Complex Analysis Notes

Before starting, I want to mention that the book "Basic Complex Analysis (3rd edition)" by Jerrold E. Marsden and Michael J. Hoffman. is used.

§1 How Complex Is It?

1.1 Basic Operations

- $(a+bi) \pm (c+di) = (a \pm c) + (b \pm d)i$
- $(a+bi)(c+di) = (ac-bd) + (ad+bc)i$
- $\dfrac{a+bi}{c+di} = \dfrac{ac+bd}{c^2+d^2} + \dfrac{bc-ad}{c^2+d^2}i$

Problem

Fix a complex number $z = x+iy$ and consider the linear mapping $\phi_z : \mathbb{R}^2 \to \mathbb{R}^2$ (that is, of $\mathbb{C} \to \mathbb{C}$) defined by $\phi_z(w) = z \cdot w$ (that is, multiplication by z). Prove that the matrix of ϕ_z in the standard basis $(1,0),(0,1)$ of $\mathbb{R}^2$ is given by

$$\begin{pmatrix} x & -y \\ y & x \end{pmatrix}.$$

Then show that $\phi_{z_1 z_2} = \phi_{z_1} \circ \phi_{z_2}$.

Let $w = a+ib$, then $z \cdot w = (x+iy)(a+ib) = (xa-yb) + (xb+ya)i$.

On the other hand,

$$\begin{pmatrix} x & -y \\ y & x \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} xa-yb \\ xb+ya \end{pmatrix}.$$

and we have

$$\phi_{z_1 z_2} = z_1 \cdot z_2 \cdot w = z_1 \cdot (z_2 \cdot w) = \phi_{z_1} \circ \phi_{z_2}.$$

1.2 What? There's More?

Proposition (De Moivre's Formula)
If $z = r(\cos\theta + i\sin\theta)$ then for some positive integer n,

$$z^n = r^n(\cos n\theta + i\sin n\theta).$$

Some notable properties of **complex conjugation** and **norm**.

- $z\overline{z} = |z|^2$.
- $\mathrm{Re}\,(z) = (z+\overline{z})/2$, $\mathrm{Im}\,(z) = (z+\overline{z})/2i$
- $|\mathrm{Re}\,(z)| \le |z|$, $|\mathrm{Im}\,(z)| \le |z|$
- Triangle Inequality: $\left|\sum_{k=1}^{n} z_k\right| \le \sum_{k=1}^{n} |z_k|$
- Cauchy-Schwarz Inequality: $\left|\sum_{k=1}^{n} z_k w_k\right| \le \sqrt{\sum_{k=1}^{n} |z_k|^2}\sqrt{\sum_{k=1}^{n} |w_k|^2}$

Problem
If $a, b \in \mathbb{C}$, prove the **parallelogram identity**: $|a-b|^2 + |a+b|^2 = 2(|a|^2+|b|^2)$.

Let $a = p + iq$ and $b = r + is$, then

$$\begin{aligned}
|a-b|^2 + |a+b|^2 &= (p-r)^2 + (q-s)^2 + (p+r)^2 + (q+s)^2 \\
&= 2(p^2+q^2+r^2+s^2) \\
&= 2(|a|^2+|b|^2)
\end{aligned}$$

Problem
Prove **Langrange's identity**:

$$\left|\sum_{k=1}^{n} z_k w_k\right|^2 = \left(\sum_{k=1}^{n} |z_k|\right)\left(\sum_{k=1}^{n} |w_k|\right) - \sum_{k<j} |z_k\overline{w_j} - z_j\overline{w_k}|.$$

We abuse the fact that $z\overline{z} = |z|^2$.

$$\begin{aligned}
\left|\sum_{k=1}^{n} z_k w_k\right|^2 &= \left(\sum_{k=1}^{n} z_k w_k\right)\left(\overline{\sum_{k=1}^{n} z_k w_k}\right) \\
&= \left(\sum_{k=1}^{n} z_k w_k\right)\left(\sum_{k=1}^{n} \overline{z_k w_k}\right) \\
&= \sum_{k=1}^{n} z_k w_k \overline{z_k w_k} + \sum_{j\neq k} z_j w_j \overline{z_k w_k} \\
&= \sum z_k w_j \overline{z_k w_j} + \sum_{j\neq k} z_j w_j \overline{z_k w_k} - \sum_{j\neq k} z_k w_j \overline{z_k w_j} \\
&= \sum |z_k|^2 |w_j|^2 + \sum_{j\neq k} z_j w_j \overline{z_k w_k} - \sum_{j\neq k} z_k w_j \overline{z_k w_j} \\
&= \left(\sum_{k=1}^{n} |z_k|\right)\left(\sum_{k=1}^{n} |w_k|\right) + \sum_{j\neq k} z_j w_j \overline{z_k w_k} - \sum_{j\neq k} z_k w_j \overline{z_k w_j}
\end{aligned}$$

For some distinct indices j, k we have

$$\begin{aligned}
z_j w_j \overline{z_k w_k} + z_k w_k \overline{z_j w_j} - z_k w_j \overline{z_k w_j} - z_j w_k \overline{z_j w_k} &= z_j \overline{w_k}(w_j \overline{z_k} - w_k \overline{z_j}) + z_k \overline{w_j}(w_k \overline{z_j} - w_j \overline{z_k}) \\
&= (w_k \overline{z_j} - w_j \overline{z_k})(z_j \overline{w_k} - z_k \overline{w_j}) \\
&= -(w_k \overline{z_j} - w_j \overline{z_k})\overline{(w_k \overline{z_j} - w_j \overline{z_k})} \\
&= -|w_k \overline{z_j} - w_j \overline{z_k}|^2
\end{aligned}$$

Summing up gives the desire result

1.3 Even Weirder Stuff

Using the fact that

$$re^{ix} = r(\cos x + i \sin x)$$

and thanks to Euler we generalize the complex numbers to even more functions.

- It's not hard to see that

$$\sin x = \frac{e^{ix} - e^{-ix}}{2i} \quad \text{and} \quad \cos x = \frac{e^{ix} + e^{-ix}}{2}$$

- Let $z = re^{i\theta}$ then $\ln z = \ln |r| + i \arg z$.
- $z^w = e^{w \ln z}$ can be determined consequently.
- Moreover, we have

$$\sinh x = -i \sin(ix) \quad \text{and} \quad \cosh x = \cos(ix)$$

which can be deduced from

$$\sinh x = \frac{e^x - e^{-x}}{2} \quad \text{and} \quad \cosh x = \frac{e^x + e^{-x}}{2}$$

Problem

Along which rays through the origin does $\lim_{z\to\infty} |e^z|$ exist?

Let $z = x + iy$, then we have $|e^z| = |e^x(\cos y + i \sin y)| = e^x$. If $x \to -\infty$ then $e^x \to 0$, but if $x \to \infty$ then $e^x \to \infty$ which the limit doesn't exist.

Hence the answers are all the rays passing through the nonnegative x plane.

Problem

Prove the identity

$$z = \tan\left[\frac{1}{i}\ln\left(\frac{1+iz}{1-iz}\right)^{1/2}\right]$$

for all real z.

$$\begin{aligned}
\tan\left[\frac{1}{i}\ln\left(\frac{1+iz}{1-iz}\right)^{1/2}\right] &= \tan\left[\frac{1}{2i}\left(\ln(1+iz) - \ln(1-iz)\right)\right] \\
&= \tan\left[\frac{1}{2i}\left(\ln|1+iz| + i(\tan^{-1} z) - \ln|1-iz| - i(\tan^{-1}(-z))\right)\right] \\
&= \tan\left[\frac{1}{2i}\left(2i(\tan^{-1} z)\right)\right] \\
&= z
\end{aligned}$$

Problem

Use the equation $\sin z = \sin x \cosh y + i \sinh y \cos x$ where $z = x + iy$ to prove that $|\sinh y| \le |\sin z| \le |\cosh y|$.

Evaluating gives

$$|\sin z| = \sqrt{\sin^2 x \cosh^2 y + \sinh^2 y \cos^2 x}$$

Using the fact that $\sinh x < \cosh x$, we have

$$\sin^2 x \sinh^2 y + \sinh^2 y \cos^2 x < \sin^2 x \cosh^2 y + \sinh^2 y \cos^2 x < \sin^2 x \cosh^2 y + \cosh^2 y \cos^2 x$$

simplifying gives the desired result.

> **Problem**
> Using polar coordinates, show that $z \mapsto z + 1/z$ maps the circle $|z| = 1$ to the interval $[-2, 2]$ on the x axis.

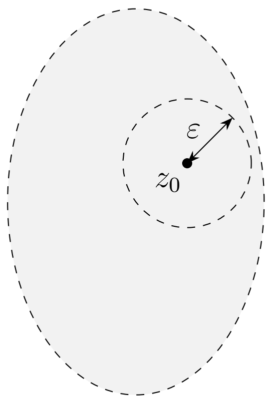
Let $z = x+iy$, then $z+\dfrac{1}{z} = x+iy+\dfrac{x-iy}{x^2+y^2}$ and since $x^2+y^2 = 1$, $z+\dfrac{1}{z} = 2x$. This means that for any complex number $z = x+iy$ on the circle, it will be mapped to $2x$.

And since x is in the interval $[-1, 1]$, hence $2x$ is in the interval $[-2, 2]$.

1.4 Topological Analysis of Complex Functions

1.4.1 Definitions

- **r Disk**: The r disk is defined by $D(z_0; r) = \{z \in \mathbb{C} | \ |z - z_0| < r\}$. The **deleted r disk** is defined by $D(z_0; r) \setminus \{z_0\}$.
- **Open Sets**: The set $A \subset \mathbb{C}$ is open when for any point z_0 in A, there exists a real number ε such that if $|z - z_0| < \varepsilon$ then $z \in A$.



- **Closed Sets**: A set F is closed if $\mathbb{C} \setminus F$ is open.
  - The empty set and $\mathbb{C}$ are both open and closed (known as **clopen sets**).
  - Let $z_1, z_2, z_3, \ldots$ are points in F and $w = \lim_{n\to\infty} z_n$, then $w \in F$.
    - **Sketch of proof**: Assume that $w \notin F$, then since $\mathbb{C} \setminus F$ is open, we can always find a disk $D(w; r)$ contained in $\mathbb{C} \setminus F$. This means that there exists some large enough n such that $z_n \in D(w; r)$ by convergence, which implies $z_n \notin F$, a contradiction.
  - The **closure** of a set S, denoted by $\overline{S}$ is the set S together with its limit points, or known as the **boundary** $\partial(S)$.
- **Limits**: The limit $\lim_{z\to z_0} f(z) = L$ exists when for all $\varepsilon > 0$, there exists $\delta > 0$ such that for all $|z - z_0| < \delta$ $(z \neq z_0)$ we have $|f(z) - L| < \varepsilon$.

