Complex Analysis Notes

Before starting, I want to mention that the book "Basic Complex Analysis (3rd edition)" by Jerrold E. Marsden and Michael J. Hoffman. is used.

§1 How Complex Is It?

1.1 Basic Operations

- $(a+bi) \pm (c+di) = (a \pm c) + (b \pm d)i$
- $(a+bi)(c+di) = (ac-bd) + (ad+bc)i$
- $\dfrac{a+bi}{c+di} = \dfrac{ac+bd}{c^2+d^2} + \dfrac{bc-ad}{c^2+d^2}i$

Problem

Fix a complex number $z = x+iy$ and consider the linear mapping $\phi_z : \mathbb{R}^2 \to \mathbb{R}^2$ (that is, of $\mathbb{C} \to \mathbb{C}$) defined by $\phi_z(w) = z \cdot w$ (that is, multiplication by z). Prove that the matrix of ϕ_z in the standard basis $(1,0), (0,1)$ of $\mathbb{R}^2$ is given by

$$\begin{pmatrix} x & -y \\ y & x \end{pmatrix}.$$

Then show that $\phi_{z_1 z_2} = \phi_{z_1} \circ \phi_{z_2}$.

Let $w = a+ib$, then $z \cdot w = (x+iy)(a+ib) = (xa-yb) + (xb+ya)i$.

On the other hand,

$$\begin{pmatrix} x & -y \\ y & x \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} xa-yb \\ xb+ya \end{pmatrix}.$$

and we have

$$\phi_{z_1 z_2} = z_1 \cdot z_2 \cdot w = z_1 \cdot (z_2 \cdot w) = \phi_{z_1} \circ \phi_{z_2}.$$

1.2 What? There's More?

Proposition (De Moivre's Formula)
If $z = r(\cos\theta + i\sin\theta)$ then for some positive integer n,

$$z^n = r^n(\cos n\theta + i\sin n\theta).$$

Some notable properties of **complex conjugation** and **norm**.

- $z\overline{z} = |z|^2$.
- $\text{Re}\,(z) = (z+\overline{z})/2$, $\text{Im}\,(z) = (z+\overline{z})/2i$
- $|\text{Re}\,(z)| \leq |z|$, $|\text{Im}\,(z)| \leq |z|$
- Triangle Inequality: $\left|\sum_{k=1}^{n} z_k\right| \leq \sum_{k=1}^{n} |z_k|$
- Cauchy-Schwarz Inequality: $\left|\sum_{k=1}^{n} z_k w_k\right| \leq \sqrt{\sum_{k=1}^{n} |z_k|^2}\sqrt{\sum_{k=1}^{n} |w_k|^2}$

Problem
If $a, b \in \mathbb{C}$, prove the **parallelogram identity**: $|a-b|^2+|a+b|^2 = 2(|a|^2+|b|^2)$.

Let $a = p + iq$ and $b = r + is$, then

$$\begin{aligned} |a-b|^2 + |a+b|^2 &= (p-r)^2 + (q-s)^2 + (p+r)^2 + (q+s)^2 \\ &= 2(p^2+q^2+r^2+s^2) \\ &= 2(|a|^2+|b|^2) \end{aligned}$$

Problem
Prove **Langrange's identity**:

$$\left|\sum_{k=1}^{n} z_k w_k\right|^2 = \left(\sum_{k=1}^{n} |z_k|\right)\left(\sum_{k=1}^{n} |w_k|\right) - \sum_{k<j} |z_k\overline{w_j} - z_j\overline{w_k}|.$$

We abuse the fact that $z\overline{z} = |z|^2$.

$$\begin{aligned}
\left|\sum_{k=1}^{n} z_k w_k\right|^2 &= \left(\sum_{k=1}^{n} z_k w_k\right)\left(\overline{\sum_{k=1}^{n} z_k w_k}\right) \\
&= \left(\sum_{k=1}^{n} z_k w_k\right)\left(\sum_{k=1}^{n} \overline{z_k w_k}\right) \\
&= \sum_{k=1}^{n} z_k w_k \overline{z_k w_k} + \sum_{j \neq k} z_j w_j \overline{z_k w_k} \\
&= \sum z_k w_j \overline{z_k w_j} + \sum_{j \neq k} z_j w_j \overline{z_k w_k} - \sum_{j \neq k} z_k w_j \overline{z_k w_j} \\
&= \sum |z_k|^2 |w_j|^2 + \sum_{j \neq k} z_j w_j \overline{z_k w_k} - \sum_{j \neq k} z_k w_j \overline{z_k w_j} \\
&= \left(\sum_{k=1}^{n} |z_k|\right)\left(\sum_{k=1}^{n} |w_k|\right) + \sum_{j \neq k} z_j w_j \overline{z_k w_k} - \sum_{j \neq k} z_k w_j \overline{z_k w_j}
\end{aligned}$$

For some distinct indices j, k we have

$$\begin{aligned}
z_j w_j \overline{z_k w_k} + z_k w_k \overline{z_j w_j} - z_k w_j \overline{z_k w_j} - z_j w_k \overline{z_j w_k} &= z_j \overline{w_k}(w_j \overline{z_k} - w_k \overline{z_j}) + z_k \overline{w_j}(w_k \overline{z_j} - w_j \overline{z_k}) \\
&= (w_k \overline{z_j} - w_j \overline{z_k})(z_j \overline{w_k} - z_k \overline{w_j}) \\
&= -(w_k \overline{z_j} - w_j \overline{z_k})\overline{(w_k \overline{z_j} - w_j \overline{z_k})} \\
&= -|w_k \overline{z_j} - w_j \overline{z_k}|^2
\end{aligned}$$

Summing up gives the desire result

1.3 Even Weirder Stuff

Using the fact that

$$re^{ix} = r(\cos x + i \sin x)$$

and thanks to Euler we generalize the complex numbers to even more functions.

- It's not hard to see that

$$\sin x = \frac{e^{ix} - e^{-ix}}{2i} \quad \text{and} \quad \cos x = \frac{e^{ix} + e^{-ix}}{2}$$

- Let $z = re^{i\theta}$ then $\ln z = \ln |r| + i \arg z$.
- $z^w = e^{w \ln z}$ can be determined consequently.
- Moreover, we have

$$\sinh x = -i \sin(ix) \quad \text{and} \quad \cosh x = \cos(ix)$$

which can be deduced from

$$\sinh x = \frac{e^x - e^{-x}}{2} \quad \text{and} \quad \cosh x = \frac{e^x + e^{-x}}{2}$$

Problem
Along which rays through the origin does $\lim_{z\to\infty} |e^z|$ exist?

Let $z = x + iy$, then we have $|e^z| = |e^x(\cos y + i \sin y)| = e^x$. If $x \to -\infty$ then $e^x \to 0$, but if $x \to \infty$ then $e^x \to \infty$ which the limit doesn't exist.

Hence the answers are all the rays passing through the nonnegative x plane.

Problem
Prove the identity

$$z = \tan\left[\frac{1}{i}\ln\left(\frac{1+iz}{1-iz}\right)^{1/2}\right]$$

for all real z.

$$\begin{aligned}
\tan\left[\frac{1}{i}\ln\left(\frac{1+iz}{1-iz}\right)^{1/2}\right] &= \tan\left[\frac{1}{2i}\left(\ln(1+iz) - \ln(1-iz)\right)\right] \\
&= \tan\left[\frac{1}{2i}\left(\ln|1+iz| + i(\tan^{-1} z) - \ln|1-iz| - i(\tan^{-1}(-z))\right)\right] \\
&= \tan\left[\frac{1}{2i}\left(2i(\tan^{-1} z)\right)\right] \\
&= z
\end{aligned}$$

Problem
Use the equation $\sin z = \sin x \cosh y + i \sinh y \cos x$ where $z = x + iy$ to prove that $|\sinh y| \le |\sin z| \le |\cosh y|$.

Evaluating gives

$$|\sin z| = \sqrt{\sin^2 x \cosh^2 y + \sinh^2 y \cos^2 x}$$

Using the fact that $\sinh x < \cosh x$, we have

$$\sin^2 x \sinh^2 y + \sinh^2 y \cos^2 x < \sin^2 x \cosh^2 y + \sinh^2 y \cos^2 x < \sin^2 x \cosh^2 y + \cosh^2 y \cos^2 x$$

simplifying gives the desired result.

Problem

Using polar coordinates, show that $z \mapsto z + 1/z$ maps the circle $|z| = 1$ to the interval $[-2, 2]$ on the x axis.

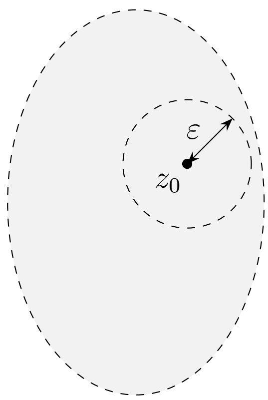
Let $z = x+iy$, then $z + \dfrac{1}{z} = x + iy + \dfrac{x - iy}{x^2 + y^2}$ and since $x^2 + y^2 = 1$, $z + \dfrac{1}{z} = 2x$. This means that for any complex number $z = x+iy$ on the circle, it will be mapped to $2x$.

And since x is in the interval $[-1, 1]$, hence $2x$ is in the interval $[-2, 2]$.

1.4 Topological Analysis of Complex Functions

1.4.1 Definitions

- **r Disk**: The r disk is defined by $D(z_0; r) = \{z \in \mathbb{C} | \, |z - z_0| < r\}$. The **deleted r disk** is defined by $D(z_0; r) \backslash \{z_0\}$.
- **Open Sets**: The set $A \subset \mathbb{C}$ is open when for any point z_0 in A, there exists a real number ε such that if $|z - z_0| < \varepsilon$ then $z \in A$.



- **Closed Sets**: A set F is closed if $\mathbb{C} \backslash F$ is open.
  - The empty set and $\mathbb{C}$ are both open and closed (known as **clopen sets**).
  - Let $z_1, z_2, z_3, \dots$ are points in F and $w = \lim_{n \to \infty} z_n$, then $w \in F$.
    * **Sketch of proof**: Assume that $w \notin F$, then since $\mathbb{C} \backslash F$ is open, we can always find a disk $D(w; r)$ contained in $\mathbb{C} \backslash F$. This means that there exists some large enough n such that $z_n \in D(w; r)$ by convergence, which implies $z_n \notin F$, a contradiction.
  - The **closure** of a set S, denoted by $\overline{S}$ is the set S together with its limit points, or known as the **boundary** $\partial(S)$.
- **Limits**: The limit $\lim_{z \to z_0} f(z) = L$ exists when for all $\varepsilon > 0$, there exists $\delta > 0$ such that for all $|z - z_0| < \delta$ $(z \neq z_0)$ we have $|f(z) - L| < \varepsilon$.

  Limits are **unique** if they exist.

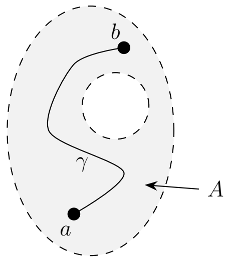
- **Continuity**: f is continuous at $z_0 \in A$ if and only if

$$\lim_{z \to z_0} f(z) = f(z_0).$$

- **Cauchy Sequence**: A sequence is *Cauchy* if for every $\varepsilon > 0$, we can find some integer N such that whenever integers m, n are greater than N, $|z_m - z_n| < \varepsilon$.

- **Path-Connected**: A set $A \in \mathbb{C}$ is path-connected if for every $a, b \in A$ there exists a *continuous map* $\gamma : [0, 1] \to A$ such that $\gamma(0) = a, \gamma(1) = b$.

  γ is a **path** joining a and b.



  Definition: A set $C \in \mathbb{C}$ is **not connected** if there are open sets U, V such that

  (a) $C \subset (U \cup V)$;

  (b) $(C \cap U \neq \varnothing) \wedge (C \cap V \neq \varnothing)$;

  (c) $C \cap U \cap V = \varnothing$.

  If a set is not "not connected", then it is **connected**.

  - **A path-connected set is connected, but a connected set may not be path-connected.**
  - Example: **Topologist's Sine Curve**

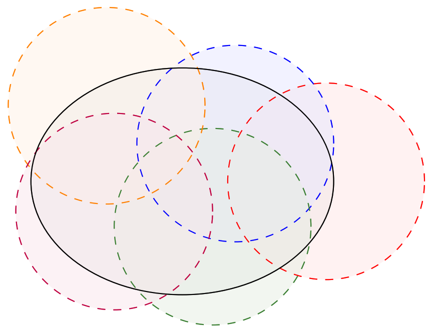
$$f(x) = \begin{cases} \sin \dfrac{1}{x} & x > 0 \\ 0 & x = 0 \end{cases}$$

    Sketch of proof: Let the two sets be A, B. WLOG let $(0, 0) \in A$.

    If some part of $\sin(1/x)$ is in A, then B should be covering the other parts. But since both sets are open, there's a point that is not covered.

    If no part of $\sin(1/x)$ is in A, then B must be covering the entire line of $\sin(1/x)$. But this is impossible since we cannot cover all points near $x = 0^+$.

- **Cover**: Let U be a collection of open sets. U is a cover of a set K if K is contained in the union of sets in U.



A **subcover** is a subset of U but can still cover K.

- **Compactness**: A set K is **compact** if every cover of K has a finite subcover.
  - **Heine-Borel Theorem**: A set K is compact if and only if K is closed and bounded.

    Sketch of proof:

    * Sufficiency:

      Boundedness: Assume that K is not bounded. Consider the set of open covers $U = \{D(O; r)\}$, (open) disks centered at the origin, then for all finite subcover U' of U, consider $R = \max(r)$ and choose some point $z \in K$ but $|z| > r$.

      Closedness: Assume that K is not closed, then there exists some $w \notin K$ such that the sequence $\{z_i\}$ in K converges to w. So the set of open covers $U = \{D(w, r)\}$ does not have a finite subcover.

    * Necessity: Assume that K is closed and bounded, then let $z \in K$ such that $|z|$ attains maximum value. Choose the open cover $D(O; |z|+1)$.

1.4.2 On Functions

- If f is a continuous function defined on a connected set C, then $f(C)$ is connected.

  Sketch of proof: FTSOC, let $A|B$ be a partition of $f(C)$. Then $f^{-1}(A)$ and $f^{-1}(B)$ are open and disjoint (since each value $f^{-1}(x)$ can only belong to either one of $f^{-1}(A)$ and $f^{-1}(B)$.)

- If f is a continuous function defined on a compact set C, then $f(C)$ is compact.

  Sketch of proof: Let U be an open cover of $f(C)$, then for each $f(z) \in U$ and $f(z) \in f(C)$, we have $z \in C$ and $z \in f^{-1}(U)$.

- **Extreme Value Theorem**: Let K be a compact set and $f : K \to \mathbb{R}$ is a continuous function, then f attains **finite** maximum and minimum values.

  Sketch of proof: K is compact implies $f(K)$ is compact, or $f(K)$ is bounded, therefore finite maximum and minimum exists.

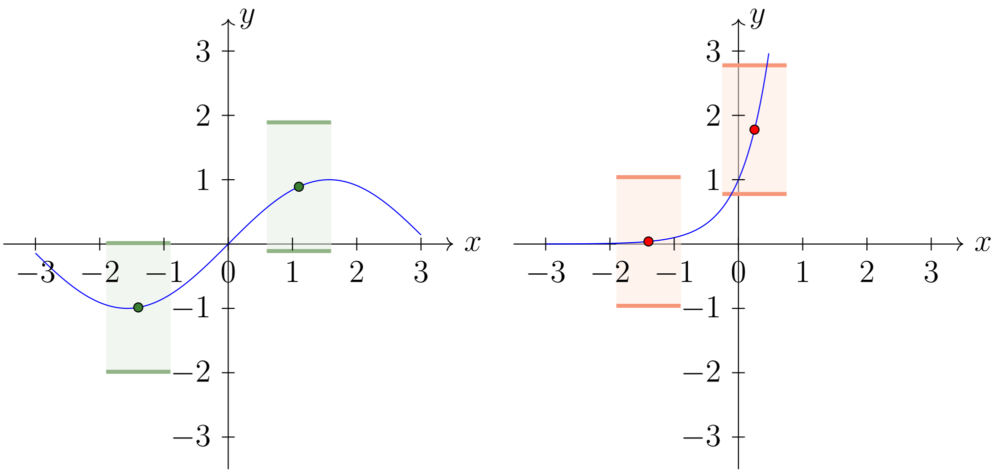
- **Distance Lemma**: Let K be a compact set and C be a closed set and $K \cap C = \varnothing$. Then there exists a number $\rho > 0$, such that whenever $z \in K$ and $w \in C$ then $|z - w| > \rho$.

  Sketch of proof: K is closed and bounded. Assume that ρ doesn't exist, $\rho \to 0$ since we can always find some $|z - w| < \rho_0$ if ρ_0 is fixed. Consider the sequences $\{z_k\}$ and $\{w_k\}$. Thus $\lim\limits_{k\to\infty} |z_k - w_k| = 0$ which means $\lim\limits_{k\to\infty} z_k = \lim\limits_{k\to\infty} w_k$.

  But since both sets are closed, we must have $\lim_{k\to\infty} z_k \in K$ and $\lim_{k\to\infty} w_k \in C$, hence a contradiction.

- **Uniform Continuity**: A function $f : A \to \mathbb{C}$ is **uniformly continuous** on A if for any $\varepsilon > 0$, there exists a $\delta > 0$ such that $|f(s) - f(t)| < \varepsilon$ whenever $s, t \in A$ and $|s - t| < \delta$.

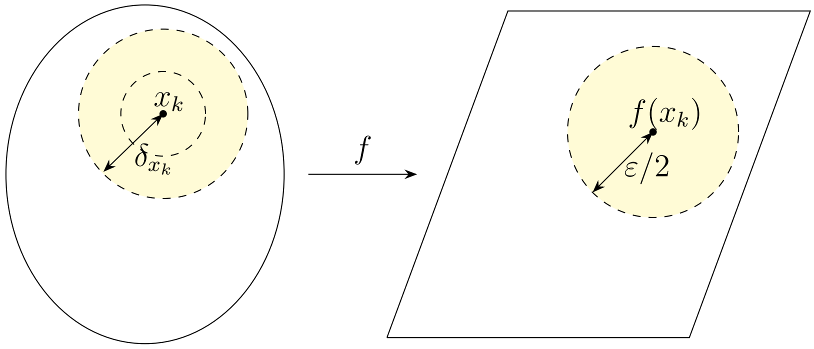
  An example $f(x) = \sin x$ and a counterexample $f(x) = 2^x$ are shown below. Choose $\delta = \varepsilon/2 = 0.5$.



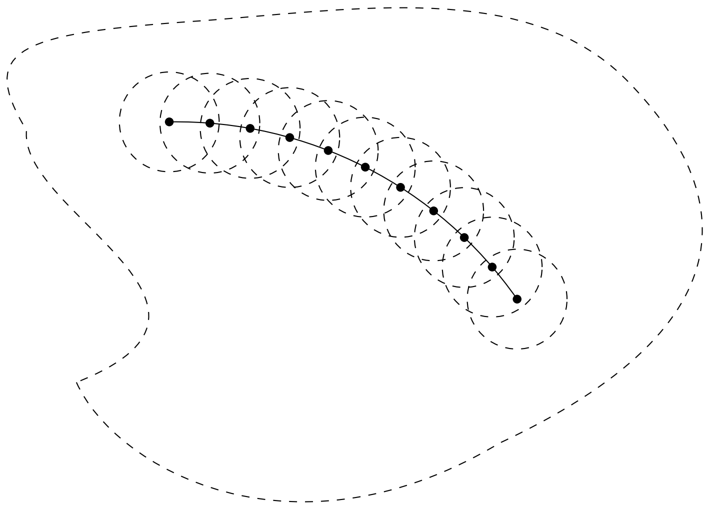
- **Heine-Cantor Theorem**: Let $f : A \to \mathbb{C}$ be a continuous function. If A is compact then $f(A)$ is uniformly continuous.

  Sketch of proof:

  – Let x be some point in A. By continuity, there is a number δ_x such that whenever for some point y satisfying $|x-y| < \delta_x$ then $|f(x)-f(y)| < \varepsilon/2$.
  – For a sequence of points x, say $\{x_i\}$, consider disks $D_k = D(x_k; \delta_{x_k}/2)$. These disks cover A by compactness. Let the minimal radius over all disks be δ.
  – For points s, t satisfying $|s - t| < \delta$, we must have t contained in some disk D_k. Thus $|t - x_k| < \delta_{x_k}/2$, implies that $|f(t) - f(x_k)| < \varepsilon/2$. We have
  $$|s - x_k| \le |s - t| + |t - x_k| < \delta + \delta_{x_k}/2 \le \delta_{x_k}$$
  – On the other hand,
  $$|f(s) - f(t)| \le |f(s) - f(x_k)| + |f(x_k) - f(t)| < \varepsilon$$



- **Path-Covering Lemma**: Suppose $\gamma : [0,1] \to K$ is a continuous path into an open subset K of $\mathbb{C}$. We can find a number $\rho > 0$ and a subdivision of $[0,1]$, namely $0 = t_0 < t_1 < t_2 < \cdots < t_{N-1} < t_N = 1$ such that
  (a) $D(\gamma(t_k);\rho) \subset G$ for all k
  (b) $\gamma(t) \in D(\gamma(t_0);\rho)$ for $t_0 \leq t \leq t_1$
  (c) $\gamma(t) \in D(\gamma(t_k);\rho)$ for $t_{k-1} \leq t \leq t_{k+1}$
  (d) $\gamma(t) \in D(\gamma(t_N);\rho)$ for $t_{N-1} \leq t \leq t_N$



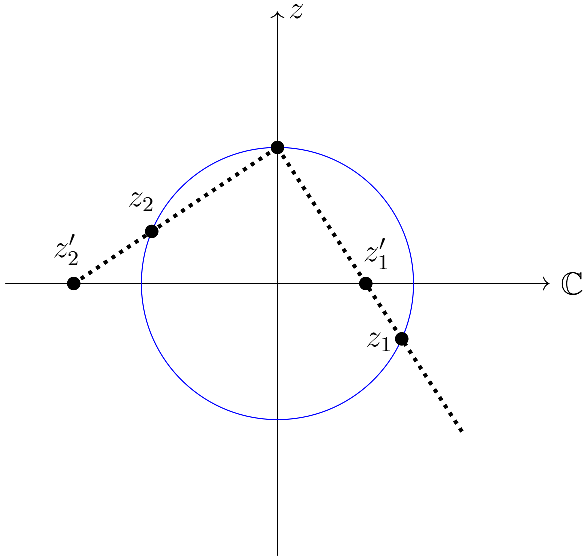
  Sketch of proof:
  - By the Distance Lemma, we can find some $\rho > 0$ such that the distance from $\gamma([0,1])$ to $\mathbb{C}\backslash K$ is at least ρ since both sets are closed.
  - By Heine-Cantor Theorem, $\gamma([0,1])$ is uniformly continuous. So for any two points s,t, if $|s-t| < \delta$ then $|f(s) - f(t)| < \rho$.
  - Choose t_k to be fine enough such that $t_{k+1} - t_k < \delta$.

- **Riemann Sphere**: We may want to define the value ∞ in the complex plane.
  - $\lim_{z\to\infty} f(z) = L$ means for any $\varepsilon > 0$, there exists $Z > 0$ such that whenever $|z| > Z$ implies $|f(z) - L| < \varepsilon$.
  - $\lim_{z\to z_0} f(z) = \infty$ means for any $R > 0$, there exists $\delta > 0$ such that whenever $|z - z_0| < \delta$ implies $|f(z)| > R$.

