Complex Analysis Notes

Before starting, I want to mention that the book "Basic Complex Analysis (3rd edition)" by Marsden, Jerrold E., and Michael J. Hoffman. is used.

§1 How Complex Is It?

1.1 Basic Operations

- $(a+bi) \pm (c+di) = (a \pm c) + (b \pm d)i$
- $(a+bi)(c+di) = (ac-bd) + (ad+bc)i$
- $\dfrac{a+bi}{c+di} = \dfrac{ac+bd}{c^2+d^2} + \dfrac{bc-ad}{c^2+d^2}i$

Problem

Fix a complex number $z = x+iy$ and consider the linear mapping $\phi_z : \mathbb{R}^2 \to \mathbb{R}^2$ (that is, of $\mathbb{C} \to \mathbb{C}$) defined by $\phi_z(w) = z \cdot w$ (that is, multiplication by z). Prove that the matrix of ϕ_z in the standard basis $(1,0), (0,1)$ of $\mathbb{R}^2$ is given by

$$\begin{pmatrix} x & -y \\ y & x \end{pmatrix}.$$

Then show that $\phi_{z_1 z_2} = \phi_{z_1} \circ \phi_{z_2}$.

Let $w = a+ib$, then $z \cdot w = (x+iy)(a+ib) = (xa-yb) + (xb+ya)i$.

On the other hand,

$$\begin{pmatrix} x & -y \\ y & x \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} xa-yb \\ xb+ya \end{pmatrix}.$$

and we have

$$\phi_{z_1 z_2} = z_1 \cdot z_2 \cdot w = z_1 \cdot (z_2 \cdot w) = \phi_{z_1} \circ \phi_{z_2}.$$

1.2 What? There's More?

Proposition (De Moivre's Formula)
If $z = r(\cos\theta + i\sin\theta)$ then for some positive integer n,

$$z^n = r^n(\cos n\theta + i\sin n\theta).$$

Some notable properties of **complex conjugation** and **norm**.

- $z\overline{z} = |z|^2$.
- $\mathrm{Re}\,(z) = (z+\overline{z})/2$, $\mathrm{Im}\,(z) = (z+\overline{z})/2i$
- $|\mathrm{Re}\,(z)| \leq |z|$, $|\mathrm{Im}\,(z)| \leq |z|$
- Triangle Inequality: $\left|\sum_{k=1}^{n} z_k\right| \leq \sum_{k=1}^{n} |z_k|$
- Cauchy-Schwarz Inequality: $\left|\sum_{k=1}^{n} z_k w_k\right| \leq \sqrt{\sum_{k=1}^{n} |z_k|^2}\sqrt{\sum_{k=1}^{n} |w_k|^2}$

Problem
If $a, b \in \mathbb{C}$, prove the **parallelogram identity**: $|a-b|^2+|a+b|^2 = 2(|a|^2+|b|^2)$.

Let $a = p + iq$ and $b = r + is$, then

$$\begin{aligned} |a-b|^2 + |a+b|^2 &= (p-r)^2 + (q-s)^2 + (p+r)^2 + (q+s)^2 \\ &= 2(p^2+q^2+r^2+s^2) \\ &= 2(|a|^2+|b|^2) \end{aligned}$$

Problem
Prove **Langrange's identity**:

$$\left|\sum_{k=1}^{n} z_k w_k\right|^2 = \left(\sum_{k=1}^{n} |z_k|\right)\left(\sum_{k=1}^{n} |w_k|\right) - \sum_{k<j} |z_k\overline{w_j} - z_j\overline{w_k}|.$$

We abuse the fact that $z\overline{z} = |z|^2$.

