}^2 \rightarrow \mathbb{R}$  is harmonic if

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$$

- **Clairaut's Theorem**:

$$\frac{\partial}{\partial x} \left( \frac{\partial f}{\partial y} \right) = \frac{\partial}{\partial y} \left( \frac{\partial f}{\partial x} \right)$$

**Sketch of proof**: A simple (but unformal) proof uses the definition of the derivative.

$$\frac{\partial}{\partial x} \left( \frac{\partial f}{\partial y} \right) = \lim_{x^* \rightarrow x} \frac{\lim_{y^* \rightarrow y} \frac{f(x^*, y^*) - f(x^*, y)}{y^* - y}}{x^* - x} - \lim_{y^* \rightarrow y} \frac{f(x, y^*) - f(x, y)}{y^* - y}$$

We can evaluate  $\frac{\partial}{\partial y} \left( \frac{\partial f}{\partial x} \right)$  a similar way.

- We now show that for some analytic function  $f = u + iv$ , then  $u$  and  $v$

are harmonic.

By utilizing the Cauchy-Riemann equations, taking partial derivative with respect to  $x$ , we have

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x} \left( \frac{\partial u}{\partial x} \right) = \frac{\partial}{\partial x} \left( \frac{\partial v}{\partial y} \right)$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y} \left( \frac{\partial u}{\partial y} \right) = -\frac{\partial}{\partial y} \left( \frac{\partial v}{\partial x} \right)$$

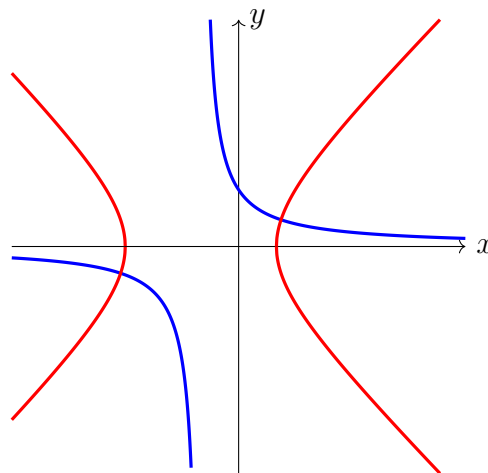
- **Harmonic Conjugate:** If  $u, v$  satisfy  $f = u + iv$ , then  $u(x, y)$  and  $v(x, y)$  are harmonic conjugates.

Let  $u(x, y)$  and  $v(x, y)$  be harmonic conjugates, then the graphs  $u(x, y) = c_1$  and  $v(x, y) = c_2$  intersect orthogonally in the Cartesian plane.

**Sketch of proof:** The dot product of two gradients equals zero

$$\nabla u \cdot \nabla v = \begin{pmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \end{pmatrix} \cdot \begin{pmatrix} \frac{\partial v}{\partial x} \\ \frac{\partial v}{\partial y} \end{pmatrix} = \frac{\partial u}{\partial x} \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \frac{\partial v}{\partial y} = 0$$

An example of  $f(z) = (z + 1)^2$  and  $u = x^2 - y^2 + 2x + 1 = 4$  and  $v = 2xy + 2y = 3$ .



**Problem**

Show, by changing variables, that the Cauchy-Riemann equations in terms of polar coordinates become

$$\frac{\partial u}{\partial r} = \frac{1}{r} \frac{\partial v}{\partial \theta} \quad \frac{\partial v}{\partial r} = -\frac{1}{r} \frac{\partial u}{\partial \theta}$$

Then show that if  $u$  is harmonic, we have

$$r^2 \frac{\partial^2 u}{\partial r^2} + r \frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial \theta^2} = 0$$

- (a) By knowing that  $f(x + iy) = u + iv$  and  $x + iy = r \cos \theta + ir \sin \theta$ , we have

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial r} = \frac{\partial u}{\partial x} \cos \theta$$

$$\frac{\partial v}{\partial \theta} = \frac{\partial v}{\partial y} \cdot \frac{\partial y}{\partial \theta} = \frac{\partial v}{\partial y} r \cos \theta$$

$$\frac{\partial v}{\partial r} = \frac{\partial v}{\partial x} \cdot \frac{\partial x}{\partial r} = \frac{\partial v}{\partial x} \cos \theta$$

$$\frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial \theta} = \frac{\partial u}{\partial y} r \cos \theta$$

Results can be shown by applying Cauchy-Riemann equations.

- (b) Evaluating the first equation gives

$$\frac{\partial^2 u}{\partial r^2} = \frac{1}{r} \frac{\partial}{\partial r} \left( \frac{\partial v}{\partial \theta} \right) - \frac{1}{r^2} \frac{\partial v}{\partial \theta} = \frac{1}{r} \frac{\partial}{\partial r} \left( \frac{\partial v}{\partial \theta} \right) - \frac{1}{r} \frac{\partial u}{\partial r}$$

while the second equation gives

$$\frac{\partial}{\partial \theta} \left( \frac{\partial v}{\partial r} \right) = -\frac{1}{r} \frac{\partial^2 u}{\partial \theta^2}$$

Eliminating  $\frac{\partial}{\partial \theta} \left( \frac{\partial v}{\partial r} \right)$  gives the desired result.

**Problem**

Show that  $\frac{\partial f}{\partial \bar{z}} = 0$ . Then, find the value of  $\frac{\partial \bar{z}}{\partial z}$ .

- (a) By using  $x = \frac{1}{2}(z + \bar{z})$  and  $y = \frac{1}{2i}(z - \bar{z})$ . A change of variable lets  $f$  depend on two independent variables  $z, \bar{z}$ . So by taking total derivative,

$$\frac{\partial f}{\partial \bar{z}} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial \bar{z}} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial \bar{z}} = \frac{1}{2} \left( \frac{\partial f}{\partial x} + i \frac{\partial f}{\partial y} \right) = 0$$

which can be shown by Cauchy-Riemann equations.

(b) By repeating the same thing,

$$\frac{\partial f}{\partial z} = \frac{1}{2} \left( \frac{\partial f}{\partial x} - i \frac{\partial f}{\partial y} \right)$$

Letting  $f = \bar{z}$  gives

$$\frac{\partial \bar{z}}{\partial z} = \frac{1}{2} (1 - i(-i)) = 0$$

### Problem

On what sets are each of the following functions harmonic?

(a)  $u(x, y) = \operatorname{Im}(z^2 + 3z + 1)$

(b)  $u(x, y) = \frac{x-1}{x^2 + y^2 - 2x + 1}$

(c)  $u(x, y) = \operatorname{Im}(z + 1/z)$

(d)  $u(x, y) = \frac{y}{(x-1)^2 + y^2}$

(a) Since  $u(x, y)$  is already the imaginary part of the function  $f(z) = z^2 + 3z + 1$ , so  $u(x, y)$  is harmonic on  $\mathbb{C}$ .

(b) Recall that  $\frac{1}{x+yi} = \frac{x}{x^2+y^2} - i \frac{y}{x^2+y^2}$ . Changing  $x$  to  $x-1$  shows that  $u(x, y)$  is the real part of  $f(z) = \frac{1}{z-1}$ . Hence  $u(x, y)$  is harmonic on  $\mathbb{C} \setminus \{1\}$ .

(c) Since  $u(x, y)$  is already the imaginary part of the function  $f(z) = z + 1/z$ , so  $u(x, y)$  is harmonic on  $\mathbb{C} \setminus \{0\}$ .

(d) Notice that  $u(x, y)$  is the imaginary part of the function  $f(z) = \frac{1}{1-z}$  hence it is harmonic on  $\mathbb{C} \setminus \{1\}$ .

### Problem

Suppose  $u$  is a twice continuously differentiable real-valued harmonic function on a disk  $D(z_0; l)$  centered at  $z_0 = x_0 + iy_0$ . For  $(x_1, y_1) \in D(z_0; r)$ , show that the equation

$$v(x_1, y_1) = c + \int_{y_0}^{y_1} \frac{\partial u}{\partial x}(x_1, y) dy - \int_{x_0}^{x_1} \frac{\partial u}{\partial y}(x, y_0) dx$$

defines a harmonic conjugate for  $u$  on  $D(z_0; r)$  with  $v(x_0, y_0) = c$ .

If  $v(x, y)$  is a harmonic conjugate, it must satisfy the Cauchy-Riemann equations.

Thus we have

$$\frac{\partial v}{\partial y}(x_1, y_1) = \frac{\partial}{\partial y} \left( c + \int_{y_0}^{y_1} \frac{\partial u}{\partial x}(x_1, y) dy - \int_{x_0}^{x_1} \frac{\partial u}{\partial y}(x, y_0) dx \right) = \frac{\partial u}{\partial x}(x_1, y_1)$$

We use the fact proven above,

$$\begin{aligned} \frac{\partial v}{\partial x}(x_1, y_1) &= \frac{\partial}{\partial x} \left( c + \int_{y_0}^{y_1} \frac{\partial u}{\partial x}(x_1, y) dy - \int_{x_0}^{x_1} \frac{\partial u}{\partial y}(x, y_0) dx \right) \\ &= \frac{\partial}{\partial x} (v(x_1, y_1) - v(x_1, y_0)) - \frac{\partial u}{\partial y}(x_1, y_0) \\ \frac{\partial v}{\partial x}(x_1, y_0) &= -\frac{\partial u}{\partial y}(x_1, y_0) \end{aligned}$$

**Remark.** An important theorem which may be worth introducing is the [Leibniz Integral Rule](#),

$$\frac{d}{dx} \int_{a(x)}^{b(x)} f(x, t) dt = f(x, b(x)) \frac{d}{dx} b(x) - f(x, a(x)) \frac{d}{dx} a(x) + \int_{a(x)}^{b(x)} \frac{\partial}{\partial x} f(x, t) dt$$

The proof uses [Fubini's Theorem](#), which states that

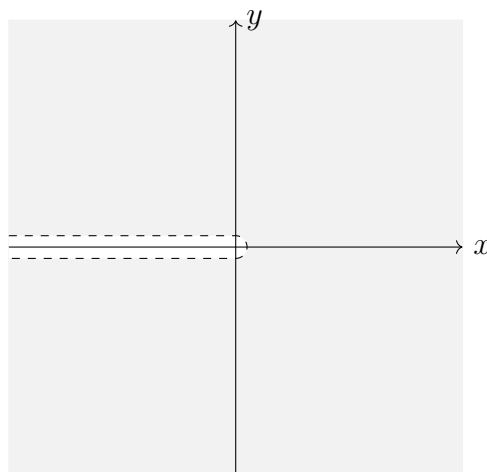
$$\int_X \left( \int_Y f(x, y) dy \right) dx = \int_Y \left( \int_X f(x, y) dx \right) dy$$

## 1.6 Differentiation of Complex Functions – Built Diff

### Principal Branch

Take  $f(z) = \ln z = \ln |z| + i \arg z$ . We may want  $-\pi < \arg z < \pi$ . This is a [principal branch](#) of the logarithm function.

This is due to the fact that  $\ln |z| + i \arg z = \ln |z| + i(\arg z + 2k\pi)$ .



**Problem**

Let  $u, v$  be real-valued functions on an open set  $A \subset \mathbb{R}^2 = \mathbb{C}$  and suppose that they satisfy the Cauchy-Riemann equations on  $A$ . Show that

(a)  $u_1 = u^2 - v^2, v_1 = 2uv$

(b)  $u_2 = e^u \cos v, v_2 = e^u \sin v$

also satisfy the Cauchy-Riemann equations on  $A$ .

(a)

$$\begin{aligned}\frac{\partial u_1}{\partial x} &= 2u \frac{\partial u}{\partial x} - 2v \frac{\partial v}{\partial x} \\ \frac{\partial v_1}{\partial y} &= 2u \frac{\partial v}{\partial y} + 2v \frac{\partial u}{\partial y} = 2u \frac{\partial u}{\partial x} - 2v \frac{\partial v}{\partial x} \\ \frac{\partial u_1}{\partial y} &= 2u \frac{\partial u}{\partial y} - 2v \frac{\partial v}{\partial y} \\ \frac{\partial v_1}{\partial x} &= 2u \frac{\partial v}{\partial x} + 2v \frac{\partial u}{\partial x} = -2u \frac{\partial u}{\partial y} + 2v \frac{\partial v}{\partial y}\end{aligned}$$

(b)

$$\begin{aligned}\frac{\partial u_2}{\partial x} &= e^u \frac{\partial u}{\partial x} \cos v - \sin v \frac{\partial v}{\partial x} e^u \\ \frac{\partial v_2}{\partial y} &= e^u \frac{\partial u}{\partial y} \sin v + \cos v \frac{\partial v}{\partial y} e^u = -e^u \frac{\partial v}{\partial x} \sin v + \cos v \frac{\partial u}{\partial x} e^u \\ \frac{\partial u_2}{\partial y} &= e^u \frac{\partial u}{\partial y} \cos v - \sin v \frac{\partial v}{\partial y} e^u = -e^u \frac{\partial v}{\partial x} \cos v - \sin v \frac{\partial u}{\partial x} e^u \\ \frac{\partial v_2}{\partial x} &= e^u \frac{\partial u}{\partial x} \sin v + \cos v \frac{\partial v}{\partial x} e^u\end{aligned}$$

**Remark.** Let  $f(z) = u + iv$ , it's not hard to see that in (a), we have  $g(z) = f(z)^2 = (u^2 - v^2) + 2uvi$  and in (b), we have  $h(z) = e^{f(z)} = e^u \cos v + ie^u \sin v$ .