– $\lim_{z\to\infty} f(z) = \infty$ means for any $Z > 0$, there exists $R > 0$ such that whenever $|z| > Z$ implies $|f(z)| > R$.

Consider the sphere $x^2 + y^2 + z^2 = 1$ in $\mathbb{R}^3$. A point z' on the plane $\mathbb{C}$ is the **stereographic projection** of some point z on the sphere through $(0, 0, 1)$.

A 2-D illustration as an example:



Problem

Show that $\lim_{z\to\infty} \frac{1}{z} = 0$.

By our definition, we must have $|z| \to \infty$. Let $z = x + iy$ so that $|z| = \sqrt{x^2 + y^2}$. Clearly, at least one of $|x|, |y|$ must tend to ∞.

So

$$\lim_{z\to\infty} \frac{1}{z} = \lim_{|x|\to\infty \text{ or } |y|\to\infty} \left(\frac{x}{x^2 + y^2} - i\frac{y}{x^2 + y^2} \right)$$

which, implies that $\lim_{z\to\infty} \frac{1}{z} = 0$.

Problem

(a) Show that

$$|\mathrm{Re}\,(z_1) - \mathrm{Re}\,(z_2)\,| \leq |z_1 - z_2| \leq |\mathrm{Re}\,(z_1) - \mathrm{Re}\,(z_2)\,| + |\mathrm{Im}\,(z_1) - \mathrm{Im}\,(z_2)\,|$$

for any two complex numbers z_1 and z_2.

(b) If $f(z) = u(x,y) + iv(x,y)$, show that

$$\lim_{z\to z_0} f(z) = \lim_{\substack{x\to x_0\\ y\to y_0}} u(x,y) + \lim_{\substack{x\to x_0\\ y\to y_0}} iv(x,y)$$

exists if both limits on the right of the equation exist. Conversely, if the limit on the left exists, show that both limits on the right exist as well and equality holds.

(a) Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ for real numbers x_i, y_i. Let $\mathcal{X} = |x_1 - x_2|$ and $\mathcal{Y} = |y_1 - y_2|$. The inequality above can be expressed as

$$\mathcal{X} \leq \sqrt{\mathcal{X}^2 + \mathcal{Y}^2} \leq \mathcal{X} + \mathcal{Y}$$

which is obvious by squaring each part in the inequality.

(b) **Necessity**: Assume that both $\lim\limits_{\substack{x\to x_0\\ y\to y_0}} u(x,y) = U$ and $\lim\limits_{\substack{x\to x_0\\ y\to y_0}} v(x,y) = V$ exists, then for some ε, there exists δ_u and δ_v such that:

- $|u(x,y) - U| < \varepsilon/2$ whenever $|(x,y) - (x_0,y_0)| < \delta_u$.
- $|v(x,y) - V| < \varepsilon/2$ whenever $|(x,y) - (x_0,y_0)| < \delta_v$.

By the limit laws, $\lim\limits_{z\to z_0} f(z) = U + iV$, then there exists $\delta = \min\{\delta_u, \delta_v\}$ such that whenever $|z - z_0| < \delta$,

$$\begin{aligned} |f(z) - Z| &= |u(x,y) + iv(x,y) - U - iV| \\ &\leq |u(x,y) - U| + |i||v(x,y) - V| \\ &< \varepsilon \end{aligned}$$

Sufficiency: Assume that $\lim\limits_{z\to z_0} f(z) = U + iV$ exists. Then for all $\varepsilon > 0$ there exists $\delta_u > 0$ such that whenever $0 < |z - z_0| < \delta_u$ then $|u(x,y) - U| < \varepsilon$ and $\delta_v > 0$ such that whenever $0 < |z - z_0| < \delta_v$ then $|v(x,y) - V| < \varepsilon$. Choose $\delta = \min\{\delta_u, \delta_v\}$.

Problem

Introduce the **chordal metric** ρ on $\bar{\mathbb{C}}$ by setting $\rho(z_1, z_2) = d(z_1', z_2')$ where z_1' and z_2' are the corresponding points on the Riemann sphere and d is the usual distance between points in $\mathbb{R}^3$.

(a) Show that $z_n \to z$ in $\mathbb{C}$ if and only if $\rho(z_n, z) \to 0$.

(b) Show that $z_n \to \infty$ if and only if $\rho(z_n, \infty) \to 0$.

(c) If $f(z) = (az + b)/(cz + d)$ and $ad - bc \neq 0$, show that f is continuous at ∞.

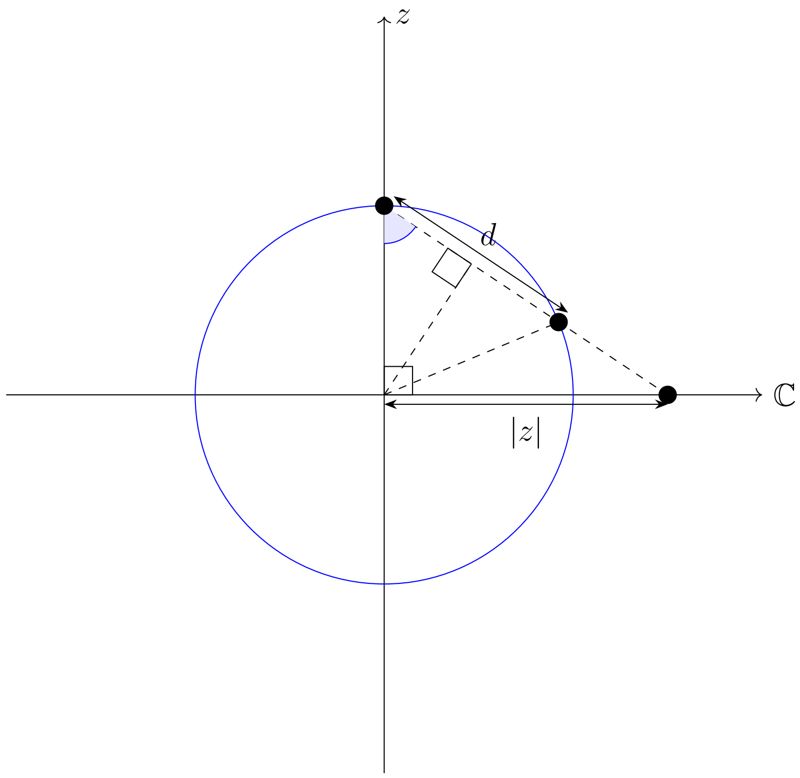
(a) If $z_n \to z$, then $\rho(z_n, z) \to \rho(z, z) = d(z', z') = 0$.

On the other hand, let $z_n' = (x_n, y_n, t_n)$ and $z' = (x, y, t)$. Then if $\rho(z_n, z) = d(z_n', z') \to 0$, we have

$$\sqrt{(x_n - x)^2 + (y_n - y)^2 + (t_n - t)^2} \to 0$$
$$(x_n - x)^2 + (y_n - y)^2 + (t_n - t)^2 \to 0$$

FTSOC, WLOG assume x_n does not converge to x, then since $(x_n - x)^2 + (y_n - y)^2 + (t_n - t)^2 \geq 0$, we have $(x_n - x)^2 + (y_n - y)^2 + (t_n - t)^2$ converges to at least $(x_n - x)^2$, contradiction.

(b) If $z_n \to \infty$, $|z_n| \to \infty$. By drawing the Riemann sphere again,



We see that $\tan\theta = |z|$ and $d = 2\cos\theta$ (θ is the measure of the angle marked in blue). Substituting gives

$$d = \frac{2}{\sqrt{|z|^2 + 1}}$$

which is obvious that $d \to 0$.

Conversely, if $d \to 0$, it can be shown that $\sqrt{|z|^2 + 1} \to \infty$ which gives $z_n \to \infty$.

(c) A function $f(z)$ is continuous at infinity if the limits when $z \to +\infty$ and $z \to -\infty$ are equal.

- $\lim_{z \to \infty} \frac{az + b}{cz + d} = \frac{a}{c}$.
- $\lim_{z \to -\infty} \frac{az + b}{cz + d} = \frac{a}{c}$.

Hence the function is continuous at ∞.

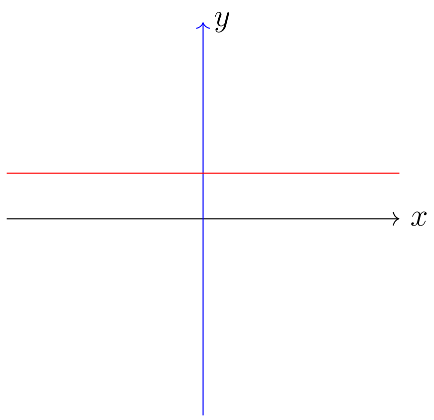
1.5 holomorphic Functions: What and Why

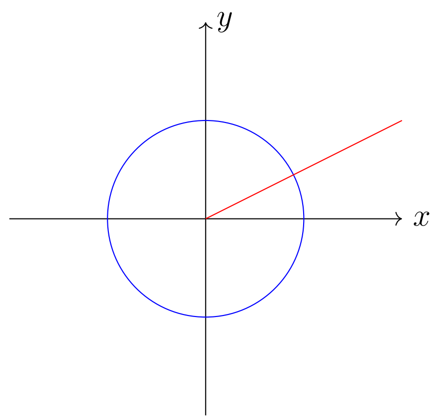
- functions that are differentiable in complex.
- "regular", "holomorphic", "holomorphic"
- a function $f(z)$ is differentiable at z_0 if $\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$ exists.

1.5.1 Differentiation

Differentiation laws apply here, including product rule, quotient rule, and chain rule.

- **Conformal Maps**: A function $f : A \to \mathbb{C}$ is conformal if it preserves **angles** between intersecting curves.
  - "conformal transformation", "angle-preserving transformation", "biholomorphic map"
  - An example of $e^z = e^{x+iy}$.



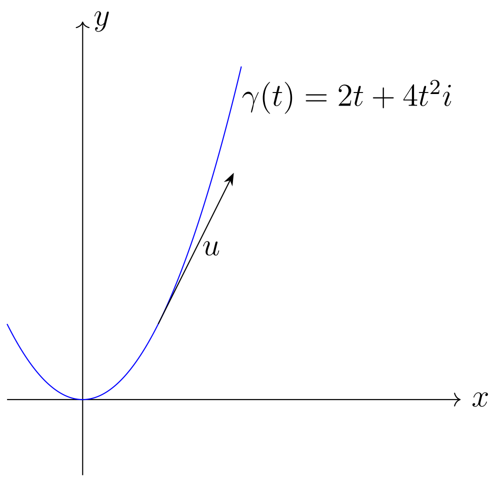


- What does it mean to differentiate complex numbers?
  - Clearly $f'(z_0) = x + iy$ is a vector $\begin{pmatrix} x \\ y \end{pmatrix}$ in the plane. Assume for some curve $\gamma(t) : \mathbb{R} \to \mathbb{C}$.

> **Claim —** Let $\gamma(t) = x(t) + iy(t)$, define $\gamma'(t) = x'(t) + iy'(t)$, then the vector $u = \begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix}$ is tangent to the curve γ at $(x(t), y(t))$.

This is trivial by noticing

$$\frac{y'(t)}{x'(t)} = \frac{\frac{\mathrm{d}y}{\mathrm{d}t}}{\frac{\mathrm{d}x}{\mathrm{d}t}} = \frac{\mathrm{d}y}{\mathrm{d}x}$$



  - **Conformal Mapping Theorem**: If $A \to \mathbb{C}$ is holomorphic and $f'(z_0) \neq 0$, then f is conformal at z_0.

Sketch of Proof: Define $\sigma(t) = f(\gamma(t))$, then obviously $\sigma(t)$ is also a curve. Taking

$$\frac{\mathrm{d}f(\gamma(t))}{\mathrm{d}t} = \frac{\mathrm{d}f(\gamma(t))}{\mathrm{d}z} \cdot \frac{\mathrm{d}z}{\mathrm{d}t}$$

Assume $\gamma(t_0) = z_0$, letting $t = t_0$ we have

$$\sigma'(t_0) = f'(z_0)\gamma'(t_0)$$

Since by our definition, $f'(z_0)$ is independent of γ, choose t_1 and t_2 so that

$$\frac{\sigma'(t_1)}{\gamma'(t_1)} = \frac{\sigma'(t_2)}{\gamma'(t_2)}$$

Taking the argument we have

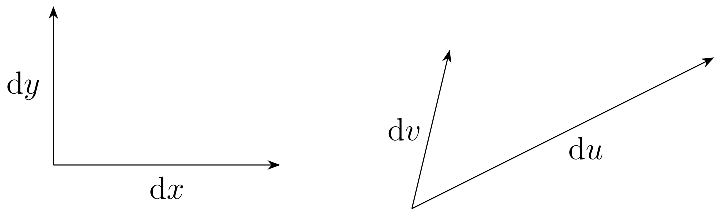
$$\arg \sigma'(t_1) - \arg \sigma'(t_2) \equiv \arg \gamma'(t_1) - \arg \gamma'(t_2) \pmod{2\pi}$$

hence it is clear that angles are preserved.

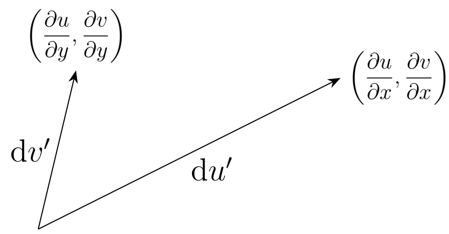
- **Cauchy-Riemann Equations**: Let $f(x, y) = u(x, y) + iv(x, y)$,
  - The **Jacobian matrix** of f is

$$\mathbf{J}_f = \begin{pmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\[2ex] \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{pmatrix}$$

What does the Jacobian matrix tell us? Recall that the transformation matrix $\mathbf{T} = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 & \dots & \mathbf{v}_n \end{bmatrix}$.



Taking ratios gives



  - **Cauchy-Riemann Theorem**:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Sketch of proof: This is due to the derivative of $f(z)$ can be approached from multiple directions.

Fix y, then when $x^* \to x$, we have

$$\begin{aligned} \lim_{x^* \to x} \frac{f(z^*) - f(z)}{z^* - z} &= \lim_{x^* \to x} \frac{u(x^*, y) + iv(x^*, y) - u(x, y) - iv(x, y)}{x^* - x} \\ &= \lim_{x^* \to x} \left(\frac{u(x^*, y) - u(x, y)}{x^* - x} + i \frac{v(x^*, y) - v(x, y)}{x^* - x} \right) \\ &= \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} \end{aligned}$$

Similarly, fix x, then when $iy^* \to iy$, we have

$$\lim_{y^* \to y} \frac{f(z^*) - f(z)}{z^* - z} = \frac{\partial v}{\partial y} - i \frac{\partial u}{\partial y}$$

Comparing both equations gives the desired result.

- Applying Cauchy-Riemann equations to the Jacobian matrix, we have

$$\mathbf{J}_f = \begin{pmatrix} \dfrac{\partial u}{\partial x} & -\dfrac{\partial v}{\partial x} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial u}{\partial x} \end{pmatrix}$$

- Recall that the multiplication of complex values is the product of matrices.

  For example, $(a+bi)(x+yi) = (ax-by) + i(ay+bx)$,

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix}\begin{pmatrix} x & -y \\ y & x \end{pmatrix} = \begin{pmatrix} ax-by & -(bx+ay) \\ bx+ay & ax-by \end{pmatrix}$$

- Inverse functions
  - It is trivial that $\dfrac{\mathrm{d}}{\mathrm{d}z} f^{-1}(z) = \dfrac{1}{f'(f^{-1}(z))}$
  - By $z = x+iy$, $f(z) = u(x,y) + iv(x,y)$,

$$f'(z) = \frac{\mathrm{d}f}{\mathrm{d}z} = \frac{\partial f}{\partial x}\frac{1}{\dfrac{\partial z}{\partial x}} = \frac{\partial f}{\partial x} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

    Thus the matrix representing $f'(z)$ is surprisingly $\mathbf{J}_f$! Moreover,

$$|f'(z)|^2 = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 = \det \mathbf{J}_f$$

  - **Inverse Function Theorem**: Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be a (continuous, differentiable) function and let $\mathbf{J}_f(\mathbf{p})$ denote the Jacobian matrix of f evaluated at point $\mathbf{p}$ (here, we may assume that $\mathbf{p}$ is a complex value on the plane), then

$$\mathbf{J}_{f^{-1}}(f(\mathbf{p})) = (\mathbf{J}_f(\mathbf{p}))^{-1}$$

    Sketch of proof: Using the two facts above, we have

$$\mathbf{J}_{f^{-1}}(z)\mathbf{J}_f(f^{-1}(z)) = \mathbf{I}$$

    Letting $z = f(\mathbf{p})$ gives the desired result.

- **Recall on linear algebra** (taking 3-D system as example)

  First, we define the **del/nabla** operator.

$$\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$$

1. **Gradient**: Denoted by ∇f – Scalar multiplication of ∇ and f.

$$\nabla f(x, y, z) = \begin{pmatrix} \dfrac{\partial f}{\partial x} \\ \dfrac{\partial f}{\partial y} \\ \dfrac{\partial f}{\partial z} \end{pmatrix} = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}$$

2. **Divergence**: Denoted by $\nabla \cdot \mathbf{f}$ – Dot product of ∇ and $\mathbf{f}$.

$$\nabla \cdot \mathbf{f} = \nabla \cdot (F_x.F_y, F_z) = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$$

3. **Curl**: Denoted by $\nabla \times \mathbf{f}$ – Cross product of ∇ and $\mathbf{f}$.

$$\nabla \times \mathbf{f} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix}$$

4. **Laplacian**: Denoted by $\nabla \cdot \nabla \mathbf{f}$ or $\nabla^2 \mathbf{f}$ – Divergence of gradient.

$$\nabla \cdot \nabla \mathbf{f} = \frac{\partial^2 F_x}{\partial x^2} + \frac{\partial^2 F_y}{\partial y^2} + \frac{\partial^2 F_z}{\partial z^2}$$

Now let's get back to the 2-D plane.

- **Harmonic Functions**: A function $f : A \subseteq \mathbb{R}^2 \to \mathbb{R}$ is harmonic if

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$$

  - **Clairaut's Theorem**:

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$$

    Sketch of proof: A simple (but unformal) proof uses the definition of the derivative.

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \lim_{x^* \to x} \frac{\displaystyle\lim_{y^* \to y} \frac{f(x^*, y^*) - f(x^*, y)}{y^* - y} - \lim_{y^* \to y} \frac{f(x, y^*) - f(x, y)}{y^* - y}}{x^* - x}$$

    We can evaluate $\dfrac{\partial}{\partial y}\left(\dfrac{\partial f}{\partial x}\right)$ a similar way.

  - We now show that for some holomorphic function $f = u + iv$, then u and

v are harmonic.

By utilizing the Cauchy-Riemann equations, taking partial derivative with respect to x, we have

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial v}{\partial y}\right)$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) = -\frac{\partial}{\partial y}\left(\frac{\partial v}{\partial x}\right)$$

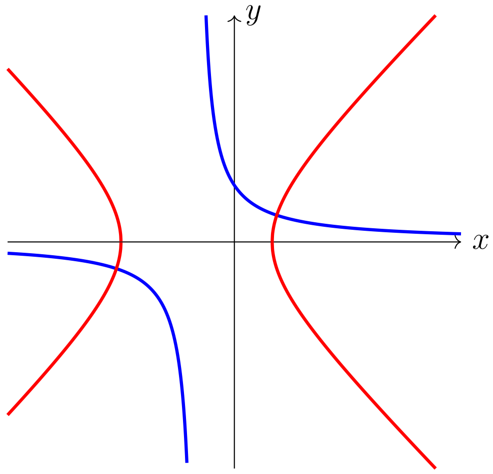
– **Harmonic Conjugate**: If u, v satisfy $f = u + iv$, then $u(x, y)$ and $v(x, y)$ are harmonic conjugates.

Let $u(x, y)$ and $v(x, y)$ be harmonic conjugates, then the graphs $u(x, y) = c_1$ and $v(x, y) = c_2$ intersect orthogonally in the Cartesian plane.

Sketch of proof: The dot product of two gradients equals zero

$$\nabla u \cdot \nabla v = \begin{pmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial u}{\partial y} \end{pmatrix} \cdot \begin{pmatrix} \dfrac{\partial v}{\partial x} \\ \dfrac{\partial v}{\partial y} \end{pmatrix} = \frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial y} = 0$$

An example of $f(z) = (z + 1)^2$ and $u = x^2 - y^2 + 2x + 1 = 4$ and $v = 2xy + 2y = 3$.



Problem

Show, by changing variables, that the Cauchy-Riemann equations in terms of polar coordinates become

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \qquad \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$$

Then show that if u is harmonic, we have

$$r^2\frac{\partial^2 u}{\partial r^2} + r\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial \theta^2} = 0$$

(a) By knowing that $f(x+iy) = u + iv$ and $x + iy = r\cos\theta + ir\sin\theta$, we have

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial r} = \frac{\partial u}{\partial x}\cos\theta$$
$$\frac{\partial v}{\partial \theta} = \frac{\partial v}{\partial y} \cdot \frac{\partial y}{\partial \theta} = \frac{\partial v}{\partial y} r\cos\theta$$
$$\frac{\partial v}{\partial r} = \frac{\partial v}{\partial x} \cdot \frac{\partial x}{\partial r} = \frac{\partial v}{\partial x}\cos\theta$$
$$\frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial \theta} = \frac{\partial u}{\partial y} r\cos\theta$$

Results can be shown by applying Cauchy-Riemann equations.

(b) Evaluating the first equation gives

$$\frac{\partial^2 u}{\partial r^2} = \frac{1}{r}\frac{\partial}{\partial r}\left(\frac{\partial v}{\partial \theta}\right) - \frac{1}{r^2}\frac{\partial v}{\partial \theta} = \frac{1}{r}\frac{\partial}{\partial r}\left(\frac{\partial v}{\partial \theta}\right) - \frac{1}{r}\frac{\partial u}{\partial r}$$

while the second equation gives

$$\frac{\partial}{\partial \theta}\left(\frac{\partial v}{\partial r}\right) = -\frac{1}{r}\frac{\partial^2 u}{\partial \theta^2}$$

Eliminating $\frac{\partial}{\partial \theta}\left(\frac{\partial v}{\partial r}\right)$ gives the desired result.

Problem

Show that $\frac{\partial f}{\partial \bar{z}} = 0$. Then, find the value of $\frac{\partial \bar{z}}{\partial z}$.

(a) By using $x = \frac{1}{2}(z + \bar{z})$ and $y = \frac{1}{2i}(z - \bar{z})$. A change of variable lets f depend on two independent variables $z, \bar{z}$. So by taking total derivative,

$$\frac{\partial f}{\partial \bar{z}} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial \bar{z}} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right) = 0$$

which can be shown by Cauchy-Riemann equations.

(b) By repeating the same thing,

$$\frac{\partial f}{\partial z} = \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right)$$

Letting $f = \bar{z}$ gives

$$\frac{\partial \bar{z}}{\partial z} = \frac{1}{2}(1 - i(-i)) = 0$$

Problem

On what sets are each of the following functions harmonic?

