Department of Mathematical Sciences

Masters Dissertation

Isometries of Riemannian Manifolds

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Declaration

This piece of work is a result of my own work except where it forms an assessment based on group project work. In the case of a group project, the work has been prepared in collaboration with other members of the group. Material from the work of others not involved in the project has been acknowledged and quotations and paraphrases suitably indicated.

Acknowledgements

I would like to thank Dr. Fernando Galaz-García for his continued supervision throughout my project. I would also like to thank all my loved ones for their guidance and support; I would be lost without them.

Abstract

This report aims to introduce and develop an understanding of Riemannian manifolds, and their Lie groups of isometries. It opens by extending the concepts of curves and surfaces to arbitrarily many dimensions, while forming the tools necessary to extend calculus to these new objects. The report continues by linking together these objects with other common structures, namely groups, metric spaces, and inner product spaces. The report concludes by proving an important theorem, first presented by Sumner Myers and Norman Steenrod. Their theorem states that the isometry group of a Riemannian manifold is a Lie group.

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Introduction

The study of differential geometry, or at least its roots, can be traced back over 2000 years to ancient Greek mathematicians and their work on the geometry of the earth. However, the theory only started to become rigorous and more fleshed out after calculus was first developed in the 1600s. Through the work of Carl Friedrich Gauss and Bernhard Riemann in the 1800s, differential geometry became a field of study in its own right.

In this report, we will be exploring Riemannian manifolds and their isometry groups. In part I, we investigate smooth manifolds and how they can be extended to two other separate concepts: Lie groups and the aforementioned Riemannian manifolds. In part II, we look at the isometries of Riemannian manifolds, and what properties the isometry group carries. This knowledge can then be applied to tackling a difficult proof of a vital theorem at the heart of Riemannian geometry: the Myers–Steenrod theorem.

Note on prior knowledge:

Throughout this report, we may often use concepts without explicitly defining/explaining them. This report is designed as a useful tool for third and fourth-year undergraduate mathematics students looking for an introduction to Riemannian geometry, and some of its important results and applications. Therefore, we assume the readers to have a good understanding of undergraduate linear algebra, group theory, calculus, real analysis and basic topology. Some key definitions are included in the appendix (part IV) to aid the reader.

Other Important Notes:

- Unless explicitly stated otherwise, definitions will be adapted from John M. Lee's [13, 14] and Manfredo do Carmo's [3].
- We will use the terms "smooth" and "differentiable" interchangeably to mean infinitely differentiable, or of class C^∞.
- We will often write "Euclidean space" to mean the real inner product space $\mathbb{R}^n$ equipped with the standard inner product (the "dot product"). Although this is not strictly accurate, any n-dimensional Euclidean space $\mathbb{E}^n$ is isomorphic to the n-dimensional real coordinate space $\mathbb{R}^n$. So, to avoid any unnecessary confusion, when we write "Euclidean space", we are referring to $\mathbb{R}^n$.
- I claim no explicit originality to any of the examples presented, however any example in this report has been worked through without any external resources or help.

Part I

Manifolds

Chapter 1

Smooth Manifolds

Smooth manifolds generalise the theory of differentiable surfaces in 3-dimensional Euclidean space to higher dimensions. Smooth manifolds are therefore, in basic terms, spaces that look locally like a part of Euclidean space $\mathbb{R}^n$, on which we can perform calculus. To define them rigorously however, we first need to introduce some other key concepts that are necessary to understand beforehand.

1.1 Smooth Manifolds & Examples

We first begin by defining a differentiable atlas.

Definition 1.1 (Differentiable atlas)**:** Let M be a set. Let I be an index set. A **differentiable atlas** on M is a collection of pairs $\{(U_\alpha, \varphi_\alpha)\}_{\alpha \in I}$ that satisfy the following conditions:

(1) Each U_α is an open subset of M and $\bigcup_{\alpha \in I} U_\alpha = M$ (i.e., $\{U_\alpha\}_{\alpha \in I}$ is a cover of M).

(2) Each $\varphi_\alpha : U_\alpha \subset M \to V_\alpha \subset \mathbb{R}^n$ is a bijective map, with each V_α being an open subset of $\mathbb{R}^n$, and, for all $\alpha, \beta \in I$, the sets $\varphi_\alpha(U_\alpha \cap U_\beta) \subset \mathbb{R}^n$ and $\varphi_\beta(U_\alpha \cap U_\beta) \subset \mathbb{R}^n$ are also open.