$$
\begin{aligned}
\left|\sum_{k=1}^{n} z_k w_k\right|^2 &= \left(\sum_{k=1}^{n} z_k w_k\right)\left(\overline{\sum_{k=1}^{n} z_k w_k}\right) \\
&= \left(\sum_{k=1}^{n} z_k w_k\right)\left(\sum_{k=1}^{n} \overline{z_k w_k}\right) \\
&= \sum_{k=1}^{n} z_k w_k \overline{z_k w_k} + \sum_{j\neq k} z_j w_j \overline{z_k w_k} \\
&= \sum z_k w_j \overline{z_k w_j} + \sum_{j\neq k} z_j w_j \overline{z_k w_k} - \sum_{j\neq k} z_k w_j \overline{z_k w_j} \\
&= \sum |z_k|^2 |w_j|^2 + \sum_{j\neq k} z_j w_j \overline{z_k w_k} - \sum_{j\neq k} z_k w_j \overline{z_k w_j} \\
&= \left(\sum_{k=1}^{n} |z_k|\right)\left(\sum_{k=1}^{n} |w_k|\right) + \sum_{j\neq k} z_j w_j \overline{z_k w_k} - \sum_{j\neq k} z_k w_j \overline{z_k w_j}
\end{aligned}
$$

For some distinct indices j, k we have

$$
\begin{aligned}
z_j w_j \overline{z_k w_k} + z_k w_k \overline{z_j w_j} - z_k w_j \overline{z_k w_j} - z_j w_k \overline{z_j w_k} &= z_j \overline{w_k}(w_j \overline{z_k} - w_k \overline{z_j}) + z_k \overline{w_j}(w_k \overline{z_j} - w_j \overline{z_k}) \\
&= (w_k \overline{z_j} - w_j \overline{z_k})(z_j \overline{w_k} - z_k \overline{w_j}) \\
&= -(w_k \overline{z_j} - w_j \overline{z_k})\overline{(w_k \overline{z_j} - w_j \overline{z_k})} \\
&= -|w_k \overline{z_j} - w_j \overline{z_k}|^2
\end{aligned}
$$

Summing up gives the desire result

1.3 Even Weirder Stuff

Using the fact that

$$re^{ix} = r(\cos x + i\sin x)$$

and thanks to Euler we generalize the complex numbers to even more functions.

- It's not hard to see that

$$\sin x = \frac{e^{ix} - e^{-ix}}{2i} \quad \text{and} \quad \cos x = \frac{e^{ix} + e^{-ix}}{2}$$

- Let $z = re^{i\theta}$ then $\ln z = \ln |r| + i \arg z$.
- $z^w = e^{w \ln z}$ can be determined consequently.
- Moreover, we have

$$\sinh x = -i\sin(ix) \quad \text{and} \quad \cosh x = \cos(ix)$$

which can be deduced from

$$\sinh x = \frac{e^x - e^{-x}}{2} \quad \text{and} \quad \cosh x = \frac{e^x + e^{-x}}{2}$$

Problem

Along which rays through the origin does $\lim_{z\to\infty} |e^z|$ exist?

Let $z = x + iy$, then we have $|e^z| = |e^x(\cos y + i \sin y)| = e^x$. If $x \to -\infty$ then $e^x \to 0$, but if $x \to \infty$ then $e^x \to \infty$ which the limit doesn't exist.

Hence the answers are all the rays passing through the nonnegative x plane.

Problem

Prove the identity

$$z = \tan\left[\frac{1}{i}\ln\left(\frac{1+iz}{1-iz}\right)^{1/2}\right]$$

for all real z.

$$\begin{aligned}
\tan\left[\frac{1}{i}\ln\left(\frac{1+iz}{1-iz}\right)^{1/2}\right] &= \tan\left[\frac{1}{2i}\left(\ln(1+iz) - \ln(1-iz)\right)\right] \\
&= \tan\left[\frac{1}{2i}\left(\ln|1+iz| + i(\tan^{-1} z) - \ln|1-iz| - i(\tan^{-1}(-z))\right)\right] \\
&= \tan\left[\frac{1}{2i}\left(2i(\tan^{-1} z)\right)\right] \\
&= z
\end{aligned}$$

Problem

Use the equation $\sin z = \sin x \cosh y + i \sinh y \cos x$ where $z = x + iy$ to prove that $|\sinh y| \le |\sin z| \le |\cosh y|$.

