The regularity of difference divisors

Baiqing Zhu[1] (ORCID)

Abstract

For a prime number $p > 2$ and a finite extension $F/\mathbb{Q}_p$, we explain the construction of the difference divisors on the unitary Rapoport–Zink spaces of hyperspecial level over $\mathcal{O}_{\breve{F}}$, and the GSpin Rapoport–Zink spaces of hyperspecial level over $\breve{\mathbb{Z}}_p$ associated to a minuscule cocharacter μ and a basic element b. We prove the regularity of the difference divisors, find the formally smooth locus of both the special cycles and the difference divisors, by a purely deformation-theoretic approach.

1 Introduction

1.1 Background

Let $n \geqslant 2$ be a positive integer. Let $\mathcal{N}_{1,n-1}$ be the unitary Rapoport–Zink space of signature $(1, n-1)$ with hyperspecial level. Kudla-Rapoport special divisors $\mathcal{Z}(x)$ on $\mathcal{N}_{1,n-1}$ have been defined and extensively studied in the works of Kudla and Rapoport [10]. Then Terstiege introduced the difference divisors on the formal scheme $\mathcal{N}_{1,n-1}$ in [23], he also proved the regularity of difference divisors in [24]. A key step in the proof is a previous result in his joint work with Rapoport and Zhang [19, Theorem 10.7], whose proof is based on the windows theory developed by Zink in [27] and [28].

For a self-dual quadratic lattice V of rank $m = n + 1$ over $\mathbb{Z}_p$, a Hodge type Rapoport–Zink space $\mathcal{N}(V)$ associated to a minuscule cocharacter μ and a basic element b has been constructed and studied in the works of Kim [8] and Howard–Pappas [7]. When V is one of the self-dual lattices of rank 4, Terstiege defined difference divisors on the formal scheme $\mathcal{N}(V)$ and studied the intersection numbers of them in [22], where he also proved the regularity of these difference divisors. The notion of difference divisors can be defined similarly on $\mathcal{N}(V)$ for general V, but the regularity of these difference divisors was previously not known to us. It's difficult to prove a similar result as [19, Theorem 10.7] since the p-divisible groups parameterized by

✉ Baiqing Zhu
    bz2393@columbia.edu

1   Columbia University Graduate School of Arts and Sciences, New York, USA

the formal scheme $\mathcal{N}(V)$ have dimension 2^n and height 2^{n+1}, so the windows theory becomes much more complicated in this case.

1.2 Main results

Let p be an odd prime. Let $\mathbb{F}$ be the algebraic closure of the finite field $\mathbb{F}_p$. Let $n \geqslant 2$ be an integer. Let $F/\mathbb{Q}_p$ be a finite extension with uniformizer ϖ in its ring of integers $\mathcal{O}_F$. Let $\breve{F}$ be the completion of the maximal unramified extension of F.

In the unitary case, let E/F be an unramified extension. The unitary Rapoport–Zink space $\mathcal{N}_{1,n-1}$ is a formally smooth formal scheme over $\mathrm{Spf}\,\mathcal{O}_{\breve{F}}$ parameterizing hermitian formal $\mathcal{O}_E$-modules of signature $(1, n-1)$ within the supersingular quasi-isogeny class (§ 2.1.1). Denote by $\mathcal{N}^u$ the formal scheme $\mathcal{N}_{1,n-1}$ for simplicity. Let $\overline{\mathbb{Y}}$ be the unique (up to isomorphism) hermitian $\mathcal{O}_E$-module of signature $(0, 1)$ over $\mathbb{F}$. Let $\mathbb{X}$ be the framing hermitian $\mathcal{O}_E$-module of signature $(1, n-1)$ over $\mathbb{F}$. The space of space quasi-homomorphisms $\mathbb{V}^u := textup{Hom}^\circ_{\mathcal{O}_E}(\overline{\mathbb{Y}}, \mathbb{X})$ carries a natural E/F-hermitian form $(\cdot, \cdot)$ (§ 2.2.1).

In the GSpin case, let V be a self-dual quadratic lattice of rank $m = n+1$ over $\mathbb{Z}_p$. Associated to V we have a local unramified Shimura–Hodge datum $(G, b, \mu, C(V))$, where $G = \mathrm{GSpin}(V)$, $b \in G(\breve{\mathbb{Q}}_p)$ is a basic element, $\mu : \mathbb{G}_m \to G$ is a certain cocharacter, and $C(V)$ is the Clifford algebra of V. This local unramified Shimura–Hodge data gives rise to a supersingular p-divisible group $\mathbb{X}_V$, and a GSpin Rapoport–Zink space $\mathrm{RZ}(G, b, \mu, C(V))$ of Hodge type, parametrizing p-divisible groups quasi-isogenous to $\mathbb{X}_V$ with crystalline Tate tensors . It is a formally smooth formal scheme over $\breve{\mathbb{Z}}_p$ (§ 2.1.2). Let $\mathcal{N}(V)$ be a connected component of the formal scheme $\mathrm{RZ}(G, b, \mu, C(V))$. The space of special quasi-homomorphisms $\mathbb{V}^o := (V \otimes_{\mathbb{Z}_p} \breve{\mathbb{Q}}_p)^{\bar{b}\circ\sigma}$ can be identified as a subspace of $\mathrm{End}^\circ(\mathbb{X}_V)$ (§ 2.2.2), where $\bar{b}$ is the image of b under the homomorphism $G(\breve{\mathbb{Q}}_p) \to \mathrm{SO}(V)(\breve{\mathbb{Q}}_p), \sigma \in \mathrm{Gal}(\breve{\mathbb{Q}}_p/\mathbb{Q}_p)$ is the arithmetic Frobenius automorphism on $\breve{\mathbb{Q}}_p$. The $\mathbb{Q}_p$-vector space $\mathbb{V}^o$ carries a natural quadratic form $q_{\mathbb{V}^o}$ induced from $V \otimes_{\mathbb{Z}_p} \breve{\mathbb{Q}}_p$. Denote by $\mathcal{N}^o$ the formal scheme $\mathcal{N}(V)$.

For any subset $L \subset \mathbb{V}^u$ (resp. $L \subset \mathbb{V}^o$). The special cycle $\mathcal{Z}(L)$ is a closed formal subscheme of $\mathcal{N}^u$ (resp. $\mathcal{N}^o$), over which each quasi-homomorphism $x \in L$ deforms to homomorphisms. When $L = \{x\}$ for a nonzero element $x \in \mathbb{V}^u$ (resp. $x \in \mathbb{V}^o$). The special cycle $\mathcal{Z}(x) := \mathcal{Z}(\{x\})$ is a Cartier divisor on the formal scheme $\mathcal{N}^u$ (resp. $\mathcal{N}^o$) (§ 2.3). There is a closed immersion of divisors $\mathcal{Z}(\varpi^{-1}x) \to \mathcal{Z}(x)$ (resp. $\mathcal{Z}(p^{-1}x) \to \mathcal{Z}(x)$), define the difference divisor associated to x to be the following Cartier divisor on $\mathcal{N}^u$ (resp. $\mathcal{N}^o$),

$$\mathcal{D}(x) := \mathcal{Z}(x) - \mathcal{Z}(\varpi^{-1}x) \quad (\text{resp. } \mathcal{D}(x) := \mathcal{Z}(x) - \mathcal{Z}(p^{-1}x)).$$

This definition recovers Terstiege's definition of difference divisors in [24] (unitary case, $F = \mathbb{Q}_p$), and in [22] (GSpin case, V is a specific rank 4 self-dual lattice over $\mathbb{Z}_p$). The main results of this article are stated in the following theorem (Theorem 5.4.1 and Corollary 5.4.3).

Theorem 1.2.1 *Let $x \in \mathbb{V}^u$ (resp. $x \in \mathbb{V}^o$) be a nonzero special quasi-homomorphism such that $(x, x) \in \mathcal{O}_E$ (resp. $q_{\mathbb{V}^o}(x) \in \mathbb{Z}_p$).*

(i) *There is an equality $\mathcal{D}(x)(\mathbb{F}) = \mathcal{Z}(x)(\mathbb{F})$.*

(ii) *The difference divisor $\mathcal{D}(x)$ is a regular formal scheme. It is formally smooth over $\mathcal{O}_{\breve{F}}$ (resp. $\breve{\mathbb{Z}}_p$) of relative dimension $n - 2$ at a point $z \in \mathcal{D}(x)(\mathbb{F})$ if and only if $z \notin \mathcal{Z}(\varpi^{-1}x)(\mathbb{F})$ (resp. $z \notin \mathcal{Z}(p^{-1}x)(\mathbb{F})$) and there exists a lift of z to $z' \in \mathcal{Z}(x)(\mathcal{O}_{\breve{F}}/\varpi^2)$ (resp. $z' \in \mathcal{Z}(x)(\breve{\mathbb{Z}}_p/p^2)$).*

(iii) *Let $z \in \mathcal{Z}(x)(\mathbb{F})$ be a point. Let $\mathcal{D}(x)_{\mathbb{F}} = \mathcal{D}(x) \times_{\mathcal{O}_{\breve{F}}} \mathbb{F}$ and $\mathcal{Z}(x)_{\mathbb{F}} = \mathcal{Z}(x) \times_{\mathcal{O}_{\breve{F}}} \mathbb{F}$ (resp. $\mathcal{D}(x)_{\mathbb{F}} = \mathcal{D}(x) \times_{\breve{\mathbb{Z}}_p} \mathbb{F}$ and $\mathcal{Z}(x)_{\mathbb{F}} = \mathcal{Z}(x) \times_{\breve{\mathbb{Z}}_p} \mathbb{F}$). Then*

$$\mathrm{Tgt}_z(\mathcal{D}(x)_{\mathbb{F}}) = \mathrm{Tgt}_z(\mathcal{Z}(x)_{\mathbb{F}}),$$

where $\mathrm{Tgt}_z(\mathcal{D}(x)_{\mathbb{F}})$ and $\mathrm{Tgt}_z(\mathcal{Z}(x)_{\mathbb{F}})$ are the tangent spaces of the formal schemes $\mathcal{D}(x)_{\mathbb{F}}$ and $\mathcal{Z}(x)_{\mathbb{F}}$ at the point z, respectively.

We give some examples of the equality $\mathcal{D}(x)(\mathbb{F}) = \mathcal{Z}(x)(\mathbb{F})$ in Theorem 1.2.1 (i). See also Example 5.4.2.

Example 1.2.2 (a) The orthogonal case: Let H be the unique rank 4 self-dual quadratic lattice over $\mathbb{Z}_p$ whose discriminant is not a square in $\mathbb{Z}_p^{\times}$. Let $x \in \mathbb{V}^o$ be a non-isotropic vector and $n = v_p(q_{\mathbb{V}^o}(x))$. Let $x' = p^{-[n/2]} \cdot x$. Let $\mathscr{B}$ be the Bruhat-Tits building of $\mathrm{PGL}_2(\mathbb{Q}_{p^2})$. The element x' induces an automorphism of $\mathscr{B}$. Let $\mathscr{B}^{x'}$ be the fixed set. For a vertex $[\Lambda]$ in the Bruhat-Tits building $\mathscr{B}$ of $\mathrm{PGL}_2(\mathbb{Q}_{p^2})$, let $d_{[\Lambda]}$ be the distance of $[\Lambda]$ to $\mathscr{B}^{x'}$.

Let $\mathcal{Z}(x)_{\mathbb{F}} = \mathcal{Z}(x) \times_{\breve{\mathbb{Z}}_p} \mathbb{F}$ and $\mathcal{D}(x)_{\mathbb{F}} = \mathcal{D}(x) \times_{\breve{\mathbb{Z}}_p} \mathbb{F}$. By [22, Theorem 3.6], we have the following equality of effective Cartier divisors on $\mathcal{N}_{\mathbb{F}}^o := \mathcal{N}^o \times_W \mathbb{F}$,

$$\mathcal{Z}(x)_{\mathbb{F}} = \sum_{[\Lambda]:d_{[\Lambda]} \leqslant (n-1)/2} (1 + p + \cdots + p^{(n-1)/2 - d_{[\Lambda]}}) \cdot \mathbb{P}_{[\Lambda]} + p^{n/2} \cdot s,$$

where $\mathbb{P}_{[\Lambda]}$ is a divisor associated to the vertex $[\Lambda]$ and isomorphic to the projective line $\mathbb{P}_{\mathbb{F}}^1$. The symbol s represents another divisor which only depends on the element x' and is nonempty only if n is even. Since $\mathcal{D}(x) = \mathcal{Z}(x) - \mathcal{Z}(p^{-1}x)$, we have

$$\mathcal{D}(x)_{\mathbb{F}} = \begin{cases} \sum\limits_{[\Lambda]:d_{[\Lambda]}=(n-1)/2} \mathbb{P}_{[\Lambda]} \\ \sum\limits_{[\Lambda]:d_{[\Lambda]} \leqslant (n-3)/2} p^{(n-1)/2 - d_{[\Lambda]}} \cdot \mathbb{P}_{[\Lambda]} + (p^{n/2} - p^{(n-2)/2}) \cdot s, & \text{if } n \geqslant 1; \\ s, & \text{if } n = 0. \end{cases}$$

Therefore $\mathcal{D}(x)(\mathbb{F}) = \mathcal{Z}(x)(\mathbb{F})$.

(b) The unitary case: Let $n = 3$ and $F = \mathbb{Q}_p$. For an element $x \in \mathbb{V}^u$, the proof of ([23, Proposition 2.13]) implies that if $z \in \mathcal{Z}(x)(\mathbb{F})$, then the local equation of the difference divisor $\mathcal{D}(x)$ is not a unit at the point z, hence $\mathcal{Z}(x)(\mathbb{F}) \subset \mathcal{D}(x)(\mathbb{F})$. Therefore $\mathcal{D}(x)(\mathbb{F}) = \mathcal{Z}(x)(\mathbb{F})$.

1.3 Strategy and novelty of the proof

We prove Theorem 1.2.1 by a purely deformation-theoretic approach.

In the unitary case, we have the Grothendieck–Messing deformation theory of hermitian $\mathcal{O}_E$-modules of signature $(1, n-1)$. In the GSpin case, similar deformation results (Lemma 3.2.1) are also available in Madapusi's work [15, Proposition 5.16] (see also the works of Li and Zhang [14, Lemma 4.6.2]).

We prove a commutative algebra result Lemma 5.1.2 which helps us to identify a sequence of regular elements in a formally finitely generated and formally smooth algebra over a discrete valuation ring.

Our final proof combines the deformation theory and the commutative algebra results. In stead of analyzing one single difference divisor, we study a sequence of difference divisors $\{\mathcal{D}(\varpi^a x)\}_{a \geqslant 0}$ (resp. $\{\mathcal{D}(p^a x)\}_{a \geqslant 0}$). We show that the local equations of difference divisors in this sequence satisfy the assumptions in Lemma 5.1.2. The verification is a series of deformation-theoretic computations based on Lemma 3.1.5 and Lemma 3.2.1, the computations are much easier compared to windows theory and works for general finite extension $F/\mathbb{Q}_p$ since both two lemmas are of Grothendieck–Messing type.

1.4 Applications

1.4.1 Linear invariance

Li and Zhang [13, Lemma 2.8.1] proved the linear invariance of the derived intersections on the Rapoport–Zink space $\mathcal{N}^u$ based on the regularity of difference divisors, which guarantees that the local arithmetic intersection numbers on $\mathcal{N}^u$ are well-defined.

The parallel result in the GSpin case is proved by Terstiege [22, Lemma 4.1, Proposition 4.2] (see also [14, Lemma 4.11.1]) using globalization. Our result can be applied to give an alternative proof of the linear invariance of the derived intersections on the Rapoport–Zink space $\mathcal{N}^o$ along the lines of [13, Lemma 2.8.1] (see [14, foodnote 3]), which guarantees that the local arithmetic intersection numbers on $\mathcal{N}^o$ are well-defined.

1.4.2 Rapoport–Zink spaces with level structures

In some cases, difference divisors can be identified with Rapoport–Zink spaces with certain level structures. We give several examples.

(a) When the valuation of the special quasi-homomorphism $x \in \mathbb{V}^u$ (resp. $x \in \mathbb{V}^o$) has valuation 0, i.e., $v_\varpi((x, x)) = 0$ (resp. $v_p(q_{\mathbb{V}^o}(x)) = 0$). Then the difference divisor $\mathcal{D}(x) = \mathcal{Z}(x)$ is isomorphic to the unitary (resp. GSpin) Rapoport–Zink space of hyperspecial level of one dimension lower ([18, Remark 4.5] and [12, Lemma 3.2.2]).

In the GSpin case, when the valuation of the special quasi-homomorphism $x \in \mathbb{V}^o$ has valuation 1, i.e., $v_p(q_{\mathbb{V}^o}(x)) = 1$. Then the difference divisor $\mathcal{D}(x) = \mathcal{Z}(x)$

is isomorphic to the GSpin Rapoport–Zink space of almost self-dual level of one dimension lower ([17, Theorem 4.5]).

(b) In the unitary case, let $n = 2$. Then $\mathcal{N}^u \simeq \mathrm{Spf}\, \mathcal{O}_{\breve{F}}[[t]]$. Let $x \in \mathbb{V}^u$ be a special quasi-homomorphism such that $v_{\varpi}((x, x)) = k \geqslant 0$. Then the difference divisor $\mathcal{D}(x)$ is isomorphic to the quasi-canonical lifting of level k with respect to the unramified extension E/F ([10, Proposition 8.1] for the case $F = \mathbb{Q}_p$ which also works for general field F).

In the GSpin case, we take $V = \mathrm{M}_2(\mathbb{Z}_p)^{\mathrm{tr}=0}$ equipped with the quadratic form given by the determinant. Then $\mathcal{N}^o \simeq \mathrm{Spf}\, \breve{\mathbb{Z}}_p[[t]]$. Let $x \in \mathbb{V}^o$ be a special quasi-homomorphism such that $v_p(q_{\mathbb{V}^o}(x)) = k \geqslant 0$. Then the difference divisor $\mathcal{D}(x)$ is isomorphic to the quasi-canonical lifting of level $[k/2]$ with respect to an unramified extension (when k is even) or ramified extension (when k is odd) ([5, (5.10)], see also [14, §5.1]).

(c) Let $V = \mathrm{M}_2(\mathbb{Z}_p)$ with the quadratic form given by the determinant. It is a self-dual $\mathbb{Z}_p$-lattice of rank 4. In this case, we have $\mathcal{N}^o \simeq \mathrm{Spf}\, \breve{\mathbb{Z}}_p[[t_1, t_2]]$, $\mathbb{V}^o$ is the unique division quaternion algebra over $\mathbb{Q}_p$. Let $x \in \mathbb{V}^o$ be an element such that $v_p(q_{\mathbb{V}^o}(x)) = k$ for some integer $k \geqslant 0$. The difference divisor $\mathcal{D}(x)$ on $\mathcal{N}^o$ is isomorphic to the Rapoport–Zink space associated with the modular curve $\mathcal{X}_0(p^k)$ (cf. [26, Theorem 6.2.3]). A key ingredient in the proof is the regularity of the difference divisor $\mathcal{D}(x)$. Notice that the definition of difference divisor is relatively simple, while the Drinfeld level structure defining the modular curve $\mathcal{X}_0(p^k)$ (cf. [9], see also [3]) is much more subtle.

1.5 The structure of the paper

In §2, we review the construction and basic properties of the unitary and GSpin Rapoport–Zink spaces. We also give the definition of the difference divisor. In § 3, we study the deformation theory of the the unitary and GSpin Rapoport–Zink spaces and special cycles on them at a given $\mathbb{F}$-point. The main results are Lemma 3.1.5 and Lemma 3.2.1. In §4, we find the formally smoothness locus of a special cycle in the deformation-theoretic language compatible with §3. In §5, we first prove a commutative algebra result (Lemma 5.1.2), then we apply it to prove Theorem 1.2.1.

1.6 Notations

Throughout this article, we fix an odd prime p. Let $F/\mathbb{Q}_p$ be a finite extension, with ring of integers $\mathcal{O}_F$, and uniformizer ϖ. Let $\mathbb{F}$ be the algebraic closure of $\mathbb{F}_p$. Let $\breve{F}$ be the completion of the maximal unramified extension of F. Let $\sigma \in \mathrm{Gal}(\breve{F}/F)$ be the arithmetic Frobenius automorphism of the field $\breve{F}$. Let $\mathcal{O}_{\breve{F}}$ be the integer ring of $\breve{F}$.

For a scheme or formal scheme S over $\mathcal{O}_{\breve{F}}$, denote by $\mathrm{NCRIS}_{\mathcal{O}_{\breve{F}}}(S/\mathrm{Spec}\,\mathcal{O}_{\breve{F}})$ the big fppf nilpotent crystalline site of S over $\mathcal{O}_{\breve{F}}$ (cf. [4, Definition B.5.7.]), the definition is the same as the crystalline site defined in the works of Berthelot, Breen and Messing [1, §1.1.1.] except that we replace the pd-structure by $\mathcal{O}_F$-pd-structure [4, Definition

B.5.1.] (Notice that they are equivalent when $F = \mathbb{Q}_p$). Denote by $\mathcal{O}_S^{\text{crys}}$ the structure sheaf in this site. For a point $z \in S(\mathbb{F})$, let $\widehat{\mathcal{O}}_{S,z}$ be the complete local ring of S at the point z. Let $\text{Nilp}_{\mathcal{O}_{\check{F}}}$ be the category of $\mathcal{O}_{\check{F}}$-schemes on which ϖ is locally nilpotent. For an object S in $\text{Nilp}_{\mathcal{O}_{\check{F}}}$, we use $\overline{S}$ to denote the scheme $S \times_{\mathcal{O}_{\check{F}}} \mathbb{F}$.

Denote by $\text{Alg}_{\mathcal{O}_{\check{F}}}$ the category of noetherian ϖ-adically complete $\mathcal{O}_{\check{F}}$-algebras. Denote by $\text{Art}_{\mathcal{O}_{\check{F}}}$ the category of local artinian $\mathcal{O}_{\check{F}}$-algebras with residue field $\mathbb{F}$. Denote by $\text{ANilp}_{\mathcal{O}_{\check{F}}}^{\text{fsm}}$ be the category of noetherian adic $\mathcal{O}_{\check{F}}$-algebras, in which ϖ is nilpotent, which are formally finitely generated and formally smooth over $\mathcal{O}_{\check{F}}/\varpi^k$ for some $k \geqslant 1$.

2 Special cycles on Rapoport–Zink spaces

2.1 Rapoport–Zink spaces

2.1.1 Unitary Rapoport–Zink space $\mathcal{N}^u$

Let E be the unramified quadratic extension of F, and $\mathcal{O}_E$ be the integer ring of E. Fix an $\mathcal{O}_F$-algebra embedding $\phi_0 : \mathcal{O}_E \to \mathcal{O}_{\check{F}}$ and denote by ϕ_1 the embedding $\sigma \circ \phi_0 : \mathcal{O}_E \to \mathcal{O}_{\check{F}}$. The embedding ϕ_0 induces an embedding between the residue fields $\mathbb{F}_{q^2} \to \mathbb{F}$, which we shall think of as the natural embedding.

Let n be a positive integer. For an $\mathcal{O}_{\check{F}}$-scheme S, a hermitian $\mathcal{O}_E$-module of signature $(1, n-1)$ over an object S in $\text{Nilp}_{\mathcal{O}_{\check{F}}}$ is a triple (X, ι, λ) where

(1) X is a formal ϖ-divisible group over S of relative height $2n$ and dimension n;
(2) $\iota : \mathcal{O}_E \to \text{End}(X)$ is an action satisfying the signature $(1, n-1)$ condition, i.e., for $\alpha \in \mathcal{O}_E$,

$$\text{char}(\iota(\alpha) : \text{Lie } X)(T) = (T - \phi_0(\alpha))(T - \phi_1(\alpha))^{n-1} \in \mathcal{O}_S[T];$$

(3) $\lambda : X \to X^\vee$ is a principal polarization such that the associated Rosati involution induces $\alpha \mapsto \sigma(\alpha)$ on $\mathcal{O}_E$ via ι.

