Lecture 1: Series Computation

1.1 Arithmetic and Geometric Series

Given an arithmetic sequence $a_n = a_1 + (n-1)d$, assume that

$$S_n = a_1 + a_2 + a_3 + \cdots + a_{n-1} + a_n$$

Or we can rewrite this as

$$\begin{aligned} S_n &= (a_1 + 0d) \quad + (a_1 + 1d) \quad + \ldots + (a_1 + (n-2)d) + (a_1 + (n-1)d) \\ S_n &= (a_1 + (n-1)d) + (a_1 + (n-2)d) + \ldots + \quad (a_1 + 1d) + \quad (a_1 + 0d) \end{aligned}$$

Summing up gives

$$2S_n = n(2a_1 + (n-1)d)$$

Simplifying gives

$$\boxed{S_n = \frac{(a_1 + a_n)n}{2}}$$

Given a geometric series $a_n = a_1 r^{n-1}$, assume that

$$S_n = a_1 + a_2 + a_3 + \cdots + a_{n-1} + a_n,$$

or we can rewrite this as

$$\begin{aligned} S_n &= a_1 + a_1 r + a_1 r^2 + \cdots + a_1 r^{n-1} \\ rS_n &= \qquad a_1 r + a_1 r^2 + \cdots + a_1 r^{n-1} + a_1 r^n \end{aligned}$$

Subtracting gives

$$(r-1)S_n = a_1(r^n - 1)$$

Simplifying yields

$$\boxed{S_n = \frac{a_1(r^n - 1)}{r - 1}}$$

If $|r| < 1$ and $n \to \infty$, then $r^n \to 0$, so we have

$$\boxed{S_\infty = \frac{a_1}{1 - r}}$$

Example 1.1.1

Prove the following equations:

1. $1 + 3 + 5 + \cdots + (2n-1) = n^2$
2. $1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \cdots = \frac{2}{3}$
3. $\frac{1}{a} + \frac{1}{a^2} + \frac{1}{a^3} + \cdots = \frac{1}{a-1}$
4. $1 + x + x^2 + x^3 + \cdots + x^n = \frac{x^{n+1}-1}{x-1}$
5. $\underbrace{24242424\ldots2424}_{n \text{ copies of } 24} = \frac{8}{33}(100^n - 1)$

Solution.

1. $a_1 = 1$, $d = 2$, the number of terms equals n,
$$S = \frac{(1 + (2n-1))n}{2} = n^2.$$
2. $a_1 = 1$, $r = -\frac{1}{2}$, so
$$S = \frac{1}{1 - \left(-\frac{1}{2}\right)} = \frac{2}{3}.$$
3. $a_1 = \frac{1}{a}$, $r = \frac{1}{a}$, so
$$S = \frac{\frac{1}{a}}{1 - \frac{1}{a}} = \frac{1}{a-1}.$$
4. $a_1 = 1$, $r = x$, the number of terms equals $n + 1$,
$$S = \frac{x^{n+1}-1}{x-1}$$
5. $$\begin{aligned}\underbrace{24242424\ldots2424}_{n \text{ copies of } 24} &= 24 + 24 \times 100 + 24 \times 100^2 + \cdots + 24 \times 100^{n-1} \\ &= \frac{24(100^n - 1)}{100 - 1} \\ &= \frac{8}{33}(100^n - 1)\end{aligned}$$

□

Example 1.1.2 (HLG 2003)

Let a_1, a_2, ..., a_n be a sequence. If a_1, $a_2 - a_1$, $a_3 - a_2$, ..., $a_n - a_{n-1}$ is a geometric sequence with $a_1 = 1$ and common ratio 0.2, find a_n.

Solution. We see that $a_n - a_{n-1} = 0.2^{n-1}$ for all $n \geq 2$. Adding the terms a_1, $a_2 - a_1$, $a_3 - a_2$, ..., $a_n - a_{n-1}$ we obtain a_n. So

$$\begin{aligned} a_1 + (a_2 - a_1) + (a_3 - a_2) + \cdots + (a_n - a_{n-1}) &= a_n \\ 1 + 0.2^1 + 0.2^2 + \cdots + 0.2^{n-1} &= a_n \\ a_n &= \frac{0.2^n - 1}{0.2 - 1} \\ &= \frac{5}{4}\left[1 - \left(\frac{1}{5}\right)^n\right] \end{aligned}$$

□

Practice Problems

Problem 1.1.3. Prove the following equations:

1. $4 + 12 + 20 + \cdots + (8n - 4) = (2n)^2$
2. $\left(2 + \frac{1}{2}\right) + \left(4 + \frac{1}{4}\right) + \left(8 + \frac{1}{8}\right) + \cdots + \left(2^n + \frac{1}{2^n}\right) = 2^{n+1} - 2^{-n} - 1$
3. $\displaystyle\sum_{k=1}^{\infty} \left[\left(\frac{1}{2}\right)^k + \left(\frac{2}{3}\right)^k + \left(\frac{3}{4}\right)^k + \cdots + \left(\frac{n}{n+1}\right)^k\right] = \frac{1}{2}(n^2 + n)$

Problem 1.1.4 (HLG 2013)**.** Given that $a_1, a_2, a_3, \ldots, a_{2012}, a_{2013}$ is an arithmetic progression with sum $a_1 + a_2 + a_3 + \cdots + a_{2012} + a_{2013} = 1098$. Find $a_7 + a_9 + a_{11} + \cdots + a_{2005} + a_{2007}$.

Problem 1.1.5 (HLG 2015)**.** Find the value of $\dfrac{2 + 4 + 8 + \cdots + 2^9}{\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^9}}$.