  Limits are **unique** if they exist.

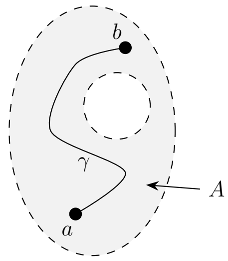
- **Continuity**: f is continuous at $z_0 \in A$ if and only if

$$\lim_{z \to z_0} f(z) = f(z_0).$$

- **Cauchy Sequence**: A sequence is *Cauchy* if for every $\varepsilon > 0$, we can find some integer N such that whenever integers m, n are greater than N, $|z_m - z_n| < \varepsilon$.

- **Path-Connected**: A set $A \in \mathbb{C}$ is path-connected if for every $a, b \in A$ there exists a *continuous map* $\gamma : [0, 1] \to A$ such that $\gamma(0) = a, \gamma(1) = b$.

  γ is a **path** joining a and b.



  Definition: A set $C \in \mathbb{C}$ is **not connected** if there are open sets U, V such that

  (a) $C \subset (U \cup V)$;
  (b) $(C \cap U \neq \varnothing) \wedge (C \cap V \neq \varnothing)$;
  (c) $C \cap U \cap V = \varnothing$.

  If a set is not "not connected", then it is **connected**.

  - **A path-connected set is connected, but a connected set may not be path-connected.**
  - Example: **Topologist's Sine Curve**

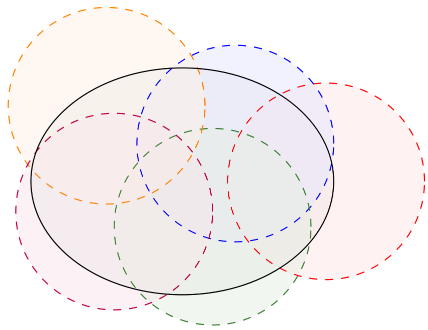
$$f(x) = \begin{cases} \sin \dfrac{1}{x} & x > 0 \\ 0 & x = 0 \end{cases}$$

    Sketch of proof: Let the two sets be A, B. WLOG let $(0, 0) \in A$.

    If some part of $\sin(1/x)$ is in A, then B should be covering the other parts. But since both sets are open, there's a point that is not covered.

    If no part of $\sin(1/x)$ is in A, then B must be covering the entire line of $\sin(1/x)$. But this is impossible since we cannot cover all points near $x = 0^+$.

- **Cover**: Let U be a collection of open sets. U is a cover of a set K if K is contained in the union of sets in U.



A **subcover** is a subset of U but can still cover K.

- **Compactness**: A set K is **compact** if every cover of K has a finite subcover.
  - **Heine-Borel Theorem**: A set K is compact if and only if K is closed and bounded.

    Sketch of proof:

    * Sufficiency:

      Boundedness: Assume that K is not bounded. Consider the set of open covers $U = \{D(O; r)\}$, (open) disks centered at the origin, then for all finite subcover U' of U, consider $R = \max(r)$ and choose some point $z \in K$ but $|z| > r$.

      Closedness: Assume that K is not closed, then there exists some $w \notin K$ such that the sequence $\{z_i\}$ in K converges to w. So the set of open covers $U = \{D(w, r)\}$ does not have a finite subcover.

    * Necessity: Assume that K is closed and bounded, then let $z \in K$ such that $|z|$ attains maximum value. Choose the open cover $D(O; |z|+1)$.

1.4.2 On Functions

- If f is a continuous function defined on a connected set C, then $f(C)$ is connected.

  Sketch of proof: FTSOC, let $A|B$ be a partition of $f(C)$. Then $f^{-1}(A)$ and $f^{-1}(B)$ are open and disjoint (since each value $f^{-1}(x)$ can only belong to either one of $f^{-1}(A)$ and $f^{-1}(B)$.)

- If f is a continuous function defined on a compact set C, then $f(C)$ is compact.

  Sketch of proof: Let U be an open cover of $f(C)$, then for each $f(z) \in U$ and $f(z) \in f(C)$, we have $z \in C$ and $z \in f^{-1}(U)$.

- **Extreme Value Theorem**: Let K be a compact set and $f : K \to \mathbb{R}$ is a continuous function, then f attains **finite** maximum and minimum values.

  Sketch of proof: K is compact implies $f(K)$ is compact, or $f(K)$ is bounded, therefore finite maximum and minimum exists.

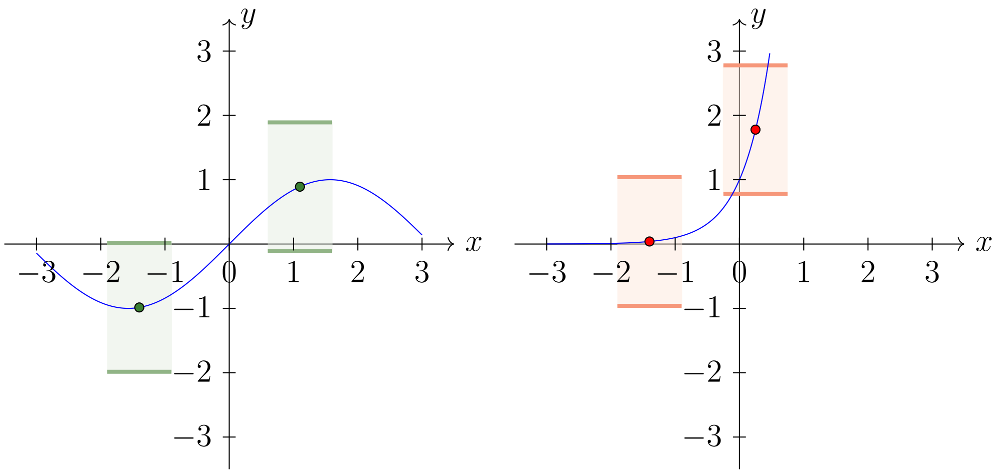
- **Distance Lemma**: Let K be a compact set and C be a closed set and $K \cap C = \varnothing$. Then there exists a number $\rho > 0$, such that whenever $z \in K$ and $w \in C$ then $|z - w| > \rho$.

  Sketch of proof: K is closed and bounded. Assume that ρ doesn't exist, $\rho \to 0$ since we can always find some $|z - w| < \rho_0$ if ρ_0 is fixed. Consider the sequences $\{z_k\}$ and $\{w_k\}$. Thus $\lim\limits_{k\to\infty} |z_k - w_k| = 0$ which means $\lim\limits_{k\to\infty} z_k = \lim\limits_{k\to\infty} w_k$.

  But since both sets are closed, we must have $\lim_{k\to\infty} z_k \in K$ and $\lim_{k\to\infty} w_k \in C$, hence a contradiction.

- **Uniform Continuity**: A function $f : A \to \mathbb{C}$ is **uniformly continuous** on A if for any $\varepsilon > 0$, there exists a $\delta > 0$ such that $|f(s) - f(t)| < \varepsilon$ whenever $s, t \in A$ and $|s - t| < \delta$.

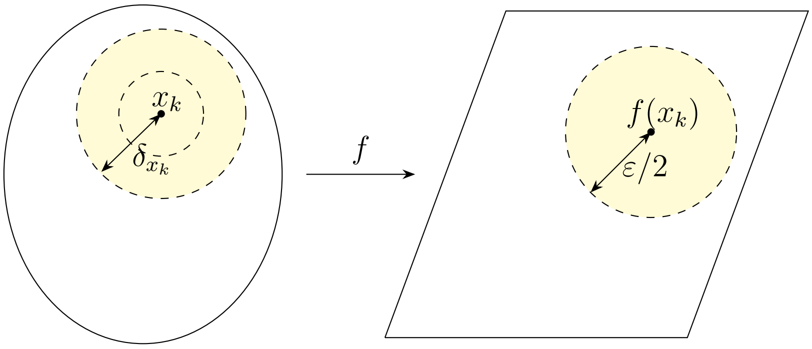
  An example $f(x) = \sin x$ and a nonexample $f(x) = 2^x$ are shown below. Choose $\delta = \varepsilon/2 = 0.5$.



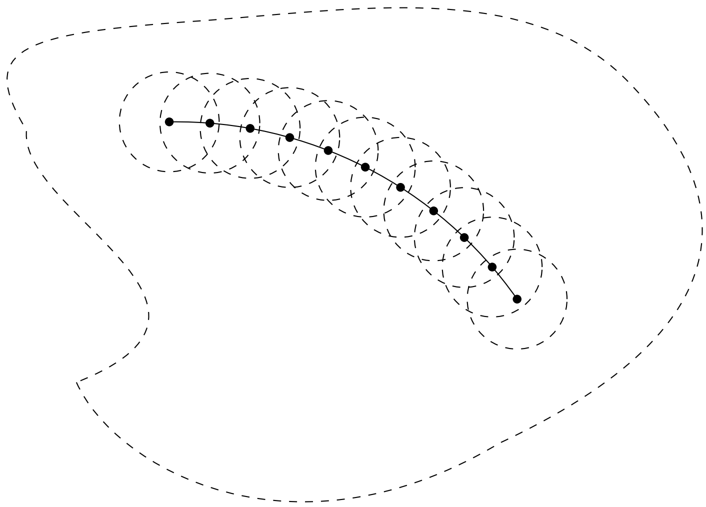
- **Heine-Cantor Theorem**: Let $f : A \to \mathbb{C}$ be a continuous function. If A is compact then $f(A)$ is uniformly continuous.