(3) For each $\alpha, \beta \in I$,

$$\varphi_\beta \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap U_\beta) \to \varphi_\beta(U_\alpha \cap U_\beta) \subset \mathbb{R}^n$$

is a smooth map between two open subsets of $\mathbb{R}^n$.

We call each function φ_α a **(coordinate) chart**, while the maps $\varphi_\beta \circ \varphi_\alpha^{-1}$ are called **coordinate changes** or **transition maps**. The integer n is called the **dimension** of the space M. See fig. 1.1 for a diagram illustrating this definition.

Remark 1.2: The coordinate changes are bijective because the inverse of $\varphi_\beta \circ \varphi_\alpha^{-1}$ is $\varphi_\alpha \circ \varphi_\beta^{-1}$.

Remark 1.3: An atlas is **finite**, **countable**, or **uncountable** if its index set I is finite, countable, or uncountable, respectively.

Example 1.4: The identity map $Id : \mathbb{R}^n \to \mathbb{R}^n$, defined by $x \mapsto x$ for $x \in \mathbb{R}^n$, is an n-dimensional chart on the set $\mathbb{R}^n$. The single pair $\{(\mathbb{R}^n, Id)\}$ is a smooth atlas for Euclidean space as it obeys all three necessary conditions. △

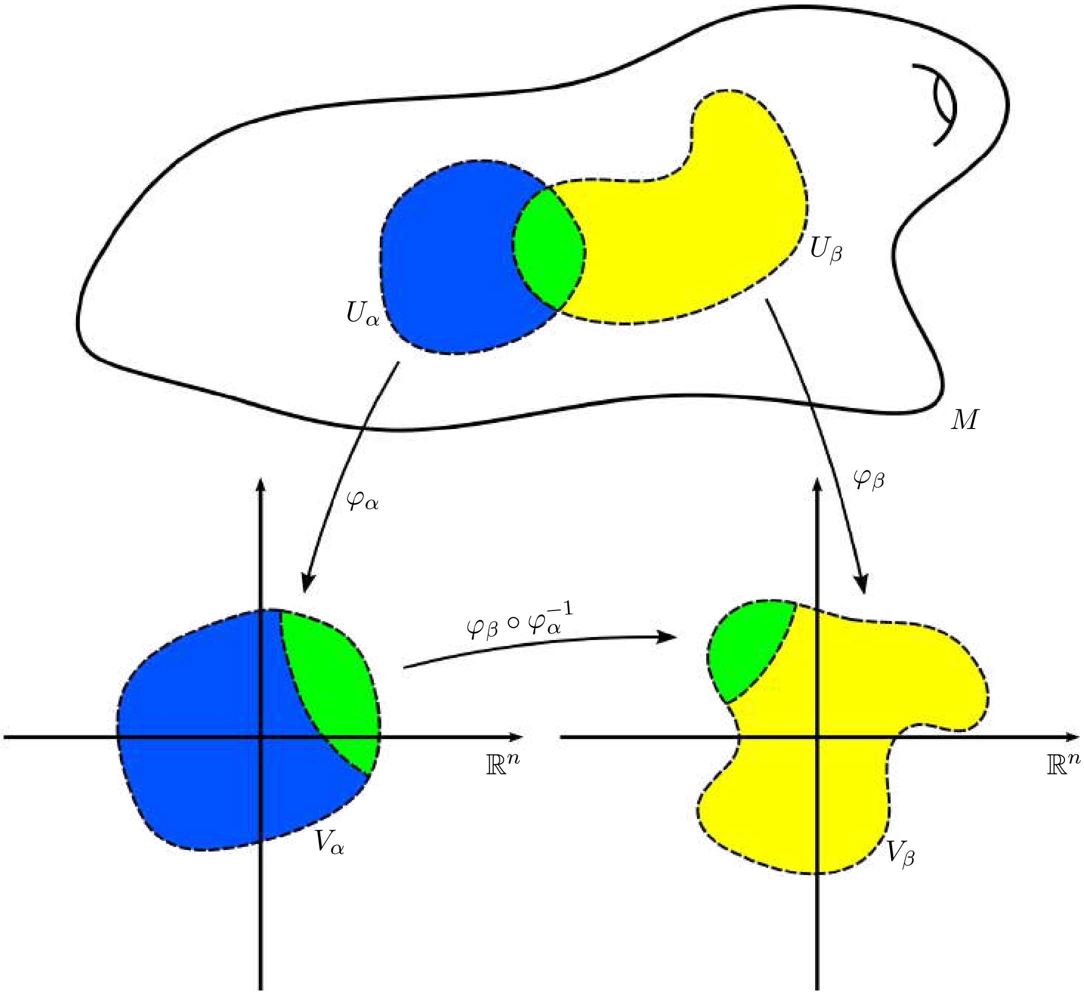

Figure 1.1: A coordinate change between two charts

Definition 1.5 (Maximal differentiable atlas)**:** A differentiable atlas is **maximal** if the family $\{(U_\alpha, \varphi_\alpha)\}_{\alpha\in I}$ is maximal with respect to the conditions (1), (2), and (3) from definition 1.1.

A maximal smooth atlas $\{(U_\alpha, \varphi_\alpha)\}_{\alpha\in I}$ on M induces a unique topology on M. It is simple to see this because each U_α is an open set within M and each φ_α is a homeomorphism[1]. The topology induced by the atlas is defined explicitly below.

Definition 1.6 (Induced topology from an atlas)**:** A subset X of M is open if and only if $\varphi_\alpha(X \cap U_\alpha) \subset V_\alpha \subset \mathbb{R}^n$ is open, for all $\alpha \in I$.

We need to introduce one final notion before we move onto the main topic of this chapter, smooth manifolds.

Definition 1.7 (Hausdorff space)**:** A topological space M is **Hausdorff** if, for every pair of distinct points $p, q \in M$, there are open subsets $U, V \subseteq M$ such that $p \in U$, $q \in V$ with $U \cap V = \emptyset$. One may also say that M has a **Hausdorff topology**.

