

Complex Analysis Notes

Before starting, I want to mention that the book “Basic Complex Analysis (3rd edition)” by Marsden, Jerrold E., and Michael J. Hoffman. is used.

§1 How Complex Is It?



1.1 Basic Operations

- $(a + bi) \pm (c + di) = (a \pm c) + (b \pm d)i$
- $(a + bi)(c + di) = (ac - bd) + (ad + bc)i$
- $\frac{a + bi}{c + di} = \frac{ac + bd}{c^2 + d^2} + \frac{bc - ad}{c^2 + d^2}i$

Problem

Fix a complex number $z = x + iy$ and consider the linear mapping $\phi_z : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ (that is, of $\mathbb{C} \rightarrow \mathbb{C}$) defined by $\phi_z(w) = z \cdot w$ (that is, multiplication by z). Prove that the matrix of ϕ_z in the standard basis $(1, 0), (0, 1)$ of \mathbb{R}^2 is given by

$$\begin{pmatrix} x & -y \\ y & x \end{pmatrix}.$$

Then show that $\phi_{z_1 z_2} = \phi_{z_1} \circ \phi_{z_2}$.

Let $w = a + ib$, then $z \cdot w = (x + iy)(a + ib) = (xa - yb) + (xb + ya)i$.

On the other hand,

$$\begin{pmatrix} x & -y \\ y & x \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} xa - yb \\ xb + ya \end{pmatrix}.$$

and we have

$$\phi_{z_1 z_2} = z_1 \cdot z_2 \cdot w = z_1 \cdot (z_2 \cdot w) = \phi_{z_1} \circ \phi_{z_2}.$$

1.2 What? There's More?

Proposition (De Moivre's Formula)

If $z = r(\cos \theta + i \sin \theta)$ then for some positive integer n ,

$$z^n = r^n(\cos n\theta + i \sin n\theta).$$

Some notable properties of **complex conjugation** and **norm**.

- $z\bar{z} = |z|^2$.
- $\operatorname{Re}(z) = (z + \bar{z})/2$, $\operatorname{Im}(z) = (z - \bar{z})/2i$
- $|\operatorname{Re}(z)| \leq |z|$, $|\operatorname{Im}(z)| \leq |z|$
- Triangle Inequality: $\left| \sum_{k=1}^n z_k \right| \leq \sum_{k=1}^n |z_k|$
- Cauchy-Schwarz Inequality: $\left| \sum_{k=1}^n z_k w_k \right| \leq \sqrt{\sum_{k=1}^n |z_k|^2} \sqrt{\sum_{k=1}^n |w_k|^2}$

Problem

If $a, b \in \mathbb{C}$, prove the **parallelogram identity**: $|a-b|^2 + |a+b|^2 = 2(|a|^2 + |b|^2)$.

Let $a = p + iq$ and $b = r + is$, then

$$\begin{aligned} |a-b|^2 + |a+b|^2 &= (p-r)^2 + (q-s)^2 + (p+r)^2 + (q+s)^2 \\ &= 2(p^2 + q^2 + r^2 + s^2) \\ &= 2(|a|^2 + |b|^2) \end{aligned}$$

Problem

Prove **Langrange's identity**:

$$\left| \sum_{k=1}^n z_k w_k \right|^2 = \left(\sum_{k=1}^n |z_k|^2 \right) \left(\sum_{k=1}^n |w_k|^2 \right) - \sum_{k < j} |z_k \bar{w}_j - z_j \bar{w}_k|.$$

We abuse the fact that $z\bar{z} = |z|^2$.

