厦门大学马来西亚分校

陈景润杯中学数学竞赛

MOCK CHEN JINGRUN'S CUP SECONDARY SCHOOL MATHEMATICS COMPETITION 2024

SENIOR CATEGORY

Date: 12/06/2024

Duration: 2 hours

Instructions and Information

考生须知

1. Do not open this question booklet until you are instructed to do so.

   未获监考老师许可之前不可翻开此题本。

2. Diagrams may NOT be drawn to scale. They are intended as aids only.

   题目所提供之图形只是示意图，不一定精准。

3. There are 30 questions in this paper. The marks carried by each question is indicated on the question.

   本试卷共含30道题，各题的分数均注明在题目上。

4. Write down your answers in the answer sheet provided. Each answer is an integer. Do not fill in units or non-numeric characters.

   在答题纸上填入答案。每题的答案为一整数，请勿填入单位或数字以外的字元。

5. Do not communicate with anyone except the invigilators during the competition. Otherwise you will be disqualified and penalized.

   在比赛期间，请勿与监考老师以外的任何人互通信息，否则将会被取消参赛资格，且被惩罚。

Question S-01 [5 points]

A physics equation is given by

$$\frac{1}{2}m(2v)^2 + mg(3h) = \frac{1}{2}m(3v)^2 + mg(2h)$$

Then $v = c\sqrt{gh}$ for some constant c. Find $140c^2$.

一个物理的公式为

$$\frac{1}{2}m(2v)^2 + mg(3h) = \frac{1}{2}m(3v)^2 + mg(2h)$$

则对于常数c, $v = c\sqrt{gh}$。求$140c^2$的值。

Question S-02 [5 points]

It is known that a point $(-4, k)$ is equidistant to the three lines $3x + 4y - 5 = 0$, $y = 2$ and $9x + 40y = h$. Find the maximum value of h.

已知点$(-4, k)$与三线$3x + 4y - 5 = 0$、$y = 2$及$9x + 40y = h$的距离相同。求h的最大值。

Question S-03 [5 points]

Find the sum of all integers n satisfying

$$\frac{(n^2 - 5n + 7)(n^2 - 5n - 9)}{(n - 5)(n - 9)} < 0$$

求所有满足

$$\frac{(n^2 - 5n + 7)(n^2 - 5n - 9)}{(n - 5)(n - 9)} < 0$$

的整数之和。

Question S-04 [5 points]

A sphere has radius R and volume V whereas another hemisphere has radius $2R$ and surface area S. Let $\dfrac{V}{RS} = \dfrac{m}{n}$, where m, n are coprime integers. Find $m + n$.

一个球体半径为R，其体积为V。另一个半球体的半径为$2R$，表面积为S。若$\dfrac{V}{RS} = \dfrac{m}{n}$，且$m, n$为互质的整数。求$m + n$。

Question S-05 [5 points]

How many ways are there to rearrange the letters in the word "BABYLONIAN" such that the vowels (A, E, I, O, U) are adjacent?

将"BABYLONIAN" 中的字母重新排列且母音（A、E、I、O、U）相邻，共有多少个方法?

Question S-06 [5 points]

Let $\triangle ABC$ be a triangle with $A = 60°$, $AB = 120$, $AC = 192$. Let the circumradius and inradius be R and r respectively. Find $\sqrt{3}(R - r)$.

已知一个三角形$\triangle ABC$，$A = 60°$、$AB = 120$、$AC = 192$。令其外接圆半径与内接圆半径为R及r，求$\sqrt{3}(R - r)$。

Question S-07 [5 points]

If $\sin x + \cos x = \dfrac{1}{3}$, then $\sin x \cos x = -\dfrac{m}{n}$ where m, n are coprime positive integers. Find $m + n$.

若$\sin x + \cos x = \dfrac{1}{3}$，则$\sin x \cos x = -\dfrac{m}{n}$，其中$m$、$n$为互质正整数。求$m + n$。

Question S-08 [5 points]

Find the maximum value of $\dfrac{2664}{81(\sin 3x+1)+360(\cos 3x+1)}$.