**Problem**

Given functions  $u(x, y)$ , find their respective harmonic conjugates.

(a)  $e^x(y \cos y + x \sin y)$

(b)  $\frac{(e^{-y} + e^y) \sin x}{2}$

(a) We want

$$\frac{\partial u}{\partial x} = e^x(y \cos y + (x+1) \sin y) = \frac{\partial v}{\partial y}$$

So by the equation above we have

$$v = e^x(y \sin y + \cos y - (x+1) \cos y) + g(x) = e^x(y \sin y - x \cos y) + g(x)$$

On the other hand, we have

$$\frac{\partial u}{\partial y} = e^x(\cos y - y \sin y + x \cos y) = -\frac{\partial v}{\partial x}$$

Solving the differential equation gives

$$v = e^x y \sin y - e^x x \cos y + h(y)$$

Thus  $g(x) = h(y) = C$  for some constant  $C$ . We have

$$v = e^x y \sin y - e^x x \cos y + C.$$

(b) We want

$$\frac{\partial u}{\partial x} = \frac{(e^{-y} + e^y) \cos x}{2} = \frac{\partial v}{\partial y}$$

So we have

$$v = \frac{(-e^{-y} + e^y) \cos x}{2} + g(x)$$

While on the other hand,

$$\frac{\partial u}{\partial y} = \frac{(-e^{-y} + e^y) \sin x}{2} = -\frac{\partial v}{\partial x}$$

Solving gives

$$v = \frac{(-e^{-y} + e^y) \cos x}{2} + h(y)$$

Comparing gives

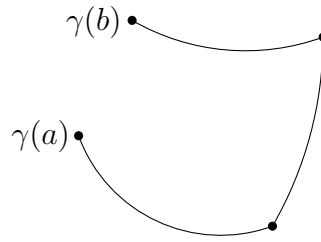
$$v = \frac{(-e^{-y} + e^y) \cos x}{2} + C$$

## §2 Cauchy's Theorem

### 2.1 Contour Integrals are Contourversial

Let  $z : [a, b] \rightarrow \mathbb{C}$  be a **curve**.

- If it is a continuous function, then it is a **smooth curve**.
- If we have  $z'(a) = z'(b)$ , then we say that the curve is **closed**.
- A curve is called **piecewise  $C^1$**  if we can divide the interval into subintervals  $a = a_0 < a_1 < a_2 < \cdots < a_n = b$  such that  $\gamma'(t)$  exists on the open intervals  $(a_k, a_{k+1})$  and continuous on  $[a_k, a_{k+1}]$ .



Let  $\gamma$  be a smooth curve, we denote the integral along  $\gamma$  as

$$\int_{\gamma} f(z) dz = \int_a^b f(z(t)) z'(t) dt$$

Sometimes if we know that  $\gamma$  is a closed curve (known as the **cyclic integral**), we can write as

$$\oint_{\gamma} f(z) dz$$

If it is known that the loop is directed clockwise or anticlockwise, it can still sometimes be written as

$$\oint_{\gamma} f(z) dz \quad \oint_{\gamma} f(z) dz$$

### Integration Properties

- In general, we have

$$\int_{\gamma} f = \sum_{i=0}^{n-1} \int_{a_i}^{a_{i+1}} f(\gamma(t)) \gamma'(t) dt$$

- Let a function be  $f = u(x, y) + iv(x, y)$ , we have

$$\int_{\gamma} f = \int_{\gamma} (u dx - v dy) + i \int_{\gamma} (u dy + v dx)$$

**Sketch of proof:** Consider

$$\begin{aligned} f(\gamma(t)) \gamma'(t) &= [u(x, y) + iv(x, y)][x'(t) + iy'(t)] \\ &= [u(x, y)x'(t) - v(x, y)y'(t)] + i[u(x, y)y'(t) + v(x, y)x'(t)] \\ \int_a^b f(\gamma(t)) \gamma'(t) dt &= \int_a^b [u(x, y)x'(t) - v(x, y)y'(t)] dt + i \int_a^b [u(x, y)y'(t) + v(x, y)x'(t)] dt \\ \int_{\gamma} f &= \int_{\gamma} (u dx - v dy) + i \int_{\gamma} (u dy + v dx) \end{aligned}$$

- An **opposite curve** of a curve  $\gamma$  is a curve (denoted as  $-\gamma$ ) traversed oppositely.





Assume that  $\gamma : [a, b] \rightarrow \mathbb{C}$  and  $-\gamma : [a, b] \rightarrow \mathbb{C}$ , we have

$$\gamma(t) = (-\gamma)(a + b - t)$$

- A **sum**  $\gamma_1 + \gamma_2$  of curves is a curve constructed by joining the endpoints of  $\gamma_1 : [a, b] \rightarrow \mathbb{C}$  and  $\gamma_2 : [b, c] \rightarrow \mathbb{C}$ . Thus

$$(\gamma_1 + \gamma_2)(t) = \begin{cases} \gamma_1(t) & t \in [a, b] \\ \gamma_2(t) & t \in [b, c] \end{cases}$$

So we must have  $\gamma_1(b) = \gamma_2(b)$ .

- We have the following list of properties, which can be proved by the definitions above

$$* \int_{\gamma} \sum_{i=1}^n c_i f_i = \sum_{i=1}^n \left( c_i \int_{\gamma} f_i \right)$$

$$* \int_{-\gamma} f = - \int_{\gamma} f$$

$$* \int_{\gamma_1 + \gamma_2 + \dots + \gamma_n} f = \sum_{i=1}^n \int_{\gamma_i} f$$

- A **reparametrization** of a piecewise smooth curve  $\gamma : [a, b] \rightarrow \mathbb{C}$  is the piecewise smooth curve  $\tilde{\gamma} : [\tilde{a}, \tilde{b}] \rightarrow \mathbb{C}$  if there exists a piecewise  $C^1$  function  $\alpha : [a, b] \rightarrow [\tilde{a}, \tilde{b}]$  with
  - $\alpha'(t) > 0$  for all  $t \in (a, b)$
  - $\alpha(a) = \tilde{a}, \alpha(b) = \tilde{b}$ ,
  - $\gamma(t) = \tilde{\gamma}(\alpha(t))$ .
- We have

$$\int_{\gamma} f = \int_{\tilde{\gamma}} f$$

**Sketch of proof:** Evaluating

$$\begin{aligned} \int_{\gamma} f &= \int_a^b f(\gamma(t)) \gamma'(t) dt \\ &= \int_a^b f(\tilde{\gamma}(\alpha(t))) \tilde{\gamma}'(\alpha(t)) \alpha'(t) dt \\ &= \int_{\tilde{a}}^{\tilde{b}} f(\tilde{\gamma}(s)) \tilde{\gamma}'(s) ds \\ &= \int_{\tilde{\gamma}} f \end{aligned}$$

- The **arc length** formula is given by

$$l(\gamma) = \int_a^b \sqrt{x'(t)^2 + y'(t)^2} dt = \int_a^b |\gamma'(t)| dt$$

Recall that the arc length formula in the Cartesian plane is

$$\int dl = \int \sqrt{dx^2 + dy^2} = \int_c^d \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$$

By recognizing that  $dx = x'(t)dt$  and  $dy = y'(t)dt$ , we have

$$\begin{aligned} \int_c^d \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx &= \int_a^b \sqrt{1 + \left(\frac{y'(t)dt}{x'(t)dt}\right)^2} x'(t)dt \\ &= \int_a^b \sqrt{x'(t)^2 + y'(t)^2} dt \end{aligned}$$

For some continuous function  $f$ , we have

$$\left| \int_a^b f(t) dt \right| \leq \int_a^b |f(t)| dt$$

**Sketch of proof:** Let  $\int_a^b f(t) dt = re^{i\theta}$ , then

$$\begin{aligned} \left| \int_a^b f(t) dt \right| &= r \\ &= \int_a^b e^{-i\theta} f(t) dt \\ &= \int_a^b \operatorname{Re}(e^{-i\theta} f(t)) dt \\ &\leq \int_a^b |\operatorname{Re}(e^{-i\theta} f(t))| dt \\ &\leq \int_a^b |e^{-i\theta} f(t)| dt \\ &= \int_a^b |f(t)| dt \end{aligned}$$

Let  $|f(z)| \leq M$  for some constant  $M > 0$  and all  $z$  on  $\gamma$ , we have

$$\left| \int_{\gamma} f(z) dz \right| \leq \int_{\gamma} |f(z)| dz = \int_a^b |f(\gamma(t))| |\gamma'(t)| dt \leq M \int_a^b |\gamma'(t)| dt = Ml(\gamma)$$

- **Fundamental Theorem of Calculus for Contour Integrals:** Recall that the fundamental theorem of calculus states that

$$\int_a^b F'(x) dx = F(b) - F(a)$$

Suppose that  $\gamma : [0, 1] \rightarrow \mathbb{C}$  is a piecewise smooth curve, we have

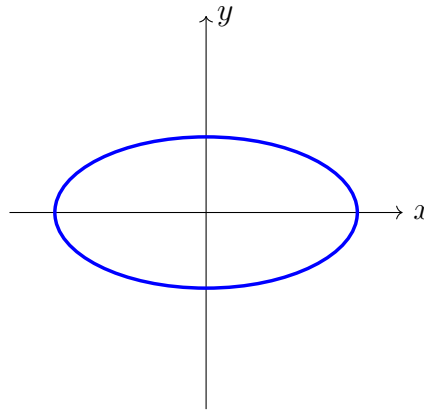
$$\int_{\gamma} F'(z) dz = F(\gamma(1)) - F(\gamma(0))$$

If it happens that  $\gamma(0) = \gamma(1)$  (which makes  $\gamma$  a loop) and  $F'(z)$  is **analytic everywhere inside**  $\gamma$ , we have

$$\oint_{\gamma} F'(z) dz = 0$$

### Example

Let's evaluate  $\int_{\gamma} z dz$  and  $\int_{\gamma_1} z dz$  where  $\gamma$  is the portion of the ellipse  $4x^2 + y^2 = 1$  joining  $z = 1/2$  to  $z = i$ , and  $\gamma_1$  is the entire ellipse, integrated counterclockwise.



A parametrization will be  $\gamma(t) = \frac{\cos t}{2} + i \sin t$  and  $t$  ranges from 0 to  $2\pi$ . Thus we have

$$\begin{aligned} \int_{\gamma} z dz &= \int_0^{\pi/2} \left( \frac{\cos t}{2} + i \sin t \right) \left( -\frac{\sin t}{2} + i \cos t \right) dt \\ &= \int_0^{\pi/2} \left( \frac{-5 \sin t \cos t}{4} + i \left( \frac{\cos^2 t - \sin^2 t}{2} \right) \right) dt \\ &= -\frac{5}{8} \end{aligned}$$

On the other hand, we have

$$\begin{aligned} \int_{\pi/2}^{\pi} \left( \frac{\cos t}{2} + i \sin t \right) \left( -\frac{\sin t}{2} + i \cos t \right) dt &= \frac{5}{8} \\ \int_{\pi}^{3\pi/2} \left( \frac{\cos t}{2} + i \sin t \right) \left( -\frac{\sin t}{2} + i \cos t \right) dt &= -\frac{5}{8} \end{aligned}$$

$$\int_{3\pi/2}^{2\pi} \left( \frac{\cos t}{2} + i \sin t \right) \left( \frac{-\sin t}{2} + i \cos t \right) dt = \frac{5}{8}$$

which indeed adds up to 0.

- **Path Independence Theorem:** Let  $f$  be a continuous function on an open connected set  $G \in \mathbb{C}$ ,

- For any closed curve  $\Gamma$ ,

$$\int_{\Gamma} f = 0$$

- A result is that for any two curves  $\gamma_1, \gamma_2$  joining  $z_0, z_1$ ,

$$\int_{\Gamma} f = \int_{\gamma_1} f + \int_{-\gamma_2} f = \int_{\gamma_1} f - \int_{\gamma_2} f = 0$$

### Problem

Evaluate  $\int_{\gamma} \sin 2z \, dz$  where  $\gamma$  is the line segment joining  $i + 1$  to  $-i$ .

Let the parametrization be  $\gamma(t) = (1 - t) + i(1 - 2t)$  where  $t \in [0, 1]$ , then

$$\begin{aligned} \int_{\gamma} \sin 2z \, dz &= \int_0^1 [\sin 2((1 - t) + i(1 - 2t))]( -1 - 2i) \, dt \\ &= (-1 - 2i) \int_0^1 [\sin((2 + 2i) - t(2 + 4i))] \, dt \\ &= \frac{-1 - 2i}{2 + 4i} (\cos(-2i) - \cos(2 + 2i)) \\ &= -\frac{1}{2} \left( \frac{e^2 + e^{-2}}{2} - \frac{e^{2i-2} + e^{2-2i}}{2} \right) \end{aligned}$$

### Problem

Evaluate  $\int_{\gamma} \bar{z}^2 \, dz$  along two paths joining  $(0, 0)$  to  $(1, 1)$  as follows:

- $\gamma$  is the straight line joining  $(0, 0)$  to  $(1, 1)$ .
- $\gamma$  is the broken line joining  $(0, 0)$  to  $(1, 0)$ , then joining  $(1, 0)$  to  $(1, 1)$ .