  Sketch of proof:

  – Let x be some point in A. By continuity, there is a number δ_x such that whenever for some point y satisfying $|x - y| < \delta_x$ then $|f(x) - f(y)| < \varepsilon/2$.
  – For a sequence of points x, say $\{x_i\}$, consider disks $D_k = D(x_k; \delta_{x_k}/2)$. These disks cover A by compactness. Let the minimal radius over all disks be δ.
  – For points s, t satisfying $|s - t| < \delta$, we must have t contained in some disk D_k. Thus $|t - x_k| < \delta_{x_k}/2$, implies that $|f(t) - f(x_k)| < \varepsilon/2$. We have
  $$|s - x_k| \le |s - t| + |t - x_k| < \delta + \delta_{x_k}/2 \le \delta_{x_k}$$
  – On the other hand,
  $$|f(s) - f(t)| \le |f(s) - f(x_k)| + |f(x_k) - f(t)| < \varepsilon$$



- **Path-Covering Lemma**: Suppose $\gamma : [0,1] \to K$ is a continuous path into an open subset K of $\mathbb{C}$. We can find a number $\rho > 0$ and a subdivision of $[0,1]$, namely $0 = t_0 < t_1 < t_2 < \cdots < t_{N-1} < t_N = 1$ such that
  (a) $D(\gamma(t_k);\rho) \subset G$ for all k
  (b) $\gamma(t) \in D(\gamma(t_0);\rho)$ for $t_0 \leq t \leq t_1$
  (c) $\gamma(t) \in D(\gamma(t_k);\rho)$ for $t_{k-1} \leq t \leq t_{k+1}$
  (d) $\gamma(t) \in D(\gamma(t_N);\rho)$ for $t_{N-1} \leq t \leq t_N$



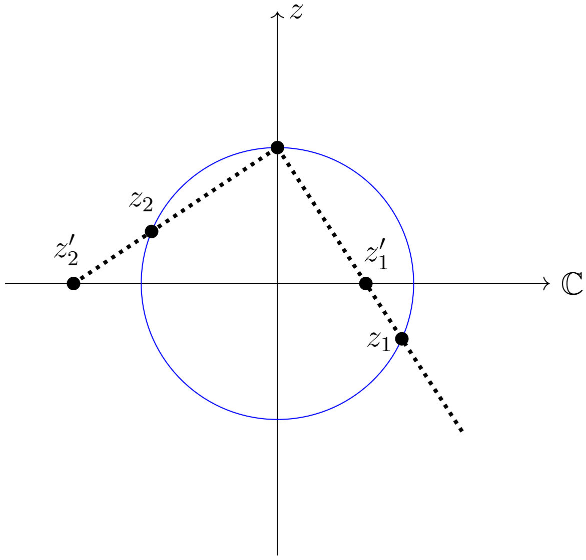
  Sketch of proof:
  - By the Distance Lemma, we can find some $\rho > 0$ such that the distance from $\gamma([0,1])$ to $\mathbb{C}\backslash K$ is at least ρ since both sets are closed.
  - By Heine-Cantor Theorem, $\gamma([0,1])$ is uniformly continuous. So for any two points s,t, if $|s-t| < \delta$ then $|f(s) - f(t)| < \rho$.
  - Choose t_k to be fine enough such that $t_{k+1} - t_k < \delta$.

- **Riemann Sphere**: We may want to define the value ∞ in the complex plane.
  - $\lim\limits_{z\to\infty} f(z) = L$ means for any $\varepsilon > 0$, there exists $Z > 0$ such that whenever $|z| > Z$ implies $|f(z) - L| < \varepsilon$.
  - $\lim\limits_{z\to z_0} f(z) = \infty$ means for any $R > 0$, there exists $\delta > 0$ such that whenever $|z - z_0| < \delta$ implies $|f(z)| > R$.

- $\lim_{z\to\infty} f(z) = \infty$ means for any $Z > 0$, there exists $R > 0$ such that whenever $|z| > Z$ implies $|f(z)| > R$.

Consider the sphere $x^2 + y^2 + z^2 = 1$ in $\mathbb{R}^3$. A point z' on the plane $\mathbb{C}$ is the **stereographic projection** of some point z on the sphere through $(0, 0, 1)$.

A 2-D illustration as an example:



Problem

Show that $\lim_{z\to\infty} \frac{1}{z} = 0$.

By our definition, we must have $|z| \to \infty$. Let $z = x + iy$ so that $|z| = \sqrt{x^2 + y^2}$. Clearly, at least one of $|x|, |y|$ must tend to ∞.

So

$$\lim_{z\to\infty} \frac{1}{z} = \lim_{|x|\to\infty \text{ or } |y|\to\infty} \left(\frac{x}{x^2 + y^2} - i\frac{y}{x^2 + y^2} \right)$$

which, implies that $\lim_{z\to\infty} \frac{1}{z} = 0$.

Problem

(a) Show that

$$|\mathrm{Re}\,(z_1) - \mathrm{Re}\,(z_2)\,| \leq |z_1 - z_2| \leq |\mathrm{Re}\,(z_1) - \mathrm{Re}\,(z_2)\,| + |\mathrm{Im}\,(z_1) - \mathrm{Im}\,(z_2)\,|$$

for any two complex numbers z_1 and z_2.

(b) If $f(z) = u(x, y) + iv(x, y)$, show that

$$\lim_{z \to z_0} f(z) = \lim_{\substack{x \to x_0 \\ y \to y_0}} u(x, y) + \lim_{\substack{x \to x_0 \\ y \to y_0}} iv(x, y)$$

exists if both limits on the right of the equation exist. Conversely, if the limit on the left exists, show that both limits on the right exist as well and equality holds.

(a) Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ for real numbers x_i, y_i. Let $\mathcal{X} = |x_1 - x_2|$ and $\mathcal{Y} = |y_1 - y_2|$. The inequality above can be expressed as

$$\mathcal{X} \leq \sqrt{\mathcal{X}^2 + \mathcal{Y}^2} \leq \mathcal{X} + \mathcal{Y}$$

which is obvious by squaring each part in the inequality.

(b) **Necessity**: Assume that both $\lim_{\substack{x \to x_0 \\ y \to y_0}} u(x, y) = U$ and $\lim_{\substack{x \to x_0 \\ y \to y_0}} v(x, y) = V$ exists, then for some ε, there exists δ_u and δ_v such that:

- $|u(x, y) - U| < \varepsilon/2$ whenever $|(x, y) - (x_0, y_0)| < \delta_u$.
- $|v(x, y) - V| < \varepsilon/2$ whenever $|(x, y) - (x_0, y_0)| < \delta_v$.

By the limit laws, $\lim_{z \to z_0} f(z) = U + iV$, then there exists $\delta = \min\{\delta_u, \delta_v\}$ such that whenever $|z - z_0| < \delta$,

$$\begin{aligned} |f(z) - Z| &= |u(x, y) + iv(x, y) - U - iV| \\ &\leq |u(x, y) - U| + |i||v(x, y) - V| \\ &< \varepsilon \end{aligned}$$

Sufficiency: Assume that $\lim_{z \to z_0} f(z) = U + iV$ exists. Then for all $\varepsilon > 0$ there exists $\delta_u > 0$ such that whenever $0 < |z - z_0| < \delta_u$ then $|u(x, y) - U| < \varepsilon$ and $\delta_v > 0$ such that whenever $0 < |z - z_0| < \delta_v$ then $|v(x, y) - V| < \varepsilon$. Choose $\delta = \min\{\delta_u, \delta_v\}$.

Problem

Introduce the **chordal metric** ρ on $\bar{\mathbb{C}}$ by setting $\rho(z_1, z_2) = d(z_1', z_2')$ where z_1' and z_2' are the corresponding points on the Riemann sphere and d is the usual distance between points in $\mathbb{R}^3$.

(a) Show that $z_n \to z$ in $\mathbb{C}$ if and only if $\rho(z_n, z) \to 0$.

(b) Show that $z_n \to \infty$ if and only if $\rho(z_n, \infty) \to 0$.

(c) If $f(z) = (az+b)/(cz+d)$ and $ad - bc \neq 0$, show that f is continuous at ∞.

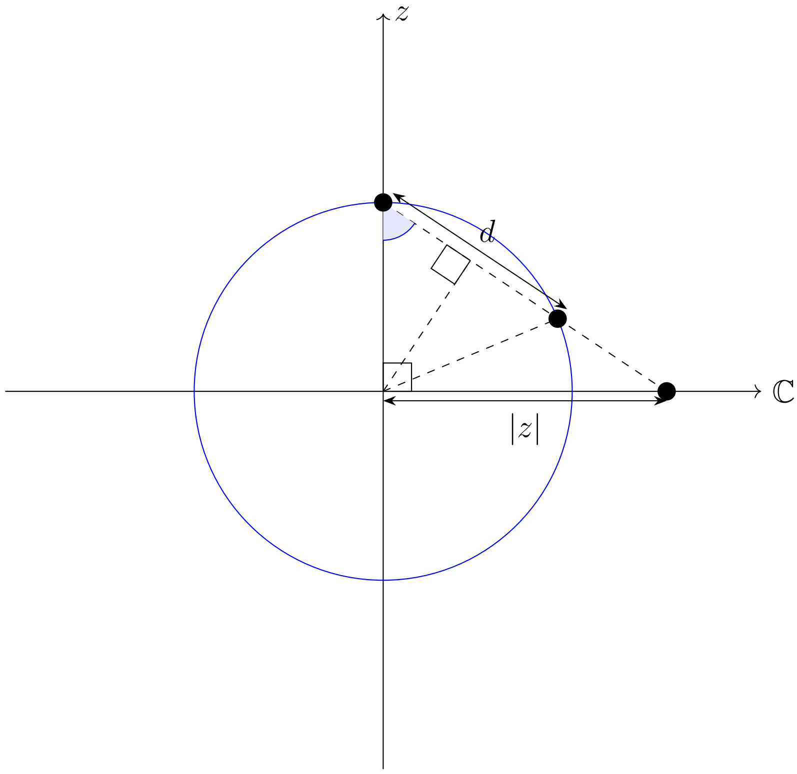
(a) If $z_n \to z$, then $\rho(z_n, z) \to \rho(z, z) = d(z', z') = 0$.

On the other hand, let $z_n' = (x_n, y_n, t_n)$ and $z' = (x, y, t)$. Then if $\rho(z_n, z) = d(z_n', z') \to 0$, we have

$$\begin{aligned} \sqrt{(x_n - x)^2 + (y_n - y)^2 + (t_n - t)^2} &\to 0 \\ (x_n - x)^2 + (y_n - y)^2 + (t_n - t)^2 &\to 0 \end{aligned}$$

FTSOC, WLOG assume x_n does not converge to x, then since $(x_n - x)^2 + (y_n - y)^2 + (t_n - t)^2 \geq 0$, we have $(x_n - x)^2 + (y_n - y)^2 + (t_n - t)^2$ converges to at least $(x_n - x)^2$, contradiction.