$$\begin{aligned}
\left| \sum_{k=1}^n z_k w_k \right|^2 &= \left(\sum_{k=1}^n z_k w_k \right) \overline{\left(\sum_{k=1}^n z_k w_k \right)} \\
&= \left(\sum_{k=1}^n z_k w_k \right) \left(\sum_{k=1}^n \overline{z_k w_k} \right) \\
&= \sum_{k=1}^n z_k w_k \overline{z_k w_k} + \sum_{j \neq k} z_j w_j \overline{z_k w_k} \\
&= \sum_{k=1}^n |z_k|^2 |w_k|^2 + \sum_{j \neq k} z_j w_j \overline{z_k w_k} - \sum_{j \neq k} z_k w_k \overline{z_j w_j} \\
&= \sum_{k=1}^n |z_k|^2 |w_k|^2 + \sum_{j \neq k} z_j w_j \overline{z_k w_k} - \sum_{j \neq k} z_k w_k \overline{z_j w_j} \\
&= \left(\sum_{k=1}^n |z_k|^2 \right) \left(\sum_{k=1}^n |w_k|^2 \right) + \sum_{j \neq k} z_j w_j \overline{z_k w_k} - \sum_{j \neq k} z_k w_k \overline{z_j w_j}
\end{aligned}$$

For some distinct indices j, k we have

$$\begin{aligned}
z_j w_j \overline{z_k w_k} + z_k w_k \overline{z_j w_j} - z_k w_k \overline{z_j w_j} - z_j w_j \overline{z_k w_k} &= z_j \overline{w_k} (w_j \overline{z_k} - w_k \overline{z_j}) + z_k \overline{w_j} (w_k \overline{z_j} - w_j \overline{z_k}) \\
&= (w_k \overline{z_j} - w_j \overline{z_k}) (z_j \overline{w_k} - z_k \overline{w_j}) \\
&= -(w_k \overline{z_j} - w_j \overline{z_k}) \overline{(w_k \overline{z_j} - w_j \overline{z_k})} \\
&= -|w_k \overline{z_j} - w_j \overline{z_k}|^2
\end{aligned}$$

Summing up gives the desired result

1.3 Even Weirder Stuff

Using the fact that

$$re^{ix} = r(\cos x + i \sin x)$$

and thanks to Euler we generalize the complex numbers to even more functions.

- It's not hard to see that

$$\sin x = \frac{e^{ix} - e^{-ix}}{2i} \quad \text{and} \quad \cos x = \frac{e^{ix} + e^{-ix}}{2}$$

- Let $z = re^{i\theta}$ then $\ln z = \ln |r| + i \arg z$.
- $z^w = e^{w \ln z}$ can be determined consequently.
- Moreover, we have

$$\sinh x = -i \sin(ix) \quad \text{and} \quad \cosh x = \cos(ix)$$

which can be deduced from

$$\sinh x = \frac{e^x - e^{-x}}{2} \quad \text{and} \quad \cosh x = \frac{e^x + e^{-x}}{2}$$

Problem

Along which rays through the origin does $\lim_{z \rightarrow \infty} |e^z|$ exist?

Let $z = x + iy$, then we have $|e^z| = |e^x(\cos y + i \sin y)| = e^x$. If $x \rightarrow -\infty$ then $e^x \rightarrow 0$, but if $x \rightarrow \infty$ then $e^x \rightarrow \infty$ which the limit doesn't exist.

Hence the answers are all the rays passing through the nonnegative x plane.

Problem

Prove the identity

$$z = \tan \left[\frac{1}{i} \ln \left(\frac{1 + iz}{1 - iz} \right)^{1/2} \right]$$

for all real z .

$$\begin{aligned} \tan \left[\frac{1}{i} \ln \left(\frac{1 + iz}{1 - iz} \right)^{1/2} \right] &= \tan \left[\frac{1}{2i} (\ln(1 + iz) - \ln(1 - iz)) \right] \\ &= \tan \left[\frac{1}{2i} (\ln |1 + iz| + i(\tan^{-1} z) - \ln |1 - iz| - i(\tan^{-1}(-z))) \right] \\ &= \tan \left[\frac{1}{2i} (2i(\tan^{-1} z)) \right] \\ &= z \end{aligned}$$

Problem

Use the equation $\sin z = \sin x \cosh y + i \sinh y \cos x$ where $z = x + iy$ to prove that $|\sinh y| \leq |\sin z| \leq |\cosh y|$.