Up to $\mathcal{O}_E$-linear quasi-isogeny compatible with the polarizations, there is a unique such triple $\mathbb{X} = (\mathbb{X}, \iota_{\mathbb{X}}, \lambda_{\mathbb{X}})$ over $S = \text{Spec } \mathbb{F}$. The unitary Rapoport–Zink space of signature $(1, n-1)$ is the following functor $\mathcal{N}_{1,n-1} : \text{Nilp}_{\mathcal{O}_{\check{F}}} \to \textbf{Sets}$: for a scheme S in the category $\text{Nilp}_{\mathcal{O}_{\check{F}}}$, the set of $\mathcal{N}_{1,n-1}(S)$ is the isomorphism classes of quadruples $(X, \iota, \lambda, \rho)$, where (X, ι, λ) is a hermitian $\mathcal{O}_E$-module of signature $(1, n-1)$ over S and $\rho : X \times_S \overline{S} \to \mathbb{X} \times_{\mathbb{F}} \overline{S}$ is an $\mathcal{O}_E$-linear quasi-isogeny of height zero such that $\rho^*((\lambda_{\mathbb{X}})_{\overline{S}}) = \lambda_{\overline{S}}$. Two quadruples $(X, \iota, \lambda, \rho)$ and $(X', \iota', \lambda', \rho')$ are isomorphic if there exists an isomorphism $f : X \to X'$ of ϖ-divisible groups such that

$$f^\vee \circ \lambda' \circ f = \lambda, \quad \rho' = f \circ \rho, \quad \iota'(\alpha) \circ f = f \circ \iota(f) \text{ for all } \alpha \in \mathcal{O}_E.$$

Theorem 2.1.1 *The functor $\mathcal{N}_{1,n-1}$ is represented by a formal scheme which is formally locally of finite type and formally smooth of relative dimension $n-1$ over $\mathrm{Spf}\,\mathcal{O}_{\breve{F}}$.*

Proof These facts about $\mathcal{N}_{1,n-1}$ are proved in [20, Theorem 2.16], [10, §2.1] and [25]. □

In the following we denote $\mathcal{N}^u := \mathcal{N}_{1,n-1}$. Let $(\mathbb{Y}, \iota_{\mathbb{Y}}, \lambda_{\mathbb{Y}})$ be the framing object for $n = 1$. Let $(\overline{\mathbb{Y}}, \iota_{\overline{\mathbb{Y}}}, \lambda_{\overline{\mathbb{Y}}}) = (\mathbb{Y}, \iota_{\mathbb{Y}} \circ \sigma, \lambda_{\mathbb{Y}})$ be its conjugate. There is a canonical lift $(\overline{Y}, \iota_{\overline{Y}}, \lambda_{\overline{Y}})$ of $(\overline{\mathbb{Y}}, \iota_{\overline{\mathbb{Y}}}, \lambda_{\overline{\mathbb{Y}}})$ over $\mathcal{O}_{\breve{F}}$ (cf. [6, Proposition 2.1]), here $\iota_{\overline{Y}} : \mathcal{O}_E \to \mathrm{End}_{\mathcal{O}_{\breve{F}}}(\overline{Y})$ is an action such that for any $\alpha \in \mathcal{O}_E$,

$$\mathrm{char}(\iota(\alpha) : \mathrm{Lie}\,\overline{Y})(T) = T - \phi_1(\alpha) \in \mathcal{O}_{\breve{F}}[T];$$

we fix an isomorphism $\rho_{\overline{Y}} : \overline{Y} \times_{\mathcal{O}_{\breve{F}}} \mathbb{F} \to \overline{\mathbb{Y}}$.

2.1.2 GSpin Rapoport–Zink space $\mathcal{N}^o$

Let $k \geq 1$ be an integer. A quadratic lattice of rank k is a pair (L, q_L) such that L is a free $\mathcal{O}_F$-module of rank k and $q_L : L \to F$ is a quadratic form on L. The quadratic form q_L induces a symmetric bilinear form $(\cdot, \cdot) : L \times L \to F$ by $(x, y) = \frac{1}{2}(q(x + y) - q(x) - q(y))$. Let $L^\sharp = \{x \in L \otimes_{\mathcal{O}_F} F : (x, L) \subset \mathcal{O}_F\}$. We say a quadratic lattice is integral if $q_L(x) \in \mathcal{O}_F$ for all $x \in L$, is self-dual if it is integral and $L = L^\sharp$. In this article we will only work with quadratic lattice over $\mathbb{Z}_p$.

Let $n \geq 2$ be an integer, and V be a self-dual quadratic lattice of rank $m = n + 1$ over $\mathbb{Z}_p$. The Clifford algebra $C(V)$ of V is the quotient of tensor algebra $T^\bullet(V)$ by the relations $v \cdot v = q(v) \cdot 1$ for all $v \in V$. It is an associated $\mathbb{Z}/2\mathbb{Z}$-graded algebra with grading given by the parity of the tensors. Let $C^+(V)$ be the even part, while $C^-(V)$ be the odd part. The canonical involution on $C(V)$ is characterized by $(v_1 \cdots v_d)^* = v_d \cdots v_1$ for all $v_1, \cdots, v_d \in V$. We fix an element $\delta \in C(V)^\times$ with $\delta^* = -\delta$, then

$$\psi_\delta(c_1, c_2) = \mathrm{Trd}(c_1 \delta c_2^*) \tag{1}$$

is a perfect symplectic form on $C(V)$.

The spinor similitude group $G = \mathrm{GSpin}(V)$ is a reductive group over $\mathbb{Z}_p$. For any $\mathbb{Z}_p$-algebra R, let $V_R = V \otimes_{\mathbb{Z}_p} R$ and $C^+(V_R) = C^+(V) \otimes_{\mathbb{Z}_p} R$, the set of R-points of G is given by

$$G(R) = \{g \in C^+(V_R)^\times : gV_R g^{-1} = V_R, g^*g \in R^\times\}.$$

The group G acts on $C(V)$ by left multiplication, yielding a closed immersion $G \to \mathrm{GL}(C(V))$. Let $(s_\alpha)_{\alpha \in I}$ be a finite set of tensors s_α in the total tensor algebra $C(V)^\otimes$ cutting out G from $\mathrm{GL}(C(V))$. One of the tensors π can be chosen as follows ([15, §1.3]): The left multiplication action of V on $C(V)$ gives a G-equivariant embedding $V \hookrightarrow \mathrm{End}(C(V))$. We identify $V \subset \mathrm{End}(C(V))$ via this embedding. Then $f \circ f = q(f) \cdot \mathrm{id}_C$ for all $f \in V$. The non-degenerate symmetric bilinear form $(\cdot, \cdot)$ on $\mathrm{End}(C(V))$ defined by

$$(f_1, f_2) := 2^{-m} \mathrm{tr}(f_1 \circ f_2)$$

extends the symmetric bilinear form on V. Let $\{e_i\}_{i=1}^m$ be an orthogonal $\mathbb{Z}_p$-basis of V, define

$$\boldsymbol{\pi} : \text{End}(C(V)) \to \text{End}(C(V)), \ \boldsymbol{\pi}(f) := \sum_{i=1}^m \frac{(f, e_i)}{(e_i, e_i)} e_i. \tag{2}$$

Fix a basis $\{x_i\}_{i=1}^m$ of V for which the inner product matrix of V has the form

$$\big((x_i, x_j)\big) = \begin{pmatrix} 0 & 1 & & & \\ 1 & 0 & & & \\ & & * & & \\ & & & \ddots & \\ & & & & * \end{pmatrix}.$$

This choice of basis determines a cocharacter $\mu : \mathbb{G}_m \to G$ (cf. [7, §4.2.1]) by

$$\mu(t) = t^{-1} x_1 x_2 + x_2 x_1.$$

We also set $b = x_3(p^{-1}x_1 + x_2) \in G(\mathbb{Q}_p)$.

Howard and Pappas [7, Proposition 4.2.6] proved that the quadruple $(G, b, \mu, C(V))$ is a local unramified Shimura–Hodge datum. Let $\breve{\mathbb{Q}}_p$ be the completion of the maximal unramified extension of $\mathbb{Q}_p$, with ring of integers $\breve{\mathbb{Z}}_p$. Set $C^\vee = \text{Hom}_{\mathbb{Z}_p}(C(V), \mathbb{Z}_p)$ with the contragredient action of G. By [7, Lemma 2.2.5], this local unramified Shimura–Hodge data gives rise to a (unique up to isomorphism) supersingular p-divisible group $\mathbb{X}_V$ over $\mathbb{F}$ whose contravariant Dieudonne module $\mathbb{D}(\mathbb{X}_V)(\breve{\mathbb{Z}}_p)$ is given by $C^\vee_{\breve{\mathbb{Z}}_p} := C^\vee \otimes_{\mathbb{Z}_p} \breve{\mathbb{Z}}_p$ with Frobenius $\mathbf{F} = b \circ \sigma$, we also obtain $t_{\alpha,0} = s_\alpha \otimes 1 \in (C^{\vee \otimes})_{\breve{\mathbb{Z}}_p} = \mathbb{D}(\mathbb{X}_V)^\otimes$, which are $\mathbf{F}$-invariant. Moreover, the perfect symplectic form ψ_δ in (1) on $C(V)$ determines a principal polarization $\lambda_0 : \mathbb{X}_V \to \mathbb{X}_V^\vee$.

Associated to the local unramified Shimura–Hodge data $(G, b, \mu, C(V))$, we have a GSpin Rapoport–Zink space $\text{RZ}(V) = \text{RZ}(G, b, \mu, C(V))$ of Hodge type ([7, §4.3], see also [8]) parametrizing p-divisible groups with crystalline Tate tensors. More precisely, it is a formal scheme over $\text{Spf} \, \breve{\mathbb{Z}}_p$ representing the functor sending $R \in \text{ANilp}_{\breve{\mathbb{Z}}_p}^{\text{fsm}}$ to the set of isomorphism classes of tuples $(X, (t_\alpha)_{\alpha \in I}, \rho)$, where

- X is a p-divisible group over $\text{Spec} \, R$.
- $(t_\alpha)_{\alpha \in I}$ is a collection of crystalline Tate tensors of X.
- $\rho : \mathbb{X}_V \times_{\mathbb{F}} R/J \to X \times_R R/J$ is a framing, i.e., a quasi-isogeny such that each t_α pulls back to $t_{\alpha,0}$ under ρ, where J is some ideal of definition of R such that $p \in J$.

The tuple $(X, (t_\alpha)_{\alpha \in I}, \rho)$ is required to satisfy additional assumptions [7, Definition 2.3.3 (ii),(iii)].

Theorem 2.1.2 *The GSpin Rapoport–Zink space* $\text{RZ}(V)$ *is formally locally of finite type and formally smooth of relative dimension* $n - 1$ *over* $\text{Spf} \, \breve{\mathbb{Z}}_p$.

Proof This is Theorem A in [7]. □

Let X^{univ} be the universal p-divisible group over $\mathrm{RZ}(V)$ with the universal quasi-isogeny $\rho^{\mathrm{univ}} : \mathbb{X}_V \times_{\mathbb{F}} \overline{\mathrm{RZ}(V)} \to X^{\mathrm{univ}} \times_{\mathrm{RZ}(V)} \overline{\mathrm{RZ}(V)}$. Let $\mathcal{N}^o$ be the connected component of $\mathrm{RZ}(V)$ such that the principal polarization λ_0 of $\mathbb{X}_V$ lifts to a principal polarization λ^{univ} on X^{univ} via the universal quasi-isogeny ρ^{univ} (cf. [14, §4.5]).

2.2 The space of special quasi-homomorphisms

2.2.1 The unitary case

Kudla and Rapoport [10, Definition 3.1] defined the space of special homomorphisms to be the E-vector space $\mathbb{V}^u := \mathrm{Hom}_{\mathcal{O}_E}(\overline{\mathbb{Y}}, \mathbb{X}) \otimes_{\mathbb{Z}} \mathbb{Q}$. There is a natural $\mathcal{O}_E$-valued σ-Hermitian form on $\mathbb{V}^u$ (cf. [10, (3.1)]) given by

$$(x, y) \mapsto \lambda_{\overline{\mathbb{Y}}}^{-1} \circ y^\vee \circ \lambda_{\mathbb{X}} \circ x \in \mathrm{End}_{\mathcal{O}_E}(\overline{\mathbb{Y}}) \xrightarrow{\sim} \mathcal{O}_E.$$

The hermitian space V^u is the unique (up to isomorphism) non-degenerate non-split E/F-hermitian space of dimension n (cf. [13, §2.2]).

2.2.2 The GSpin case

The inclusion $V \subset C(V)^{\mathrm{op}}$ (where V acts on $C(V)$ via right multiplication) realizes $V \subset \mathrm{End}_{\mathbb{Z}_p}(C^\vee)$. Tensoring with $\breve{\mathbb{Q}}_p$ gives a subspace $V_{\breve{\mathbb{Q}}_p} \subset \mathrm{End}_{\breve{\mathbb{Q}}_p}(C^\vee_{\breve{\mathbb{Q}}_p})$. Define the σ-linear operator $\Phi = \overline{b} \circ \sigma$ on $V_{\breve{\mathbb{Q}}_p}$, where $\overline{b} \in \mathrm{SO}(V)(\breve{\mathbb{Q}}_p)$ is the image of $b \in G(\breve{\mathbb{Q}}_p)$ under the natural quotient map $G = \mathrm{GSpin}(V) \to \mathrm{SO}(V)$. Then $(V_{\breve{\mathbb{Q}}_p}, \Phi)$ is an isocrystal. The Φ-fixed vectors form a distinguished $\mathbb{Q}_p$-vector subspace

$$\mathbb{V}^o := V_{\breve{\mathbb{Q}}_p}^\Phi \subset \mathrm{End}^\circ(\mathbb{X}) := \mathrm{End}(\mathbb{X}) \otimes_{\mathbb{Z}} \mathbb{Q},$$

called the space of special quasi-homomorphisms of $\mathbb{X}_V$ ([7, §4.3.1], see also [14, §4.2]).

Let $\breve{q}$ be the base change of the quadratic form q on V to $V_{\breve{\mathbb{Q}}_p}$. For any $x \in V_{\breve{\mathbb{Q}}_p}$, we have $x \circ x = \breve{q}(x) \cdot \mathrm{id}_{\mathbb{X}_V}$ in $\mathrm{End}_{\breve{\mathbb{Q}}_p}(C^\vee_{\breve{\mathbb{Q}}_p})$ by the construction of the Clifford algebra $C(V)$. Let $q_{\mathbb{V}^o}$ be the restriction of the quadratic form $\breve{q}$ on $V_{\breve{\mathbb{Q}}_p}$ to $\mathbb{V}^o$. Therefore $x \circ x = q_{\mathbb{V}^o}(x) \cdot \mathrm{id}_{\mathbb{X}}$ for $x \in \mathbb{V}^o$. The quadratic form $q_{\mathbb{V}^o}$ values in $\mathbb{Q}_p$ and the quadratic space $\mathbb{V}^o$ is the unique (up to isomorphism of quadratic spaces) non-degenerate quadratic space with the same dimension and discriminant, but opposite Hasse invariant as V (cf. [7, Proposition 4.2.5, §4.3.1]).

2.3 Special cycles and difference divisors

Definition 2.3.1 In the unitary case, for any subset $L \subset \mathbb{V}^u$, define the special cycle $\mathcal{Z}(L)$ in $\mathcal{N}^u$ as the following subfunctor: for any object S in $\mathrm{Nilp}_{\mathcal{O}_{\breve{F}}}$, the set $\mathcal{Z}(L)(S)$ consists of elements $(X, \iota_X, \lambda_X, \rho_X) \in \mathcal{N}^u(S)$ such that the quasi-homomorphism

$\rho_X^{-1} \circ x \circ \rho_{\overline{Y}} : \overline{Y} \times_S \overline{S} \to X \times_S \overline{S}$ extends to a homomorphism from $\overline{Y}$ to X for all $x \in L$.

In the GSpin case, for any subset $L \subset \mathbb{V}^o$, define the special cycle $\mathcal{Z}(L) \subset \mathcal{N}^o$ to be the closed formal subscheme cut out by the condition

$$\rho^{\mathrm{univ}} \circ x \circ (\rho^{\mathrm{univ}})^{-1} \subset \mathrm{End}(X^{\mathrm{univ}}),$$

for all $x \in L$.

Remark 2.3.2 We use the same symbol $\mathcal{Z}(L)$ for special cycles, although they may lie in different formal schemes. In the following discussion, we will always make clear on which space the special cycle lie in.

Proposition 2.3.3 *Let* $x \in \mathbb{V}^u$ *(resp.* $x \in \mathbb{V}^o$*) be a nonzero element such that* $(x, x) \in \mathcal{O}_E$ *(resp.* $q_{\mathbb{V}^o}(x) \in \mathbb{Z}_p$*), the special cycle* $\mathcal{Z}(x)$ *is an effective Cartier divisor in* $\mathcal{N}^u$ *(resp.* $\mathcal{N}^o$*) and flat over* W.

Proof For the unitary case, this is proved in [10, Proposition 3.5, Lemma 3.7]. For the GSpin case, this is proved in [14, Proposition 4.10.1] $\qquad\qquad\square$

Definition 2.3.4 Let $x \in \mathbb{V}^u$ (resp. $x \in \mathbb{V}^o$) be a nonzero element such that $(x, x) \in \mathcal{O}_E$ (resp. $q_{\mathbb{V}^o}(x) \in \mathbb{Z}_p$). For a point $z \in \mathcal{N}^u(\mathbb{F})$ (resp. $\mathcal{N}^o(\mathbb{F})$), let $f_{x,z} \in \widehat{\mathcal{O}}_{\mathcal{N}^u,z}$ (resp. $\in \widehat{\mathcal{O}}_{\mathcal{N}^o,z}$) be the local equation of the closed formal subscheme $\mathcal{Z}(x)$ at z, i.e., $\widehat{\mathcal{O}}_{\mathcal{Z}(x),z} \simeq \widehat{\mathcal{O}}_{\mathcal{N}^u,z}/(f_{x,z})$ (resp. $\widehat{\mathcal{O}}_{\mathcal{Z}(x),z} \simeq \widehat{\mathcal{O}}_{\mathcal{N}^o,z}/(f_{x,z})$). Notice that the element $f_{x,z}$ is well-defined up to units in $\widehat{\mathcal{O}}_{\mathcal{N}^u,z}$ (resp. $\widehat{\mathcal{O}}_{\mathcal{N}^o,z}$).

Let $x \in \mathbb{V}^u$ ($x \in \mathbb{V}^o$) be a nonzero element such that $(x, x) \in \mathcal{O}_E$ (resp. $q_{\mathbb{V}^o}(x) \in \mathbb{Z}_p$), we have a natural closed immersion of closed formal schemes $\mathcal{Z}(\varpi^{-1}x) \hookrightarrow \mathcal{Z}(x)$ (resp. $\mathcal{Z}(p^{-1}x) \hookrightarrow \mathcal{Z}(x)$) by Definition 2.3.1. The closed immersion $\mathcal{Z}(\varpi^{-1}x) \hookrightarrow \mathcal{Z}(x)$ (resp. $\mathcal{Z}(p^{-1}x) \hookrightarrow \mathcal{Z}(x)$) implies the divisibility $f_{\varpi^{-1}x,z} | f_{x,z}$ at every point $z \in \mathcal{N}^u(\mathbb{F})$ (resp. $\mathcal{N}^o(\mathbb{F})$).

Definition 2.3.5 Let $x \in \mathbb{V}^u$ (resp. $x \in \mathbb{V}^o$) be an element such that $(x, x) \in \mathcal{O}_E$ (resp. $q_{\mathbb{V}^o}(x) \in \mathbb{Z}_p$). Define the difference divisor associated to x to be the following Cartier divisor on $\mathcal{N}^u$ (resp. $\mathcal{N}^o$),

$$\mathcal{D}(x) := \mathcal{Z}(x) - \mathcal{Z}(\varpi^{-1}x) \quad (\text{resp. } \mathcal{D}(x) := \mathcal{Z}(x) - \mathcal{Z}(p^{-1}x)).$$

i.e., at a point $z \in \mathcal{Z}(x)(\mathbb{F})$, the difference divisor $\mathcal{D}(x)$ is cut out by $d_{x,z} := f_{x,z}/f_{\varpi^{-1}x,z} \in \widehat{\mathcal{O}}_{\mathcal{N}^u,z}$ (resp. $\widehat{\mathcal{O}}_{\mathcal{N}^o,z}$).

3 Deformation theory

3.1 The unitary case

3.1.1 Preliminaries on linear algebra

Definition 3.1.1 Let R be a commutative ring. Let D be a locally free R-module of finite rank n, a hyperplane P in D is a locally free R-submodule of D of rank $n - 1$, a line L in D is a locally free R-submodule of D of rank 1.

Definition 3.1.2 Let R be a commutative ring. Let D_1 and D_2 be two R-modules. An R-bilinear pairing $\langle \cdot, \cdot \rangle : D_1 \times D_2 \to R$ is perfect if the induced maps $D_1 \to D_2^\vee$, $d_1 \mapsto \langle d_1, \cdot \rangle$ and $D_2 \to D_1^\vee$, $d_1 \mapsto \langle d_1, \cdot \rangle$ are isomorphisms.

Lemma 3.1.3 *Let R be a commutative ring. Let D_1 and D_2 be two locally free R-module of finite rank n. Let $\langle \cdot, \cdot \rangle : D_1 \times D_2 \to R$ be an R-bilinear pairing. Then we have a bijection,*

$$\{ \text{Lines in } D_1.\} \longleftrightarrow \{ \text{Hyperplanes in } D_2.\},$$
$$L \longmapsto P = \{x \in D : \langle l, x \rangle = 0, \ \text{for all } l \in L.\};$$
$$L = \{l \in D_1 : \langle l, x \rangle = 0, \ \text{for all } x \in P.\}. \longleftarrow\!\!\!\shortmid P$$

Proof This can be proved by purely linear algebra. $\qquad\qquad\square$

3.1.2 Deformation theory

For an object $S \in \mathrm{Alg}_{\mathcal{O}_{\breve{F}}}$ and any $z \in \mathcal{N}^u(S)$, the point z corresponds to a hermitian $\mathcal{O}_E$-module $(X_z, \iota_z, \lambda_z, \rho_z)$ of signature $(1, n - 1)$ as in §2.1.1. Let $\mathbb{D}(X_z)$ be the Dieudonne crystal of X_z in the site $\mathrm{NCRIS}_{\mathcal{O}_{\breve{F}}}(S/\mathrm{Spec}\,\mathcal{O}_{\breve{F}})$, it is a rank $2n$ $\mathcal{O}_S^{\mathrm{crys}}$-module crystal with $\mathcal{O}_E$-action. Under the isomorphism $\mathcal{O}_E \otimes_{\mathcal{O}_F} \mathcal{O}_{\breve{F}} \simeq \mathcal{O}_{\breve{F}} \times \mathcal{O}_{\breve{F}}$, we have

$$\mathcal{O}_E \otimes_{\mathcal{O}_F} \mathcal{O}_S^{\mathrm{crys}} \simeq \mathcal{O}_S^{\mathrm{crys}} \times \mathcal{O}_S^{\mathrm{crys}}.$$

Accordingly, the crystal $\mathbb{D}(X_z)$ has the following decomposition

$$\mathbb{D}(X_z) = \mathbb{D}_0(X_z) \oplus \mathbb{D}_1(X_z),$$

where both of $\mathbb{D}_0(X_z)$ and $\mathbb{D}_1(X_z)$ are $\mathcal{O}_S^{\mathrm{crys}}$-modules of rank n, and $\mathcal{O}_E$ acts on $\mathbb{D}_i(X_z)$ by multiplication through $\phi_i : \mathcal{O}_E \to \mathcal{O}_{\breve{F}}$ for $i = 0, 1$.