(a) $u(x, y) = \operatorname{Im}(z^2 + 3z + 1)$

(b) $u(x, y) = \dfrac{x-1}{x^2 + y^2 - 2x + 1}$

(c) $u(x, y) = \operatorname{Im}(z + 1/z)$

(d) $u(x, y) = \dfrac{y}{(x-1)^2 + y^2}$

(a) Since $u(x.y)$ is already the imaginary part of the function $f(z) = z^2 + 3z + 1$, so $u(x, y)$ is harmonic on $\mathbb{C}$.

(b) Recall that $\dfrac{1}{x + yi} = \dfrac{x}{x^2 + y^2} - i\dfrac{y}{x^2 + y^2}$. Changing x to $x - 1$ shows that $u(x, y)$ is the real part of $f(z) = \dfrac{1}{z-1}$. Hence $u(x, y)$ is harmonic on $\mathbb{C} \setminus \{1\}$.

(c) Since $u(x.y)$ is already the imaginary part of the function $f(z) = z + 1/z$, so $u(x, y)$ is harmonic on $\mathbb{C} \setminus \{0\}$.

(d) Notice that $u(x, y)$ is the imaginary part of the function $f(z) = \dfrac{1}{1-z}$ hence it is harmonic on $\mathbb{C} \setminus \{1\}$.

Problem

Suppose u is a twice continuously differentiable real-valued harmonic function on a disk $D(z_0; l)$ centered at $z_0 = x_0 + iy_0$. For $(x_1, y_1) \in D(z_0; r)$, show that the equation

$$v(x_1, y_1) = c + \int_{y_0}^{y_1} \frac{\partial u}{\partial x}(x_1, y)\mathrm{d}y - \int_{x_0}^{x_1} \frac{\partial u}{\partial y}(x, y_0)\mathrm{d}x$$

defines a harmonic conjugate for u on $D(z_0; r)$ with $v(x_0, y_0) = c$.

If $v(x, y)$ is a harmonic conjugate, it must satisfy the Cauchy-Riemann equations.

Thus we have

$$\frac{\partial v}{\partial y}(x_1, y_1) = \frac{\partial}{\partial y}\left(c + \int_{y_0}^{y_1} \frac{\partial u}{\partial x}(x_1, y)\mathrm{d}y - \int_{x_0}^{x_1} \frac{\partial u}{\partial y}(x, y_0)\mathrm{d}x\right) = \frac{\partial u}{\partial x}(x_1, y_1)$$

We use the fact proven above,

$$\begin{aligned}\frac{\partial v}{\partial x}(x_1, y_1) &= \frac{\partial}{\partial x}\left(c + \int_{y_0}^{y_1} \frac{\partial u}{\partial x}(x_1, y)\mathrm{d}y - \int_{x_0}^{x_1} \frac{\partial u}{\partial y}(x, y_0)\mathrm{d}x\right) \\ &= \frac{\partial}{\partial x}(v(x_1, y_1) - v(x_1, y_0)) - \frac{\partial u}{\partial y}(x_1, y_0) \\ \frac{\partial v}{\partial x}(x_1, y_0) &= -\frac{\partial u}{\partial y}(x_1, y_0)\end{aligned}$$

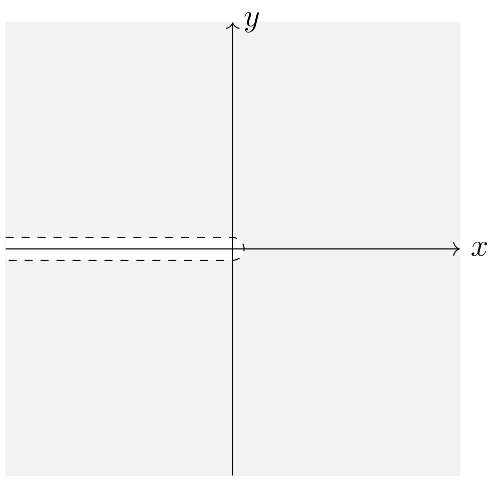
> **Remark.** An important theorem which may be worth introducing is the **Leibniz Integral Rule**,
>
> $$\frac{\mathrm{d}}{\mathrm{d}x}\int_{a(x)}^{b(x)} f(x,t)\mathrm{d}t = f(x, b(x))\frac{\mathrm{d}}{\mathrm{d}x}b(x) - f(x, a(x))\frac{\mathrm{d}}{\mathrm{d}x}a(x) + \int_{a(x)}^{b(x)} \frac{\partial}{\partial x}f(x,t)\mathrm{d}t$$
>
> The proof uses **Fubini's Theorem**, which states that
>
> $$\int_X \left(\int_Y f(x,y)\mathrm{d}y\right)\mathrm{d}x = \int_Y \left(\int_X f(x,y)\mathrm{d}x\right)\mathrm{d}y$$

1.6 Differentiation of Complex Functions – Built Diff

Principal Branch

Take $f(z) = \ln z = \ln|z| + i\arg z$. We may want $-\pi < \arg z < \pi$. This is a **principal branch** of the logarithm function.

This is due to the fact that $\ln|z| + i\arg z = \ln|z| + i(\arg z + 2k\pi)$.



Problem

Let u, v be real-valued functions on an open set $A \subset \mathbb{R}^2 = \mathbb{C}$ and suppose that they satisfy the Cauchy-Riemann equations on A. Show that

(a) $u_1 = u^2 - v^2$, $v_1 = 2uv$

(b) $u_2 = e^u \cos v$, $v_2 = e^u \sin v$

also satisfy the Cauchy-Riemann equations on A.

(a)

$$\frac{\partial u_1}{\partial x} = 2u\frac{\partial u}{\partial x} - 2v\frac{\partial v}{\partial x}$$

$$\frac{\partial v_1}{\partial y} = 2u\frac{\partial v}{\partial y} + 2v\frac{\partial u}{\partial y} = 2u\frac{\partial u}{\partial x} - 2v\frac{\partial v}{\partial x}$$

$$\frac{\partial u_1}{\partial y} = 2u\frac{\partial u}{\partial y} - 2v\frac{\partial v}{\partial y}$$

$$\frac{\partial v_1}{\partial x} = 2u\frac{\partial v}{\partial x} + 2v\frac{\partial u}{\partial x} = -2u\frac{\partial u}{\partial y} + 2v\frac{\partial v}{\partial y}$$

(b)

$$\frac{\partial u_2}{\partial x} = e^u\frac{\partial u}{\partial x}\cos v - \sin v\frac{\partial v}{\partial x}e^u$$

$$\frac{\partial v_2}{\partial y} = e^u\frac{\partial u}{\partial y}\sin v + \cos v\frac{\partial v}{\partial y}e^u = -e^u\frac{\partial v}{\partial x}\sin v + \cos v\frac{\partial u}{\partial x}e^u$$

$$\frac{\partial u_2}{\partial y} = e^u\frac{\partial u}{\partial y}\cos v - \sin v\frac{\partial v}{\partial y}e^u = -e^u\frac{\partial v}{\partial x}\cos v - \sin v\frac{\partial u}{\partial x}e^u$$

$$\frac{\partial v_2}{\partial x} = e^u\frac{\partial u}{\partial x}\sin v + \cos v\frac{\partial v}{\partial x}e^u$$

Remark. Let $f(z) = u + iv$, it's not hard to see that in (a), we have $g(z) = f(z)^2 = (u^2 - v^2) + 2uvi$ and in (b), we have $h(z) = e^{f(z)} = e^u \cos v + ie^u \sin v$.

Problem

Given functions $u(x, y)$, find their respective harmonic conjugates.

(a) $e^x(y \cos y + x \sin y)$

(b) $\dfrac{(e^{-y} + e^y)\sin x}{2}$

(a) We want

$$\frac{\partial u}{\partial x} = e^x(y\cos y + (x+1)\sin y) = \frac{\partial v}{\partial y}$$

So by the equation above we have

$$v = e^x(y\sin y + \cos y - (x+1)\cos y) + g(x) = e^x(y\sin y - x\cos y) + g(x)$$

On the other hand, we have

$$\frac{\partial u}{\partial y} = e^x(\cos y - y\sin y + x\cos y) = -\frac{\partial v}{\partial x}$$

Solving the differential equation gives

$$v = e^x y\sin y - e^x x\cos y + h(y)$$

Thus $g(x) = h(y) = C$ for some constant C. We have

$$v = e^x y\sin y - e^x x\cos y + C.$$

(b) We want

$$\frac{\partial u}{\partial x} = \frac{(e^{-y}+e^y)\cos x}{2} = \frac{\partial v}{\partial y}$$

So we have

$$v = \frac{(-e^{-y}+e^y)\cos x}{2} + g(x)$$

While on the other hand,

$$\frac{\partial u}{\partial y} = \frac{(-e^{-y}+e^y)\sin x}{2} = -\frac{\partial v}{\partial x}$$

Solving gives

$$v = \frac{(-e^{-y}+e^y)\cos x}{2} + h(y)$$

Comparing gives

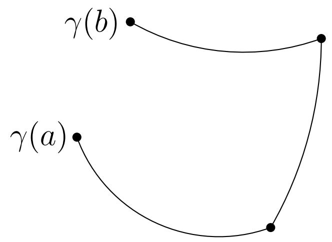
$$v = \frac{(-e^{-y}+e^y)\cos x}{2} + C$$

§2 Cauchy's Theorem

2.1 Contour Integrals are Contourversial

Let $z : [a,b] \to \mathbb{C}$ be a **curve**.

- If it is a continuous function, then it is a **smooth curve**.
- If we have $z'(a) = z'(b)$, then we say that the curve is **closed**.
- A curve is called **piecewise C^1** if we can divide the interval into subintervals $a = a_0 < a_1 < a_2 < \cdots < a_n = b$ such that $\gamma'(t)$ exists on the open intervals $(a_k, a_k + 1)$ and continuous on $[a_k, a_k + 1]$.



Let γ be a smooth curve, we denote the integral along γ as

$$\int_\gamma f(z)\mathrm{d}z = \int_a^b f(z(t))z'(t)\mathrm{d}t$$

Sometimes if we know that γ is a closed curve (known as the **cyclic integral**), we can write as

$$\oint_\gamma f(z)\mathrm{d}z$$

If it is known that the loop is directed clockwise or anticlockwise, it can still sometimes be written as

$$\ointclockwise_\gamma f(z)\mathrm{d}z \qquad \ointctrclockwise_\gamma f(z)\mathrm{d}z$$

Integration Properties

- In general, we have

$$\int_\gamma f = \sum_{i=0}^{n-1} \int_{a_i}^{a_{i+1}} f(\gamma(t))\gamma'(t)\mathrm{d}t$$

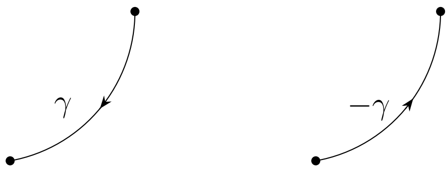
- Let a function be $f = u(x, y) + iv(x, y)$, we have

$$\int_\gamma f = \int_\gamma (u\mathrm{d}x - v\mathrm{d}y) + i\int_\gamma (u\mathrm{d}y + v\mathrm{d}x)$$

  Sketch of proof: Consider

$$\begin{aligned} f(\gamma(t))\gamma'(t) &= [u(x, y) + iv(x, y)][x'(t) + iy'(t)] \\ &= [u(x, y)x'(t) - v(x, y)y'(t)] + i[u(x, y)y'(t) + v(x, y)x'(t)] \\ \int_a^b f(\gamma(t))\gamma'(t)\mathrm{d}t &= \int_a^b [u(x, y)x'(t) - v(x, y)y'(t)]\mathrm{d}t + i\int_a^b [u(x, y)y'(t) + v(x, y)x'(t)]\mathrm{d}t \\ \int_\gamma f &= \int_\gamma (u\mathrm{d}x - v\mathrm{d}y) + i\int_\gamma (u\mathrm{d}y + v\mathrm{d}x) \end{aligned}$$

- An **opposite curve** of a curve γ is a curve (denoted as $-\gamma$) traversed oppositely.



Assume that $\gamma : [a, b] \to \mathbb{C}$ and $-\gamma : [a, b] \to \mathbb{C}$, we have

$$\gamma(t) = (-\gamma)(a + b - t)$$

- A **sum** $\gamma_1 + \gamma_2$ of curves is a curve constructed by joining the endpoints of $\gamma_1 : [a, b] \to \mathbb{C}$ and $\gamma_2 : [b, c] \to \mathbb{C}$. Thus

$$(\gamma_1 + \gamma_2)(t) = \begin{cases} \gamma_1(t) & t \in [a, b] \\ \gamma_2(t) & t \in [b, c] \end{cases}$$

  So we must have $\gamma_1(b) = \gamma_2(b)$.

- We have the following list of properties, which can be proved by the definitions above
  * $\displaystyle\int_\gamma \sum_{i=1}^n c_i f_i = \sum_{i=1}^n \left(c_i \int_\gamma f_i \right)$
  * $\displaystyle\int_{-\gamma} f = -\int_\gamma f$
  * $\displaystyle\int_{\gamma_1+\gamma_2+\cdots+\gamma_n} f = \sum_{i=1}^n \int_{\gamma_i} f$

- A **reparametrization** of a piecewise smooth curve $\gamma : [a, b] \to \mathbb{C}$ is the piecewise smooth curve $\tilde{\gamma} : [\tilde{a}, \tilde{b}] \to \mathbb{C}$ if there exists a piecewise C^1 function $\alpha : [a, b] \to [\tilde{a}, \tilde{b}]$ with
  - $\alpha'(t) > 0$ for all $t \in (a, b)$
  - $\alpha(a) = \tilde{a}$, $\alpha(b) = \tilde{b}$,
  - $\gamma(t) = \tilde{\gamma}(\alpha(t))$.

- We have

$$\int_\gamma f = \int_{\tilde{\gamma}} f$$

  Sketch of proof: Evaluating

$$\begin{aligned} \int_\gamma f &= \int_a^b f(\gamma(t))\gamma'(t)\mathrm{d}t \\ &= \int_a^b f(\tilde{\gamma}(\alpha(t)))\tilde{\gamma}'(\alpha(t))\alpha'(t)\mathrm{d}t \\ &= \int_{\tilde{a}}^{\tilde{b}} f(\tilde{\gamma}(s))\tilde{\gamma}'(s)\mathrm{d}s \\ &= \int_{\tilde{\gamma}} f \end{aligned}$$

- The **arc length** formula is given by

$$l(\gamma) = \int_a^b \sqrt{x'(t)^2 + y'(t)^2}\mathrm{d}t = \int_a^b |\gamma'(t)|\mathrm{d}t$$

Recall that the arc length formula in the Cartesian plane is

$$\int \mathrm{d}l = \int \sqrt{\mathrm{d}x^2 + \mathrm{d}y^2} = \int_c^d \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\mathrm{d}x$$

By recognizing that $\mathrm{d}x = x'(t)\mathrm{d}t$ and $\mathrm{d}y = y'(t)\mathrm{d}t$, we have

$$\begin{aligned}\int_c^d \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\mathrm{d}x &= \int_a^b \sqrt{1 + \left(\frac{y'(t)\mathrm{d}t}{x'(t)\mathrm{d}t}\right)^2}x'(t)\mathrm{d}t \\ &= \int_a^b \sqrt{x'(t)^2 + y'(t)^2}\mathrm{d}t\end{aligned}$$

For some continuous function f, we have

$$\left|\int_a^b f(t)\mathrm{d}t\right| \leq \int_a^b |f(t)|\mathrm{d}t$$

Sketch of proof: Let $\int_a^b f(t)\mathrm{d}t = re^{i\theta}$, then

$$\begin{aligned}\left|\int_a^b f(t)\mathrm{d}t\right| &= r \\ &= \int_a^b e^{-i\theta} f(t)\mathrm{d}t \\ &= \int_a^b \mathrm{Re}\left(e^{-i\theta} f(t)\right)\mathrm{d}t \\ &\leq \int_a^b \left|\mathrm{Re}\left(e^{-i\theta} f(t)\right)\right|\mathrm{d}t \\ &\leq \int_a^b \left|e^{-i\theta} f(t)\right|\mathrm{d}t \\ &= \int_a^b |f(t)|\,\mathrm{d}t\end{aligned}$$

Let $|f(z)| \leq M$ for some constant $M > 0$ and all z on γ, we have

$$\left|\int_\gamma f(z)\mathrm{d}z\right| \leq \int_\gamma |f(z)|\mathrm{d}z = \int_a^b |f(\gamma(t))||\gamma'(t)|\mathrm{d}t \leq M\int_a^b |\gamma'(t)|\mathrm{d}t = Ml(\gamma)$$

- **Fundamental Theorem of Calculus for Contour Integrals**: Recall that the fundamental theorem of calculus states that

$$\int_a^b F'(x)\mathrm{d}x = F(b) - F(a)$$

Suppose that $\gamma : [0,1] \to \mathbb{C}$ is a piecewise smooth curve, we have

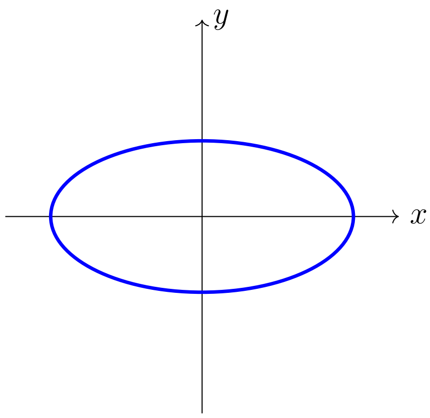
$$\int_\gamma F'(z)\mathrm{d}z = F(\gamma(1)) - F(\gamma(0))$$

If it happens that $\gamma(0) = \gamma(1)$ (which makes γ a loop) and $F'(z)$ **is holomorphic everywhere inside** γ, we have

$$\oint_\gamma F'(z)\mathrm{d}z = 0$$

Example

Let's evaluate $\displaystyle\int_\gamma z\mathrm{d}z$ and $\displaystyle\int_{\gamma_1} z\mathrm{d}z$ where γ is the portion of the ellipse $4x^2 + y^2 = 1$ joining $z = 1/2$ to $z = i$, and γ_1 is the entire ellipse, integrated counterclockwise.



A parametrization will be $\gamma(t) = \dfrac{\cos t}{2} + i\sin t$ and t ranges from 0 to 2π. Thus we have

$$\begin{aligned}\int_\gamma z\mathrm{d}z &= \int_0^{\pi/2} \left(\frac{\cos t}{2} + i\sin t\right)\left(\frac{-\sin t}{2} + i\cos t\right)\mathrm{d}t \\ &= \int_0^{\pi/2} \left(\frac{-5\sin t\cos t}{4} + i\left(\frac{\cos^2 t - \sin^2 t}{2}\right)\right)\mathrm{d}t \\ &= -\frac{5}{8}\end{aligned}$$

On the other hand, we have

$$\int_{\pi/2}^{\pi} \left(\frac{\cos t}{2} + i\sin t\right)\left(\frac{-\sin t}{2} + i\cos t\right)\mathrm{d}t = \frac{5}{8}$$

$$\int_{\pi}^{3\pi/2} \left(\frac{\cos t}{2} + i\sin t\right)\left(\frac{-\sin t}{2} + i\cos t\right)\mathrm{d}t = -\frac{5}{8}$$

$$\int_{3\pi/2}^{2\pi} \left(\frac{\cos t}{2} + i\sin t\right)\left(\frac{-\sin t}{2} + i\cos t\right) \mathrm{d}t = \frac{5}{8}$$

which indeed adds up to 0.

- **Path Independence Theorem**: Let f be a continuous function on an open connected set $G \in \mathbb{C}$,
  - For any closed curve Γ,
  $$\int_\Gamma f = 0$$
  - A result is that for any two curves γ_1, γ_2 joining z_0, z_1,
  $$\int_\Gamma f = \int_{\gamma_1} f + \int_{-\gamma_2} f = \int_{\gamma_1} f - \int_{\gamma_2} f = 0$$

Problem

Evaluate $\int_\gamma \sin 2z \ \mathrm{d}z$ where γ is the line segment joining $i+1$ to $-i$.

Let the parametrization be $\gamma(t) = (1-t) + i(1-2t)$ where $t \in [0,1]$, then

$$\begin{aligned}
\int_\gamma \sin 2z \ \mathrm{d}z &= \int_0^1 [\sin 2((1-t) + i(1-2t))](-1-2i) \ \mathrm{d}t \\
&= (-1-2i)\int_0^1 [\sin((2+2i) - t(2+4i))] \ \mathrm{d}t \\
&= \frac{-1-2i}{2+4i}(\cos(-2i) - \cos(2+2i)) \\
&= -\frac{1}{2}\left(\frac{e^2+e^{-2}}{2} - \frac{e^{2i-2}+e^{2-2i}}{2}\right)
\end{aligned}$$

Problem

Evaluate $\int_\gamma \bar{z}^2 \ \mathrm{d}z$ along two paths joining $(0,0)$ to $(1,1)$ as follows:

(a) γ is the straight line joining $(0,0)$ to $(1,1)$.

(b) γ is the broken line joining $(0,0)$ to $(1,0)$, then joining $(1,0)$ to $(1,1)$.

(a) We know that $\bar{z} = \mathrm{Re}\,(z) - \mathrm{Im}\,(z)$, consider the parametrization $\gamma : [0,1] \to \mathbb{C}$ be $\gamma(t) = t + it$,

$$\begin{aligned}
\int_\gamma \bar{z}^2 \ \mathrm{d}z &= \int_0^1 (t - it)^2(1+i) \ \mathrm{d}t \\
&= (1+i)(1-i)^2 \int_0^1 t^2 \ \mathrm{d}t
\end{aligned}$$

$$= \frac{2-2i}{3}$$

(b) Similarly, let $\gamma_1 : [0,1] \to \mathbb{C}$ be $\gamma_1(t) = t$ and $\gamma_2 : [0,1] \to \mathbb{C}$ be $\gamma_2(t) = 1 + it$.

$$\begin{aligned}
\int_\gamma \bar{z}^2 \, \mathrm{d}z &= \int_{\gamma_1} \bar{z}^2 \, \mathrm{d}z + \int_{\gamma_2} \bar{z}^2 \, \mathrm{d}z \\
&= \int_0^1 t^2 \, \mathrm{d}t + \int_0^1 (1-it)^2 \, \mathrm{d}t \\
&= \int_0^1 (1-2it) \, \mathrm{d}t \\
&= 1 - i
\end{aligned}$$

Problem

Prove that

(a) $\left| \int_C \frac{\mathrm{d}z}{1+z^2} \right| \leq \frac{\pi}{3}$ where C is the arc of the circle $|z| = 2$ in the first quadrant.

(b) $\left| \int_\gamma \frac{\sin z}{z^2} \mathrm{d}z \right| \leq 2\pi e$ where γ is the unit circle.

(a) Consider the parametrization $C : [0, \pi/2] \to \mathbb{C}$ defined by $C(t) = 2(\cos t + i \sin t)$.

Since we have $\left| \frac{1}{1+z^2} \right| = \frac{1}{\sqrt{17 + 8\cos 2t}} \leq \frac{1}{3}$,

$$\begin{aligned}
\left| \int_C \frac{\mathrm{d}z}{1+z^2} \right| &\leq \int_C \left| \frac{\mathrm{d}z}{1+z^2} \right| \\
&\leq \frac{1}{3} \cdot 2 \cdot \frac{\pi}{2} \\
&= \frac{\pi}{3}
\end{aligned}$$

(b) Consider the paremetrization $\gamma : [0, 2\pi] \to \mathbb{C}$ defined by $\gamma(t) = \cos t + i \sin t$.