Evaluating gives

$$|\sin z| = \sqrt{\sin^2 x \cosh^2 y + \sinh^2 y \cos^2 x}$$

Using the fact that $\sinh x < \cosh x$, we have

$$\sin^2 x \sinh^2 y + \sinh^2 y \cos^2 x < \sin^2 x \cosh^2 y + \sinh^2 y \cos^2 x < \sin^2 x \cosh^2 y + \cosh^2 y \cos^2 x$$

simplifying gives the desired result.

Problem

Using polar coordinates, show that $z \mapsto z + 1/z$ maps the circle $|z| = 1$ to the interval $[-2, 2]$ on the x axis.

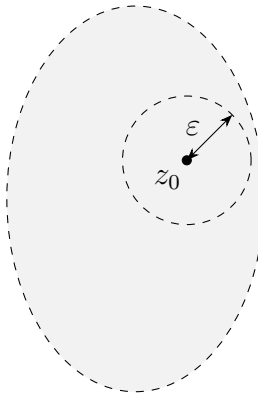
Let $z = x + iy$, then $z + \frac{1}{z} = x + iy + \frac{x - iy}{x^2 + y^2}$ and since $x^2 + y^2 = 1$, $z + \frac{1}{z} = 2x$. This means that for any complex number $z = x + iy$ on the circle, it will be mapped to $2x$.

And since x is in the interval $[-1, 1]$, hence $2x$ is in the interval $[-2, 2]$.

1.4 Topological Analysis of Complex Functions

1.4.1 Definitions

- **r Disk**: The r disk is defined by $D(z_0; r) = \{z \in \mathbb{C} \mid |z - z_0| < r\}$. The **deleted r disk** is defined by $D(z_0; r) \setminus \{z_0\}$.
- **Open Sets**: The set $A \subset \mathbb{C}$ is open when for any point z_0 in A , there exists a real number ε such that if $|z - z_0| < \varepsilon$ then $z \in A$.



- **Closed Sets**: A set F is closed if $\mathbb{C} \setminus F$ is open.
 - The empty set and \mathbb{C} are both open and closed (known as **clopen sets**).
 - Let z_1, z_2, z_3, \dots are points in F and $w = \lim_{n \rightarrow \infty} z_n$, then $w \in F$.
 - * **Sketch of proof**: Assume that $w \notin F$, then since $\mathbb{C} \setminus F$ is open, we can always find a disk $D(w; r)$ contained in $\mathbb{C} \setminus F$. This means that there exists some large enough n such that $z_n \in D(w; r)$ by convergence, which implies $z_n \notin F$, a contradiction.
 - The **closure** of a set S , denoted by \overline{S} is the set S together with its limit points, or known as the **boundary** $\partial(S)$.
- **Limits**: The limit $\lim_{z \rightarrow z_0} f(z) = L$ exists when for all $\varepsilon > 0$, there exists $\delta > 0$ such that for all $|z - z_0| < \delta$ ($z \neq z_0$) we have $|f(z) - L| < \varepsilon$.

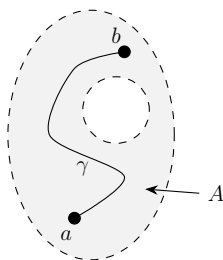
Limits are **unique** if they exist.

- **Continuity**: f is continuous at $z_0 \in A$ if and only if

$$\lim_{z \rightarrow z_0} f(z) = f(z_0).$$

- **Cauchy Sequence**: A sequence is *Cauchy* if for every $\varepsilon > 0$, we can find some integer N such that whenever integers m, n are greater than N , $|z_m - z_n| < \varepsilon$.
- **Path-Connected**: A set $A \in \mathbb{C}$ is path-connected if for every $a, b \in A$ there exists a *continuous map* $\gamma : [0, 1] \rightarrow A$ such that $\gamma(0) = a, \gamma(1) = b$.

γ is a **path** joining a and b .



