Complex Analysis Notes

Before starting, I want to mention that the book "Basic Complex Analysis (3rd edition)" by Jerrold E. Marsden and Michael J. Hoffman. is used.

§1 How Complex Is It?

1.1 Basic Operations

- $(a + bi) \pm (c + di) = (a \pm c) + (b \pm d)i$

- $(a + bi)(c + di) = (ac - bd) + (ad + bc)i$

- $\dfrac{a + bi}{c + di} = \dfrac{ac + bd}{c^2 + d^2} + \dfrac{bc - ad}{c^2 + d^2}i$

Problem

Fix a complex number $z = x + iy$ and consider the linear mapping $\phi_z : \mathbb{R}^2 \to \mathbb{R}^2$ (that is, of $\mathbb{C} \to \mathbb{C}$) defined by $\phi_z(w) = z \cdot w$ (that is, multiplication by z). Prove that the matrix of ϕ_z in the standard basis $(1,0),(0,1)$ of $\mathbb{R}^2$ is given by

$$\begin{pmatrix} x & -y \\ y & x \end{pmatrix}.$$

Then show that $\phi_{z_1 z_2} = \phi_{z_1} \circ \phi_{z_2}$.

Let $w = a + ib$, then $z \cdot w = (x + iy)(a + ib) = (xa - yb) + (xb + ya)i$.

On the other hand,

$$\begin{pmatrix} x & -y \\ y & x \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} xa - yb \\ xb + ya \end{pmatrix}.$$

and we have
$$\phi_{z_1 z_2} = z_1 \cdot z_2 \cdot w = z_1 \cdot (z_2 \cdot w) = \phi_{z_1} \circ \phi_{z_2}.$$

1.2 What? There's More?

> **Proposition** (De Moivre's Formula)
> If $z = r(\cos\theta + i\sin\theta)$ then for some positive integer n,
> $$z^n = r^n(\cos n\theta + i\sin n\theta).$$

Some notable properties of **complex conjugation** and **norm**.

- $z\overline{z} = |z|^2$.

- $\operatorname{Re}(z) = (z + \overline{z})/2$, $\operatorname{Im}(z) = (z + \overline{z})/2i$

- $|\operatorname{Re}(z)| \leq |z|$, $|\operatorname{Im}(z)| \leq |z|$

- Triangle Inequality: $\left|\sum_{k=1}^{n} z_k\right| \leq \sum_{k=1}^{n} |z_k|$

- Cauchy-Schwarz Inequality: $\left|\sum_{k=1}^{n} z_k w_k\right| \leq \sqrt{\sum_{k=1}^{n} |z_k|^2} \sqrt{\sum_{k=1}^{n} |w_k|^2}$

> **Problem**
> If $a, b \in \mathbb{C}$, prove the **parallelogram identity**: $|a-b|^2 + |a+b|^2 = 2(|a|^2 + |b|^2)$.

Let $a = p + iq$ and $b = r + is$, then
$$\begin{aligned}
|a-b|^2 + |a+b|^2 &= (p-r)^2 + (q-s)^2 + (p+r)^2 + (q+s)^2 \\
&= 2(p^2 + q^2 + r^2 + s^2) \\
&= 2(|a|^2 + |b|^2)
\end{aligned}$$

> **Problem**
> Prove **Langrange's identity**:
> $$\left|\sum_{k=1}^{n} z_k w_k\right|^2 = \left(\sum_{k=1}^{n} |z_k|\right)\left(\sum_{k=1}^{n} |w_k|\right) - \sum_{k<j} |z_k \overline{w_j} - z_j \overline{w_k}|.$$

We abuse the fact that $z\overline{z} = |z|^2$.

$$
\begin{aligned}
\left| \sum_{k=1}^{n} z_k w_k \right|^2 &= \left(\sum_{k=1}^{n} z_k w_k \right) \overline{\left(\sum_{k=1}^{n} z_k w_k \right)} \\
&= \left(\sum_{k=1}^{n} z_k w_k \right) \left(\sum_{k=1}^{n} \overline{z_k w_k} \right) \\
&= \sum_{k=1}^{n} z_k w_k \overline{z_k w_k} + \sum_{j \neq k} z_j w_j \overline{z_k w_k} \\
&= \sum z_k w_j \overline{z_k w_j} + \sum_{j \neq k} z_j w_j \overline{z_k w_k} - \sum_{j \neq k} z_k w_j \overline{z_k w_j} \\
&= \sum |z_k|^2 |w_j|^2 + \sum_{j \neq k} z_j w_j \overline{z_k w_k} - \sum_{j \neq k} z_k w_j \overline{z_k w_j} \\
&= \left(\sum_{k=1}^{n} |z_k| \right) \left(\sum_{k=1}^{n} |w_k| \right) + \sum_{j \neq k} z_j w_j \overline{z_k w_k} - \sum_{j \neq k} z_k w_j \overline{z_k w_j}
\end{aligned}
$$