Since

$$\begin{aligned}
\left| \frac{\sin z}{z} \right| &= \left| \frac{e^{-\sin t + i\cos t} - e^{\sin t - i \cos t}}{2i(\cos 2t + i \sin 2t)} \right| \\
&= \frac{1}{2} \sqrt{\left(e^{-\sin t} - e^{\sin t}\right)^2 (\cos\cos t)^2 + \left(e^{-\sin t} + e^{\sin t}\right)^2 (\sin \cos t)^2} \\
&= \frac{1}{2} \sqrt{e^{-2\sin t} + e^{2\sin t} - 2\cos(2\cos t)} \\
&\leq \frac{1}{2} \left(e^{\sin t} + e^{-sint}\right)
\end{aligned}$$

$$
\begin{aligned}
&\le \frac{1}{2}(e+e) \\
&\le e
\end{aligned}
$$

On the other hand, since $l(\gamma) = 2\pi$,

$$
\begin{aligned}
\left| \int_\gamma \frac{\sin z}{z^2} \mathrm{d}z \right| &\le \int_\gamma \left| \frac{\sin z}{z^2} \right| \mathrm{d}z \\
&\le e \cdot 2\pi
\end{aligned}
$$

Problem

Show that the arc length $l(\gamma)$ of a curve γ is unchanged if γ is reparametrized.

Consider the reparametrization $\tilde{\gamma} : [\tilde{a}, \tilde{b}] \to \mathbb{C}$ defined by $\gamma(t) = \tilde{\gamma}(\alpha(t))$, $\alpha(a) = \tilde{a}$, $\alpha(b) = \tilde{b}$ and $\alpha'(t) > 0$ for all $t \in (a, b)$.

Then

$$
\begin{aligned}
l(\gamma) &= \int_a^b |\gamma'(t)| \mathrm{d}t \\
&= \int_a^b |\tilde{\gamma}(\alpha(t))| \alpha'(t) \mathrm{d}t \\
&= \int_{\tilde{a}}^{\tilde{b}} |\tilde{\gamma}(t)| \mathrm{d}t \\
&= l(\tilde{\gamma})
\end{aligned}
$$

2.2 Cauchy's Theorem

Cauchy's Theorem states that if γ is a closed curve intersecting itself **only at its endpoints**, then

$$\int_\gamma f = 0$$

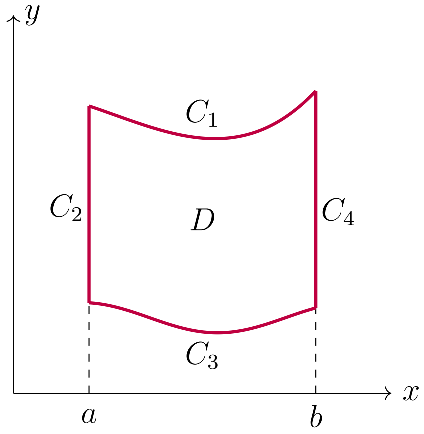
Green's Theorem: For continuously differentiable functions $P(x, y)$ and $Q(x, y)$, we have

$$\oint_\gamma P(x, y) \mathrm{d}x + Q(x, y) \mathrm{d}y = \iint_A \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \mathrm{d}A$$

> **Remark.** Green's Theorem is a special case of the **Kelvin-Stokes Theorem** (or sometimes known as the Fundamental Theorem of Curls), stated that for some smooth oriented surface Σ in $\mathbb{R}^3$ with boundary $\partial\Sigma$,
>
> $$\iint_\Sigma \left(\left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z} \right) \mathrm{d}y\, \mathrm{d}z + \left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x} \right) \mathrm{d}z\, \mathrm{d}x + \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right) \mathrm{d}x\, \mathrm{d}y \right)$$

$$= \oint_{\partial\Sigma} \Big(F_x \,\mathrm{d}x + F_y \,\mathrm{d}y + F_z \,\mathrm{d}z \Big).$$

Sketch of proof:



By integrating along C_1, we have

$$\int_{C_1} P(x,y)\mathrm{d}x = \int_b^a P(x, g_1(x))\mathrm{d}x$$

Similarly,

$$\int_{C_3} P(x,y)\mathrm{d}x = \int_a^b P(x, g_3(x))\mathrm{d}x$$

On the other hand, $\displaystyle\int_{C_2} P(x,y)\mathrm{d}x = \int_{C_4} P(x,y)\mathrm{d}x = 0$. As a result we have

$$\begin{aligned}
\int_{C_1+C_2+C_3+C_4} P(x,y)\mathrm{d}x &= \int_{C_1} P(x,y)\mathrm{d}x + \int_{C_2} P(x,y)\mathrm{d}x + \int_{C_3} P(x,y)\mathrm{d}x + \int_{C_4} P(x,y)\mathrm{d}x \\
&= \int_a^b P(x, g_3(x))\mathrm{d}x - \int_a^b P(x, g_1(x))\mathrm{d}x \\
&= \int_a^b [P(x, g_3(x)) - P(x, g_1(x))]\mathrm{d}x \\
&= \int_a^b \int_{g_1(x)}^{g_3(x)} \frac{\partial P}{\partial y}\mathrm{d}y\mathrm{d}x \\
&= -\iint_D \frac{\partial P}{\partial y}\mathrm{d}A
\end{aligned}$$

Similarly one can get

$$\int_C Q(x,y)\mathrm{d}y = \iint_D \frac{\partial Q}{\partial x}\mathrm{d}A$$

Yet, summing up gives our result

$$\oint_\gamma P(x,y)\mathrm{d}x + Q(x,y)\mathrm{d}y = \iint_A \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \mathrm{d}A$$

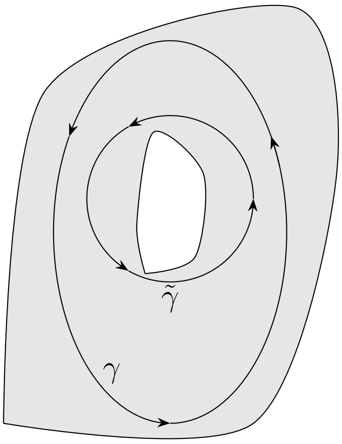
We can express our complex integral

$$\int_\gamma f = \iint_A \left[-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right] \mathrm{d}y\mathrm{d}x + i \iint_A \left[\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} \right] \mathrm{d}y\mathrm{d}x$$

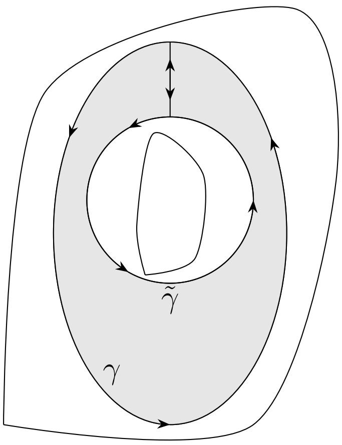
Deformation Theorem: Let f be holomorphic on a region A and let γ be a simple closed curve in A. We assume that we can γ can be deformed to another simple closed curve $\tilde{\gamma}$ without passing outside A.

We say γ is **homotopic** to $\tilde{\gamma}$ in A. Then we have

$$\int_\gamma f = \int_{\tilde{\gamma}} f$$

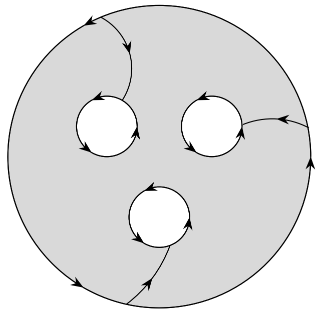


This is due to the fact that, we can construct some curve γ_0 and $-\gamma_0$, thus the curve $\displaystyle\int_{\gamma+\gamma_0-\tilde{\gamma}+(-\gamma_0)} f = 0$.



In fact, let γ be a simple closed curve with f holomorphic between γ and $\gamma_1, \gamma_2, \ldots, \gamma_n$, then

$$\int_\gamma f = \sum_{i=1}^{n} \int_{\gamma_i} f$$



Similarly we can "build bridges" between each closed curve.

Problem

Evaluate

(a) $\displaystyle\int_\gamma (z^3+3)\mathrm{d}z$, where γ is the unit circle.

(b) $\displaystyle\int_\gamma \cos[3+1/(z-3)]\mathrm{d}z$, where γ is a circle of radius 3 centered at $5i+1$.

(c) $\displaystyle\int_\gamma \sqrt{z^2-1}\mathrm{d}z$ where γ is the circle of radius $1/2$ centered at 0.

(d) $\displaystyle\int_\gamma \frac{2z^2-15z+30}{z^3-10z^2+32z-32}\mathrm{d}z$ where γ is the circle $|z|=3$.

(a) Since z^3+3 is holomorphic everywhere in γ, we have

$$\int_\gamma (z^3+3)\mathrm{d}z = 0$$

(b) Similarly we have

$$\int_\gamma \cos[3+1/(z-3)]\mathrm{d}z = 0$$

(c) Consider the parametrization $\gamma(t) = \dfrac{1}{2}e^{i\theta}$ for $\theta \in [0, 2\pi]$. We have

$$\begin{aligned}
\int_\gamma \sqrt{z^2-1}\mathrm{d}z &= \int_0^{2\pi} \sqrt{\frac{1}{4}e^{2i\theta}-1}\cdot\frac{1}{2}e^{i\theta}\mathrm{d}\theta \\
&= \int_0^{\pi} \sqrt{\frac{1}{4}e^{2i\theta}-1}\cdot\frac{1}{2}e^{i\theta}\mathrm{d}\theta + \int_\pi^{2\pi} \sqrt{\frac{1}{4}e^{2i\theta}-1}\cdot\frac{1}{2}e^{i\theta}\mathrm{d}\theta \\
&= \int_0^{\pi} \sqrt{\frac{1}{4}e^{2i\theta}-1}\cdot\frac{1}{2}e^{i\theta}\mathrm{d}\theta - \int_0^{\pi} \sqrt{\frac{1}{4}e^{2i\theta}-1}\cdot\frac{1}{2}e^{i\theta}\mathrm{d}\theta \\
&= 0
\end{aligned}$$

(d) By using partial fractions gives

$$\int_{\gamma} \frac{2z^2 - 15z + 30}{z^3 - 10z^2 + 32z - 32} \mathrm{d}z = \int_{\gamma} \left(\frac{2}{z-2} + \frac{1}{(z-4)^2} \right) \mathrm{d}z$$

We have $\int_{\gamma} \frac{1}{(z-4)^2} \mathrm{d}z = 0$, and $\int_{\gamma} \frac{1}{z-2} \mathrm{d}z = \int_{|z-2|=1} \frac{1}{z-2} \mathrm{d}z = 2\pi i$, we obtain

$$\int_{\gamma} \left(\frac{2}{z-2} + \frac{1}{(z-4)^2} \right) \mathrm{d}z = 4\pi i$$

Problem

Let f be entire. Evaluate

$$\int_0^{2\pi} f(z_0 + re^{i\theta}) e^{ki\theta} \mathrm{d}\theta$$

for k an integer, $k \geq 1$.

Let $\gamma(\theta) = z_0 + re^{i\theta}$ with $\theta \in [0, 2\pi]$, then since f is entire, f is holomorphic everywhere in $\mathbb{C}$. So $f(\gamma(\theta))e^{ki\theta}$ is holomorphic everywhere in γ. By applying Cauchy's Theorem, we have

$$\int_0^{2\pi} f(z_0 + re^{i\theta}) e^{ki\theta} \mathrm{d}\theta = 0$$

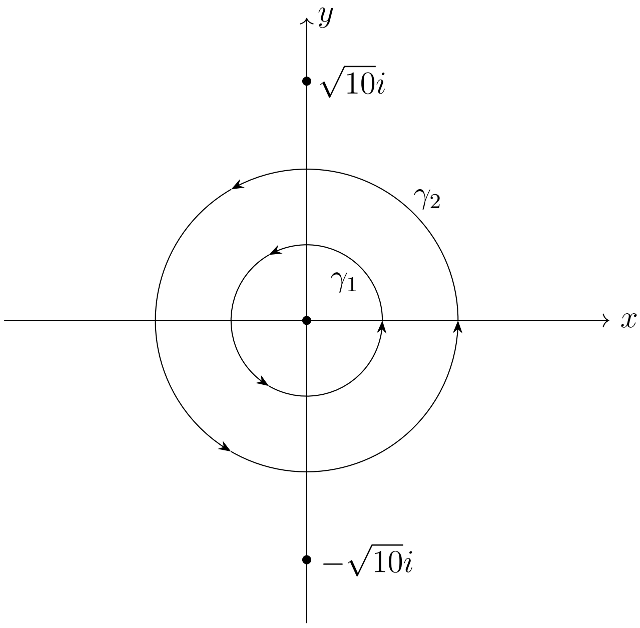
Problem

Let γ_1 be the circle of the radius 1 and let γ_2 be the circle of radius 2 (traversed counterclockwise and centered at the origin). Show that

$$\int_{\gamma_1} \frac{\mathrm{d}z}{z^3(z^2+10)} = \int_{\gamma_2} \frac{\mathrm{d}z}{z^3(z^2+10)}$$

Let $f(z) = \frac{1}{z^3(z^2+10)}$, and γ' is a curve connecting γ_1 and γ_2, then we have

$$\int_{\gamma_1 + \gamma' - \gamma_2 + (-\gamma')} f = 0 \Longleftrightarrow \int_{\gamma_1} f = \int_{\gamma_2} f$$



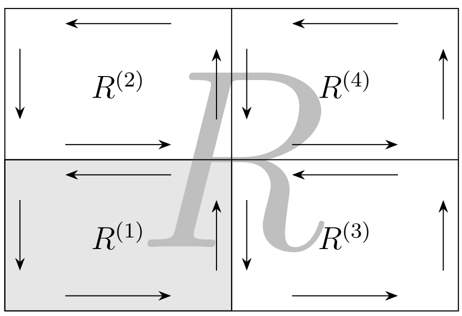
2.3 More on Cauchy's Theorem

- **For rectangles**: Let R be a rectangular path with sides parallel to the axes and f is a function defined and holomorphic on an open set G containing R and its interior, then $\int_R f = 0$.

  Sketch of proof: We must be aware that we do not know if f' is continuous. In fact, we will use Cauchy's Theorem to prove that f' is **automatically continuous**.

  Let R be a rectangle in $G \subset \mathbb{C}$, and P, Δ be the perimeter and diagonal length of R respectively. Divide R into $R^{(1)}, R^{(2)}, R^{(3)}, R^{(4)}$. We have

$$\int_R f = \int_{R^{(1)}} f + \int_{R^{(2)}} f + \int_{R^{(3)}} f + \int_{R^{(4)}} f$$



  Applying inequality gives

$$\left|\int_R f\right| = \left|\int_{R^{(1)}} f\right| + \left|\int_{R^{(2)}} f\right| + \left|\int_{R^{(3)}} f\right| + \left|\int_{R^{(4)}} f\right|$$

For each $R^{(i)}$, WLOG let $R^{(1)}$ be the subrectangle with $\left|\int_{R^{(1)}} f\right| \geq \frac{1}{4}\left|\int_R f\right|$, let R_1 be this subrectangle. We construct $R_2, R_3, R_4, \ldots$ similarly. Therefore, we have

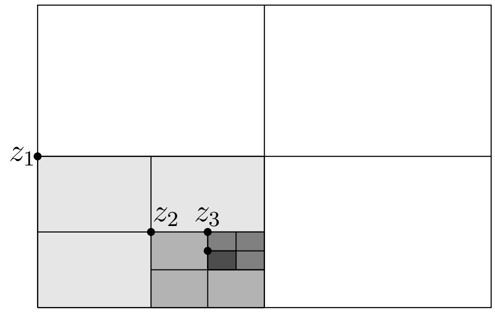
$$\left|\int_{R_n} f\right| \geq \frac{1}{4^n}\left|\int_R f\right|$$

On the other hand, we have

(i) Perimeter of R_n, $P(R_n) = P/2^n$.

(ii) Diagonal of R_n, $\Delta(R_n) = \Delta/2^n$.

Let z_n be the upper left vertex of R_n, then we see that whenever $m > n$, we have $|z_m - z_n| \leq P(R_n) = \Delta/2^n$.



Thus $\{z_n\}$ forms a Cauchy sequence that must converge to some point w_0. Let z be a point in R_n, then we have $|z - w_0| \leq \Delta(R_n)$.

For some z in R_n, fix ε so that

$$\left|\frac{f(z) - f(w_0)}{z - w_0} - f'(w_0)\right| < \varepsilon$$

We choose δ and large enough n such that $|z - w_0| \leq \Delta(R_n) < \delta$, we have

$$|f(z) - f(w_0) - (z - w_0)f'(w_0)| < \varepsilon|z - w_0| \leq \varepsilon\Delta(R_n)$$

And since we have $\int_{R_n} 1\mathrm{d}z = \int_{R_n} (z - w_0)\mathrm{d}z = 0$,

$$\begin{aligned}
\left|\int_R f\right| &\leq 4^n \left|\int_{R_n} f\right| \\
&= 4^n \left|\int_{R_n} f(z)\mathrm{d}z - f(w_0)\int_{R_n} 1\mathrm{d}z - f'(w_0)\int_{R_n} (z - w_0)\mathrm{d}z\right| \\
&\leq 4^n \int_{R_n} |f(z) - f(w_0) - (z - w_0)f'(w_0)|\,\mathrm{d}z \\
&\leq 4^n(\varepsilon\Delta(R_n))P(R_n) \\
&= \varepsilon\Delta P
\end{aligned}$$

Since this is true for all ε, choose ε small, we have $|\int_R f| = 0$ which gives $\int_R f = 0$.

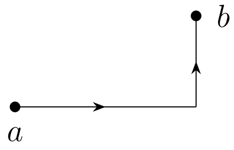
- **For disks**: Suppose $f : D \to \mathbb{C}$ is holomorphic on an (open) disk $D =$

$D(z_0, \rho) \subset \mathbb{C}$, then

(i) There exists a function $f : D \to \mathbb{C}$, the **antiderivative** of f on D, satisfying $F'(z) = f(z)$ for all z in D.

(ii) For any closed curve Γ in D, $\int_\Gamma f = 0$.

We let $\langle\!\langle a, b \rangle\!\rangle$ denote the **polygonal path** from a to b, first parallel to the x axis then parallel to the y axis.

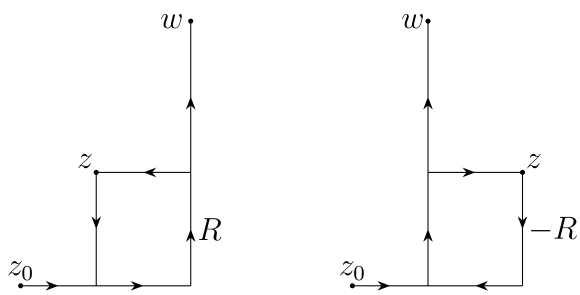


Sketch of Proof: We may want to show that there exists a function $F(z)$ such that $F'(z) = f(z)$ for all $z \in D(z_0; \rho)$.

Fix z and $\varepsilon > 0$ so that we can choose some $\delta > 0$ and whenever $\xi \in D(z; \delta)$ we have $|f(z) - f(\xi)| < \varepsilon$.

In any case, we have

$$\int_{\langle\!\langle z_0, z \rangle\!\rangle} f(\xi)\mathrm{d}\xi \pm \int_R f(\xi)\mathrm{d}\xi + \int_{\langle\!\langle z, w \rangle\!\rangle} f(\xi)\mathrm{d}\xi = \int_{\langle\!\langle z_0, w \rangle\!\rangle} f(\xi)\mathrm{d}\xi$$



Thus we have

$$F(z) + \int_{\langle\!\langle z, w \rangle\!\rangle} f(\xi)\mathrm{d}\xi = F(w)$$

For any ε,

$$\begin{aligned}
\left|\frac{F(w) - F(z)}{w - z} - f(z)\right| &= \frac{1}{|w - z|}\left|\int_{\langle\!\langle z, w \rangle\!\rangle} f(\xi)\mathrm{d}\xi - (w - z)f(z)\right| \\
&= \frac{1}{|w - z|}\left|\int_{\langle\!\langle z, w \rangle\!\rangle} [f(\xi) - f(z)]\mathrm{d}\xi\right| \\
&\leq \frac{1}{|w - z|}\int_{\langle\!\langle z, w \rangle\!\rangle} |f(\xi) - f(z)|\mathrm{d}\xi \\
&\leq \frac{1}{|w - z|}\varepsilon \cdot 2|w - z| = 2\varepsilon
\end{aligned}$$

Choose ε small, we have

$$\lim_{w \to z} \frac{F(w) - F(z)}{w - z} = f(z)$$

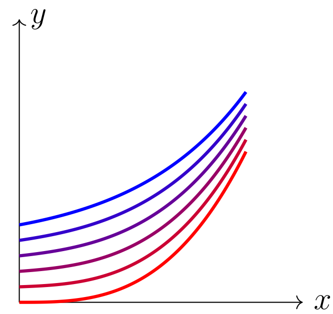
This proves (i) of the theorem, followed by $\int_\gamma f = 0$ by the Path Independence Theorem.

- **Homotopy**: Let $f, g : \mathbb{R} \to \mathbb{R}^2$ be two functions, the map $H(x,t)$ with $H : \mathbb{R} \times [0,1] \to \mathbb{R}^2$ is a **homotopy**.

  Two functions f and g are said to be **homotopic** $f \simeq g$ if there exists a homotopy H that maps f to g.
  Say $f(x) := \left(x, \frac{1}{2}e^{x-1} + \frac{1}{2}\right)$ and $g(x) := \left(x, \frac{1}{6}x^3\right)$, then

  $$H(x,t) = \left(x, (1-t)\left(\frac{1}{6}x^3\right) + t\left(\frac{1}{2}e^{x-1} + \frac{1}{2}\right)\right)$$

  is a homotopy.

  In general, for functions f and g, the function

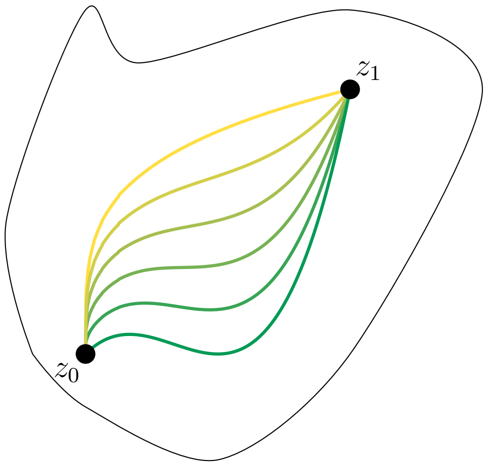
  $$H(x,t) = (1-t)f(x) + tg(x)$$

  is a homotopy.

  Now let $\gamma_1 : [0,1] \to G$ and $\gamma_2 : [0,1] \to G$ be two continuous curves with the same endpoints, i.e. from z_0 to z_1. Thus a **homotopy with fixed endpoints** $H : [0,1] \times [0,1] \to G$ can be defined as
  - $H(0,t) = \gamma_1(t)$ for all $0 \leq t \leq 1$.
  - $H(1,t) = \gamma_2(t)$ for all $0 \leq t \leq 1$.
  - $H(s,0) = z_0$ for all $0 \leq s \leq 1$.
  - $H(s,1) = z_1$ for all $0 \leq s \leq 1$.