Example 1.8: The set of real numbers $\mathbb{R}$ equipped with the standard metric topology is a Hausdorff space. This can easily be shown. Let $p, q \in \mathbb{R}$ be two distinct points and

[1] See A.2.3 for definition.

let $\delta = \frac{1}{4}|p-q|$. Clearly, $U = (p-\delta, q+\delta)$ and $V = (q-\delta, q+\delta)$ are open subsets of $\mathbb{R}$ containing p and q, respectively. To show that U and V are disjoint, we will aim for a contradiction. Let $x \in U \cap V$. Then,

$$|p-q| = |p+x-x-q| \leq |p-x| + |x-q| < \delta + \delta = 2\delta = \frac{1}{2}|p-q|.$$

This is clearly a contradiction as $|p-q| \not< \frac{1}{2}|p-q|$, hence U and V are disjoint and the real numbers $\mathbb{R}$ are Hausdorff. $\triangle$

Definition 1.9 (Smooth manifold)**:** A set M is an n-dimensional **smooth manifold** if it has a countable, maximal smooth atlas (whose charts have the codomain $\mathbb{R}^n$) which induces a Hausdorff topology on M. The maximal smooth atlas is called a **smooth structure** on M.

For simplicity, we may write M^n to mean a smooth manifold M of dimension n.

Remark 1.10: We require the maximality of the atlas because we want each smooth structure (in this case a maximal atlas) to define uniquely what it means for a function on a manifold to be smooth (we will visit this later). Two non-maximal atlases may be identical in this sense, hence maximality is needed to ensure this uniqueness.

Example 1.11: The simplest example of an n-dimensional smooth manifold is Euclidean space $\mathbb{R}^n$ equipped with the atlas seen in example 1.4: $\{(\mathbb{R}^n, Id)\}$. $\triangle$

Example 1.12: Let $U \subset \mathbb{R}^n$ be an open subset. Define $Id_U : U \to \mathbb{R}^n$ by $Id_U(u) := u$ for all $u \in U$. Then U equipped with the atlas $\{(U, Id_U)\}$ is an n-dimensional smooth manifold. $\triangle$

Example 1.13 (More simple examples)**:** Figure 1.2, below, shows four examples of smooth manifolds: the n-sphere S^n (see example 1.19), a torus, a double torus, and the upper sheet of a hyperboloid (which is the underlying manifold of hyperbolic space $\mathbb{H}^n$). $\triangle$

