

MA1100T Quick Notes

§1 Logic

1.1 Statement vs. Proposition vs. Predicate

- A **statement** is a sentence.
- A **proposition** is a statement that is either true or false, but not both.
- A **predicate** is an assignment of truth values to elements of some domain.

1.2 Implications

“On Wednesdays, we wear pink.” – *Mean Girls*

Denoted as $p \rightarrow q$, if it is Wednesday today, then I should probably wear pink. If it is not Wednesday, it doesn't mean I cannot wear pink, so we have

p	q	$p \rightarrow q$
0	1	1

Moreover, $p \rightarrow q \equiv \neg p \vee q$.

1.3 Logical Equivalence

- Two propositional formulas are **logically equivalence** if \forall assignment of truth values of propositions, they have the same truth value.
- Two propositional formulas are **not logically equivalence** if \exists assignment of truth values of propositions, they have different truth values.

1.4 If and only if

$$p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$$

p	q	$p \leftrightarrow q$
1	1	1
1	0	0
0	1	0
0	0	1

1.4.1 Necessity and Sufficiency

- If $p \rightarrow q$ then p is **sufficient** for q .
- If $p \leftarrow q$ then p is **necessary** for q .

Example (SJTU Mathematics Contest 2023)

If α : “The equation of a hyperbola is $x^2 - y^2 = a^2$, $a > 0$ ” and β : “The asymptotes of this hyperbola form an angle of $\pi/2$ ”.

α	β	α implies β ?	β implies α ?
1	1	True	False since $(x-1)^2 - y^2 = a^2$ also works
1	0	False since β is true	False since the angle equals $\pi/2$
0	1	True, consider $(x-1)^2 - y^2 = a^2$	False since $(x-1)^2 - y^2 = a^2$ also works
0	0	True, consider $x^2 - 4y^2 = a^2$ and $\theta = 52.13^\circ$	True, if $\theta \neq \pi/2$, the equation will not be $x^2 - y^2 = a^2$

As we can see, “ α implies β ” is tally to “ $\alpha \rightarrow \beta$ ” but “ β implies α ” does not tally to “ $\beta \rightarrow \alpha$ ” so for β , α is **sufficient** but not **necessary**.

1.5 Tautology & Contradiction

- If $F_1 \equiv F_2$, then $F_1 \leftrightarrow F_2$ is a **tautology**.
- If $F_1 \not\equiv F_2$, then $F_1 \leftrightarrow F_2$ is a **contradiction**.

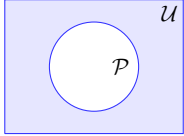
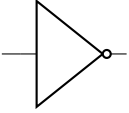
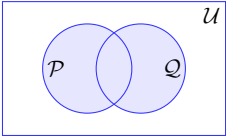
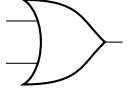
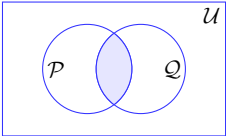
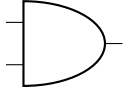
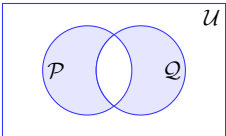
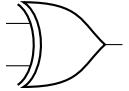
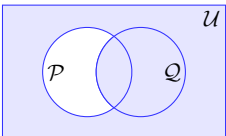
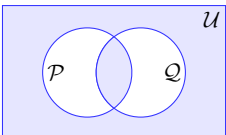
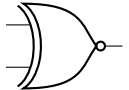
1.6 Useful Denial

F_2 is a **useful denial** of F_1 iff $F_1 \equiv \neg F_2$.

1.7 Boolean Algebra in Words

- $p \rightarrow q$: shown in §1.2
- $p \wedge \neg p$ is a contradiction: Nothing can be both true and false at the same time.
- $p \vee \neg p$ is a tautology: Something must either be true or false right?
- $\neg(P \vee Q) \equiv (\neg P) \wedge (\neg Q)$: Let say if I ask you: “Tea or coffee?”. If you want tea but not coffee, then P is true and Q is false, negating lets you drink coffee but not tea, you only drank coffee in the end.
- $\neg(\forall x)(P(x)) \equiv (\exists x)(\neg P(x))$: I don’t drink coffee every day, I drank tea on Wednesday.
- $\neg(\exists x)(P(x)) \equiv (\forall x)(\neg P(x))$: Haha you can’t find me drinking coffee for all days – I drank tea every day.