- We know that  $\bar{z} = \operatorname{Re}(z) - i \operatorname{Im}(z)$ , consider the parametrization  $\gamma : [0, 1] \rightarrow \mathbb{C}$  be  $\gamma(t) = t + it$ ,

$$\begin{aligned} \int_{\gamma} \bar{z}^2 \, dz &= \int_0^1 (t - it)^2 (1 + i) \, dt \\ &= (1 + i)(1 - i)^2 \int_0^1 t^2 \, dt \end{aligned}$$

$$= \frac{2 - 2i}{3}$$

(b) Similarly, let  $\gamma_1 : [0, 1] \rightarrow \mathbb{C}$  be  $\gamma_1(t) = t$  and  $\gamma_2 : [0, 1] \rightarrow \mathbb{C}$  be  $\gamma_2(t) = 1 + it$ .

$$\begin{aligned} \int_{\gamma} \bar{z}^2 \, dz &= \int_{\gamma_1} \bar{z}^2 \, dz + \int_{\gamma_2} \bar{z}^2 \, dz \\ &= \int_0^1 t^2 \, dt + \int_0^1 (1 - it)^2 \, dt \\ &= \int_0^1 (1 - 2it) \, dt \\ &= 1 - i \end{aligned}$$

### Problem

Prove that

- (a)  $\left| \int_C \frac{dz}{1 + z^2} \right| \leq \frac{\pi}{3}$  where  $C$  is the arc of the circle  $|z| = 2$  in the first quadrant.
- (b)  $\left| \int_{\gamma} \frac{\sin z}{z^2} dz \right| \leq 2\pi e$  where  $\gamma$  is the unit circle.

(a) Consider the parametrization  $C : [0, \pi/2] \rightarrow \mathbb{C}$  defined by  $C(t) = 2(\cos t + i \sin t)$ .

$$\text{Since we have } \left| \frac{1}{1 + z^2} \right| = \frac{1}{\sqrt{17 + 8 \cos 2t}} \leq \frac{1}{3},$$

$$\begin{aligned} \left| \int_C \frac{dz}{1 + z^2} \right| &\leq \int_C \left| \frac{dz}{1 + z^2} \right| \\ &\leq \frac{1}{3} \cdot 2 \cdot \frac{\pi}{2} \\ &= \frac{\pi}{3} \end{aligned}$$

(b) Consider the parametrization  $\gamma : [0, 2\pi] \rightarrow \mathbb{C}$  defined by  $\gamma(t) = \cos t + i \sin t$ .

Since

$$\begin{aligned} \left| \frac{\sin z}{z} \right| &= \left| \frac{e^{-\sin t + i \cos t} - e^{\sin t - i \cos t}}{2i(\cos 2t + i \sin 2t)} \right| \\ &= \frac{1}{2} \sqrt{(e^{-\sin t} - e^{\sin t})^2 (\cos 2t)^2 + (e^{-\sin t} + e^{\sin t})^2 (\sin 2t)^2} \\ &= \frac{1}{2} \sqrt{e^{-2 \sin t} + e^{2 \sin t} - 2 \cos(2 \cos t)} \\ &\leq \frac{1}{2} (e^{\sin t} + e^{-\sin t}) \end{aligned}$$

$$\begin{aligned} &\leq \frac{1}{2}(e + e) \\ &\leq e \end{aligned}$$

On the other hand, since  $l(\gamma) = 2\pi$ ,

$$\begin{aligned} \left| \int_{\gamma} \frac{\sin z}{z^2} dz \right| &\leq \int_{\gamma} \left| \frac{\sin z}{z^2} \right| dz \\ &\leq e \cdot 2\pi \end{aligned}$$

### Problem

Show that the arc length  $l(\gamma)$  of a curve  $\gamma$  is unchanged if  $\gamma$  is reparametrized.

Consider the reparametrization  $\tilde{\gamma} : [\tilde{a}, \tilde{b}] \rightarrow \mathbb{C}$  defined by  $\gamma(t) = \tilde{\gamma}(\alpha(t))$ ,  $\alpha(a) = \tilde{a}$ ,  $\alpha(b) = \tilde{b}$  and  $\alpha'(t) > 0$  for all  $t \in (a, b)$ .

Then

$$\begin{aligned} l(\gamma) &= \int_a^b |\gamma'(t)| dt \\ &= \int_a^b |\tilde{\gamma}(\alpha(t))| \alpha'(t) dt \\ &= \int_{\tilde{a}}^{\tilde{b}} |\tilde{\gamma}(t)| dt \\ &= l(\tilde{\gamma}) \end{aligned}$$

## 2.2 Cauchy's Theorem

**Cauchy's Theorem** states that if  $\gamma$  is a closed curve intersecting itself **only at its endpoints**, then

$$\int_{\gamma} f = 0$$

**Green's Theorem:** For continuously differentiable functions  $P(x, y)$  and  $Q(x, y)$ , we have

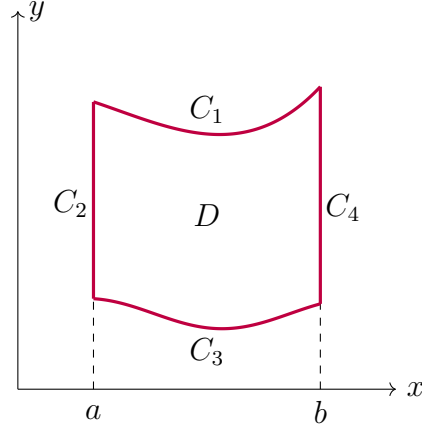
$$\oint_{\gamma} P(x, y) dx + Q(x, y) dy = \iint_A \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

**Remark.** Green's Theorem is a special case of the **Kelvin-Stokes Theorem** (or sometimes known as the Fundamental Theorem of Curls), stated that for some smooth oriented surface  $\Sigma$  in  $\mathbb{R}^3$  with boundary  $\partial\Sigma$ ,

$$\iint_{\Sigma} \left( \left( \frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z} \right) dy dz + \left( \frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x} \right) dz dx + \left( \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right) dx dy \right)$$

$$= \oint_{\partial \Sigma} (F_x dx + F_y dy + F_z dz).$$

**Sketch of proof:**



By integrating along  $C_1$ , we have

$$\int_{C_1} P(x, y) dx = \int_b^a P(x, g_1(x)) dx$$

Similarly,

$$\int_{C_3} P(x, y) dx = \int_a^b P(x, g_3(x)) dx$$

On the other hand,  $\int_{C_2} P(x, y) dx = \int_{C_4} P(x, y) dx = 0$ . As a result we have

$$\begin{aligned} \int_{C_1+C_2+C_3+C_4} P(x, y) dx &= \int_{C_1} P(x, y) dx + \int_{C_2} P(x, y) dx + \int_{C_3} P(x, y) dx + \int_{C_4} P(x, y) dx \\ &= \int_a^b P(x, g_3(x)) dx - \int_b^a P(x, g_1(x)) dx \\ &= \int_a^b [P(x, g_3(x)) - P(x, g_1(x))] dx \\ &= \int_a^b \int_{g_1(x)}^{g_3(x)} \frac{\partial P}{\partial y} dy dx \\ &= - \iint_D \frac{\partial P}{\partial y} dA \end{aligned}$$

Similarly one can get

$$\int_C Q(x, y) dy = \iint_D \frac{\partial Q}{\partial x} dA$$

Yet, summing up gives our result

$$\oint_{\gamma} P(x, y) dx + Q(x, y) dy = \iint_A \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

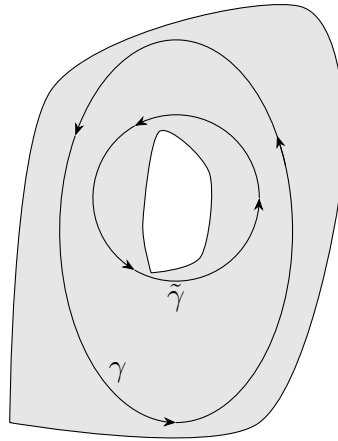
We can express our complex integral

$$\int_{\gamma} f = \iint_A \left[ -\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right] dydx + i \iint_A \left[ \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} \right] dydx$$

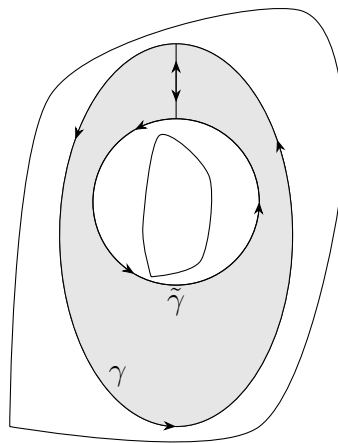
**Deformation Theorem:** Let  $f$  be analytic on a region  $A$  and let  $\gamma$  be a simple closed curve in  $A$ . We assume that we can  $\gamma$  can be deformed to another simple closed curve  $\tilde{\gamma}$  without passing outside  $A$ .

We say  $\gamma$  is **homotopic** to  $\tilde{\gamma}$  in  $A$ . Then we have

$$\int_{\gamma} f = \int_{\tilde{\gamma}} f$$



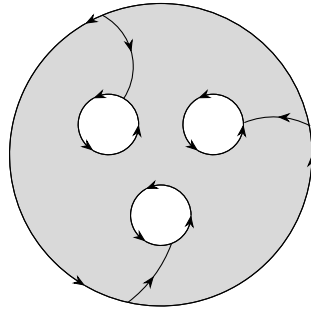
This is due to the fact that, we can construct some curve  $\gamma_0$  and  $-\gamma_0$ , thus the curve  $\int_{\gamma+\gamma_0-\tilde{\gamma}+(-\gamma_0)} f = 0$ .



In fact, let  $\gamma$  be a simple closed curve with  $f$  holomorphic between  $\gamma$  and  $\gamma_1, \gamma_2, \dots, \gamma_n$ , then

$$\int_{\gamma} f = \sum_{i=1}^n \int_{\gamma_i} f$$





Similarly we can “build bridges” between each closed curve.

### Problem

Evaluate

- (a)  $\int_{\gamma} (z^3 + 3)dz$ , where  $\gamma$  is the unit circle.
- (b)  $\int_{\gamma} \cos[3 + 1/(z - 3)]dz$ , where  $\gamma$  is a circle of radius 3 centered at  $5i + 1$ .
- (c)  $\int_{\gamma} \sqrt{z^2 - 1}dz$  where  $\gamma$  is the circle of radius  $1/2$  centered at 0.
- (d)  $\int_{\gamma} \frac{2z^2 - 15z + 30}{z^3 - 10z^2 + 32z - 32}dz$  where  $\gamma$  is the circle  $|z| = 3$ .

- (a) Since  $z^3 + 3$  is analytic everywhere in  $\gamma$ , we have

$$\int_{\gamma} (z^3 + 3)dz = 0$$

- (b) Similarly we have

$$\int_{\gamma} \cos[3 + 1/(z - 3)]dz = 0$$

- (c) Consider the parametrization  $\gamma(t) = \frac{1}{2}e^{i\theta}$  for  $\theta \in [0, 2\pi]$ . We have

$$\begin{aligned} \int_{\gamma} \sqrt{z^2 - 1}dz &= \int_0^{2\pi} \sqrt{\frac{1}{4}e^{2i\theta} - 1} \cdot \frac{1}{2}e^{i\theta}d\theta \\ &= \int_0^{\pi} \sqrt{\frac{1}{4}e^{2i\theta} - 1} \cdot \frac{1}{2}e^{i\theta}d\theta + \int_{\pi}^{2\pi} \sqrt{\frac{1}{4}e^{2i\theta} - 1} \cdot \frac{1}{2}e^{i\theta}d\theta \\ &= \int_0^{\pi} \sqrt{\frac{1}{4}e^{2i\theta} - 1} \cdot \frac{1}{2}e^{i\theta}d\theta - \int_0^{\pi} \sqrt{\frac{1}{4}e^{2i\theta} - 1} \cdot \frac{1}{2}e^{i\theta}d\theta \\ &= 0 \end{aligned}$$

(d) By using partial fractions gives

$$\int_{\gamma} \frac{2z^2 - 15z + 30}{z^3 - 10z^2 + 32z - 32} dz = \int_{\gamma} \left( \frac{2}{z-2} + \frac{1}{(z-4)^2} \right) dz$$

We have  $\int_{\gamma} \frac{1}{(z-4)^2} dz = 0$ , and  $\int_{\gamma} \frac{1}{z-2} dz = \int_{|z-2|=1} \frac{1}{z-2} dz = 2\pi i$ , we obtain

$$\int_{\gamma} \left( \frac{2}{z-2} + \frac{1}{(z-4)^2} \right) dz = 4\pi i$$

### Problem

Let  $f$  be entire. Evaluate

$$\int_0^{2\pi} f(z_0 + re^{i\theta}) e^{ki\theta} d\theta$$

for  $k$  an integer,  $k \geq 1$ .