Problem 1.1.6. Given that $n = \sqrt{\underbrace{111\ldots1}_{\text{200 digits of 1}} - \underbrace{222\ldots2}_{\text{100 digits of 1}}}$ is an integer, find the sum of the digits of n.

Problem 1.1.7 (CJR 2018)**.** The repeating decimal $2.2\dot{4}\dot{5} = 2.245454545\ldots$ is equal to (in fraction)?

Problem 1.1.8 (CJR 2023)**.** Given that

$$\lim_{n\to\infty} \sum_{k=1}^{n} \left(\frac{a}{a+2}\right)^k = 123.$$

Find the value of a.

Problem 1.1.9 (OMK 2019)**.** We call a sequence of five numbers a *good* sequence if it is an arithmetic progression that contains the terms 20 and 19. For example, these two sequences are good sequences:

$$18, 19, 20, 21, 22;$$

$$20, 19\frac{2}{3}, 19\frac{1}{3}, 19, 18\frac{2}{3}.$$

For each good sequence, we take the sum of all terms in the sequence. Then we add the sums over all possible good sequences. What will be the result?

Problem 1.1.10. Given that a polynomial $P(x) = 2023x^{2023} + 2022x^{2022} + ... + 3x^3 + 2x^2 + x$ can be factored into 2023 linear factors, namely $k(x - \alpha_1)(x - \alpha_2)...(x - \alpha_{2023})$. Then, given a polynomial $Q(x)$ of degree 2023 and leading coefficient 1 such that the roots of the equation $Q(x) = 0$ are

$$1 - \frac{\alpha_1 + \alpha_2}{1 + \alpha_2},\ 1 - \frac{\alpha_2 + \alpha_3}{1 + \alpha_3},\ 1 - \frac{\alpha_3 + \alpha_4}{1 + \alpha_4}, ..., 1 - \frac{\alpha_{2022} + \alpha_{2023}}{1 + \alpha_{2023}},\ 1 - \frac{\alpha_{2023} + \alpha_1}{1 + \alpha_1}$$

It is known that $Q(0)$ is an integer. Find the last three digits of $|Q(0)|$.

1.2 Telescoping Series

Assume that

$$\sum_{k=1}^{n} g(k)$$

Assume that we can express $g(x) = f(x+1) - f(x)$, so that we obtain

$$\begin{aligned}\sum_{k=1}^{n} g(k) &= \sum_{k=1}^{n} (f(k+1) - f(k)) \\ &= f(n+1) - f(n) + f(n) - f(n-1) + f(n-1) - f(n-2) + \cdots + f(2) - f(1) \\ &= f(n+1) - f(1)\end{aligned}$$

This works for products too! For

$$\prod_{k=1}^{n} g(k)$$

we may obtain $g(x) = \dfrac{f(x+1)}{f(x)}$, and yet we have

$$\begin{aligned}\prod_{k=1}^{n} g(k) &= \prod_{k=1}^{n} \frac{f(k+1)}{f(k)} \\ &= \frac{f(n+1)}{f(n)} \cdot \frac{f(n)}{f(n-1)} \cdot \frac{f(n-1)}{f(n-2)} \cdot \cdots \cdot \frac{f(2)}{f(1)} \\ &= \frac{f(n+1)}{f(1)}\end{aligned}$$

Example 1.2.1 (HLG 2017)

Let

$$S=\sum_{k=1}^{99}\sqrt{1+\frac{1}{k^2}+\frac{1}{(k+1)^2}}.$$

Given that $S=m-\frac{1}{n}$, where m, n are positive integers. Find the value of $m+n$.

Solution. First identify this to be a telescoping series, then assume that $g(x)=\sqrt{1+\frac{1}{x^2}+\frac{1}{(x+1)^2}}$. Then, we see that

$$\begin{aligned}
g(x)&=\sqrt{1+\frac{1}{x^2}+\frac{1}{(x+1)^2}}\\
&=\sqrt{\frac{x^4+2x^3+3x^2+2x+1}{x^2(x+1)^2}}\\
&=\sqrt{\frac{(x^2+x+1)^2}{(x^2+x)^2}}\\
&=\frac{x^2+x+1}{x^2+x}\\
&=1+\frac{1}{x^2+x}\\
&=1+\frac{1}{x}-\frac{1}{x+1}
\end{aligned}$$

Yet, we let $f(x)=\frac{1}{x}$, then we have

$$\begin{aligned}
S&=\sum_{k=1}^{99}\sqrt{1+\frac{1}{k^2}+\frac{1}{(k+1)^2}}\\
&=\sum_{k=1}^{99}g(k)\\
&=\sum_{k=1}^{99}(1+f(k)-f(k+1))\\
&=\sum_{k=1}^{99}1+\sum_{k=1}^{99}(f(k)-f(k+1))\\
&=99+f(1)-f(100)\\
&=99+\frac{1}{1}-\frac{1}{100}\\
&=100-\frac{1}{100}
\end{aligned}$$

So the answer is $m+n=100+100=200$. $\square$

Example 1.2.2 (Purple Comet Math Meet High School 2022)

The product

$$\left(\frac{1+1}{1^2+1}+\frac{1}{4}\right)\left(\frac{2+1}{2^2+1}+\frac{1}{4}\right)\left(\frac{3+1}{3^2+1}+\frac{1}{4}\right)\cdots\left(\frac{2022+1}{2022^2+1}+\frac{1}{4}\right)$$

can be written as $\frac{q}{2^r \cdot s}$, where r is a positive integer, and q and s are relatively prime odd positive integers. Find s.