(b) If $z_n \to \infty$, $|z_n| \to \infty$. By drawing the Riemann sphere again,



We see that $\tan\theta = |z|$ and $d = 2\cos\theta$ (θ is the measure of the angle marked in blue). Substituting gives

$$d = \frac{2}{\sqrt{|z|^2 + 1}}$$

which is obvious that $d \to 0$.

Conversely, if $d \to 0$, it can be shown that $\sqrt{|z|^2 + 1} \to \infty$ which gives $z_n \to \infty$.

(c) A function $f(z)$ is continuous at infinity if the limits when $z \to +\infty$ and $z \to -\infty$ are equal.

- $\lim_{z \to \infty} \frac{az + b}{cz + d} = \frac{a}{c}$.
- $\lim_{z \to -\infty} \frac{az + b}{cz + d} = \frac{a}{c}$.

Hence the function is continuous at ∞.

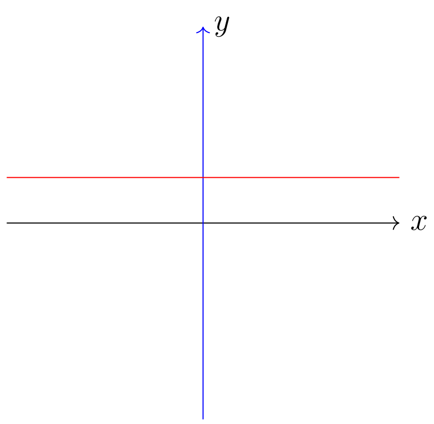
1.5 Analytic Functions: What and Why

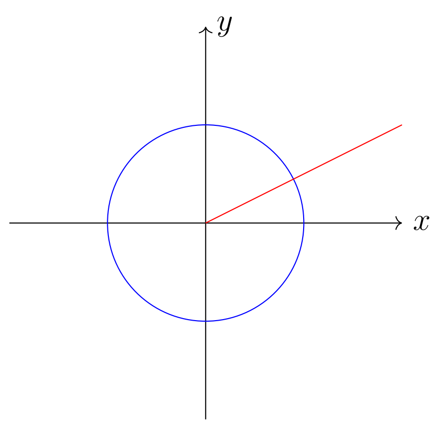
- functions that are differentiable in complex.
- "regular", "holomorphic", "analytic"
- a function $f(z)$ is differentiable at z_0 if $\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$ exists.

1.5.1 Differentiation

Differentiation laws apply here, including product rule, quotient rule, and chain rule.

- **Conformal Maps**: A function $f : A \to \mathbb{C}$ is conformal if it preserves **angles** between intersecting curves.
  - "conformal transformation", "angle-preserving transformation", "biholomorphic map"
  - An example of $e^z = e^{x+iy}$.



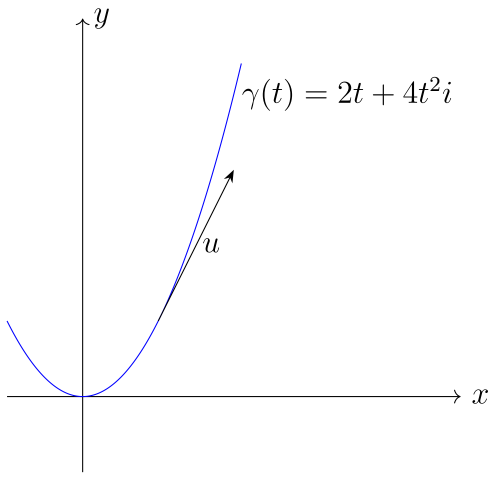


- What does it mean to differentiate complex numbers?
  - Clearly $f'(z_0) = x + iy$ is a vector $\begin{pmatrix} x \\ y \end{pmatrix}$ in the plane. Assume for some curve $\gamma(t) : \mathbb{R} \to \mathbb{C}$.

    Claim — Let $\gamma(t) = x(t) + iy(t)$, define $\gamma'(t) = x'(t) + iy'(t)$, then the vector $u = \begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix}$ is tangent to the curve γ at $(x(t), y(t))$.

    This is trivial by noticing

$$\frac{y'(t)}{x'(t)} = \frac{\frac{\mathrm{d}y}{\mathrm{d}t}}{\frac{\mathrm{d}x}{\mathrm{d}t}} = \frac{\mathrm{d}y}{\mathrm{d}x}$$



  - **Conformal Mapping Theorem**: If $A \to \mathbb{C}$ is analytic and $f'(z_0) \neq 0$, then f is conformal at z_0.

    Sketch of Proof: Define $\sigma(t) = f(\gamma(t))$, then obviously $\sigma(t)$ is also a curve. Taking

$$\frac{\mathrm{d}f(\gamma(t))}{\mathrm{d}t} = \frac{\mathrm{d}f(\gamma(t))}{\mathrm{d}z} \cdot \frac{\mathrm{d}z}{\mathrm{d}t}$$

    Assume $\gamma(t_0) = z_0$, letting $t = t_0$ we have

$$\sigma'(t_0) = f'(z_0)\gamma'(t_0)$$

    Since by our definition, $f'(z_0)$ is independent of γ, choose t_1 and t_2 so that

$$\frac{\sigma'(t_1)}{\gamma'(t_1)} = \frac{\sigma'(t_2)}{\gamma'(t_2)}$$

    Taking the argument we have

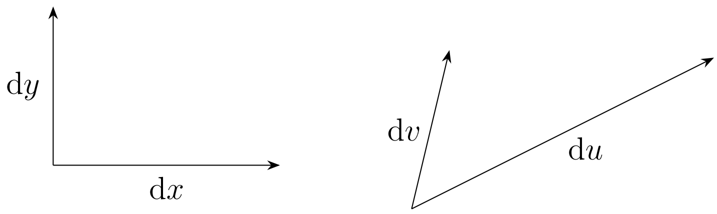
$$\arg \sigma'(t_1) - \arg \sigma'(t_2) \equiv \arg \gamma'(t_1) - \arg \gamma'(t_2) \pmod{2\pi}$$

    hence it is clear that angles are preserved.

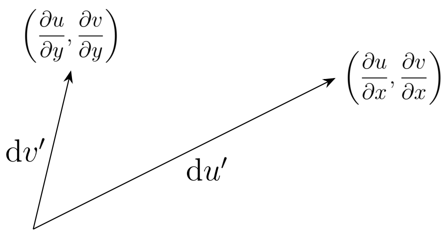
- **Cauchy-Riemann Equations**: Let $f(x,y) = u(x,y) + iv(x,y)$,
  - The **Jacobian matrix** of f is

$$\mathbf{J}_f = \begin{pmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\[2ex] \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{pmatrix}$$

What does the Jacobian matrix tell us? Recall that the transformation matrix $\mathbf{T} = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 & \dots & \mathbf{v}_n \end{bmatrix}$.



Taking ratios gives



  - **Cauchy-Riemann Theorem**:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Sketch of proof: This is due to the derivative of $f(z)$ can be approached from multiple directions.

Fix y, then when $x^* \to x$, we have

$$\begin{aligned}
\lim_{x^* \to x} \frac{f(z^*) - f(z)}{z^* - z} &= \lim_{x^* \to x} \frac{u(x^*, y) + iv(x^*, y) - u(x,y) - iv(x,y)}{x^* - x} \\
&= \lim_{x^* \to x} \left(\frac{u(x^*, y) - u(x,y)}{x^* - x} + i \frac{v(x^*, y) - v(x,y)}{x^* - x} \right) \\
&= \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x}
\end{aligned}$$

Similarly, fix x, then when $iy^* \to iy$, we have

$$\lim_{y^* \to y} \frac{f(z^*) - f(z)}{z^* - z} = \frac{\partial v}{\partial y} - i \frac{\partial u}{\partial y}$$

Comparing both equations gives the desired result.

- Applying Cauchy-Riemann equations to the Jacobian matrix, we have

$$\mathbf{J}_f = \begin{pmatrix} \dfrac{\partial u}{\partial x} & -\dfrac{\partial v}{\partial x} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial u}{\partial x} \end{pmatrix}$$

- Recall that the multiplication of complex values is the product of matrices.

  For example, $(a+bi)(x+yi) = (ax-by) + i(ay+bx)$,

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix} \begin{pmatrix} x & -y \\ y & x \end{pmatrix} = \begin{pmatrix} ax-by & -(bx+ay) \\ bx+ay & ax-by \end{pmatrix}$$

- Inverse functions
  - It is trivial that $\dfrac{\mathrm{d}}{\mathrm{d}z} f^{-1}(z) = \dfrac{1}{f'(f^{-1}(z))}$
  - By $z = x+iy$, $f(z) = u(x,y) + iv(x,y)$,

$$f'(z) = \frac{\mathrm{d}f}{\mathrm{d}z} = \frac{\partial f}{\partial x}\frac{1}{\dfrac{\partial z}{\partial x}} = \frac{\partial f}{\partial x} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

    Thus the matrix representing $f'(z)$ is surprisingly $\mathbf{J}_f$! Moreover,

$$|f'(z)|^2 = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 = \det \mathbf{J}_f$$

  - **Inverse Function Theorem**: Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be a (continuous, differentiable) function and let $\mathbf{J}_f(\mathbf{p})$ denote the Jacobian matrix of f evaluated at point $\mathbf{p}$ (here, we may assume that $\mathbf{p}$ is a complex value on the plane), then

$$\mathbf{J}_{f^{-1}}(f(\mathbf{p})) = (\mathbf{J}_f(\mathbf{p}))^{-1}$$

    Sketch of proof: Using the two facts above, we have

$$\mathbf{J}_{f^{-1}}(z)\mathbf{J}_f(f^{-1}(z)) = \mathbf{I}$$

    Letting $z = f(\mathbf{p})$ gives the desired result.