For simplicity, let $\mathbf{D}_z = \mathbb{D}_0(X_z)$ and $\overline{\mathbf{D}}_z = \mathbb{D}_1(X_z)$. Let $0 \subset \mathrm{Fil}^1\mathbb{D}(X_z)(S) \subset \mathbb{D}(X_z)(S)$ be the Hodge filtration. Let $\mathrm{Fil}^1\mathbf{D}_z(S) = \mathrm{Fil}^1\mathbb{D}(X_z)(S) \cap \mathbf{D}_z(S)$ and $\mathrm{Fil}^1\overline{\mathbf{D}}_z(S) = \mathrm{Fil}^1\mathbb{D}(X_z)(S) \cap \overline{\mathbf{D}}_z(S)$. The compatibility with the $\mathcal{O}_E$-action implies that

$$\mathrm{Fil}^1\mathbb{D}(X_z)(S) = \mathrm{Fil}^1\mathbf{D}_z(S) \oplus \mathrm{Fil}^1\overline{\mathbf{D}}_z(S).$$

Recall that there is a principal polarization $\lambda_z : X_z \to X_z^\vee$. It induces an isomorphism of crystals $\mathbb{D}(\lambda_z) : \mathbb{D}(X_z) \to \mathbb{D}(X_z^\vee)$. Notice that $\mathbb{D}(X_z^\vee) \simeq \mathbb{D}(X_z)^\vee$ by [1,

Theorem 5.1.8]. Hence $\mathbb{D}(\lambda_z)$ induces a perfect symplectic pairing between crystals $\langle \cdot, \cdot \rangle : \mathbb{D}(X_z) \times \mathbb{D}(X_z) \to \mathcal{O}_S^{\text{crys}}$ (cf. [2, Chapter 2]). Moreover, the Hodge filtration $\text{Fil}^1 \mathbb{D}(X_z)(S)$ is totally isotropic under this symplectic pairing (cf. [1, Proposition 5.1.10]).

Lemma 3.1.4 *Let $R \to S$ be a surjection in $\text{Alg}_{\mathcal{O}_{\breve{F}}}$ whose kernel admits divided powers. The perfect symplectic pairing defined by the principal polarization λ_z induces a perfect pairing*

$$\langle \cdot, \cdot \rangle : \mathbf{D}_z(R) \times \overline{\mathbf{D}}_z(R) \to R.$$

Moreover, the intersection $\text{Fil}^1 \mathbf{D}_z(S)$ is a hyperplane in $\mathbf{D}_z(S)$, $\text{Fil}^1 \overline{\mathbf{D}}_z(S)$ is a line in $\overline{\mathbf{D}}_z(S)$, and they are annihilators of each other under the perfect R-linear pairing $\langle \cdot, \cdot \rangle$.

Proof The principal polarization λ_z induces a perfect symplectic R-linear pairing

$$\langle \cdot, \cdot \rangle : \mathbb{D}(X_z)(R) \times \mathbb{D}(X_z)(R) \to R.$$

By definition, for all elements $a \in \mathcal{O}_E$, we have $\langle \mathbb{D}(\iota_z(a))(x), y \rangle = \langle x, \mathbb{D}(\iota_z(\sigma(a)))(y) \rangle$ for all $x, y \in \mathbb{D}(X_z)(R)$. Let $\varepsilon \in \mathcal{O}_E^\times$ be an element such that $\sigma(\varepsilon) = -\varepsilon$, we get

$$\langle \varepsilon x, y \rangle = \langle x, -\varepsilon y \rangle, \quad \text{for all } x, y \in \mathbf{D}_z(R);$$
$$\langle -\varepsilon x, y \rangle = \langle x, \varepsilon y \rangle, \quad \text{for all } x, y \in \overline{\mathbf{D}}_z(R).$$

Hence $\langle x, y \rangle = 0$ if $x, y \in \mathbf{D}_z(R)$ or $x, y \in \overline{\mathbf{D}}_z(R)$. Therefore the perfect symplectic R-linear pairing $\langle \cdot, \cdot \rangle$ restricts to a perfect R-linear pairing $\langle \cdot, \cdot \rangle : \mathbf{D}_z(R) \times \overline{\mathbf{D}}_z(R) \to R$.

For the second claim, notice that there is an exact sequence of locally free S-modules,

$$0 \longrightarrow \text{Fil}^1 \mathbb{D}(X_z)(S) \longrightarrow \mathbb{D}(X_z)(S) \longrightarrow \text{Lie } X_z \longrightarrow 0.$$

The signature $(1, n-1)$ condition on $\text{Lie } X_z$ implies that the intersection $\text{Fil}^1 \mathbf{D}_z(S)$ is a hyperplane in $\mathbf{D}_z(S)$, $\text{Fil}^1 \overline{\mathbf{D}}_z(S)$ is a line in $\overline{\mathbf{D}}_z(S)$. We also know that $\text{Fil}^1 \mathbf{D}_z(S)$ is totally isotropic under the pairing $\langle \cdot, \cdot \rangle$, hence $\text{Fil}^1 \mathbf{D}_z(S)$ and $\text{Fil}^1 \overline{\mathbf{D}}_z(S)$ are annihilators of each other. $\square$

Let $\mathbb{D}(\overline{\mathbb{Y}})$ be the (covariant) Dieudonne module of $\overline{\mathbb{Y}}$ over $\mathcal{O}_{\breve{F}}$, it is a free $\mathcal{O}_{\breve{F}}$-module of rank 2. We have a decomposition $\mathbb{D}(\overline{\mathbb{Y}}) = \mathbb{D}_0(\overline{\mathbb{Y}}) \oplus \mathbb{D}_1(\overline{\mathbb{Y}})$, both $\mathbb{D}_0(\overline{\mathbb{Y}})$ and $\mathbb{D}_1(\overline{\mathbb{Y}})$ are free $\mathcal{O}_{\breve{F}}$-module of rank 1 by the signature condition. Let $\overline{1}_0$ be an $\mathcal{O}_{\breve{F}}$-generator of $\mathbb{D}_0(\overline{\mathbb{Y}})$. Let $\overline{1}_1 = \varpi^{-1} \Phi \overline{1}_0$ where Φ is the Frobenius morphism on the Dieudonne module $\mathbb{D}(\overline{\mathbb{Y}})$. We have $\mathbb{D}_1(\overline{\mathbb{Y}}) = \mathcal{O}_{\breve{F}} \cdot \overline{1}_1$ ([10, Remark 2.5]).

For a point $z \in \mathcal{N}^u(\mathbb{F})$. The (covariant) Dieudonne module $\mathbb{D}(X_z)$ of X_z also has a decomposition $\mathbb{D}_0(X_z) \oplus \mathbb{D}_1(X_z)$ as $\mathcal{O}_{\breve{F}}$-module. Let $\mathbf{D}_z = \mathbb{D}_0(X_z)$ and $\overline{\mathbf{D}}_z = \mathbb{D}_1(X_z)$. Let $x \in \mathbb{V}^u$ be a special quasi-homomorphism, it induces an $\breve{F}$-linear homomorphism $\mathbb{D}(x) : \mathbb{D}(\overline{\mathbb{Y}}) \otimes_{\mathcal{O}_{\breve{F}}} \breve{F} \to \mathbb{D}(X_z) \otimes_{\mathcal{O}_{\breve{F}}} \breve{F}$, sending $\mathbb{D}_i(\overline{\mathbb{Y}}) \otimes_{\mathcal{O}_{\breve{F}}} \breve{F}$ to $\mathbb{D}_i(X_z) \otimes_{\mathcal{O}_{\breve{F}}} \breve{F}$ for $i = 0, 1$. Define

$$i_{\text{crys},z} : \mathbb{V}^u \longrightarrow \mathbf{D}_z \otimes_{\mathcal{O}_{\breve{F}}} \breve{F}, \quad x \longmapsto x_{\text{crys},z} := \mathbb{D}(x)(\overline{1}_0) \in \mathbf{D}_z \otimes_{\mathcal{O}_{\breve{F}}} \breve{F}. \quad (3)$$

More generally, for $S \in \mathrm{Alg}_{\mathcal{O}_{\breve{F}}}$, let $z \in \mathcal{Z}(L)(S)$. For an arbitrary element $x \in L$, there is an isogeny $\overline{Y}_S \to X_z$, where $\overline{Y}$ is the canonical lift of $\overline{Y}$ and $\overline{Y}_S = \overline{Y} \times_{\mathcal{O}_{\breve{F}}} S$, lifting the element $x \in \mathbb{V}^u = \mathrm{Hom}^\circ_{\mathcal{O}_E}(\overline{\mathbb{Y}}, \mathbb{X})$. Let $\mathbb{D}(\overline{Y}_S)$ be the Dieudonne crystal of $\overline{Y}_S$, then $\mathbb{D}(\overline{Y}_S) \simeq \mathbb{D}(\overline{\mathbb{Y}}) \otimes_{\mathcal{O}_{\breve{F}}} \mathcal{O}_S^{\mathrm{crys}}$. Similarly, we have a decomposition $\mathbb{D}(\overline{Y}_S) = \mathbb{D}_0(\overline{Y}_S) \oplus \mathbb{D}_1(\overline{Y}_S)$, here $\mathbb{D}_i(\overline{Y}_S) \simeq \mathbb{D}_i(\overline{\mathbb{Y}}) \otimes_{\mathcal{O}_{\breve{F}}} \mathcal{O}_S^{\mathrm{crys}}$ for $i = 0, 1$, then the element $\overline{1}_0$ we defined above is also a generator of the $\mathcal{O}_S^{\mathrm{crys}}$-module $\mathbb{D}_0(\overline{Y}_S)$.

Moreover, the Hodge filtration on $\mathbb{D}(\overline{Y}_S)$ equals to $0 \subset \mathbb{D}_0(\overline{Y}_S) \subset \mathbb{D}(\overline{Y}_S)$. Let $R \to S$ be a surjection in $\mathrm{Alg}_{\mathcal{O}_{\breve{F}}}$ whose kernel admits a nilpotent $\mathcal{O}_F$-pd-structure, the element x induces a morphism of crystals $\mathbb{D}(x) : \mathbb{D}(\overline{Y}_S) \to \mathbb{D}(X_z)$. Let $x_{\mathrm{crys},z}(R) = \mathbb{D}(x)(\overline{1}_0) \in \mathbf{D}_z(R)$.

Lemma 3.1.5 *Let $R \to S$ be a surjection in $\mathrm{Alg}_{\mathcal{O}_{\breve{F}}}$ whose kernel admits nilpotent $\mathcal{O}_F$-pd-structure.*
(i) Let $z_0 \in \mathcal{N}^u(\mathbb{F})$ and $\widehat{\mathcal{N}}^u_{z_0}$ be the completion of $\mathcal{N}^u$ at z_0. Let $z \in \widehat{\mathcal{N}}^u_{z_0}(S)$. Then there is a natural bijection

$$\left\{ \text{Lifts } \tilde{z} \in \widehat{\mathcal{N}}^u_{z_0}(R) \text{ of } z. \right\} \xrightarrow{\sim} \left\{ \text{Lines in } \overline{\mathbf{D}}_z(R) \text{ lifting the line } \mathrm{Fil}^1 \overline{\mathbf{D}}_z(S) \right\}.$$

which maps a lift $\tilde{z} \in \widehat{\mathcal{N}}^u_{z_0}(R)$ to the line $\mathrm{Fil}^1 \overline{\mathbf{D}}_{\tilde{z}}(R) \subset \overline{\mathbf{D}}_{\tilde{z}}(R) \simeq \overline{\mathbf{D}}_z(R)$.
(ii) Let $L \subset \mathbb{V}^u$ be a $\mathcal{O}_F$-lattice of rank $r \geq 1$. Let $z_0 \in \mathcal{Z}(L)(\mathbb{F})$ and $\widehat{\mathcal{Z}(L)}_{z_0}$ be the completion of $\mathcal{Z}(L)$ at z_0, then there is a natural bijection compatible with the bijection in (i)

$$\left\{ \text{Lifts } \tilde{z} \in \widehat{\mathcal{Z}(L)}_{z_0}(R) \text{ of } z. \right\} \xrightarrow{\sim} \left\{ \begin{array}{l} \text{Lines in } \overline{\mathbf{D}}(R) \text{ lifting } \mathrm{Fil}^1 \overline{\mathbf{D}}_z(S) \text{ and} \\ \text{orthogonal to } x_{\mathrm{crys},z}(R) \text{ for all } x \in L. \\ \text{under the pairing } \langle \cdot, \cdot \rangle. \end{array} \right\}.$$

Proof We first prove (i). We construct the reverse of the map stated in the lemma. Let $L \subset \overline{\mathbf{D}}_z(R)$ be a line lifting the line $\mathrm{Fil}^1 \overline{\mathbf{D}}_z(S)$. Let $P \subset \mathbf{D}_z(R)$ be the annihilator of L under the perfect pairing $\langle \cdot, \cdot \rangle : \mathbb{D}(X_z)(R) \times \mathbb{D}(X_z)(R) \to R$. The R-module P is a hyperplane by Lemma 3.1.1, lifting the hyperplane $\mathrm{Fil}^1 \mathbf{D}_z(S)$. By $H = L \oplus P \subset \mathbb{D}(X_z)(R)$, it is a locally free S-module of rank n lifting the Hodge filtration $\mathrm{Fil}^1 \mathbb{D}(X_z)(S)$. By Grothendieck–Messing deformation theory, it corresponds to a ϖ-divisible group $\tilde{X}_z$ over R lifting X_z. Notice that the Hodge filtration $0 \subset H \subset \mathrm{Fil}^1 \mathbb{D}(X_z)(R)$ is stable under $\mathcal{O}_E$-action, hence ι_z lifts to $\tilde{\iota}_z : \mathcal{O}_E \to \mathrm{End}(\tilde{X}_z)$. The signature $(1, n-1)$ condition on $\mathrm{Lie}\, \tilde{X}_z$ is automatic by construction. Hence we construct a point $\tilde{z} \in \widehat{\mathcal{N}}^u_{z_0}(R)$ which lifts z. It's easy to see that this map is the reverse of the map stated in the lemma.

Now we prove (ii). Let $\tilde{z} \in \widehat{\mathcal{N}}^u_{z_0}(R)$ be a lift of the point z. If $\tilde{z} \in \widehat{\mathcal{Z}(L)}_{z_0}(R)$. Let $x \in L$ be an arbitrary element. The morphism $\mathbb{D}(x)(R)$ preserves the Hodge filtrations, i.e., it maps $\mathbb{D}_0(\overline{Y}_R)(R) \simeq \mathbb{D}_0(\overline{Y}_S)(R)$ to $\mathrm{Fil}^1 \mathbb{D}(X_{\tilde{z}})(R)$. Hence $x_{\mathrm{crys},z}(R) = D(x)(R)(\overline{1}_0) \in \mathrm{Fil}^1 \mathbb{D}(X_{\tilde{z}})(R) \cap \mathbf{D}_{\tilde{z}}(R) = \mathrm{Fil}^1 \mathbf{D}_{\tilde{z}}(R)$. Notice that the hyperplane $\mathrm{Fil}^1 \mathbf{D}_{\tilde{z}}(R)$ is the annihilator of the line $\mathrm{Fil}^1 \overline{\mathbf{D}}_{\tilde{z}}(R)$. Therefore the line corresponding to $\tilde{z}$ under the bijection in (i) is orthogonal to $x_{\mathrm{crys},z}(R)$ for all $x \in L$.

If we have a line $L \subset \overline{\mathbf{D}}_z(R)$ lifting the line $\mathrm{Fil}^1\overline{\mathbf{D}}_z(S)$ and orthogonal to $x_{\mathrm{crys},z}(R)$ for all $x \in L$. Let $P \subset \mathbf{D}_z(R)$ be the annihilator of L under the perfect pairing. The orthogonal condition is equivalent to $x_{\mathrm{crys},z}(R) \in P$ for all $x \in L$. Let $\tilde{z} \in \widehat{\mathcal{N}}^u_{z_0}(R)$ be the point corresponding to the line L. The Hodge filtration of $X_{\tilde{z}}$ is given by $0 \subset L \oplus P \subset \mathbb{D}(X_z)(R)$. Let $x \in L$ be an element, the morphism $\mathbb{D}(x)(R) : \mathbb{D}(\overline{Y}_R)(R) \rightarrow \mathbb{D}(X_{\tilde{z}})(R)$ preserves the Hodge filtration since $\mathbb{D}(x)(\overline{1}_0) = x_{\mathrm{crys},z}(R) \in P \subset \mathrm{Fil}^1\mathbb{D}(X_{\tilde{z}})(R)$. Hence x is an isogeny for all $x \in L$, i.e., $\tilde{z} \in \widehat{\mathcal{Z}(L)}_{z_0}(R)$. $\qquad\square$

3.2 The GSpin case

3.2.1 Crystals over $\mathcal{N}^o$

There is a rank m $\mathcal{O}^{\mathrm{crys}}_{\mathcal{N}^o}$-module crystal $\mathbf{V}_{\mathrm{crys}}$ in the site $\mathrm{NCRIS}_{\breve{\mathbb{Z}}_p}(\mathcal{N}^o/\mathrm{Spf}\,\breve{\mathbb{Z}}_p)$. The construction of $\mathbf{V}_{\mathrm{crys}}$ is based on the element π in (2). By [8, Theorem 4.9.1], we obtain from π a universal crystalline Tate tensor π_{crys} on the universal p-divisible group X^{univ} over $\mathcal{N}^o$, which induces a projector of crystals

$$\pi_{\mathrm{crys}} : \mathrm{End}(\mathbb{D}(X^{\mathrm{univ}})) \rightarrow \mathrm{End}(\mathbb{D}(X^{\mathrm{univ}})).$$

Let $\mathbf{V}_{\mathrm{crys}}$ be the image of this projector, it is a rank m $\mathcal{O}^{\mathrm{crys}}_{\mathcal{N}^o}$-module crystal ([14, §4.3]).

For any $S \in \mathrm{Alg}_{\breve{\mathbb{Z}}_p}$ and any $z \in \mathcal{N}^o(S)$, we similarly have a projector of crystals

$$\pi_{\mathrm{crys},z} : \mathrm{End}(\mathbb{D}(X_z)) \rightarrow \mathrm{End}(\mathbb{D}(X_z)).$$

whose image $\mathbf{V}_{\mathrm{crys},z} := \mathrm{im}(\pi_{\mathrm{crys},z})$ is a crystal of $\mathcal{O}^{\mathrm{crys}}_S$-modules of rank m. Here X_z denotes the p-divisible group over S obtained by the base change of X^{univ} to z.

Let $R \rightarrow S$ be a surjection in $\mathrm{Alg}_{\breve{\mathbb{Z}}_p}$ whose kernel admits nilpotent divided powers, we have $\mathbf{V}_{\mathrm{crys},z}(R)$ a projective R-module of rank m. It is equipped with a non-degenerate symmetric R-bilinear form $(\cdot,\cdot)$. The projective S-module $\mathbf{V}_{\mathrm{crys},z}(S)$ is equipped with a Hodge filtration $\mathrm{Fil}^1\mathbf{V}_{\mathrm{crys},z}(S)$, which is an isotropic S-line ([14, §4.3]). The S-line $\mathrm{Fil}^1\mathbf{V}_{\mathrm{crys},z}(S)$ is isotropic because it is contained in $\mathrm{Fil}^1\mathrm{End}(\mathbb{D}(X_z))$ which consists of endomorphisms mapping the Hodge filtration $\mathrm{Fil}^1\mathbb{D}(X_z)$ to 0, and $\mathbb{D}(X_z)$ to $\mathrm{Fil}^1\mathbb{D}(X_z)$. Therefore the composition $v \circ v = 0$ for every element $v \in \mathrm{Fil}^1\mathbf{V}_{\mathrm{crys},z}(S)$.

Moreover, let $L \subset \mathbb{V}^o$ be a subset, for $z \in \mathcal{Z}(L)(S)$ and an arbitrary element $x \in L$, the crystalline realization $x_{\mathrm{crys},z}(R) \in \mathrm{End}(\mathbb{D}(X_z))(R)$ lies in the image of $\pi_{\mathrm{crys},z}(R)$, hence it is an element in $\mathbf{V}_{\mathrm{crys},z}(R)$ (cf. [14, §4.6]).

Lemma 3.2.1 *Let $R \rightarrow S$ be a surjection in $\mathrm{Art}_{\breve{\mathbb{Z}}_p}$ whose kernel admits nilpotent divided powers.*

(i) Let $z_0 \in \mathcal{N}^o(\mathbb{F})$ and $\widehat{\mathcal{N}}_{z_0}^o$ be the completion of $\mathcal{N}^o$ at z_0. Let $z \in \widehat{\mathcal{N}}_{z_0}^o(S)$. Then there is a natural bijection

$$\left\{ Lifts\ \tilde{z} \in \widehat{\mathcal{N}}_{z_0}^o(R)\ of\ z. \right\} \xrightarrow{\sim} \left\{ \begin{array}{c} Isotropic\ R - lines\ in\ \mathbf{V}_{\mathrm{crys},z}(R) \\[1em] lifting\ Fil^{\,1}\mathbf{V}_{crys,z}(S). \end{array} \right\},$$

which maps a lift $\tilde{z} \in \widehat{\mathcal{N}}_{z_0}^o(R)$ to the Hodge filtration $\mathrm{Fil}^1\mathbf{V}_{\mathrm{crys},\tilde{z}}(R) \subset \mathbf{V}_{\mathrm{crys},\tilde{z}}(R) \simeq \mathbf{V}_{\mathrm{crys},z}(R)$.

(ii) Let $L \subset \mathbb{V}^o$ be a $\mathbb{Z}_p$-lattice of rank $r \geqslant 1$. Let $z_0 \in \mathcal{Z}(L)(\mathbb{F})$ and $\widehat{\mathcal{Z}(L)}_{z_0}$ be the completion of $\mathcal{Z}(L)$ at z_0, then there is a natural bijection compatible with the bijection in (i)

$$\left\{ Lifts\ \tilde{z} \in \widehat{\mathcal{Z}(L)}_{z_0}(R)\ of\ z. \right\}$$
$$\xrightarrow{\sim} \left\{ \begin{array}{c} Isotropic\ R - lines\ in\ \mathbf{V}_{crys,z}(R)\ lifting\ Fil^{\,1}\mathbf{V}_{\mathrm{crys},z}(S) \\ and\ orthogonal\ to\ x_{crys,z}(R)\ for\ any\ x \in L. \end{array} \right\}.$$

Proof This is [14, Lemma 4.6.2]. □

Especially, let $z \in \mathcal{N}^o(\mathbb{F})$ be a point. Taking successive surjection $\breve{\mathbb{Z}}_p/(p^{n+1}) \to \breve{\mathbb{Z}}_p/(p^n) \to \mathbb{F}$, we define $\mathbf{V}_z := \varprojlim_n \mathbf{V}_{\mathrm{crys},z}(\breve{\mathbb{Z}}_p/p^n)$, it is a free $\breve{\mathbb{Z}}_p$-module of rank m.

We define a map $i_{\mathrm{crys},z} : \mathbb{V}^o \to \mathbf{V}_z \otimes_{\breve{\mathbb{Z}}_p} \breve{\mathbb{Q}}_p$ in the following way: For an element $x \in \mathbb{V}^o$, there exists an integer k such that $p^k x$ lifts to an isogeny of X_z under the quasi-isogeny $\rho_z : \mathbb{X}_V \to X_z$. Let $x' = p^k x$, then $z \in \mathcal{Z}(x')(\mathbb{F})$. Let $x'_{\mathrm{crys},z} = \varprojlim_n x'_{\mathrm{crys},z}(\breve{\mathbb{Z}}_p/p^n) \in \mathbf{V}_z$, define

$$i_{\mathrm{crys},z}(x) = p^{-k} x'_{\mathrm{crys},z}. \tag{4}$$

The definition is independent of the integer k we choose. For any $x \in \mathbb{V}^o$, denote by $x_{\mathrm{crys},z}$ the element $i_{\mathrm{crys},z}(x)$. In this way we defined a map $i_{\mathrm{crys},z} : \mathbb{V}^o \to \mathbf{V}_z \otimes_{\breve{\mathbb{Z}}_p} \breve{\mathbb{Q}}_p$.