Solution. Let $g(x)=\frac{x+1}{x^2+1}+\frac{1}{4}$, then we have

$$\begin{aligned} g(x) &= \frac{x+1}{x^2+1}+\frac{1}{4} \\ &= \frac{x^2+4x+5}{4(x^2+1)} \\ &= \frac{(x+2)^2+1}{4(x^2+1)} \end{aligned}$$

Then assume that $f(x)=x^2+1$, so that $g(x)=\frac{f(x+2)}{4f(x)}$. Yet we obtain

$$\begin{aligned} \prod_{k=1}^{2022}\left(\frac{k+1}{k^2+1}+\frac{1}{4}\right) &= \prod_{k=1}^{2022} g(k) \\ &= \prod_{k=1}^{2022}\frac{f(k+2)}{4f(k)} \\ &= \frac{f(3)}{4f(1)}\cdot\frac{f(4)}{4f(2)}\cdot\frac{f(5)}{4f(3)}\cdots\frac{f(2023)}{4f(2021)}\cdot\frac{f(2024)}{4f(2022)} \\ &= \frac{f(2023)f(2024)}{4^{2022}f(1)f(2)} \\ &= \frac{(2023^2+1)(2024^2+1)}{4^{2022}(1^2+1)(2^2+1)} \\ &= \frac{(2023^2+1)(2024^2+1)}{2^{4045}\cdot 5} \end{aligned}$$

A quick check shows that 2023^2+1 is a multiple of 5, so $s=1$. □

Practice Problems

Problem 1.2.3. Compute the following series:

1. $1\times 1!+2\times 2!+3\times 3!+\cdots+n\times n!$
2. $\frac{1}{3\times 5}+\frac{1}{5\times 7}+\frac{1}{7\times 9}+\cdots+\frac{1}{(2n-1)(2n+1)}$
3. $\frac{1}{2!}+\frac{2}{3!}+\frac{3}{4!}+\cdots+\frac{n-1}{n!}$

4. $\dfrac{1\times 2!}{2^3}+\dfrac{2\times 3!}{2^4}+\dfrac{3\times 4!}{2^5}+\cdots+\dfrac{n\times(n+1)!}{2^{n+2}}$

5. (CJR 2021) $120\displaystyle\sum_{k=1}^{112}\frac{(-1)^{k+1}}{\sqrt{4k^2-1}(\sqrt{2k+1}-\sqrt{2k-1})}$

6. (HLG 2014) $\left(1+\dfrac{2}{3}\right)\left(1+\dfrac{2}{5}\right)\left(1+\dfrac{2}{7}\right)\cdots\left(1+\dfrac{2}{1355}\right)\left(1+\dfrac{2}{1357}\right)$

Problem 1.2.4 (CJR 2017)**.** If $S_n=\dfrac{1}{1\times 2\times 4}+\dfrac{1}{2\times 3\times 5}+\dfrac{1}{3\times 4\times 6}+\cdots+\dfrac{1}{n\times(n+1)\times(n+3)}$, find $\lim\limits_{n\to\infty}(360S_n)$.

Problem 1.2.5 (CJR 2020)**.** Let

$$A_1=\{\log_{2k-1}(2k+1)\mid k \text{ is an integer, } 2\le k\le 1093\},$$
$$A_2=\{\log_{2k+1}(2k-1)\mid k \text{ is an integer, } 2\le k\le 364\}$$

Define P_1 to be the product of all the elements in A_1, and P_2 the product of all the elements in A_2. Find $\dfrac{P_1}{P_2}$.

Problem 1.2.6 (CJR 2020)**.** Given that $\displaystyle\sum_{k=1}^{n}\frac{1}{\sqrt{3k+1}+\sqrt{3k-2}}=17$, find the value of n.

Problem 1.2.7 (HLG 2011)**.** Assume that a is an integer greater than 1, find $\displaystyle\sum_{n=0}^{a}\frac{1}{(a^2+n)(a^2+n-1)}$.

Problem 1.2.8 (HLG 2014)**.** Find the value of $\dfrac{3}{1!+2!+3!}+\dfrac{4}{2!+3!+4!}+\cdots+\dfrac{50}{48!+49!+50!}$

Problem 1.2.9 (HLG 2015)**.** If n is a positive integer such that

$$\frac{1}{\sqrt{2}+\sqrt{1}}+\frac{1}{\sqrt{3}+\sqrt{2}}+\frac{1}{\sqrt{4}+\sqrt{3}}+\cdots+\frac{1}{\sqrt{n}+\sqrt{n-1}}=10,$$

Problem 1.2.10 (HLG 2018)**.** Let n be an integer larger than 3, and let α_n and β_n be the two roots of the equation $x^2+(n^2-3)x+3n=0$. Find the value of

$$\frac{3}{(\alpha_4-3)(\beta_4-3)}+\frac{3}{(\alpha_5-3)(\beta_5-3)}+\cdots+\frac{3}{(\alpha_{99}-3)(\beta_{99}-3)}.$$

Problem 1.2.11 (HLG 2019)**.** Given that

$$N=625\left(1-\frac{9}{5^2}\right)\left(1-\frac{9}{8^2}\right)\left(1-\frac{9}{11^2}\right)\cdots\left(1-\frac{9}{125^2}\right).$$

Find the sum of the digits of N.

Problem 1.2.12. Assume that $\dbinom{n}{r}$ is the number of ways to choose r things from n

distinct things. Find the value of

$$\frac{\binom{4}{2}}{\binom{4}{4}} + \frac{\binom{5}{2}}{\binom{5}{4}} + \frac{\binom{6}{2}}{\binom{6}{4}} + ... + \frac{\binom{9}{2}}{\binom{9}{4}} + \frac{\binom{10}{2}}{\binom{10}{4}}.$$

Problem 1.2.13 (Purple Comet Math Meet High School 2021)**.** The product

$$\left(\frac{1}{2^3-1}+\frac{1}{2}\right)\left(\frac{1}{3^3-1}+\frac{1}{2}\right)\left(\frac{1}{4^3-1}+\frac{1}{2}\right)\cdots\left(\frac{1}{100^3-1}+\frac{1}{2}\right)$$

can be written as $\frac{r}{s2^t}$, where r, s, and t are positive integers and r and s are odd and relatively prime. Find $r+s+t$.