Let  $\gamma(\theta) = z_0 + re^{i\theta}$  with  $\theta \in [0, 2\pi]$ , then since  $f$  is entire,  $f$  is holomorphic everywhere in  $\mathbb{C}$ . So  $f(\gamma(\theta))e^{ki\theta}$  is holomorphic everywhere in  $\gamma$ . By applying Cauchy's Theorem, we have

$$\int_0^{2\pi} f(z_0 + re^{i\theta}) e^{ki\theta} d\theta = 0$$

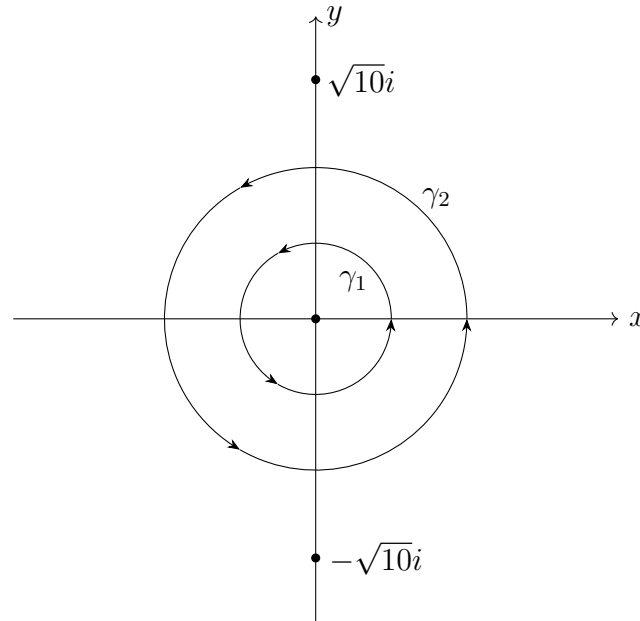
### Problem

Let  $\gamma_1$  be the circle of the radius 1 and let  $\gamma_2$  be the circle of radius 2 (traversed counterclockwise and centered at the origin). Show that

$$\int_{\gamma_1} \frac{dz}{z^3(z^2 + 10)} = \int_{\gamma_2} \frac{dz}{z^3(z^2 + 10)}$$

Let  $f(z) = \frac{1}{z^3(z^2 + 10)}$ , and  $\gamma'$  is a curve connecting  $\gamma_1$  and  $\gamma_2$ , then we have

$$\int_{\gamma_1 + \gamma' - \gamma_2 + (-\gamma')} f = 0 \iff \int_{\gamma_1} f = \int_{\gamma_2} f$$



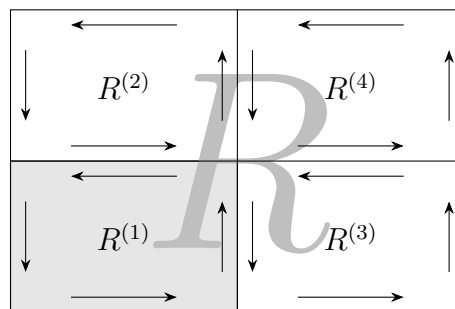
### 2.3 More on Cauchy's Theorem

- **For rectangles:** Let  $R$  be a rectangular path with sides parallel to the axes and  $f$  is a function defined and holomorphic on an open set  $G$  containing  $R$  and its interior, then  $\int_R f = 0$ .

**Sketch of proof:** We must be aware that we do not know if  $f'$  is continuous. In fact, we will use Cauchy's Theorem to prove that  $f'$  is **automatically continuous**.

Let  $R$  be a rectangle in  $G \subset \mathbb{C}$ , and  $P, \Delta$  be the perimeter and diagonal length of  $R$  respectively. Divide  $R$  into  $R^{(1)}, R^{(2)}, R^{(3)}, R^{(4)}$ . We have

$$\int_R f = \int_{R^{(1)}} f + \int_{R^{(2)}} f + \int_{R^{(3)}} f + \int_{R^{(4)}} f$$



Applying inequality gives

$$\left| \int_R f \right| = \left| \int_{R^{(1)}} f \right| + \left| \int_{R^{(2)}} f \right| + \left| \int_{R^{(3)}} f \right| + \left| \int_{R^{(4)}} f \right|$$

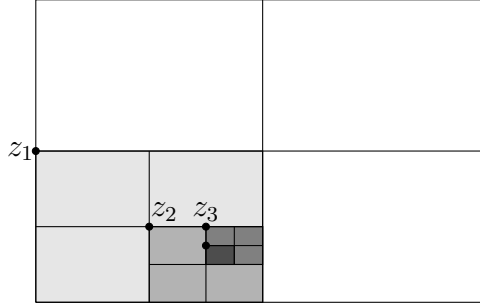
For each  $R^{(i)}$ , WLOG let  $R^{(1)}$  be the subrectangle with  $\left| \int_{R^{(1)}} f \right| \geq \frac{1}{4} \left| \int_R f \right|$ , let  $R_1$  be this subrectangle. We construct  $R_2, R_3, R_4, \dots$  similarly. Therefore, we have

$$\left| \int_{R_n} f \right| \geq \frac{1}{4^n} \left| \int_R f \right|$$

On the other hand, we have

- (i) Perimeter of  $R_n$ ,  $P(R_n) = P/2^n$ .
- (ii) Diagonal of  $R_n$ ,  $\Delta(R_n) = \Delta/2^n$ .

Let  $z_n$  be the upper left vertex of  $R_n$ , then we see that whenever  $m > n$ , we have  $|z_m - z_n| \leq P(R_n) = \Delta/2^n$ .



Thus  $\{z_n\}$  forms a Cauchy sequence that must converge to some point  $w_0$ . Let  $z$  be a point in  $R_n$ , then we have  $|z - w_0| \leq \Delta(R_n)$ .

For some  $z$  in  $R_n$ , fix  $\varepsilon$  so that

$$\left| \frac{f(z) - f(w_0)}{z - w_0} - f'(w_0) \right| < \varepsilon$$

We choose  $\delta$  and large enough  $n$  such that  $|z - w_0| \leq \Delta(R_n) < \delta$ , we have

$$|f(z) - f(w_0) - (z - w_0)f'(w_0)| < \varepsilon|z - w_0| \leq \varepsilon\Delta(R_n)$$

And since we have  $\int_{R_n} 1dz = \int_{R_n} (z - w_0)dz = 0$ ,

$$\begin{aligned} \left| \int_R f \right| &\leq 4^n \left| \int_{R_n} f \right| \\ &= 4^n \left| \int_{R_n} f(z)dz - f(w_0) \int_{R_n} 1dz - f'(w_0) \int_{R_n} (z - w_0)dz \right| \\ &\leq 4^n \int_{R_n} |f(z) - f(w_0) - (z - w_0)f'(w_0)| dz \\ &\leq 4^n (\varepsilon\Delta(R_n)) P(R_n) \\ &= \varepsilon\Delta P \end{aligned}$$

Since this is true for all  $\varepsilon$ , choose  $\varepsilon$  small, we have  $|\int_R f| = 0$  which gives  $\int_R f = 0$ .

- **For disks:** Suppose  $f : D \rightarrow \mathbb{C}$  is holomorphic on an (open) disk  $D =$

$D(z_0, \rho) \subset \mathbb{C}$ , then

- (i) There exists a function  $f : D \rightarrow \mathbb{C}$ , the **antiderivative** of  $f$  on  $D$ , satisfying  $F'(z) = f(z)$  for all  $z$  in  $D$ .
- (ii) For any closed curve  $\Gamma$  in  $D$ ,  $\int_{\Gamma} f = 0$ .

We let  $\langle\langle a, b \rangle\rangle$  denote the **polygonal path** from  $a$  to  $b$ , first parallel to the  $x$  axis then parallel to the  $y$  axis.

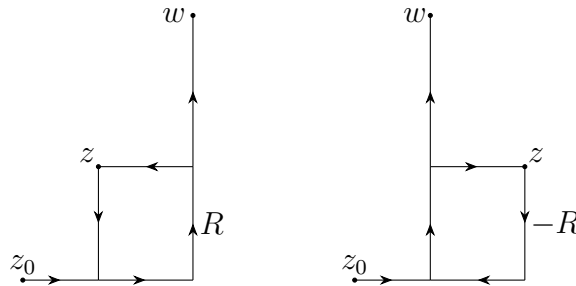


**Sketch of Proof:** We may want to show that there exists a function  $F(z)$  such that  $F'(z) = f(z)$  for all  $z \in D(z_0; \rho)$ .

Fix  $z$  and  $\varepsilon > 0$  so that we can choose some  $\delta > 0$  and whenever  $\xi \in D(z; \delta)$  we have  $|f(z) - f(\xi)| < \varepsilon$ .

In any case, we have

$$\int_{\langle\langle z_0, z \rangle\rangle} f(\xi) d\xi \pm \int_R f(\xi) d\xi + \int_{\langle\langle z, w \rangle\rangle} f(\xi) d\xi = \int_{\langle\langle z_0, w \rangle\rangle} f(\xi) d\xi$$



Thus we have

$$F(z) + \int_{\langle\langle z, w \rangle\rangle} f(\xi) d\xi = F(w)$$

For any  $\varepsilon$ ,

$$\begin{aligned} \left| \frac{F(w) - F(z)}{w - z} - f(z) \right| &= \frac{1}{|w - z|} \left| \int_{\langle\langle z, w \rangle\rangle} f(\xi) d\xi - (w - z)f(z) \right| \\ &= \frac{1}{|w - z|} \left| \int_{\langle\langle z, w \rangle\rangle} [f(\xi) - f(z)] d\xi \right| \\ &\leq \frac{1}{|w - z|} \int_{\langle\langle z, w \rangle\rangle} |f(\xi) - f(z)| d\xi \\ &\leq \frac{1}{|w - z|} \varepsilon \cdot 2|w - z| = 2\varepsilon \end{aligned}$$

Choose  $\varepsilon$  small, we have

$$\lim_{w \rightarrow z} \frac{F(w) - F(z)}{w - z} = f(z)$$

This proves (i) of the theorem, followed by  $\int_{\gamma} f = 0$  by the Path Independence Theorem.

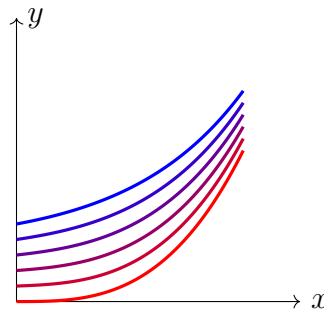
- **Homotopy**: Let  $f, g : \mathbb{R} \rightarrow \mathbb{R}^2$  be two functions, the map  $H(x, t)$  with  $H : \mathbb{R} \times [0, 1] \rightarrow \mathbb{R}^2$  is a **homotopy**.

Two functions  $f$  and  $g$  are said to be **homotopic**  $f \simeq g$  if there exists a homotopy  $H$  that maps  $f$  to  $g$ .

Say  $f(x) := \left(x, \frac{1}{2}e^{x-1} + \frac{1}{2}\right)$  and  $g(x) := \left(x, \frac{1}{6}x^3\right)$ , then

$$H(x, t) = \left(x, (1-t) \left(\frac{1}{6}x^3\right) + t \left(\frac{1}{2}e^{x-1} + \frac{1}{2}\right)\right)$$

is a homotopy.



In general, for functions  $f$  and  $g$ , the function

$$H(x, t) = (1-t)f(x) + tg(x)$$

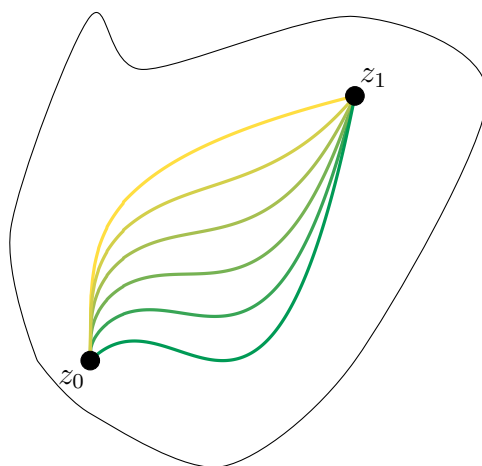
is a homotopy.

Now let  $\gamma_1 : [0, 1] \rightarrow G$  and  $\gamma_2 : [0, 1] \rightarrow G$  be two continuous curves with the same endpoints, i.e. from  $z_0$  to  $z_1$ . Thus a **homotopy with fixed endpoints**  $H : [0, 1] \times [0, 1] \rightarrow G$  can be defined as

- $H(0, t) = \gamma_1(t)$  for all  $0 \leq t \leq 1$ .
- $H(1, t) = \gamma_2(t)$  for all  $0 \leq t \leq 1$ .
- $H(s, 0) = z_0$  for all  $0 \leq s \leq 1$ .
- $H(s, 1) = z_1$  for all  $0 \leq s \leq 1$ .