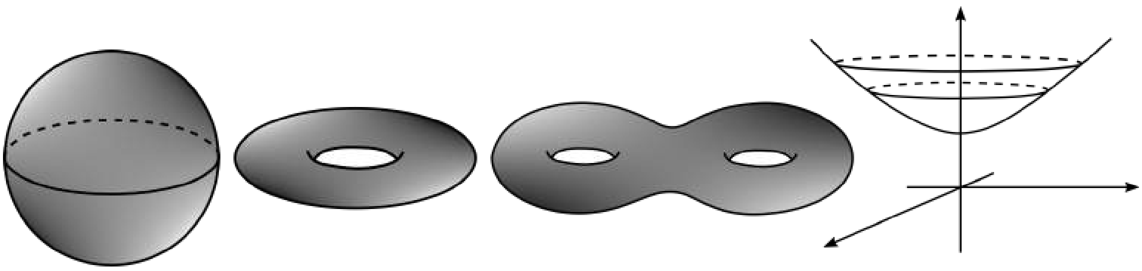

Figure 1.2: Four examples of smooth manifolds

Example 1.14 (Real projective space)**:** The real projective space, denoted $\mathbb{R}P^n$, is the space of all lines that pass through the origin $0 = (0, \ldots, 0) \in \mathbb{R}^{n+1}$. The space can be formed by taking the quotient $\mathbb{R}^{n+1}/\{0\}$, with the equivalence relation $x \sim cx$ for non-zero $c \in \mathbb{R}$. This space is an example of an n-dimensional smooth manifold. $\triangle$

Definition 1.15 (Connected smooth manifold)**:** A smooth manifold M is a **connected smooth manifold** if the only subsets of M that are both open and closed are $\emptyset$ and M itself. Equivalently, M is connected if it cannot be divided into two disjoint non-empty open sets. Otherwise, M is disconnected.

Note: We define the notion of connectedness as we will require manifolds to be connected in many of our theorems and results. The notion of connectedness can be applied to general topological spaces, see appendix A.2.

Remark 1.16: Every connected smooth manifold is also path-connected.

Example 1.17 (Disjoint subsets of $\mathbb{R}^n$)**:** Let

$$M^- = \{x = (x_1, \dots, x_n) \in \mathbb{R}^n \,|\, x_1 < 0\} \text{ and } M^+ = \{x = (x_1, \dots, x_n) \in \mathbb{R}^n \,|\, x_1 > 0\}.$$

These are both open subsets of Euclidean space, hence they are both n-dimensional smooth manifolds when equipped with the corresponding atlases (see example 1.12). Define $M = M^- \cup M^+$. Then M, equipped with the smooth atlas $\{(M^-, Id_{M^-}), (M^+, Id_{M^+})\}$, is also a smooth manifold as it clearly fits the necessary conditions.

However, M *can* be divided into two disjoint non-empty open sets, namely M^- and M^+. Hence M is a *disconnected* smooth manifold. △

Example 1.18 (Disjoint unions)**:** Let M_1, M_2 be any two smooth manifolds. Define $M = M_1 \sqcup M_2$.[2] Then M is a disconnected smooth manifold. △