Problem 1.2.14. Compute the following series:

1. (AIME 2008)

$$2[\cos(a)\sin(a)+\cos(4a)\sin(2a)+\cos(9a)\sin(3a)+\cdots+\cos(n^2a)\sin(na)]$$

2. (AIME 2000)

$$\frac{1}{\sin 45^\circ \sin 46^\circ}+\frac{1}{\sin 47^\circ \sin 48^\circ}+\cdots+\frac{1}{\sin 133^\circ \sin 134^\circ}$$

3. $$\left(\frac{1+\tan 1^\circ \cot 1^\circ}{1-\tan 1^\circ \tan 1^\circ}\right)\left(\frac{1+\tan 1^\circ \cot 2^\circ}{1-\tan 1^\circ \tan 2^\circ}\right)\cdots\left(\frac{1+\tan 1^\circ \cot 44^\circ}{1-\tan 1^\circ \tan 44^\circ}\right)$$

1.3 Sums of Powers

Define $S_p = \sum_{k=1}^{n} k^p$, then we should have

$$\begin{aligned}
S_3+3S_2+3S_1+S_0 &= \sum_{k=1}^{n}k^3+3\sum_{k=1}^{n}k^2+3\sum_{k=1}^{n}k^1+\sum_{k=1}^{n}k^0\\
&= \sum_{k=1}^{n}(k^3+3k^2+3k+1)\\
&= \sum_{k=1}^{n}(k+1)^3\\
&= (S_3-1^3)+(n+1)^3
\end{aligned}$$

Since it is known that $S_1 = \frac{1}{2}n^2+\frac{1}{2}n$ and $S_0 = n$, we get $S_2 = \frac{1}{3}n^3+\frac{1}{2}n^2+\frac{1}{6}n$.

Likewise, we may conclude that

$$\sum_{k=0}^{p+1} \binom{p+1}{k} S_k = S_{p+1} + (n+1)^{p+1}$$

By substituting appropriate values of p, the formula of S_p can be obtained, some include:

p	S_p
3	$\frac{1}{4}n^4 + \frac{1}{2}n^3 + \frac{1}{4}n^2$
4	$\frac{1}{5}n^5 + \frac{1}{2}n^4 + \frac{1}{3}n^3 - \frac{1}{30}n$
5	$\frac{1}{6}n^6 + \frac{1}{2}n^5 + \frac{5}{12}n^4 - \frac{1}{12}n^2$
6	$\frac{1}{7}n^7 + \frac{1}{2}n^6 + \frac{1}{2}n^5 - \frac{1}{6}n^3 + \frac{1}{42}n$
7	$\frac{1}{8}n^8 + \frac{1}{2}n^7 + \frac{7}{12}n^6 - \frac{7}{24}n^4 + \frac{1}{12}n^2$

Some patterns include:

- The leading term of S_p is $\dfrac{1}{p+1}n^{p+1}$.
- The second term is $\dfrac{1}{2}n^p$.
- There's no constant term.

The power sum is also closely related to the **Bernoulli Numbers**.

Example 1.3.1 (HLG 2010)

Compute $\sum_{m=1}^{n} m(m+1)(m+2)$ for all positive integers n.

Solution.

$$\begin{aligned}
\sum_{m=1}^{n} m(m+1)(m+2) &= \sum_{m=1}^{n} (m^3 + 3m^2 + 2m) \\
&= \sum_{m=1}^{n} m^3 + 3\sum_{m=1}^{n} m^2 + 2\sum_{m=1}^{n} m \\
&= \frac{1}{4}n^4 + \frac{1}{2}n^3 + \frac{1}{4}n^2 + 3\left(\frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n\right) + 2\left(\frac{1}{2}n^2 + \frac{1}{2}n\right) \\
&= \frac{n(n+1)(n+2)(n+3)}{4}
\end{aligned}$$

□

Practice Problems

Problem 1.3.2. Compute the following series:

1. (CJR 2017) $1+2-3+4+5-6+7+8-9+\cdots+100+101-102$.
2. $2^2+4^2+6^2+\cdots+200^2$.
3. $1\times 2^2+2\times 3^2+3\times 4^2+\cdots+100\times 101^2$.

Problem 1.3.3 (CJR 2022). Given that $a_1, a_2, a_3, \ldots, a_{2022}$ is a sequence such that for all $1 \leq k \leq 2022$,

$$a_k + a_{2023-k} = k(2023-k) + 11.$$

Let $S = a_1 + a_2 + \cdots + a_{2022}$ be the sum of the sequence. Find the last three digits of S.

Problem 1.3.4 (CJR 2023). Given that n is a positive integer and

$$1^2-2^2+3^2-4^2+\cdots+(2n-1)^2-(2n)^2=-5050.$$

find the value of n.

Problem 1.3.5. Prove that

$$\sum_{k=1}^{n} k(k+1)(k+2)(k+3) = \frac{1}{5}n(n+1)(n+2)(n+3)(n+4)$$

.

1.4 Sums of Binomial Coefficients

We define the **Binomial Coefficients** $\binom{n}{k}$ to be the number of ways to choose k things from n different things. Hence, we have

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

Moreover, we define $\binom{n}{k} = 0$ if $k > n$. Now, let us take a look at series related to combinations.

Theorem 1.4.1 (Binomial Theorem)

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

Letting $y = 1$, gives

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$$

We will discuss more about the derivation of Binomial Theorem using combinatorics.