For some distinct indices j, k we have

$$
\begin{aligned}
z_j w_j \overline{z_k w_k} + z_k w_k \overline{z_j w_j} - z_k w_j \overline{z_k w_j} - z_j w_k \overline{z_j w_k} &= z_j \overline{w_k} (w_j \overline{z_k} - w_k \overline{z_j}) + z_k \overline{w_j} (w_k \overline{z_j} - w_j \overline{z_k}) \\
&= (w_k \overline{z_j} - w_j \overline{z_k})(z_j \overline{w_k} - z_k \overline{w_j}) \\
&= -(w_k \overline{z_j} - w_j \overline{z_k}) \overline{(w_k \overline{z_j} - w_j \overline{z_k})} \\
&= -|w_k \overline{z_j} - w_j \overline{z_k}|^2
\end{aligned}
$$

Summing up gives the desire result

1.3 **Even Weirder Stuff**

Using the fact that
$$
re^{ix} = r(\cos x + i \sin x)
$$
and thanks to Euler we generalize the complex numbers to even more functions.

- It's not hard to see that

$$
\sin x = \frac{e^{ix} - e^{-ix}}{2i} \quad \text{and} \quad \cos x = \frac{e^{ix} + e^{-ix}}{2}
$$

- Let $z = re^{i\theta}$ then $\ln z = \ln |r| + i \arg z$.

- $z^w = e^{w \ln z}$ can be determined consequently.

- Moreover, we have

$$
\sinh x = -i \sin(ix) \quad \text{and} \quad \cosh x = \cos(ix)
$$

which can be deduced from

$$\sinh x = \frac{e^x - e^{-x}}{2} \quad \text{and} \quad \cosh x = \frac{e^x + e^{-x}}{2}$$

> **Problem**
>
> Along which rays through the origin does $\lim_{z \to \infty} |e^z|$ exist?

Let $z = x + iy$, then we have $|e^z| = |e^x(\cos y + i \sin y)| = e^x$. If $x \to -\infty$ then $e^x \to 0$, but if $x \to \infty$ then $e^x \to \infty$ which the limit doesn't exist.

Hence the answers are all the rays passing through the nonnegative x plane.

> **Problem**
>
> Prove the identity
> $$z = \tan \left[\frac{1}{i} \ln \left(\frac{1 + iz}{1 - iz} \right)^{1/2} \right]$$
> for all real z.

$$\tan \left[\frac{1}{i} \ln \left(\frac{1 + iz}{1 - iz} \right)^{1/2} \right] = \tan \left[\frac{1}{2i} \left(\ln(1 + iz) - \ln(1 - iz) \right) \right]$$

$$= \tan \left[\frac{1}{2i} \left(\ln|1 + iz| + i(\tan^{-1} z) - \ln|1 - iz| - i(\tan^{-1}(-z)) \right) \right]$$

$$= \tan \left[\frac{1}{2i} \left(2i(\tan^{-1} z) \right) \right]$$

$$= z$$

> **Problem**
>
> Use the equation $\sin z = \sin x \cosh y + i \sinh y \cos x$ where $z = x + iy$ to prove that $|\sinh y| \leq |\sin z| \leq |\cosh y|$.

Evaluating gives

$$|\sin z| = \sqrt{\sin^2 x \cosh^2 y + \sinh^2 y \cos^2 x}$$

Using the fact that $\sinh x < \cosh x$, we have

$$\sin^2 x \sinh^2 y + \sinh^2 y \cos^2 x < \sin^2 x \cosh^2 y + \sinh^2 y \cos^2 x < \sin^2 x \cosh^2 y + \cosh^2 y \cos^2 x$$

simplifying gives the desired result.

> **Problem**
>
> Using polar coordinates, show that $z \mapsto z + 1/z$ maps the circle $|z| = 1$ to the interval $[-2, 2]$ on the x axis.

Let $z = x + iy$, then $z + \dfrac{1}{z} = x + iy + \dfrac{x - iy}{x^2 + y^2}$ and since $x^2 + y^2 = 1$, $z + \dfrac{1}{z} = 2x$. This means that for any complex number $z = x + iy$ on the circle, it will be mapped to $2x$.

And since x is in the interval $[-1, 1]$, hence $2x$ is in the interval $[-2, 2]$.

1.4 Topological Analysis of Complex Functions

1.4.1 Definitions

- r **Disk**: The r disk is defined by $D(z_0; r) = \{z \in \mathbb{C} |\ |z - z_0| < r\}$. The **deleted r disk** is defined by $D(z_0; r) \backslash \{z_0\}$.