Lemma 3.2.2 *Let $z \in \mathcal{N}^o(\mathbb{F})$ be a point. Let $L \subset \mathbb{V}^o$ be a subset. Then $z \in \mathcal{Z}(L)(\mathbb{F})$ if and only if $i_{\mathrm{crys},z}(L) \subset \mathbf{V}_z$.*

Proof It follows from the construction the map $i_{\mathrm{crys},z}$ that the condition $z \in \mathcal{Z}(L)(\mathbb{F})$ implies $i_{\mathrm{crys},z}(L) \subset \mathbf{V}_z$. Now we prove the converse direction. Suppose that $i_{\mathrm{crys},z}(L) \subset \mathbf{V}_z$. Let $x \in L$. We have $x_{\mathrm{crys},z} \in \mathbf{V}_z \subset \mathrm{End}_{\breve{\mathbb{Z}}_p}(\mathbb{D}(X_z)_{\breve{\mathbb{Z}}_p})$ where $\mathbb{D}(X_z)_{\breve{\mathbb{Z}}_p}$ is the Dieudonne module of the p-divisible group X_z. Let $\rho_z : \mathbb{X}_V \to X_z$ be the framing quasi-isogeny. Then $x_{\mathrm{crys},z} = \mathbb{D}(\rho_z \circ x \circ \rho_z^{-1})$ by construction. By Dieudonne theory, the quasi-isogeny $\rho_z \circ x \circ \rho_z^{-1}$ is indeed an isogeny because the Dieudonne module $\mathbb{D}(X_z)$ is stable under the map $\mathbb{D}(\rho_z \circ x \circ \rho_z^{-1}) = x_{\mathrm{crys},z}$. Hence $z \in \mathcal{Z}(x)(\mathbb{F})$ for all $x \in L$. Therefore $z \in \mathcal{Z}(L)(\mathbb{F})$. □

Remark 3.2.3 The analogous statement in the unitary case is not true. Let $z \in \mathcal{N}^u(\mathbb{F})$ be a point. Let $L \subset \mathbb{V}^u$ be a subset. Then $z \in \mathcal{Z}(L)(\mathbb{F})$ implies that $i_{\text{crys},z}(L) \subset \mathbf{D}_z$ by the construction of the map $i_{\text{crys},z}$. However, the reverse direction is not true. We refer the reader to Example 5.3.4 (b). In that case, let $x' = \varpi^{-1}x$. We have $x'_{\text{crys},z} \in \mathbf{D}_z$ but $z \notin \mathcal{Z}(x')(\mathbb{F})$.

In the work of Li and Zhu [12], they defined a non-degenerate bilinear form $(\cdot, \cdot)_{\mathbf{D}_z}$ on the $\check{\mathbb{Q}}_p$-vector space $\mathbf{D}_z \otimes_{\check{\mathbb{Z}}_p} \check{\mathbb{Q}}_p$ ((2.2.0.3) in loc. cit., see also [10, §2.1]) in the following way:

$$(x, y)_{\mathbf{D}_z} := (\varpi \delta)^{-1} \langle x, Fy \rangle \quad \text{for } x, y \in \mathbf{D}_z \otimes_{\check{\mathbb{Z}}_p} \check{\mathbb{Q}}_p,$$

where δ is a fixed element in $\mathcal{O}_E^\times$ such that $\sigma(\delta) = -\delta$, and F is the Frobenius morphism on the Dieudonne module $\mathbb{D}(X_z) = \mathbf{D}_z \oplus \overline{\mathbf{D}}_z$. Let $\mathbf{D}_z^\vee$ be the dual lattice of $\mathbf{D}_z$ under the bilinear form $(\cdot, \cdot)_{\mathbf{D}_z}$. Notice that $\mathbf{D}_z \not\subseteq \mathbf{D}_z^\vee$ because there exists an element $x \in \mathbf{D}_z$ such that $(x, x)_{\mathbf{D}_z} \notin \check{\mathbb{Z}}_p$. In [12, Corollary 3.1.4], they proved that $z \in \mathcal{Z}(L)(\mathbb{F})$ if and only if $i_{\text{crys},z}(L) \subset \mathbf{D}_z^\vee$.

3.2.2 Strong divisibility

Let $z \in \mathcal{N}^o(\mathbb{F})$ be a point. By §3.2.1, the $\check{\mathbb{Z}}_p$-module $\mathbf{V}_z$ is a free $\check{\mathbb{Z}}_p$-module of rank m, with a nondegenerate $\check{\mathbb{Z}}_p$-bilinear pairing $(\cdot, \cdot)$. The $\check{\mathbb{Q}}_p$-vector space $\mathbf{V}_{z,\check{\mathbb{Q}}_p} := \mathbf{V}_z \otimes_{\check{\mathbb{Z}}_p} \check{\mathbb{Q}}_p$ is equipped with a Frobenius-linear action $\Phi_z : \mathbf{V}_{z,\check{\mathbb{Q}}_p} \to \mathbf{V}_{z,\check{\mathbb{Q}}_p}$ in the following way: let X_z be the base change of X^{univ} to z, then the rational Dieudonne module $\mathbb{D}(X_z)_{\check{\mathbb{Q}}_p} := \mathbb{D}(X_z)(\check{\mathbb{Z}}_p) \otimes_{\check{\mathbb{Z}}_p} \check{\mathbb{Q}}_p$ has a natural Frobenius-linear isomorphism $\mathbf{F}_z : \mathbb{D}(X_z)_{\check{\mathbb{Q}}_p} \to \mathbb{D}(X_z)_{\check{\mathbb{Q}}_p}$ induced by the relative Frobenius morphism on X_z. The morphism $\mathbf{F}_z$ induces a Frobenius-linear isomorphism $\mathbf{F}_z^\otimes$ on the tensor algebra $\mathbb{D}(X_z)_{\check{\mathbb{Q}}_p}^\otimes$. In particular, let $\mathbf{F}_z^{(2,2)}$ be its restriction on $\mathbb{D}(X_z)_{\check{\mathbb{Q}}_p}^{\otimes(2,2)} \simeq \text{End}_{\check{\mathbb{Q}}_p}\left(\text{End}_{\check{\mathbb{Q}}_p}(\mathbb{D}(X_z)_{\check{\mathbb{Q}}_p})\right)$. We have

$$\mathbf{F}_z^{(2,2)} : \text{End}_{\check{\mathbb{Q}}_p}(\mathbb{D}(X_z)_{\check{\mathbb{Q}}_p}) \to \text{End}_{\check{\mathbb{Q}}_p}(\mathbb{D}(X_z)_{\check{\mathbb{Q}}_p}), \quad f \mapsto \mathbf{F}_z \circ f \circ \mathbf{F}_z^{-1}.$$

By the definition of $\mathcal{N}^o$ in §2.1.2, the crystalline Tate tensor $\pi_{\text{crys},z}$ is Frobenius invariant. Hence $\mathbf{F}_z^{(2,2)}$ acts on $\mathbf{V}_{z,\check{\mathbb{Q}}_p} = \text{im}\, \pi_{\text{crys},z}(\text{End}_{\check{\mathbb{Q}}_p}(\mathbb{D}(X_z)_{\check{\mathbb{Q}}_p}))$ as a Frobenius-linear isomorphism. We use Φ_z to denote this action.

Let $\tilde{z} \in \mathcal{N}^o(\check{\mathbb{Z}}_p)$ be a lift of z to $\check{\mathbb{Z}}_p$. By discussion in §3.2.1, the point $\tilde{z}$ induces an isotropic line $L^1 = \varprojlim_n \text{Fil}^1 \mathbf{V}_{\text{crys},z}(\check{\mathbb{Z}}_p/p^n) \subset \mathbf{V}_z$. Let $L^0 := (L^1)^\perp \subset \mathbf{V}_z$, we have a filtration $0 \subset L^1 \subset L^0 \subset \mathbf{V}_z$, the following lemma proves that this filtration is strongly divisible in the sense of [11, §3.1] (see also [15, §4.8]).

Lemma 3.2.4 *The Frobenius-linear map Φ_z on $\mathbf{V}_{z,\check{\mathbb{Q}}_p}$ induces the following isomorphism of $\check{\mathbb{Z}}_p$-lattices,*

$$\Phi_z : p^{-1}L^1 + L^0 + p\mathbf{V}_z \xrightarrow{\sim} \mathbf{V}_z$$

Proof The proof is basically the same as that of [15, §4.8]. Here we give a self-contained argument without referring to results of [11, §4.2]. For simplicity, we use H to denote the Dieudonne module $\mathbb{D}(X_z)(\breve{\mathbb{Z}}_p) \simeq \mathbb{D}(X_{\bar{z}})(\breve{\mathbb{Z}}_p)$. Let $H_{\mathbb{F}} = H \otimes_{\breve{\mathbb{Z}}_p} \mathbb{F}$. Let $\mathcal{V}_z = p \cdot \mathbf{F}_z^{-1}$ be the Verschiebung morphism on H. Then the supersinglarity of X_z implies that $\mathbf{F}_z^2(H) = pH$, hence $\mathcal{V}_z(H) = \mathbf{F}_z(H)$.

Let $0 \subset \text{Fil}^1 H_{\mathbb{F}} \subset H_{\mathbb{F}}$ be the Hodge filtration given by X_z. It is well-known that this filtration is equal to $0 \subset \mathcal{V}_z(H)/pH \subset H_{\mathbb{F}}$ (see Oda's work [16, Corollary 5.11]). By Grothendieck–Messing theory, the Hodge filtration $0 \subset \text{Fil}^1 H \subset H$ determined by the p-divisible group $X_{\bar{z}}$ lifts the filtration $0 \subset \text{Fil}^1 H_{\mathbb{F}} \subset H_{\mathbb{F}}$. Therefore we have $\mathbf{F}_z(H) = \mathcal{V}_z(H) = \text{Fil}^1 H + pH$. Combining this with $\mathbf{F}_z^2(H) = pH$, we get

$$\mathbf{F}_z\left(p^{-1}\text{Fil}^1 H + H\right) = H. \tag{5}$$

We still use $\mathcal{V}_z$ to denote the base change morphism $\mathcal{V}_z \otimes_{\breve{\mathbb{Z}}_p} \breve{\mathbb{Q}}_p : H_{\breve{\mathbb{Q}}_p} \to H_{\breve{\mathbb{Q}}_p}$. It's a Frobenius-conjugate linear isomorphism. Then $\mathcal{V}_z^\vee : H_{\breve{\mathbb{Q}}_p}^\vee \to H_{\breve{\mathbb{Q}}_p}^\vee$, $f \mapsto f \circ \mathcal{V}_z$ is a Frobenius-linear isomorphism, we define $\mathbf{F}_z^\vee$ similarly. The filtration $0 \subset \text{Fil}^1 H \subset H$ induces a filtration on $H^\vee : 0 \to \text{Fil}^0 H^\vee \to H^\vee$ where $\text{Fil}^0 H^\vee \simeq (H/\text{Fil}^1 H)^\vee$. The equality (5) induces

$$\mathcal{V}_z^\vee\left(p^{-1}\text{Fil}^0 H^\vee + H^\vee\right) = H^\vee. \tag{6}$$

It's easy to check that the Frobenius-linear action $\mathbf{F}_z^{(2,2)}$ on $\text{End}_{\breve{\mathbb{Q}}_p}(H_{\breve{\mathbb{Q}}_p}) \simeq H_{\breve{\mathbb{Q}}_p} \otimes_{\breve{\mathbb{Q}}_p} H_{\breve{\mathbb{Q}}_p}^\vee$ we defined before equals to $p^{-1}\mathbf{F}_z \otimes \mathcal{V}_z^\vee$. Therefore (5) and (6) together imply that

$$\mathbf{F}_z^{(2,2)}\left(p^{-1}\text{Fil}^1\text{End}(H) + \text{Fil}^0\text{End}(H) + p\text{End}(H)\right) = \text{End}(H).$$

Applying the projector $\boldsymbol{\pi}_{\text{crys},z}$, notice that it is Frobenius invariant and respects the Hodge filtration, we get

$$\Phi_z\left(p^{-1}L^1 + L^0 + p\mathbf{V}_z\right) = \mathbf{V}_z.$$

$\square$

Lemma 3.2.5 *Let* $\nu_p : \breve{\mathbb{Q}}_p \to \mathbb{Z} \cup \{\infty\}$ *be the* p-adic valuation of $\breve{\mathbb{Q}}_p$. *Let* $\mathbf{V}$ *be a free* $\breve{\mathbb{Z}}_p$-module of finite rank equipped with a non-degenerate bilinear pairing $(\cdot, \cdot)$. *Let* $L^1 \subset \mathbf{V}$ *be an isotropic line, and* $L^0 = (L^1)^\perp \subset \mathbf{V}$. *Let* $l \in \mathbf{V}$ *be a generator of the isotropic line* L^1, *then*

$$L^0 + p\mathbf{V} = \{x \in \mathbf{V} : \nu_p((x, l)) \geqslant 1\}.$$

Proof By the definition of L^0, we have $\nu_p((x, l)) \geqslant 1$ for any $x \in L^0 + p\mathbf{V}$. Now if $x \in \mathbf{V}$ satisfies the condition that $\nu_p((x, l)) \geqslant 1$, there exists $x' \in \mathbf{V}$ such that $(l, x') = p^{-1}(l, x)$ since $l \notin p\mathbf{V}$ and $\mathbf{V}$ is a self-dual quadratic lattice. Therefore $x - p \cdot x' \in L^0$, hence $x \in L^0 + p\mathbf{V}$. $\square$

4 Formally smoothness locus of the special cycle

4.1 Formally smoothness over a discrete valuation ring

Lemma 4.1.1 *Let $n \geqslant 2$ be an integer. Let $R = \mathcal{O}[[t_1, \cdots, t_{n-1}]]$ where $\mathcal{O}$ is a discrete valuation ring of characteristic $(0, p)$ with uniformizer π and residue field $\mathbf{k}$. The ring R has maximal ideal $\mathfrak{m}_R := (\pi, t_1, \cdots, t_{n-1})$. Let $f_1, f_2, \cdots, f_r$ be r elements in $\mathfrak{m}_R$. The quotient ring $\overline{R} = R/(f_1, \cdots, f_r)$ is formally smooth over $\mathcal{O}$ of relative dimension $n - r - 1$ if and only if its base change $\overline{R}_\mathbf{k} = \overline{R} \otimes_\mathcal{O} \mathbf{k}$ is formally smooth over $\mathbf{k}$ of relative dimension $n - r - 1$.*

Proof Let $\mathbf{0} : R \to \mathcal{O}$ be the continuous homomorphism which sends all the t_j to 0. Let $J = (\frac{\partial f_i}{\partial t_j}(\mathbf{0}))_{\substack{1 \leqslant i \leqslant r \\ 1 \leqslant j \leqslant n-1}}$ be the Jacobian matrix of $f_1, f_2, \cdots, f_r$ at $\mathbf{0}$. The quotient ring $\overline{R}$ is formally smooth over $\mathcal{O}$ of relative dimension $n - r - 1$ if and only if the matrix J has a $r \times r$-minor which is invertible in $\mathcal{O}$, because in this case we can rearrange a system of parameters $t_1', \cdots, t_{n-1}'$ of R such that $t_i' = f_i$ for $1 \leqslant i \leqslant r$.

Let $J_\mathbf{k} = J \otimes_\mathcal{O} \mathbf{k}$ be the base change of J to $\mathbf{k}$, the matrix J has a $r \times r$-minor which is invertible in $\mathcal{O}$ is equivalent to the rank of $J_\mathbf{k}$ is r, which is further equivalent to $\overline{R}_\mathbf{k}$ is formally smooth over $\mathbf{k}$ of relative dimension $n - r - 1$, hence the quotient ring $\overline{R} = R/(f_1, \cdots, f_r)$ is formally smooth over $\mathcal{O}$ of relative dimension $n - r - 1$ if and only if $\overline{R}_\mathbf{k}$ is formally smooth over $\mathbf{k}$ of relative dimension $n - r - 1$. $\square$

4.2 The unitary case

For a point $z \in \mathcal{N}^u(\mathbb{F})$, let $\widehat{\mathcal{N}^u_z}$ be the completion of the formal scheme $\mathcal{N}^u$ at z. Recall that the Dieudonne module $\mathbb{D}(X_z)$ of X_z has a decomposition $\mathbb{D}(X_z) = \mathbf{D}_z \oplus \overline{\mathbf{D}}_z$ as in §3.1.2. Both $\mathbf{D}_z$ and $\overline{\mathbf{D}}_z$ are free $\mathcal{O}_{\breve{F}}$-modules of rank n. Let $L \subset \mathbb{V}^u$ be an $\mathcal{O}_E$-lattice. For a point $z \in \mathcal{Z}(L)(\mathbb{F})$, we defined in (3) an $\mathcal{O}_F$-linear map $i_{\mathrm{crys},z} : L \to \mathbf{D}_z$. We say the map $i_{\mathrm{crys},z}$ is primitive at z if the induced map $\overline{i_{\mathrm{crys},z}} : L/\varpi L \to \mathbf{D}_z/\varpi \mathbf{D}_z$ is injective.

Lemma 4.2.1 *Let $L \subset \mathbb{V}^u$ be an $\mathcal{O}_E$-lattice of rank $1 \leqslant r \leqslant n - 1$ and $z \in \mathcal{Z}(L)(\mathbb{F})$, then $\mathcal{Z}(L)$ is formally smooth over $\mathcal{O}_{\breve{F}}$ of relative dimension $n - r - 1$ at z if and only if the map $i_{\mathrm{crys},z}$ is primitive at z.*

Proof The special cycle $\mathcal{Z}(L)$ is cut out by r equations in the deformation space $\widehat{\mathcal{N}^u_z} \simeq \mathrm{Spf}\, \mathcal{O}_{\mathcal{N}^u,z}$ by Proposition 2.3.3. Therefore the special cycle $\mathcal{Z}(L)$ is formally smooth over $\mathcal{O}_{\breve{F}}$ of relative dimension $n - r - 1$ at z if and only if $\mathcal{Z}(L)_\mathbb{F} := \mathcal{Z}(L) \times_{\mathcal{O}_{\breve{F}}} \mathbb{F}$ is formally smooth over $\mathbb{F}$ of relative dimension $n - r - 1$ at z by Lemma 4.1.1. Let $\mathcal{N}^u_\mathbb{F} = \mathcal{N}^u \times_{\mathcal{O}_{\breve{F}}} \mathbb{F}$. Let $l \in \overline{\mathbf{D}}_z$ be an element such that its image $\overline{l}$ in $\overline{\mathbf{D}}_z/\varpi \overline{\mathbf{D}}_z \simeq \overline{\mathbf{D}}_z(\mathbb{F})$ is a generator of the line $\mathrm{Fil}^1\mathbb{D}(X_z)(\mathbb{F}) \cap \overline{\mathbf{D}}_z(\mathbb{F})$. Let $\mathbb{F}[\epsilon] = \mathbb{F}[X]/X^2$.

By Schlessinger's criterion in [21, Theorem 2.11], the tangent space $\mathrm{Tgt}_z(\mathcal{N}^u_\mathbb{F})$ of $\mathcal{N}^u_\mathbb{F}$ at z can be identified with the set $\widehat{\mathcal{N}^u_z}(\mathbb{F}[\epsilon])$. This set has a natural bijection to the set of lines in $\overline{\mathbf{D}}_z(\mathbb{F}) \otimes_\mathbb{F} \mathbb{F}[\epsilon]$ which lifts the line $\mathbb{F} \cdot \overline{l}$ in $\overline{\mathbf{D}}_z(\mathbb{F})$ by Lemma 3.1.5. Therefore any such line has a generator of the form $\overline{l} + \epsilon \cdot w$ for some element $w \in \overline{\mathbf{D}}_z(\mathbb{F})$. Two

elements $\bar{l} + \epsilon \cdot w$ and $\bar{l} + \epsilon \cdot w'$ generate the same line if and only if $w - w' \in \mathbb{F} \cdot \bar{l}$, hence $\mathrm{Tgt}_z(\mathcal{N}_{\mathbb{F}}^u) \simeq \overline{\mathbf{D}}_z(\mathbb{F})/\mathbb{F} \cdot \bar{l}$, and $\dim_{\mathbb{F}} \mathrm{Tgt}_z(\mathcal{N}_{\mathbb{F}}^u) = n - 1$.

The tangent space $\mathrm{Tgt}_z(\mathcal{Z}(L)_{\mathbb{F}})$ of $\mathcal{Z}(L)_{\mathbb{F}}$ at z can be identified with the set $\widehat{\mathcal{Z}(L)}_z(\mathbb{F}[\epsilon])$. For all $x \in L$, let $\overline{x_{\mathrm{crys},z}}$ be the image of $x_{\mathrm{crys},z}$ in $\mathbf{D}_z(\mathbb{F}) \simeq \mathbf{D}_z/\varpi \mathbf{D}_z$. The set $\widehat{\mathcal{Z}(L)}_z(\mathbb{F}[\epsilon])$ has a natural bijection to the set of lines in $\overline{\mathbf{D}}_z(\mathbb{F}) \otimes_W \mathbb{F}[\epsilon]$ which lifts the line $\mathbb{F} \cdot \bar{l}$ in $\overline{\mathbf{D}}_z(\mathbb{F})$ and orthogonal to $\overline{x_{\mathrm{crys},z}}$ for all $x \in L$. Let $\bar{l} + \epsilon \cdot w$ be a generator of a line in the set $\widehat{\mathcal{Z}(L)}_z(\mathbb{F}[\epsilon])$, then $\langle \overline{x_{\mathrm{crys},z}}, \bar{l} + \epsilon \cdot w \rangle = \epsilon \langle \overline{x_{\mathrm{crys},z}}, w \rangle = 0$, hence $\langle \overline{x_{\mathrm{crys},z}}, w \rangle = 0$. Therefore $\mathrm{Tgt}_z(\mathcal{Z}(L)_{\mathbb{F}})$ is the subspace of $\overline{\mathbf{D}}_{\mathbb{F}}/\mathbb{F} \cdot \bar{l}$ orthogonal to the image of L in $\mathbf{D}_z(\mathbb{F})$ under the perfect pairing $\langle \cdot, \cdot \rangle : \mathbf{D}_z(\mathbb{F}) \times \overline{\mathbf{D}}_z(\mathbb{F}) \to \mathbb{F}$. Let $\overline{L} \subset \mathbf{D}_z/\varpi \mathbf{D}_z$ be the $\mathbb{F}$-linear subspace spanned by the image of L in $\mathbf{D}_z(\mathbb{F})$, we have

$$\dim_{\mathbb{F}} \mathrm{Tgt}_z(\mathcal{Z}(L)_{\mathbb{F}}) = n - 1 - \dim_{\mathbb{F}}(\overline{L}). \tag{7}$$

therefore $\mathcal{Z}(L)$ is formally smooth over $\mathcal{O}_{\breve{F}}$ of relative dimension $n - r - 1$ at z if and only if $\dim_{\mathbb{F}}(\overline{L}) = r$, which is equivalent to the fact that the map $\overline{i_{\mathrm{crys},z}} : L/\varpi L \to \mathbf{D}_z/\varpi \mathbf{D}_z$ is injective, i.e., the map $i_{\mathrm{crys},z}$ is primitive. $\qquad \square$

4.3 The GSpin case

For a point $z \in \mathcal{N}^o(\mathbb{F})$, let $\widehat{\mathcal{N}}_z^o$ be the completion of the formal scheme $\mathcal{N}^o$ at z. Let $\mathbf{V}_z = \mathbf{V}_{\mathrm{crys},z}$ be the free $\breve{\mathbb{Z}}_p$-module of rank m defined in §3.2.1. Let $L \subset \mathbb{V}^o$ be an $\mathbb{Z}_p$-lattice. For a point $z \in \mathcal{Z}(L)(\mathbb{F})$, we defined in (4) a $\mathbb{Z}_p$-linear map $i_{\mathrm{crys},z} : L \to \mathbf{V}_z$. We say the map $i_{\mathrm{crys},z}$ is primitive at z if the induced map $\overline{i_{\mathrm{crys},z}} : L/pL \to \mathbf{V}_z/p\mathbf{V}_z$ is injective.