Example 1.4.1

Prove the following identities:

1. (Recursive Formula) $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$.
2. (Hockey Stick Identity) $\binom{n}{n} + \binom{n+1}{n} + \binom{n+2}{n} + \cdots + \binom{r}{n} = \binom{r+1}{n+1}$
3. (Vandermonde's Identity)

$$\binom{m}{0}\binom{n}{r} + \binom{m}{1}\binom{n}{r-1} + \binom{m}{2}\binom{n}{r-2} + \cdots + \binom{m}{r}\binom{n}{0}$$

Solution.

1.

$$\begin{aligned}\binom{n-1}{k-1} + \binom{n-1}{k} &= \frac{(n-1)!}{(k-1)!((n-1)-(k-1))!} + \frac{(n-1)!}{k!(n-1-k)!} \\ &= \frac{(n-1)!}{(k-1)!(n-k)!} + \frac{(n-1)!}{k!(n-k-1)!}\end{aligned}$$

Gentle reminder: always factor out the common factors,

$$\begin{aligned}\frac{(n-1)!}{(k-1)!(n-k)!} + \frac{(n-1)!}{k!(n-k-1)!} &= \frac{(n-1)!}{(k-1)!(n-k-1)!}\left(\frac{1}{n-k} + \frac{1}{k}\right) \\ &= \frac{(n-1)!}{(k-1)!(n-k-1)!} \cdot \frac{n}{k(n-k)} \\ &= \frac{n!}{k!(n-k)!} \\ &= \binom{n}{k}\end{aligned}$$

2.

$$\begin{aligned}&\binom{n}{n} + \binom{n+1}{n} + \binom{n+2}{n} + \cdots + \binom{r}{n} \\ =& \binom{n+1}{n+1} + \binom{n+1}{n} + \binom{n+2}{n} + \cdots + \binom{r}{n} \\ =& \binom{n+1}{n+1} + \binom{n+1}{n} + \binom{n+2}{n} + \cdots + \binom{r}{n} \\ =& \binom{n+2}{n+1} + \binom{n+2}{n} + \cdots + \binom{r}{n} \\ =& \binom{n+2}{n} + \binom{n+2}{n} + \binom{n+3}{n} + \cdots + \binom{r}{n} \\ =& \binom{n+3}{n+1} + \binom{n+3}{n} + \cdots + \binom{r}{n}\end{aligned}$$

Repeat this process until we get $\binom{r+1}{n+1}$.

3. Consider the polynomials $(1+x)^m$ and $(1+x)^n$

$$(1+x)^m(1+x)^n = \left(1+\binom{m}{1}x+\cdots+\binom{m}{m}x^m\right)\left(1+\binom{n}{1}x+\cdots+\binom{n}{n}x^n\right)$$
$$(1+x)^{m+n} = \left(1+\binom{m+n}{1}x+\binom{m+n}{1}x^2\cdots+\binom{m+n}{m+n}x^{m+n}\right)$$

By comparing the coefficients of x^r, we get

$$\binom{m}{0}\binom{n}{r}+\binom{m}{1}\binom{n}{r-1}+\binom{m}{2}\binom{n}{r-2}+\cdots+\binom{m}{r}\binom{n}{0}$$

as desired.

□

Example 1.4.2 (HLG 2014)

Find the value of $\binom{999}{0}-\binom{999}{2}+\binom{999}{4}-\binom{999}{6}+\cdots+\binom{999}{996}-\binom{999}{998}$.

Solution. Consider the polynomial

$$(1+x)^{999} = \binom{999}{0}+\binom{999}{1}x+\binom{999}{2}x^2+\cdots+\binom{999}{998}x^{998}+\binom{999}{999}x^{999}.$$

Let $S_k = \sum\limits_{\substack{n\equiv k \pmod 4\\ 0\le n\le 999}}\binom{999}{n}$ for simplicity, then by letting $x=i$ and $x=-i$ gives

$$\begin{cases} S_0+iS_1-S_2-iS_3 = (1+i)^{999}\\ S_0-iS_1-S_2+iS_3 = (1-i)^{999}\end{cases}$$

Summing up gives $2(S_0-S_2)=(1+i)^{999}+(1-i)^{999}=2^{500}$, so we have $S_0-S_2=2^{499}$. □

As we can see from the example above, to compute the sum

$$\sum_{\substack{m\equiv k \pmod p\\ 0\le m\le n}}\binom{n}{m}$$

we utilize the polynomial $(1+x)^n$ and substitute x by the p'th roots of unity.

Example 1.4.3

Compute $\binom{999}{0}+\binom{999}{3}+\binom{999}{6}+\cdots+\binom{999}{999}$.

Solution. Assume S_0, S_1, S_2 the similar way, then consider the cube root $\omega \neq 1$ of 1, so we have

$$\begin{cases} S_0 + S_1 + S_2 = (1+1)^{999} \\ S_0 + \omega S_1 + \omega^2 S_2 = (1+\omega)^{999} \\ S_0 + \omega^2 S_1 + \omega S_3 = (1+\omega^2)^{999} \end{cases}$$

By using the fact that $1+\omega+\omega^2 = 0$ and $\omega^3 = 1$, summing up the three equations yields

$$\begin{aligned} 3S_0 + (1+\omega^2_\omega)(S_1+S_2) &= 2^{999} + (1+\omega)^{999} + (1+\omega^2)^{999} \\ 3S_0 &= 2^{999} + (-\omega^2)^{999} + (-\omega)^{999} \\ 3S_0 &= 2^{999} - \omega^{1998} - \omega^{999} \\ 3S_0 &= 2^{999} - 2 \\ S_0 &= \frac{2^{999}-2}{3} \end{aligned}$$