- **Open Sets**: The set $A \subset \mathbb{C}$ is open when for any point z_0 in A, there exists a real number ε such that if $|z - z_0| < \varepsilon$ then $z \in A$.
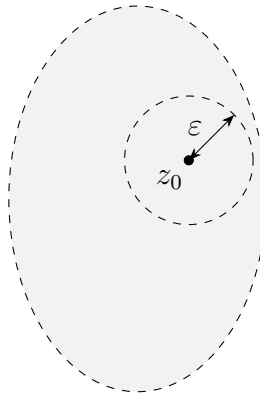
- **Closed Sets**: A set F is closed if $\mathbb{C} \backslash F$ is open.
  - The empty set and $\mathbb{C}$ are both open and closed (known as **clopen sets**).
  - Let $z_1, z_2, z_3, \ldots$ are points in F and $w = \lim\limits_{n \to \infty} z_n$, then $w \in F$.
    * **Sketch of proof**: Assume that $w \notin F$, then since $\mathbb{C} \backslash F$ is open, we can always find a disk $D(w; r)$ contained in $\mathbb{C} \backslash F$. This means that there exists some large enough n such that $z_n \in D(w; r)$ by convergence, which implies $z_n \notin F$, a contradiction.
  - The **closure** of a set S, denoted by $\overline{S}$ is the set S together with its limit points, or known as the **boundary** $\partial(S)$.

- **Limits**: The limit $\lim\limits_{z \to z_0} f(z) = L$ exists when for all $\varepsilon > 0$, there exists $\delta > 0$ such that for all $|z - z_0| < \delta$ ($z \neq z_0$) we have $|f(z) - L| < \varepsilon$.
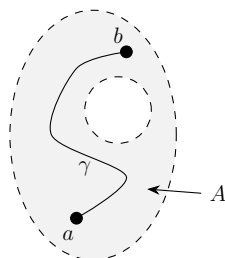
  Limits are **unique** if they exist.

- **Continuity**: f is continuous at $z_0 \in A$ if and only if

$$\lim_{z \to z_0} f(z) = f(z_0).$$

- **Cauchy Sequence**: A sequence is *Cauchy* if for every $\varepsilon > 0$, we can find some integer N such that whenever integers m, n are greater than N, $|z_m - z_n| < \varepsilon$.

- **Path-Connected**: A set $A \in \mathbb{C}$ is path-connected if for every $a, b \in A$ there exists a *continuous map* $\gamma : [0, 1] \to A$ such that $\gamma(0) = a, \gamma(1) = b$.

  γ is a **path** joining a and b.

Definition: A set $C \in \mathbb{C}$ is **not connected** if there are open sets U, V such that

(a) $C \subset (U \cup V)$;

(b) $(C \cap U \neq \varnothing) \wedge (C \cap V \neq \varnothing)$;

(c) $C \cap U \cap V = \varnothing$.

If a set is not "not connected", then it is **connected**.

- **A path-connected set is connected, but a connected set may not be path-connected.**

- Example: **Topologist's Sine Curve**

$$f(x) = \begin{cases} \sin \dfrac{1}{x} & x > 0 \\ 0 & x = 0 \end{cases}$$

  Sketch of proof: Let the two sets be A, B. WLOG let $(0, 0) \in A$.

  If some part of $\sin(1/x)$ is in A, then B should be covering the other parts. But since both sets are open, there's a point that is not covered.

  If no part of $\sin(1/x)$ is in A, then B must be covering the entire line of $\sin(1/x)$. But this is impossible since we cannot cover all points near $x = 0^+$.

- **Cover**: Let U be a collection of open sets. U is a cover of a set K if K is contained in the union of sets in U.
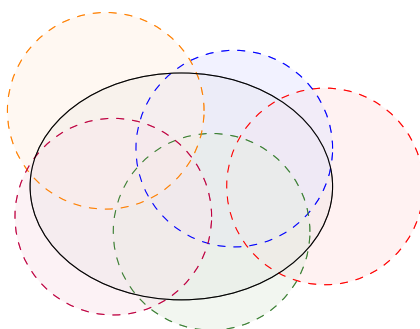
A **subcover** is a subset of U but can still cover K.

- **Compactness**: A set K is **compact** if every cover of K has a finite subcover.
  - **Heine-Borel Theorem**: A set K is compact if and only if K is closed and bounded.

    Sketch of proof:
    * Sufficiency:

      Boundedness: Assume that K is not bounded. Consider the set of open covers $U = \{D(O; r)\}$, (open) disks centered at the origin, then for all finite subcover U' of U, consider $R = \max(r)$ and choose some point $z \in K$ but $|z| > r$.