Definition: A set $C \in \mathbb{C}$ is **not connected** if there are open sets U, V such that

- $C \subset (U \cup V)$;
- $(C \cap U \neq \emptyset) \wedge (C \cap V \neq \emptyset)$;
- $C \cap U \cap V = \emptyset$.

If a set is not “not connected”, then it is **connected**.

- A **path-connected set is connected**, but a **connected set may not be path-connected**.
- Example: **Topologist’s Sine Curve**

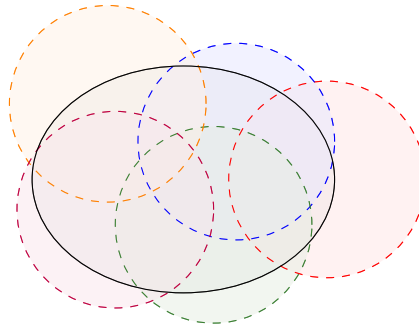
$$f(x) = \begin{cases} \sin \frac{1}{x} & x > 0 \\ 0 & x = 0 \end{cases}$$

Sketch of proof: Let the two sets be A, B . WLOG let $(0, 0) \in A$.

If some part of $\sin(1/x)$ is in A , then B should be covering the other parts. But since both sets are open, there’s a point that is not covered.

If no part of $\sin(1/x)$ is in A , then B must be covering the entire line of $\sin(1/x)$. But this is impossible since we cannot cover all points near $x = 0^+$.

- **Cover**: Let U be a collection of open sets. U is a cover of a set K if K is contained in the union of sets in U .



A **subcover** is a subset of U but can still cover K .

- **Compactness**: A set K is **compact** if every cover of K has a finite subcover.
 - **Heine-Borel Theorem**: A set K is compact if and only if K is closed and bounded.

Sketch of proof:

* Sufficiency:

Boundedness: Assume that K is not bounded. Consider the set of open covers $U = \{D(O; r)\}$, (open) disks centered at the origin, then for all finite subcover U' of U , consider $R = \max(r)$ and choose some point $z \in K$ but $|z| > R$.

Closedness: Assume that K is not closed, then there exists some $w \notin K$ such that the sequence $\{z_i\}$ in K converges to w . So the set of open covers $U = \{D(w, r)\}$ does not have a finite subcover.

* Necessity: Assume that K is closed and bounded, then let $z \in K$ such that $|z|$ attains maximum value. Choose the open cover $D(O; |z| + 1)$.

1.4.2 On Functions

- If f is a continuous function defined on a connected set C , then $f(C)$ is connected.

Sketch of proof: FTSOC, let $A|B$ be a partition of $f(C)$. Then $f^{-1}(A)$ and $f^{-1}(B)$ are open and disjoint (since each value $f^{-1}(x)$ can only belong to either one of $f^{-1}(A)$ and $f^{-1}(B)$.)

- If f is a continuous function defined on a compact set C , then $f(C)$ is compact.

Sketch of proof: Let U be an open cover of $f(C)$, then for each $f(z) \in U$ and $f(z) \in f(C)$, we have $z \in C$ and $z \in f^{-1}(U)$.

- **Extreme Value Theorem**: Let K be a compact set and $f : K \rightarrow \mathbb{R}$ is a continuous function, then f attains **finite** maximum and minimum values.

Sketch of proof: K is compact implies $f(K)$ is compact, or $f(K)$ is bounded, therefore finite maximum and minimum exists.