Lemma 4.3.1 *Let $L \subset \mathbb{V}^o$ be a $\mathbb{Z}_p$-lattice of rank $r \geqslant 1$ and $z \in \mathcal{Z}(L)(\mathbb{F})$, then $\mathcal{Z}(L)$ is formally smooth over $\breve{\mathbb{Z}}_p$ of relative dimension $n - r - 1$ at z if and only if the following two assertions hold,*

- *(i) The map $i_{crys,z}$ is primitive at z;*
- *(ii) There exists a lift of z to $z' \in \mathcal{Z}(L)(\breve{\mathbb{Z}}_p/p^2)$.*

Proof The special cycle $\mathcal{Z}(L)$ is cut out by r equations in the deformation space $\widehat{\mathcal{N}}_z^o \simeq \mathrm{Spf}\, \mathcal{O}_{\mathcal{N},z}$ by Proposition 2.3.3. Therefore the special cycle $\mathcal{Z}(L)$ is formally smooth over $\breve{\mathbb{Z}}_p$ of relative dimension $n - r - 1$ at z if and only if $\mathcal{Z}(L)_{\mathbb{F}} := \mathcal{Z}(L) \times_{\breve{\mathbb{Z}}_p} \mathbb{F}$ is formally smooth over $\mathbb{F}$ of relative dimension $n - r - 1$ at z by Lemma 4.1.1. Let $\mathcal{N}_{\mathbb{F}}^o = \mathcal{N}^o \times_{\breve{\mathbb{Z}}_p} \mathbb{F}$. Let $l \in \mathbf{V}_z$ be an isotropic element such that its image $\bar{l} \in \mathbf{V}_z/p\mathbf{V}_z \simeq \mathbf{V}_{\mathrm{crys},z}(\mathbb{F})$ generates the line $\mathrm{Fil}^1 \mathbf{V}_{\mathrm{crys},z}(\mathbb{F})$. Let $(\cdot, \cdot)$ be the bilinear form on $\mathbf{V}_{\mathrm{crys},z}(\mathbb{F})$. Let $\mathbb{F}[\epsilon] = \mathbb{F}[X]/X^2$.

By Schlessinger's criterion in [21, Theorem 2.11], the tangent space $\mathrm{Tgt}_z(\mathcal{N}_{\mathbb{F}}^o)$ (resp. $\mathrm{Tgt}_z(\mathcal{Z}(L)_{\mathbb{F}})$) of $\mathcal{N}_{\mathbb{F}}^o$ (resp. $\mathcal{Z}(L)_{\mathbb{F}}$) at z can be identified with the set $\widehat{\mathcal{N}}_z^o(\mathbb{F}[\epsilon])$. This set has a natural bijection to the set of isotropic lines in $\mathbf{V}_{\mathrm{crys}}(\mathbb{F}[\epsilon]) \simeq \mathbf{V}_{\mathrm{crys},z}(\mathbb{F}) \otimes_{\mathbb{F}} \mathbb{F}[\epsilon]$ which lifts the line $\mathbb{F} \cdot \bar{l}$ in $\mathbf{V}_{\mathrm{crys},z}(\mathbb{F})$ by Lemma 3.2.1. Therefore any such line has a

generator of the form $\bar{l} + \epsilon \cdot w$ for some $w \in \mathbf{V}_{\mathrm{crys},z}(\mathbb{F})$, note that

$$0 = (\bar{l} + \epsilon \cdot w, \bar{l} + \epsilon \cdot w) = (\bar{l}, \bar{l}) + 2\epsilon(\bar{l}, w) = 2\epsilon(\bar{l}, w),$$

hence the fact that the line generated by $\bar{l} + \epsilon \cdot w$ is isotropic is equivalent to the vector $w \in \mathbf{V}_{\mathrm{crys},z}(\mathbb{F})$ is orthogonal to $\bar{l}$. Two elements $\bar{l} + \epsilon \cdot w$ and $\bar{l} + \epsilon \cdot w'$ generate the same line if and only if $w' - w \in \mathbb{F} \cdot \bar{l}$, hence $\mathrm{Tgt}_z(\mathcal{N}_{\mathbb{F}}^o) \simeq \{\bar{l}\}^{\perp}/\mathbb{F} \cdot \bar{l}$, and $\dim_{\mathbb{F}} \mathrm{Tgt}_z(\mathcal{N}_{\mathbb{F}}^o) = n - 1$. There is a bilinear pairing on the space $\{\bar{l}\}^{\perp}/\mathbb{F} \cdot \bar{l}$ induced by $(\cdot, \cdot)$, we denote it by $\overline{(\cdot, \cdot)}$. It is also non-degenerate since because the pairing $(\cdot, \cdot)$ is non-degenerate.

The tangent space $\mathrm{Tgt}_z(\mathcal{Z}(L)_{\mathbb{F}})$ of $\mathcal{Z}(L)_{\mathbb{F}}$ at z can be identified with the set $\widehat{\mathcal{Z}(L)}_z(\mathbb{F}[\epsilon])$. For all $x \in L$, let $\overline{x_{\mathrm{crys},z}}$ be the image of $x_{\mathrm{crys},z}$ in $\mathbf{V}_{\mathrm{crys},z}(\mathbb{F})$. The set $\widehat{\mathcal{Z}(L)}_z(\mathbb{F}[\epsilon])$ has a natural bijection to the set of isotropic lines in $\mathbf{V}_{\mathrm{crys},z}(\mathbb{F}) \otimes_{\mathbb{F}} \mathbb{F}[\epsilon]$ which lifts the line $\mathbb{F} \cdot \bar{l}$ in $\mathbf{V}_{\mathrm{crys},z}(\mathbb{F})$ and orthogonal to $\overline{x_{\mathrm{crys},z}}$ for any $x \in L$. Let $\bar{l} + \epsilon \cdot w$ be a generator of a line in the set $\widehat{\mathcal{Z}(L)}_z(\mathbb{F}[\epsilon])$, then $(\overline{x_{\mathrm{crys},z}}, \bar{l} + \epsilon \cdot w) = \epsilon(\overline{x_{\mathrm{crys},z}}, w) = 0$, hence $(\overline{x_{\mathrm{crys},z}}, w) = 0$. Therefore $\mathrm{Tgt}_z(\mathcal{Z}(L)_{\mathbb{F}})$ is the subspace of $\{\bar{l}\}^{\perp}/\mathbb{F} \cdot \bar{l}$ orthogonal to the image of L in $\mathbf{V}_{\mathrm{crys},z}(\mathbb{F})$ under the pairing $\overline{(\cdot, \cdot)}$. Let $\overline{L} \subset \mathbf{V}_z/p\mathbf{V}_z$ be the $\mathbb{F}$-linear subspace spanned by the image of L in $\mathbf{V}_{\mathrm{crys},z}(\mathbb{F})$, it is contained in the subspace $\{\bar{l}\}^{\perp}$ of $\mathbf{V}_{\mathrm{crys},z}(\mathbb{F})$, we have

$$\mathrm{Tgt}_z(\mathcal{Z}(L)_{\mathbb{F}}) = \left((\overline{L} + \mathbb{F} \cdot \bar{l})/\mathbb{F} \cdot \bar{l} \right)^{\perp} \subset \{\bar{l}\}^{\perp}/\mathbb{F} \cdot \bar{l}.$$

Therefore $\dim_{\mathbb{F}} \mathrm{Tgt}_z(\mathcal{Z}(L)_{\mathbb{F}}) = n - 1 - \dim_{\mathbb{F}} \left((\overline{L} + \mathbb{F} \cdot \bar{l})/\mathbb{F} \cdot \bar{l} \right)$ since the pairing $\overline{(\cdot, \cdot)}_{\mathbb{F}}$ is non-degenerate on $\{\bar{l}\}^{\perp}/\mathbb{F} \cdot \bar{l}$. Therefore $\mathcal{Z}(L)$ is formally smooth over $\check{\mathbb{Z}}_p$ of relative dimension $n - r - 1$ at z if and only if $\dim_{\mathbb{F}} \left((\overline{L} + \mathbb{F} \cdot \bar{l})/\mathbb{F} \cdot \bar{l} \right) = r$, which is further equivalent to $\dim_{\mathbb{F}} \overline{L} = r$ and $\bar{l} \notin \overline{L}$. The condition $\dim_{\mathbb{F}} \overline{L} = r$ is equivalent to (i) which says that the map $i_{\mathrm{crys},z}$ is primitive at z.

For one direction, if the formal scheme $\mathcal{Z}(L)$ is formally smooth over $\check{\mathbb{Z}}_p$ of relative dimension $n - r - 1$ at z, clearly $i_{\mathrm{crys},z}$ is primitive by the above discussion. The formally smoothness of $\mathcal{Z}(L)$ over $\check{\mathbb{Z}}_p$ also implies that an $\mathbb{F}$-point $z \in \mathcal{Z}(\mathbb{F})$ can be lifted to a point $z' \in \mathcal{Z}(L)(\check{\mathbb{Z}}_p/p^2)$. Hence (i) and (ii) are true.

Let's consider the other direction. Assume that (i) and (ii) are true, we want to show that the element $\bar{l}$ doesn't belong to the space $\overline{L}$. Let $z' \in \mathcal{Z}(L)(\check{\mathbb{Z}}_p/p^2)$ be a lift of z to $\check{\mathbb{Z}}_p/\varpi^2$. By the formally smoothness of $\mathcal{N}^o$ at z, there exists a point $\tilde{z} \in \widehat{\mathcal{N}}_z^o(\check{\mathbb{Z}}_p)$ lifting the point $z' \in \widehat{\mathcal{N}}_z^o(\check{\mathbb{Z}}_p/p^2)$. The point $\tilde{z}$ corresponds to an isotropic line L^1. Let $l \in \mathbf{V}_z$ be a generator of L^1 whose image in $\mathbf{V}_{\mathrm{crys},z}(\mathbb{F})$ is $\bar{l}$. Let $\{x_i\}_{i=1}^r$ be an $\mathbb{Z}_p$-basis of L. If $\bar{l}$ belongs to the space $\overline{L}$, there exist $a_i \in \check{\mathbb{Z}}_p$ for $1 \leqslant i \leqslant r$ and $v \in \mathbf{V}_z$ such that

$$l = \sum_{i=1}^r a_i x_{i,\mathrm{crys},z} + p \cdot v. \tag{8}$$

Notice that $\bar{l} = \sum_{i=1}^r \overline{a_i} \cdot \overline{x_{i,\mathrm{crys},z}} \neq 0$, hence there exists at least one i such that $a_i \in \check{\mathbb{Z}}_p^{\times}$ and $x_{i,\mathrm{crys},z} \notin p\mathbf{V}_z$. Let Φ_z be the Frobenius action on $\mathbf{V}_z$. If $v \in L^0 + p\mathbf{V}_z$,

then $\Phi_z(v) \in \mathbf{V}_z$ by Lemma 3.2.4 The same lemma also implies that $\Phi_z(l) \in p\mathbf{V}_z$. Therefore $\Phi_z(\sum_{i=1}^r a_i x_{i,\mathrm{crys},z}) = \Phi_z(l) - p\Phi_z(v) \in p\mathbf{V}_z$. However,

$$\Phi_z\left(\sum_{i=1}^r a_i x_{i,\mathrm{crys},z}\right) = \sum_{i=1}^r \sigma(a_i) \cdot \Phi_z(x_{i,\mathrm{crys},z}) = \sum_{i=1}^r \sigma(a_i) \cdot x_{i,\mathrm{crys},z},$$

hence $\Phi_z(\sum_{i=1}^r a_i x_{i,\mathrm{crys},z}) \notin p\mathbf{V}_z$ because there exists an integer $1 \leqslant i \leqslant r$ such that $\sigma(a_i) \in \check{\mathbb{Z}}_p^\times$ and $x_{i,\mathrm{crys},z} \notin p\mathbf{V}_z$, hence $v \notin L^0 + p\mathbf{V}_z$. Therefore $(l, v) \in \check{\mathbb{Z}}_p^\times$ by Lemma 3.2.5.

Recall that v_p is the p-adic valuation on $\check{\mathbb{Q}}_p$. We claim that $v_p((l, x_{i,\mathrm{crys},z})) \geqslant 2$ for all $1 \leqslant i \leqslant r$. Since the point $\tilde{z} \in \mathcal{N}^o(\check{\mathbb{Z}}_p)$ is a lift of z', the image of l in $\mathbf{V}_{\mathrm{crys},z}(\check{\mathbb{Z}}_p/p^2) \simeq \mathbf{V}_z/p^2\mathbf{V}_z$ generates the line $\mathrm{Fil}^1\mathbf{V}_{\mathrm{crys},z}(\check{\mathbb{Z}}_p/p^2)$ corresponding to z'. By definition, the image of $x_{i,\mathrm{crys},z} \in \mathbf{V}_z$ in $\mathbf{V}_{\mathrm{crys},z}(\check{\mathbb{Z}}_p/p^2)$ is $x_{i,\mathrm{crys},z}(\check{\mathbb{Z}}_p/p^2)$. By Lemma 3.2.1, the line $\mathrm{Fil}^1\mathbf{V}_{\mathrm{crys},z}(\check{\mathbb{Z}}_p/p^2)$ is orthogonal to $x_{\mathrm{crys},z}(\check{\mathbb{Z}}_p/p^2)$, which is equivalent to $(l, x_{i,\mathrm{crys},z}) = 0 \in \check{\mathbb{Z}}_p/(p^2)$, i.e., $v_p((l, x_{i,\mathrm{crys},z})) \geqslant 2$ for all $1 \leqslant i \leqslant r$. Therefore $v_p(\sum_{i=1}^r a_i(l, x_{i,\mathrm{crys},z})) \geqslant 2$. However, the pairing $(l, \sum_{i=1}^r a_i x_{i,\mathrm{crys},z}) = (l, l - p \cdot v) = -p(l, v)$ by (8), hence $v_p((l, \sum_{i=1}^r a_i x_{i,\mathrm{crys},z})) = 1$ since $(l, v) \in \check{\mathbb{Z}}_p^\times$ by previous discussion, this is a contradiction. Therefore the element $\bar{l}$ doesn't belong to the space $\overline{L}$ if (i) and (ii) are true, hence the formal scheme $\mathcal{Z}(L)$ is formally smooth over $\check{\mathbb{Z}}_p$ of relative dimension $n - r - 1$ at z. □

Remark 4.3.2 In the above proof, we only use the inclusion part of the strong divisibility Lemma 3.2.4, i.e., $\Phi_z\left(p^{-1}L^1 + L^0 + p\mathbf{V}_z\right) \subset \mathbf{V}_z$.

5 Regularity of the difference divisor

5.1 Commutative algebra preparation

Lemma 5.1.1 Let $n \geqslant 2$ be an integer. Let $R = \mathcal{O}[[t_1, \cdots, t_{n-1}]]$ where $\mathcal{O}$ is a discrete valuation ring of characteristic $(0, p)$ with uniformizer π and π-adic valuation v_π. The ring R has maximal ideal $\mathfrak{m}_R := (\pi, t_1, \cdots, t_{n-1})$. In the following the symbol (unit) means an element in $R^\times$.

- Let $g \in \mathfrak{m}_R$. If for any ring homomorphism $f : R \to \mathcal{O}$, we have $v_\pi(f(g)) = 1$, then $g \equiv (\mathrm{unit}) \cdot \pi \pmod{\mathfrak{m}_R^2}$.
- Let $d \in \mathfrak{m}_R \backslash \mathfrak{m}_R^2$. Let h be an element in $\mathfrak{m}_R$ such that $h \not\equiv (\mathrm{unit}) \cdot \pi \pmod{\mathfrak{m}_R^2}$. Then there exists a ring homomorphism $f : R \to \mathcal{O}$ such that $f(d) \neq 0$ and $v_\pi(f(h)) \geqslant 2$.

Proof Notice that any ring homomorphism $f : R \to \mathcal{O}$ maps t_i into the maximal ideal $(\pi) \subset \mathcal{O}$. For the first assertion, let $g \equiv a_0\pi + \sum_{j=1}^{n-1} a_j t_j \pmod{\mathfrak{m}_R^2}$ where $a_j \in \mathcal{O}$ for every $0 \leqslant j \leqslant n-1$. If $v_\pi(a_0) \geqslant 1$, consider the continuous homomorphism $f : R \to \mathcal{O}$ such that $f(t_j) = \pi^2$ for every $0 \leqslant j \leqslant n-1$, then $v_\pi(f(g)) \geqslant 2$, which is a contradiction. Therefore $v_\pi(a_0) = 0$. If $g \not\equiv (\mathrm{unit}) \cdot \pi \pmod{\mathfrak{m}_R^2}$, then there exists

at least one $j \geqslant 1$ such that $v_\pi(a_j) = 0$. Suppose that $v_\pi(a_{j_0}) = 0$ for some $j_0 \geqslant 1$, consider the continuous homomorphism $f : R \to \mathcal{O}$ such that $f(t_{j_0}) = -a_{j_0}^{-1}a_0\pi$ and $f(t_j) = \pi^2$ for $j \neq j_0$, then $v_\pi(f(g)) \geqslant 2$, which is a contradiction, hence $g \equiv (\text{unit}) \cdot \pi \pmod{\mathfrak{m}_R^2}$.

For the second assertion, we consider the following two cases.

Case 1. $d \equiv (\text{unit}) \cdot \pi \pmod{\mathfrak{m}_R^2}$. Since $h \not\equiv (\text{unit}) \cdot \pi \pmod{\mathfrak{m}_R^2}$, there exists a continuous ring homomorphism $f : R \to \mathcal{O}$ such that $v_\pi(f(h)) \geqslant 2$ by the proof of the first assertion, for this f, we have $v_\pi(f(d)) = 1$, hence $f(d) \neq 0$.

Case 2. $d \not\equiv (\text{unit}) \cdot \pi \pmod{\mathfrak{m}_R^2}$, then we can choose another system of uniformizers $t_1, \cdots, t_{n-1}$ such that $d = t_1$. Let $h \equiv b_0\pi + \sum_{j=1}^{n-1} b_j t_j \pmod{\mathfrak{m}_R^2}$ where $b_j \in \mathcal{O}$ for every $0 \leqslant j \leqslant n-1$. If $v_\pi(b_0) \geqslant 1$, consider the continuous homomorphism $f : R \to \mathcal{O}$ such that $f(t_j) = \pi^2$ for every $0 \leqslant j \leqslant n-1$, then $v_\pi(f(h)) \geqslant 2$ and $f(d) = f(t_1) \neq 0$. If $v_\pi(b_0) = 0$, then there exists at least one $j \geqslant 1$ such that $v_\pi(b_j) = 0$ since $h \not\equiv (\text{unit}) \cdot \pi \pmod{\mathfrak{m}_R^2}$. Suppose that $v_\pi(b_{j_0}) = 0$ for some $j_0 \geqslant 1$, consider the continuous homomorphism $f : R \to \mathcal{O}$ such that $f(t_{j_0}) = -b_{j_0}^{-1}b_0\pi$ and $f(t_j) = \pi^2$ for $j \neq j_0$, then $v_\pi(f(g)) \geqslant 2$ and $f(d) = f(t_1) \neq 0$. $\qed$

Lemma 5.1.2 *Let $n \geqslant 2$ be an integer. Let $R = \mathcal{O}[[t_1, \cdots, t_{n-1}]]$ where $\mathcal{O}$ is a discrete valuation ring of characteristic $(0, p)$ with uniformizer π and π-adic valuation v_π. Let $(d_a)_{a \geqslant 0}$ be a sequence of elements in R such that $d_a \in \mathfrak{m}_R := (\pi, t_1, \cdots, t_{n-1})$ for any $a \geqslant 0$ and $d_0 \in \mathfrak{m}_R \backslash \mathfrak{m}_R^2$. Let $f_a = \prod_{i=0}^a d_i$ and $\mathcal{Z}(f_a) := \mathrm{Spf}\, R/f_a$ be the closed formal subscheme of $\mathrm{Spf}\, R$. For any morphism $z : \mathrm{Spf}\, \mathcal{O} \to \mathrm{Spf}\, R$, we use $z^\sharp$ to denote the corresponding ring homomorphism $R \to \mathcal{O}$. If for all $a \geqslant 1$ and all morphisms $z : \mathrm{Spf}\, \mathcal{O} \to \mathrm{Spf}\, R$, there exists a Cartesian diagram*

$$
\begin{array}{ccc}
\mathrm{Spec}\ \mathcal{O}/(\pi^a \cdot z^\sharp(d_0)) & \longrightarrow & \mathrm{Spf}\ \mathcal{O} \\
\downarrow & & \downarrow{\scriptstyle z} \\
\mathcal{Z}(f_a) & \longrightarrow & \mathrm{Spf}\ R.
\end{array}
$$

then $d_a \equiv (\text{unit}) \cdot \pi \pmod{\mathfrak{m}_R^2}$ for all $a \geqslant 1$.

Proof Let's assume the contrary that there exists $a \geqslant 1$ such that $d_a \not\equiv (\text{unit}) \cdot \pi \pmod{\mathfrak{m}_R^2}$. Let $k \geqslant 1$ be the least integer such that $d_i \equiv (\text{unit}) \cdot \pi \pmod{\mathfrak{m}_R^2}$ for all $1 \leqslant i < k$ and $d_k \not\equiv (\text{unit}) \cdot \pi \pmod{\mathfrak{m}_R^2}$, then by Lemma 5.1.1 there exists a morphism $z : \mathrm{Spf}\, \mathcal{O} \to \mathrm{Spf}\, R$ such that $z^\sharp(d_0) \neq 0$ and $v_\pi(z^\sharp(d_k)) \geqslant 2$, then $\mathcal{Z}(f_k) \times_{\mathrm{Spf}\, R, z} \mathrm{Spf}\, \mathcal{O}$ is cut out in $\mathrm{Spf}\, \mathcal{O}$ by (π^m) for some integer m such that $k + v_\pi(z^\sharp(d_0)) + 1 \leqslant m < \infty$, hence the fiber product $\mathcal{Z}(f_k) \times_{\mathrm{Spf}\, R, z} \mathrm{Spf}\, \mathcal{O}$ is not isomorphic to $\mathrm{Spec}\, \mathcal{O}/(\pi^k \cdot z^\sharp(d_0))$, which is a contradiction. Therefore such k doesn't exist, hence $d_a \equiv (\text{unit}) \cdot \pi \pmod{\mathfrak{m}_R^2}$ for all $a \geqslant 1$. $\qed$

5.2 Local equations of special divisors

In the following, we always fix a geometric point $z \in \mathcal{N}^u(\mathbb{F})$ (resp. $\mathcal{N}^o(\mathbb{F})$). Denote by $\widehat{\mathcal{N}}^u_z$ (resp. $\widehat{\mathcal{N}}^o_z$) the formal completion of the formal scheme $\mathcal{N}^u$ (resp. $\mathcal{N}^o$) at the point z. Denote by $\widehat{\mathcal{O}}_{\mathcal{N}^u,z}$ (resp. $\widehat{\mathcal{O}}_{\mathcal{N}^o,z}$) the complete local ring of the formal scheme $\mathcal{N}^u$ (resp. $\mathcal{N}^o$) at this point. Denote by $\mathbf{D}$ and $\overline{\mathbf{D}}$ (resp. $\mathbf{V}$) the $\mathcal{O}_{\breve{F}}$-module (resp. $\breve{\mathbb{Z}}_p$-module) $\mathbf{D}_z$ and $\overline{\mathbf{D}}_z$ (resp. $\mathbf{V}_z$) defined in §3.1.2 (resp. §3.2.1).