      Closedness: Assume that K is not closed, then there exists some $w \notin K$ such that the sequence $\{z_i\}$ in K converges to w. So the set of open covers $U = \{D(w, r)\}$ does not have a finite subcover.

    * Necessity: Assume that K is closed and bounded, then let $z \in K$ such that $|z|$ attains maximum value. Choose the open cover $D(O; |z| + 1)$.

1.4.2 On Functions

- If f is a continuous function defined on a connected set C, then $f(C)$ is connected.

  Sketch of proof: FTSOC, let $A|B$ be a partition of $f(C)$. Then $f^{-1}(A)$ and $f^{-1}(B)$ are open and disjoint (since each value $f^{-1}(x)$ can only belong to either one of $f^{-1}(A)$ and $f^{-1}(B)$.)

- If f is a continuous function defined on a compact set C, then $f(C)$ is compact.

  Sketch of proof: Let U be an open cover of $f(C)$, then for each $f(z) \in U$ and $f(z) \in f(C)$, we have $z \in C$ and $z \in f^{-1}(U)$.

- **Extreme Value Theorem**: Let K be a compact set and $f : K \to \mathbb{R}$ is a continuous function, then f attains **finite** maximum and minimum values.

  Sketch of proof: K is compact implies $f(K)$ is compact, or $f(K)$ is bounded, therefore finite maximum and minimum exists.

- **Distance Lemma**: Let K be a compact set and C be a closed set and $K \cap C = \varnothing$. Then there exists a number $\rho > 0$, such that whenever $z \in K$ and $w \in C$ then $|z - w| > \rho$.

  Sketch of proof: K is closed and bounded. Assume that ρ doesn't exist, $\rho \to 0$ since we can always find some $|z - w| < \rho_0$ if ρ_0 is fixed. Consider the sequences $\{z_k\}$ and $\{w_k\}$. Thus $\lim_{k \to \infty} |z_k - w_k| = 0$ which means $\lim_{k \to \infty} z_k = \lim_{k \to \infty} w_k$.

  But since both sets are closed, we must have $\lim_{k \to \infty} z_k \in K$ and $\lim_{k \to \infty} w_k \in C$, hence a contradiction.

- **Uniform Continuity**: A function $f : A \to \mathbb{C}$ is **uniformly continuous** on A if for any $\varepsilon > 0$, there exists a $\delta > 0$ such that $|f(s) - f(t)| < \varepsilon$ whenever $s, t \in A$ and $|s - t| < \delta$.