  For example, consider $\gamma_1(t) = t + i\sqrt[4]{t}$ and $\gamma_2(t) = t + i(4t^3 - 4t^2 + t)$, we have

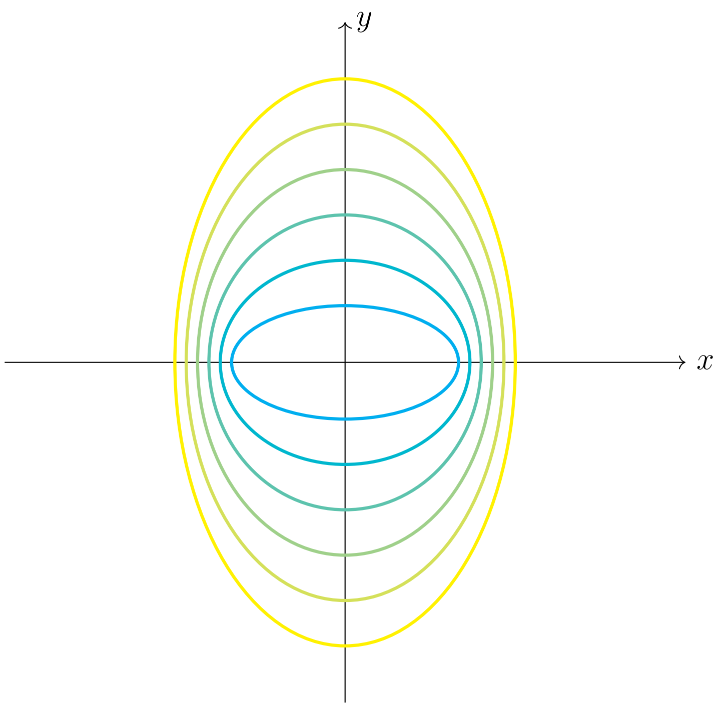
  $$H(s,t) = (1-s)(t + i\sqrt[4]{t}) + s(t + i(4t^3 - 4t^2 + t))$$



Moreover, if $z_0 = z_1$, they are **homotopic as closed curves**. For instance, consider the curves $\gamma_1(t) = 2\cos t + i \sin t$ and $\gamma_2(t) = 3\cos t + 5i \sin t$. Then the function

$$H(s,t) = (2+s)\cos t + (1+4s)i \sin t$$

is a homotopy.

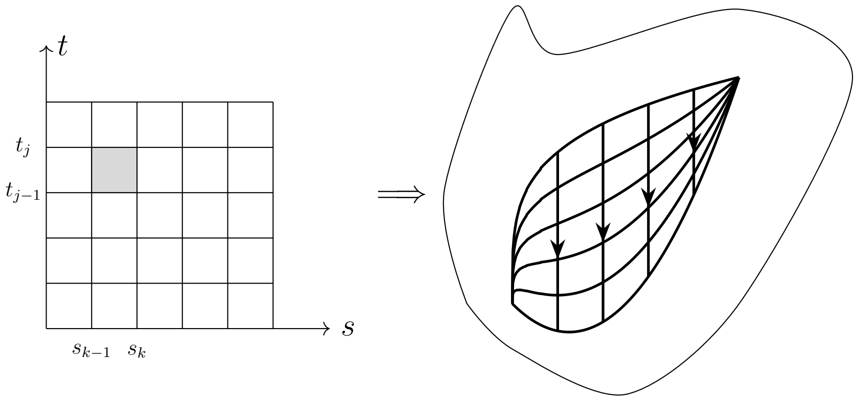


- A connected set G is called **simply connected** if every closed curve γ in G is homotopic as a closed curve to some constant curve in G.
- A set G is called **convex** if it contains the straight line segment between every pair of points in G.

  A convex region is simply connected.
- A homotopy $H : [0,1] \times [0,1] \to G$ is **smooth** if the **intermediate curves** $\gamma_s(t)$ for each s and the **cross curves** $\lambda_t(s)$ for each t are piecewise C^1 curves.

- **Deformation Theorem**: Suppose that f is an holomorphic function on **open set** G and γ_1 and γ_2 are piecewise C^1 curves in G, then
  (a) If γ_1 and γ_2 are paths with fixed endpoints in G, then
  $$\int_{\gamma_1} f = \int_{\gamma_2} f$$
  (b) If γ_1 and γ_2 are homotopic as closed curves in G, then
  $$\int_{\gamma_1} f = \int_{\gamma_2} f$$

  Sketch of Proof: First we will prove (a). partition the square into n^2 regions with $0 = s_0 < s_1 < s_2 < \cdots < s_n = 1$ and $0 = t_0 < t_1 < t_2 < \cdots < t_n = 1$



  Now since $\mathbb{C}\backslash G$ is closed. On the other hand, the image of H on the interval $[s_k, s_{k+1}] \times [t_j, t_{j+1}]$ is compact on G. By the Distance Lemma, this image stays some positive distance from $\mathbb{C}\backslash G$. In other words. whenever $z \in G$ we have $|H(s,t) - z| < \rho$ for some positive value ρ.

  But we know that a continuous function on a compact set is uniformly continuous, therefore there exists a number δ such that whenever the distance between (s,t) and some point (s', t') is less than δ we have $|H(s,t) - H(s',t')| < \rho$.

  Consider partitioning the intervals s and t into n equal parts. Choose n large so that $\delta > \sqrt{2}/n$, the diagonal of the square. Thus the subsquare R_{kj} is mapped to some region in G, which is in some disk $D(H(s_k, t_j))$ in G. Let the closed curve Γ_{kj}.

  Therefore, by summing up all curves. we have
  $$\sum_{j=0}^{n-1}\sum_{i=0}^{n-1}\int_{\Gamma_{kj}} f = \int_{\lambda_0} f + \int_{\gamma_2} f - \int_{\lambda_1} f - \int_{\gamma_1} = 0$$
  By the Cauchy's Theorem for a disk, the sum of the integrals are 0.

  (a) If γ_1 and γ_2 have fixed endpoints, then $\int_{\lambda_0} f = \int_{\lambda_1} f = 0$ since they are constant curves.

(b) If γ_1 and γ_2 are closed curves, we have $\int_{\lambda_0} f = \int_{\lambda_1} f$. In either case, we have

$$\int_{\gamma_1} f = \int_{\gamma_2} f$$

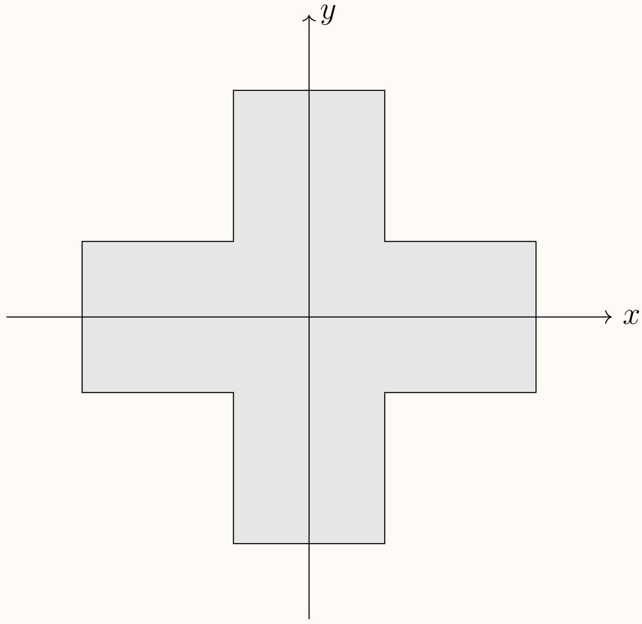
Problem

A region A is called **star-shaped with respect to z_0** if it contains the line segment between each of its points and z_0, that is, if $z \in A$ and $0 \leq s \leq 1$ imply that $sz_0 + (1-s)z \in A$. The region is called *star-shaped* if there is at least one such point in A.

(a) Show that a star-shaped set is simply connected.

(b) Show that a set A is convex if and only if it is star-shaped with respect to each of its points.

(c) Let G be the region built as a union of two rectangular regions $G = \{z$ such that $|\mathrm{Re}\,(z)\,| < 1$ and $|\mathrm{Im}\,(z)\,| < 3\} \cup \{z$ such that $|\mathrm{Re}\,(z)\,| < 3$ and $|\mathrm{Im}\,(z)\,| < 1\}$. Show that G is star-shaped.



(a) Let z_0 be such point that makes A star-shaped, as defined. I claim that every closed curve in A is homotopic to this point.

By the definition, let C be a closed curve, then for every $z \in C$, and for every $s \in [0, 1]$, we have

$$sz_0 + (1-s)z \in A$$

Traversing all $z \in C$, we have

$$sz_0 + (1-s)C \in A$$

which is a homotopy in A.

(b) ($\Rightarrow$) If A is convex, then by definition every $z_0, z_1 \in A$, we have $sz_0+(1-s)z_1 \in A$ for all $s \in [0,1]$. Fix z_0 so that for every $z \in A$, $sz_0+(1-s)z \in A$ as desired.

($\Leftarrow$) If A is star shaped with respect to each of its points, then for every point $z_0 \in A$, and for every point $z \in A$, $s \in [0,1]$, we have $sz_0 + (1-s)z \in A$.

(c) Now, we want to show that the region G defined is star-shaped. Consider the origin O, I claim that every point $z \in G$, and for every $s \in [0,1]$, the point sz is in G.

Let $z = x + iy$. By symmetry, consider the region $0 \le x < 3$ and $0 \le y < 1$. Since we have $sx \le x$ and $sy \le y$, therefore $0 < \le sx < 3$ and $0 \le sy < 3$, so the point sz is indeed in G.

Problem

Evaluate the following:

(a) $\displaystyle\int_{|z|=\frac{1}{2}} \frac{\mathrm{d}z}{(1-z)^3}$

(b) $\displaystyle\int_{|z-1|=\frac{1}{2}} \frac{\mathrm{d}z}{(1-z)^3}$

(c) $\displaystyle\int_{|z+1|=\frac{1}{2}} \frac{\mathrm{d}z}{(1-z)^3}$

(a) Note that the function $1/(1-z)^3$ is holomorphic everywhere except at $z = 1$, thence

$$\int_{|z|=\frac{1}{2}} \frac{\mathrm{d}z}{(1-z)^3} = 0$$

(b) This is the case when $z = 1$ is in the curve, we have

$$\int_{|z-1|=\frac{1}{2}} \frac{\mathrm{d}z}{(1-z)^3} = \int_0^{2\pi} \frac{1}{\left(1-\left(\frac{1}{2}e^{i\theta}+1\right)\right)^3} \cdot \frac{i}{2}e^{i\theta}\mathrm{d}z = 0$$

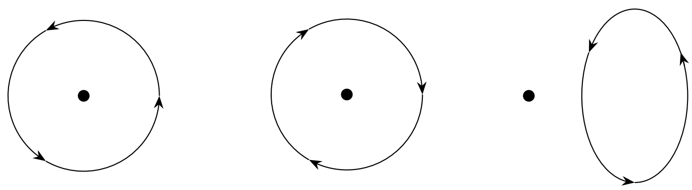
(c) Similarly, we have

$$\int_{|z+1|=\frac{1}{2}} \frac{\mathrm{d}z}{(1-z)^3} = 0$$

2.4 Caucheeeee's Integral Formula

- **Winding Number**: The winding number of a curve γ (or known as the **index** is defined by

$$I(\gamma; z_0) = \frac{1}{2\pi i}\int_\gamma \frac{\mathrm{d}z}{z-z_0}$$

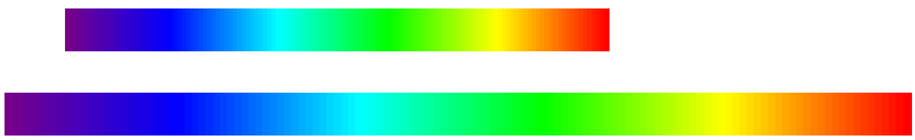


$I = 1$ $\quad I = -1$ $\quad I = 0$

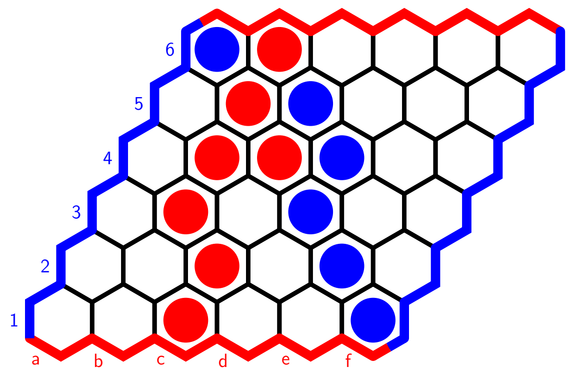
- The **Jordan Curve Theorem** states that a closed curve divides the plane **uniquely** into three regions – the curve itself, the interior (bounded) and the exterior.

  An elegant proof is dedicated to the **Brouwer Fixed Point Theorem**, stating that for every continuous function f, mapping a **compact convex set** to itself, then there exists some c such that $f(c) = c$, a **fixed point** of f.

  A visualization is of this theorem is to stretch an elastic band long enough such that it covers the original band entirely. Then there exists a point on the band for which its position (coordinates) does not change.



  This turns out that, proving that a game of Hex cannot end in a draw (**Hex Theorem**) is related to the Fixed Point Theorem.



- Now we return to the winding number. Given two homotopic curves γ and $\tilde{\gamma}$, we have
$$I(\gamma; z_0) = I(\tilde{\gamma}; z_0)$$

- We would like to show that the integral
$$\frac{1}{2\pi i} \int_\gamma \frac{\mathrm{d}z}{z - z_0}$$
  is an integer when γ is a closed curve.

Sketch of Proof: Consider the function

$$g(t) = \int_a^t \frac{\gamma'(s)}{\gamma(s) - z_0} \mathrm{d}s$$

so that $g(b) = I$.

Thus we have

$$\begin{aligned} g'(t) &= \frac{\gamma'(t)}{\gamma(t) - z_0} \\ \gamma'(t) - g'(t)\gamma(t) &= -z_0 g'(t) \\ \frac{\mathrm{d}}{\mathrm{d}t}\left(e^{-g(t)}\gamma(t)\right) &= \frac{\mathrm{d}}{\mathrm{d}t} z_0 e^{-g(t)} \\ e^{-g(t)}\gamma(t) &= z_0 e^{-g(t)} + \gamma(a) - z_0 \end{aligned}$$

By the fact that $\gamma(b) = \gamma(a)$, letting $t = b$, gives

$$e^{-g(b)} = 1$$

which implies $g(b) = 2n\pi i$, the result then follows.

- **Cauchy's Integral Formula**: Let f be a holomorphic function on A and let γ be a closed curve homotopic to some point in A. Let z_0 be a point not on γ, then

$$f(z_0)I(\gamma; z_0) = \frac{1}{2\pi i} \oint_\gamma \frac{f(z)}{z - z_0} \mathrm{d}z$$

Sketch of Proof: Consider the function

$$g(z) = \begin{cases} \dfrac{f(z) - f(z_0)}{z - z_0} & \text{if } z \neq z_0 \\ f'(z_0) & \text{if } z = z_0 \end{cases}$$

Therefore g is continuous at z_0 and holomorphic except perhaps at z_0. Thus by using Cauchy's theorem,

$$\begin{aligned} 0 &= \int_\gamma g(z) \mathrm{d}z \\ &= \int_\gamma \frac{f(z)}{z - z_0} \mathrm{d}z - f(z_0) \int_\gamma \frac{1}{z - z_0} \mathrm{d}z \\ &= \int_\gamma \frac{f(z)}{z - z_0} \mathrm{d}z - 2\pi i f(z_0) I(\gamma; z_0) \end{aligned}$$

- **Cauchy-Type Integrals**: Rewrite Cauchy's integral as

$$f(z)I(\gamma; z) = \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{\zeta - z} \mathrm{d}\zeta = G(z)$$

Then by differentiating both sides with respect to z,

$$G^{(k)}(z) = \frac{k!}{2\pi i}\int_\gamma \frac{f(\zeta)}{(\zeta - z)^{k+1}}\mathrm{d}\zeta$$

Since $I(\gamma; z)$ is constant except when z crosses the curve (which is not the case here), thus

$$f^{(k)}(z_0)I(\gamma; z_0) = \frac{k!}{2\pi i}\int_\gamma \frac{f(\zeta)}{(\zeta - z_0)^{k+1}}\mathrm{d}\zeta$$

Example

Now we want to witness how powerful the Cauchy Integral is. Consider the integral

$$\int_0^{10\pi} \frac{4e^{4i\theta} + 3e^{2i\theta}}{(4e^{2i\theta} - 1)^2}\mathrm{d}\theta$$

This can be interpreted as the parametrization $z = 2e^{i\theta}$ and integrated along the circle with radius 2 and centered at the origin 5 times. We can rewrite the integral as

$$-\frac{i}{8}\int_\gamma \frac{z^3 + 3z}{(z^2 - 1)^2}\mathrm{d}z = -\frac{i}{8}\left(\int_\gamma \frac{z-1}{(z+1)^2}\mathrm{d}z + \int_\gamma \frac{z+1}{(z-1)^2}\mathrm{d}z\right)$$

By using the formulas, we have

$$(1)(5) = \frac{1!}{2\pi i}\int_\gamma \frac{z-1}{(z+1)^2}\mathrm{d}z$$

$$(1)(5) = \frac{1!}{2\pi i}\int_\gamma \frac{z+1}{(z-1)^2}\mathrm{d}z$$

Substituting back into our integral we have

$$-\frac{i}{8}(10\pi i + 10\pi i) = 2.5\pi$$

THIS IS QUICKER THAN LIGHT!!!

- **Cauchy's Inequality**: Let f be holomorphic on A and γ be a circle with radius R and center z_0. Suppose that both γ and the disk $|z - z_0| < R$ also lies in A. If for all z on γ we have $|f(z)| \leq M$, then for natural numbers k, we have

$$|f^{(k)}(z_0)| \leq \frac{k!}{R^k}M$$

Sketch of Proof:

$$|f^{(k)}(z_0)| = \frac{k!}{2\pi}\left|\int_\gamma \frac{f(\zeta)}{(\zeta - z_0)^{k+1}}\mathrm{d}\zeta\right| \leq \frac{k!}{2\pi}\cdot\frac{M}{R^{k+1}}\cdot l(\gamma) = \frac{k!}{R^k}M$$

- **Liouville's Theorem**: If f is holomorphic everywhere and there is a constant M such that for all $z \in \mathbb{C}$, $|f(z)| \leq M$, then f is constant.

  This is the special case of Cauchy's inequality with $k = 1$ and $R \to \infty$.

- **Morera's Theorem**: Let f be continuous in A, and for any closed curve γ in A, $\int_\gamma f = 0$, then f is holomorphic in A, and there exists some holomorphic function F in A such that $f = F'$.

Problem

Let f be holomorphic on a region A and let γ be a closed curve in A. For any $z_0 \in A$ not on γ, show that

$$\int_\gamma \frac{f'(\zeta)}{\zeta - z_0} \mathrm{d}\zeta = \int_\gamma \frac{f(\zeta)}{(\zeta - z_0)^2} \mathrm{d}\zeta$$

By

$$f'(z_0) = \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{(\zeta - z_0)^2} \mathrm{d}\zeta$$

$$f'(z_0) = \frac{1}{2\pi i} \int_\gamma \frac{f'(\zeta)}{\zeta - z_0} \mathrm{d}\zeta$$

Equating $2\pi i f'(z_0)$ gives the desired result.

> **Remark.** In general, for positive integers $k \leq m$, the integral
>
> $$k! \int_\gamma \frac{f^{(m-k)}(\zeta)}{(\zeta - z_0)^{k+1}} \mathrm{d}\zeta$$
>
> is constant and equals $2\pi i \cdot I(\gamma; z_0) f^{(m)}(z_0)$ for all k.

Problem

Prove that

$$\int_0^\pi e^{\cos\theta} \cos(\sin\theta) \mathrm{d}\theta = \pi$$

by considering

$$\int_\gamma \frac{e^z}{z} \mathrm{d}z$$

where γ is the unit circle.

We have

$$e^0 = \frac{1}{2\pi i} \int_\gamma \frac{e^z}{z} \mathrm{d}z$$

By parametrization, $z = \cos\theta + i\sin\theta$, so $e^z = e^{\cos\theta}(\cos\sin\theta + i\sin\sin\theta)$ and

$\mathrm{d}z = iz\mathrm{d}\theta$. Thus the integral becomes

$$\frac{1}{2\pi i}\int_\gamma \frac{e^z}{z}\mathrm{d}z = \frac{1}{2\pi i}\int_0^{2\pi} \frac{e^{\cos\theta}(\cos\sin\theta + i\sin\sin\theta)}{z} iz\mathrm{d}\theta$$

So we have

$$2\pi = \int_0^{2\pi} e^{\cos\theta}\cos\sin\theta\mathrm{d}\theta + i\int_0^{2\pi} e^{\cos\theta}\sin\sin\theta\mathrm{d}\theta$$

The former is a real value, hence the latter equals 0. It remains to show that

$$\pi = \int_\pi^{2\pi} e^{\cos\theta}\cos\sin\theta\mathrm{d}\theta$$

Letting $\theta' = 2\pi - \theta$ gives the desired result. $\square$

Problem

Let f be holomorphic inside and on the circle $\gamma : |z - z_0| = R$. Prove that

$$\frac{f(z_1) - f(z_2)}{z_1 - z_2} - f'(z_0) = \frac{1}{2\pi i}\int_\gamma \left[\frac{1}{(z - z_1)(z - z_2)} - \frac{1}{(z - z_0)^2}\right] f(z)\mathrm{d}z$$

for z_1, z_2 inside γ.

By the standard Cauchy's Integral,

$$-f'(z_0) = -\frac{1}{2\pi i}\int_\gamma \frac{f(z)}{(z - z_0)^2}\mathrm{d}z$$

On the other hand,

$$\begin{aligned}\frac{1}{2\pi i}\int_\gamma \frac{1}{(z - z_1)(z - z_2)}\mathrm{d}z &= \frac{1}{2\pi i}\int_\gamma \left(\frac{1}{z - z_1} - \frac{1}{z - z_2}\right)\frac{f(z)}{z_1 - z_2}\mathrm{d}z \\ &= \frac{f(z_1) - f(z_2)}{z_1 - z_2}\end{aligned}$$

as desired.