Proposition 5.2.1 *Let $x \in \mathbb{V}^u$ (resp. $x \in \mathbb{V}^o$) be a nonzero special quasi-homomorphism such that $(x, x) \in \mathcal{O}_E$ (resp. $q_{\mathbb{V}^o}(x) \in \mathbb{Z}_p$). Let $z \in \mathcal{N}^u(\mathbb{F})$ (resp. $z \in \mathcal{N}^o(\mathbb{F})$) be a point. Let $\tilde{z} \in \mathcal{N}^u(\mathcal{O}_{\breve{F}})$ (resp. $\mathcal{N}^o(\breve{\mathbb{Z}}_p)$) be a lift of z to $\mathcal{O}_{\breve{F}}$ and $\tilde{z}^{\sharp} : \widehat{\mathcal{O}}_{\mathcal{N}^u,z} \to \mathcal{O}_{\breve{F}}$ (resp. $\tilde{z}^{\sharp} : \widehat{\mathcal{O}}_{\mathcal{N}^o,z} \to \breve{\mathbb{Z}}_p$) be the corresponding ring homomorphism. Let $f_{x,z} \in \widehat{\mathcal{O}}_{\mathcal{N}^u,z}$ (resp. $\widehat{\mathcal{O}}_{\mathcal{N}^o,z}$) be the equation of the Cartier divisor $\mathcal{Z}(x)$ at z. Assume that $x_{\mathrm{crys},z} \in \mathbf{V}$ (resp. $x_{\mathrm{crys},z} \in \mathbf{D}$).*

- *In the unitary case, let $\langle \cdot, \cdot \rangle : \mathbf{D} \times \overline{\mathbf{D}} \to \mathcal{O}_{\breve{F}}$ be the perfect pairing. The lift $\tilde{z}$ corresponds to a line $L^1 \subset \overline{\mathbf{D}}$. Let l be a generator of L^1, then*

$$(\tilde{z}^{\sharp}(f_{x,z})) = (\langle x_{\mathrm{crys},z}, l \rangle) \subset \mathcal{O}_{\breve{F}}. \tag{9}$$

- *In the GSpin case, let $(\cdot, \cdot) : \mathbf{V} \times \mathbf{V} \to \breve{\mathbb{Z}}_p$ be the non-degenerate bilinear pairing on $\mathbf{V}$. The lift $\tilde{z}$ corresponds to an isotropic line $L^1 \subset \mathbf{V}$. Let l be a generator of L^1, then*

$$(\tilde{z}^{\sharp}(f_{x,z})) = \big((x_{\mathrm{crys},z}, l)\big) \subset \breve{\mathbb{Z}}_p. \tag{10}$$

Proof We first consider the unitary case. Let $\widehat{\mathcal{Z}(x)}_z$ be the completion of the special cycle $\mathcal{Z}(x)$ at z, there exists a Cartesian diagram,

$$
\begin{array}{ccc}
\mathrm{Spf}\, \mathcal{O}_{\breve{F}}/(\tilde{z}^{\sharp}(f_{x,z})) & \longrightarrow & \mathrm{Spf}\, \mathcal{O}_{\breve{F}} \\
\downarrow & & \downarrow{\scriptstyle \tilde{z}} \\
\widehat{\mathcal{Z}(x)}_z & \longrightarrow & \widehat{\mathcal{N}}^u_z.
\end{array}
$$

Let $m = v_{\varpi}(\tilde{z}^{\sharp}(f_{x,z})) \in \mathbb{Z}_{\geqslant 0} \cup \{\infty\}$, $t = v_{\varpi}(\langle x_{\mathrm{crys},z}, l \rangle) \in \mathbb{Z}_{\geqslant 0} \cup \{\infty\}$, the equality in (9) is equivalent to $t = m$.

If $m = 0$, we have $\mathrm{Spf}\, \mathcal{O}_{\breve{F}}/(\tilde{z}^{\sharp}(f_{x,z})) = \varnothing$, then $f_{x,z}$ is a unit in the ring $\widehat{\mathcal{O}}_{\mathcal{N}^u,z}$ $z \notin \mathcal{Z}(x)(\mathbb{F})$ because otherwise $\tilde{z}^{\sharp}(f_{x,z}) \in (\varpi) \subset \mathcal{O}_{\breve{F}}$. Hence $z \notin \mathcal{Z}(x)(\mathbb{F})$. We want to show that $t = 0$. Let's assume the converse, i.e., $t \geqslant 1$. Let $\mathbb{D}(x) : \mathbb{D}(\overline{\mathbb{Y}}) \otimes_{\mathcal{O}_{\breve{F}}} \breve{F} \to \mathbb{D}(X_z) \otimes_{\mathcal{O}_{\breve{F}}} \breve{F}$ be the map on the Dieudonne modules induced by the quasi-homomorphism x. Denote by Φ the Frobenius morphism on $\mathbb{D}(\overline{\mathbb{Y}})$ and $\mathbb{D}(X_z)$. Let $\bar{l} \in \overline{\mathbf{D}}/\varpi\overline{\mathbf{D}}$ be the image of l. Let $\overline{x_{\mathrm{crys},z}}$ be the image of $x_{\mathrm{crys},z}$ in $\mathbf{D}/\varpi\mathbf{D}$. The Hodge filtration of X_z is given by $0 \subset \mathbb{F} \cdot \bar{l} \oplus \{\bar{l}\}^{\perp} \subset \overline{\mathbf{D}}/\varpi\overline{\mathbf{D}} \oplus \mathbf{D}/\varpi\mathbf{D}$, where $\{\bar{l}\}^{\perp} \subset \mathbf{D}/\varpi\mathbf{D}$ is the kernel of the linear map $\langle \bar{l}, \cdot \rangle : \mathbf{D}/\varpi\mathbf{D} \to \mathbb{F}$. The condition $t \geqslant 1$ implies that $\overline{x_{\mathrm{crys},z}} \in \{\bar{l}\}^{\perp}$. Therefore $\Phi(x_{\mathrm{crys},z}) \in \varpi\mathbb{D}(X_z)$. Then $\mathbb{D}(x)(\bar{1}_1) =$

$\varpi^{-1}\mathbb{D}(x)(\Phi\bar{1}_0) = \varpi^{-1}\Phi x_{\mathrm{crys},z} \in \mathbb{D}(X_z)$. Therefore $\mathbb{D}(x)$ maps $\mathbb{D}(\overline{\mathbb{Y}})$ to $\mathbb{D}(X_z)$, i.e., x is a homomorphism by the theory of Dieudonne modules. Therefore $z \in \mathcal{Z}(x)(\mathbb{F})$, which contradicts to the assumption. Hence $t = 0$.

If $m = \infty$, i.e., the point $\tilde{z} : \mathrm{Spf}\,\mathcal{O}_{\breve{F}} \to \widehat{\mathcal{N}}_z^u$ factors through $\widehat{\mathcal{Z}(x)}_z$. If t is a finite number. The morphism $\mathcal{O}_{\breve{F}}/(\varpi^{t+1}) \to \mathbb{F}$ is a surjection in $\mathrm{Art}_{\mathcal{O}_{\breve{F}}}$ whose kernel admits a nilpotent $\mathcal{O}_F$-pd-structure. The line $\mathcal{O}_{\breve{F}}/(\varpi^{t+1}) \cdot l$ corresponds to the point $\tilde{z}_{t+1} : \mathrm{Spec}\,\mathcal{O}_{\breve{F}}/(\varpi^{t+1}) \to \mathrm{Spf}\,\mathcal{O}_{\breve{F}} \xrightarrow{\tilde{z}} \widehat{\mathcal{N}}_z^u$ under the bijection (i) in Lemma 3.1.5. It also factors through $\widehat{\mathcal{Z}(x)}_z$. By definition, the image of $x_{\mathrm{crys},z}$ in $\mathbf{D}/\varpi^{t+1}\mathbf{D}$ equals to $x_{\mathrm{crys},z}(\mathcal{O}_{\breve{F}}/(\varpi^{t+1}))$. Therefore $\langle x_{\mathrm{crys},z}(\mathcal{O}_{\breve{F}}/(\varpi^{t+1})), l \rangle \equiv \langle x_{\mathrm{crys},z}, l \rangle \equiv 0 \bmod \varpi^{t+1}$ by Lemma 3.1.5. This contradicts to the fact that $\nu_\varpi(\langle x_{\mathrm{crys},z}, l \rangle) = t$. Hence $t = \infty = m$.

If $1 \leqslant m < \infty$. The morphism $\mathcal{O}_{\breve{F}}/(\varpi^m) \to \mathbb{F}$ is a surjection in $\mathrm{Art}_{\mathcal{O}_{\breve{F}}}$ whose kernel admits a nilpotent $\mathcal{O}_F$-pd-structure. The line $\mathcal{O}_{\breve{F}}/(\varpi^m) \cdot l$ corresponds to the point $\tilde{z}_m : \mathrm{Spec}\,\mathcal{O}_{\breve{F}}/(\varpi^m) \to \mathrm{Spf}\,\mathcal{O}_{\breve{F}} \xrightarrow{\tilde{z}} \widehat{\mathcal{N}}_z^u$ under the bijection (i) in Lemma 3.1.5, which factors through $\widehat{\mathcal{Z}(x)}_z$. By definition, the image of $x_{\mathrm{crys},z}$ in $\mathbf{D}/\varpi^m\mathbf{D}$ equals to $x_{\mathrm{crys},z}(\mathcal{O}_{\breve{F}}/(\varpi^m))$. Therefore $\langle x_{\mathrm{crys},z}(\mathcal{O}_{\breve{F}}/(\varpi^m)), l \rangle \equiv \langle x_{\mathrm{crys},z}, l \rangle \equiv 0 \bmod \varpi^m$ by Lemma 3.1.5. Hence $t \geqslant m$.

On the other hand, the morphism $\mathcal{O}_{\breve{F}}/(\varpi^t) \to \mathbb{F}$ is a surjection in $\mathrm{Art}_{\mathcal{O}_{\breve{F}}}$ whose kernel admits a nilpotent $\mathcal{O}_F$-pd-structure. The line $\mathcal{O}_{\breve{F}}/(\varpi)^t \cdot l$ corresponds to the point $\tilde{z}_t : \mathrm{Spec}\,\mathcal{O}_{\breve{F}}/(\varpi^t) \to \mathrm{Spf}\,\mathcal{O}_{\breve{F}} \xrightarrow{\tilde{z}} \widehat{\mathcal{N}}_z^u$ under the bijection (i) in Lemma 3.1.5. The point $\tilde{z}_t \in \widehat{\mathcal{N}}_z^u(\mathcal{O}_{\breve{F}}/(\varpi^t))$ factors through $\widehat{\mathcal{Z}(x)}_z$ because $\langle x_{\mathrm{crys},z}(\mathcal{O}_{\breve{F}}/(\varpi^t)), l \rangle \equiv \langle x_{\mathrm{crys},z}, l \rangle \equiv 0 \pmod{\varpi^t}$ by Lemma 3.1.5. Therefore we have a morphism $\mathrm{Spec}\,\mathcal{O}_{\breve{F}}/(\varpi^t) \to \widehat{\mathcal{Z}(x)}_z \times_{\widehat{\mathcal{N}}_z^u} \mathrm{Spf}\,\mathcal{O}_{\breve{F}} \simeq \mathrm{Spec}\,\mathcal{O}_{\breve{F}}/(\varpi^m)$ by the universal property of fiber product, hence $m \geqslant t$. Therefore we conclude that $m = t$. Now we consider the GSpin case, the proof is similar. Let $\widehat{\mathcal{Z}(x)}_z$ be the completion of the special cycle $\mathcal{Z}(x)$ at z, there exists a Cartesian diagram,

$$\begin{array}{ccc} \mathrm{Spf}\,\breve{\mathbb{Z}}_p/(\tilde{z}^\sharp(f_{x,z})) & \longrightarrow & \mathrm{Spf}\,\breve{\mathbb{Z}}_p \\ \downarrow & & \downarrow{\scriptstyle \tilde{z}} \\ \widehat{\mathcal{Z}(x)}_z & \longrightarrow & \widehat{\mathcal{N}}_z^o. \end{array}$$

Let $m = \nu_p(\tilde{z}^\sharp(f_{x,z})) \in \mathbb{Z}_{\geqslant 0} \cup \{\infty\}$, $t = \nu_p(\langle x_{\mathrm{crys},z}, l \rangle) \in \mathbb{Z}_{\geqslant 0} \cup \{\infty\}$, the equality in (10) is equivalent to $t = m$.

We first show that the number m is always nonzero. If $m = 0$, we have $z \notin \mathcal{Z}(x)(\mathbb{F})$ by similar arguments in the unitary case. By Lemma 3.2.2, the assumption $x_{\mathrm{crys},z} \in \mathbf{V}$ implies that $z \in \mathcal{Z}(x)(\mathbb{F})$, which is a contradiction. Hence m is nonzero.

If $m = \infty$, i.e., the point $\tilde{z} : \mathrm{Spf}\,\breve{\mathbb{Z}}_p \to \widehat{\mathcal{N}}_z^o$ factors through $\widehat{\mathcal{Z}(x)}_z$. If t is a finite number. The morphism $\breve{\mathbb{Z}}_p/(p^{t+1}) \to \mathbb{F}$ is a surjection in $\mathrm{Art}_{\breve{\mathbb{Z}}_p}$ whose kernel admits a nilpotent divided power structure. The isotropic line $\breve{\mathbb{Z}}_p/(p^{t+1}) \cdot l$ corresponds to the

point $\tilde{z}_{t+1} : \operatorname{Spec} \breve{\mathbb{Z}}_p/(p^{t+1}) \to \operatorname{Spf} \breve{\mathbb{Z}}_p \xrightarrow{\tilde{z}} \widehat{\mathcal{N}}_z^o$ under the bijection (i) in Lemma 3.2.1. It also factors through $\widehat{\mathcal{Z}(x)}_z$. By definition, the image of $x_{\mathrm{crys},z}$ in $\mathbf{V}/p^{t+1}\mathbf{V}$ equals to $x_{\mathrm{crys},z}(\breve{\mathbb{Z}}_p/(p^{t+1}))$. Therefore $(x_{\mathrm{crys},z}(\breve{\mathbb{Z}}_p/(p^{t+1})), l) \equiv (x_{\mathrm{crys},z}, l) \equiv 0 \bmod p^{t+1}$ by Lemma 3.2.1. This contradicts to the fact that $\nu_p((x_{\mathrm{crys},z}, l)) = t$. Hence $t = \infty = m$.

If $m < \infty$. The morphism $\breve{\mathbb{Z}}_p/(p^m) \to \mathbb{F}$ is a surjection in $\operatorname{Art}_{\breve{\mathbb{Z}}_p}$ whose kernel admits a nilpotent divided power structure. The isotropic line $\breve{\mathbb{Z}}_p/(p^m) \cdot l$ corresponds to the point $\tilde{z}_m : \operatorname{Spec} \breve{\mathbb{Z}}_p/(p^m) \to \operatorname{Spf} \breve{\mathbb{Z}}_p \xrightarrow{\tilde{z}} \widehat{\mathcal{N}}_z^o$ under the bijection (i) in Lemma 3.2.1, which factors through $\widehat{\mathcal{Z}(x)}_z$. By definition, the image of $x_{\mathrm{crys},z}$ in $\mathbf{V}/p^m\mathbf{V}$ equals to $x_{\mathrm{crys},z}(\breve{\mathbb{Z}}_p/(p^m))$. Therefore $(x_{\mathrm{crys},z}(\breve{\mathbb{Z}}_p/(p^m)), l) \equiv (x_{\mathrm{crys},z}, l) \equiv 0 \bmod p^m$ by Lemma 3.2.1. Hence $t \geqslant m$.

On the other hand, the morphism $\breve{\mathbb{Z}}_p/(p^t) \to \mathbb{F}$ is a surjection in $\operatorname{Art}_{\breve{\mathbb{Z}}_p}$ whose kernel admits a nilpotent divided power structure. The isotropic line $\breve{\mathbb{Z}}_p/(p)^t \cdot l$ corresponds to the point $\tilde{z}_t : \operatorname{Spec} \breve{\mathbb{Z}}_p/(p^t) \to \operatorname{Spf} \breve{\mathbb{Z}}_p \xrightarrow{\tilde{z}} \widehat{\mathcal{N}}_z^u$ under the bijection (i) in Lemma 3.2.1. The point $\tilde{z}_t \in \widehat{\mathcal{N}}_z^u(\breve{\mathbb{Z}}_p/(p^t))$ factors through $\widehat{\mathcal{Z}(x)}_z$ because $(x_{\mathrm{crys},z}(\breve{\mathbb{Z}}_p/(p^t)), l) \equiv (x_{\mathrm{crys},z}, l) \equiv 0 \pmod{p^t}$ by Lemma 3.1.5. Therefore we have a morphism $\operatorname{Spec} \breve{\mathbb{Z}}_p/(p^t) \to \widehat{\mathcal{Z}(x)}_z \times_{\widehat{\mathcal{N}}_z^u} \operatorname{Spf} \breve{\mathbb{Z}}_p \simeq \operatorname{Spec} \breve{\mathbb{Z}}_p/(p^m)$ by the universal property of fiber product, hence $m \geqslant t$. Therefore we conclude that $m = t$. $\qquad\square$

5.3 Comparison of several quantities

Definition 5.3.1 Let $x \in \mathbb{V}^u$ (resp. $x \in \mathbb{V}^o$) be a nonzero special quasi-homomorphism such that $(x, x) \in \mathcal{O}_E$ (resp. $q_{\mathbb{V}^o}(x) \in \mathbb{Z}_p$). Let $z \in \mathcal{Z}(x)(\mathbb{F})$ be a point. Let $\widehat{\mathcal{N}}_z^u$ (resp. $\widehat{\mathcal{N}}_z^o$) be the formal completion of $\mathcal{N}^u$ (resp. $\mathcal{N}^o$) at z. Let $\widehat{\mathcal{Z}(x)}_z$ be the formal completion of $\mathcal{Z}(x)$ at z. Define

$l_{z,x} =$ the largest number l such that $z \in \mathcal{Z}(\varpi^{-l+1}x)(\mathbb{F})$ (resp. $z \in \mathcal{Z}(p^{-l+1}x)(\mathbb{F})$).

$m_{z,x} =$ the largest number m such that $x_{\mathrm{crys},z} \in \varpi^m\mathbf{D}$ (resp. $x_{\mathrm{crys},z} \in p^m\mathbf{V}$).

$n_{z,x} =$ the largest number n (could be infinity) such that there exists a point

$$z' \in \widehat{\mathcal{Z}(x)}_z(\mathcal{O}_{\breve{F}}/(\varpi^n)) \text{ (resp. } z' \in \widehat{\mathcal{Z}(x)}_z(\breve{\mathbb{Z}}_p/(p^n))).$$

For a point $\tilde{z} \in \widehat{\mathcal{N}}_z^u(\mathcal{O}_{\breve{F}})$ (resp. $\tilde{z} \in \widehat{\mathcal{N}}_z^o(\breve{\mathbb{Z}}_p)$), let $l \in \overline{\mathbf{D}}$ (resp. $l \in \mathbf{V}$) be a generator of the line (resp. isotropic line) $L^1 \subset \overline{\mathbf{D}}$ (resp. $L^1 \subset \mathbf{V}$) corresponding to the point $\tilde{z}$. Define

$$t_x(\tilde{z}) = \nu_\varpi(\langle x_{\mathrm{crys},z}, l \rangle) \text{ (resp. } \nu_p((x_{\mathrm{crys},z}, l))).$$

Remark 5.3.2 In the set up of the above definition, the smoothness criterion (Lemmas 4.2.1 and 4.3.1) of the formal scheme $\mathcal{Z}(x)$ at a point $z \in \mathcal{Z}(x)(\mathbb{F})$ can be interpreted in the following way:

- In the unitary case, the formal scheme $\mathcal{Z}(x)$ is formally smooth over $\mathcal{O}_{\breve{F}}$ of relative dimension $n - 2$ at the point z if and only if $m_{z,x} = 0$.
- In the GSpin case, the formal scheme $\mathcal{Z}(x)$ is formally smooth over $\breve{\mathbb{Z}}_p$ of relative dimension $n - 2$ at the point z if and only if $m_{z,x} = 0$ and $n_{z,x} \geq 2$.

Proposition 5.3.3 *Let* $x \in \mathbb{V}^u$ *(resp.* $x \in \mathbb{V}^o$*) be a nonzero special quasi-homomorphism such that* $(x, x) \in \mathcal{O}_E$ *(resp.* $q_{\mathbb{V}^o}(x) \in \mathbb{Z}_p$*). Let* $z \in \mathcal{Z}(x)(\mathbb{F})$ *be a point. We have*

$$l_{z,x} = \min_{\tilde{z} \in \widehat{\mathcal{N}}_z^u(\mathcal{O}_{\breve{F}})} \{t_x(\tilde{z})\} \left(resp. \min_{\tilde{z} \in \widehat{\mathcal{N}}_z^o(\mathcal{O}_{\breve{F}})} \{t_x(\tilde{z})\} \right).$$

$$n_{z,x} = \max_{\tilde{z} \in \widehat{\mathcal{N}}_z^u(\mathcal{O}_{\breve{F}})} \{t_x(\tilde{z})\} \left(resp. \max_{\tilde{z} \in \widehat{\mathcal{N}}_z^o(\mathcal{O}_{\breve{F}})} \{t_x(\tilde{z})\} \right).$$

Moreover, we have $l_{z,x} = m_{z,x} + 1$ *in the GSpin case, while* $l_{z,x} - m_{z,x}$ *can be 0 or 1 in the unitary case.*

Proof We first prove the equality about $n_{z,x}$. Then we prove the equality about $l_{z,x}$ while which the statements about the relationships between $l_{z,x}$ and $m_{z,x}$ will be clear. We separate the unitary and the GSpin case.

In the unitary case. For a point $\tilde{z} \in \widehat{\mathcal{N}}_z^u(\mathcal{O}_{\breve{F}})$, denote by $l_{\tilde{z}} \in \overline{\mathbf{D}}$ be a generator of the line $L^1 \subset \overline{\mathbf{D}}$ corresponding to the point $\tilde{z}$. For an integer n, let $f_{\varpi^n x, z} \in \widehat{\mathcal{O}}_{\mathcal{N}^u, z}$ be the local equation of the special divisor $\mathcal{Z}(\varpi^n x)$ at z. By Proposition 5.2.1, we have

$$(\tilde{z}^\sharp(f_{\varpi^n x})) = (\langle \varpi^n x_{\text{crys},z}, l_{\tilde{z}} \rangle) = (\varpi^{n + t_x(\tilde{z})}) \subset (\varpi)$$

as long as $z \in \mathcal{Z}(\varpi^n x)(\mathbb{F})$. Notice that the point $z \in \mathcal{Z}(\varpi^{-l_{z,x}+1} x)(\mathbb{F})$. Let $n = -l_{z,x} + 1$, we get $-l_{z,x} + 1 + t_x(\tilde{z}) \geq 1$, i.e., $l_{z,x} \leq t_x(\tilde{z})$. Let $n = 0$, the fiber product $\widehat{\mathcal{Z}(x)}_z \times_{\widehat{\mathcal{N}}_z^u, \tilde{z}} \text{Spf } \mathcal{O}_{\breve{F}} \simeq \text{Spec } \mathcal{O}_{\breve{F}}/(\varpi^{t_x(\tilde{z})})$ defines a point $z' \in \widehat{\mathcal{Z}(x)}_z(\varpi^{t_x(\tilde{z})})$, hence $n_{z,x} \geq t_x(\tilde{z})$. Therefore we have proved that $l_{z,x} \leq t_x(\tilde{z}) \leq n_{z,x}$ for all points $\tilde{z} \in \widehat{\mathcal{N}}_z^u(\mathcal{O}_{\breve{F}})$. Hence

$$l_{z,x} \leq \min_{\tilde{z} \in \widehat{\mathcal{N}}_z^u(\mathcal{O}_{\breve{F}})} \{t_x(\tilde{z})\} \leq \max_{\tilde{z} \in \widehat{\mathcal{N}}_z^u(\mathcal{O}_{\breve{F}})} \{t_x(\tilde{z})\} \leq n_{z,x}.$$

We first prove $n_{z,x} = \max_{\tilde{z} \in \widehat{\mathcal{N}}_z^u(\mathcal{O}_{\breve{F}})} \{t_x(\tilde{z})\}$. Let $M = \max_{\tilde{z} \in \widehat{\mathcal{N}}_z^u(\mathcal{O}_{\breve{F}})} \{t_x(\tilde{z})\}$. If $n_{z,x} = \infty$. For all integer n, there exists a point $z' \in \widehat{\mathcal{Z}(x)}_z(\mathcal{O}_{\breve{F}}/(\varpi^n))$. Let $\tilde{z} \in \widehat{\mathcal{N}}_z^u(\mathcal{O}_{\breve{F}})$ be a lift of z' to $\mathcal{O}_{\breve{F}}$, the existence of $\tilde{z}$ follows from the fact that $\widehat{\mathcal{O}}_{\mathcal{N}^u, z}$ is formally smooth over $\mathcal{O}_{\breve{F}}$. Then we have $\langle l_{\tilde{z}}, x_{\text{crys},z} \rangle \equiv 0 \mod \varpi^n$ by Lemma 3.1.5. Hence $t_x(\tilde{z}) = v_\varpi(\langle l_{\tilde{z}}, x_{\text{crys},z} \rangle) \geq n$. Therefore $M = \infty = n_{z,x}$. If $n_{z,x} < \infty$, there exists a point $z' \in \widehat{\mathcal{Z}(x)}_z(\mathcal{O}_{\breve{F}}/(\varpi^{n_{z,x}}))$. Let $\tilde{z} \in \widehat{\mathcal{N}}_z^u(\mathcal{O}_{\breve{F}})$ be a lift of z' to $\mathcal{O}_{\breve{F}}$. Then

we have $\langle l_{\tilde{z}}, x_{\mathrm{crys},z}\rangle \equiv 0 \bmod \varpi^{n_{z,x}}$ by Lemma 3.1.5. Hence $M \geqslant n_{z,x}$. We already have $M \leqslant n_{z,x}$. Hence $n_{z,x} = M = \max_{\tilde{z} \in \widehat{\mathcal{N}}_z^u(\mathcal{O}_{\breve{F}})}\{t_x(\tilde{z})\}$.