- **Recall on linear algebra** (taking 3-D system as example)

  First, we define the **del/nabla** operator.

$$\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$$

1. **Gradient**: Denoted by ∇f – Scalar multiplication of ∇ and f.

$$\nabla f(x,y,z) = \begin{pmatrix} \dfrac{\partial f}{\partial x} \\ \dfrac{\partial f}{\partial y} \\ \dfrac{\partial f}{\partial z} \end{pmatrix} = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}$$

2. **Divergence**: Denoted by $\nabla \cdot \mathbf{f}$ – Dot product of ∇ and $\mathbf{f}$.

$$\nabla \cdot \mathbf{f} = \nabla \cdot (F_x.F_y, F_z) = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$$

3. **Curl**: Denoted by $\nabla \times \mathbf{f}$ – Cross product of ∇ and $\mathbf{f}$.

$$\nabla \times \mathbf{f} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix}$$

4. **Laplacian**: Denoted by $\nabla \cdot \nabla \mathbf{f}$ or $\nabla^2 \mathbf{f}$ – Divergence of gradient.

$$\nabla \cdot \nabla \mathbf{f} = \frac{\partial^2 F_x}{\partial x^2} + \frac{\partial^2 F_y}{\partial y^2} + \frac{\partial^2 F_z}{\partial z^2}$$

Now let's get back to the 2-D plane.

- **Harmonic Functions**: A function $f : A \subseteq \mathbb{R}^2 \to \mathbb{R}$ is harmonic if

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$$

  – **Clairaut's Theorem**:

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$$

  Sketch of proof: A simple (but unformal) proof uses the definition of the derivative.

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \lim_{x^* \to x} \frac{\displaystyle\lim_{y^* \to y} \frac{f(x^*, y^*) - f(x^*, y)}{y^* - y} - \lim_{y^* \to y} \frac{f(x, y^*) - f(x, y)}{y^* - y}}{x^* - x}$$

  We can evaluate $\dfrac{\partial}{\partial y}\left(\dfrac{\partial f}{\partial x}\right)$ a similar way.

  – We now show that for some analytic function $f = u + iv$, then u and v

are harmonic.

By utilizing the Cauchy-Riemann equations, taking partial derivative with respect to x, we have

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial v}{\partial y}\right)$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) = -\frac{\partial}{\partial y}\left(\frac{\partial v}{\partial x}\right)$$

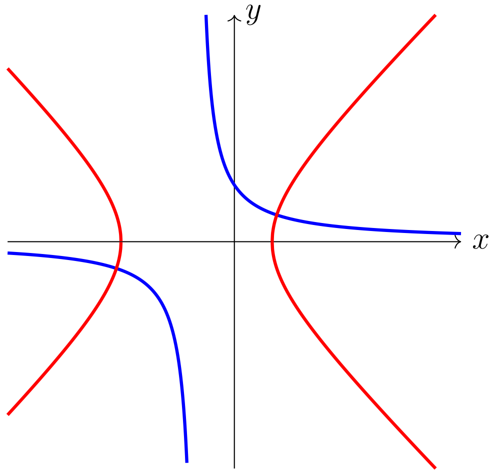
- **Harmonic Conjugate**: If u, v satisfy $f = u + iv$, then $u(x,y)$ and $v(x,y)$ are harmonic conjugates.

  Let $u(x,y)$ and $v(x,y)$ be harmonic conjugates, then the graphs $u(x,y) = c_1$ and $v(x,y) = c_2$ intersect orthogonally in the Cartesian plane.

  Sketch of proof: The dot product of two gradients equals zero

$$\nabla u \cdot \nabla v = \begin{pmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \end{pmatrix} \cdot \begin{pmatrix} \frac{\partial v}{\partial x} \\ \frac{\partial v}{\partial y} \end{pmatrix} = \frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial y} = 0$$

  An example of $f(z) = (z+1)^2$ and $u = x^2 - y^2 + 2x + 1 = 4$ and $v = 2xy + 2y = 3$.



Problem

Show, by changing variables, that the Cauchy-Riemann equations in terms of polar coordinates become

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \qquad \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$$

Then show that if u is harmonic, we have

$$r^2\frac{\partial^2 u}{\partial r^2} + r\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial \theta^2} = 0$$

(a) By knowing that $f(x+iy) = u + iv$ and $x + iy = r\cos\theta + ir\sin\theta$, we have

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial r} = \frac{\partial u}{\partial x}\cos\theta$$
$$\frac{\partial v}{\partial \theta} = \frac{\partial v}{\partial y}\cdot\frac{\partial y}{\partial \theta} = \frac{\partial v}{\partial y}r\cos\theta$$
$$\frac{\partial v}{\partial r} = \frac{\partial v}{\partial x}\cdot\frac{\partial x}{\partial r} = \frac{\partial v}{\partial x}\cos\theta$$
$$\frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial \theta} = \frac{\partial u}{\partial y}r\cos\theta$$

Results can be shown by applying Cauchy-Riemann equations.

(b) Evaluating the first equation gives

$$\frac{\partial^2 u}{\partial r^2} = \frac{1}{r}\frac{\partial}{\partial r}\left(\frac{\partial v}{\partial \theta}\right) - \frac{1}{r^2}\frac{\partial v}{\partial \theta} = \frac{1}{r}\frac{\partial}{\partial r}\left(\frac{\partial v}{\partial \theta}\right) - \frac{1}{r}\frac{\partial u}{\partial r}$$

while the second equation gives

$$\frac{\partial}{\partial \theta}\left(\frac{\partial v}{\partial r}\right) = -\frac{1}{r}\frac{\partial^2 u}{\partial \theta^2}$$

Eliminating $\frac{\partial}{\partial \theta}\left(\frac{\partial v}{\partial r}\right)$ gives the desired result.

Problem

Show that $\frac{\partial f}{\partial \bar{z}} = 0$. Then, find the value of $\frac{\partial \bar{z}}{\partial z}$.

(a) By using $x = \frac{1}{2}(z+\bar{z})$ and $y = \frac{1}{2i}(z-\bar{z})$. A change of variable lets f depend on two independent variables $z, \bar{z}$. So by taking total derivative,

$$\frac{\partial f}{\partial \bar{z}} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial \bar{z}} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right) = 0$$

which can be shown by Cauchy-Riemann equations.

(b) By repeating the same thing,

$$\frac{\partial f}{\partial z} = \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right)$$

Letting $f = \bar{z}$ gives

$$\frac{\partial \bar{z}}{\partial z} = \frac{1}{2}\left(1 - i(-i)\right) = 0$$

Problem

On what sets are each of the following functions harmonic?

(a) $u(x, y) = \operatorname{Im}\left(z^2 + 3z + 1\right)$

(b) $u(x, y) = \dfrac{x-1}{x^2 + y^2 - 2x + 1}$

(c) $u(x, y) = \operatorname{Im}\left(z + 1/z\right)$

(d) $u(x, y) = \dfrac{y}{(x-1)^2 + y^2}$

(a) Since $u(x.y)$ is already the imaginary part of the function $f(z) = z^2 + 3z + 1$, so $u(x, y)$ is harmonic on $\mathbb{C}$.

(b) Recall that $\dfrac{1}{x + yi} = \dfrac{x}{x^2 + y^2} - i\dfrac{y}{x^2 + y^2}$. Changing x to $x - 1$ shows that $u(x, y)$ is the real part of $f(z) = \dfrac{1}{z-1}$. Hence $u(x, y)$ is harmonic on $\mathbb{C} \setminus \{1\}$.

(c) Since $u(x.y)$ is already the imaginary part of the function $f(z) = z + 1/z$, so $u(x, y)$ is harmonic on $\mathbb{C} \setminus \{0\}$.

(d) Notice that $u(x, y)$ is the imaginary part of the function $f(z) = \dfrac{1}{1-z}$ hence it is harmonic on $\mathbb{C} \setminus \{1\}$.

Problem

Suppose u is a twice continuously differentiable real-valued harmonic function on a disk $D(z_0; l)$ centered at $z_0 = x_0 + iy_0$. For $(x_1, y_1) \in D(z_0; r)$, show that the equation

$$v(x_1, y_1) = c + \int_{y_0}^{y_1} \frac{\partial u}{\partial x}(x_1, y)\mathrm{d}y - \int_{x_0}^{x_1} \frac{\partial u}{\partial y}(x, y_0)\mathrm{d}x$$

defines a harmonic conjugate for u on $D(z_0; r)$ with $v(x_0, y_0) = c$.

If $v(x, y)$ is a harmonic conjugate, it must satisfy the Cauchy-Riemann equations.

Thus we have

$$\frac{\partial v}{\partial y}(x_1, y_1) = \frac{\partial}{\partial y}\left(c + \int_{y_0}^{y_1} \frac{\partial u}{\partial x}(x_1, y)\mathrm{d}y - \int_{x_0}^{x_1} \frac{\partial u}{\partial y}(x, y_0)\mathrm{d}x\right) = \frac{\partial u}{\partial x}(x_1, y_1)$$

We use the fact proven above,

$$\begin{aligned}
\frac{\partial v}{\partial x}(x_1, y_1) &= \frac{\partial}{\partial x}\left(c + \int_{y_0}^{y_1} \frac{\partial u}{\partial x}(x_1, y)\mathrm{d}y - \int_{x_0}^{x_1} \frac{\partial u}{\partial y}(x, y_0)\mathrm{d}x\right) \\
&= \frac{\partial}{\partial x}(v(x_1, y_1) - v(x_1, y_0)) - \frac{\partial u}{\partial y}(x_1, y_0) \\
\frac{\partial v}{\partial x}(x_1, y_0) &= -\frac{\partial u}{\partial y}(x_1, y_0)
\end{aligned}$$

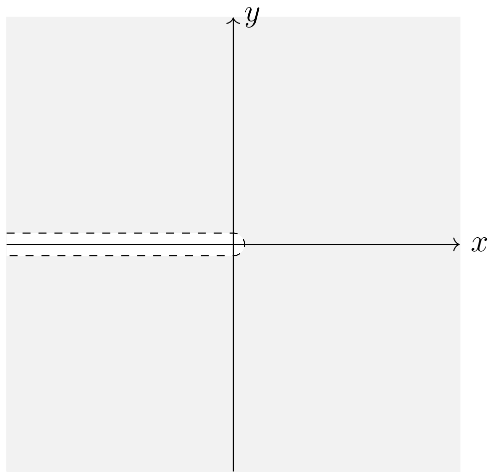
> **Remark.** An important theorem which may be worth introducing is the **Leibniz Integral Rule**,
>
> $$\frac{\mathrm{d}}{\mathrm{d}x}\int_{a(x)}^{b(x)} f(x,t)\mathrm{d}t = f(x, b(x))\frac{\mathrm{d}}{\mathrm{d}x}b(x) - f(x, a(x))\frac{\mathrm{d}}{\mathrm{d}x}a(x) + \int_{a(x)}^{b(x)} \frac{\partial}{\partial x} f(x,t)\mathrm{d}t$$
>
> The proof uses **Fubini's Theorem**, which states that
>
> $$\int_X \left(\int_Y f(x,y)\mathrm{d}y\right)\mathrm{d}x = \int_Y \left(\int_X f(x,y)\mathrm{d}x\right)\mathrm{d}y$$

1.6 Differentiation of Complex Functions – Built Diff

Principal Branch

Take $f(z) = \ln z = \ln|z| + i\arg z$. We may want $-\pi < \arg z < \pi$. This is a **principal branch** of the logarithm function.