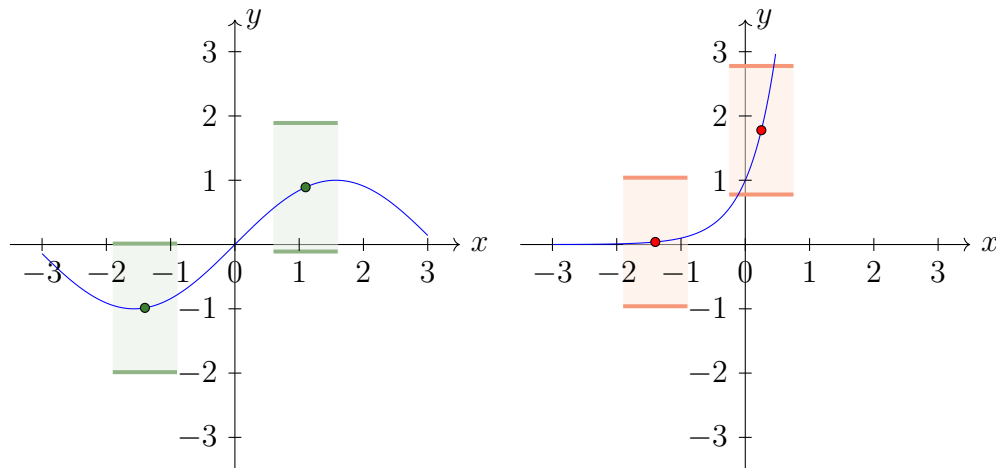
- **Distance Lemma:** Let K be a compact set and C be a closed set and $K \cap C = \emptyset$. Then there exists a number $\rho > 0$, such that whenever $z \in K$ and $w \in C$ then $|z - w| > \rho$.

Sketch of proof: K is closed and bounded. Assume that ρ doesn't exist, $\rho \rightarrow 0$ since we can always find some $|z - w| < \rho_0$ if ρ_0 is fixed. Consider the sequences $\{z_k\}$ and $\{w_k\}$. Thus $\lim_{k \rightarrow \infty} |z_k - w_k| = 0$ which means $\lim_{k \rightarrow \infty} z_k = \lim_{k \rightarrow \infty} w_k$.

But since both sets are closed, we must have $\lim_{k \rightarrow \infty} z_k \in K$ and $\lim_{k \rightarrow \infty} w_k \in C$, hence a contradiction.

- **Uniform Continuity:** A function $f : A \rightarrow \mathbb{C}$ is **uniformly continuous** on A if for any $\varepsilon > 0$, there exists a $\delta > 0$ such that $|f(s) - f(t)| < \varepsilon$ whenever $s, t \in A$ and $|s - t| < \delta$.

An example $f(x) = \sin x$ and a nonexample $f(x) = 2^x$ are shown below. Choose $\delta = \varepsilon/2 = 0.5$.



- **Heine-Cantor Theorem:** Let $f : A \rightarrow \mathbb{C}$ be a continuous function. If A is compact then $f(A)$ is uniformly continuous.

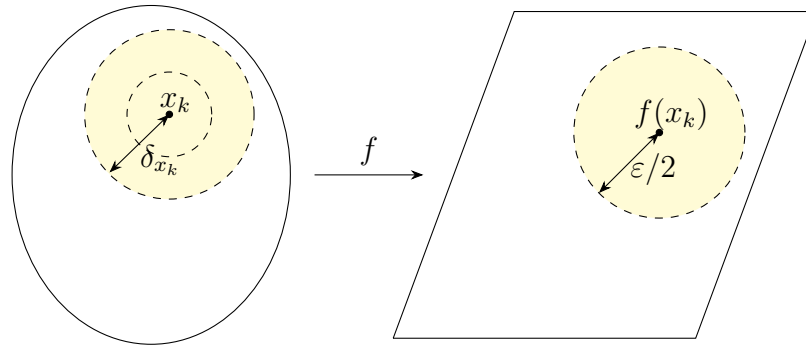
Sketch of proof:

- Let x be some point in A . By continuity, there is a number δ_x such that whenever for some point y satisfying $|x - y| < \delta_x$ then $|f(x) - f(y)| < \varepsilon/2$.
- For a sequence of points x , say $\{x_i\}$, consider disks $D_k = D(x_k; \delta_{x_k}/2)$. These disks cover A by compactness. Let the minimal radius over all disks be δ .
- For points s, t satisfying $|s - t| < \delta$, we must have t contained in some disk D_k . Thus $|t - x_k| < \delta_{x_k}/2$, implies that $|f(t) - f(x_k)| < \varepsilon/2$. We have