For example, consider  $\gamma_1(t) = t + i\sqrt[4]{t}$  and  $\gamma_2(t) = t + i(4t^3 - 4t^2 + t)$ , we have

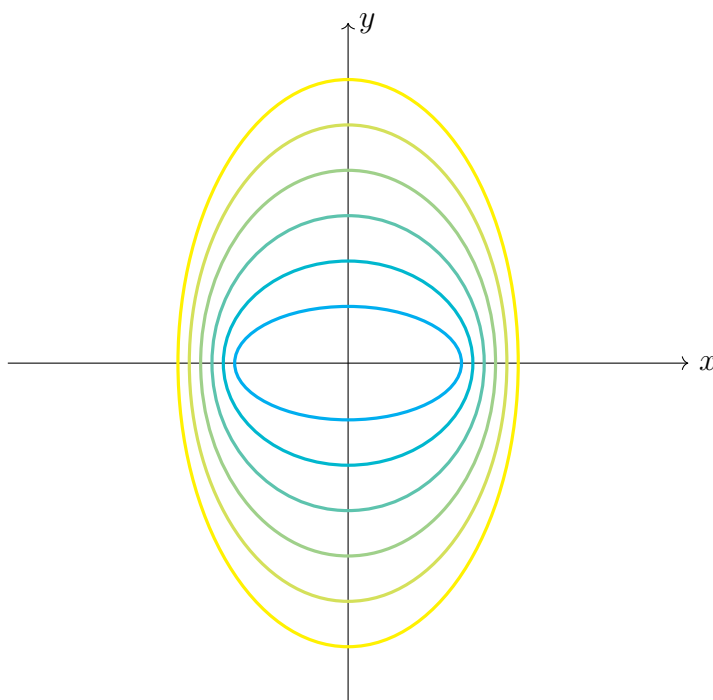
$$H(s, t) = (1-s)(t + i\sqrt[4]{t}) + s(t + i(4t^3 - 4t^2 + t))$$



Moreover, if  $z_0 = z_1$ , they are **homotopic as closed curves**. For instance, consider the curves  $\gamma_1(t) = 2 \cos t + i \sin t$  and  $\gamma_2(t) = 3 \cos t + 5i \sin t$ . Then the function

$$H(s, t) = (2 + s) \cos t + (1 + 4s)i \sin t$$

is a homotopy.



- A connected set  $G$  is called **simply connected** if every closed curve  $\gamma$  in  $G$  is homotopic as a closed curve to some constant curve in  $G$ .
- A set  $G$  is called **convex** if it contains the straight line segment between every pair of points in  $G$ .

A convex region is simply connected.

- A homotopy  $H : [0, 1] \times [0, 1] \rightarrow G$  is **smooth** if the **intermediate curves**  $\gamma_s(t)$  for each  $s$  and the **cross curves**  $\lambda_t(s)$  for each  $t$  are piecewise  $C^1$  curves.

- **Deformation Theorem:** Suppose that  $f$  is an analytic function on **open set**  $G$  and  $\gamma_1$  and  $\gamma_2$  are piecewise  $C^1$  curves in  $G$ , then

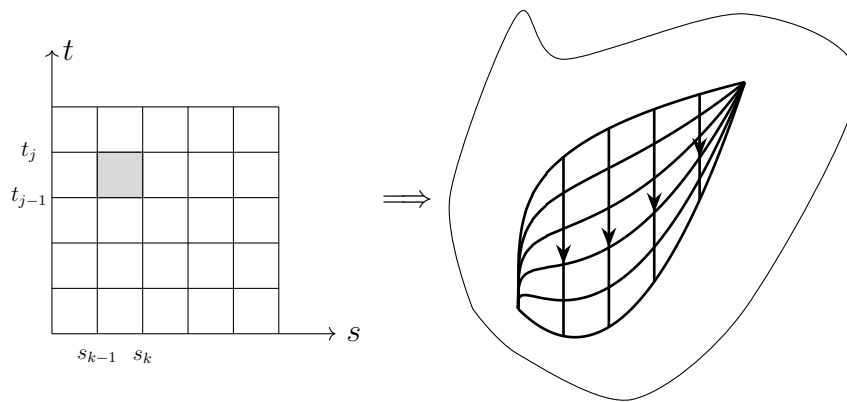
(a) If  $\gamma_1$  and  $\gamma_2$  are paths with fixed endpoints in  $G$ , then

$$\int_{\gamma_1} f = \int_{\gamma_2} f$$

(b) If  $\gamma_1$  and  $\gamma_2$  are homotopic as closed curves in  $G$ , then

$$\int_{\gamma_1} f = \int_{\gamma_2} f$$

**Sketch of Proof:** First we will prove (a). partition the square into  $n^2$  regions with  $0 = s_0 < s_1 < s_2 < \cdots < s_n = 1$  and  $0 = t_0 < t_1 < t_2 < \cdots < t_n = 1$



Now since  $\mathbb{C} \setminus G$  is closed. On the other hand, the image of  $H$  on the interval  $[s_k, s_{k+1}] \times [t_j, t_{j+1}]$  is compact on  $G$ . By the Distance Lemma, this image stays some positive distance from  $\mathbb{C} \setminus G$ . In other words. whenever  $z \in G$  we have  $|H(s, t) - z| < \rho$  for some positive value  $\rho$ .

But we know that a continuous function on a compact set is uniformly continuous, therefore there exists a number  $\delta$  such that whenever the distance between  $(s, t)$  and some point  $(s', t')$  is less than  $\delta$  we have  $|H(s, t) - H(s', t')| < \rho$ .

Consider partitioning the intervals  $s$  and  $t$  into  $n$  equal parts. Choose  $n$  large so that  $\delta > \sqrt{2}/n$ , the diagonal of the square. Thus the subsquare  $R_{kj}$  is mapped to some region in  $G$ , which is in some disk  $D(H(s_k, t_j))$  in  $G$ . Let the closed curve  $\Gamma_{kj}$ .

Therefore, by summing up all curves. we have

$$\sum_{j=0}^{n-1} \sum_{i=0}^{n-1} \int_{\Gamma_{kj}} f = \int_{\lambda_0} f + \int_{\gamma_2} f - \int_{\lambda_1} f - \int_{\gamma_1} f = 0$$

By the Cauchy's Theorem for a disk, the sum of the integrals are 0.

(a) If  $\gamma_1$  and  $\gamma_2$  have fixed endpoints, then  $\int_{\lambda_0} f = \int_{\lambda_1} f = 0$  since they are constant curves.



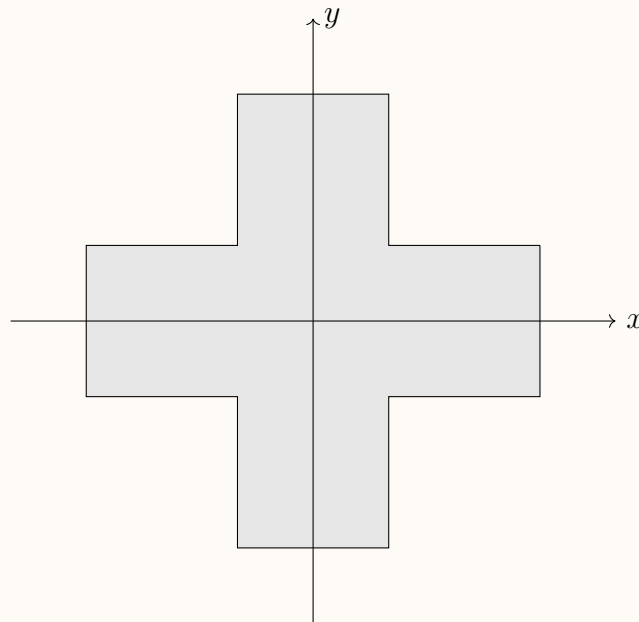
(b) If  $\gamma_1$  and  $\gamma_2$  are closed curves, we have  $\int_{\gamma_0} f = \int_{\gamma_1} f$ . In either case, we have

$$\int_{\gamma_1} f = \int_{\gamma_2} f$$

### Problem

A region  $A$  is called **star-shaped with respect to  $z_0$**  if it contains the line segment between each of its points and  $z_0$ , that is, if  $z \in A$  and  $0 \leq s \leq 1$  imply that  $sz_0 + (1-s)z \in A$ . The region is called *star-shaped* if there is at least one such point in  $A$ .

- (a) Show that a star-shaped set is simply connected.
- (b) Show that a set  $A$  is convex if and only if it is star-shaped with respect to each of its points.
- (c) Let  $G$  be the region built as a union of two rectangular regions  $G = \{z \text{ such that } |\operatorname{Re}(z)| < 1 \text{ and } |\operatorname{Im}(z)| < 3\} \cup \{z \text{ such that } |\operatorname{Re}(z)| < 3 \text{ and } |\operatorname{Im}(z)| < 1\}$ . Show that  $G$  is star-shaped.



- (a) Let  $z_0$  be such point that makes  $A$  star-shaped, as defined. I claim that every closed curve in  $A$  is homotopic to this point.

By the definition, let  $C$  be a closed curve, then for every  $z \in C$ , and for every  $s \in [0, 1]$ , we have

$$sz_0 + (1-s)z \in A$$

Traversing all  $z \in C$ , we have

$$sz_0 + (1-s)C \in A$$

which is a homotopy in  $A$ .

(b) ( $\Rightarrow$ ) If  $A$  is convex, then by definition every  $z_0, z_1 \in A$ , we have  $sz_0 + (1-s)z_1 \in A$  for all  $s \in [0, 1]$ . Fix  $z_0$  so that for every  $z \in A$ ,  $sz_0 + (1-s)z \in A$  as desired.

( $\Leftarrow$ ) If  $A$  is star shaped with respect to each of its points, then for every point  $z_0 \in A$ , and for every point  $z \in A$ ,  $s \in [0, 1]$ , we have  $sz_0 + (1-s)z \in A$ .

(c) Now, we want to show that the region  $G$  defined is star-shaped. Consider the origin  $O$ , I claim that every point  $z \in G$ , and for every  $s \in [0, 1]$ , the point  $sz$  is in  $G$ .

Let  $z = x + iy$ . By symmetry, consider the region  $0 \leq x < 3$  and  $0 \leq y < 1$ . Since we have  $sx \leq x$  and  $sy \leq y$ , therefore  $0 \leq sx < 3$  and  $0 \leq sy < 1$ , so the point  $sz$  is indeed in  $G$ .

### Problem

Evaluate the following:

(a)  $\int_{|z|=\frac{1}{2}} \frac{dz}{(1-z)^3}$

(b)  $\int_{|z-1|=\frac{1}{2}} \frac{dz}{(1-z)^3}$

(c)  $\int_{|z+1|=\frac{1}{2}} \frac{dz}{(1-z)^3}$

(a) Note that the function  $1/(1-z)^3$  is analytic everywhere except at  $z = 1$ , thence

$$\int_{|z|=\frac{1}{2}} \frac{dz}{(1-z)^3} = 0$$

(b) This is the case when  $z = 1$  is in the curve, we have

$$\int_{|z-1|=\frac{1}{2}} \frac{dz}{(1-z)^3} = \int_0^{2\pi} \frac{1}{\left(1 - \left(\frac{1}{2}e^{i\theta} + 1\right)\right)^3} \cdot \frac{i}{2}e^{i\theta} dz = 0$$

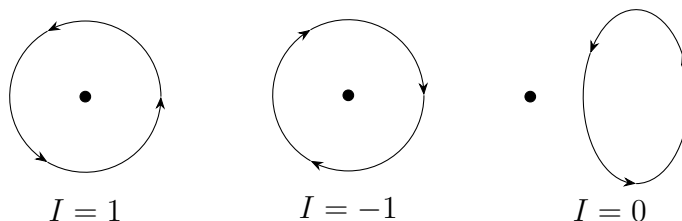
(c) Similarly, we have

$$\int_{|z+1|=\frac{1}{2}} \frac{dz}{(1-z)^3} = 0$$

## 2.4 Caucheeee's Integral Formula

- **Winding Number:** The winding number of a curve  $\gamma$  (or known as the **index**) is defined by

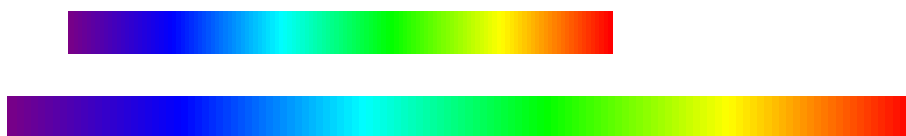
$$I(\gamma; z_0) = \frac{1}{2\pi i} \int_{\gamma} \frac{dz}{z - z_0}$$



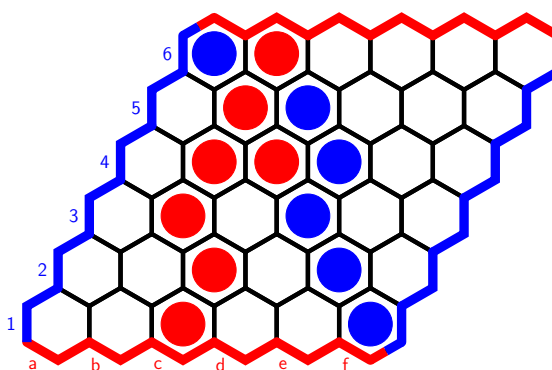
- The **Jordan Curve Theorem** states that a closed curve divides the plane **uniquely** into three regions – the curve itself, the interior (bounded) and the exterior.