$\square$

Practice Problems

Problem 1.4.4. Prove the following identities:

1. $\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} = 2^n$
2. $\binom{n}{0}^2 + \binom{n}{1}^2 + \binom{n}{2}^2 + \cdots + \binom{n}{n}^2 = \binom{2n}{n}$
3. $0\binom{n}{0} + 1\binom{n}{1} + 2\binom{n}{2} + \cdots + n\binom{n}{n} = n \cdot 2^{n-1}$
4. (HLG 2001) $\binom{n}{0} + 3\binom{n}{1} + 5\binom{n}{2} + \cdots + (2n+1)\binom{n}{n} = (n+1) \cdot 2^n$
5. $0^2\binom{n}{0} + 1^2\binom{n}{1} + 2^2\binom{n}{2} + \cdots + n^2\binom{n}{n} = n(n+1) \cdot 2^{n-2}$
6. $0\binom{n}{0}^2 + 1\binom{n}{1}^2 + 2\binom{n}{2}^2 + \cdots + n\binom{n}{n}^2 = \frac{n}{2}\binom{2n}{n}$

Problem 1.4.5. Prove that

$$\sum_{k=1}^{1012} \binom{2025}{2k-1} = \sum_{k=0}^{1012} \binom{2025}{2k} = \sum_{k=0}^{1012} \binom{2025}{k} + \frac{1}{2}\binom{2025}{1013} = \sum_{k=1014}^{2025} \binom{2025}{k} + \frac{1}{2}\binom{2025}{1013}$$

Problem 1.4.6. Prove that

$$\sum_{k=q}^{n} \binom{n}{k}\binom{k}{q} = 2^{n-q}\binom{n}{q}$$

Problem 1.4.7 (CJR 2020)**.** Find the value of n such that

$$\sum_{k=1}^{n} \binom{2n+1}{k} = 2^{586} - 1$$

Problem 1.4.8 (AIME 2017). For nonnegative integers a and b with $a+b \leq 6$, let $T(a,b) = \binom{6}{a}\binom{6}{b}\binom{6}{a+b}$. Let S denote the sum of all $T(a,b)$, where a and b are nonnegative integers with $a+b \leq 6$. Find the remainder when S is divided by 1000.

Problem 1.4.9 (SGP Open 2010 Round 1). Let

$$A = \left(\binom{2010}{0} - \binom{2010}{-1}\right)^2 + \left(\binom{2010}{1} - \binom{2010}{0}\right)^2 + \left(\binom{2010}{2} - \binom{2010}{1}\right)^2 + \cdots + \left(\binom{2010}{1005} - \binom{2010}{1004}\right)^2$$

Determine the minimum integer s such that

$$sA \geq \binom{4020}{2010}.$$

(Note: For $r < 0$, define $\binom{n}{r} = 0$.)

Problem 1.4.10 (SGP Open 2012 Round 1). Evaluate $\frac{-1}{2^{2011}}\sum_{k=0}^{1006}(-1)^k 3^k \binom{2012}{2k}$

Problem 1.4.11. By using the Hockey Stick Identity, prove that

$$\sum_{k=1}^{n} k(k+1)(k+2)\dots(k+p) = \frac{n(n+1)(n+2)\dots(n+p)(n+p+1)}{p+2}.$$

Problem 1.4.12. Let n, p be positive integers such that $n \gg p$ and p is a prime, assume that $\omega \neq 1$ be a root of the equation $x^p = 1$, prove that

$$\sum_{\substack{p|k \\ 0\leq k\leq n}} \binom{n}{k} = \frac{1}{p}\sum_{k=0}^{p-1}(1+\omega^k)^p.$$

Problem 1.4.13 (IDN 2014). Suppose that k, m, n are positive integers with $k \leq n$. Prove that:

$$\sum_{r=0}^{m} \frac{k\binom{m}{r}\binom{n}{k}}{(r+k)\binom{m+n}{r+k}} = 1$$

1.5 Others

Strategy: Always identify the general term of the series before summoning the "sigma magic".

Example 1.5.1 (Power-Sum - Geometric Series, HLG 2017)

Find $\frac{1}{2} + 4 \times \frac{1}{4} + 9 \times \frac{1}{8} + 16 \times \frac{1}{16} + 25 \times \frac{1}{32} + \cdots + \frac{n^2}{2^n} + \dots$

Solution. Always assume $S = \frac{1}{2} + 4 \times \frac{1}{4} + 9 \times \frac{1}{8} + 16 \times \frac{1}{16} + 25 \times \frac{1}{32} + \cdots + \frac{n^2}{2^n} + \ldots$, then we see that

$$\begin{array}{rcllllll}
S &=& \frac{1}{2} &+4 \times \frac{1}{4} &+9 \times \frac{1}{8} &+16 \times \frac{1}{16} &+25 \times \frac{1}{32} &+ \ldots \\
\frac{1}{2}S &=& & \frac{1}{4} &+4 \times \frac{1}{8} &+ 9 \times \frac{1}{16} &+16 \times \frac{1}{32} &+ \ldots \\
\hline
\frac{1}{2}S &=& \frac{1}{2} &+3 \times \frac{1}{4} &+5 \times \frac{1}{8} &+ 7 \times \frac{1}{16} &+ 9 \times \frac{1}{32} &+ \ldots \\
\frac{1}{4}S &=& & \frac{1}{4} &+3 \times \frac{1}{8} &+ 5 \times \frac{1}{16} &+ 7 \times \frac{1}{32} &+ \ldots \\
\hline
\frac{1}{4}S &=& \frac{1}{2} &+2 \times \frac{1}{4} &+2 \times \frac{1}{8} &+ 2 \times \frac{1}{16} &+ 2 \times \frac{1}{32} &+ \ldots
\end{array}$$

We end up with a geometric series, so $\frac{1}{4}S = \frac{1}{2} + 2\left(\frac{\frac{1}{4}}{1 - \frac{1}{2}}\right)$ so $S = 6$. □

Example 1.5.2 (Gaussian Pairing - HLG 2013)

Given that $f(x) = \frac{9^x}{9^x + 27}$. Find the value of $S = f\left(\frac{1}{9}\right) + f\left(\frac{2}{9}\right) + f\left(\frac{3}{9}\right) + \cdots + f\left(\frac{25}{9}\right) + f\left(\frac{26}{9}\right)$.