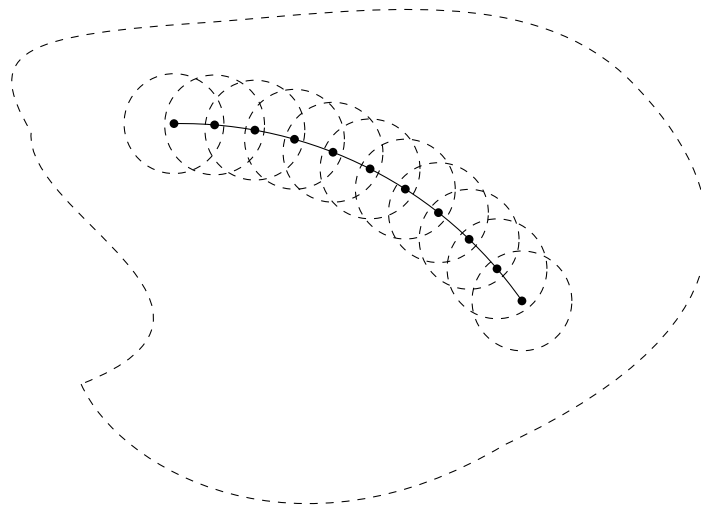
$$|s - x_k| \leq |s - t| + |t - x_k| < \delta + \delta_{x_k}/2 \leq \delta_{x_k}$$

- On the other hand,

$$|f(s) - f(t)| \leq |f(s) - f(x_k)| + |f(x_k) - f(t)| < \varepsilon$$



- **Path-Covering Lemma:** Suppose $\gamma : [0, 1] \rightarrow K$ is a continuous path into an open subset K of \mathbb{C} . We can find a number $\rho > 0$ and a subdivision of $[0, 1]$, namely $0 = t_0 < t_1 < t_2 < \dots < t_{N-1} < t_N = 1$ such that
 - (a) $D(\gamma(t_k); \rho) \subset G$ for all k
 - (b) $\gamma(t) \in D(\gamma(t_0); \rho)$ for $t_0 \leq t \leq t_1$
 - (c) $\gamma(t) \in D(\gamma(t_k); \rho)$ for $t_{k-1} \leq t \leq t_{k+1}$
 - (d) $\gamma(t) \in D(\gamma(t_N); \rho)$ for $t_{N-1} \leq t \leq t_N$



Sketch of proof:

- By the Distance Lemma, we can find some $\rho > 0$ such that the distance from $\gamma([0, 1])$ to $\mathbb{C} \setminus K$ is at least ρ since both sets are closed.
- By Heine-Cantor Theorem, $\gamma([0, 1])$ is uniformly continuous. So for any two points s, t , if $|s - t| < \delta$ then $|f(s) - f(t)| < \rho$.
- Choose t_k to be fine enough such that $t_{k+1} - t_k < \delta$.
- **Riemann Sphere:** We may want to define the value ∞ in the complex plane.
 - $\lim_{z \rightarrow \infty} f(z) = L$ means for any $\varepsilon > 0$, there exists $Z > 0$ such that whenever $|z| > Z$ implies $|f(z) - L| < \varepsilon$.
 - $\lim_{z \rightarrow z_0} f(z) = \infty$ means for any $R > 0$, there exists $\delta > 0$ such that whenever $|z - z_0| < \delta$ implies $|f(z)| > R$.

- $\lim_{z \rightarrow \infty} f(z) = \infty$ means for any $Z > 0$, there exists $R > 0$ such that whenever $|z| > Z$ implies $|f(z)| > R$.

Consider the sphere $x^2 + y^2 + z^2 = 1$ in \mathbb{R}^3 . A point z' on the plane \mathbb{C} is the **stereographic projection** of some point z on the sphere through $(0, 0, 1)$.

A 2-D illustration as an example:

