Black Magic

Brandon Wang

Summer 2024

Example Problems

Example 1 (USA Junior MO 2011/4): A word is defined as any finite string of letters. A word is a palindrome if it reads the same backwards and forwards. Let a sequence of words $W_0, W_1, W_2, \ldots$ be defined as follows: $W_0 = a, W_1 = b$, and for $n \geq 2$, W_n is the word formed by writing W_{n-2} followed by W_{n-1}. Prove that for any $n \geq 1$, the word formed by writing $W_1, W_2, W_3, \ldots, W_n$ in succession is a palindrome.

Example 2 (USA TST 2018/4): Let n be a positive integer and let $S \subseteq \{0,1\}^n$ be a set of binary strings of length n. Given an odd number $x_1, \ldots, x_{2k+1} \in S$ of binary strings (not necessarily distinct), their majority is defined as the binary string $y \in \{0,1\}^n$ for which the i^{th} bit of y is the most common bit among the i^{th} bits of $x_1, \ldots, x_{2k+1}$. (For example, if $n = 4$ the majority of 0000, 0000, 1101, 1100, 0101 is 0100.)

Suppose that for some positive integer k, S has the property P_k that the majority of any $2k+1$ binary strings in S (possibly with repetition) is also in S. Prove that S has the same property P_k for all positive integers k.

Example 3 (Hall): Let G be a bipartite graph on A, B. Show that there exists an A-perfect matching (i.e. an injection $f : A \to B$ so that $(a, f(a))$ is an edge for all a) if and only if for all subsets S of A, the set

$$\{b \in B : (a,b) \text{ is an edge for some } a \in A\}$$

has size at least $|S|$.

Counting by Two

Problem 4 (EGMO 2021/5 Generalized): A plane has a special point O called the origin. Let P be a set of n points in the plane such that no three points in P lie on a line and no two points in P lie on a line through the origin.

A triangle with vertices in P is *fat* if O is strictly inside the triangle. Find the maximum number of fat triangles.

Problem 5 (ELMO 2017/5 Generalized): For n odd, the edges of K_n are each labeled with $1, 2$, or 3 such that any triangle has sum of labels at least 5. Determine the minimum possible average of all $\binom{n}{2}$ **labels.**

Problem 6 (CMC 2020/4 Generalized, via William Wang): Let n be an odd positive integer. Some of the unit squares of an $n \times n$ unit-square board are colored green. It turns out the green squares can be partitioned into k connected components $C_1, \ldots, C_k$, where two vertices are considered connected if and only if they share a vertex. Let $d(C_i)$ denote the diameter (maximum number of king moves needed to get from a to b for $a, b in C_i$) of C_i. Show that

$$d(C_1) + \cdots + d(C_k) \leq \frac{n^2 - 1}{2}.$$

Problem 7 (2018 C5): Let n be a positive integer. The organising committee of a tennis tournament is to schedule the matches for n players so that every two players play once, each day exactly one match is played, and each player arrives to the tournament site the day of his first match, and departs the day of his last match. For every day a player is present on the tournament, the committee has to pay 1 coin to the hotel. The organisers want to design the schedule so as to minimise the total cost of all players' stays. Determine this minimum cost.

Induct on Everything

Problem 8 (2013 C3): A crazy physicist discovered a new kind of particle wich he called an imon, after some of them mysteriously appeared in his lab. Some pairs of imons in the lab can be entangled, and each imon can participate in many entanglement relations. The physicist has found a way to perform the following two kinds of operations with these particles, one operation at a time.

(i) If some imon is entangled with an odd number of other imons in the lab, then the physicist can destroy it.

(ii) At any moment, he may double the whole family of imons in the lab by creating a copy I' of each imon I. During this procedure, the two copies I' and J' become entangled if and only if the original imons I and J are entangled, and each copy I' becomes entangled with its original imon I; no other entanglements occur or disappear at this moment.

Prove that the physicist may apply a sequence of much operations resulting in a family of imons, no two of which are entangled.

Problem 9 (2006 C4): A cake has the form of an $n \times n$ square composed of n^2 unit squares. Strawberries lie on some of the unit squares so that each row or column contains exactly one strawberry; call this arrangement $\mathcal{A}$.