Evaluating gives

$$|\sin z| = \sqrt{\sin^2 x \cosh^2 y + \sinh^2 y \cos^2 x}$$

Using the fact that $\sinh x < \cosh x$, we have

$$\sin^2 x \sinh^2 y + \sinh^2 y \cos^2 x < \sin^2 x \cosh^2 y + \sinh^2 y \cos^2 x < \sin^2 x \cosh^2 y + \cosh^2 y \cos^2 x$$

simplifying gives the desired result.

Problem

Using polar coordinates, show that $z \mapsto z + 1/z$ maps the circle $|z| = 1$ to the interval $[-2, 2]$ on the x axis.

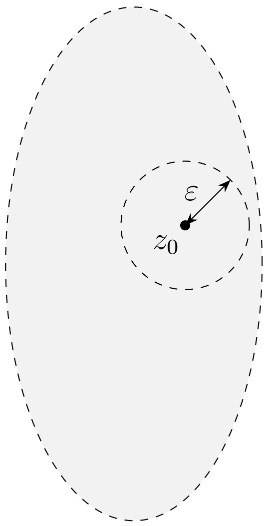
Let $z = x+iy$, then $z+\frac{1}{z} = x+iy+\frac{x-iy}{x^2+y^2}$ and since $x^2+y^2 = 1$, $z+\frac{1}{z} = 2x$. This means that for any complex number $z = x+iy$ on the circle, it will be mapped to $2x$.

And since x is in the interval $[-1, 1]$, hence $2x$ is in the interval $[-2, 2]$.

1.4 Topological Analysis of Complex Functions

1.4.1 Definitions

- **r Disk**: The r disk is defined by $D(z_0; r) = \{z \in \mathbb{C}|\ |z - z_0| < r\}$. The **deleted r disk** is defined by $D(z_0; r)\backslash \{z_0\}$.
- **Open Sets**: The set $A \subset \mathbb{C}$ is open when for any point z_0 in A, there exists a real number ε such that if $|z - z_0| < \varepsilon$ then $z \in A$.



- **Closed Sets**: A set F is closed if $\mathbb{C}\backslash F$ is open.
  - The empty set and $\mathbb{C}$ are both open and closed (known as **clopen sets**).
  - Let $z_1, z_2, z_3, \ldots$ are points in F and $w = \lim_{n\to\infty} z_n$, then $w \in F$.
    * **Sketch of proof**: Assume that $w \notin F$, then since $\mathbb{C}\backslash F$ is open, we can always find a disk $D(w; r)$ contained in $\mathbb{C}\backslash F$. This means that there exists some large enough n such that $z_n \in D(w; r)$ by convergence, which implies $z_n \notin F$, a contradiction.
  - The **closure** of a set S, denoted by $\overline{S}$ is the set S together with its limit points, or known as the **boundary** $\partial(S)$.
- **Limits**: The limit $\lim_{z\to z_0} f(z) = L$ exists when for all $\varepsilon > 0$, there exists $\delta > 0$ such that for all $|z - z_0| < \delta$ $(z \neq z_0)$ we have $|f(z) - L| < \varepsilon$.

Limits are **unique** if they exist.

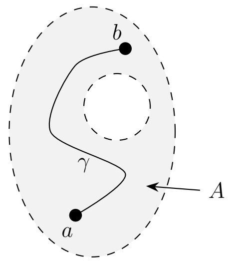
- **Continuity**: f is continuous at $z_0 \in A$ if and only if

$$\lim_{z \to z_0} f(z) = f(z_0).$$

- **Cauchy Sequence**: A sequence is *Cauchy* if for every $\varepsilon > 0$, we can find some integer N such that whenever integers m, n are greater than N, $|z_m - z_n| < \varepsilon$.

- **Path-Connected**: A set $A \in \mathbb{C}$ is path-connected if for every $a, b \in A$ there exists a *continuous map* $\gamma : [0, 1] \to A$ such that $\gamma(0) = a, \gamma(1) = b$.