Problem

Show that:

1.
$$3\sum_{k=1}^{\infty}\left|\int_{|z-k|=1/2}\frac{1}{z^2(z-k)}\mathrm{d}z\right| = \pi^3$$

2.
$$\frac{1}{2\pi}\sum_{k=0}^{\infty}\frac{1}{k!\binom{2k}{k}}\left|\int_{|z|=\pi}\frac{z^{2k}}{(z-3)^{k+1}}\mathrm{d}z\right| = e^3$$

3.
$$-\frac{1}{2\pi i}\sum_{k=0}^{4n}k!\int_{|z|=1}\frac{\cos z}{z^{k+1}}\mathrm{d}z = i^3$$

1. From
$$\int_{|z-k|=1/2}\frac{1}{z^2(z-k)}\mathrm{d}z = \int_{|z-k|=1/2}\frac{1/z^2}{z-k}\mathrm{d}z = \frac{2\pi i}{k^2}$$
We have
$$3\sum_{k=1}^{\infty}\left|\frac{2\pi i}{k^2}\right| = \pi^3$$

2. From
$$\int_{|z|=\pi}\frac{z^{2k}}{(z-3)^{k+1}}\mathrm{d}z = \frac{2\pi i}{k!}\left(\frac{\mathrm{d}^k}{\mathrm{d}z^k}z^{2k}\bigg|_{z=3}\right) = \frac{2\pi i}{k!}\frac{(2k)!3^k}{k!} = 2\pi i\binom{2k}{k}\cdot 3^k$$
We have
$$\frac{1}{2\pi}\sum_{k=0}^{\infty}\frac{1}{k!\binom{2k}{k}}\left|2\pi i\binom{2k}{k}\cdot 3^k\right| = \sum_{k=0}^{\infty}\frac{3^k}{k!} = e^3$$

3. From
$$\int_{|z|=1}\frac{\cos z}{z^{k+1}}\mathrm{d}z = \frac{2\pi i}{k!}\left(\frac{\mathrm{d}^k}{\mathrm{d}z^k}\cos z\bigg|_{z=0}\right)$$
We have
$$-\frac{1}{2\pi i}\sum_{k=0}^{4n}k!\int_{|z|=1}\frac{\cos z}{z^{k+1}}\mathrm{d}z = -\sum_{k=0}^{4n}\frac{\mathrm{d}^k}{\mathrm{d}z^k}\cos z\bigg|_{z=0} = -1 = i^3$$

2.5 Maximum Modulus Theorem & Harmonic Functions

Maximum Modulus Theorem

- **Gauss' Mean Value Theorem**: Let γ be a circle around z_0 with radius R, then
$$f(z_0) = \frac{1}{2\pi}\int_0^{2\pi} f(z_0 + Re^{i\theta})\mathrm{d}\theta$$

A geometric interpretation of this formula is that, the value of an holomorphic function at z_0 equals the average of values around the circle.

By the concept of "Riemann Sums", consider dissecting the circumference of the circle into n equal arcs. Each arc is $2\pi/n$ radians apart. Thus we have

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} f\left(z_0 + Re^{i\frac{2k\pi}{n}}\right) = \frac{1}{2\pi}\int_0^{2\pi} f(z_0 + Re^{i\theta})\mathrm{d}\theta$$

- **Closure**: The closure of a set $\mathrm{cl}(A)$ consists A and the limit points of the convergent sequences of points in A.

  Some properties of closure include:
  - $A \subset \mathrm{cl}(A)$
  - A is closed iff $A = \mathrm{cl}(A)$
  - If $A \subset C$ for some closed C, then $\mathrm{cl}(C)$
  - $\mathrm{cl}(A)$ is closed

- Let
$$\mathrm{limit}(A) = \{w|\text{ there is a sequence in } A \text{ convergent to } w\}$$

  We have
  - $A \subset \mathrm{limit}(A)$
  - $\mathrm{cl}(A) = \mathrm{limit}(A)$
  - $\mathrm{limit}(\mathrm{limit}(A)) = \mathrm{cl}(\mathrm{cl}(A)) = \mathrm{limit}(A) = \mathrm{cl}(A)$

  Sketch of Proof: We want to prove that
  1. $\mathrm{cl}(A) = \mathrm{limit}(A)$
  2. $\mathrm{cl}(A) \subseteq \mathrm{cl}(\mathrm{cl}(A))$
  3. $\mathrm{cl}(\mathrm{cl}(A)) \subseteq \mathrm{cl}(A)$

  $\mathrm{cl}(A) = \mathrm{limit}(A)$ almost follows by definition, since $\mathrm{cl}(A) = A \cup \mathrm{limit}(A)$ and $A \subset \mathrm{limit}(A)$.

  Since if X is closed then $X \subseteq \mathrm{cl}(X)$, it follows that $\mathrm{cl}(A) \subseteq \mathrm{cl}(\mathrm{cl}(A))$.

  Now let $z_1, z_2, z_3, \ldots$ be a sequence in $\mathrm{limit}(A)$ so that they converge to w, we want to show that $w \in \mathrm{limit}(A)$. Since each z_i is in $\mathrm{limit}(A)$, there exists a sequence w_i in A so that $\lim\limits_{n\to\infty} |z_n - w_n| = 0$. So $\lim\limits_{n\to\infty} w_n = w$.

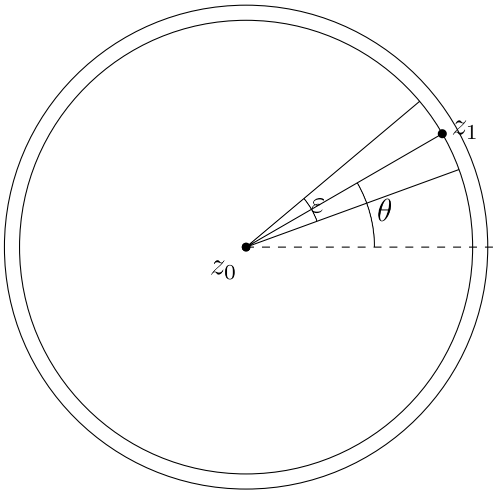
- **Boundary**: The boundary of a set $\mathrm{bd}(A)$ is defined by
$$\mathrm{bd}(A) = \mathrm{cl}(A) \cap \mathrm{cl}(\mathbb{C}\backslash A)$$

  On the other hand, we have
$$\mathrm{cl}(A) = \mathrm{bd}(A) \cup A$$

- **Local Maximum Modulus Principle**: Let f be holomorphic on A and $|f|$ has relative maximum at $z_0 \in A$, then f is constant in some neighbourhood of z_0.

  Sketch of Proof: Assume that on some disk $D_0 = D(z_0; r_0)$, $|f(z_0+r_0e^{i\theta})| \le |f(z_0)|$ for all θ. If there does not exist a strict inequality then we are done. Assume otherwise, there exists some z_1 with $|f(z_1)| < |f(z_0)|$. By assumption we have $z_1 = z_0 + re^{i\theta}$ with $r < r_0$.

  Since f is continuous, there exists $\varepsilon > 0$ and $\delta > 0$ such that whenever $|\theta - a| < \varepsilon$ we have
  $$|f(z_0 + re^{ia})| < |f(z_0)| - \delta$$

  Thus we have
  $$\begin{aligned}|f(z_0)| &= \left|\frac{1}{2\pi}\int_{-\pi}^{\pi} f(z_0+re^{ia})\mathrm{d}a\right| \\ &\le \left|\frac{1}{2\pi}\int_{-\pi}^{-\varepsilon} f(z_0+re^{ia})\mathrm{d}a\right| + \left|\frac{1}{2\pi}\int_{-\varepsilon}^{\varepsilon} f(z_0+re^{ia})\mathrm{d}a\right| + \left|\frac{1}{2\pi}\int_{\varepsilon}^{\pi} f(z_0+re^{ia})\mathrm{d}a\right| \\ &\le \frac{1}{2\pi}\int_{-\pi}^{-\varepsilon} |f(z_0)|\,\mathrm{d}a + \frac{1}{2\pi}\int_{-\varepsilon}^{\varepsilon} (|f(z_0)| - \delta)\mathrm{d}a + \frac{1}{2\pi}\int_{\varepsilon}^{\pi} |f(z_0)|\,\mathrm{d}a \\ &= |f(z_0)| - \frac{\varepsilon\delta}{\pi}\end{aligned}$$

  Hence a contradiction. This means that no point in D_0 have modulus strictly less than $|f(z_0)|$, thus f is constant on some disk D.

- **Global Maximum Modulus Principle**: Let $A \subseteq \mathbb{C}$ be an **open connected bounded** set and suppose $f : \mathrm{cl}(A) \to \mathbb{C}$ is holomorphic on A and continuous on $\mathrm{cl}(A)$, then $|f|$ has a finite maximum value on $\mathrm{cl}(A)$, attained at some point on $\mathrm{bd}(A)$. If this value is attained in the interior of A, then f is constant on $\mathrm{cl}(A)$.

  Sketch of Proof: According to the **Extreme Value Theorem**, there exists a maximum value for $|f|$, say M. Assume there exists some $a \in A$ such that $|f(a)| = M$. Consider the sets

$$A_1 = \{z \in A | f(z) = f(a)\}, \qquad A_2 = A \backslash \mathrm{cl}(A_1)$$

Assume that $z \in A$ but $z \notin A_2$, then $z \in \mathrm{cl}(A_1)$. We can choose some sequence in A_1 so it converges to z. Since each point $f(w)$ in the sequence equals $f(a)$, $f(z) = f(a)$ so $z \in A_1$. This shows that $A_1 \cup A_2 = A$ and $A_1 \cap A_2 = \varnothing$.

Since $A_2 = A \backslash \mathrm{cl}(A)$, A_2 is open. Since A is open, we can find some disk in A with its radius small enough so that it is in A_1 as well (This is the result of the local maximum modulus principle). Thus A_1 is open. But this means that A_1 and A_2 disconnect A.

We know that A_1 is nonempty, so $A_2 = \varnothing$ and $A = A_1$ as desired.

- **Schwarz Lemma**: Let f be holomorphic on the open unit disk $A : (0; 1)$ with $f(0) = 0$ and $|f(z)| \leq 1$ for all $z \in A$, then
  (a) $|f'(0)| \leq 1$ and $|f(z)| \leq |z|$ for all $z \in A$.
  (b) If $|f'(0)| = 1$ or if there is a point $z_0 \neq 0$ such that $|f(z_0)| = |z_0|$, then there is a constant c with $|c| = 1$ and $f(z) = cz$ for all $z \in A$.

  Sketch of Proof: Consider the function $g(z) = f(z)/z$, so $g(0) = f'(0)$. Define
  $$A_r = \{z |\ 0 < r < 1, |z| \leq r\}$$
  Then g is holomorphic on A_r. On $|z| = r$, we have
  $$|g(z)| \leq |f(z)/z| \leq 1/r$$
  Thus by the Maximum Modulus Principle, we have $|g(z)| \leq 1/r$ for all $|z| \leq r$. We now have
  $$|f(z)| \leq |z|/r$$
  Fix z, let $r \to 1$, we have $|f(z)| \leq |z|$ as desired. Since $|g(z)| \leq 1$ so $|f'(0)| \leq 1$.

  Assume that there exists some point z_0 so that $|f(z_0)| = |z_0|$. If $z_0 = 0$ then $|f'(0)| = |g(0)| = 1$ or in other words, $|g(z)|$ is maximized. If $z_0 \neq 0$ we still get $|g(z_0)| = 1$.

  Since this value $|g(z_0)|$ is independent of r, so by the Maximum Modulus Principle g is constant on A. We can easily check that the maximum value is c and that $|c| = 1$.

- **Phragmén–Lindelöf Principle**: Suppose that we are given a function f and an **unbounded region** S, we want to show that $|f| \leq M$ is bounded on S.

  The strategy is to introduce some multiplicative factor h_ε and a bounded subregion $S_{\mathrm{bdd}} \subset S$ with $\varepsilon > 0$ such that
  1. $|fh_\varepsilon| \leq M$ is holomorphic and bounded on the boundary $\mathrm{bd}(S_{\mathrm{bdd}})$.

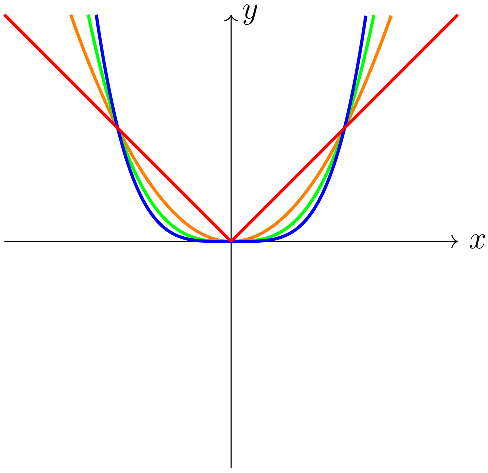
2. We can deduce $|fh_\varepsilon| \leq M$ for $z \in S \backslash \mathrm{cl}(S_{\mathrm{bdd}})$.

And that we can extend our conclusion to all $z \in S$.

2.5.1 Harmonic Functions

- **Smoothness**: A function is said to have class of at least C^k if the k'th derivative of the function exists.

  For instance, the function $|x|^{k+1}$ is C^k. The graphs below are when $k = 0, 1, 2, 3$.

- **Antiderivative Theorem**: Let f be defined and holomorphic on a simply connected region A, there is a holomorphic function F, the **antiderivative**, defined on A that is unique up to an **additive constant** and that $F'(z) = f(z)$ for all $z \in A$.

- Let $A \subseteq \mathbb{C}$ be a region and let u be a twice continuously differentiable harmonic function on A, then u is C^∞. In a neighborhood of each point $z_0 \in A$, u is the real part of some holomorphic function.

  Sketch of Proof: Consider the function
  $$g = U + iV = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}$$
  Then obviously
  $$\frac{\partial U}{\partial y} = -\frac{\partial V}{\partial x}$$
  and
  $$\frac{\partial U}{\partial x} = \frac{\partial V}{\partial y}$$
  since $\nabla^2 u = 0$. Therefore g is holomorphic since it satisfies the Cauchy-Riemann equations.

  Since A is simply connected there exists a holomorphic function f on A

so that $f' = g$ by the Antiderivative Theorem. Let $f = \tilde{u} + i\tilde{v}$ then

$$f' = \frac{\partial \tilde{u}}{\partial x} - i\frac{\partial \tilde{u}}{\partial y} = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}$$

Thus $\tilde{u}$ and u differ by some constant, which can be made such that $\tilde{u} = u$.

Since f is C^∞, it follows that u is also C^∞ on A.

- **Mean Value Property for Harmonic Functions**: Let u be harmonic on a region A containing a circle with radius r around $z_0 = x_0 + iy_0$, and its interior, then

  $$u(x_0, y_0) = \frac{1}{2\pi}\int_0^{2\pi} u(z_0 + re^{i\theta})\mathrm{d}\theta$$

  Sketch of Proof: We know that there is a function f defined on a region containing this circle. Taking the real part and the imaginary part of the function gives

  $$f(z_0) = \frac{1}{2\pi}\int_0^{2\pi} f(z_0 + re^{i\theta})\mathrm{d}\theta$$

- **Dirichlet Problem**: Let A be an open bounded region and u_0 be a given continuous function on $\mathrm{bd}(A)$. Find a real-valued function u on $\mathrm{cl}(A)$ that is continuous on $\mathrm{cl}(A)$ and harmonic on A and equals u_0 on $\mathrm{bd}(A)$.

  To solve the Dirichlet problem, we may want to introduce the Maximum Principle for Harmonic Functions.

- **Local Maximum Principle for Harmonic Functions**: Let u be harmonic on A. Suppose u has a relative maximum at $z_0 \in A$, then u is constant in some neighborhood of z_0.

  Sketch of Proof: Since u is harmonic, there exists some f such that $u = \mathrm{Re}\,(f)$. Consider the function

  $$e^{f(z)}$$

  We have $\left|e^{f(z)}\right| = e^{u(z)}$. Since the function e^x is strictly increasing, thus $e^{u(z)}$ attains maximum value. Thus by the Maximum Modulus principle $e^{f(z)}$ is constant in some neighborhood of z_0, so u is also constant.

- **Global Maximum Principle for Harmonic Functions**: Given $A \subset \mathbb{C}$ an **open connected bounded** set. Let $u : \mathrm{cl}(A) \to \mathbb{R}$ be continuous and harmonic on A and assume M the maximum value of u on $\mathrm{bd}(A)$, then

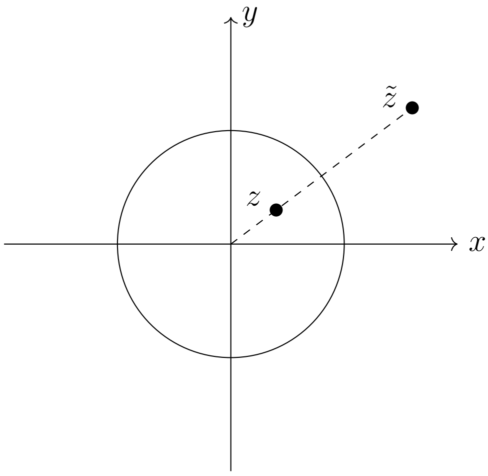
  (a) $u(x, y) \leq M$ for all $(x, y) \in A$.

  (b) if $u(x, y) = M$ for some $(x, y) \in A$, then u is constant on A.

- **Poisson's Formula**: Let u be defined and continuous on the disk $|z| \leq r$ and is harmonic on the open disk $D(0, r) = \{z|\ |z| < r\}$. For $\rho < r$, we have

  $$u(\rho e^{i\theta}) = \frac{r^2 - \rho^2}{2\pi}\int_0^{2\pi} \frac{u(re^{i\theta})}{r^2 - 2r\rho\cos(\phi - \theta) + \rho^2}\mathrm{d}\theta$$

Sketch of Proof: Consider some ζ such that $0 < |\zeta| = s < r$. Define $\tilde{z}$ satisfying $z\tilde{z} = s^2$, the image of inversion of z with respect to $|z| = s$.



Thus we have

$$f(z) = \frac{1}{2\pi i}\int_{|z|=s}\frac{f(\zeta)}{\zeta - z}\mathrm{d}\zeta$$

and

$$\int_{|z|=s}\frac{f(\zeta)}{\zeta - \tilde{z}}\mathrm{d}\zeta = 0$$

Subtracting gives

$$f(z) = \frac{1}{2\pi i}\int_{|z|=s} f(\zeta)\left(\frac{1}{\zeta - z} - \frac{1}{\zeta - \tilde{z}}\right)\mathrm{d}\zeta$$

Notice that $\zeta\bar{\zeta} = s^2$ and $\tilde{z} = s^2/\bar{z}$, thus we have

$$\begin{aligned}
\frac{1}{\zeta - z} - \frac{1}{\zeta - \tilde{z}} &= \frac{1}{\zeta - z} - \frac{\bar{z}}{\zeta\bar{z} - s^2} \\
&= \frac{1}{\zeta - z} - \frac{\bar{z}}{\zeta\bar{z} - \zeta\bar{\zeta}} \\
&= \frac{1}{\zeta - z} - \frac{\bar{z}}{\zeta(\bar{z} - \bar{\zeta})} \\
&= \frac{1}{\zeta - z} - \frac{\bar{z}}{\zeta(\overline{z - \zeta})} \\
&= \frac{|\zeta|^2 - |z|^2}{\zeta|\zeta - z|^2}
\end{aligned}$$

Expressing $\zeta = se^{i\theta}$ and $z = \rho e^{i\phi}$ gives

$$f(\rho e^{i\phi}) = \frac{1}{2\pi}\int_0^{2\pi}\frac{f(se^{i\theta})(s^2 - \rho^2)}{|se^{i\theta} - \rho e^{i\phi}|^2}\mathrm{d}\theta = \frac{1}{2\pi}\int_0^{2\pi}\frac{f(se^{i\theta})(s^2 - \rho^2)}{s^2 + \rho^2 - 2s\rho\cos(\phi - \theta)}\mathrm{d}\theta$$

Fix z, since for $s > p$, the function $\dfrac{f(se^{i\theta})(s^2 - \rho^2)}{s^2 + \rho^2 - 2s\rho\cos(\phi - \theta)}$ is continuous for all $s > p$ and $0 \le \theta \le 2\pi$, hence letting $s \to r$ and taking the real part of f we have

$$u(\rho e^{i\theta}) = \frac{r^2 - \rho^2}{2\pi} \int_0^{2\pi} \frac{u(re^{i\theta})}{r^2 - 2r\rho\cos(\phi - \theta) + \rho^2} \mathrm{d}\theta$$

- Poisson's formula allows us to construct a solution to the Dirichlet problem.

Problem

Find the maximum of:

1. $|e^z|$ on $|z| \le 1$
2. $|\cos z|$ on $[0, 2\pi] \times [0, 2\pi]$

1. We want to find the maximum of $|e^z|$ on the circle $|z| = 1$. Let $z = \cos\theta + i\sin\theta$, then

$$|e^z| = |e^{\cos\theta}| \le e$$

Hence the maximum value is e.

2. Consider z on the boundary $[0, 2\pi] \times [0, 2\pi]$. Letting $z = x + iy$ gives

$$|\cos z| = \frac{1}{2}|e^z + e^{-z}| = \frac{1}{2}\sqrt{(e^{2y} + e^{-2y}) + 2\cos 2x}$$

Hence $|\cos z|$ attains maximum value on $z = 2\pi i$, which gives $|\cos z| = \dfrac{e^{2\pi} + e^{-2\pi}}{2}$.

Problem

(a) Let the mapping T be defined by $T(z) = R(z - z_0)/(R^2 - \bar{z}_0 z)$. Show that for $|z_0| < R$, T takes the open disk of radius R one to one onto the disk of radius 1 and takes z_0 to the origin.

(b) Suppose that f is holomorphic on the open disk $|z| < R$ and that $|f(z)| < M$ for $|z| < R$. Suppose also that $f(z_0) = w_0$. Show that

$$\left|\frac{M[f(z) - w_0]}{M^2 - \bar{w}_0 f(z)}\right| \le \left|\frac{R(z - z_0)}{R^2 - \bar{z}_0 z}\right|.$$

(a) Consider all z such that $|z| = R$. Let $z = Re^{i\phi}$ and $z_0 = re^{i\theta}$. We have

$$\begin{aligned} |T(z)| &= \left|\frac{R(z - z_0)}{R^2 - \bar{z}_0 z}\right| \\ &= \frac{R|Re^{i\phi} - re^{i\theta}|}{|R^2 - Rre^{i(\phi - \theta)|}} \end{aligned}$$

$$
\begin{aligned}
&= \frac{|Re^{i\phi} - re^{i\theta}|}{|Re^{i\theta} - re^{i\phi}|} \\
&= \sqrt{\frac{(R\cos\phi - r\cos\theta)^2 + (R\sin\phi - r\sin\theta)^2}{(R\cos\theta - r\cos\phi)^2 + (R\sin\theta - r\sin\phi)^2}} \\
&= 1
\end{aligned}
$$

Hence for every point in $|z| < R$ we have $|T(z)| < 1$ by the Maximum Modulus Principle. Obviously $T(z_0) = 0$. It remains to show that $T(z)$ is injective. Assume that $T(z_1) = T(z_2)$, then

$$
\begin{aligned}
\frac{R(z_1 - z_0)}{R^2 - \bar{z}_0 z_1} &= \frac{R(z_2 - z_0)}{R^2 - \bar{z}_0 z_2} \\
(z_1 - z_0)(R^2 - \bar{z}_0 z_2) &= (z_2 - z_0)(R^2 - \bar{z}_0 z_1) \\
(R^2 - |z_0|^2)(z_1 - z_2) &= 0
\end{aligned}
$$

But we know that $|z_0| < R$ so $z_1 = z_2$ as desired.

(b) The strategy here is to consider two mappings $T : D_R(0; R) \to D_1(0; 1)$ and $F : D_M(0; M) \to D_1(0; 1)$ defined by

$$
T(z) = \frac{R(z - z_0)}{R^2 - \bar{z}_0 z}, \qquad F(z) = \frac{M(z - f(z_0))}{M^2 - \overline{f(z_0)} z}
$$

We have proved that in (a), $|T(z)| \leq 1$ and $|F(z)| \leq 1$. Consider the composition $F \circ f \circ T^{-1} : D_1(0; 1) \to D_1(0; 1)$. This way we can apply Schwarz lemma.

$$
\begin{aligned}
|F(f(T^{-1}(z)))| &\leq |z| \\
|F(f(z))| &\leq |T(z)| \\
\left| \frac{M(f(z) - f(z_0))}{M^2 - \overline{f(z_0)} f(z)} \right| &\leq \left| \frac{R(z - z_0)}{R^2 - \bar{z}_0 z} \right|
\end{aligned}
$$

as desired.