An elegant proof is dedicated to the **Brouwer Fixed Point Theorem**, stating that for every continuous function  $f$ , mapping a **compact convex set** to itself, then there exists some  $c$  such that  $f(c) = c$ , a **fixed point** of  $f$ .

A visualization of this theorem is to stretch an elastic band long enough such that it covers the original band entirely. Then there exists a point on the band for which its position (coordinates) does not change.



This turns out that, proving that a game of Hex cannot end in a draw (**Hex Theorem**) is related to the Fixed Point Theorem.



- Now we return to the winding number. Given two homotopic curves  $\gamma$  and  $\tilde{\gamma}$ , we have

$$I(\gamma; z_0) = I(\tilde{\gamma}; z_0)$$

- We would like to show that the integral

$$\frac{1}{2\pi i} \int_{\gamma} \frac{dz}{z - z_0}$$

is an integer when  $\gamma$  is a closed curve.

**Sketch of Proof:** Consider the function

$$g(t) = \int_a^t \frac{\gamma'(s)}{\gamma(s) - z_0} ds$$

so that  $g(b) = I$ .

Thus we have

$$\begin{aligned} g'(t) &= \frac{\gamma'(t)}{\gamma(t) - z_0} \\ \gamma'(t) - g'(t)\gamma(t) &= -z_0 g'(t) \\ \frac{d}{dt} (e^{-g(t)} \gamma(t)) &= \frac{d}{dt} z_0 e^{-g(t)} \\ e^{-g(t)} \gamma(t) &= z_0 e^{-g(t)} + \gamma(a) - z_0 \end{aligned}$$

By the fact that  $\gamma(b) = \gamma(a)$ , letting  $t = b$ , gives

$$e^{-g(b)} = 1$$

which implies  $g(b) = 2n\pi i$ , the result then follows.

- **Cauchy's Integral Formula:** Let  $f$  be a holomorphic function on  $A$  and let  $\gamma$  be a closed curve homotopic to some point in  $A$ . Let  $z_0$  be a point not on  $\gamma$ , then

$$f(z_0)I(\gamma; z_0) = \frac{1}{2\pi i} \oint_{\gamma} \frac{f(z)}{z - z_0} dz$$

**Sketch of Proof:** Consider the function

$$g(z) = \begin{cases} \frac{f(z) - f(z_0)}{z - z_0} & \text{if } z \neq z_0 \\ f'(z_0) & \text{if } z = z_0 \end{cases}$$

Therefore  $g$  is continuous at  $z_0$  and holomorphic except perhaps at  $z_0$ . Thus by using Cauchy's theorem,

$$\begin{aligned} 0 &= \int_{\gamma} g(z) dz \\ &= \int_{\gamma} \frac{f(z)}{z - z_0} dz - f(z_0) \int_{\gamma} \frac{1}{z - z_0} dz \\ &= \int_{\gamma} \frac{f(z)}{z - z_0} dz - 2\pi i f(z_0) I(\gamma; z_0) \end{aligned}$$

- **Cauchy-Type Integrals:** Rewrite Cauchy's integral as

$$f(z)I(\gamma; z) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(\zeta)}{\zeta - z} d\zeta = G(z)$$

Then by differentiating both sides with respect to  $z$ ,

$$G^{(k)}(z) = \frac{k!}{2\pi i} \int_{\gamma} \frac{f(\zeta)}{(\zeta - z)^{k+1}} d\zeta$$

Since  $I(\gamma; z)$  is constant except when  $z$  crosses the curve (which is not the case here), thus

$$f^{(k)}(z_0)I(\gamma; z_0) = \frac{k!}{2\pi i} \int_{\gamma} \frac{f(\zeta)}{(\zeta - z_0)^{k+1}} d\zeta$$

### Example

Now we want to witness how powerful the Cauchy Integral is. Consider the integral

$$\int_0^{10\pi} \frac{4e^{4i\theta} + 3e^{2i\theta}}{(4e^{2i\theta} - 1)^2} d\theta$$

This can be interpreted as the parametrization  $z = 2e^{i\theta}$  and integrated along the circle with radius 2 and centered at the origin 5 times. We can rewrite the integral as

$$-\frac{i}{8} \int_{\gamma} \frac{z^3 + 3z}{(z^2 - 1)^2} dz = -\frac{i}{8} \left( \int_{\gamma} \frac{z - 1}{(z + 1)^2} dz + \int_{\gamma} \frac{z + 1}{(z - 1)^2} dz \right)$$

By using the formulas, we have

$$(1)(5) = \frac{1!}{2\pi i} \int_{\gamma} \frac{z - 1}{(z + 1)^2} dz$$

$$(1)(5) = \frac{1!}{2\pi i} \int_{\gamma} \frac{z + 1}{(z - 1)^2} dz$$

Substituting back into our integral we have

$$-\frac{i}{8}(10\pi i + 10\pi i) = 2.5\pi$$

THIS IS QUICKER THAN LIGHT!!!

- **Cauchy's Inequality:** Let  $f$  be holomorphic on  $A$  and  $\gamma$  be a circle with radius  $R$  and center  $z_0$ . Suppose that both  $\gamma$  and the disk  $|z - z_0| < R$  also lies in  $A$ . If for all  $z$  on  $\gamma$  we have  $|f(z)| \leq M$ , then for natural numbers  $k$ , we have

$$|f^{(k)}(z_0)| \leq \frac{k!}{R^k} M$$

**Sketch of Proof:**

$$|f^{(k)}(z_0)| = \frac{k!}{2\pi} \left| \int_{\gamma} \frac{f(\zeta)}{(\zeta - z_0)^{k+1}} d\zeta \right| \leq \frac{k!}{2\pi} \cdot \frac{M}{R^{k+1}} \cdot l(\gamma) = \frac{k!}{R^k} M$$

- **Liouville's Theorem:** If  $f$  is analytic everywhere and there is a constant  $M$  such that for all  $z \in \mathbb{C}$ ,  $|f(z)| \leq M$ , then  $f$  is constant.

This is the special case of Cauchy's inequality with  $k = 1$  and  $R \rightarrow \infty$ .

- **Morera's Theorem:** Let  $f$  be continuous in  $A$ , and for any closed curve  $\gamma$  in  $A$ ,  $\int_{\gamma} f = 0$ , then  $f$  is holomorphic in  $A$ , and there exists some holomorphic function  $F$  in  $A$  such that  $f = F'$ .

### Problem

Let  $f$  be analytic on a region  $A$  and let  $\gamma$  be a closed curve in  $A$ . For any  $z_0 \in A$  not on  $\gamma$ , show that

$$\int_{\gamma} \frac{f'(\zeta)}{\zeta - z_0} d\zeta = \int_{\gamma} \frac{f(\zeta)}{(\zeta - z_0)^2} d\zeta$$

By

$$f'(z_0) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(\zeta)}{(\zeta - z_0)^2} d\zeta$$

$$f'(z_0) = \frac{1}{2\pi i} \int_{\gamma} \frac{f'(\zeta)}{\zeta - z_0} d\zeta$$

Equating  $2\pi i f'(z_0)$  gives the desired result.

**Remark.** In general, for positive integers  $k \leq m$ , the integral

$$k! \int_{\gamma} \frac{f^{(m-k)}(\zeta)}{(\zeta - z_0)^{k+1}} d\zeta$$

is constant and equals  $2\pi i \cdot I(\gamma; z_0) f^{(m)}(z_0)$  for all  $k$ .

### Problem

Prove that

$$\int_0^{\pi} e^{\cos \theta} \cos(\sin \theta) d\theta = \pi$$

by considering

$$\int_{\gamma} \frac{e^z}{z} dz$$

where  $\gamma$  is the unit circle.

We have

$$e^0 = \frac{1}{2\pi i} \int_{\gamma} \frac{e^z}{z} dz$$

By parametrization,  $z = \cos \theta + i \sin \theta$ , so  $e^z = e^{\cos \theta} (\cos \sin \theta + i \sin \sin \theta)$  and

$dz = izd\theta$ . Thus the integral becomes

$$\frac{1}{2\pi i} \int_{\gamma} \frac{e^z}{z} dz = \frac{1}{2\pi i} \int_0^{2\pi} \frac{e^{\cos \theta} (\cos \sin \theta + i \sin \sin \theta)}{z} iz d\theta$$

So we have

$$2\pi = \int_0^{2\pi} e^{\cos \theta} \cos \sin \theta d\theta + i \int_0^{2\pi} e^{\cos \theta} \sin \sin \theta d\theta$$

The former is a real value, hence the latter equals 0. It remains to show that

$$\pi = \int_{\pi}^{2\pi} e^{\cos \theta} \cos \sin \theta d\theta$$

Letting  $\theta' = 2\pi - \theta$  gives the desired result. □

### Problem

Let  $f$  be analytic inside and on the circle  $\gamma : |z - z_0| = R$ . Prove that

$$\frac{f(z_1) - f(z_2)}{z_1 - z_2} - f'(z_0) = \frac{1}{2\pi i} \int_{\gamma} \left[ \frac{1}{(z - z_1)(z - z_2)} - \frac{1}{(z - z_0)^2} \right] f(z) dz$$

for  $z_1, z_2$  inside  $\gamma$ .

By the standard Cauchy's Integral,

$$-f'(z_0) = -\frac{1}{2\pi i} \int_{\gamma} \frac{f(z)}{(z - z_0)^2} dz$$

On the other hand,

$$\begin{aligned} \frac{1}{2\pi i} \int_{\gamma} \frac{1}{(z - z_1)(z - z_2)} dz &= \frac{1}{2\pi i} \int_{\gamma} \left( \frac{1}{z - z_1} - \frac{1}{z - z_2} \right) \frac{f(z)}{z_1 - z_2} dz \\ &= \frac{f(z_1) - f(z_2)}{z_1 - z_2} \end{aligned}$$

as desired.

**Problem**

Show that:

1.

$$3 \sum_{k=1}^{\infty} \left| \int_{|z-k|=1/2} \frac{1}{z^2(z-k)} dz \right| = \pi^3$$

2.

$$\frac{1}{2\pi} \sum_{k=0}^{\infty} \frac{1}{k! \binom{2k}{k}} \left| \int_{|z|=\pi} \frac{z^{2k}}{(z-3)^{k+1}} dz \right| = e^3$$

3.

$$-\frac{1}{2\pi i} \sum_{k=0}^{4n} k! \int_{|z|=1} \frac{\cos z}{z^{k+1}} dz = i^3$$

1. From

$$\int_{|z-k|=1/2} \frac{1}{z^2(z-k)} dz = \int_{|z-k|=1/2} \frac{1/z^2}{z-k} dz = \frac{2\pi i}{k^2}$$

We have

$$3 \sum_{k=1}^{\infty} \left| \frac{2\pi i}{k^2} \right| = \pi^3$$

2. From

$$\int_{|z|=\pi} \frac{z^{2k}}{(z-3)^{k+1}} dz = \frac{2\pi i}{k!} \left( \frac{d^k}{dz^k} z^{2k} \Big|_{z=3} \right) = \frac{2\pi i (2k)! 3^k}{k! k!} = 2\pi i \binom{2k}{k} \cdot 3^k$$

We have

$$\frac{1}{2\pi} \sum_{k=0}^{\infty} \frac{1}{k! \binom{2k}{k}} \left| 2\pi i \binom{2k}{k} \cdot 3^k \right| = \sum_{k=0}^{\infty} \frac{3^k}{k!} = e^3$$

3. From

$$\int_{|z|=1} \frac{\cos z}{z^{k+1}} dz = \frac{2\pi i}{k!} \left( \frac{d^k}{dz^k} \cos z \Big|_{z=0} \right)$$

We have

$$-\frac{1}{2\pi i} \sum_{k=0}^{4n} k! \int_{|z|=1} \frac{\cos z}{z^{k+1}} dz = - \sum_{k=0}^{4n} \frac{d^k}{dz^k} \cos z \Big|_{z=0} = -1 = i^3$$

## 2.5 Maximum Modulus Theorem & Harmonic Functions

### Maximum Modulus Theorem

- **Gauss' Mean Value Theorem:** Let  $\gamma$  be a circle around  $z_0$  with radius  $R$ , then

$$f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + Re^{i\theta}) d\theta$$



A geometric interpretation of this formula is that, the value of an analytic function at  $z_0$  equals the average of values around the circle.

By the concept of “Riemann Sums”, consider dissecting the circumference of the circle into  $n$  equal arcs. Each arc is  $2\pi/n$  radians apart. Thus we have

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n f\left(z_0 + Re^{i\frac{2k\pi}{n}}\right) = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + Re^{i\theta}) d\theta$$

- **Closure:** The closure of a set  $\text{cl}(A)$  consists  $A$  and the limit points of the convergent sequences of points in  $A$ .