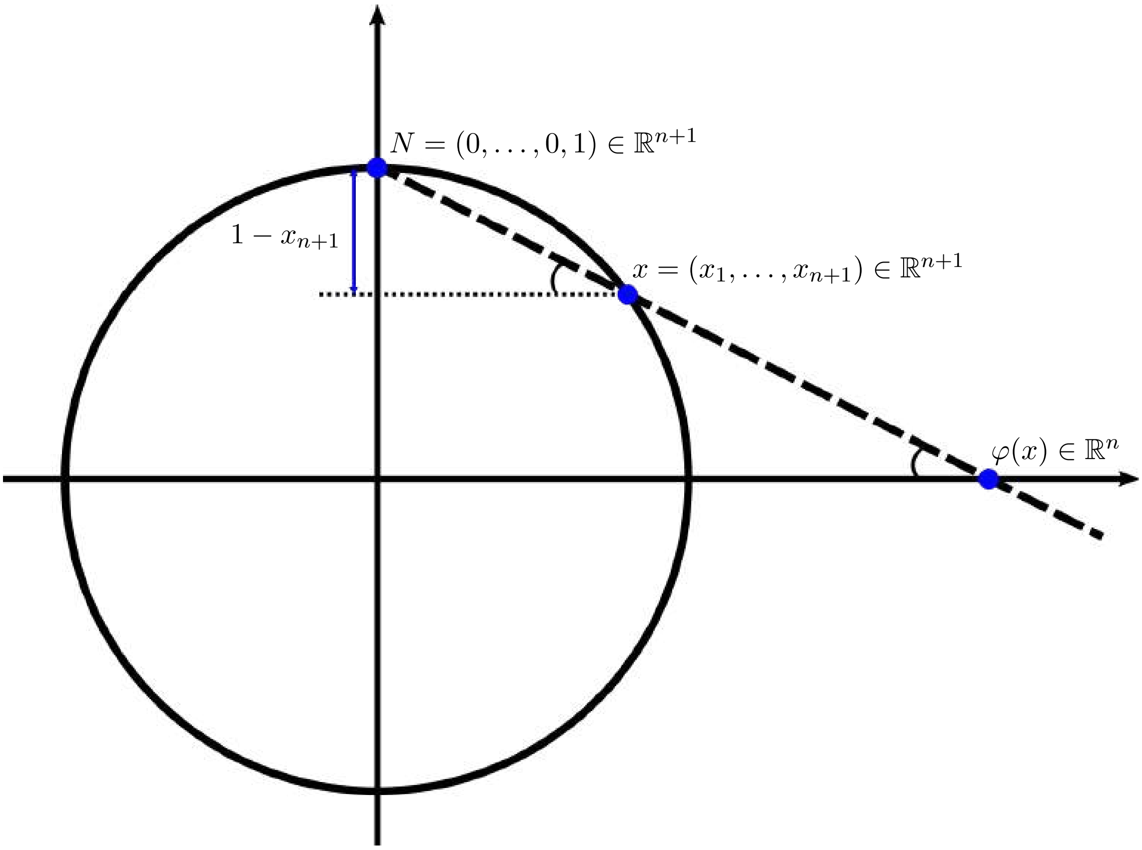

Figure 1.3: Stereographic projection in $(n+1)$ dimensions

Example 1.19 (The n-sphere is a smooth manifold)**:** Let $S^n = \{x = (x_1, \dots, x_{n+1}) \in \mathbb{R}^{n+1} : ||x|| = 1\}$ be the unit n-sphere, where $|| \bullet ||$ is the standard Euclidean norm. It is important to note that the dimension of S^n is in fact n, and not $n+1$ which is the

[2]See [24] for the definition of "$\sqcup$" (the reference uses the notation "$\cup^*$", rather than "$\sqcup$").

dimension of the Euclidean space in which S^n is embedded. To show that S^n is a smooth manifold, we will construct the atlas explicitly, with the charts being given by *stereographic projections*.

Firstly, define $N := (0, \ldots, 0, 1) \in \mathbb{R}^{n+1}$ and $S := (0, \ldots, 0, -1) \in \mathbb{R}^{n+1}$ (i.e. N and S are the north and south poles of S^n, respectively). Then, define $U_N := S^n \setminus \{N\}$, $U_S := S^n \setminus \{S\}$, $V_N = V_S = \mathbb{R}^n$. One can easily see that $S^n = U_N \cup U_S$.

Define the map $\varphi_N : U_N \to V_N$ as follows: for $p \in U_N$, $\varphi_N(p)$ is the intersection of the line passing through p and N with the plane $\{x_{n+1} = 0\}$. Define $\varphi_S : U_S \to V_S$ similarly. We claim that S^n equipped with the atlas $\{(U_\alpha, \varphi_\alpha)\}_{\alpha \in \{N,S\}}$ is a smooth manifold.

We can construct φ_N explicitly. Using Figure 1.3, one can use the similarity of triangles to see that

$$\varphi_N(x) = \frac{1}{1 - x_{n+1}}(x_1, \ldots, x_n) \in V_N, \text{ for } x \in U_N.$$

Then, similarly for φ_S,

$$\varphi_S(x) = \frac{1}{1 + x_{n+1}}(x_1, \ldots, x_n) \in V_S, \text{ for } x \in U_S.$$

Two computations will show:

$$\varphi_N^{-1}(y) = \frac{1}{1 + ||y||^2}(2y_1, 2y_2, \ldots, 2y_n, ||y||^2 - 1), \text{ for } y = (y_1, \ldots, y_n) \in V_N,$$
$$\varphi_S^{-1}(y) = \frac{1}{1 + ||y||^2}(2y_1, 2y_2, \ldots, 2y_n, 1 - ||y||^2), \text{ for } y = (y_1, \ldots, y_n) \in V_S.$$

Hence, both φ_N and φ_S are diffeomorphisms as they are bijective and smooth, with smooth inverses.