Next we prove $l_{z,x} = \min_{\tilde{z} \in \widehat{\mathcal{N}}_z^u(\mathcal{O}_{\breve{F}})}\{t_x(\tilde{z})\}$. Let $N = \min_{\tilde{z} \in \widehat{\mathcal{N}}_z^u(\mathcal{O}_{\breve{F}})}\{t_x(\tilde{z})\}$. Let $x' = \varpi^{-l_{z,x}+1}x$. Then we have $z \in \mathcal{Z}(x')(\mathbb{F})$ but $z \notin \mathcal{Z}(\varpi^{-1}x')(\mathbb{F})$. There are two possibilities.

(a) The element $\varpi^{-1}x'_{\mathrm{crys},z} \notin \mathbf{D}$. Let $\tilde{z} \in \widehat{\mathcal{N}}_z^u(\mathcal{O}_{\breve{F}})$ be a point. Notice that $t_{x'}(\tilde{z}) \geqslant 1$ since $z \in \mathcal{Z}(x')(\mathbb{F})$. If $t_{x'}(\tilde{z}) = 1$, then $t_x(\tilde{z}) = t_{x'}(\tilde{z}) + l_{z,x} - 1 = l_{z,x}$, hence $l_{z,x} \geqslant N$. If $t_{x'}(\tilde{z}) \geqslant 2$, let $l \in \overline{\mathbf{D}}$ be an element such that $v_{\varpi}(\langle l, x_{\mathrm{crys},z}\rangle) = 0$, the existence of l follows from the fact that $x_{\mathrm{crys},z} \notin \varpi\mathbf{D}$ and the pairing $\langle \cdot, \cdot \rangle : \overline{\mathbf{D}} \times \mathbf{D} \to \mathcal{O}_{\breve{F}}$ is perfect. Let $l' = l_{\tilde{z}} + \varpi \cdot l$. Then $\langle l', x_{\mathrm{crys},z}\rangle = \langle l_{\tilde{z}}, x_{\mathrm{crys},z}\rangle + \varpi \langle l, x_{\mathrm{crys},z}\rangle$. Hence $v_{\varpi}(\langle l', x_{\mathrm{crys},z}\rangle) = 1$. Let $\tilde{z}' \in \widehat{\mathcal{N}}_z^u(\mathcal{O}_{\breve{F}})$ be the point corresponding to the line generated by l' in Lemma 3.1.5. Then $t_{x'}(\tilde{z}') = 1$, then $t_x(\tilde{z}') = t_{x'}(\tilde{z}') + l_{z,x} - 1 = l_{z,x}$, hence $l_{z,x} \geqslant N$.
Notice that in this case $m_{z,x'} = 0$, therefore $m_{z,x} = m_{z,x'} + l_{z,x} - 1 = l_{z,x} - 1$. We give an example of case (a) in Example 5.3.4.

(b) The element $\varpi^{-1}x'_{\mathrm{crys},z} \in \mathbf{D}$. Let $\tilde{z} \in \widehat{\mathcal{N}}_z^u(\mathcal{O}_{\breve{F}})$ be a point, then $v_{\varpi}(\langle l_{\tilde{z}}, \varpi^{-1}x'_{\mathrm{crys},z}\rangle) = 0$ because otherwise $z \in \mathcal{Z}(\varpi^{-1}x')(\mathbb{F})$ by Proposition 5.2.1. Hence $t_{x'}(\tilde{z}') = 1$, then $t_x(\tilde{z}') = t_{x'}(\tilde{z}') + l_{z,x} - 1 = l_{z,x}$, hence $l_{z,x} \geqslant N$.
In this case we can conclude that $l_{z,x} = m_{z,x}$: the assumption $\varpi^{-1}x'_{\mathrm{crys},z} \in \mathbf{D}$ implies that $m_{z,x'} = m_{z,x} - l_{z,x} + 1 \geqslant 1$, hence $m_{z,x} \geqslant l_{z,x}$. If $m_{z,x} \geqslant l_{z,x} \geqslant 1$, we have $\varpi^{-1}x'_{\mathrm{crys},z} \in \varpi\mathbf{D}$. Then $v_{\varpi}(\langle l_{\tilde{z}}, \varpi^{-1}x'_{\mathrm{crys},z}\rangle) \geqslant 1$. Proposition 5.2.1 implies that $z \in \mathcal{Z}(\varpi^{-1}x')(\mathbb{F})$, which is a contradiction. Therefore $l_{z,x} = m_{z,x}$. We give an example of case (b) in Example 5.3.4.

In both cases, we have $l_{z,x} \geqslant N$. We already have $N \geqslant l_{z,x}$, hence $l_{z,x} = N = \min_{\tilde{z} \in \widehat{\mathcal{N}}_z^u(\mathcal{O}_{\breve{F}})}\{t_x(\tilde{z})\}$. Notice that $l_{z,x} - m_{z,x}$ can be 0 or 1.

The proof for the GSpin case is basically the same. For a point $\tilde{z} \in \widehat{\mathcal{N}}_z^o(\breve{\mathbb{Z}}_p)$, denote by $l_{\tilde{z}} \in \mathbf{V}$ be a generator of the line $L^1 \subset \mathbf{V}$ corresponding to the point $\tilde{z}$. For an integer n, let $f_{p^n x,z} \in \widehat{\mathcal{O}}_{\mathcal{N}^o,z}$ be the local equation of the special divisor $\mathcal{Z}(p^n x)$ at z. By Proposition 5.2.1, we have

$$(\tilde{z}^{\sharp}(f_{p^n x})) = ((p^n x_{\mathrm{crys},z}, l_{\tilde{z}})) = (p^{n+t_x(\tilde{z})}) \subset (p)$$

as long as $z \in \mathcal{Z}(p^n x)(\mathbb{F})$. Notice that the point $z \in \mathcal{Z}(p^{-l_{z,x}+1}x)(\mathbb{F})$. Let $n = -l_{z,x} + 1$, we get $-l_{z,x} + 1 + t_x(\tilde{z}) \geqslant 1$, i.e., $l_{z,x} \leqslant t_x(\tilde{z})$. Let $n = 0$, the fiber product $\widehat{\mathcal{Z}(x)}_z \times_{\widehat{\mathcal{N}}_z^o,\tilde{z}} \mathrm{Spf}\,\mathbb{Z}_p \simeq \mathrm{Spec}\,\breve{\mathbb{Z}}_p/(p^{t_x(\tilde{z})})$ defines a point $z' \in \widehat{\mathcal{Z}(x)}_z(p^{t_x(\tilde{z})})$, hence $n_{z,x} \geqslant t_x(\tilde{z})$. Therefore we have proved that $l_{z,x} \leqslant t_x(\tilde{z}) \leqslant n_{z,x}$ for all points $\tilde{z} \in \widehat{\mathcal{N}}_z^o(\breve{\mathbb{Z}}_p)$. Hence

$$l_{z,x} \leqslant \min_{\tilde{z} \in \widehat{\mathcal{N}}_z^o(\breve{\mathbb{Z}}_p)}\{t_x(\tilde{z})\} \leqslant \max_{\tilde{z} \in \widehat{\mathcal{N}}_z^o(\breve{\mathbb{Z}}_p)}\{t_x(\tilde{z})\} \leqslant n_{z,x}.$$

We first prove $n_{z,x} = \max_{\tilde{z} \in \widehat{\mathcal{N}}_z^o(\breve{\mathbb{Z}}_p)}\{t_x(\tilde{z})\}$. Let $M = \max_{\tilde{z} \in \widehat{\mathcal{N}}_z^o(\breve{\mathbb{Z}}_p)}\{t_x(\tilde{z})\}$. If $n_{z,x} = \infty$. For all integers n, there exists a point $z' \in \widehat{\mathcal{Z}(x)}_z(\breve{\mathbb{Z}}_p/(p^n))$. Let $\tilde{z} \in \widehat{\mathcal{N}}_z^o(\breve{\mathbb{Z}}_p)$

be a lift of z' to $\breve{\mathbb{Z}}_p$, the existence of $\tilde{z}$ follows from the fact that $\widehat{\mathcal{O}}_{\mathcal{N}^o,z}$ is formally smooth over $\breve{\mathbb{Z}}_p$. Then we have $(l_{\tilde{z}}, x_{\text{crys},z}) \equiv 0 \bmod p^n$ by Lemma 3.1.5. Hence $t_x(\tilde{z}) = v_p((l_{\tilde{z}}, x_{\text{crys},z})) \geqslant n$. Therefore $M = \infty = n_{z,x}$. If $n_{z,x} < \infty$, there exists a point $z' \in \widehat{\mathcal{Z}(x)}_z(\breve{\mathbb{Z}}_p/(p^{n_{z,x}}))$. Let $\tilde{z} \in \widehat{\mathcal{N}}_z^o(\breve{\mathbb{Z}}_p)$ be a lift of z' to $\breve{\mathbb{Z}}_p$. Then we have $(l_{\tilde{z}}, x_{\text{crys},z}) \equiv 0 \bmod p^{n_{z,x}}$ by Lemma 3.1.5. Hence $M \geqslant n_{z,x}$. We already have $M \leqslant n_{z,x}$. Hence $n_{z,x} = M = \max_{\tilde{z} \in \widehat{\mathcal{N}}_z^o(\breve{\mathbb{Z}}_p)}\{t_x(\tilde{z})\}$.

Next we prove $l_{z,x} = \min_{\tilde{z} \in \widehat{\mathcal{N}}_z^o(\breve{\mathbb{Z}}_p)}\{t_x(\tilde{z})\}$. Let $N = \min_{\tilde{z} \in \widehat{\mathcal{N}}_z^o(\breve{\mathbb{Z}}_p)}\{t_x(\tilde{z})\}$. Let $x' = p^{-l_{z,x}+1}x$. Then we have $z \in \mathcal{Z}(x')(\mathbb{F})$ but $z \notin \mathcal{Z}(p^{-1}x')(\mathbb{F})$. We must have the element $p^{-1}x'_{\text{crys},z} \notin \mathbf{V}$ because otherwise $z \in \mathcal{Z}(p^{-1}x')(\mathbb{F})$ by Lemma 3.2.2.

Let $\tilde{z} \in \widehat{\mathcal{N}}_z^o(\breve{\mathbb{Z}}_p)$ be a point. Notice that $t_{x'}(\tilde{z}) \geqslant 1$ since $z \in \mathcal{Z}(x')(\mathbb{F})$. If $t_{x'}(\tilde{z}) = 1$, then $t_x(\tilde{z}) = t_{x'}(\tilde{z}) + l_{z,x} - 1 = l_{z,x}$, hence $l_{z,x} \geqslant N$. If $t_{x'}(\tilde{z}) \geqslant 2$, let $l \in \mathbf{V}$ be an element such that $v_p((l, x_{\text{crys},z})) = 0$, the existence of l follows from the fact that $x_{\text{crys},z} \notin \mathbf{V}$ and the pairing $(\cdot, \cdot) : \mathbf{V} \times \mathbf{V} \to \breve{\mathbb{Z}}_p$ is perfect. Let $l' = l_{\tilde{z}} + p \cdot l$. Then $(l', x_{\text{crys},z}) = (l_{\tilde{z}}, x_{\text{crys},z}) + p(l, x_{\text{crys},z})$. Hence $v_p((l', x_{\text{crys},z})) = 1$. Let $\tilde{z}' \in \widehat{\mathcal{N}}_z^o(\breve{\mathbb{Z}}_p)$ be the point corresponding to the line generated by l' in Lemma 3.2.1. Then $t_{x'}(\tilde{z}') = 1$, then $t_x(\tilde{z}') = t_{x'}(\tilde{z}') + l_{z,x} - 1 = l_{z,x}$, hence $l_{z,x} \geqslant N$. Since we already have $N \geqslant l_{z,x}$, we get $l_{z,x} = N = \min_{\tilde{z} \in \widehat{\mathcal{N}}_z^o(\breve{\mathbb{Z}}_p)}\{t_x(\tilde{z})\}$.

Notice that $m_{z,x'} = 0$, therefore $m_{z,x} = m_{z,x'} + l_{z,x} - 1 = l_{z,x} - 1$. □

Example 5.3.4 In the unitary case, the difference $l_{z,x} - m_{z,x}$ can be 0 (case (b)) or 1 (case (a)). We give examples for both cases.

(a) Let $n = 2$. Let x be a special quasi-homomorphism such that $v_\varpi((x, x)) = 0$. Then $\mathcal{N}^u \simeq \operatorname{Spf} \mathcal{O}_{\breve{F}}[[t]]$. We have $\mathcal{Z}(x) \simeq\simeq \operatorname{Spf} \mathcal{O}_{\breve{F}}$ (cf. [10, Proposition 8.1]). It is formally smooth over $\mathcal{O}_{\breve{F}}$, hence $m_{z,x} = 0$ by Lemma 4.2.1. Therefore $\varpi^{-1}x_{\text{crys},z} \notin \mathbf{D}$. We have $l_{z,x} = 1$ and $m_{z,x} = 0$

(b) Let $n = 2$. Let x be a special quasi-homomorphism such that $v_\varpi((x, x)) = 1$. Then $\mathcal{N}^u \simeq \operatorname{Spf} \mathcal{O}_{\breve{F}}[[t]]$. We have $\mathcal{Z}(x) \simeq \operatorname{Spf} W_1$ (cf. [10, Proposition 8.1]). Here W_1 is a ramified extension of $\mathcal{O}_{\breve{F}}$ of degree $q + 1$. It is not formally smooth over $\mathcal{O}_{\breve{F}}$, hence $m_{z,x} \geqslant 1$ by Lemma 4.2.1, i.e., $\varpi^{-1}x_{\text{crys},z} \in \mathbf{D}$. We have $l_{z,x} = m_{z,x} = 1$ by the above proof.

Lemma 5.3.5 *Let $x \in \mathbb{V}^u$ (resp. $x \in \mathbb{V}^o$) be a nonzero special quasi-homomorphism such that $(x, x) \in \mathcal{O}_E$ (resp. $q_{\mathbb{V}^o}(x) \in \mathbb{Z}_p$). Let $z \in \mathcal{Z}(x)(\mathbb{F})$ be a point. The formal scheme $\mathcal{Z}(x)$ is formally smooth over $\mathcal{O}_{\breve{F}}$ (resp. $\breve{\mathbb{Z}}_p$) of relative dimension $n - 2$ at the point z if and only if $l_{z,x} = 1$ and $n_{z,x} \geqslant 2$.*

Proof We first prove the "only if" direction: if the formal scheme $\mathcal{Z}(x)$ is formally smooth over $\mathcal{O}_{\breve{F}}$ (resp. $\breve{\mathbb{Z}}_p$) of relative dimension $n - 2$ at the point z. Then $n_{z,x} = \infty$ by the formally smoothness. We also have $m_{z,x} = 0$ by Remark 5.3.2. By Proposition 5.3.3, we have $l_{z,x} = m_{z,x} + 1 = 1$ in the orthogonal case. And $l_{z,x} = 0$ or 1 in the unitary case. However, since $z \in \mathcal{Z}(x)(\mathbb{F})$, we must have $l_{z,x} \geqslant 1$. Therefore in both cases we have $l_{z,x} = 1$.

Now we prove the "if" direction. By Remark 5.3.2, we only need to prove the two conditions $l_{z,x} = 1$ and $n_{z,x} \geqslant 2$ imply the equality $m_{z,x} = 0$. Let's assume the converse, i.e., $m_{z,x} \geqslant 1$. Therefore $\varpi^{-1}x_{\text{crys},z} \in \mathbf{D}$ (resp. $p^{-1}x_{\text{crys},z} \in \mathbf{V}$).

Since $n_{z,x} \geqslant 2$, there exists a point $\tilde{z} \in \mathcal{N}^u(\mathcal{O}_{\breve{F}})$ (resp. $\tilde{z} \in \mathcal{N}^o(\breve{\mathbb{Z}}_p)$) such that $t_x(\tilde{z}) \geqslant 2$. Let $l \in \overline{\mathbf{D}}$ (resp. $l \in \mathbf{V}$) be a generator of the line $L^1 \subset \overline{\mathbf{D}}$ (resp. the isotropic line $L^1 \subset \mathbf{V}$) corresponding to the point $\tilde{z} \in \mathcal{N}^u(\mathcal{O}_{\breve{F}})$ (resp. $\tilde{z} \in \mathcal{N}^o(\breve{\mathbb{Z}}_p)$), then $v_{\varpi}(\langle x_{\text{crys},z}, l \rangle) \geqslant 2$ (resp. $v_p((x_{\text{crys},z}, l)) \geqslant 2$), hence $v_{\varpi}(\langle \varpi^{-1}x_{\text{crys},z}, l \rangle) \geqslant 1$ (resp. $v_p((p^{-1}x_{\text{crys},z}, l)) \geqslant 1$).

Let $f_{\varpi^{-1}x,z} \in \widehat{\mathcal{O}}_{\mathcal{N}^u,z}$ (resp. $f_{p^{-1}x,z} \in \widehat{\mathcal{O}}_{\mathcal{N}^o,z}$) be the local equation of $\mathcal{Z}(\varpi^{-1}x)$ (resp. $\mathcal{Z}(p^{-1}x)$) at the point z. By Lemma 5.2.1, the ideal $(\tilde{z}^{\sharp}(f_{\varpi^{-1}x,z})) = (\langle \varpi^{-1}x_{\text{crys},z}, l \rangle) \subset \mathcal{O}_{\breve{F}}$ (resp. $(\tilde{z}^{\sharp}(f_{p^{-1}x,z})) = ((p^{-1}x_{\text{crys},z}, l)) \subset \breve{\mathbb{Z}}_p$), hence $f_{\varpi^{-1}x,z} \in \widehat{\mathcal{O}}_{\mathcal{N}^u,z}$ is not a unit (resp. $f_{p^{-1}x,z} \in \widehat{\mathcal{O}}_{\mathcal{N}^o,z}$ is not a unit). Therefore $z \in \mathcal{Z}(\varpi^{-1}x)(\mathbb{F})$ (resp. $z \in \mathcal{Z}(p^{-1}x)(\mathbb{F})$), i.e., $l_{z,x} \geqslant 2$, which is a contradiction. Therefore we have $m_{z,x} = 0$. Then the formal scheme $\mathcal{Z}(x)$ is formally smooth over $\mathcal{O}_{\breve{F}}$ (resp. $\breve{\mathbb{Z}}_p$) of relative dimension $n - 2$ at the point z by Lemma 4.2.1 (resp. the extra condition $n_{z,x} \geqslant 2$ and Lemma 4.3.1). □

5.4 Proof of the main results

Theorem 5.4.1 *Let $x \in \mathbb{V}^u$ (resp. $x \in \mathbb{V}^o$) be a nonzero special quasi-homomorphism such that $(x, x) \in \mathcal{O}_E$ (resp. $q_{\mathbb{V}^o}(x) \in \mathbb{Z}_p$).*

(i) There is an equality $\mathcal{D}(x)(\mathbb{F}) = \mathcal{Z}(x)(\mathbb{F})$.

Let $z \in \mathcal{N}^u(\mathbb{F})$ (resp. $z \in \mathcal{N}^o(\mathbb{F})$) be a point such that $z \in \mathcal{Z}(x)(\mathbb{F})$ but $z \notin \mathcal{Z}(\varpi^{-1}x)(\mathbb{F})$ (resp. $z \in \mathcal{Z}(x)(\mathbb{F})$ but $z \notin \mathcal{Z}(p^{-1}x)(\mathbb{F})$).

(ii) The point $z \in \mathcal{D}(\varpi^a x)(\mathbb{F})$ (resp. $z \in \mathcal{D}(p^a x)(\mathbb{F})$) for all $a \geqslant 0$.

(iii) For all $a \geqslant 1$, the difference divisor $\mathcal{D}(\varpi^a x)$ (resp. $\mathcal{D}(p^a x)$) is regular but not formally smooth over $\mathcal{O}_{\breve{F}}$ (resp. $\breve{\mathbb{Z}}_p$) at z.

(iv) For $a = 0$, the difference divisor $\mathcal{D}(x)$ is regular at z, it is formally smooth over $\mathcal{O}_{\breve{F}}$ (resp. $\breve{\mathbb{Z}}_p$) at z if and only if $n_{z,x} \geqslant 2$.

Proof of the unitary case We first prove (ii), (iii) and (iv), then come back to prove (i).

Let $\mathfrak{m}_z = (\varpi, t_1, \cdots, t_{n-1})$ be the maximal ideal of the complete local ring $\widehat{\mathcal{O}}_{\mathcal{N}^u,z}$. Notice that the set $\widehat{\mathcal{N}}_z^u(\mathcal{O}_{\breve{F}})$ has a bijection to the set of $\mathcal{O}_{\breve{F}}$-algebra homomorphisms $\mathcal{O}_{\breve{F}}[[t_1, \cdots, t_{n-1}]] \to \mathcal{O}_{\breve{F}}$ by sending a point $\tilde{z}$ to $\mathcal{O}_{\breve{F}}$-algebra homomorphism $\tilde{z}^{\sharp} : \mathcal{O}_{\breve{F}}[[t_1, \cdots, t_{n-1}]] \to \mathcal{O}_{\breve{F}}$.

For $a \geqslant 0$, let $f_{\varpi^a x} \in \widehat{\mathcal{O}}_{\mathcal{N}^u,z}$ be the local equation of the special divisor $\mathcal{Z}(\varpi^a x)$ at z, and let $d_{\varpi^a x} \in \widehat{\mathcal{O}}_{\mathcal{N}^u,z}$ be the local equation of the difference divisor $\mathcal{D}(\varpi^a x)$ at z. Notice that the point $z \in \mathcal{D}(\varpi^a x)(\mathbb{F})$ if and only if $d_{\varpi^a x} \in \mathfrak{m}_z$. Since $z \notin \mathcal{Z}(\varpi^{-1}x)$, we can assume $f_x = d_x$.

We first prove $d_x \in \mathfrak{m}_z \backslash \mathfrak{m}_z^2$, which is equivalent to $f_x \in \mathfrak{m}_z \backslash \mathfrak{m}_z^2$. There are two cases.