Some properties of closure include:

- $A \subset \text{cl}(A)$
- $A$  is closed iff  $A = \text{cl}(A)$
- If  $A \subset C$  for some closed  $C$ , then  $\text{cl}(C)$
- $\text{cl}(A)$  is closed
- Let  $\text{limit}(A) = \{w \mid \text{there is a sequence in } A \text{ convergent to } w\}$

We have

- $A \subset \text{limit}(A)$
- $\text{cl}(A) = \text{limit}(A)$
- $\text{limit}(\text{limit}(A)) = \text{cl}(\text{cl}(A)) = \text{limit}(A) = \text{cl}(A)$

**Sketch of Proof:** We want to prove that

1.  $\text{cl}(A) = \text{limit}(A)$
2.  $\text{cl}(A) \subseteq \text{cl}(\text{cl}(A))$
3.  $\text{cl}(\text{cl}(A)) \subseteq \text{cl}(A)$

$\text{cl}(A) = \text{limit}(A)$  almost follows by definition, since  $\text{cl}(A) = A \cup \text{limit}(A)$  and  $A \subset \text{limit}(A)$ .

Since if  $X$  is closed then  $X \subseteq \text{cl}(X)$ , it follows that  $\text{cl}(A) \subseteq \text{cl}(\text{cl}(A))$ .

Now let  $z_1, z_2, z_3, \dots$  be a sequence in  $\text{limit}(A)$  so that they converge to  $w$ , we want to show that  $w \in \text{limit}(A)$ . Since each  $z_i$  is in  $\text{limit}(A)$ , there exists a sequence  $w_i$  in  $A$  so that  $\lim_{n \rightarrow \infty} |z_n - w_n| = 0$ . So  $\lim_{n \rightarrow \infty} w_n = w$ .

- **Boundary:** The boundary of a set  $\text{bd}(A)$  is defined by

$$\text{bd}(A) = \text{cl}(A) \cap \text{cl}(\mathbb{C} \setminus A)$$

On the other hand, we have

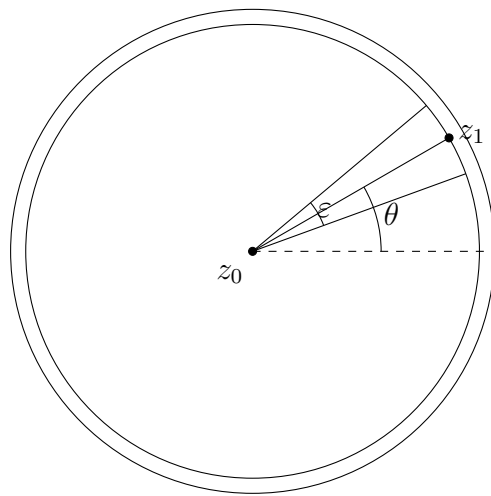
$$\text{cl}(A) = \text{bd}(A) \cup A$$

- **Local Maximum Modulus Principle:** Let  $f$  be holomorphic on  $A$  and  $|f|$  has relative maximum at  $z_0 \in A$ , then  $f$  is constant in some neighbourhood of  $z_0$ .

**Sketch of Proof:** Assume that on some disk  $D_0 = D(z_0; r_0)$ ,  $|f(z_0 + r_0 e^{i\theta})| \leq |f(z_0)|$  for all  $\theta$ . If there does not exist a strict inequality then we are done. Assume otherwise, there exists some  $z_1$  with  $|f(z_1)| < |f(z_0)|$ . By assumption we have  $z_1 = z_0 + r e^{i\theta}$  with  $r < r_0$ .

Since  $f$  is continuous, there exists  $\varepsilon > 0$  and  $\delta > 0$  such that whenever  $|\theta - a| < \varepsilon$  we have

$$|f(z_0 + r e^{ia})| < |f(z_0)| - \delta$$



Thus we have

$$\begin{aligned} |f(z_0)| &= \left| \frac{1}{2\pi} \int_{-\pi}^{\pi} f(z_0 + r e^{ia}) da \right| \\ &\leq \left| \frac{1}{2\pi} \int_{-\pi}^{-\varepsilon} f(z_0 + r e^{ia}) da \right| + \left| \frac{1}{2\pi} \int_{-\varepsilon}^{\varepsilon} f(z_0 + r e^{ia}) da \right| + \left| \frac{1}{2\pi} \int_{\varepsilon}^{\pi} f(z_0 + r e^{ia}) da \right| \\ &\leq \frac{1}{2\pi} \int_{-\pi}^{-\varepsilon} |f(z_0)| da + \frac{1}{2\pi} \int_{-\varepsilon}^{\varepsilon} (|f(z_0)| - \delta) da + \frac{1}{2\pi} \int_{\varepsilon}^{\pi} |f(z_0)| da \\ &= |f(z_0)| - \frac{\varepsilon \delta}{\pi} \end{aligned}$$

Hence a contradiction. This means that no point in  $D_0$  have modulus strictly less than  $|f(z_0)|$ , thus  $f$  is constant on some disk  $D$ .

- **Global Maximum Modulus Principle:** Let  $A \subseteq \mathbb{C}$  be an **open connected** set and suppose  $f : \text{cl}(A) \rightarrow \mathbb{C}$  is analytic on  $A$  and continuous on  $\text{cl}(A)$ , then  $|f|$  has a finite maximum value on  $\text{cl}(A)$ , attained at some point on  $\text{bd}(A)$ . If this value is attained in the interior of  $A$ , then  $f$  is constant on  $\text{cl}(A)$ .

**Sketch of Proof:** According to the **Extreme Value Theorem**, there exists a maximum value for  $|f|$ , say  $M$ . Assume there exists some  $a \in A$  such

that  $|f(a)| = M$ . Consider the sets

$$A_1 = \{z \in A \mid f(z) = f(a)\}, \quad A_2 = A \setminus \text{cl}(A_1)$$

Assume that  $z \in A$  but  $z \notin A_2$ , then  $z \in \text{cl}(A_1)$ . We can choose some sequence in  $A_1$  so it converges to  $z$ . Since each point  $f(w)$  in the sequence equals  $f(a)$ ,  $f(z) = f(a)$  so  $z \in A_1$ . This shows that  $A_1 \cup A_2 = A$  and  $A_1 \cap A_2 = \emptyset$ .

Since  $A_2 = A \setminus \text{cl}(A_1)$ ,  $A_2$  is open. Since  $A$  is open, we can find some disk in  $A$  with its radius small enough so that it is in  $A_1$  as well (This is the result of the local maximum modulus principle). Thus  $A_1$  is open. But this means that  $A_1$  and  $A_2$  disconnect  $A$ .

We know that  $A_1$  is nonempty, so  $A_2 = \emptyset$  and  $A = A_1$  as desired.

- **Schwarz Lemma:** Let  $f$  be holomorphic on the open unit disk  $A : (0; 1)$  with  $f(0) = 0$  and  $|f(z)| \leq 1$  for all  $z \in A$ , then
  - (a)  $|f'(0)| \leq 1$  and  $|f(z)| \leq |z|$  for all  $z \in A$ .
  - (b) If  $|f'(0)| = 1$  or if there is a point  $z_0 \neq 0$  such that  $|f(z_0)| = |z_0|$ , then there is a constant  $c$  with  $|c| = 1$  and  $f(z) = cz$  for all  $z \in A$ .

**Sketch of Proof:** Consider the function  $g(z) = f(z)/z$ , so  $g(0) = f'(0)$ . Define

$$A_r = \{z \mid 0 < r < 1, |z| \leq r\}$$

Then  $g$  is holomorphic on  $A_r$ . On  $|z| = r$ , we have

$$|g(z)| \leq |f(z)/z| \leq 1/r$$

Thus by the Maximum Modulus Principle, we have  $|g(z)| \leq 1/r$  for all  $|z| \leq r$ . We now have

$$|f(z)| \leq |z|/r$$

Fix  $z$ , let  $r \rightarrow 1$ , we have  $|f(z)| \leq |z|$  as desired. Since  $|g(z)| \leq 1$  so  $|f'(0)| \leq 1$ .

Assume that there exists some point  $z_0$  so that  $|f(z_0)| = |z_0|$ . If  $z_0 = 0$  then  $|f'(0)| = |g(0)| = 1$  or in other words,  $|g(z)|$  is maximized. If  $z_0 \neq 0$  we still get  $|g(z_0)| = 1$ .

Since this value  $|g(z_0)|$  is independent of  $r$ , so by the Maximum Modulus Principle  $g$  is constant on  $A$ . We can easily check that the maximum value is  $c$  and that  $|c| = 1$ .

- **Phragmén–Lindelöf Principle:** Suppose that we are given a function  $f$  and an **unbounded region**  $S$ , we want to show that  $|f| \leq M$  is bounded on  $S$ .

The strategy is to introduce some multiplicative factor  $h_\varepsilon$  and a bounded subregion  $S_{\text{bdd}} \subset S$  with  $\varepsilon > 0$  such that

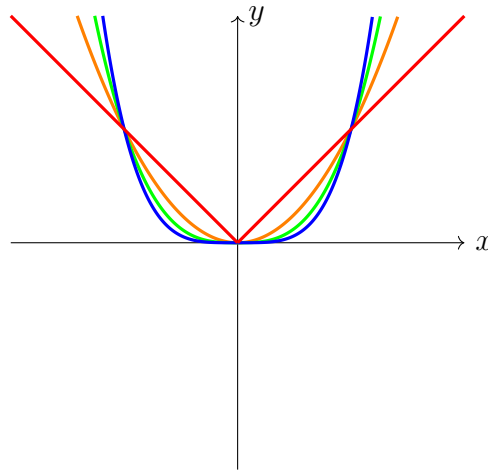
1.  $|fh_\varepsilon| \leq M$  is holomorphic and bounded on the boundary  $\text{bd}(S_{\text{bdd}})$ .
2. We can deduce  $|fh_\varepsilon| \leq M$  for  $z \in S \setminus \text{cl}(S_{\text{bdd}})$ .

And that we can extend our conclusion to all  $z \in S$ .

### 2.5.1 Harmonic Functions

- **Smoothness:** A function is said to have class of at least  $C^k$  if the  $k$ 'th derivative of the function exists.

For instance, the function  $|x|^{k+1}$  is  $C^k$ . The graphs below are when  $k = 0, 1, 2, 3$ .



- **Antiderivative Theorem:** Let  $f$  be defined and holomorphic on a simply connected region  $A$ , there is a holomorphic function  $F$ , the **antiderivative**, defined on  $A$  that is unique up to an **additive constant** and that  $F'(z) = f(z)$  for all  $z \in A$ .
- Let  $A \subseteq \mathbb{C}$  be a region and let  $u$  be a twice continuously differentiable harmonic function on  $A$ , then  $u$  is  $C^\infty$ . In a neighborhood of each point  $z_0 \in A$ ,  $u$  is the real part of some holomorphic function.

**Sketch of Proof:** Consider the function

$$g = U + iV = \frac{\partial u}{\partial x} - i \frac{\partial u}{\partial y}$$

Then obviously

$$\frac{\partial U}{\partial y} = -\frac{\partial V}{\partial x}$$

and

$$\frac{\partial U}{\partial x} = \frac{\partial V}{\partial y}$$

since  $\nabla^2 u = 0$ . Therefore  $g$  is analytic since it satisfies the Cauchy-Riemann equations.

Since  $A$  is simply connected there exists a holomorphic function  $f$  on  $A$

so that  $f' = g$  by the Antiderivative Theorem. Let  $f = \tilde{u} + i\tilde{v}$  then

$$f' = \frac{\partial \tilde{u}}{\partial x} - i \frac{\partial \tilde{u}}{\partial y} = \frac{\partial u}{\partial x} - i \frac{\partial u}{\partial y}$$

Thus  $\tilde{u}$  and  $u$  differ by some constant, which can be made such that  $\tilde{u} = u$ .

Since  $f$  is  $C^\infty$ , it follows that  $u$  is also  $C^\infty$  on  $A$ .

- **Mean Value Property for Harmonic Functions:** Let  $u$  be harmonic on a region  $A$  containing a circle with radius  $r$  around  $z_0 = x_0 + iy_0$ , and its interior, then

$$u(x_0, y_0) = \frac{1}{2\pi} \int_0^{2\pi} u(z_0 + re^{i\theta}) d\theta$$

**Sketch of Proof:** We know that there is a function  $f$  defined on a region containing this circle. Taking the real part and the imaginary part of the function gives

$$f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{i\theta}) d\theta$$

- **Dirichlet Problem:** Let  $A$  be an open bounded region and  $u_0$  be a given continuous function on  $\text{bd}(A)$ . Find a real-valued function  $u$  on  $\text{cl}(A)$  that is continuous on  $\text{cl}(A)$  and harmonic on  $A$  and equals  $u_0$  on  $\text{bd}(A)$ .

To solve the Dirichlet problem, we may want to introduce the Maximum Principle for Harmonic Functions.

- **Local Maximum Principle for Harmonic Functions:** Let  $u$  be harmonic on  $A$ . Suppose  $u$  has a relative maximum at  $z_0 \in A$ , then  $u$  is constant in some neighborhood of  $z_0$ .

**Sketch of Proof:** Since  $u$  is harmonic, there exists some  $f$  such that  $u = \text{Re}(f)$ . Consider the function

$$e^{f(z)}$$

We have  $|e^{f(z)}| = e^{u(z)}$ . Since the function  $e^x$  is strictly increasing, thus  $e^{u(z)}$  attains maximum value. Thus by the Maximum Modulus principle  $e^{f(z)}$  is constant in some neighborhood of  $z_0$ , so  $u$  is also constant.