1.8 Boolean, Set Theory, Bitwise Operations, Logic Gates

Boolean	Set Theory	Bitwise	Logic Gates
$\neg p$	 <p>p^c</p> <p>Complement</p>	$\text{NOT } x = (2^{\lfloor \log_2 x \rfloor + 1} - 1) - x$ $\text{NOT } 10110101 = 01001010$	
$p \vee q$	 <p>$p \cup q$</p> <p>Union</p>	$101101 \text{ OR } 011001 = 111101$ $\begin{array}{r} 101101 \\ \text{OR } 011001 \\ \hline 111101 \end{array}$	
$p \wedge q$	 <p>$p \cap q$</p> <p>Intersection</p>	$101101 \text{ AND } 011001 = 001001$ $\begin{array}{r} 101101 \\ \text{AND } 011001 \\ \hline 001001 \end{array}$	
$p \oplus q$	 <p>$p \Delta q$</p> <p>Symmetric Difference</p>	$101101 \text{ XOR } 011001 = 110100$ $\begin{array}{r} 101101 \\ \text{XOR } 011001 \\ \hline 110100 \end{array}$	
$p \rightarrow q$	 <p>$p \rightarrow q$</p> <p>Implies</p>	$101101 \text{ THEN } 011001 = 011011$ $\begin{array}{r} 101101 \\ \text{THEN } 011001 \\ \hline 011011 \end{array}$	
$p \leftrightarrow q$	 <p>$p \leftrightarrow q$</p> <p>Equivalent</p>	$101101 \text{ XNOR } 011001 = 001011$ $\begin{array}{r} 101101 \\ \text{XNOR } 011001 \\ \hline 001011 \end{array}$	

1.9 More on Quantifiers

- The **universal quantification** $\forall xP(x)$ states that “**for all** x such that $P(x)$ is true”.
- The **existential quantification** $\exists xP(x)$ states that “**there exists** x such that $P(x)$ is true”.

1.9.1 Equivalent Formulae on Finite Sets

Let the domain of $P(x)$ be a set with finite cardinality, assign the elements $x_1, x_2, x_3, \dots, x_k$,

- $\forall xP(x) \equiv \bigwedge_{i=1}^k P(x_i)$
- $\exists xP(x) \equiv \bigvee_{i=1}^k P(x_i)$

By the inductive process, we can also deduce De Morgan’s law.

$$\neg \left(\bigwedge_{i=1}^k P(x_i) \right) \equiv \bigvee_{i=1}^k (\neg P(x_i))$$

1.9.2 Quantifiers Overloading

Consider the following sentence

“Everyone who takes a break can have a snack.”

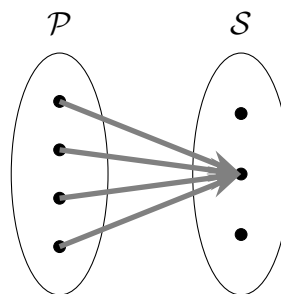
Let \mathcal{P} be the set of people, and \mathcal{S} be the set of all snacks. We say that $H(p, s)$ equals “person p had snack s ”. Consider the following formulae:

(a)
$$(\exists s)(\forall p)H(p, s)$$

In words, we have

“There exists a snack s (probably KitKat) such that every person p , p ate s .”

It is an **all-to-one** situation here.



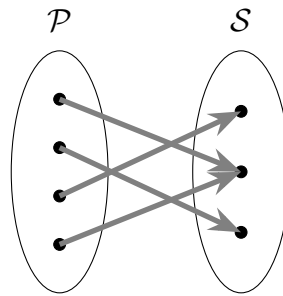
(b)

$$(\forall p)(\exists s)H(p, s)$$

In words, we have

“For all people p , there exists a snack s such that p ate s .”