  An example $f(x) = \sin x$ and a nonexample $f(x) = 2^x$ are shown below. Choose $\delta = \varepsilon/2 = 0.5$.
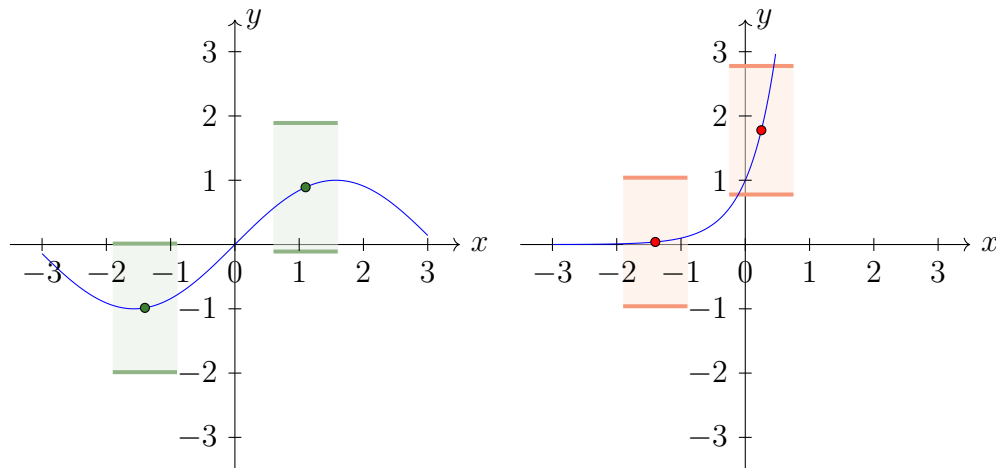
- **Heine-Cantor Theorem**: Let $f : A \to \mathbb{C}$ be a continuous function. If A is compact then $f(A)$ is uniformly continuous.
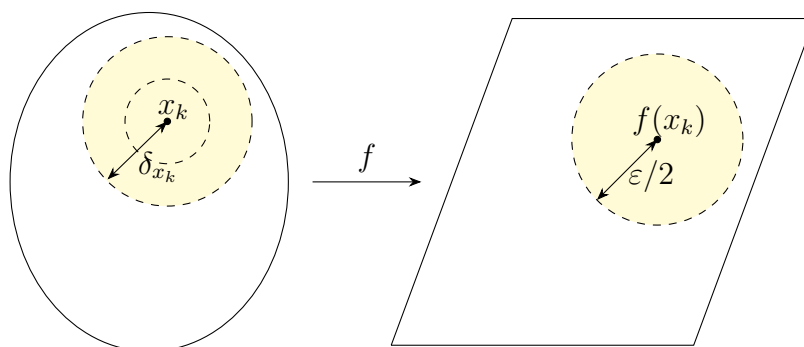
  Sketch of proof:
  - Let x be some point in A. By continuity, there is a number δ_x such that whenever for some point y satisfying $|x - y| < \delta_x$ then $|f(x) - f(y)| < \varepsilon/2$.
  - For a sequence of points x, say $\{x_i\}$, consider disks $D_k = D(x_k; \delta_{x_k}/2)$. These disks cover A by compactness. Let the minimal radius over all disks be δ.
  - For points s, t satisfying $|s - t| < \delta$, we must have t contained in some disk D_k. Thus $|t - x_k| < \delta_{x_k}/2$, implies that $|f(t) - f(x_k)| < \varepsilon/2$. We have
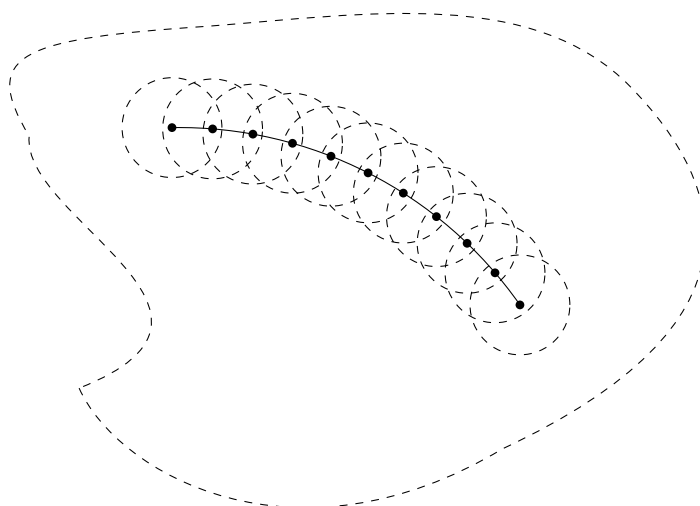 $$|s - x_k| \le |s - t| + |t - x_k| < \delta + \delta_{x_k}/2 \le \delta_{x_k}$$
  - On the other hand,
  $$|f(s) - f(t)| \le |f(s) - f(x_k)| + |f(x_k) - f(t)| < \varepsilon$$

- **Path-Covering Lemma**: Suppose $\gamma : [0,1] \to K$ is a continuous path into an open subset K of $\mathbb{C}$. We can find a number $\rho > 0$ and a subdivision of $[0,1]$, namely $0 = t_0 < t_1 < t_2 < \cdots < t_{N-1} < t_N = 1$ such that

  (a) $D(\gamma(t_k); \rho) \subset G$ for all k

  (b) $\gamma(t) \in D(\gamma(t_0); \rho)$ for $t_0 \leq t \leq t_1$

  (c) $\gamma(t) \in D(\gamma(t_k); \rho)$ for $t_{k-1} \leq t \leq t_{k+1}$

 (d) $\gamma(t) \in D(\gamma(t_N); \rho)$ for $t_{N-1} \leq t \leq t_N$

    Sketch of proof:

  – By the Distance Lemma, we can find some $\rho > 0$ such that the distance from $\gamma([0,1])$ to $\mathbb{C} \backslash K$ is at least ρ since both sets are closed.

  – By Heine-Cantor Theorem, $\gamma([0,1])$ is uniformly continuous. So for any two points s, t, if $|s - t| < \delta$ then $|f(s) - f(t)| < \rho$.

  – Choose t_k to be fine enough such that $t_{k+1} - t_k < \delta$.

- **Riemann Sphere**: We may want to define the value ∞ in the complex plane.

  – $\lim\limits_{z \to \infty} f(z) = L$ means for any $\varepsilon > 0$, there exists $Z > 0$ such that whenever $|z| > Z$ implies $|f(z) - L| < \varepsilon$.

  – $\lim\limits_{z \to z_0} f(z) = \infty$ means for any $R > 0$, there exists $\delta > 0$ such that whenever $|z - z_0| < \delta$ implies $|f(z)| > R$.

– $\lim\limits_{z \to \infty} f(z) = \infty$ means for any $Z > 0$, there exists $R > 0$ such that whenever $|z| > Z$ implies $|f(z)| > R$.