Solution. We see that $f(x) + f(3 - x) = 1$, so we have

$$\begin{aligned}
&f\left(\frac{1}{9}\right) + f\left(\frac{2}{9}\right) + f\left(\frac{3}{9}\right) + \cdots + f\left(\frac{25}{9}\right) + f\left(\frac{26}{9}\right) \\
&= \left(f\left(\frac{1}{9}\right) + f\left(\frac{26}{9}\right)\right) + \left(f\left(\frac{2}{9}\right) + f\left(\frac{25}{9}\right)\right) + \cdots + \left(f\left(\frac{13}{9}\right) + f\left(\frac{14}{9}\right)\right) \\
&= 13
\end{aligned}$$

□

Example 1.5.3 (HLG 2004)

Compute

$$\left\lfloor \frac{1}{4} \right\rfloor + \left\lfloor \frac{1}{4} + \frac{1}{100} \right\rfloor + \left\lfloor \frac{1}{4} + \frac{2}{100} \right\rfloor + \cdots + \left\lfloor \frac{1}{4} + \frac{99}{100} \right\rfloor = 1$$

Solution. Since we have $\left\lfloor \frac{1}{4} + \frac{k}{100} \right\rfloor = 0$ for $k = 0, 1, 2, \ldots, 74$, and $\left\lfloor \frac{1}{4} + \frac{m}{100} \right\rfloor = 0$ for

$m = 75, 76, 77, \ldots, 99$. So

$$\begin{aligned}
&\left\lfloor \frac{1}{4} \right\rfloor + \left\lfloor \frac{1}{4} + \frac{1}{100} \right\rfloor + \left\lfloor \frac{1}{4} + \frac{2}{100} \right\rfloor + \cdots + \left\lfloor \frac{1}{4} + \frac{99}{100} \right\rfloor \\
= &\left\lfloor \frac{1}{4} + \frac{75}{100} \right\rfloor + \left\lfloor \frac{1}{4} + \frac{76}{100} \right\rfloor + \cdots + \left\lfloor \frac{1}{4} + \frac{99}{100} \right\rfloor \\
= &\ 25.
\end{aligned}$$

□

Practice Problems

Problem 1.5.4. Compute the following series:

1. $$\sum_{k=1}^{n} \frac{k}{2^k}$$

2. $$\sum_{k=1}^{9999} \frac{3\sqrt{\log(100000-k)}}{\sqrt{\log(100000-k)} + \sqrt{\log(k+90000)}}$$

3. $$\sum_{k=1}^{100} \left(\left\lfloor \sqrt{k} \right\rfloor + \left\lceil \sqrt{k} \right\rceil \right)$$

4. $$\sum_{k=1}^{1000} k \left\lfloor \log_2 k \right\rfloor$$

Problem 1.5.5 (CJR 2018)**.** Find $\left\lfloor \frac{3^2}{1} \right\rfloor + \left\lfloor \frac{4^2}{2} \right\rfloor + \left\lfloor \frac{5^2}{3} \right\rfloor + \left\lfloor \frac{6^2}{4} \right\rfloor + \cdots + \left\lfloor \frac{42^2}{40} \right\rfloor$.

Problem 1.5.6 (CJR 2022)**.** Given that r is a rational number such that $0 < r < 1$ and

$$\sum_{n=1}^{\infty} n^2 r^n = 180,$$

find the value of $180r$.

Problem 1.5.7 (HLG 2002)**.** An array of $n \times n$ numbers is shown below, for $n \geq 4$:

$$\begin{array}{cccc}
a_{11} & a_{12} & \ldots & a_{1n} \\
a_{21} & a_{22} & \ldots & a_{2n} \\
\ldots & \ldots & \ldots & \ldots \\
a_{n1} & a_{n2} & \ldots & a_{nn}
\end{array}$$

It is known that each row is an arithmetic sequence, each column is a geometric sequence and their common ratio are equal. Given that $a_{24} = 1$, $a_{42} = \frac{1}{8}$, $a_{43} = \frac{3}{16}$, find $a_{11} + a_{22} + a_{33} + a_{44} + \cdots + a_{nn}$ in terms of n.

Problem 1.5.8 (HLG 2015). Find the value of $\sum_{n=1}^{128} \lfloor \log_2 n \rfloor$.

Problem 1.5.9 (HLG 2016). Find the sum of the infinite series $1 - \frac{2}{3} + \frac{3}{3^2} - \frac{4}{3^3} + \frac{5}{3^4} - \frac{6}{3^5} + \ldots$.

Problem 1.5.10 (HLG 2016). If $S = 1 + \frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{4^3} + \cdots + \frac{1}{n^3} + \ldots$, find the value of $1 + \frac{1}{3^3} + \frac{1}{5^3} + \cdots + \frac{1}{(2n-1)^3} + \ldots$ in terms of S.

> **Remark 1.5.1.** The value $S = \zeta(3)$ is the Apery Constant and $\eta(3) = 1 + \frac{1}{3^3} + \frac{1}{5^3} + \cdots + \frac{1}{(2n-1)^3} + \ldots$, where $\zeta(z)$ is the Riemann Zeta Function and $\eta(z)$ the Eta Function.