Let $\mathcal{B}$ be another such arrangement. Suppose that every grid rectangle with one vertex at the top left corner of the cake contains no fewer strawberries of arrangement $\mathcal{B}$ than of arrangement $\mathcal{A}$. Prove that arrangement $\mathcal{B}$ can be obtained from $\mathcal{A}$ by performing a number of switches, defined as follows:

A switch consists in selecting a grid rectangle with only two strawberries, situated at its top right corner and bottom left corner, and moving these two strawberries to the other two corners of that rectangle.

Problem 10 (2009 C8): For any integer $n \geq 2$, we compute the integer $h(n)$ by applying the following procedure to its decimal representation. Let r be the rightmost digit of n.

- If $r = 0$, then the decimal representation of $h(n)$ results from the decimal representation of n by removing this rightmost digit 0.
- If $1 \leq r \leq 9$ we split the decimal representation of n into a maximal right part R that solely consists of digits not less than r and into a left part L that either is empty or ends with a digit strictly smaller than r. Then the decimal representation of $h(n)$ consists of the decimal representation of L, followed by two copies of the decimal representation of $R - 1$. For instance, for the number $17,151,345,543$, we will have $L = 17,151$, $R = 345,543$ and $h(n) = 17,151,345,542,345,542$.

Prove that, starting with an arbitrary integer $n \geq 2$, iterated application of h produces the integer 1 after finitely many steps.

More Problems

Exercise 11 (2016 C1): In Lineland there are $n \geq 1$ towns, arranged in a road running from left to right. Each town has a left bulldozer (put to the left of the town and facing left) and a right bulldozer (put to the right of the town and facing right). The sizes of the $2n$ bulldozers are distinct. Every time when a left and right bulldozer confront each other, the larger bulldozer pushes the smaller one off the road. On the other hand, bulldozers are quite unprotected at their rears; so, if a bulldozer reaches the rear-end of another one, the first one pushes the second one off the road, regardless of their sizes.

Let A and B be two towns, with B to the right of A. We say that town A can sweep town B away if the right bulldozer of A can move over to B pushing off all bulldozers it meets. Similarly town B can sweep town A away if the left bulldozer of B can move over to A pushing off all bulldozers of all towns on its way.

Prove that there is exactly one town that cannot be swept away by any other one.

Problem 12 (2009 C3): Let n be a positive integer. Given a sequence $\varepsilon_1, ..., \varepsilon_{n-1}$ with $\varepsilon_i \in \{0, 1\}$, the sequences $a_0, ..., a_n$ and $b_0, ..., b_n$ are constructed with the following rules:

$$a_0 = b_0 = 1, a_1 = b_1 = 7,$$

$$a_{i+1} = \begin{cases} 2a_{i-1} + 3a_i \text{ if } \varepsilon_i = 0 \\ 3a_{i-1} + a_i \text{ if } \varepsilon_i = 1 \end{cases}$$

and

$$b_{i+1} = \begin{cases} 2b_{i-1} + 3b_i \text{ if } \varepsilon_{n-i} = 0 \\ 3b_{i-1} + b_i \text{ if } \varepsilon_{n-i} = 0 \end{cases}$$

for $i = 1, ..., n - 1$. Prove that $a_n = b_n$.

Problem 13 (2017 N7): An ordered pair (x, y) of integers is a *primitive point* if the greatest common divisor of x and y is 1. Given a finite set S of primitive points, prove that there exists a positive integer n and integers $a_0, a_1, ..., a_n$ such that, for each (x, y) in S, we have

$$a_0x^n + a_1x^{n-1}y + \cdots + a_{n-1}xy^{n-1} + a_ny^n = 1.$$

Problem 14 (2009 C7): Let $a_1, a_2, ..., a_n$ be distinct positive integers and let M be a set of $n - 1$ positive integers not containing $a_1 + a_2 + \cdots + a_n$. A grasshopper is to jump along the real axis, starting at the point 0 and making n jumps to the right with lengths $a_1, a_2, ..., a_n$ in some order. Prove that the order can be chosen in such a way that the grasshopper never lands on any point in M.

Problem 15 (2017 A7): Let $a_0, a_1, a_2, ...$ be a sequence of integers and $b_0, b_1, b_2, ...$ a sequence of *positive* such that $a_0 = 0, a_1 = 1$, and

$$a_{n+1} = \begin{cases} a_nb_n + a_{n-1} \text{ if } b_{n-1} = 1 \\ a_nb_n - a_{n-1} \text{ if } b_{n-1} > 1 \end{cases}$$

for $n = 1, 2, ...$ Prove that at least one of the numbers a_{2017} and a_{2018} must be greater than or equal to 2017.