This is due to the fact that $\ln|z| + i\arg z = \ln|z| + i(\arg z + 2k\pi)$.



Problem

Let u, v be real-valued functions on an open set $A \subset \mathbb{R}^2 = \mathbb{C}$ and suppose that they satisfy the Cauchy-Riemann equations on A. Show that

(a) $u_1 = u^2 - v^2$, $v_1 = 2uv$

(b) $u_2 = e^u \cos v$, $v_2 = e^u \sin v$

also satisfy the Cauchy-Riemann equations on A.

(a)

$$\frac{\partial u_1}{\partial x} = 2u\frac{\partial u}{\partial x} - 2v\frac{\partial v}{\partial x}$$

$$\frac{\partial v_1}{\partial y} = 2u\frac{\partial v}{\partial y} + 2v\frac{\partial u}{\partial y} = 2u\frac{\partial u}{\partial x} - 2v\frac{\partial v}{\partial x}$$

$$\frac{\partial u_1}{\partial y} = 2u\frac{\partial u}{\partial y} - 2v\frac{\partial v}{\partial y}$$

$$\frac{\partial v_1}{\partial x} = 2u\frac{\partial v}{\partial x} + 2v\frac{\partial u}{\partial x} = -2u\frac{\partial u}{\partial y} + 2v\frac{\partial v}{\partial y}$$

(b)

$$\frac{\partial u_2}{\partial x} = e^u\frac{\partial u}{\partial x}\cos v - \sin v\frac{\partial v}{\partial x}e^u$$

$$\frac{\partial v_2}{\partial y} = e^u\frac{\partial u}{\partial y}\sin v + \cos v\frac{\partial v}{\partial y}e^u = -e^u\frac{\partial v}{\partial x}\sin v + \cos v\frac{\partial u}{\partial x}e^u$$

$$\frac{\partial u_2}{\partial y} = e^u\frac{\partial u}{\partial y}\cos v - \sin v\frac{\partial v}{\partial y}e^u = -e^u\frac{\partial v}{\partial x}\cos v - \sin v\frac{\partial u}{\partial x}e^u$$

$$\frac{\partial v_2}{\partial x} = e^u\frac{\partial u}{\partial x}\sin v + \cos v\frac{\partial v}{\partial x}e^u$$

Remark. Let $f(z) = u + iv$, it's not hard to see that in (a), we have $g(z) = f(z)^2 = (u^2 - v^2) + 2uvi$ and in (b), we have $h(z) = e^{f(z)} = e^u \cos v + ie^u \sin v$.

Problem

Given functions $u(x, y)$, find their respective harmonic conjugates.

(a) $e^x(y\cos y + x\sin y)$

(b) $\dfrac{(e^{-y} + e^y)\sin x}{2}$

(a) We want

$$\frac{\partial u}{\partial x} = e^x(y\cos y + (x+1)\sin y) = \frac{\partial v}{\partial y}$$

So by the equation above we have

$$v = e^x(y\sin y + \cos y - (x+1)\cos y) + g(x) = e^x(y\sin y - x\cos y) + g(x)$$

On the other hand, we have

$$\frac{\partial u}{\partial y} = e^x(\cos y - y\sin y + x\cos y) = -\frac{\partial v}{\partial x}$$

Solving the differential equation gives

$$v = e^x y\sin y - e^x x\cos y + h(y)$$

Thus $g(x) = h(y) = C$ for some constant C. We have

$$v = e^x y\sin y - e^x x\cos y + C.$$

(b) We want

$$\frac{\partial u}{\partial x} = \frac{(e^{-y} + e^y)\cos x}{2} = \frac{\partial v}{\partial y}$$

So we have

$$v = \frac{(-e^{-y} + e^y)\cos x}{2} + g(x)$$

While on the other hand,

$$\frac{\partial u}{\partial y} = \frac{(-e^{-y} + e^y)\sin x}{2} = -\frac{\partial v}{\partial x}$$

Solving gives

$$v = \frac{(-e^{-y} + e^y)\cos x}{2} + h(y)$$

Comparing gives

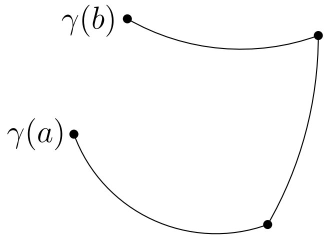
$$v = \frac{(-e^{-y} + e^y)\cos x}{2} + C$$

§2 Cauchy's Theorem

2.1 Contour Integrals are Contourversial

Let $z : [a, b] \to \mathbb{C}$ be a **curve**.

- If it is a continuous function, then it is a **smooth curve**.
- If we have $z'(a) = z'(b)$, then we say that the curve is **closed**.
- A curve is called **piecewise C^1** if we can divide the interval into subintervals $a = a_0 < a_1 < a_2 < \cdots < a_n = b$ such that $\gamma'(t)$ exists on the open intervals $(a_k, a_k + 1)$ and continuous on $[a_k, a_k + 1]$.



Let γ be a smooth curve, we denote the integral along γ as

$$\int_\gamma f(z)\mathrm{d}z = \int_a^b f(z(t))z'(t)\mathrm{d}t$$

Sometimes if we know that γ is a closed curve (known as the **cyclic integral**), we can write as

$$\oint_\gamma f(z)\mathrm{d}z$$

If it is known that the loop is directed clockwise or anticlockwise, it can still sometimes be written as

$$\ointclockwise_\gamma f(z)\mathrm{d}z \qquad \ointctrclockwise_\gamma f(z)\mathrm{d}z$$

Integration Properties

- In general, we have

$$\int_\gamma f = \sum_{i=0}^{n-1} \int_{a_i}^{a_{i+1}} f(\gamma(t))\gamma'(t)\mathrm{d}t$$

- Let a function be $f = u(x,y) + iv(x,y)$, we have

$$\int_\gamma f = \int_\gamma (u\mathrm{d}x - v\mathrm{d}y) + i\int_\gamma (u\mathrm{d}y + v\mathrm{d}x)$$

  Sketch of proof: Consider

$$\begin{aligned} f(\gamma(t))\gamma'(t) &= [u(x,y) + iv(x,y)][x'(t) + iy'(t)] \\ &= [u(x,y)x'(t) - v(x,y)y'(t)] + i[u(x,y)y'(t) + v(x,y)x'(t)] \\ \int_a^b f(\gamma(t))\gamma'(t)\mathrm{d}t &= \int_a^b [u(x,y)x'(t) - v(x,y)y'(t)]\mathrm{d}t + i\int_a^b [u(x,y)y'(t) + v(x,y)x'(t)]\mathrm{d}t \\ \int_\gamma f &= \int_\gamma (u\mathrm{d}x - v\mathrm{d}y) + i\int_\gamma (u\mathrm{d}y + v\mathrm{d}x) \end{aligned}$$

- An **opposite curve** of a curve γ is a curve (denoted as $-\gamma$) traversed oppositely.

Assume that $\gamma : [a, b] \to \mathbb{C}$ and $-\gamma : [a, b] \to \mathbb{C}$, we have

$$\gamma(t) = (-\gamma)(a + b - t)$$

- A **sum** $\gamma_1 + \gamma_2$ of curves is a curve constructed by joining the endpoints of $\gamma_1 : [a, b] \to \mathbb{C}$ and $\gamma_2 : [b, c] \to \mathbb{C}$. Thus

$$(\gamma_1 + \gamma_2)(t) = \begin{cases} \gamma_1(t) & t \in [a, b] \\ \gamma_2(t) & t \in [b, c] \end{cases}$$

  So we must have $\gamma_1(b) = \gamma_2(b)$.

- We have the following list of properties, which can be proved by the definitions above
  * $\displaystyle\int_\gamma \sum_{i=1}^n c_i f_i = \sum_{i=1}^n \left(c_i \int_\gamma f_i \right)$
  * $\displaystyle\int_{-\gamma} f = -\int_\gamma f$
  * $\displaystyle\int_{\gamma_1+\gamma_2+\cdots+\gamma_n} f = \sum_{i=1}^n \int_{\gamma_i} f$

- A **reparametrization** of a piecewise smooth curve $\gamma : [a, b] \to \mathbb{C}$ is the piecewise smooth curve $\tilde{\gamma} : [\tilde{a}, \tilde{b}] \to \mathbb{C}$ if there exists a piecewise C^1 function $\alpha : [a, b] \to [\tilde{a}, \tilde{b}]$ with
  - $\alpha'(t) > 0$ for all $t \in (a, b)$
  - $\alpha(a) = \tilde{a}$, $\alpha(b) = \tilde{b}$,
  - $\gamma(t) = \tilde{\gamma}(\alpha(t))$.