It is a **one-to-one** situation here.



Moreover, we can stack quantifiers.

- x is irrational: $(\forall p \in \mathbb{Z})(\forall q \in \mathbb{Z})(x \neq p/q) \equiv (\nexists (p, q) \in \mathbb{Z}^2)(x = p/q)$.
- x is rational: $(\exists p \in \mathbb{Z})(\exists q \in \mathbb{Z})(x = p/q) \equiv \neg(\forall (p, q) \in \mathbb{Z}^2)(x \neq p/q)$.

§2 Proofs

No notes GG.

2.1 Tutorial 1

Problem 2.1

In this problem we consider binary logical operators in propositional logic.

1. How many are there?
2. Prove that there are exactly two binary logical operators \star in propositional logic such that $p \star (p \vee q)$ is a tautology.

1. $2^{2 \times 2} = 16$ of them
- 2.

p	q	$p \vee q$	$p \star (p \vee q)$
1	1	1	1
1	0	1	1
0	1	1	1
0	0	0	1

Since we know

p	q	$p \star q$
1	1	1
0	1	1
0	0	1

But we don't know the case when p is true and q is false – there are two possibilities, hence there exists two such binary logical operators.

Problem 2.2

Define predicates $P(x)$ and $Q(x)$ with the same domain such that the formulas $\exists x(P(x) \rightarrow Q(x))$ and $(\exists xP(x)) \rightarrow (\exists xQ(x))$ have different truth values. Be sure to include brief justifications for the truth values of the two formulas (i.e., write down which one is true, which one is false, and why they are each true or false.)

Since $\exists x(P(x) \rightarrow Q(x))$ is a *useful denial* of $(\exists xP(x)) \rightarrow (\exists xQ(x))$, we must have

$$\begin{aligned}\neg \exists x(P(x) \rightarrow Q(x)) &\equiv (\exists xP(x)) \rightarrow (\exists xQ(x)) \\ \forall x(\neg(P(x) \rightarrow Q(x))) &\equiv (\exists xP(x)) \rightarrow (\exists xQ(x)) \\ \forall x(\neg(\neg P(x) \vee Q(x))) &\equiv (\neg \exists xP(x)) \vee (\exists xQ(x)) \\ \forall x(P(x) \wedge \neg Q(x)) &\equiv (\neg \exists xP(x)) \vee (\exists xQ(x))\end{aligned}$$

If $\forall x(P(x) \wedge \neg Q(x))$ is true, then $\forall x, P(x)$ and $\neg Q(x)$ are true, but this implies $\neg \exists xP(x)$ and $\exists xQ(x)$ are false, hence $(\neg \exists xP(x)) \vee (\exists xQ(x))$ is false, no solution.

If $\forall x(P(x) \wedge \neg Q(x))$ is false, then $\neg \exists xP(x)$ and $\exists xQ(x)$ are false.

Set $P(x)$ be “ x is odd” and $Q(x)$ be “ x is prime”. Let $9 \in \mathcal{U}$ so $\exists xP(x)$. Next choose $4 \in \mathcal{U}$ so $\neg \exists xQ(x)$ is true and $\forall x(P(x) \wedge \neg Q(x))$ is false.

Therefore an answer is $\boxed{P(x): \text{“}x \text{ is odd”}, Q(x): \text{“}x \text{ is prime”}, \mathcal{U} = \{4, 9\}}.$

The statement $\exists x(P(x) \rightarrow Q(x))$ is true whereas $(\exists xP(x)) \rightarrow (\exists xQ(x))$ is false.

Problem 2.3

Same question as the previous, except with the formulas $\forall x(P(x) \vee Q(x))$ and $(\forall xP(x)) \vee (\forall xQ(x))$.