求$\dfrac{2664}{81(\sin 3x+1)+360(\cos 3x+1)}$的最大值。

Question S-09 [5 points]

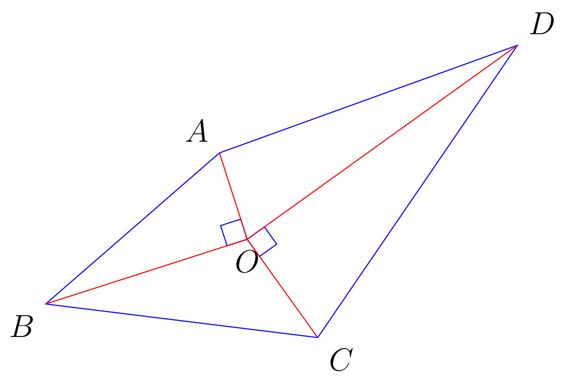
Find the least positive integer k such that $\sqrt[3]{486k}$ is an integer.

求最小的正整数使得$\sqrt[3]{486k}$是一个整数。

Question S-10 [5 points]

In the figure below, $\triangle AOB$ and $\triangle COD$ are right triangles with $\dfrac{AO}{BO}=\dfrac{3}{7}$, $\dfrac{DO}{CO}=\dfrac{11}{4}$. If the area of $\triangle AOD$ is 561, find the area of $\triangle BOC$.

下图中，$\triangle AOB$与$\triangle COD$是直角三角形且$\dfrac{AO}{BO}=\dfrac{3}{7}$，$\dfrac{DO}{CO}=\dfrac{11}{4}$。若$\triangle AOD$的面积为561，求$\triangle BOC$的面积。



Question S-11 [5 points]

Let $\log_a x=9$. If $\dfrac{\log_{x^x} a^a+\log_{x^a} a^x}{\log_{a^x} x^a+\log_{a^a} x^x}=\dfrac{1}{n}$, find the value of $5n$.

设$\log_a x=9$。若$\dfrac{\log_{x^x} a^a+\log_{x^a} a^x}{\log_{a^x} x^a+\log_{a^a} x^x}=\dfrac{1}{n}$，求$5n$的值。

Question S-12 [5 points]

Let

$$2+\cfrac{1+\sqrt{3}}{2+\cfrac{1+\sqrt{3}}{2+\cfrac{1+\sqrt{3}}{2+\cfrac{1+\sqrt{3}}{\ddots}}}}=a+\sqrt{b}+\sqrt{c}$$

where $a, 2b, 2c$ are integers and $b < c$. Find the value of $20b + 24c$.

若

$$2+\cfrac{1+\sqrt{3}}{2+\cfrac{1+\sqrt{3}}{2+\cfrac{1+\sqrt{3}}{2+\cfrac{1+\sqrt{3}}{\ddots}}}}=a+\sqrt{b}+\sqrt{c}$$

其中a、$2b$、$2c$为整数且$b < c$。求$20b + 24c$的值。

Question S-13 [5 points]

Find the coefficient of ab^2c^3d in the expansion of $(a + b + c)^3(b + c + d)^4$.

求$(a + b + c)^3(b + c + d)^4$中$ab^2c^3d$的系数。

Question S-14 [5 points]

Let

$$(x + 1)(x + 2)(x + 3) = Ax(x + 1)(x + 2) + Bx(x + 1) + Cx + D$$

Find the value of $24A + 6B + 2C + D$.

已知

$$(x + 1)(x + 2)(x + 3) = Ax(x + 1)(x + 2) + Bx(x + 1) + Cx + D$$

求$24A + 6B + 2C + D$的值。

Question S-15 [5 points]

Given that the equation $123x^3 - 321x^2 + cx - 666 = 0$ has three roots α, β, γ satisfying

$$\alpha\beta\gamma + \alpha + \beta + \gamma = \alpha\beta + \beta\gamma + \gamma\alpha + 1$$

Find the value of c.

已知方程式$123x^3 - 321x^2 + cx - 666 = 0$的三根$\alpha$、$\beta$、$\gamma$满足

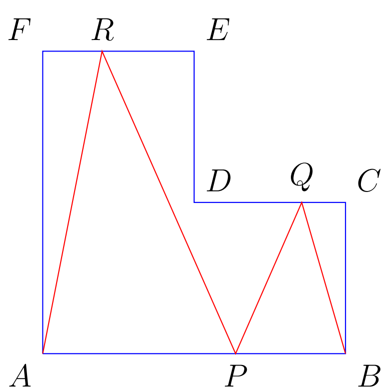
$$\alpha\beta\gamma + \alpha + \beta + \gamma = \alpha\beta + \beta\gamma + \gamma\alpha + 1$$

求c的值。

Question S-16 [5 points]

The figure below shows an L-shaped polygon $ABCDEF$ with all angles equal right angles. Points P, Q, R are on AB, CD, EF respectively. Let $AB = 2BC = 2CD = 2DE = 2EF = AF = \sqrt{360}$, find the minimum value of $AR + RP + PQ + QB$.