- We have

$$\int_\gamma f = \int_{\tilde{\gamma}} f$$

  Sketch of proof: Evaluating

$$\begin{aligned} \int_\gamma f &= \int_a^b f(\gamma(t))\gamma'(t)\mathrm{d}t \\ &= \int_a^b f(\tilde{\gamma}(\alpha(t)))\tilde{\gamma}'(\alpha(t))\alpha'(t)\mathrm{d}t \\ &= \int_{\tilde{a}}^{\tilde{b}} f(\tilde{\gamma}(s))\tilde{\gamma}'(s)\mathrm{d}s \\ &= \int_{\tilde{\gamma}} f \end{aligned}$$

- The **arc length** formula is given by

$$l(\gamma) = \int_a^b \sqrt{x'(t)^2 + y'(t)^2}\mathrm{d}t = \int_a^b |\gamma'(t)|\mathrm{d}t$$

Recall that the arc length formula in the Cartesian plane is

$$\int \mathrm{d}l = \int \sqrt{\mathrm{d}x^2 + \mathrm{d}y^2} = \int_c^d \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\mathrm{d}x$$

By recognizing that $\mathrm{d}x = x'(t)\mathrm{d}t$ and $\mathrm{d}y = y'(t)\mathrm{d}t$, we have

$$\begin{aligned}\int_c^d \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\mathrm{d}x &= \int_a^b \sqrt{1 + \left(\frac{y'(t)\mathrm{d}t}{x'(t)\mathrm{d}t}\right)^2}x'(t)\mathrm{d}t \\ &= \int_a^b \sqrt{x'(t)^2 + y'(t)^2}\mathrm{d}t\end{aligned}$$

For some continuous function f, we have

$$\left|\int_a^b f(t)\mathrm{d}t\right| \leq \int_a^b |f(t)|\mathrm{d}t$$

Sketch of proof: Let $\int_a^b f(t)\mathrm{d}t = re^{i\theta}$, then

$$\begin{aligned}\left|\int_a^b f(t)\mathrm{d}t\right| &= r \\ &= \int_a^b e^{-i\theta}f(t)\mathrm{d}t \\ &= \int_a^b \mathrm{Re}\left(e^{-i\theta}f(t)\right)\mathrm{d}t \\ &\leq \int_a^b \left|\mathrm{Re}\left(e^{-i\theta}f(t)\right)\right|\mathrm{d}t \\ &\leq \int_a^b \left|e^{-i\theta}f(t)\right|\mathrm{d}t \\ &= \int_a^b |f(t)|\,\mathrm{d}t\end{aligned}$$

Let $|f(z)| \leq M$ for some constant $M > 0$ and all z on γ, we have

$$\left|\int_\gamma f(z)\mathrm{d}z\right| \leq \int_\gamma |f(z)|\mathrm{d}z = \int_a^b |f(\gamma(t))||\gamma'(t)|\mathrm{d}t \leq M\int_a^b |\gamma'(t)|\mathrm{d}t = Ml(\gamma)$$

- **Fundamental Theorem of Calculus for Contour Integrals**: Recall that the fundamental theorem of calculus states that

$$\int_a^b F'(x)\mathrm{d}x = F(b) - F(a)$$

Suppose that $\gamma : [0,1] \to \mathbb{C}$ is a piecewise smooth curve, we have

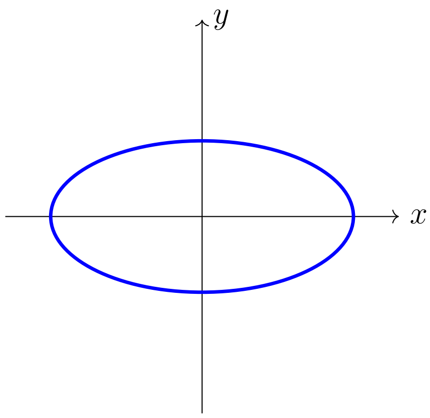
$$\int_\gamma F'(z)\mathrm{d}z = F(\gamma(1)) - F(\gamma(0))$$

If it happens that $\gamma(0) = \gamma(1)$ (which makes γ a loop) and $F'(z)$ **is analytic everywhere inside** γ, we have

$$\oint_\gamma F'(z)\mathrm{d}z = 0$$

Example

Let's evaluate $\displaystyle\int_\gamma z\mathrm{d}z$ and $\displaystyle\int_{\gamma_1} z\mathrm{d}z$ where γ is the portion of the ellipse $4x^2 + y^2 = 1$ joining $z = 1/2$ to $z = i$, and γ_1 is the entire ellipse, integrated counterclockwise.



A parametrization will be $\gamma(t) = \dfrac{\cos t}{2} + i\sin t$ and t ranges from 0 to 2π. Thus we have

$$\begin{aligned}
\int_\gamma z\mathrm{d}z &= \int_0^{\pi/2} \left(\frac{\cos t}{2} + i\sin t\right)\left(\frac{-\sin t}{2} + i\cos t\right)\mathrm{d}t \\
&= \int_0^{\pi/2} \left(\frac{-5\sin t\cos t}{4} + i\left(\frac{\cos^2 t - \sin^2 t}{2}\right)\right)\mathrm{d}t \\
&= -\frac{5}{8}
\end{aligned}$$

On the other hand, we have

$$\int_{\pi/2}^{\pi} \left(\frac{\cos t}{2} + i\sin t\right)\left(\frac{-\sin t}{2} + i\cos t\right)\mathrm{d}t = \frac{5}{8}$$

$$\int_{\pi}^{3\pi/2} \left(\frac{\cos t}{2} + i\sin t\right)\left(\frac{-\sin t}{2} + i\cos t\right)\mathrm{d}t = -\frac{5}{8}$$

$$\int_{3\pi/2}^{2\pi}\left(\frac{\cos t}{2}+i\sin t\right)\left(\frac{-\sin t}{2}+i\cos t\right)\,\mathrm{d}t=\frac{5}{8}$$

which indeed adds up to 0.

- **Path Independence Theorem**: Let f be a continuous function on an open connected set $G \in \mathbb{C}$,
  - For any closed curve Γ,
  $$\int_\Gamma f = 0$$
  - A result is that for any two curves γ_1, γ_2 joining z_0, z_1,
  $$\int_\Gamma f = \int_{\gamma_1} f + \int_{-\gamma_2} f = \int_{\gamma_1} f - \int_{\gamma_2} f = 0$$

Problem

Evaluate $\int_\gamma \sin 2z \,\mathrm{d}z$ where γ is the line segment joining $i+1$ to $-i$.

Let the parametrization be $\gamma(t) = (1-t) + i(1-2t)$ where $t \in [0,1]$, then

$$\begin{aligned}
\int_\gamma \sin 2z\,\mathrm{d}z &= \int_0^1 [\sin 2((1-t)+i(1-2t))](-1-2i)\,\mathrm{d}t\\
&= (-1-2i)\int_0^1 [\sin((2+2i)-t(2+4i))]\,\mathrm{d}t\\
&= \frac{-1-2i}{2+4i}(\cos(-2i)-\cos(2+2i))\\
&= -\frac{1}{2}\left(\frac{e^2+e^{-2}}{2}-\frac{e^{2i-2}+e^{2-2i}}{2}\right)
\end{aligned}$$

Problem

Evaluate $\int_\gamma \bar{z}^2\,\mathrm{d}z$ along two paths joining $(0,0)$ to $(1,1)$ as follows:

(a) γ is the straight line joining $(0,0)$ to $(1,1)$.

(b) γ is the broken line joining $(0,0)$ to $(1,0)$, then joining $(1,0)$ to $(1,1)$.

(a) We know that $\bar{z} = \mathrm{Re}\,(z) - \mathrm{Im}\,(z)$, consider the parametrization $\gamma : [0,1] \to \mathbb{C}$ be $\gamma(t) = t + it$,

$$\begin{aligned}
\int_\gamma \bar{z}^2\,\mathrm{d}z &= \int_0^1 (t-it)^2(1+i)\,\mathrm{d}t\\
&= (1+i)(1-i)^2\int_0^1 t^2\,\mathrm{d}t
\end{aligned}$$

$$= \frac{2-2i}{3}$$

(b) Similarly, let $\gamma_1 : [0,1] \to \mathbb{C}$ be $\gamma_1(t) = t$ and $\gamma_2 : [0,1] \to \mathbb{C}$ be $\gamma_2(t) = 1 + it$.

$$\begin{aligned}\int_\gamma \bar{z}^2 \, \mathrm{d}z &= \int_{\gamma_1} \bar{z}^2 \, \mathrm{d}z + \int_{\gamma_2} \bar{z}^2 \, \mathrm{d}z \\ &= \int_0^1 t^2 \, \mathrm{d}t + \int_0^1 (1-it)^2 \, \mathrm{d}t \\ &= \int_0^1 (1-2it) \, \mathrm{d}t \\ &= 1 - i\end{aligned}$$

Problem

Prove that

(a) $\left| \int_C \frac{\mathrm{d}z}{1+z^2} \right| \leq \frac{\pi}{3}$ where C is the arc of the circle $|z| = 2$ in the first quadrant.

(b) $\left| \int_\gamma \frac{\sin z}{z^2} \mathrm{d}z \right| \leq 2\pi e$ where γ is the unit circle.

(a) Consider the parametrization $C : [0, \pi/2] \to \mathbb{C}$ defined by $C(t) = 2(\cos t + i \sin t)$.

Since we have $\left| \frac{1}{1+z^2} \right| = \frac{1}{\sqrt{17 + 8\cos 2t}} \leq \frac{1}{3}$,

$$\begin{aligned}\left| \int_C \frac{\mathrm{d}z}{1+z^2} \right| &\leq \int_C \left| \frac{\mathrm{d}z}{1+z^2} \right| \\ &\leq \frac{1}{3} \cdot 2 \cdot \frac{\pi}{2} \\ &= \frac{\pi}{3}\end{aligned}$$

(b) Consider the parematrization $\gamma : [0, 2\pi] \to \mathbb{C}$ defined by $\gamma(t) = \cos t + i \sin t$.

Since

$$\begin{aligned}\left| \frac{\sin z}{z} \right| &= \left| \frac{e^{-\sin t + i\cos t} - e^{\sin t - i \cos t}}{2i(\cos 2t + i \sin 2t)} \right| \\ &= \frac{1}{2}\sqrt{\left(e^{-\sin t} - e^{\sin t}\right)^2 (\cos\cos t)^2 + \left(e^{-\sin t} + e^{\sin t}\right)^2 (\sin \cos t)^2} \\ &= \frac{1}{2}\sqrt{e^{-2\sin t} + e^{2\sin t} - 2\cos(2\cos t)} \\ &\leq \frac{1}{2}\left(e^{\sin t} + e^{-sint}\right)\end{aligned}$$

$$\begin{aligned} &\le \frac{1}{2}(e+e) \\ &\le e \end{aligned}$$

On the other hand, since $l(\gamma) = 2\pi$,

$$\begin{aligned} \left| \int_\gamma \frac{\sin z}{z^2} \mathrm{d}z \right| &\le \int_\gamma \left| \frac{\sin z}{z^2} \right| \mathrm{d}z \\ &\le e \cdot 2\pi \end{aligned}$$

Problem

Show that the arc length $l(\gamma)$ of a curve γ is unchanged if γ is reparametrized.