Negating gives

$$\begin{aligned}\neg(\forall x)(P(x) \vee Q(x)) &\equiv (\exists x\neg(P(x) \vee Q(x))) \\ \exists x(\neg(P(x) \vee Q(x))) &\equiv (\exists x\neg P(x)) \wedge (\exists x\neg Q(x)) \\ \exists x(\neg P(x) \wedge \neg Q(x)) &\equiv (\exists x\neg P(x)) \wedge (\exists x\neg Q(x))\end{aligned}$$

Assume $\forall x P(x)$ and $\forall x Q(x)$ are false, then $\exists x(\neg P(x) \wedge \neg Q(x))$ is false. This implies that $\forall x(P(x) \vee Q(x))$ is true.

Set $P(x)$ be “ x is odd” and $Q(x)$ be “ x is prime”. Let $9 \in \mathcal{U}$ so $\forall x Q(x)$ is false. Next, choose $4 \in \mathcal{U}$ so $\forall x P(x)$ is false and $\forall x(P(x) \vee Q(x))$ is true.

Therefore an answer is $\boxed{P(x): \text{“}x \text{ is odd”}, Q(x): \text{“}x \text{ is prime”}, \mathcal{U} = \{2, 9\}}$.

The statement $\forall x(P(x) \vee Q(x))$ is true whereas $(\forall x P(x)) \vee (\forall x Q(x))$ is false.

Problem 2.4

Prove that if m and n are integers such that mn is even, then either m is even or n is even. In your proof, point out where you are using the following facts (1): every integer is either even or odd; (2) no integer is both even and odd.

We will prove by contradiction. For the sake of contradiction, assume m, n are odd integers by (1). We can express $m = 2k + 1$ and $n = 2\ell + 1$ for some integer k, ℓ . Thus

$$mn = (2k + 1)(2\ell + 1) = 2(2k\ell + k + \ell) + 1$$

which is odd by (2) since it is not divisible by 2.

Problem 2.5

Prove that for every two distinct rational numbers, there is an irrational number in between them. (Recall we proved in lecture that $\sqrt{2}$ is irrational. If you want to claim any other number is irrational, you have to prove your claim first.)

Assume that the two rational numbers are a/b and c/d respectively (a, b, c, d are integers throughout this problem), and that $a/b < c/d$. Choose the number $\frac{a}{b} + \frac{\sqrt{2}}{C}$ for sufficiently large C .

Claim — For some rational number r/s where r, s are coprime integers, and an integer C , the number $\frac{r}{s} + \frac{\sqrt{2}}{C}$ is irrational.

For the sake of contradiction, assume that $\frac{r}{s} + \frac{\sqrt{2}}{C} = \frac{m}{n}$ for some coprime integers m, n . A quick simplification yields

$$2(ns)^2 = C^2(ms - nr)^2$$

This implies that $C(ms - nr)$ is even. In any case, we have $\nu_2(2(ns)^2)$ is odd but $\nu_2(C^2(ms - nr)^2)$ is even, hence a contradiction.

Problem 2.6

Prove or disprove: The sum of two irrational numbers is irrational.

Choose $1 \pm \sqrt{2}$, thus $(1 + \sqrt{2}) + (1 - \sqrt{2}) = 2$ is rational.

§3 Induction

3.1 Some Examples

Problem (McCarthy)

Let x be an integer, and

$$f(x) = \begin{cases} x - 10 & x > 100; \\ f(f(x + 11)) & x \leq 100 \end{cases}$$

Then $f(x) = 91$ for all $x \leq 101$.

If $90 \leq x \leq 99$, then $f(x) = f(f(x + 11)) = f(x + 1)$.

If $x = 100$, then $f(100) = f(f(111)) = f(101) = 91$.

This concludes that for all integers $90 \leq x \leq 100$ we have $f(x) = 91$.