Consider the sphere $x^2 + y^2 + z^2 = 1$ in $\mathbb{R}^3$. A point z' on the plane $\mathbb{C}$ is the **stereographic projection** of some point z on the sphere through $(0, 0, 1)$.

A 2-D illustration as an example:
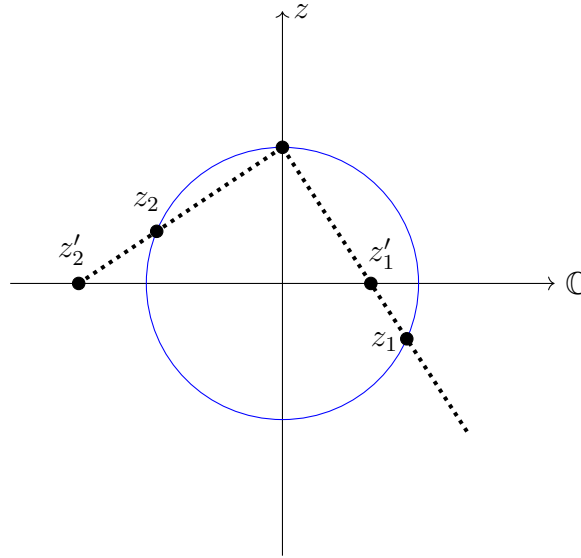
Problem

Show that $\lim\limits_{z \to \infty} \dfrac{1}{z} = 0$.

By our definition, we must have $|z| \to \infty$. Let $z = x + iy$ so that $|z| = \sqrt{x^2 + y^2}$. Clearly, at least one of $|x|, |y|$ must tend to ∞.

So
$$\lim_{z \to \infty} \frac{1}{z} = \lim_{|x| \to \infty \text{ or } |y| \to \infty} \left(\frac{x}{x^2 + y^2} - i\frac{y}{x^2 + y^2} \right)$$
which, implies that $\lim\limits_{z \to \infty} \dfrac{1}{z} = 0$.

Problem

(a) Show that

$$|\mathrm{Re}\,(z_1) - \mathrm{Re}\,(z_2)| \leq |z_1 - z_2| \leq |\mathrm{Re}\,(z_1) - \mathrm{Re}\,(z_2)| + |\mathrm{Im}\,(z_1) - \mathrm{Im}\,(z_2)|$$

for any two complex numbers z_1 and z_2.

(b) If $f(z) = u(x, y) + iv(x, y)$, show that

$$\lim_{z \to z_0} f(z) = \lim_{\substack{x \to x_0 \\ y \to y_0}} u(x, y) + \lim_{\substack{x \to x_0 \\ y \to y_0}} iv(x, y)$$

exists if both limits on the right of the equation exist. Conversely, if the limit on the left exists, show that both limits on the right exist as well and equality holds.

(a) Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ for real numbers x_i, y_i. Let $\mathcal{X} = |x_1 - x_2|$ and $\mathcal{Y} = |y_1 - y_2|$. The inequality above can be expressed as

$$\mathcal{X} \leq \sqrt{\mathcal{X}^2 + \mathcal{Y}^2} \leq \mathcal{X} + \mathcal{Y}$$

which is obvious by squaring each part in the inequality.

(b) **Necessity**: Assume that both $\lim\limits_{\substack{x \to x_0 \\ y \to y_0}} u(x, y) = U$ and $\lim\limits_{\substack{x \to x_0 \\ y \to y_0}} v(x, y) = V$ exists, then for some ε, there exists δ_u and δ_v such that:

- $|u(x, y) - U| < \varepsilon/2$ whenever $|(x, y) - (x_0, y_0)| < \delta_u$.
- $|v(x, y) - V| < \varepsilon/2$ whenever $|(x, y) - (x_0, y_0)| < \delta_v$.

By the limit laws, $\lim\limits_{z \to z_0} f(z) = U + iV$, then there exists $\delta = \min\{\delta_u, \delta_v\}$ such that whenever $|z - z_0| < \delta$,

$$\begin{aligned} |f(z) - Z| &= |u(x, y) + iv(x, y) - U - iV| \\ &\leq |u(x, y) - U| + |i||v(x, y) - V| \\ &< \varepsilon \end{aligned}$$

Sufficiency: Assume that $\lim\limits_{z \to z_0} f(z) = U + iV$ exists. Then for all $\varepsilon > 0$ there exists $\delta_u > 0$ such that whenever $0 < |z - z_0| < \delta_u$ then $|u(x, y) - U| < \varepsilon$ and $\delta_v > 0$ such that whenever $0 < |z - z_0| < \delta_v$ then $|v(x, y) - V| < \varepsilon$. Choose $\delta = \min\{\delta_u, \delta_v\}$.