Now, we must show that the transition maps are smooth. Indeed, for $y \in V_S$,

$$\begin{aligned}
\varphi_N \circ \varphi_S^{-1}(y) &= \varphi_N\left(\frac{1}{1 + ||y||^2}\left(2y_1, \ldots, 2y_n, 1 - ||y||^2\right)\right) \\
&= \frac{1}{1 - \frac{1-||y||^2}{1+||y||^2}} \frac{1}{1 + ||y||^2}(2y_1, \ldots, 2y_n) \\
&= \frac{1 + ||y||^2}{2||y||^2} \frac{1}{1 + ||y||^2}(2y_1, \ldots, 2y_n) \\
&= \frac{1}{||y||^2}(y_1, \ldots, y_n)
\end{aligned}$$

This is clearly smooth in $\mathbb{R}^n \setminus \{0\}$. Similarly, $\varphi_S \circ \varphi_N^{-1}$ is smooth.

Therefore, we have that $\{(U_\alpha, \varphi_\alpha)\}_{\alpha \in \{N,S\}}$ is a countable smooth atlas. This is because $U_N \cup U_S = S^n$, the charts are bijective and their images are clearly open, and the transition maps are smooth.

We can also see that S^n is Hausdorff as it is a subspace of $\mathbb{R}^{n+1}$ which is a Hausdorff space[3].

[3] See [13, Proposition A.17(g)].

Hence, S^n equipped with the smooth structure $\{(U_\alpha, \varphi_\alpha)\}_{\alpha\in\{N,S\}}$ is a smooth manifold. Moreover, it is a *connected* smooth manifold as it cannot be separated into two disjoint non-empty open sets. △

1.2 Real-World Atlas Example

To help explain atlases and their charts further, we can look at real-world maps of Earth and real-world atlases.[4] Real-world atlases are collections of maps of Earth, showing what the planet looks like locally in a flat plane. Similarly, an atlas in the mathematical sense is a collection of maps (charts) on a manifold, showing what this manifold looks locally like in Euclidean space. So, relating manifolds to Earth, we can imagine the earth as a manifold, the open subsets U_α are cities/countries/continents etc., and 2-dimensional maps are the subsets of Euclidean space – V_α.

We can, rather crudely, model Earth as a 2-sphere (within 3-dimensional Euclidean space). To take this model in 3-dimensional space and project it onto a 2-dimensional map, we can use a process similar to the stereographic projection we used in example 1.19, namely *Mercator projection*. In 3-dimensional stereographic projection, we take a point at the top of the sphere and project it down onto a 2-dimensional plane. In Mercator projection however, we take the center of the sphere (the origin) and project outwards onto a cylinder tangential to the "equator" of the sphere. Then, this cylinder is "unravelled" to form a 2-dimensional map. This will yield the coordinate charts, φ_α, in a similar way to the method shown in example 1.19. This method can be seen in action in fig. 1.4, below.