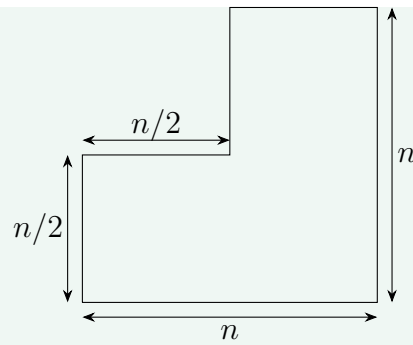
And by the inductive process, let's say $90 - 11k \leq x \leq 100 - 11k$ for some nonnegative integer k . We've proved that case $k = 0$. For other k , $f(f(x)) = f(x + 11)$ which can be proved by induction.

Problem (Golomb, 1965)

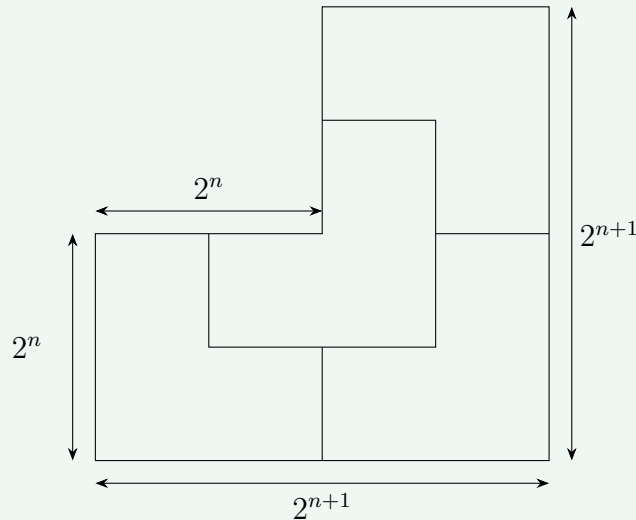
Prove that any $2^n \times 2^n$ board with one square removed can be tiled with L-shaped triominoes.

Indeed, we have to use induction twice, as follows:

Claim — Define an ℓ -block of size n to be a shape on the board as follows:



then an ℓ -block of size 2^{n+1} can be tiled by ℓ -blocks of size 2^n . A construction can easily show this



Base Case: $n = 1$ is just an L-shaped triomino.

Inductive Step: is shown in the construction above.

Now we have to deal with the entire square.

Claim — Any $2^n \times 2^n$ board with one square removed can be tiled with L-shaped triominoes.

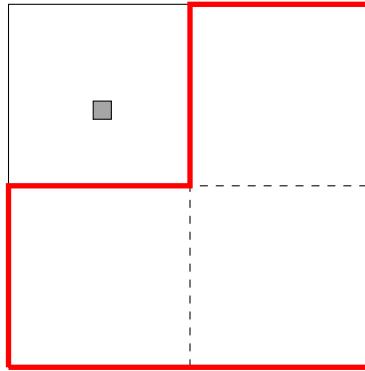
We'll proceed by induction.

Assume that for some positive integer n , any $2^n \times 2^n$ board with one square removed can be tiled with L-shaped triominoes.

Base Case: $n = 1$ can be tiled with only one L-shaped triomino.

Inductive Step: Let the square removed be X . Divide the $2^{n+1} \times 2^{n+1}$ board into four equal smaller $2^n \times 2^n$ boards. By the pigeonhole principle, exactly one of the $2^n \times 2^n$ boards must contain X . Let this $2^n \times 2^n$ board be B . The three other $2^n \times 2^n$ boards form an ℓ -block of size 2^{n+1} , which can be tiled.

On the other hand, B can be tiled according to our inductive hypothesis.



3.1.1 A False Induction Proof

“Assume that for any $n \geq 1$ horses, they have the same color.” Taking the first n horses and last n horses, we can conclude that $n + 1$ horses have the same color.

The problem lies

3.1.2 Well-Ordering Principle

Theorem (Well-Ordering Principle)

An **ordered set** is said to be **well-ordered** if each and every nonempty subset has a smallest or least element.

This is the foundation of proof by infinite descent.

3.2 Beyond Induction: Infinite Descent

Problem (IMO 1988)

Let a and b be positive integers such that $ab + 1$ divides $a^2 + b^2$. Show that

$$\frac{a^2 + b^2}{ab + 1}$$

is the square of an integer.