下图为一个L形的多边形$ABCDEF$，其中全部内角及外角为直角。点P、Q、R在AB、CD、EF上。若$AB = 2BC = 2CD = 2DE = 2EF = AF = \sqrt{360}$，求$AR + RP + PQ + QB$的最小值。



Question S-17 [5 points]

Given

$$9=\frac{a_1}{1}=\frac{a_1+a_2}{2}=\frac{a_1+a_2+a_3}{3}=\cdots=\frac{a_1+a_2+\cdots+a_n}{n}$$

$$a_1+2a_2+3a_3+\cdots+na_n=14877$$

Find the value of $n+224$.

已知

$$9=\frac{a_1}{1}=\frac{a_1+a_2}{2}=\frac{a_1+a_2+a_3}{3}=\cdots=\frac{a_1+a_2+\cdots+a_n}{n}$$

$$a_1+2a_2+3a_3+\cdots+na_n=14877$$

求$n+224$的值。

Question S-18 [5 points]

If the circle $x^2-2x+y^2-6y+k=0$ is rotated 90 degrees clockwise about the origin, then the circle is tangent to $y=\frac{3}{4}x$. Find the value of $120\sqrt{k}$.

若圆$x^2-2x+y^2-6y+k=0$绕原点顺时针旋转90度后，此圆与$y=\frac{3}{4}x$相切。求$120\sqrt{k}$的值。

Question S-19 [5 points]

Let P be a regular pentagon with 3 points evenly spaced on each side (so that there are 10 points in total). Let there be N ways to color the points using one of 10 colors such that each color is used exactly once, find the last three digits of N. (Two colorings are considered identical if they differ up to rotation.)

设P为一个正五边形，每个边上有3个距离均相等的点（所以总共有10个点）。若将所有点用10个颜色涂色，每个颜色恰好用一次，共有N个方法，求N的最后三位数。（一个方法若经过旋转后与另一个方法相等，则视为同一个方法。）

Question S-20 [5 points]

Let a, b be real number such that

$$\lim_{x\to 1}\frac{\sqrt{a+bx}-\sqrt{ax+b}}{a+b-x}=9$$

Find the value of $|4ab|$.

已知a、b为实数且

$$\lim_{x\to 1}\frac{\sqrt{a+bx}-\sqrt{ax+b}}{a+b-x}=9$$

求$|4ab|$的值。

Question S-21 [6 points]

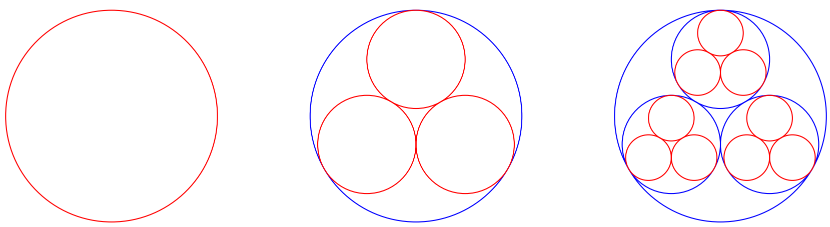
Let $S=-\sum_{n=1}^{\infty}\frac{(-2)^n(n+3)}{(n+1)!}$, find the value of $111S$.

已知$S=-\sum_{n=1}^{\infty}\frac{(-2)^n(n+3)}{(n+1)!}$，求$111S$的值。

Question S-22 [6 points]

The figures below show externally tangent circles. In each step, we pick a red circle and draw three externally tangent circles in it. The circle then becomes blue. This process continues indefinitely. Assume that the largest circle has radius 10, and the sum of areas of all the circles equals $C\pi$. Find the value of $\lfloor C\rfloor$.

