MA1100T Quick Notes

§1 Logic

1.1 Statement vs. Proposition vs. Predicate

- A **statement** is a sentence.
- A **proposition** is a statement that is either true or false, but not both.
- A **predicate** is an assignment of truth values to elements of some domain.

1.2 Implications

> "On Wednesdays, we wear pink." – *Mean Girls*

Denoted as $p \to q$, if it is Wednesday today, then I should probably wear pink. If it is not Wednesday, it doesn't mean I cannot wear pink, so we have

p	q	$p \to q$
0	1	1

Moreover, $p \to q \equiv \neg p \vee q$.

1.3 Logical Equivalence

- Two propositional formulas are **logically equivalence** if $\forall$ assignment of truth values of propositions, they have the same truth value.
- Two propositional formulas are **not logically equivalence** if $\exists$ assignment of truth values of propositions, they have different truth values.

1.4 If and only if

$$p \leftrightarrow q \equiv (p \to q) \wedge (q \to p)$$

p	q	$p \leftrightarrow q$
1	1	1
1	0	0
0	1	0
0	0	1

1.4.1 Necessity and Sufficiency

- If $p \to q$ then p is **sufficient** for q.
- If $p \leftarrow q$ then p is **necessary** for q.

> **Example** (SJTU Mathematics Contest 2023)
> If α: "The equation of a hyperbola is $x^2 - y^2 = a^2$, $a > 0$" and β: "The asymptotes of this hyperbola form and angle of $\pi/2$".

α	β	α implies β?	β implies α?
1	1	True	False since $(x-1)^2 - y^2 = a^2$ also works
1	0	False since β is true	False since the angle equals $\pi/2$
0	1	True, consider $(x-1)^2 - y^2 = a^2$	False since $(x-1)^2 - y^2 = a^2$ also works
0	0	True, consider $x^2 - 4y^2 = a^2$ and $\theta = 52.13°$	True, if $\theta \neq \pi/2$, the equation will not be $x^2 - y^2 = a^2$

As we can see, "α implies β" is tally to "$\alpha \to \beta$" but "β implies α" does not tally to "$\beta \to \alpha$" so for β, α is **sufficient** but not **necessary**.

1.5 Tautology & Contradiction

- If $F_1 \equiv F_2$, then $F_1 \leftrightarrow F_2$ is a **tautology**.
- If $F_1 \not\equiv F_2$, then $F_1 \leftrightarrow F_2$ is a **contradiction**.

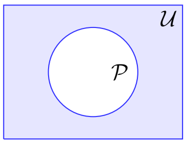
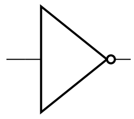
1.6 Useful Denial

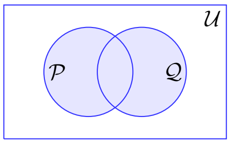
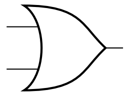
F_2 is a **useful denial** of F_1 iff $F_1 \equiv \neg F_2$.

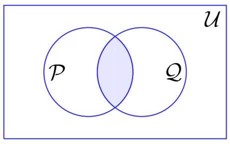
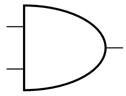
1.7 Boolean Algebra in Words

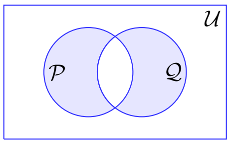
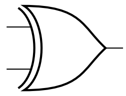
- $p \to q$: shown in §1.2
- $p \land \neg p$ is a contradiction: Nothing can be both true and false at the same time.
- $p \lor \neg p$ is a tautology: Something must either be true or false right?
- $\neg(P \lor Q) \equiv (\neg P) \land (\neg Q)$: Let say if I ask you: "Tea or coffee?". If you want tea but not coffee, then P is true and Q is false, negating lets you drink coffee but not tea, you only drank coffee in the end.
- $\neg(\forall x)(P(x)) \equiv (\exists x)(\neg P(x))$: I don't drink coffee every day, I drank tea on Wednesday.
- $\neg(\exists x)(P(x)) \equiv (\forall x)(\neg P(x))$: Haha you can't find me drinking coffee for all days – I drank tea every day.