FTSOC, let $\frac{a^2 + b^2}{ab + 1} = k$ for some nonsquare integer k . Rearranging gives

$$a^2 - kab + (b^2 - k) = 0$$

Assume that (a, b) is a solution and WLOG assume $a \geq b$. FTSOC let $a + b$ be minimal. Then (a', b) is also a solution by the quadratic equation in a . By the Vieta's formulas, we have

$$\begin{cases} a + a' = kb \\ aa' = b^2 - k \end{cases}$$

By the first equation, a' is an integer. And by the second equation

$$b^2 - k \geq aa' \geq ba' \Rightarrow b > b - \frac{b}{k} \geq a'$$

Therefore $a > a'$ which leads to a contradiction.

3.3 Tutorial 2

Problem 3.1

For each of the following sentences, explain (in one sentence) whether it is true or false. The scope of x and y is taken to be \mathbb{R} .

1. $\forall x \exists y (x^2 = y)$
2. $\forall y \exists x (x^2 = y)$
3. $\exists x \forall y (x^2 = y)$
4. $\exists y \forall x (x^2 = y)$
5. $\forall x \forall y (x^2 = y)$
6. $\exists x \exists y (x^2 = y)$

1. **True**, the value of y is uniquely determined by the value of x^2 .
2. **False**, choose a negative y .
3. **False**, $x^2 = y$ but $x^2 \neq y + 1$ for every real y .
4. **False**, if $y = 0$ choose positive x , conversely if $y^2 > 0$ choose $x = 0$.
5. **False**, $1^2 \neq 0$.
6. **True**, choose $x = y = 0$.

Problem 3.2

Prove that for every nonzero integer a and b , there is a positive integer c such that a and b both divide c . (Remember, we don't know yet that the lcm exists.) Then prove that there is a smallest positive integer c such that a and b both divide c .

- (i) Choose $c = |ab| = |a||b|$.

Claim — For all nonzero integer n and a positive integer m , n divides m if and only if $|n|$ divides m .

If n is positive, then there's nothing to consider.

If $n < 0$ divides m , then m/n is an integer. $|n|/m = -n/m$ which is an integer so $|n|$ divides m .

If $|n|/m = -n/m$ is an integer, then n/m is an integer as well.

So by (i) we know that there exists such c . Moreover, we know that $c/|a|$ and $c/|b|$ are natural numbers. Assume otherwise, c does not have a smallest value, then $c/|a|$ does not have a smallest value, contradicting to the Well-Ordering principle.

Problem 3.3

Use induction to prove that for every n in \mathbb{N} , 3 divides $n^3 - n$. (If you want to use facts about division modulo 3 you have to prove them first. The proof I have in mind doesn't use such facts.)

Base Case: $n = 0$, then $n^3 - n = 0 = 3 \times 0$ so 3 divides $n^3 - n$.

Inductive Step: Assume that $n^3 - n$ is divisible by 3 for some n in \mathbb{N} . Let $n^3 - n = 3k$ for some integer k . We hope to prove that $(n+1)^3 - (n+1)$ is divisible by 3.

$$\begin{aligned} (n+1)^3 - (n+1) &= n^3 + 3n^2 + 2n \\ &= (n^3 + 3n^2 + 2n + 2(3k)) - 2(3k) \\ &= (3n^3 + 3n^2) - 2(3k) \\ &= 3(n^3 + n^2 - 2k) \end{aligned}$$

Problem 3.4

Define a sequence $\{a_n\}_n$ of real numbers by $a_0 = 0$ and $a_{n+1} = (a_n)^2 + \frac{1}{4}$ for $n \in \mathbb{N}$. Prove that for all n in \mathbb{N}^+ , we have $0 < a_n < 1$.

It is easy to show that $a_{n+1} > 0$ by $(a_n)^2 + \frac{1}{4} \geq \frac{1}{4} > 0$ for all $n \in \mathbb{N}$.

Claim — $a_n < \frac{1}{2}$ for all $n \in \mathbb{N}$.