  γ is a **path** joining a and b.



  Definition: A set $C \in \mathbb{C}$ is **not connected** if there are open sets U, V such that

  (a) $C \subset (U \cup V)$;

  (b) $(C \cap U \neq \varnothing) \wedge (C \cap V \neq \varnothing)$;

  (c) $C \cap U \cap V = \varnothing$.

  If a set is not "not connected", then it is **connected**.

  - **A path-connected set is connected, but a connected set may not be path-connected.**
  - Example: **Topologist's Sine Curve**

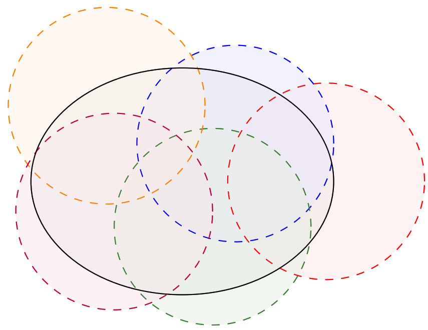
$$f(x) = \begin{cases} \sin \dfrac{1}{x} & x > 0 \\ 0 & x = 0 \end{cases}$$

    Sketch of proof: Let the two sets be A, B. WLOG let $(0, 0) \in A$.

    If some part of $\sin(1/x)$ is in A, then B should be covering the other parts. But since both sets are open, there's a point that is not covered.

    If no part of $\sin(1/x)$ is in A, then B must be covering the entire line of $\sin(1/x)$. But this is impossible since we cannot cover all points near $x = 0^+$.

- **Cover**: Let U be a collection of open sets. U is a cover of a set K if K is contained in the union of sets in U.



A **subcover** is a subset of U but can still cover K.

- **Compactness**: A set K is **compact** if every cover of K has a finite subcover.
  - **Heine-Borel Theorem**: A set K is compact if and only if K is closed and bounded.

    Sketch of proof:

    * Sufficiency:

      Boundedness: Assume that K is not bounded. Consider the set of open covers $U = \{D(O; r)\}$, (open) disks centered at the origin, then for all finite subcover U' of U, consider $R = \max(r)$ and choose some point $z \in K$ but $|z| > r$.

      Closedness: Assume that K is not closed, then there exists some $w \notin K$ such that the sequence $\{z_i\}$ in K converges to w. So the set of open covers $U = \{D(w, r)\}$ does not have a finite subcover.

    * Necessity: Assume that K is closed and bounded, then let $z \in K$ such that $|z|$ attains maximum value. Choose the open cover $D(O; |z|+1)$.

1.4.2 On Functions

- If f is a continuous function defined on a connected set C, then $f(C)$ is connected.

  Sketch of proof: FTSOC, let $A|B$ be a partition of $f(C)$. Then $f^{-1}(A)$ and $f^{-1}(B)$ are open and disjoint (since each value $f^{-1}(x)$ can only belong to either one of $f^{-1}(A)$ and $f^{-1}(B)$.)

- If f is a continuous function defined on a compact set C, then $f(C)$ is compact.

  Sketch of proof: Let U be an open cover of $f(C)$, then for each $f(z) \in U$ and $f(z) \in f(C)$, we have $z \in C$ and $z \in f^{-1}(U)$.

- **Extreme Value Theorem**: Let K be a compact set and $f : K \to \mathbb{R}$ is a continuous function, then f attains **finite** maximum and minimum values.

  Sketch of proof: K is compact implies $f(K)$ is compact, or $f(K)$ is bounded, therefore finite maximum and minimum exists.

- **Distance Lemma**: Let K be a compact set and C be a closed set and $K \cap C = \varnothing$. Then there exists a number $\rho > 0$, such that whenever $z \in K$ and $w \in C$ then $|z - w| > \rho$.

  Sketch of proof: K is closed and bounded. Assume that ρ doesn't exist, $\rho \to 0$ since we can always find some $|z - w| < \rho_0$ if ρ_0 is fixed.