> **Remark.** The problem (b) is a generalization of the **Schwarz-Pick Theorem**.

Problem

Find harmonic conjugates for each of the following functions

(a) $u(x, y) = \sinh x \sin y$

(b) $u(x, y) = \ln \sqrt{x^2 + y^2}$

(c) $u(x, y) = e^x \cos y$

(a) We have

$$\frac{\partial u}{\partial x} = \cosh x \sin y = \frac{\partial v}{\partial y}$$

so $v = -\cosh x \cos y + C_1(x)$. On the other hand,

$$\frac{\partial u}{\partial y} = \sinh x \cos y = -\frac{\partial v}{\partial x}$$

so $v = -\cosh x \cos y + C_2(y)$. Thus

$$v = -\cosh x \cos y + C$$

for some constant C.

(b) We have

$$\frac{\partial u}{\partial x} = \frac{x}{x^2+y^2} = \frac{\partial v}{\partial y}$$

so $v = \tan^{-1}\left(\frac{y}{x}\right) + C_1(x)$. On the other hand,

$$\frac{\partial u}{\partial y} = \frac{y}{x^2+y^2} = -\frac{\partial v}{\partial x}$$

so $v = -\tan^{-1}\left(\frac{x}{y}\right) + C_2(y) = \tan^{-1}\left(\frac{y}{x}\right) - \frac{\pi}{2} + C_2(y)$. Thus

$$v = \tan^{-1}\left(\frac{y}{x}\right) + C$$

for some constant C.

(c) We have

$$\frac{\partial u}{\partial x} = e^x \cos y = \frac{\partial v}{\partial y}$$

so $v = e^x \sin y + C_1(x)$. On the other hand,

$$\frac{\partial u}{\partial y} = -e^x \sin y = -\frac{\partial v}{\partial x}$$

so $v = e^x \sin y + C_2(y)$. Thus

$$v = e^x \sin y + C$$

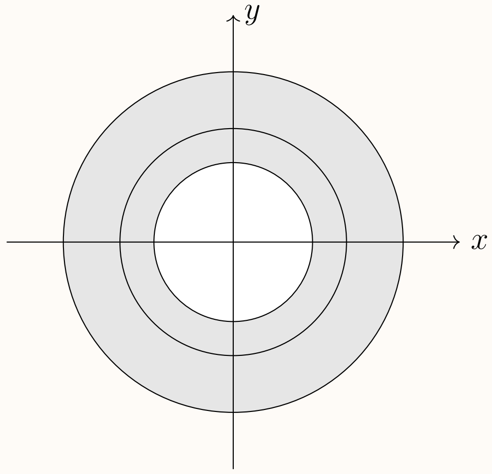
for some constant C.

Remark. These functions $u(x, y)$ are derived from (a) $\cos(x + iy)$, (b) $\ln(x + iy)$, (c) e^z.

Problem

Prove **Hadamard's Three-circle Theorem**: Let f be holomorphic on a region containing the set R. Let $R = \{z | \ r_1 \leq |z| \leq r_3\}$ and assume $0 < r_1 < r_2 < r_3$. Let M_1, M_2, M_3 be the maxima of $|f|$ on the circles $|z| = r_1, r_2, r_3$, respectively. Then we have the inequality

$$M_2^{\ln(r_3/r_1)} \leq M_1^{\ln(r_3/r_2)} M_3^{\ln(r_2/r_1)}.$$



Consider the function

$$g(z) = z^\lambda f(z)$$

given $\lambda = -\dfrac{\ln(M_3/M_1)}{\ln(r_3/r_1)}$, rewriting gives $r_1^\lambda M_1 = r_3^\lambda M_3$.

By the Maximum Modulus principle, we have

$$|g(z)| = r_3^\lambda |f(z)| \leq r_3^\lambda M_3 = r_1^\lambda M_1$$

Since the maximum may not be obtained in the region R, we have

$$r_2^\lambda M_2 \leq r_1^\lambda M_1$$

Substituting the definition of λ gives the desired result.

2.6 Review Exercises

Problem

Evaluate $\displaystyle\int_\gamma \frac{\mathrm{d}z}{1+z^2}$, where γ is a circle of radius 2 and center 0.

We have

$$\int_\gamma \frac{\mathrm{d}z}{1+z^2} = \frac{1}{2i} \int_\gamma \left(\frac{-1}{z+i} + \frac{1}{z-i} \right) \mathrm{d}z = 0$$

Problem

Let f be entire and let $|f(z)| \leq M$ for z on the circle $|z| = R$; let R be fixed. Prove that

$$\left|f^{(k)}(re^{i\theta})\right| \leq \frac{k!M}{(R-r)^k}, \quad k = 0, 1, 2, \ldots$$

for all $0 \leq r < R$.

Since by the Maximum Modulus principle, the maximum of $|f(z)|$ is found on the boundary of $|z| \leq R$. Thus, for every circle with radius $R - r > 0$, we have $|f(z')| \leq |f(z)| \leq M$ for $|z'| \leq R$, integrating gives

$$f^{(k)}(re^{i\theta}) = \frac{k!}{2\pi i} \int_{|\zeta|=R-r} \frac{f(\zeta)}{(\zeta - re^{i\theta})^{k+1}} \mathrm{d}\zeta$$

Since $|\zeta - z| \geq R - r$, we have

$$\begin{aligned}
\left|\frac{k!}{2\pi i} \int_{|\zeta|=R-r} \frac{f(\zeta)}{(\zeta - re^{i\theta})^{k+1}} \mathrm{d}\zeta\right| &= \frac{k!}{2\pi} \left|\int_{|\zeta|=R-r} \frac{f(\zeta)}{(\zeta - re^{i\theta})^{k+1}} \mathrm{d}\zeta\right| \\
&\leq \frac{k!}{2\pi} \cdot \frac{M}{(R-r)^{k+1}} \cdot 2\pi(R-r) \\
&= \frac{k!M}{(R-r)^k}
\end{aligned}$$

Problem

Let f and g be holomorphic in a region A and let $g'(z) \neq 0$ for all $z \in A$; let g be one to one and let γ be a closed curve in A. Then for z not on γ, prove that

$$f(z)I(\gamma; z) = \frac{g'(z)}{2\pi i} \int_\gamma \frac{f(\zeta)}{g(\zeta) - g(z)} \mathrm{d}\zeta$$

Since f and g are holomorphic in A, so they are homotopic to the constant curve $\gamma(t) = z$. Thus we have

$$\begin{aligned}
f(z)I(\gamma; z) &= \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{\zeta - z} \mathrm{d}\zeta \\
&= \lim_{z_0 \to z} \frac{1}{2\pi i} \int_{\gamma = z_0} \frac{f(\zeta)}{z_0 - z} \mathrm{d}\zeta \\
&= \lim_{z_0 \to z} \frac{1}{2\pi i} \int_{\gamma = z_0} \frac{f(\zeta)(g(z_0) - g(z))}{(z_0 - z)(g(z_0) - g(z))} \mathrm{d}\zeta \\
&= \frac{g'(z)}{2\pi i} \int_\gamma \frac{f(\zeta)}{g(\zeta) - g(z)} \mathrm{d}\zeta
\end{aligned}$$

Problem

Show that Poisson's Formula may be written

$$u(z) = \mathrm{Re}\left(\frac{1}{2\pi i}\int_{\gamma_r}\frac{\zeta+z}{\zeta-z}u(\zeta)\frac{\mathrm{d}\zeta}{\zeta}\right)$$

Use this to write a formula for the harmonic conjugate of u.

By assuming $z = \rho e^{i\phi}$ and $\zeta = re^{i\theta}$,

$$\begin{aligned}
\mathrm{Re}\left(\frac{1}{2\pi i}\int_{\gamma_r}\frac{\zeta+z}{\zeta-z}u(\zeta)\frac{\mathrm{d}\zeta}{\zeta}\right) &= \mathrm{Re}\left(\frac{1}{2\pi i}\int_0^{2\pi}\frac{re^{i\theta}+\rho e^{i\phi}}{re^{i\theta}-\rho e^{i\phi}}u(re^{i\theta})\frac{rie^{i\theta}\mathrm{d}\theta}{re^{i\theta}}\right)\\
&= \mathrm{Re}\left(\frac{1}{2\pi}\int_0^{2\pi}\frac{re^{i\theta}+\rho e^{i\phi}}{re^{i\theta}-\rho e^{i\phi}}u(re^{i\theta})\mathrm{d}\theta\right)\\
&= \mathrm{Re}\left(\frac{1}{2\pi}\int_0^{2\pi}\frac{(r\cos\theta+\rho\cos\phi)+i(r\sin\theta+\rho\cos\phi)}{(r\cos\theta+\rho\cos\phi)-i(r\sin\theta+\rho\cos\phi)}u(re^{i\theta})\mathrm{d}\theta\right)\\
&= \frac{1}{2\pi}\int_0^{2\pi}\frac{r^2-\rho^2}{r^2+\rho^2-2r\rho\cos(\phi-\theta)}u(re^{i\theta})\mathrm{d}\theta
\end{aligned}$$

In fact, the harmonic conjugate of u is just

$$\mathrm{Im}\left(\frac{1}{2\pi i}\int_{\gamma_r}\frac{\zeta+z}{\zeta-z}u(\zeta)\frac{\mathrm{d}\zeta}{\zeta}\right)$$

Problem

If f is holomorphic on and inside the unit disk, then show that

$$f(re^{i\phi}) = \frac{1}{2\pi}\int_0^{2\pi}\frac{f(e^{i\theta})}{1-re^{i(\phi-\theta)}}\mathrm{d}\theta, \quad r<1$$

From

$$f(z) = \frac{1}{2\pi i}\int_{|z|=1}\frac{f(\zeta)}{\zeta-z}\mathrm{d}\zeta$$

Letting $z = re^{i\phi}$ and $\zeta = e^{i\theta}$ gives

$$\begin{aligned}
f(re^{i\theta}) &= \frac{1}{2\pi i}\int_0^{2\pi}\frac{f(e^{i\theta})}{e^{i\theta}-re^{i\phi}}\cdot ie^{i\theta}\mathrm{d}\theta\\
&= \frac{1}{2\pi}\int_0^{2\pi}\frac{f(e^{i\theta})}{1-re^{i(\phi-\theta)}}\mathrm{d}\theta
\end{aligned}$$

Problem

Discuss the following sketch of a proof for Cauchy's Theorem: Suppose f is holomorphic in a convex region G containing 0 and γ is a closed curve in G. Define $F(t) = t\int_\gamma f(tz)\mathrm{d}z$ for $0 \le t \le 1$. Cauchy's Theorem is that $F(1) = 0$. Compute that $F'(t) = \int_\gamma f(tz)\mathrm{d}z + t\int_\gamma zf'(tz)\mathrm{d}z$, and integrate the second integral by parts to obtain

$$F'(t) = \int_\gamma f(tz)\mathrm{d}z + t\left\{\left[\frac{zf(tz)}{t}\right]_\gamma - \frac{1}{t}\int_\gamma f(tz)\mathrm{d}z\right\} = 0$$

so that $F(1) = F(0) = 0$. (Morse 1953)

By using the product rule, we have

$$F'(t) = \int_\gamma f(tz)\mathrm{d}z \cdot \frac{\mathrm{d}}{\mathrm{d}t}t + \left(\frac{\mathrm{d}}{\mathrm{d}t}\int_\gamma f(tz)\mathrm{d}z\right)\cdot t = \int_\gamma f(tz)\mathrm{d}z + t\int_\gamma zf'(tz)\mathrm{d}z$$

Now we integrate the second integral using IBP.

$$\begin{array}{ccc} & \mathrm{D} & \mathrm{I} \\ + & z & f'(tz) \\ - & 1 & \frac{1}{t}f(tz) \end{array}$$

Thus

$$\int_\gamma z'(tz)\mathrm{d}z = \left[\frac{zf(tz)}{t}\right]_\gamma - \frac{1}{t}\int_\gamma f(tz)\mathrm{d}z$$

Substituting back we have

$$F'(t) = \int_\gamma f(tz)\mathrm{d}z + t\left\{\left[\frac{zf(tz)}{t}\right]_\gamma - \frac{1}{t}\int_\gamma f(tz)\mathrm{d}z\right\} = [zf(tz)]_\gamma$$

Since γ is a closed loop, we have $[zf(tz)]_\gamma = 0$.

Obviously $F(0) = 0$, for $t = 1$, $F(1) = \int_\gamma f(z)\mathrm{d}z = 0$ since γ is a closed loop.

Problem

Prove **Harnack's Inequality**: If u is harmonic and nonnegative for $|z| \le R$, then

$$u(0)\frac{R-|z|}{R+|z|} \le u(z) \le u(0)\frac{R+|z|}{R-|z|}.$$

Using Poisson's Formula, we have

$$u(z) = \frac{R^2-|z|^2}{2\pi}\int_0^{2\pi}\frac{u(Re^{i\theta})}{R^2+|z|^2-2R|z|\cos\theta}\mathrm{d}\theta$$

Setting $z = 0$,

$$u(0) = \frac{1}{2\pi}\int_0^{2\pi} u(Re^{i\theta})\mathrm{d}\theta$$

On the other hand, notice that

$$(R - |z|)^2 \le R^2 + |z|^2 - 2R|z|\cos\theta \le (R + |z|)^2$$

Substituting this to Poisson's Formula, giving

$$\frac{R^2 - |z|^2}{2\pi}\int_0^{2\pi} \frac{u(Re^{i\theta})}{(R + |z|)^2}\mathrm{d}\theta \le u(z) \le \frac{R^2 - |z|^2}{2\pi}\int_0^{2\pi} \frac{u(Re^{i\theta})}{(R - |z|)^2}\mathrm{d}\theta$$
$$\frac{R - |z|}{R + |z|}u(0) \le u(z) \le \frac{R + |z|}{R - |z|}u(0)$$

§3 Series Representation of Holomorphic Functions

3.1 Convergent Series of holomorphic Functions

- A sequence $\{z_n\}n \ge 0$ is said to be **convergent** to some number z_0 if for each $\varepsilon > 0$, there exists some integer N such that for every $n > N$, we have $|z_n - z_0| < \varepsilon$.

  A series is said to be convergent to S if the partial sum $s_n = a_1 + a_2 + \cdots + a_n$ converges to S.

  If a sequence or series is not convergent, then it is **divergent**.

- The limit of convergence of a sequence is **unique**.

- **Cauchy's Criterion for Series** states that a sequence converges if and only if it is a Cauchy sequence, that is, for every $\epsilon > 0$, there exists an index N, such that for all $m, n > N$, we have $|z_m - z_n| < \varepsilon$.

  Cauchy's criterion states that,

  > The series $\sum_{k=1}^{\infty} a_k$ converges if and only if for every $\varepsilon > 0$, there exists some N, such that for all $n > N$ and $p \in \mathbb{N}$, we have
  >
  > $$\left|\sum_{k=n+1}^{n+p} a_k\right| < \varepsilon$$

  A corollary of the criterion is that

  > If $\sum_{k=1}^{\infty} a_k$ converges then $a_k \to 0$.

- A series $\sum_{k=1}^{\infty} a_k$ is said to **converge absolutely** if $\sum_{k=1}^{\infty} |a_k|$ converges.
- **Convergent Tests.** Some of the common convergent tests include
  - **Geometric Series**: If $|r| < 1$ then the series $\sum_{k=0}^{\infty} r^k$ converges to $\frac{1}{1-r}$.
  - **Comparison Test**: Given $0 \leq a_k \leq b_k$ for some sequences $\{a_k\}$ and $\{b_k\}$. If $\sum_{k=1}^{\infty} b_k$ converges then $\sum_{k=1}^{\infty} a_k$ converges. If $\sum_{k=1}^{\infty} a_k$ diverges then $\sum_{k=1}^{\infty} b_k$ diverges.
  - **p-series Test**: The sum $\sum_{n=1}^{\infty} n^{-p}$ converges if and only if $p > 1$.

    If $p = 1$, then

$$\begin{aligned} & \frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}+\dots \\ \geq & \frac{1}{1}+\frac{1}{2}+\left(\frac{1}{4}+\frac{1}{4}\right)+\left(\frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}\right)+\dots \\ = & \infty \end{aligned}$$

    If $p > 1$, then

$$\begin{aligned} & \frac{1}{1^p}+\frac{1}{2^p}+\frac{1}{3^p}+\frac{1}{4^p}+\frac{1}{5^p}+\frac{1}{6^p}+\frac{1}{7^p}+\frac{1}{8^p}+\dots \\ \geq & \frac{1}{1^p}+\left(\frac{1}{2^p}+\frac{1}{2^p}\right)+\left(\frac{1}{4^p}+\frac{1}{4^p}+\frac{1}{4^p}+\frac{1}{4^p}\right)+\frac{1}{8^p}+\dots \\ = & \frac{1}{1-2^{1-p}} \end{aligned}$$

  - **Ratio Test**: If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$ exists and strictly less than 1, then $\sum_{n=1}^{\infty} a_n$ converges absolutely; if the limit is strictly greater than 1, the series diverges; if the limit equals 1, the test in inconclusive.

    If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = r < 1$ then there exists some N such that $n \geq N$ implies $\left|\frac{a_{n+1}}{a_n}\right| < r' < 1$ for some $r < r'$. For $M > N$, we have

$$|a_M| \leq r'|a_{M-1}| \leq (r')^2|a_{M-2}| \leq \cdots \leq (r')^{M-N}|a_N|$$

    If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = r > 1$ then there exists some N such that $n \geq N$ implies $\left|\frac{a_{n+1}}{a_n}\right| > r'$ for some $1 < r' < r$. This means that $|a_{n+1}| > |a_n|$ for all

$n \geq N$, hence $|a_n| \to \infty$.

- **Root Test**: If $\lim_{n\to\infty} |a_n|^{1/n}$ exists and strictly less than 1, then $\sum_{k=1}^{\infty} a_k$ converges absolutely; if the limit is strictly greater than 1, the series diverges; if the limit equals 1, the test in inconclusive.

  If $\lim_{n\to\infty} |a_n|^{1/n} = r < 1$ then there exists some N such that $n \geq N$ implies $|a_n|^{1/n} < r' < 1$ for some $r < r'$. Thus the partial infinite sum is bounded by a geometric series with common ratio r'.

  If $\lim_{n\to\infty} |a_n|^{1/n} = r > 1$ then there exists some N such that $n \geq N$ implies $|a_n|^{1/n} > r'$ for some $1 < r' < r$, hence $|a_n| = r^n \to \infty$.

- **Integral Test**: The sum $\sum_{k=1}^{\infty} f(k)$ converges if and only if $\int_1^{\infty} f(x)\mathrm{d}x$ converges.

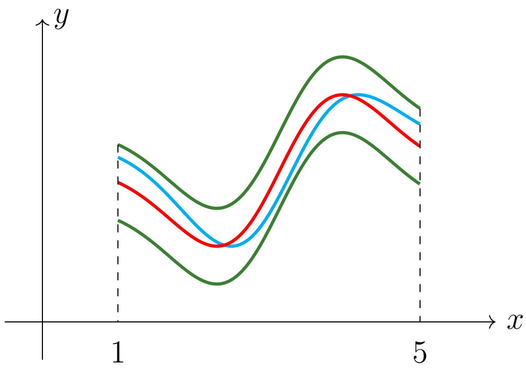
- **Alternating Series Test**: The sum $\sum_{k=1}^{\infty} (-1)^k a_k$ converges if

  1. $|a_{k+1}| \leq |a_k|$;
  2. $\lim_{k\to\infty} a_k = 0$.

- Let $\{f_n\}$ be a sequence of functions from a set A to $\mathbb{C}$. This sequence is said to **converge pointwise** if for every point z in A, the sequence $\{f_n(z)\}$ converges.

  This sequence of functions is said to **converge uniformly** to some function f if for any $\varepsilon > 0$, there exists some N such that whenever $n \geq N$ then $|f_n(z) - f(z)| < \varepsilon$ for all $z \in A$.

  For example, consider the set of functions $\{f_n\} = \sin(\sin x - x + 1/n) + 2$ in the interval $[1, 5]$.

  A graph of $f(x) = \sin(\sin x - x) + 2$ (red) with $n = 3$ and $\varepsilon = 1/2$.

  Similarly, a series $\sum_{k=1}^{\infty} f_k(z)$ is said to converge pointwise if $s_k(z) = f_1(z) +$

$f_2(z)+\cdots+f_k(z)$ converges pointwise. This series is said to converge uniformly if $s_k(z)$ converges uniformly.

- **Cauchy's Criterion**
  (i) A sequence of functions $\{f_n\}$ converges uniformly on A if and only if for each $\varepsilon > 0$, there exists some N such that whenever $n \geq N$, we have $|f_n(z) - f_{n+p}(z)| < \varepsilon$ for all $z \in A$ and $p = 1, 2, 3, \ldots$.

  (ii) A series $\displaystyle\sum_{n=1}^{\infty} f_n$ converges uniformly on A if and only if for each $\varepsilon > 0$, there exists some N such that whenever $n \geq N$, we have $\left|\displaystyle\sum_{k=n+1}^{n+p} f_k(z)\right| < \varepsilon$ for all $z \in A$ and $p = 1, 2, 3, \ldots$.

  Sketch of Proof: We will prove (i) as (ii) can be deduced similarly.

  ($\Rightarrow$) Assume that $\{f_n\}$ converges uniformly, for every $\varepsilon/2$, there exists N such that whenever $n \geq N$, $|f_n(z) - f(z)| < \varepsilon/2$. Thus for every $p = 1, 2, 3, \ldots$,

$$\varepsilon > |f_n(z) - f(z)| + |f(z) - f_{n+p}(z)| \geq |f_n(z) - f_{n+p}(z)|$$

  ($\Leftarrow$) Assume the converse, choose p large so that $|f_{n+p}(z) - f_(z)| < \varepsilon/2$ by pointwise convergence, thus

$$\varepsilon > |f_n(z) - f_{n+p}(z)| \geq |f_n(z) - f(z)|$$

- A uniform limit of a sequence of continuous functions is continuous.

  Sketch of Proof: For some point $z_0 \in A$, given $\varepsilon > 0$, it suffices to show that there exists $\delta > 0$ such that if $|z - z_0| < \delta$ then $|f(z) - f(z_0)| < \varepsilon$.

  We choose some N such that $|f_N(z) - f_N(z_0)| < \varepsilon/3$ for all $z \in A$. Since f_N is continuous, there exists some δ such that $|z - z_0| < \delta$ implies $|f_N(z) - f_N(z_0)| < \varepsilon/3$. Thus we have

$$\begin{aligned} |f(z) - f(z_0)| &\leq |f(z) - f_N(z)| + |f_N(z) - f_N(z_0)| + |f_N(z_0) - f_(z_0)| \\ &< \varepsilon \end{aligned}$$

- **Weierstrass M Test**: Suppose that $\{g_n\}$ is a sequence of functions on $A \subset \mathbb{C}$ and $\{M_n\}$ a sequence of nonnegative real constants. Suppose for each index n,
  (i) $|g_n(z)| \leq M_n$ for all $z \in A$;

  (ii) $\displaystyle\sum_{n=1}^{\infty} M_n$ converges.

  Then $\displaystyle\sum_{n=1}^{\infty} g_n(z)$ converges absolutely and uniformly on A.