We will prove by induction.

Base Case: $a_0 = 0 < \frac{1}{2}$.

Inductive Step: Assume that for some $n \in \mathbb{N}$, we have $a_n < \frac{1}{2}$, we wish to prove that $a_{n+1} < \frac{1}{2}$.

By

$$a_{n+1} = (a_n)^2 + \frac{1}{4} < \left(\frac{1}{2}\right)^2 + \frac{1}{4} = \frac{1}{2}$$

Hence each term $a_n, n \in \mathbb{N}^+$ satisfy $0 < a_n < \frac{1}{2} < 1$.

Problem 3.5

What rule(s) of addition for real numbers (among associativity and commutativity) guarantees that $(a + b) + (c + d) = a + (b + (c + d))$? Use induction to prove that for every $n \in \mathbb{N}^+$ and every sequence of real numbers $\{a_i\}_{i=0}^n$ and $\{b_i\}_{i=0}^n$, we have $\sum_{i=0}^n (a_i + b_i) = \sum_{i=0}^n a_i + \sum_{i=0}^n b_i$. In your proof, specify which rules of addition you use and where.

Left associativity:

$$(a + b) + (c + d) = a + b + (c + d)$$

Right associativity:

$$a + b + (c + d) = a + (b + (c + d))$$

We will proceed by induction.

Base Case: $(a_1) + (b_1) = (a_1 + b_1)$ which is obviously true.

Inductive Step: We assume that $\sum_{i=0}^n (a_i + b_i) = \sum_{i=0}^n a_i + \sum_{i=0}^n b_i$, we want

$$\sum_{i=0}^n (a_i + b_i) + (a_{n+1} + b_{n+1}) = \left(\sum_{i=0}^n a_i + a_{n+1} \right) + \left(\sum_{i=0}^n b_i + b_{n+1} \right)$$

Let $p \stackrel{L}{=} q$ if q can be shown equal to p by using left associativity. Similarly define $p \stackrel{R}{=} q$ for right associativity and $p \stackrel{C}{=} q$ for commutativity.

From the right, we have

$$\begin{aligned} \left(\sum_{i=0}^n a_i + a_{n+1} \right) + \left(\sum_{i=0}^n b_i + b_{n+1} \right) &\stackrel{L}{=} \sum_{i=0}^n a_i + a_{n+1} + \left(\sum_{i=0}^n b_i + b_{n+1} \right) \\ &\stackrel{R}{=} \sum_{i=0}^n a_i + \left(a_{n+1} + \left(\sum_{i=0}^n b_i + b_{n+1} \right) \right) \end{aligned}$$

While from the left, using our inductive hypothesis, we have

$$\begin{aligned} \sum_{i=0}^n (a_i + b_i) + (a_{n+1} + b_{n+1}) &= \sum_{i=0}^n a_i + \sum_{i=0}^n b_i + (a_{n+1} + b_{n+1}) \\ &\stackrel{R}{=} \sum_{i=0}^n a_i + \left(\sum_{i=0}^n b_i + (a_{n+1} + b_{n+1}) \right) \\ &\stackrel{C}{=} \sum_{i=0}^n a_i + \left((a_{n+1} + b_{n+1}) + \sum_{i=0}^n b_i \right) \end{aligned}$$

$$\begin{aligned}
&\stackrel{L}{=} \sum_{i=0}^n a_i + \left(a_{n+1} + b_{n+1} + \sum_{i=0}^n b_i \right) \\
&\stackrel{R}{=} \sum_{i=0}^n a_i + \left(a_{n+1} + \left(b_{n+1} + \sum_{i=0}^n b_i \right) \right) \\
&\stackrel{C}{=} \sum_{i=0}^n a_i + \left(a_{n+1} + \left(\sum_{i=0}^n b_i + b_{n+1} \right) \right)
\end{aligned}$$

So we have shown that both sides equal

$$\sum_{i=0}^n a_i + \left(a_{n+1} + \left(\sum_{i=0}^n b_i + b_{n+1} \right) \right)$$

and therefore we are done.