Consider the reparametrization $\tilde{\gamma} : [\tilde{a}, \tilde{b}] \to \mathbb{C}$ defined by $\gamma(t) = \tilde{\gamma}(\alpha(t))$, $\alpha(a) = \tilde{a}$, $\alpha(b) = \tilde{b}$ and $\alpha'(t) > 0$ for all $t \in (a, b)$.

Then

$$\begin{aligned} l(\gamma) &= \int_a^b |\gamma'(t)| \mathrm{d}t \\ &= \int_a^b |\tilde{\gamma}(\alpha(t))| \alpha'(t) \mathrm{d}t \\ &= \int_{\tilde{a}}^{\tilde{b}} |\tilde{\gamma}(t)| \mathrm{d}t \\ &= l(\tilde{\gamma}) \end{aligned}$$

2.2 Cauchy's Theorem

Cauchy's Theorem states that if γ is a closed curve intersecting itself **only at its endpoints**, then

$$\int_\gamma f = 0$$

Green's Theorem: For continuously differentiable functions $P(x,y)$ and $Q(x,y)$, we have

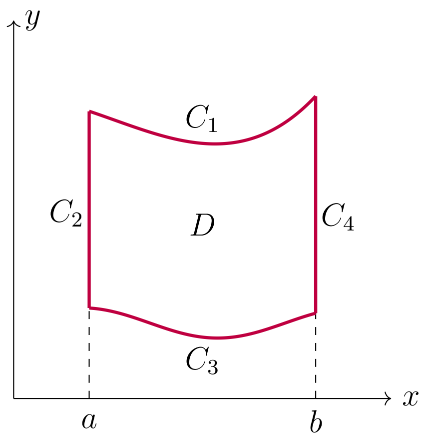
$$\oint_\gamma P(x,y)\mathrm{d}x + Q(x,y)\mathrm{d}y = \iint_A \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \mathrm{d}A$$

Remark. Green's Theorem is a special case of the **Kelvin-Stokes Theorem** (or sometimes known as the Fundamental Theorem of Curls), stated that for some smooth oriented surface Σ in $\mathbb{R}^3$ with boundary $\partial\Sigma$,

$$\iint_\Sigma \left(\left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z} \right) \mathrm{d}y\, \mathrm{d}z + \left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x} \right) \mathrm{d}z\, \mathrm{d}x + \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right) \mathrm{d}x\, \mathrm{d}y \right)$$

$$= \oint_{\partial\Sigma} \Big(F_x \, \mathrm{d}x + F_y \, \mathrm{d}y + F_z \, \mathrm{d}z \Big).$$

Sketch of proof:



By integrating along C_1, we have

$$\int_{C_1} P(x,y)\mathrm{d}x = \int_b^a P(x, g_1(x))\mathrm{d}x$$

Similarly,

$$\int_{C_3} P(x,y)\mathrm{d}x = \int_a^b P(x, g_3(x))\mathrm{d}x$$

On the other hand, $\displaystyle\int_{C_2} P(x,y)\mathrm{d}x = \int_{C_4} P(x,y)\mathrm{d}x = 0.$ As a result we have

$$\begin{aligned}
\int_{C_1+C_2+C_3+C_4} P(x,y)\mathrm{d}x &= \int_{C_1} P(x,y)\mathrm{d}x + \int_{C_2} P(x,y)\mathrm{d}x + \int_{C_3} P(x,y)\mathrm{d}x + \int_{C_4} P(x,y)\mathrm{d}x \\
&= \int_a^b P(x, g_3(x))\mathrm{d}x - \int_a^b P(x, g_1(x))\mathrm{d}x \\
&= \int_a^b [P(x, g_3(x)) - P(x, g_1(x))]\mathrm{d}x \\
&= \int_a^b \int_{g_1(x)}^{g_3(x)} \frac{\partial P}{\partial y}\mathrm{d}y\mathrm{d}x \\
&= -\iint_D \frac{\partial P}{\partial y}\mathrm{d}A
\end{aligned}$$

Similarly one can get

$$\int_C Q(x,y)\mathrm{d}y = \iint_D \frac{\partial Q}{\partial x}\mathrm{d}A$$

Yet, summing up gives our result

$$\oint_\gamma P(x,y)\mathrm{d}x + Q(x,y)\mathrm{d}y = \iint_A \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \mathrm{d}A$$

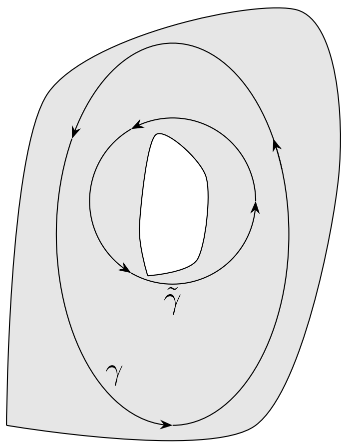
We can express our complex integral

$$\int_\gamma f = \iint_A \left[-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right] \mathrm{d}y\mathrm{d}x + i \iint_A \left[\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right] \mathrm{d}y\mathrm{d}x$$

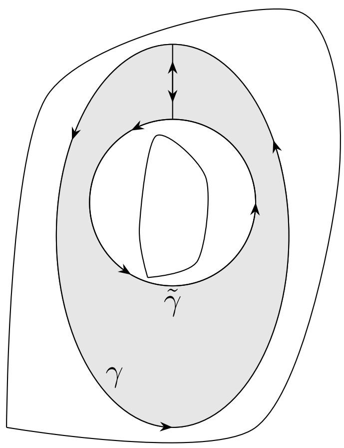
Deformation Theorem: Let f be analytic on a region A and let γ be a simple closed curve in A. We assume that we can γ can be deformed to another simple closed curve $\tilde{\gamma}$ without passing outside A.

We say γ is **homotopic** to $\tilde{\gamma}$ in A. Then we have

$$\int_\gamma f = \int_{\tilde{\gamma}} f$$



This is due to the fact that, we can construct some curve γ_0 and $-\gamma_0$, thus the curve $\displaystyle\int_{\gamma+\gamma_0-\tilde{\gamma}+(-\gamma_0)} f = 0$.



Problem

Evaluate

(a) $\int_\gamma (z^3+3)\mathrm{d}z$, where γ is the unit circle.

(b) $\int_\gamma \cos[3+1/(z-3)]\mathrm{d}z$, where γ is a circle of radius 3 centered at $5i+1$.

(c) $\int_\gamma \sqrt{z^2-1}\mathrm{d}z$ where γ is the circle of radius $1/2$ centered at 0.

(d) $\int_\gamma \frac{2z^2-15z+30}{z^3-10z^2+32z-32}\mathrm{d}z$ where γ is the circle $|z|=3$.

(a) Since z^3+3 is analytic everywhere in γ, we have

$$\int_\gamma (z^3+3)\mathrm{d}z = 0$$

(b) Similarly we have

$$\int_\gamma \cos[3+1/(z-3)]\mathrm{d}z = 0$$

(c) Consider the parametrization $\gamma(t) = \frac{1}{2}e^{i\theta}$ for $\theta \in [0, 2\pi]$. We have

$$\begin{aligned}\int_\gamma \sqrt{z^2-1}\mathrm{d}z &= \int_0^{2\pi} \sqrt{\frac{1}{4}e^{2i\theta}-1}\cdot\frac{1}{2}e^{i\theta}\mathrm{d}\theta \\ &= \int_0^{\pi} \sqrt{\frac{1}{4}e^{2i\theta}-1}\cdot\frac{1}{2}e^{i\theta}\mathrm{d}\theta + \int_\pi^{2\pi} \sqrt{\frac{1}{4}e^{2i\theta}-1}\cdot\frac{1}{2}e^{i\theta}\mathrm{d}\theta \\ &= \int_0^{\pi} \sqrt{\frac{1}{4}e^{2i\theta}-1}\cdot\frac{1}{2}e^{i\theta}\mathrm{d}\theta - \int_0^{\pi} \sqrt{\frac{1}{4}e^{2i\theta}-1}\cdot\frac{1}{2}e^{i\theta}\mathrm{d}\theta \\ &= 0\end{aligned}$$

(d) By using partial fractions gives

$$\int_\gamma \frac{2z^2-15z+30}{z^3-10z^2+32z-32}\mathrm{d}z = \int_\gamma \left(\frac{2}{z-2}+\frac{1}{(z-4)^2}\right)\mathrm{d}z$$

We have $\int_\gamma \frac{1}{(z-4)^2}\mathrm{d}z = 0$, and $\int_\gamma \frac{1}{z-2}\mathrm{d}z = \int_{|z-2|=1} \frac{1}{z-2}\mathrm{d}z = 2\pi i$, we obtain

$$\int_\gamma \left(\frac{2}{z-2}+\frac{1}{(z-4)^2}\right)\mathrm{d}z = 4\pi i$$

Problem

Let f be entire. Evaluate

$$\int_0^{2\pi} f(z_0 + re^{i\theta})e^{ki\theta}\mathrm{d}\theta$$

for k an integer, $k \geq 1$.

Let $\gamma(\theta) = z_0 + re^{i\theta}$ with $\theta \in [0, 2\pi]$, then since f is entire, f is holomorphic everywhere in $\mathbb{C}$. So $f(\gamma(\theta))e^{ki\theta}$ is holomorphic everywhere in γ. By applying Cauchy's Theorem, we have

$$\int_0^{2\pi} f(z_0 + re^{i\theta})e^{ki\theta}\mathrm{d}\theta = 0$$

Problem

Let γ_1 be the circle of the radius 1 and let γ_2 be the circle of radius 2 (traversed counterclockwise and centered at the origin). Show that

$$\int_{\gamma_1} \frac{\mathrm{d}z}{z^3(z^2+10)} = \int_{\gamma_2} \frac{\mathrm{d}z}{z^3(z^2+10)}$$

Let $f(z) = \dfrac{1}{z^3(z^2+10)}$, and γ' is a curve connecting γ_1 and γ_2, then we have

$$\int_{\gamma_1+\gamma'-\gamma_2+(-\gamma')} f = 0 \Longleftrightarrow \int_{\gamma_1} f = \int_{\gamma_2} f$$