下图所示为互切的圆形。在一步内，我们选择一个红色的圆并在其中画三个互相外切的圆。这个红色的圆变成蓝色。我们重复这个步骤无数次。假设最大的圆半径为10，所有圆的面积之和为$C\pi$，求$\lfloor C\rfloor$的值。



Question S-23 [6 points]

Let the graph of the equation $y=\sqrt{x}$ be ℓ and the image of ℓ rotated about $\left(\frac{1}{2},\frac{1}{2}\right)$ by 180° be ℓ'. Assume that the area of the region enclosed by ℓ and ℓ' be A, find the value of $345A$.

设ℓ为方程式$y=\sqrt{x}$的图像，ℓ'为ℓ绕$\left(\frac{1}{2},\frac{1}{2}\right)$旋转$180^\circ$后得到的图像。若$\ell$与$\ell'$所围成的区域面积为$A$，求$345A$的值。

Question S-24 [6 points]

Given that a,b,c,d are positive real numbers and $a^2+b^2+c^2+d^2=750$. Find the maximum value of $a+2b+3c+4d$.

已知a、b、c、d为四个正实数且$a^2+b^2+c^2+d^2=750$。求$a+2b+3c+4d$的最大值。

Question S-25 [6 points]

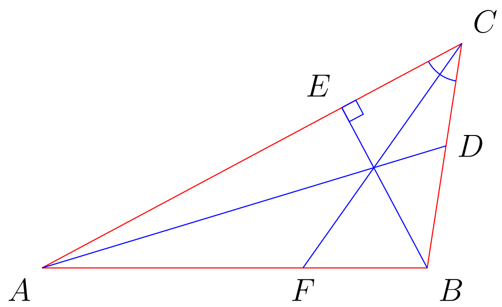
6 balls of 5 different colors are distributed to 2 students, two of the balls with the same color are identical. If it is possible for a student not to receive any ball, how many ways are there to do so?

将5种不同颜色的6个球分配给2名学生，其中的2个相同颜色的球完全相同。如果一名学生可以不收到任何球，有多少种分配方式?

Question S-26 [8 points]

The figure below shows a triangle $\triangle ABC$ with $AB = 391$, $BC = 230$. Points D, E, F are on AB, BC, CA respectively such that $BE \perp AC$, CF bisects $\angle ACB$ and AD, BE, CF passes through a common point. If $CE = 138$, find BD.

下图所示为一个三角形$\triangle ABC$，$AB = 391$，$BC = 230$。D、E、F在AB、BC、CA上使得$BE \perp AC$，CF是$\angle ACB$的角平分线，以及AD、BE、CF通过同一个点。若$CE = 138$，求BD。



Question S-27 [8 points]

Given that $a_0 = 1$ and $a_1 = 3$ and for all $n \geq 1$,

$$a_{n+1} = \frac{b_n}{a_n} + a_{n-1}, \quad b_n + b_{n+1} = 2n^2$$

Find the value of $a_{22}a_{23}$.

已知$a_0 = 1$，$a_1 = 3$，且对于所有$n \geq 1$，

$$a_{n+1} = \frac{b_n}{a_n} + a_{n-1}, \quad b_n + b_{n+1} = 2n^2$$

求$a_{22}a_{23}$的值。

Question S-28 [8 points]

A 6-digit number is said to be *special* if every digit appears an equal number of times. For example, 215125, 633366, 123456 are special but 183331, 987655 are not. Let there be N special numbers. Find the last three digits of N.

我们说一个6位数特别如果每个数字出现的次数一样。例如，215125、633366、123456是特别的，可是183331、987655不是。假设有N个特别的数字，求N的最后三位数。

Question S-29 [8 points]

Determine the number of positive integer solutions (x, y, z) to the inequality

$$2024 < 446x^2 + 1553y + 1999z \leq 20240$$

求不等式

$$2024 < 446x^2 + 1553y + 1999z \leq 20240$$

的正整数解(x, y, z)的个数。

Question S-30 [8 points]

Let

$$S = 123^{456^{789}} + 234^{567^{891}} + 345^{678^{912}} + \cdots + 912^{345^{678}}$$

where the numbers from 1 to 9 cycle once. Find the remainder when S is divided by 165.

设

$$S = 123^{456^{789}} + 234^{567^{891}} + 345^{678^{912}} + \cdots + 912^{345^{678}}$$

其中1至9各个数字循环一次。求S除以165的余数。