Problem 1.5.11 (OMK 2017). Given a positive integer n. Consider all subsets of $\{1, 2, 3, \ldots, n\}$ except the empty set. For each subset, consider a fraction $\frac{1}{d}$, where d is the product of all elements in the subset. Let S_n be the sum of such fractions taken ovre all subsets.
Example: For $n = 3$, the nonempty subsets of $\{2, 2, 3\}$ are $\{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}$. Therefore,

$$S_3 = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{1\cdot 2} + \frac{1}{1\cdot 3} + \frac{1}{2\cdot 3} + \frac{1}{1\cdot 2\cdot 3} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{2} + \frac{1}{3} + \frac{1}{6} + \frac{1}{6} = 3.$$

Prove that $S_n = n$ for all positive integers n.

Problem 1.5.12 (AIME 2013). For $\pi \le \theta < 2\pi$, let

$$P = \frac{1}{2}\cos\theta - \frac{1}{4}\sin 2\theta - \frac{1}{8}\cos 3\theta + \frac{1}{16}\sin 4\theta + \frac{1}{32}\cos 5\theta - \frac{1}{64}\sin 6\theta - \frac{1}{128}\cos 7\theta + \ldots$$

and

$$Q = 1 - \frac{1}{2}\sin\theta - \frac{1}{4}\cos 2\theta + \frac{1}{8}\sin 3\theta + \frac{1}{16}\cos 4\theta - \frac{1}{32}\sin 5\theta - \frac{1}{64}\cos 6\theta + \frac{1}{128}\sin 7\theta + \ldots$$

so that $\frac{P}{Q} = \frac{2\sqrt{2}}{7}$. Then $\sin\theta = -\frac{m}{n}$ where m and n are relatively prime positive integers. Find $m + n$.

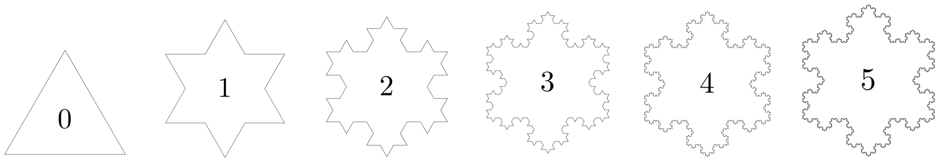
Problem 1.5.13 (AIME 2017). For every $m \ge 2$, let $Q(m)$ be the least positive integer with the following property: For every $n \ge Q(m)$, there is always a perfect cube k^3 in the range $n < k^3 \le m \cdot n$. Find the remainder when

$$\sum_{m=2}^{2017} Q(m)$$

is divided by 1000.

Problem 1.5.14 (APMO 2000). Compute the sum $S = \sum_{i=0}^{101} \frac{x_i^3}{1 - 3x_i + 3x_i^2}$ for $x_i = \frac{i}{101}$.

Problem 1.5.15 (SGP Senior 2013 Round 1). Given an equilateral triangle of side 10, divide each side into three equal parts, construct an equilateral triangle on the middle part, and then delete the middle part. Repeat this step for each side of the resulting polygon. Find S^2, where S is the area of region obtained by repeating this procedure infinitely many times.



Problem 1.5.16 (Putnam 2001). For any positive integer n, let $\langle n \rangle$ denote the closest integer to $\sqrt{n}$. Evaluate

$$\sum_{n=1}^{\infty} \frac{2^{\langle n \rangle} + 2^{-\langle n \rangle}}{2^n}.$$

Problem 1.5.17 (Putnam 2016). Evaluate

$$\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} \sum_{n=0}^{\infty} \frac{1}{k2^n + 1}.$$

1.6 Riemann Sum

Definition 1.6.1 (Riemann Sum)

$$\int_a^b f(x)\mathrm{d}x = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x_i.$$

Corollary 1.6.2

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} f\left(\frac{i}{n}\right) = \int_0^1 f(x)\mathrm{d}x.$$

Example 1.6.3

$$\lim_{n\to\infty} \frac{\cos\left(\frac{1}{n}\right) + \cos\left(\frac{2}{n}\right) + \cdots + \cos\left(\frac{n}{n}\right)}{n}.$$

Solution. The series is equivalent to

$$\int_0^1 \cos x\mathrm{d}x = \sin 1.$$

□

Practice Problems

Problem 1.6.1. Calculate the series:

1. $$\lim_{n\to\infty}\left(\frac{n}{1^2+n^2}+\frac{n}{2^2+n^2}+\frac{n}{3^2+n^2}+\cdots+\frac{n}{n^2+n^2}\right)$$

2. $$\lim_{n\to\infty}\left(\frac{n+1}{1^2+n^2}+\frac{n+1}{2^2+n^2}+\frac{n+1}{3^2+n^2}+\cdots+\frac{n+1}{n^2+n^2}\right)$$

3. $$\lim_{n\to\infty}\left(\frac{4n+1}{1^2+n^2}+\frac{4n+1}{2^2+n^2}+\frac{4n+1}{3^2+n^2}+\cdots+\frac{4n+1}{n^2+n^2}\right)$$

4. $$\lim_{n\to\infty}\left(\frac{e^{1/n}+e^{2/n}+e^{3/n}+\cdots+e^{n/n}}{n}\right)$$

5. $$\lim_{n\to\infty}\left(\sqrt{\frac{1}{n^2}-\frac{1^2}{n^4}}+\sqrt{\frac{1}{n^2}-\frac{2^2}{n^4}}+\sqrt{\frac{1}{n^2}-\frac{3^2}{n^4}}+\cdots+\sqrt{\frac{1}{n^2}-\frac{n^2}{n^4}}\right)$$

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