- If $n_{z,x} \leqslant 1$. Notice that $l_{z,x} = 1$ since $z \notin \mathcal{Z}(\varpi^{-1}x)(\mathbb{F})$. By Proposition 5.3.3, $l_{z,x} \leqslant t_x(\tilde{z}) \leqslant n_{z,x}$ for all points $\tilde{z} \in \widehat{\mathcal{N}}_z^u(\mathcal{O}_{\breve{F}})$, then we must have $l_{z,x} = t_x(\tilde{z}) = n_{z,x} = 1$. By Lemma 5.2.1, the number $v_{\varpi}(\tilde{z}^{\sharp}(f_x)) = t_x(\tilde{z}) = 1$ for all $\mathcal{O}_{\breve{F}}$-algebra homomorphism $\tilde{z}^{\sharp} : \mathcal{O}_{\breve{F}}[[t_1, \cdots, t_{n-1}]] \to \mathcal{O}_{\breve{F}}$, hence $f_x \equiv (\text{unit}) \cdot \varpi \pmod{\mathfrak{m}_z^2}$ by Lemma 5.1.1, then $d_x = f_x \in \mathfrak{m}_z \backslash \mathfrak{m}_z^2$. Notice that in this case the difference divisor $\mathcal{D}(x)$ is regular but not formally smooth over $\mathcal{O}_{\breve{F}}$ at z.

- If $n_{z,x} \geqslant 2$. Notice that we already have $l_{z,x} = 1$, then by Lemma 5.3.5 the formal scheme $\mathcal{Z}(x)$ is smooth over $\mathcal{O}_{\breve{F}}$ of relative dimension $n-2$ at the point z, hence $f_x \in \mathfrak{m}_z \backslash \mathfrak{m}_z^2$. Notice that in this case the difference divisor $\mathcal{D}(x)$ is formally smooth over $\mathcal{O}_{\breve{F}}$ at z.

Now we already show that $d_x \in \mathfrak{m}_z \backslash \mathfrak{m}_z^2$. Our next goal is to prove $d_{\varpi^a x} \equiv \text{(unit)} \cdot \varpi \mod \mathfrak{m}_z^2$ for all $a \geqslant 1$. Let $\tilde{z} \in \widehat{\mathcal{N}}_z^u(\mathcal{O}_{\breve{F}})$ be a point, by Lemma 5.2.1,

$$(\tilde{z}^\sharp(f_{\varpi^a x})) = (\langle \varpi^a \cdot x_{\text{crys,z}}, l \rangle) = (\varpi^a \cdot \langle x_{\text{crys,z}}, l \rangle) = (\varpi^a \cdot \tilde{z}^\sharp(f_x)). \qquad (11)$$

Let $\widehat{\mathcal{Z}(\varpi^a x)}_z$ be the completion of the formal scheme $\mathcal{Z}(\varpi^a x)$ at z, there exists a Cartesian diagram by (11),

$$
\begin{array}{ccc}
\text{Spf } \mathcal{O}_{\breve{F}}/(\varpi^a \cdot \tilde{z}^\sharp(f_x)) & \longrightarrow & \text{Spf } \mathcal{O}_{\breve{F}} \\
\downarrow & & \downarrow{\tilde{z}} \\
\widehat{\mathcal{Z}(\varpi^a x)}_z & \longrightarrow & \widehat{\mathcal{N}}_z^u.
\end{array}
$$

Recall that by the definition of difference divisors, we have $f_{\varpi^a x} = \prod_{i=0}^{a} d_{\varpi^i x}$ for all integer $a \geqslant 0$, hence Lemma 5.1.2 implies that $d_{\varpi^a x} \equiv \text{(unit)} \cdot \varpi \pmod{\mathfrak{m}_z^2}$ for all $a \geqslant 1$ since $d_x \in \mathfrak{m}_z \backslash \mathfrak{m}_z^2$.

For all integer $a \geqslant 0$, the point $z \in \mathcal{D}(\varpi^a x)(\mathbb{F})$ because the local equation $d_{\varpi^a x} \in \mathfrak{m}_z$. Moreover, the local equation $d_x \in \mathfrak{m}_z \backslash \mathfrak{m}_z^2$ implies that the difference divisor $\mathcal{D}(\varpi^a x)$ is regular at the point z.

For $a = 0$, we have proved that the difference divisor $\mathcal{D}(x)$ is formally smooth over $\mathcal{O}_{\breve{F}}$ at z if and only if $n_{z,x} \geqslant 2$. For all $a \geqslant 1$, the equation $d_{\varpi^a x} \equiv \text{(unit)} \cdot \varpi \pmod{\mathfrak{m}_z^2}$, hence the difference divisor $\mathcal{D}(\varpi^a x)$ is not formally smooth at the point z.

Now we have shown (ii), (iii) and (iv), let's come back to prove (i). The one-side inclusion $\mathcal{D}(x)(\mathbb{F}) \subset \mathcal{Z}(x)(\mathbb{F})$ follows from definition. Conversely, if $z' \in \mathcal{Z}(x)(\mathbb{F})$. Let $x' = \varpi^{-l_{z',x}+1}x$, then $z' \in \mathcal{Z}(x')(\mathbb{F})$ but $z' \notin \mathcal{Z}(\varpi^{-1}x')(\mathbb{F})$. (ii) implies that $z' \in \mathcal{D}(\varpi^{l_{z',x}-1}x')(\mathbb{F}) = \mathcal{D}(x)(\mathbb{F})$. Therefore we conclude that $\mathcal{D}(x)(\mathbb{F}) = \mathcal{Z}(x)(\mathbb{F})$. $\qquad \square$

Proof of the GSpin case The proof is similar to the unitary case above. We first prove (ii), (iii) and (iv), then come back to prove (i).

Let $\mathfrak{m}_z = (p, t_1, \cdots, t_{n-1})$ be the maximal ideal of the complete local ring $\widehat{\mathcal{O}}_{\mathcal{N}^o, z}$. Notice that the set $\widehat{\mathcal{N}}_z^o(\breve{\mathbb{Z}}_p)$ has a bijection to the set of $\breve{\mathbb{Z}}_p$-algebra homomorphisms $\breve{\mathbb{Z}}_p[[t_1, \cdots, t_{n-1}]] \to \breve{\mathbb{Z}}_p$ by sending a point $\tilde{z}$ to $\breve{\mathbb{Z}}_p$-algebra homomorphism $\tilde{z}^\sharp : \breve{\mathbb{Z}}_p[[t_1, \cdots, t_{n-1}]] \to \breve{\mathbb{Z}}_p$.

For $a \geqslant 0$, let $f_{p^a x} \in \widehat{\mathcal{O}}_{\mathcal{N}^o, z}$ be the local equation of the special divisor $\mathcal{Z}(p^a x)$ at z, and let $d_{p^a x} \in \widehat{\mathcal{O}}_{\mathcal{N}^o, z}$ be the local equation of the difference divisor $\mathcal{D}(p^a x)$ at z. Notice that the point $z \in \mathcal{D}(p^a x)(\mathbb{F})$ if and only if $d_{p^a x} \in \mathfrak{m}_z$. Since $z \notin \mathcal{Z}(p^{-1}x)$, we can assume $f_x = d_x$.

We first prove $d_x \in \mathfrak{m}_z \backslash \mathfrak{m}_z^2$, which is equivalent to $f_x \in \mathfrak{m}_z \backslash \mathfrak{m}_z^2$. There are two cases.

- If $n_{z,x} \leqslant 1$. Notice that $l_{z,x} = 1$ since $z \notin \mathcal{Z}(p^{-1}x)(\mathbb{F})$. By Proposition 5.3.3, $l_{z,x} \leqslant t_x(\tilde{z}) \leqslant n_{z,x}$ for all points $\tilde{z} \in \widehat{\mathcal{N}}_z^o(\mathbb{Z}_p)$, then we must have $l_{z,x} = t_x(\tilde{z}) = n_{z,x} = 1$. By Lemma 5.2.1, the number $\nu_p(\tilde{z}^\sharp(f_x)) = t_x(\tilde{z}) = 1$ for all $\mathbb{Z}_p$-algebra homomorphism $\tilde{z}^\sharp : \mathbb{Z}_p[[t_1, \cdots, t_{n-1}]] \to \mathbb{Z}_p$, hence $f_x \equiv (\text{unit}) \cdot p \pmod{\mathfrak{m}_z^2}$ by Lemma 5.1.1, then $d_x = f_x \in \mathfrak{m}_z \setminus \mathfrak{m}_z^2$. Notice that in this case the difference divisor $\mathcal{D}(x)$ is regular but not formally smooth over $\mathbb{Z}_p$ at z.
- If $n_{z,x} \geqslant 2$. Notice that we already have $l_{z,x} = 1$, then by Lemma 5.3.5 the formal scheme $\mathcal{Z}(x)$ is smooth over $\mathbb{Z}_p$ of relative dimension $n-2$ at the point z, hence $f_x \in \mathfrak{m}_z \setminus \mathfrak{m}_z^2$. Notice that in this case the difference divisor $\mathcal{D}(x)$ is formally smooth over $\mathbb{Z}_p$ at z.

Now we already show that $d_x \in \mathfrak{m}_z \setminus \mathfrak{m}_z^2$. Our next goal is to prove $d_{p^a x} \equiv (\text{unit}) \cdot p \bmod \mathfrak{m}_z^2$ for all $a \geqslant 1$. Let $\tilde{z} \in \widehat{\mathcal{N}}_z^o(\mathbb{Z}_p)$ be a point, by Lemma 5.2.1,

$$(\tilde{z}^\sharp(f_{p^a x})) = (\langle p^a \cdot x_{\text{crys},z}, l \rangle) = (p^a \cdot \langle x_{\text{crys},z}, l \rangle) = (p^a \cdot \tilde{z}^\sharp(f_x)). \quad (12)$$

Let $\widehat{\mathcal{Z}(p^a x)}_z$ be the completion of the formal scheme $\mathcal{Z}(p^a x)$ at z, there exists a Cartesian diagram by (12),

$$
\begin{array}{ccc}
\mathrm{Spf}\ \mathbb{Z}_p/(p^a \cdot \tilde{z}^\sharp(f_x)) & \longrightarrow & \mathrm{Spf}\ \mathbb{Z}_p \\
\downarrow & & \downarrow{\scriptstyle \tilde{z}} \\
\widehat{\mathcal{Z}(p^a x)}_z & \longrightarrow & \widehat{\mathcal{N}}_z^o.
\end{array}
$$

Recall that by the definition of difference divisors, we have $f_{p^a x} = \prod_{i=0}^a d_{p^i x}$ for all integer $a \geqslant 0$, hence Lemma 5.1.2 implies that $d_{p^a x} \equiv (\text{unit}) \cdot p \pmod{\mathfrak{m}_z^2}$ for all $a \geqslant 1$ since $d_x \in \mathfrak{m}_z \setminus \mathfrak{m}_z^2$.

For all integer $a \geqslant 0$, the point $z \in \mathcal{D}(p^a x)(\mathbb{F})$ because the local equation $d_{p^a x} \in \mathfrak{m}_z$. Moreover, the local equation $d_x \in \mathfrak{m}_z \setminus \mathfrak{m}_z^2$ implies that the difference divisor $\mathcal{D}(p^a x)$ is regular at the point z.

For $a = 0$, we have proved that the difference divisor $\mathcal{D}(x)$ is formally smooth over $\mathbb{Z}_p$ at z if and only if $n_{z,x} \geqslant 2$. For all $a \geqslant 1$, the equation $d_{p^a x} \equiv (\text{unit}) \cdot p \pmod{\mathfrak{m}_z^2}$, hence the difference divisor $\mathcal{D}(p^a x)$ is not formally smooth at the point z.

Now we have shown (ii), (iii) and (iv), let's come back to prove (i). The one-side inclusion $\mathcal{D}(x)(\mathbb{F}) \subset \mathcal{Z}(x)(\mathbb{F})$ follows from definition. Conversely, if $z' \in \mathcal{Z}(x)(\mathbb{F})$. Let $x' = p^{-l_{z',x}+1}x$, then $z' \in \mathcal{Z}(x')(\mathbb{F})$ but $z' \notin \mathcal{Z}(p^{-1}x')(\mathbb{F})$. (ii) implies that $z' \in \mathcal{D}(p^{l_{z',x}-1}x')(\mathbb{F}) = \mathcal{D}(x)(\mathbb{F})$. Therefore we conclude that $\mathcal{D}(x)(\mathbb{F}) = \mathcal{Z}(x)(\mathbb{F})$. □

Example 5.4.2 We give some examples of the equality $\mathcal{D}(x)(\mathbb{F}) = \mathcal{Z}(x)(\mathbb{F})$.

(a) The orthogonal case: Let H be the unique rank 4 self-dual quadratic lattice over $\mathbb{Z}_p$ whose discriminant is not a square in $\mathbb{Z}_p^\times$. Let $x \in \mathbb{V}^o$ be a non-isotropic vector and $n = \nu_p(q_{\mathbb{V}^o}(x))$. Let $x' = p^{-[n/2]} \cdot x$. Let $\mathscr{B}$ be the Bruhat-Tits building of

$PGL_2(\mathbb{Q}_{p^2})$. The element x' induces an automorphism of $\mathscr{B}$. Let $\mathscr{B}^{x'}$ be the fixed set. For a vertex $[\Lambda]$ in the Bruhat-Tits building $\mathscr{B}$ of $PGL_2(\mathbb{Q}_{p^2})$, let $d_{[\Lambda]}$ be the distance of $[\Lambda]$ to $\mathscr{B}^{x'}$.

Let $\mathcal{Z}(x)_{\mathbb{F}} = \mathcal{Z}(x) \times_{\breve{\mathbb{Z}}_p} \mathbb{F}$ and $\mathcal{D}(x)_{\mathbb{F}} = \mathcal{D}(x) \times_{\breve{\mathbb{Z}}_p} \mathbb{F}$. By [22, Theorem 3.6], we have the following equality of effective Cartier divisors on $\mathcal{N}_{\mathbb{F}}^o := \mathcal{N}^o \times_W \mathbb{F}$,

$$\mathcal{Z}(x)_{\mathbb{F}} = \sum_{[\Lambda]: d_{[\Lambda]} \leqslant (n-1)/2} (1 + p + \cdots + p^{(n-1)/2 - d_{[\Lambda]}}) \cdot \mathbb{P}_{[\Lambda]} + p^{n/2} \cdot s,$$

where $\mathbb{P}_{[\Lambda]}$ is a divisor associated to the vertex $[\Lambda]$ and isomorphic to the projective line $\mathbb{P}_{\mathbb{F}}^1$. The symbol s represents another divisor which only depends on the element x' and is nonempty only if n is even. Since $\mathcal{D}(x) = \mathcal{Z}(x) - \mathcal{Z}(p^{-1}x)$, we have

$$\mathcal{D}(x)_{\mathbb{F}} = \begin{cases} \sum\limits_{[\Lambda]: d_{[\Lambda]} = (n-1)/2} \mathbb{P}_{[\Lambda]} \\ + \sum\limits_{[\Lambda]: d_{[\Lambda]} \leqslant (n-3)/2} p^{(n-1)/2 - d_{[\Lambda]}} \cdot \mathbb{P}_{[\Lambda]} + (p^{n/2} - p^{(n-2)/2}) \cdot s, & \text{if } n \geqslant 1; \\ s, & \text{if } n = 0. \end{cases}$$

Therefore $\mathcal{D}(x)(\mathbb{F}) = \mathcal{Z}(x)(\mathbb{F})$.

(b) The unitary case: Let $n = 3$ and $F = \mathbb{Q}_p$. For an element $x \in \mathbb{V}^u$, the proof of ([23, Proposition 2.13]) implies that if $z \in \mathcal{Z}(x)(\mathbb{F})$, then the local equation of the difference divisor $\mathcal{D}(x)$ is not a unit at the point z, hence $\mathcal{Z}(x)(\mathbb{F}) \subset \mathcal{D}(x)(\mathbb{F})$. Therefore $\mathcal{D}(x)(\mathbb{F}) = \mathcal{Z}(x)(\mathbb{F})$.

Corollary 5.4.3 *Let $x \in \mathbb{V}^u$ (resp. $x \in \mathbb{V}^o$) be an element such that $(x, x) \in \mathcal{O}_E$ (resp. $q_{\mathbb{V}^o}(x) \in \mathcal{O}_F$).*

(i) *The difference divisor $\mathcal{D}(x)$ is a regular formal scheme.*

(ii) *The formal scheme $\mathcal{D}(x)$ is formally smooth over $\mathcal{O}_{\breve{F}}$ (resp. $\breve{\mathbb{Z}}_p$) of relative dimension $n - 2$ at a point $z \in \mathcal{D}(x)(\mathbb{F})$ if and only if $l_{z,x} = 1$ and $n_{z,x} \geqslant 2$.*

(iii) *Let $z \in \mathcal{Z}(x)(\mathbb{F})$ be a point. Let $\mathcal{D}(x)_{\mathbb{F}} = \mathcal{D}(x) \times_{\mathcal{O}_{\breve{F}}} \mathbb{F}$ and $\mathcal{Z}(x)_{\mathbb{F}} = \mathcal{Z}(x) \times_{\mathcal{O}_{\breve{F}}} \mathbb{F}$ (resp. $\mathcal{D}(x)_{\mathbb{F}} = \mathcal{D}(x) \times_{\breve{\mathbb{Z}}_p} \mathbb{F}$ and $\mathcal{Z}(x)_{\mathbb{F}} = \mathcal{Z}(x) \times_{\breve{\mathbb{Z}}_p} \mathbb{F}$). Then*

$$Tgt_z(\mathcal{D}(x)_{\mathbb{F}}) = Tgt_z(\mathcal{Z}(x)_{\mathbb{F}}),$$

where $Tgt_z(\mathcal{D}(x)_{\mathbb{F}})$ and $Tgt_z(\mathcal{Z}(x)_{\mathbb{F}})$ are the tangent spaces of the formal scheme $\mathcal{D}(x)_{\mathbb{F}}$ and $\mathcal{Z}(x)_{\mathbb{F}}$ at the point z, respectively.

Proof Let $x' = \varpi^{-l_{z,x}+1}x$ throughout this proof. Notice that $z \in \mathcal{Z}(x')(\mathbb{F})$ but $z \notin \mathcal{Z}(\varpi^{-1}x')(\mathbb{F})$ (resp. $z \notin \mathcal{Z}(p^{-1}x')(\mathbb{F})$).

We first prove (i). Let $z \in \mathcal{D}(x)(\mathbb{F})$ be a point. By Theorem 5.4.1 (iii) and (iv), the difference divisor $\mathcal{D}(x) = \mathcal{D}(\varpi^{l_{z,x}-1}x')$ (resp. $\mathcal{D}(x) = \mathcal{D}(p^{l_{z,x}-1}x')$) is regular at z. Since z is an arbitrary point, we conclude that the formal scheme $\mathcal{D}(x)$ is regular.

Now we prove (ii). Suppose that the difference divisor $\mathcal{D}(x)$ is formally smooth over $\mathcal{O}_{\breve{F}}$ (resp. $\breve{\mathbb{Z}}_p$) at the point $z \in \mathcal{D}(x)(\mathbb{F})$. If $l_{z,x} \geqslant 2$, the difference divisor $\mathcal{D}(x) = \mathcal{D}(\varpi^{l_{z,x}-1}x')$ (resp. $\mathcal{D}(x) = \mathcal{D}(p^{l_{z,x}-1}x')$) is not smooth over $\mathcal{O}_{\breve{F}}$ (resp. $\breve{\mathbb{Z}}_p$)

by Theorem 5.4.1 (ii), hence we must have $l_{z,x} = 1$. In the case $l_{z,x} = 1$, we get $n_{z,x} \geqslant 2$ by Theorem 5.4.1 (iii). For the converse direction. Let's assume $l_{z,x} = 1$ and $n_{z,x} \geqslant 2$. Theorem 5.4.1 (iii) implies that the difference divisor $\mathcal{D}(x)$ is formally smooth over $\mathcal{O}_{\breve{F}}$ (resp. $\breve{\mathbb{Z}}_p$) at z. This concludes the whole proof.

Finally we prove (iii). Let $\widehat{\mathcal{D}(x)}_z$ and $\widehat{\mathcal{Z}(x)}_z$ be the completion of the formal scheme $\mathcal{D}(x)$ and $\mathcal{Z}(x)$ at the point z respectively. By Schlessinger's criterion in [21, Theorem 2.11],

$$\mathrm{Tgt}_z(\mathcal{D}(x)_{\mathbb{F}}) = \widehat{\mathcal{D}(x)}_z(\mathbb{F}[\epsilon]), \ \ \mathrm{Tgt}_z(\mathcal{Z}(x)_{\mathbb{F}}) = \widehat{\mathcal{Z}(x)}_z(\mathbb{F}[\epsilon])$$

where $\mathbb{F}[\epsilon] = \mathbb{F}[X]/(X^2)$. The closed immersion $\mathcal{D}(x) \to \mathcal{Z}(x)$ implies that there is an injection $\mathrm{Tgt}_z(\mathcal{D}(x)_{\mathbb{F}}) \to \mathrm{Tgt}_z(\mathcal{Z}(x)_{\mathbb{F}})$. We want to show that this is bijective. We separate the proof into two cases.

(a) The point $z \notin \mathcal{Z}(\varpi^{-1}x)(\mathbb{F})$ (resp. $z \notin \mathcal{Z}(p^{-1}x)(\mathbb{F})$). Then we have $\widehat{\mathcal{D}(x)}_z \simeq \widehat{\mathcal{Z}(x)}_z$. The bijitivity is obvious.

(b) The point $z \in \mathcal{Z}(\varpi^{-1}x)(\mathbb{F})$ (resp. $z \in \mathcal{Z}(p^{-1}x)(\mathbb{F})$). Let $d_{x,z} \in \widehat{\mathcal{O}}_{\mathcal{N}^u,z}$ (resp. $d_{x,z} \in \widehat{\mathcal{O}}_{\mathcal{N}^o,z}$) be the local equation of the difference divisor. Since $l_{z,x} \geqslant 2$, the difference divisor $\mathcal{D}(x)$ is not formally smooth over $\mathcal{O}_{\breve{F}}$ (resp. $\breve{\mathbb{Z}}_p$) at the z, hence $d_{x,z} \equiv \varpi \bmod \mathfrak{m}_z^2$ (resp. $d_{x,z} \equiv p \bmod \mathfrak{m}_z^2$). Let $\tilde{z} \in \widehat{\mathcal{N}}_z^u(\mathbb{F}[\epsilon])$ (resp. $\tilde{z} \in \widehat{\mathcal{N}}_z^o(\mathbb{F}[\epsilon])$) be a point. Let $\tilde{z}^\sharp : \widehat{\mathcal{O}}_{\mathcal{N}^u,z} \to \mathbb{F}[\epsilon]$ (resp. $\tilde{z}^\sharp : \widehat{\mathcal{O}}_{\mathcal{N}^o,z} \to \mathbb{F}[\epsilon]$) be the corresponding ring homomorphism. We have $\tilde{z}^\sharp(d_{x,z}) = 0$ because $\tilde{z}^\sharp(\varpi) = 0$ (resp. $\tilde{z}^\sharp(p) = 0$) and $\tilde{z}^\sharp(\mathfrak{m}_z^2) \subset (\epsilon^2) = 0$. Hence $\widehat{\mathcal{D}(x)}_z(\mathbb{F}[\epsilon]) = \widehat{\mathcal{N}}_z^u(\mathbb{F}[\epsilon]) \supset \widehat{\mathcal{Z}(x)}_z(\mathbb{F}[\epsilon])$ (resp. $\widehat{\mathcal{D}(x)}_z(\mathbb{F}[\epsilon]) = \widehat{\mathcal{N}}_z^o(\mathbb{F}[\epsilon]) \supset \widehat{\mathcal{Z}(x)}_z(\mathbb{F}[\epsilon])$). Therefore $\widehat{\mathcal{D}(x)}_z(\mathbb{F}[\epsilon]) = \widehat{\mathcal{Z}(x)}_z(\mathbb{F}[\epsilon])$.

Therefore we have proved that $\mathrm{Tgt}_z(\mathcal{D}(x)_{\mathbb{F}}) = \mathrm{Tgt}_z(\mathcal{Z}(x)_{\mathbb{F}})$. □

Data availability No new data were created or analysed in this study. Data sharing is not applicable to this article.

Declarations

Conflict of interest Baiqing Zhu declares no conflict of interest.

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