> **Problem**
>
> Introduce the **chordal metric** ρ on $\bar{\mathbb{C}}$ by setting $\rho(z_1, z_2) = d(z_1', z_2')$ where z_1' and z_2' are the corresponding points on the Riemann sphere and d is the usual distance between points in $\mathbb{R}^3$.
>
> (a) Show that $z_n \to z$ in $\mathbb{C}$ if and only if $\rho(z_n, z) \to 0$.
>
> (b) Show that $z_n \to \infty$ if and only if $\rho(z_n, \infty) \to 0$.
>
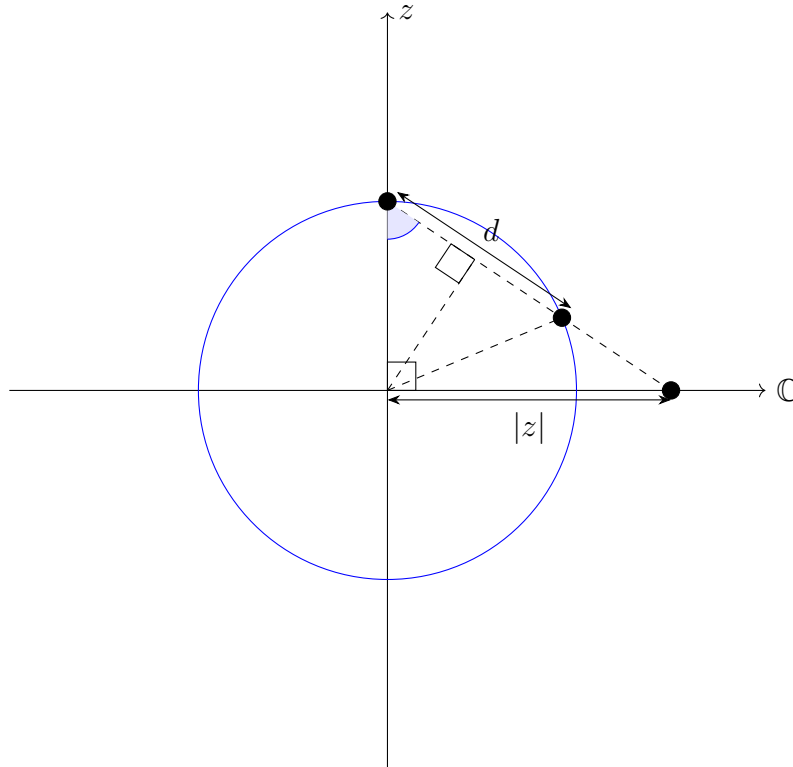> (c) If $f(z) = (az + b)/(cz + d)$ and $ad - bc \neq 0$, show that f is continuous at ∞.

(a) If $z_n \to z$, then $\rho(z_n, z) \to \rho(z, z) = d(z', z') = 0$.

On the other hand, let $z_n' = (x_n, y_n, t_n)$ and $z' = (x, y, t)$. Then if $\rho(z_n, z) = d(z_n', z') \to 0$, we have

$$\sqrt{(x_n - x)^2 + (y_n - y)^2 + (t_n - t)^2} \to 0$$
$$(x_n - x)^2 + (y_n - y)^2 + (t_n - t)^2 \to 0$$

FTSOC, WLOG assume x_n does not converge to x, then since $(x_n - x)^2 + (y_n - y)^2 + (t_n - t)^2 \geq 0$, we have $(x_n - x)^2 + (y_n - y)^2 + (t_n - t)^2$ converges to at least $(x_n - x)^2$, contradiction.

(b) If $z_n \to \infty$, $|z_n| \to \infty$. By drawing the Riemann sphere again,

We see that $\tan \theta = |z|$ and $d = 2 \cos \theta$ (θ is the measure of the angle marked

in blue). Substituting gives

$$d = \frac{2}{\sqrt{|z|^2 + 1}}$$

which is obvious that $d \to 0$.

Conversely, if $d \to 0$, it can be shown that $\sqrt{|z|^2 + 1} \to \infty$ which gives $z_n \to \infty$.

(c) A function $f(z)$ is continuous at infinity if the limits when $z \to +\infty$ and $z \to -\infty$ are equal.

- $\lim\limits_{z \to \infty} \dfrac{az + b}{cz + d} = \dfrac{a}{c}$.

- $\lim\limits_{z \to -\infty} \dfrac{az + b}{cz + d} = \dfrac{a}{c}$.

Hence the function is continuous at ∞.

1.5 Analytic Functions: What and Why

- functions that are differentiable in complex.