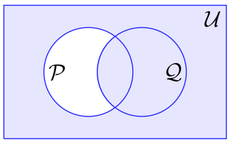
1.8 Boolean, Set Theory, Bitwise Operations, Logic Gates

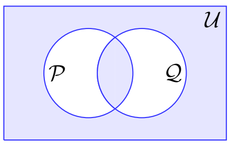
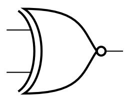
Boolean	Set Theory	Bitwise	Logic Gates
$\neg p$	 $\mathcal{P}^{\complement}$ Complement	$\texttt{NOT}\ \ x = \left(2^{\lfloor \log_2 x \rfloor + 1} - 1\right) - x$ `NOT 10110101 = 01001010`	

$p \lor q$	
	$\mathcal{P} \cup \mathcal{Q}$
	Union
		`101101 OR 011001 = 111101`
		`101101`
		`OR 011001`
		`111101`
			

$p \land q$	
	$\mathcal{P} \cap \mathcal{Q}$
	Intersection
		`101101 AND 011001 = 001001`
		`101101`
		`AND 011001`
		`001001`
			

$p \oplus q$	
	$\mathcal{P} \triangle \mathcal{Q}$
	Symmetric Difference
		`101101 XOR 011001 = 110100`
		`101101`
		`XOR 011001`
		`110100`
			

$p \to q$	
	$\mathcal{P} \to \mathcal{Q}$
	Implies
		`101101 THEN 011001 = 011011`
		`101101`
		`THEN 011001`
		`011011`	

$p \leftrightarrow q$	
	$\mathcal{P} \leftrightarrow \mathcal{Q}$
	Equivalent
		`101101 XNOR 011001 = 001011`
		`101101`
		`XNOR 011001`
		`001011`
			

1.9 More on Quantifiers

- The **universal quantification** $\forall x P(x)$ states that "**for all** x such that $P(x)$ is true".
- The **existential quantification** $\exists x P(x)$ states that "**there exists** x such that $P(x)$ is true".

1.9.1 Equivalent Formulae on Finite Sets

Let the domain of $P(x)$ be a set with finite cardinality, assign the elements $x_1, x_2, x_3, \ldots, x_k$,

- $\forall x P(x) \equiv \bigwedge_{i=1}^{k} P(x_i)$
- $\exists x P(x) \equiv \bigvee_{i=1}^{k} P(x_i)$

By the inductive process, we can also deduce De Morgan's law.

$$\neg\left(\bigwedge_{i=1}^{k} P(x_i)\right) \equiv \bigvee_{i=1}^{k}(\neg P(x_i))$$

1.9.2 Quantifiers Overloading

Consider the following sentence

"Everyone who takes a break can have a snack."

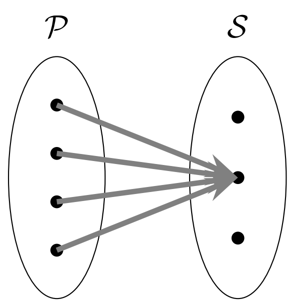
Let $\mathcal{P}$ be the set of people, and $\mathcal{S}$ be the set of all snacks. We say that $H(p,s)$ equals "person p had snack s". Consider the following formulae:

(a)
$$(\exists s)(\forall p)H(p,s)$$

In words, we have

"There exists a snack s (probably KitKat) such that every person p, p ate s."

It is an **all-to-one** situation here.



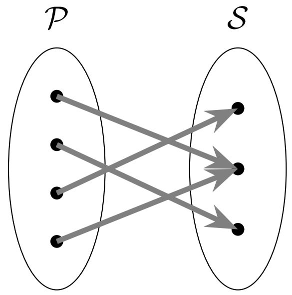
(b)

$$(\forall p)(\exists s)H(p, s)$$

In words, we have

"For all people p, there exists a snack s such that p ate s."

It is a **one-to-one** situation here.



Moreover, we can stack quantifiers.

- x is irrational: $(\forall p \in \mathbb{Z})(\forall q \in \mathbb{Z})(x \neq p/q) \equiv (\nexists (p, q) \in \mathbb{Z}^2)(x = p/q)$.
- x is rational: $(\exists p \in \mathbb{Z})(\exists q \in \mathbb{Z})(x = p/q) \equiv \neg(\forall (p, q) \in \mathbb{Z}^2)(x \neq p/q)$.