Problem 3.6

Suppose S is a nonempty subset of \mathbb{Z} which is bounded above by some integer m , i.e., $s < m$ for all $s \in S$. Use well-ordering to prove that S has a largest element.

Step 1: If $m > 0$ then for every element in S , we subtract it by m so that we end up with another set of integers T . If $m \leq 0$ then just let $T = S$. Thus T is a set of nonpositive integers.

Step 2: Similarly, for every element in T , multiply by -1 so that we end up with another set of integers U . Each element in U is nonnegative, hence it is a subset of \mathbb{N} .

By the well-ordering principle, U has a smallest element. So T has a largest element and S has a largest element since S, T, U are one to one mappings.

Remark. One can consider the sets of all values $m - s$.

Problem 3.7

Define the predicate $P(n, m)$ to say that there is a finite sequence of positive integers $\{a_i\}_{i=0}^j$ such that

$$\frac{n}{m} = \frac{1}{a_0 + \frac{1}{a_1 + \frac{1}{\ddots + \frac{1}{a_j}}}}.$$

The domain of $P(n, m)$ is the set $\{(n, m) : n, m \in \mathbb{N}^+, n < m\}$. For example, $P(7, 9)$ is true, as witnessed by the sequence $a_0 = 1, a_1 = 3, a_2 = 2$. $P(1, 4)$ is true, as witnessed by $a_0 = 4$. (Make sure you understand these computational examples before proceeding.) The goal of this question is to prove that $P(n, m)$ holds for all m and n (in its domain, of course).

1. Suppose $n < m$ are positive integers such that n does not divide m . Prove that there is a largest positive integer a such that $\frac{n}{m} < \frac{1}{a}$.
2. In the context of the previous part, prove that $0 < m - an < n$.
3. Suppose $n < m$ are positive integers such that for all positive integers $n' < n$, $P(n', n)$ holds. Prove that $P(n, m)$ holds.
4. Convince yourself that we have a proof that $P(n, m)$ holds for all m and n .

1. Rearranging gives $a < m/n$, which by [Problem 3.6](#) there is a largest element.

Remark. By $n/m = 1/a$ we can show that a exists and $a > 0$.

2. For the left part of the inequality, rearranging inequality gives

$$\begin{aligned} a &< m/n \\ an &< m \\ 0 &< m - an \end{aligned}$$

For the right part of the inequality, since m/n is not an integer, there exists a positive integer k such that $k < m/n < k + 1$.

If $a < k$ then $a + 1 \leq k < m/n$ which is a contradiction since a is largest positive integer less than m/n . Therefore $k = a$.

Hence

$$\begin{aligned} m/n &< a + 1 \\ m &< an + n \\ m - an &< n \end{aligned}$$

3. By rewriting n/m ,

$$\begin{aligned}\frac{n}{m} &= \frac{1}{\frac{m}{n}} \\ &= \frac{1}{a + \frac{m - an}{n}}\end{aligned}$$

Since $m - an < n$, $P(m - an, n)$ holds. So in this case, let $a_0 = a$, which clearly shows that $P(n, m)$ holds.

4. **Base Case:** A quick check shows that $P(1, 2), P(1, 3), P(2, 3)$ holds.

$$\frac{1}{2} = \frac{1}{1 + \frac{1}{1}} \quad \frac{1}{3} = \frac{1}{2 + \frac{1}{1}} \quad \frac{2}{3} = \frac{1}{1 + \frac{1}{2}}$$

Inductive Step: Now assume that $m = 4$ for example. Then by our induction hypothesis for every integer $2 \leq n < 4$, the statement $P(n, m)$ holds. Assume that for all integers $1 \leq k < n$, $P(k, n)$ holds. Consider $P(k', n + 1)$ where k' is an integer $1 \leq k' < n + 1$. If $n = 1$, choose $a_0 = m$.

Therefore, we have proved that for all integers $1 \leq n < m$, the statement $P(n, m)$ holds.