- "regular", "holomorphic", "analytic"

- a function $f(z)$ is differentiable at z_0 if $\lim\limits_{z \to z_0} \dfrac{f(z) - f(z_0)}{z - z_0}$ exists.

1.5.1 Differentiation

Differentiation laws apply here, including product rule, quotient rule, and chain rule.

- **Conformal Maps**: A function $f : A \to \mathbb{C}$ is conformal if it preserves **angles** between intersecting curves.
    - "conformal transformation", "angle-preserving transformation", "biholomorphic map"
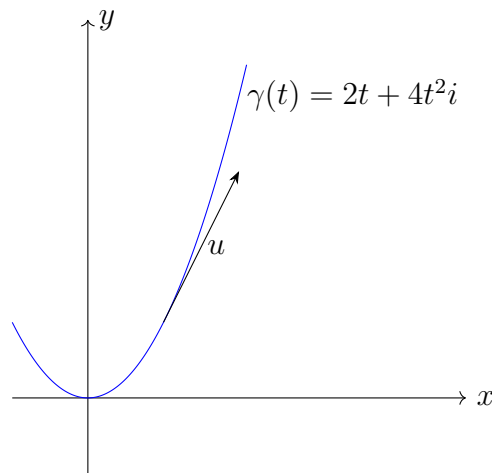   - An example of $e^z = e^{x+iy}$.
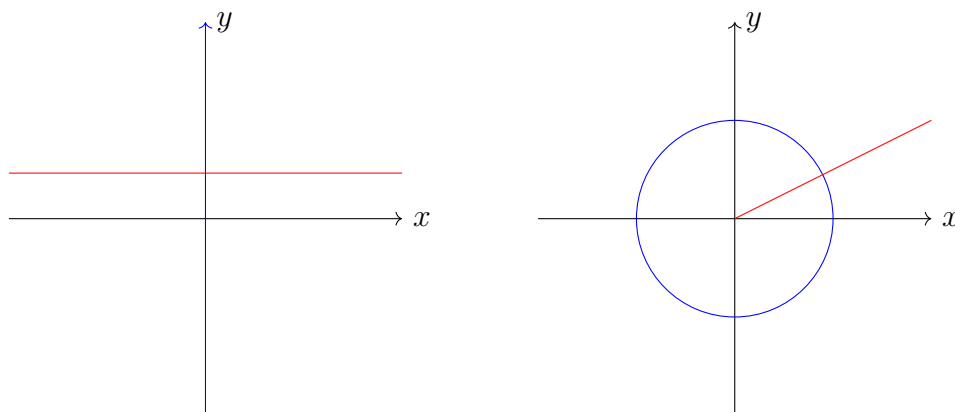
- What does it mean to differentiate complex numbers?

  - Clearly $f'(z_0) = x + iy$ is a vector $\begin{pmatrix} x \\ y \end{pmatrix}$ in the plane. Assume for some curve $\gamma(t) : \mathbb{R} \to \mathbb{C}$.

  > **Claim** — Let $\gamma(t) = x(t) + iy(t)$, define $\gamma'(t) = x'(t) + iy'(t)$, then the vector $u = \begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix}$ is tangent to the curve γ at $(x(t), y(t))$.

  This is trivial by noticing

  $$\frac{y'(t)}{x'(t)} = \frac{\dfrac{\mathrm{d}y}{\mathrm{d}t}}{\dfrac{\mathrm{d}x}{\mathrm{d}t}} = \frac{\mathrm{d}y}{\mathrm{d}x}$$

  $\gamma(t) = 2t + 4t^2 i$

  - **Conformal Mapping Theorem**: If $A \to \mathbb{C}$ is analytic and $f'(z_0) \neq 0$, then f is conformal at z_0.

  Sketch of Proof: Define $\sigma(t) = f(\gamma(t))$, then obviously $\sigma(t)$ is also a curve. Taking

  $$\frac{\mathrm{d}f(\gamma(t))}{\mathrm{d}t} = \frac{\mathrm{d}f(\gamma(t))}{\mathrm{d}z} \cdot \frac{\mathrm{d}z}{\mathrm{d}t}$$

  Assume $\gamma(t_0) = z_0$, letting $t = t_0$ we have

  $$\sigma'(t_0) = f'(z_0)\gamma'(t_0)$$

  Since by our definition, $f'(z_0)$ is independent of γ, choose t_1 and t_2 so that

  $$\frac{\sigma'(t_1)}{\gamma'(t_1)} = \frac{\sigma'(t_2)}{\gamma'(t_2)}$$

  Taking the argument we have

  $$\arg \sigma'(t_1) - \arg \sigma'(t_2) \equiv \arg \gamma'(t_1) - \arg \gamma'(t_2) \pmod{2\pi}$$

  hence it is clear that angles are preserved.