- **Global Maximum Principle for Harmonic Functions:** Given  $A \subset \mathbb{C}$  an **open connected bounded** set. Let  $u : \text{cl}(A) \rightarrow \mathbb{R}$  be continuous and harmonic on  $A$  and assume  $M$  the maximum value of  $u$  on  $\text{bd}(A)$ , then

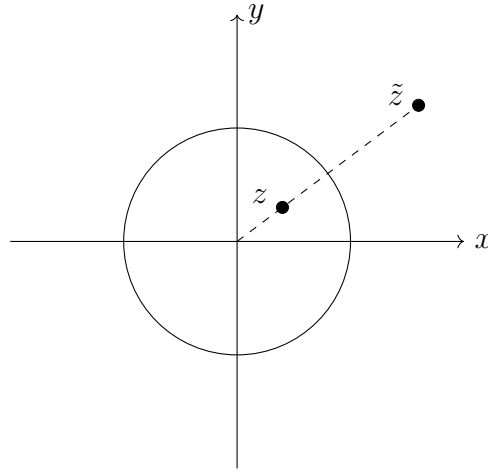
(a)  $u(x, y) \leq M$  for all  $(x, y) \in A$ .

(b) if  $u(x, y) = M$  for some  $(x, y) \in A$ , then  $u$  is constant on  $A$ .

- **Poisson's Formula:** Let  $u$  be defined and continuous on the disk  $|z| \leq r$  and is harmonic on the open disk  $D(0, r) = \{z \mid |z| < r\}$ . For  $\rho < r$ , we have

$$u(\rho e^{i\theta}) = \frac{r^2 - \rho^2}{2\pi} \int_0^{2\pi} \frac{u(re^{i\theta})}{r^2 - 2r\rho \cos(\phi - \theta) + \rho^2} d\theta$$

**Sketch of Proof:** Consider some  $\zeta$  such that  $0 < |\zeta| = s < r$ . Define  $\tilde{z}$  satisfying  $z\tilde{z} = s^2$ , the image of inversion of  $z$  with respect to  $|z| = s$ .



Thus we have

$$f(z) = \frac{1}{2\pi i} \int_{|z|=s} \frac{f(\zeta)}{\zeta - z} d\zeta$$

and

$$\int_{|z|=s} \frac{f(\zeta)}{\zeta - \tilde{z}} d\zeta = 0$$

Subtracting gives

$$f(z) = \frac{1}{2\pi i} \int_{|z|=s} f(\zeta) \left( \frac{1}{\zeta - z} - \frac{1}{\zeta - \tilde{z}} \right) d\zeta$$

Notice that  $\zeta\bar{\zeta} = s^2$  and  $\tilde{z} = s^2/\bar{z}$ , thus we have

$$\begin{aligned} \frac{1}{\zeta - z} - \frac{1}{\zeta - \tilde{z}} &= \frac{1}{\zeta - z} - \frac{\bar{z}}{\zeta\bar{z} - s^2} \\ &= \frac{1}{\zeta - z} - \frac{\bar{z}}{\zeta\bar{z} - \zeta\bar{\zeta}} \\ &= \frac{1}{\zeta - z} - \frac{\bar{z}}{\zeta(\bar{z} - \bar{\zeta})} \\ &= \frac{1}{\zeta - z} - \frac{\bar{z}}{\zeta(\overline{z - \zeta})} \\ &= \frac{|\zeta|^2 - |z|^2}{\zeta|\zeta - z|^2} \end{aligned}$$

Expressing  $\zeta = se^{i\theta}$  and  $z = \rho e^{i\phi}$  gives

$$f(\rho e^{i\phi}) = \frac{1}{2\pi} \int_0^{2\pi} \frac{f(se^{i\theta})(s^2 - \rho^2)}{|se^{i\theta} - \rho e^{i\phi}|^2} d\theta = \frac{1}{2\pi} \int_0^{2\pi} \frac{f(se^{i\theta})(s^2 - \rho^2)}{s^2 + \rho^2 - 2s\rho \cos(\phi - \theta)} d\theta$$

Fix  $z$ , since for  $s > \rho$ , the function  $\frac{f(se^{i\theta})(s^2 - \rho^2)}{s^2 + \rho^2 - 2s\rho \cos(\phi - \theta)}$  is continuous for

all  $s > p$  and  $0 \leq \theta \leq 2\pi$ , hence letting  $s \rightarrow r$  and taking the real part of  $f$  we have

$$u(\rho e^{i\theta}) = \frac{r^2 - \rho^2}{2\pi} \int_0^{2\pi} \frac{u(re^{i\theta})}{r^2 - 2r\rho \cos(\phi - \theta) + \rho^2} d\theta$$

- Poisson's formula allows us to construct a solution to the Dirichlet problem.

### Problem

Find the maximum of:

1.  $|e^z|$  on  $|z| \leq 1$
2.  $|\cos z|$  on  $[0, 2\pi] \times [0, 2\pi]$

1. We want to find the maximum of  $|e^z|$  on the circle  $|z| = 1$ . Let  $z = \cos \theta + i \sin \theta$ , then

$$|e^z| = |e^{\cos \theta}| \leq e$$

Hence the maximum value is  $e$ .

2. Consider  $z$  on the boundary  $[0, 2\pi] \times [0, 2\pi]$ . Letting  $z = x + iy$  gives

$$|\cos z| = \frac{1}{2} |e^z + e^{-z}| = \frac{1}{2} \sqrt{(e^{2y} + e^{-2y}) + 2 \cos 2x}$$

Hence  $|\cos z|$  attains maximum value on  $z = 2\pi i$ , which gives  $|\cos z| = \frac{e^{2\pi} + e^{-2\pi}}{2}$ .

### Problem

- (a) Let the mapping  $T$  be defined by  $T(z) = R(z - z_0)/(R^2 - \bar{z}_0 z)$ . Show that for  $|z_0| < R$ ,  $T$  takes the open disk of radius  $R$  one to one onto the disk of radius 1 and takes  $z_0$  to the origin.
- (b) Suppose that  $f$  is analytic on the open disk  $|z| < R$  and that  $|f(z)| < M$  for  $|z| < R$ . Suppose also that  $f(z_0) = w_0$ . Show that

$$\left| \frac{M[f(z) - w_0]}{M^2 - \bar{w}_0 f(z)} \right| \leq \left| \frac{R(z - z_0)}{R^2 - \bar{z}_0 z} \right|.$$

- (a) Consider all  $z$  such that  $|z| = R$ . Let  $z = Re^{i\phi}$  and  $z_0 = re^{i\theta}$ . We have

$$\begin{aligned} |T(z)| &= \left| \frac{R(z - z_0)}{R^2 - \bar{z}_0 z} \right| \\ &= \frac{R|Re^{i\phi} - re^{i\theta}|}{|R^2 - Rre^{i(\phi-\theta)}|} \end{aligned}$$

$$\begin{aligned}
&= \frac{|Re^{i\phi} - re^{i\theta}|}{|Re^{i\theta} - re^{i\phi}|} \\
&= \sqrt{\frac{(R \cos \phi - r \cos \theta)^2 + (R \sin \phi - r \sin \theta)^2}{(R \cos \theta - r \cos \phi)^2 + (R \sin \theta - r \sin \phi)^2}} \\
&= 1
\end{aligned}$$

Hence for every point in  $|z| < R$  we have  $|T(z)| < 1$  by the Maximum Modulus Principle. Obviously  $T(z_0) = 0$ . It remains to show that  $T(z)$  is injective. Assume that  $T(z_1) = T(z_2)$ , then

$$\begin{aligned}
\frac{R(z_1 - z_0)}{R^2 - \bar{z}_0 z_1} &= \frac{R(z_2 - z_0)}{R^2 - \bar{z}_0 z_2} \\
(z_1 - z_0)(R^2 - \bar{z}_0 z_2) &= (z_2 - z_0)(R^2 - \bar{z}_0 z_1) \\
(R^2 - |z_0|^2)(z_1 - z_2) &= 0
\end{aligned}$$

But we know that  $|z_0| < R$  so  $z_1 = z_2$  as desired.

- (b) The strategy here is to consider two mappings  $T : D_R(0; R) \rightarrow D_1(0; 1)$  and  $F : D_M(0; M) \rightarrow D_1(0; 1)$  defined by

$$T(z) = \frac{R(z - z_0)}{R^2 - \bar{z}_0 z}, \quad F(z) = \frac{M(z - f(z_0))}{M^2 - \overline{f(z_0)}z}$$

We have proved that in (a),  $|T(z)| \leq 1$  and  $|F(z)| \leq 1$ . Consider the composition  $F \circ f \circ T^{-1} : D_1(0; 1) \rightarrow D_1(0; 1)$ . This way we can apply Schwarz lemma.

$$\begin{aligned}
|F(f(T^{-1}(z)))| &\leq |z| \\
|F(f(z))| &\leq |T(z)| \\
\left| \frac{M(f(z) - f(z_0))}{M^2 - \overline{f(z_0)}f(z)} \right| &\leq \left| \frac{R(z - z_0)}{R^2 - \bar{z}_0 z} \right|
\end{aligned}$$

as desired.

**Remark.** The problem (b) is a generalization of the [Schwarz-Pick Theorem](#).

### Problem

Find harmonic conjugates for each of the following functions

- (a)  $u(x, y) = \sinh x \sin y$
- (b)  $u(x, y) = \ln \sqrt{x^2 + y^2}$
- (c)  $u(x, y) = e^x \cos y$



(a) We have

$$\frac{\partial u}{\partial x} = \cosh x \sin y = \frac{\partial v}{\partial y}$$

so  $v = -\cosh x \cos y + C_1(x)$ . On the other hand,

$$\frac{\partial u}{\partial y} = \sinh x \cos y = -\frac{\partial v}{\partial x}$$

so  $v = -\cosh x \cos y + C_2(y)$ . Thus

$$v = -\cosh x \cos y + C$$

for some constant  $C$ .

(b) We have

$$\frac{\partial u}{\partial x} = \frac{x}{x^2 + y^2} = \frac{\partial v}{\partial y}$$

so  $v = \tan^{-1} \left( \frac{y}{x} \right) + C_1(x)$ . On the other hand,

$$\frac{\partial u}{\partial y} = \frac{y}{x^2 + y^2} = -\frac{\partial v}{\partial x}$$

so  $v = -\tan^{-1} \left( \frac{x}{y} \right) + C_2(y) = \tan^{-1} \left( \frac{y}{x} \right) - \frac{\pi}{2} + C_2(y)$ . Thus

$$v = \tan^{-1} \left( \frac{y}{x} \right) + C$$

for some constant  $C$ .

(c) We have

$$\frac{\partial u}{\partial x} = e^x \cos y = \frac{\partial v}{\partial y}$$

so  $v = e^x \sin y + C_1(x)$ . On the other hand,

$$\frac{\partial u}{\partial y} = -e^x \sin y = -\frac{\partial v}{\partial x}$$

so  $v = e^x \sin y + C_2(y)$ . Thus

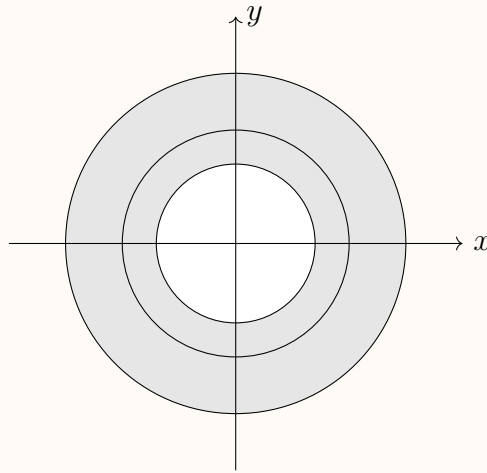
$$v = e^x \sin y + C$$

for some constant  $C$ .

**Problem**

Prove **Hadamard's Three-circle Theorem**: Let  $f$  be analytic on a region containing the set  $R$ . Let  $R = \{z \mid r_1 \leq |z| \leq r_3\}$  and assume  $0 < r_1 < r_2 < r_3$ . Let  $M_1, M_2, M_3$  be the maxima of  $|f|$  on the circles  $|z| = r_1, r_2, r_3$ , respectively. Then we have the inequality

$$M_2^{\ln(r_3/r_1)} \leq M_1^{\ln(r_3/r_2)} M_3^{\ln(r_2/r_1)}.$$



Consider the function

$$g(z) = z^\lambda f(z)$$

given  $\lambda = -\frac{\ln(M_3/M_1)}{\ln(r_3/r_1)}$ , rewriting gives  $r_1^\lambda M_1 = r_3^\lambda M_3$ .

By the Maximum Modulus principle, we have

$$|g(z)| = r_3^\lambda |f(z)| \leq r_3^\lambda M_3 = r_1^\lambda M_1$$

Since the maximum may not be obtained in the region  $R$ , we have

$$r_2^\lambda M_2 \leq r_1^\lambda M_1$$

Substituting the definition of  $\lambda$  gives the desired result.