Sketch of Proof: Since $\sum_{n=1}^{\infty} M_n$ converges, for every $\varepsilon > 0$, there exists some N such that for all $n \geq N$ and $p \in \mathbb{N}$ we have

$$\sum_{k=n+1}^{n+p} M_n < \varepsilon$$

Thus we can show that

$$\left| \sum_{k=n+1}^{n+p} g_k(z) \right| \leq \sum_{k=n+1}^{n+p} |g_k(z)| \leq \sum_{k=n+1}^{n+p} M_n < \varepsilon$$

The result follows by Cauchy's Criterion.

- Let $\gamma : [a,b] \to A$ be a curve in $A \subset \mathbb{C}$ and let $\{f_n\}$ be a sequence of functions defined on $\gamma([a,b])$ which converge uniformly to f on $\gamma([a,b])$, then

$$\int_\gamma f_n \qquad \text{converges to} \qquad \int_\gamma f$$

Sketch of Proof: For every $\varepsilon > 0$, there exists N such that for all $n \geq N$, $|f_n(z) - f(z)| < \varepsilon$ for all $z \in \gamma$. Thus

$$\left| \int_\gamma f_n - \int_\gamma f \right| \leq \int_\gamma |f_n(z) - f(z)| \mathrm{d}z < \varepsilon l(\gamma)$$

Consequently, we have

$$\int_\gamma \left(\sum_{n=1}^{\infty} g_n(z) \right) \mathrm{d}z = \sum_{n=1}^{\infty} \int_\gamma g_n(z) \mathrm{d}z$$

- **Analytic Convergence Theorem**
  - (i) Let $A \subset \mathbb{C}$ be an open set and $\{f_n\}$ be a sequence of holomorphic functions on A. If f_n converges to f uniformly on every **closed disk** in A then f is holomorphic. Additionally, f_n' converges to f' pointwise on A on uniformly on every **closed disk** in A.
  - (ii) If $\{g_n\}$ is a sequence of holomorphic functions on an open set $A \subset \mathbb{C}$ and $g(z) = \sum_{k=1}^{\infty} g_k(z)$ converges uniformly on every closed disk in A, then g is analytic on A and $g'(z) = \sum_{k=1}^{\infty} g_k'(z)$ pointwise on A and uniformly on every closed disk in A.

Sketch of Proof: Let $z_0 \in A$ and $\{z| \ \ |z - z_0| \leq r\}$ be a closed disk entirely contained in A. Moreover, consider the open disk $D(z_0; r)$. $f_n \to f$ is uniform in the set $\{z| \ \ |z - z_0| \leq r\}$ thus $f_n \to f$ is uniform in $D(z_0; r)$.

By the fact that a uniform limit of continuous functions is continuous, f is continuous in $D(z_0; r)$. For any closed curve γ in $D(z_0; r)$, we have $\int_\gamma f_n = 0$ and hence $\int_\gamma f = 0$, which, by Morera's Theorem f is holomorphic.

To show $f_n' \to f'$ is uniform, let $B = \{z| \ \ |z - z_0| \leq r\}$. Consider a circle γ centered at z_0 with radius $\rho > r$ that is contained in A. Since $f_n \to f$ is uniform by assumption, then for all $\varepsilon > 0$ there exists some N such that whenever $n \geq N$ we have $|f_n(z) - f(z)| < \varepsilon$. Thus

$$\begin{aligned} |f_n'(z) - f'(z)| &= \left| \frac{1}{2\pi i} \int_\gamma \frac{f_n(\zeta) - f(\zeta)}{(\zeta - z)^2} \mathrm{d}\zeta \right| \\ &\leq \frac{1}{2\pi} \cdot \frac{\varepsilon}{(\rho - r)^2} \cdot l(\gamma) \end{aligned}$$

which is a constant considering $|\zeta - z| \geq \rho - r$.

- The **Riemann Zeta Function** ζ is defined by

  $$\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z}$$

  This function is analytic and converges absolutely on $\{z(= x + iy) \mid \mathrm{Re}(z) > 1\}$, as

  $$\sum_{n=1}^{\infty} \left| \frac{1}{n^z} \right| = \sum_{n=1}^{\infty} \left| \frac{1}{n^x} \right|$$

  converges by p-series test. Furthermore, $\zeta'(z)$ converges absolutely by comparison test.

Problem

Prove that $\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z}$ does not converge uniformly on $A = \{z \mid \mathrm{Re}(z) > 1\}$.

Let $z = x + iy$ and assume that for all $\varepsilon > 0$ there exists such N such that whenever $n \geq N$, we have

$$\left| \zeta(z) - \sum_{k=n}^{\infty} k^{-z} \right| < \varepsilon$$

which simplifies to

$$\left| (n+1)^{-x} + (n+2)^{-x} + (n+3)^{-x} + \dots \right| < \varepsilon$$

But we cannot bound this sum independent of x, that is, when $x \to 1$ the sum "diverges".

Problem

Show that $\sum_{n=1}^{\infty} \frac{1}{n!z^n}$ is analytic on $\mathbb{C}\setminus\{0\}$. Compute its integral around the unit circle.

We shall use the Weierstrass M test. Assume that D is a disk in $\mathbb{C}\setminus\{0\}$, then there exists some $\delta > 0$ such that $|z| \leq \delta$. Thus define $M_n = 1/(n!\delta^n)$, we have

$$\left|\sum_{n=1}^{\infty} \frac{1}{n!z^n}\right| \leq \sum_{n=1}^{\infty} \frac{1}{|n!z^n|} \leq \sum_{n=1}^{\infty} \frac{1}{n!\delta^n} = e^{\delta} - 1$$

Hence the sum is analytic on D, by the Analytic Convergence Theorem. To compute the integral, we have

$$\int_0^{2\pi} \sum_{n=1}^{\infty} \frac{1}{n!e^{in\theta}} \mathrm{d}\theta = 0$$

by the Cauchy's Integral Formula.

Problem

Let f be an analytic function on the disk $D(0;2)$ such that $|f(z)| \leq 7$ for all $z \in D(0;2)$. Prove that there exists a $\delta > 0$ such that if $z_1, z_2 \in D(0;1)$, and if $|z_1 - z_2| < \delta$, then $|f(z_1) - f(z_2)| < 1/10$. Find a numerical value of δ independent of f that has this property.

I claim that $\delta = 1/141$ satisfies the problem. By the Cauchy's Integration Formula, for every $z_1, z_2 \in D(0;1)$, we have

$$\begin{aligned} |f(z_1) - f(z_2)| &= \frac{1}{2\pi}\left|\int_{|\zeta|=1} \frac{f(\zeta)}{\zeta - z_1} - \frac{f(\zeta)}{\zeta - z_2}\mathrm{d}\zeta\right| \\ &= \frac{1}{2\pi}\left|\int_{|\zeta|=1} f(\zeta)\cdot\frac{z_1 - z_2}{(\zeta - z_1)(\zeta - z_2)}\mathrm{d}\zeta\right| \end{aligned}$$

Since $|f(\zeta)| \leq 7$, $|z_1 - z_2| < \delta$, $1 \leq 2 - |z_1| < |\zeta - z_1|$, we have

$$\begin{aligned} \frac{1}{2\pi}\left|\int_{|\zeta|=2} f(\zeta)\cdot\frac{z_1 - z_2}{(\zeta - z_1)(\zeta - z_2)}\mathrm{d}\zeta\right| &\leq \frac{1}{2\pi}\int_{|\zeta|=2}\left|f(\zeta)\cdot\frac{z_1 - z_2}{(\zeta - z_1)(\zeta - z_2)}\right|\mathrm{d}\zeta \\ &\leq \frac{1}{2\pi}\cdot 7\delta\cdot 2\pi(2) \\ &< 14\delta \\ &= \frac{14}{141} \\ &< \frac{1}{10} \end{aligned}$$

3.2 Paowerrr Series and Taylor's Theorem

- A **power series** is given by

$$\sum_{n=0}^{\infty} a_n(z-z_0)^n$$

There is a unique $R \geq 0$, the **radius of convergence**, such that if $|z-z_0| < R$ then the series converges, and if $|z-z_0| > R$ then the series diverges. The circle $|z-z_0| = R$ is called the **circle of convergence**.

- The **Taylor Series** of a function f is given by

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z-z_0)^n$$

- Taking $z_0 = 0$, we obtain the **Maclaurin Series**

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} z^n$$

This shows that the power series expansion of a function is unique.

- **Abel-Weierstrass Lemma**: Suppose $r_0 \geq 0$ and for some constant M, $|a_n| r_0^n \leq M$ holds for all $n \geq 0$, then for some $r < r_0$, the series $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ **converges uniformly and absolute** on the closed disk $D_r = \{z| \ \ |z-z_0| \leq r\}$.

Sketch of Proof: We see that for all $z \in D_r$,

$$|a_n(z-z_0)^n| = |a_n| r^n = |a_n| r_0^n \left(\frac{r}{r_0}\right)^n \leq M \left(\frac{r}{r_0}\right)^n$$

The result follows by the Weierstrass M test $M_n = M\left(\frac{r}{r_0}\right)^n$.

- **Proof of Convergence of Power Series**: First we show the existence of R. In which case, if $R = 0$ then the series converge, so there must be at least a value of R.

By definition of R, assume $R > 0$ and let $r_0 < R$, there is an r_1 so that $r_0 < r_1 \leq R$ such that $\sum |a_n| r_1^n$ converges. Then by comparison test $\sum |a_n| r_0^n$ converges. Since $\sum |a_n| r_0^n$ is bounded, by the Abel-Weierstrass lemma this series converges uniformly and absolutely on $A_r = \{z| \ \ |z-z_0| \leq r\}$. By choosing values of r_0, we see that the power series converges in $D(z_0; R)$.

Suppose that there exists z_1 such that $|z_1 - z_0| > R$ and the series $\sum a_n(z_1 - z_0)^n$ converges. Since it converges to zero it is bounded in absolute value. Thus by the Abel-Weierstrass Lemma if $R < r < |z_1 - z_0|$ then $\sum |a_n| r^n$ converges. But this contradicts the maximality of R.

- A power series $f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n$ is an analytic function on the inside of its circle of convergence.

  Moreover, the derivative $f'(z) = \sum_{n=0}^{\infty} na_n(z - z_0)^{n-1}$ has the same circle of convergence as $f(z)$.

  Sketch of Proof: By the Analytic Convergence Theorem we know $\sum_{n=0}^{\infty} na_n(z-z_0)^{n-1}$ converges on $D(z_0; R)$. Assume a point z_1 exists such that $|z_1 - z_0| = r_0 > R$ and the series converges.

  By the convergence of f, $\sum_{n=0}^{\infty} na_n(z - z_0)^{n-1}$ converges absolutely, so $na_n r_0^{n-1}$ is bounded, as well as $na_n r_0^{n-1}(r_0/n)$, so for every z such that $|z - z_0| = r_0$ the series converges. But this contradicts the maximality of R.

- **Convergent Tests for Power Series**
  - **Ratio Test**: If $\lim_{n\to\infty} \frac{|a_n|}{|a_{n+1}|}$ exists then it equals R.
  - **Root Test**: If $\rho = \lim_{n\to\infty} \sqrt[n]{|a_n|}$ exists then $1/\rho = R$.

  Refining the root test, the **Hadamard's Formula** states that

$$\rho = \limsup_{n\to\infty} \sqrt[n]{|a_n|}$$

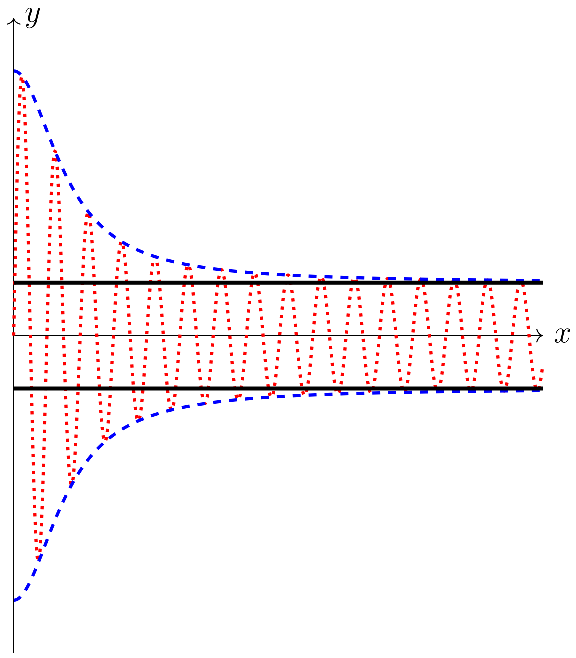
- **Recap – Limit Inferior and Limit Superior**:
  - The **limit inferior** of a sequence is defined as

$$\liminf_{n\to\infty} s_n = \varliminf_{n\to\infty} s_n := \lim_{n\to\infty} \inf_{m\geq n} s_m$$

  - The **limit superior** of a sequence is defined as

$$\limsup_{n\to\infty} s_n = \varlimsup_{n\to\infty} s_n := \lim_{n\to\infty} \sup_{m\geq n} s_m$$

  The graph of the function (sequence) $\frac{x^2+5}{x^2+1} \sin 20x$ illustrates the limit inferior and superior.



- **Taylor's Theorem**: Let f be analytic on the open set A in $\mathbb{C}$. Let $z_0 \in A$ and let $D_r = \{z| \ \ |z - z_0| < r\}$ in A, then for every $z \in D_r$, we have

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n$$

Sketch of Proof: We integrate f along the circle with radius $0 < \sigma < r$ centered at z_0. Thus

$$\begin{aligned}
f(z) &= \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{\zeta - z} \mathrm{d}\zeta \\
&= \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{\zeta - z_0} \cdot \frac{1}{1 - \dfrac{z - z_0}{\zeta - z_0}} \mathrm{d}\zeta \\
&= \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{\zeta - z_0} \cdot \sum_{n=0}^{\infty} \left(\frac{z - z_0}{\zeta - z_0}\right)^n \mathrm{d}\zeta \\
&= \frac{1}{2\pi i} \int_\gamma \sum_{n=0}^{\infty} \frac{(z - z_0)^n f(\zeta)}{(\zeta - z_0)^{n+1}} \mathrm{d}\zeta \\
&= \frac{1}{2\pi i} \sum_{n=0}^{\infty} (z - z_0)^n \int_\gamma \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} \mathrm{d}\zeta \\
&= \frac{1}{2\pi i} \sum_{n=0}^{\infty} (z - z_0)^n \cdot \frac{2\pi i \cdot f^{(n)}(z_0)}{n!} \\
&= \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} \cdot (z - z_0)^n
\end{aligned}$$

Note that $|z - z_0| < |\zeta - z_0|$ hence we can apply the geometric series formula.

- A function f is said to have **zero of order n** at c if

$$f(c) = f'(c) = f''(c) = \cdots = f^{(n-1)}(c) = 0$$
$$f^{(n)} \neq 0$$

- Suppose $f : \Omega \to \mathbb{C}$ is holomorphic on an open set Ω in $\mathbb{C}$ and suppose $c \in \Omega$ with $f(c) = 0$. Let $D(c; r)$ be an open disk, contained in Ω. Exactly one of these two things must occur:
  - For every $z \in D(c; r)$, $f(z) = 0$, or
  - There is a positive integer n such that

$$f(c) = f'(c) = f''(c) = \cdots = f^{(n-1)})(c) = 0 \quad \text{and} \quad f^{(n)}(c) \neq 0$$

  In the latter case there exists a function $\varphi(z)$ analytic in $D(c; r)$ with $\varphi(c) \neq 0$ and $f(z) = (z - c)^n \varphi(z)$ for all $z \in D(c; r)$ and a radius $\rho > 0$ such that $f(z) = 0$ for all $z \in D(c; \rho)$.

- We say that the zeros of $f : \Omega \to \mathbb{C}$ is **isolated** if for every $c \in \Omega$ such that $f(c) = 0$ there exists some $r > 0$ such that $D(c; r)$ contains only one zero (that is c itself).

- **Local Isolation of Zeros**: Suppose $f : \Omega \to \mathbb{C}$ be holomorphic on an open set Ω in $\mathbb{C}$ and for some $c \in \Omega$, if there is a sequence $z_1, z_2, z_3, \ldots$ converges to c and $f(z_k) = 0$ for all k, then $f(z) = 0$ for all z in the largest disk centered at c and contained in Ω.

- Let $f(z) = \displaystyle\sum_{n=0}^{\infty} \frac{H_n(t) z^n}{n!}$ and $f(z) = e^{2tz - z^2}$ then $H_n(t)$ are the **Physicist's Hermite Polynomials** while $f(z)$ is the **generating function**. Some values are given by

$$\begin{aligned}
H_0(t) &= 1 \\
H_1(t) &= 2t \\
H_2(t) &= 4t^2 - 2 \\
H_3(t) &= 8t^3 - 12t \\
H_4(t) &= 16t^4 - 48x^2 + 12 \\
&\vdots
\end{aligned}$$

Problem

Compute the Taylor series of the following:

1. $e^z \sin z$ around 0
2. e^{z^2} around 0
3. $\sin z^2$ around 0

1. By expanding the series,

$$\begin{aligned}
e^z \sin z &= \left(1 + \frac{z}{1!} + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!} + \frac{z^5}{5!} + \dots\right)\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \dots\right) \\
&= z + \frac{z^2}{1!} + \left(\frac{1}{2!} - \frac{1}{3!}\right) z^3 + \left(\frac{1}{3!} - \frac{1}{1!3!}\right) z^4 + \left(\frac{1}{5!} - \frac{1}{2!3!} + \frac{1}{1!4!}\right) z^5 + \dots \\
&= z + z^2 + \frac{1}{6} z^3 + \frac{7}{40} z^5 + \dots
\end{aligned}$$

2.

$$e^{z^2} = 1 + \frac{z^2}{1!} + \frac{z^4}{2!} + \frac{z^6}{3!} + \frac{z^8}{4!} + \dots$$

3.

$$\sin z^2 = z^2 - \frac{z^6}{3!} + \frac{z^{10}}{5!} - \frac{z^{14}}{7!} + \dots$$

Problem

Suppose $f(z) = \sum_{n=0\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$ converge for $|z| < R$. For $|z| < R^2$, define $F(z)$ by selecting r with $|z|/R < r < R$ and setting

$$F(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{\zeta} g\left(\frac{z}{\zeta}\right) \mathrm{d}\zeta,$$

where γ is the circle of radius r centered at the origin.

(a) Show that the value of $F(z)$ does not depend on r so long as $|z|/R < r < R$.

(b) Show that $F(z) = \sum_{n=0}^{\infty} a_n b_n z^n$.

(a) Consider the coefficient of z^k, denoted by $[z^k]$ of $F(z)$.

$$\begin{aligned}
[z^k] &= \frac{1}{2\pi i} \int_{|\zeta|=r} \frac{a_0 + a_1\zeta + a_2\zeta^2 + a_3\zeta^3 + \dots}{\zeta} \cdot b_k \left(\frac{1}{\zeta^k}\right) \mathrm{d}\zeta \\
&= \frac{b_k}{2\pi i} \int_0^{2\pi} \frac{a_0 + a_1 re^{i\theta} + a_2 re^{2i\theta} + a_3 re^{3i\theta} + \dots}{r^{k+1} e^{(k+1)i\theta}} rie^{i\theta} \mathrm{d}\theta
\end{aligned}$$

$$= \frac{b_k}{2\pi} \int_0^{2\pi} \frac{a_0 + a_1 re^{i\theta} + a_2 re^{2i\theta} + a_3 re^{3i\theta} + \dots}{r^k e^{ki\theta}} \mathrm{d}\theta$$

Since $|z|/R < r$ so $g(z/\zeta)$ converges; since $|z| < R$ so $f(\zeta)$ converges.

Clearly $e^{2\pi i} = 1$ and $e^0 = 1$, so if there exists some $e^{i\theta}$ term, the integral of it equals zero. Hence the integral becomes

$$\frac{b_k}{2\pi} \int_0^{2\pi} a_k \mathrm{d}\theta = a_k b_k$$

Hence F does not depend on r.

(b) Summing up the coefficients of z^k gives

$$F(z) = \sum_{n=0}^{\infty} [z^n] z^n = \sum_{n=0}^{\infty} a_n b_n z^n$$

as desired.

Problem

Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ converge for $|z| < R$. If $0 < r < R$, show that $f(z) = \sum_{n=0}^{\infty} a_n r^n e^{in\theta}$, where $z = re^{i\theta}$ and

$$a_n = \frac{1}{2\pi r^n} \int_0^{2\pi} f(re^{i\theta}) e^{-in\theta} \mathrm{d}\theta.$$

Also show

$$\frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^2 \mathrm{d}\theta = \sum_{n=0}^{\infty} |a_n|^2 r^{2n}.$$

The second equation is referred to as **Parseval's Theorem**, and we say that $f(z) = \sum_{n=0}^{\infty} a_n r^n e^{in\theta}$ expresses the Taylor series as a **Fourier series**.

Similar to the problem above, $f(z) = \sum_{n=0}^{\infty} a_n r^n e^{in\theta}$ can be shown by direct substitution, and the integral

$$\begin{aligned} \frac{1}{2\pi r^n} \int_0^{2\pi} f(re^{i\theta}) e^{-in\theta} \mathrm{d}\theta &= \frac{1}{2\pi r^n} \int_0^{2\pi} \frac{a_0 + a_1 re^{i\theta} + a_2 r^2 e^{2i\theta} + \dots}{e^{in\theta}} \mathrm{d}\theta \\ &= \frac{1}{2\pi r^n} \int_0^{2\pi} a_n r^n \mathrm{d}\theta \\ &= a_n \end{aligned}$$

Now using the fact that $f\bar{f} = |f|^2$, we have

$$\frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^2 \mathrm{d}\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi}\left(a_0 + a_1 re^{i\theta} + a_2 r^2 e^{2i\theta} + \dots\right)\left(\overline{a_0} + \overline{a_1} re^{-i\theta} + \overline{a_2} r^2 e^{-2i\theta} + \dots\right) \mathrm{d}\theta$$
$$= \frac{1}{2\pi}\int_0^{2\pi}\left(a_0\overline{a_0} + a_1\overline{a_1} r^2 + a_2\overline{a_2} r^4 + \dots\right) \mathrm{d}\theta$$
$$= \sum_{n=0}^{\infty} |a_n|^2 r^{2n}$$

Remark. In general, Parseval's theorem states that for complex functions $A(x) = \sum_{n=-\infty}^{\infty} a_n e^{2n\pi i(x/T)}$, $B(x) = \sum_{n=-\infty}^{\infty} b_n e^{2n\pi i(x/T)}$ on $\mathbb{R}$ of period T and are square integrable, then

$$\sum_{n=-\infty}^{\infty} a_n \overline{b_n} = \frac{1}{T}\int_{-T/2}^{T/2} A(x)\overline{B(x)}\mathrm{d}x$$

Remark. The Fourier series of a function f in the sine-cosine form

$$f_N(x) = a_0 + \sum_{n=1}^{N}\left(a_n \cos\left(2\pi\frac{n}{T}x\right) + b_n \sin\left(2\pi\frac{n}{T}x\right)\right)$$

or in the exponential form

$$f_N(x) = \sum_{n=-N}^{N} C_n e^{2\pi i \frac{n}{T} x}$$

Problem

Let $H_n(x)$ be the Hermite polynomials. Show that $H_{n+1}(x) = 2xH_n(x) - 2nH_{n-1}(x)$.

Problem

Find a function f such that (for all x)

(a) $f'(x) = x + 2f(x)$

(b) $f'(x) = xf(x)$

3.3 Laurent Series and Singularities