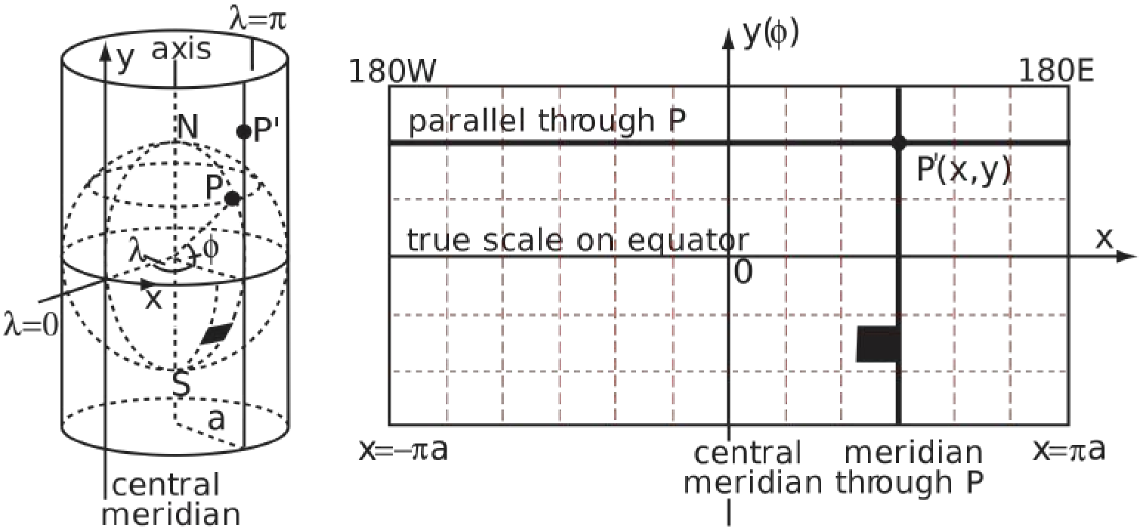

Figure 1.4: The geometry of the Mercator projection
(Peter Mercator, Public domain, via Wikimedia Commons, see [15])

As a more direct example, fig. 1.5 shows a 2-dimensional map of Durham, a hilly city. This map was obtained using the Mercator projection chart from the earth (a smooth manifold) to a flat surface (a subset of $\mathbb{R}^2$).

[4]When using the term "real-world", we are referring to maps and atlases in the geographical sense.

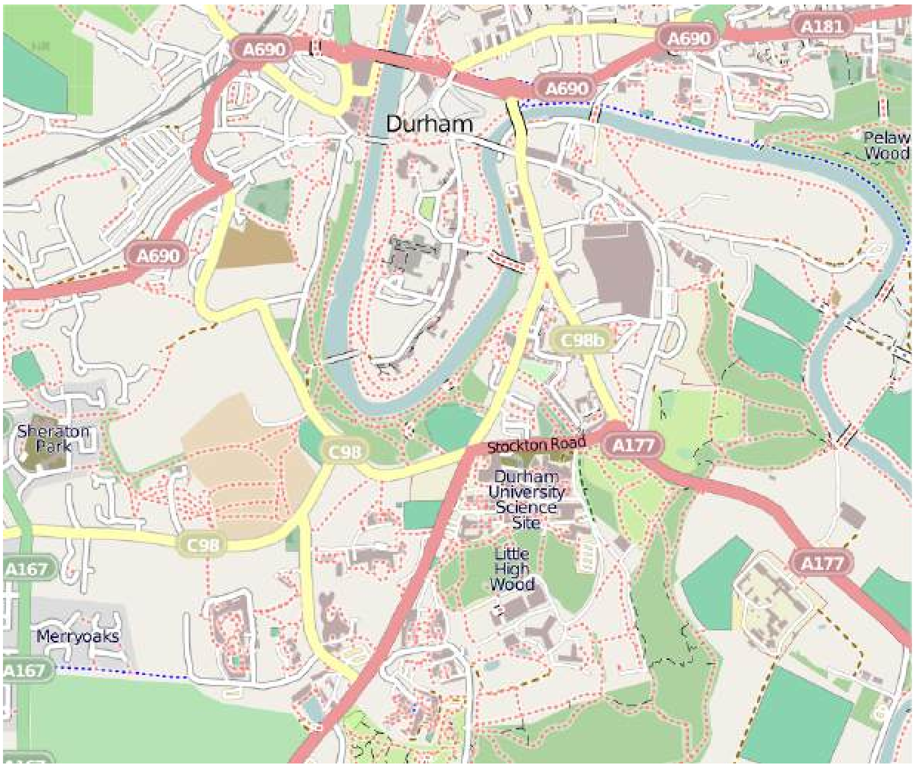

Figure 1.5: A map(chart) of Durham city, extracted from OpenStreetMap
(Wereon, CC BY-SA 2.0, via Wikimedia Commons, see [25])

1.3 Smooth Manifolds are Locally Euclidean

We will now discuss a property of smooth manifolds and why this property makes them "nice" to work with. These points are based on material from [13, Chapter 1].

Each chart φ_α is a homeomorphism which takes open subsets of M to open subsets of $\mathbb{R}^n$. Since we know that each φ_α is bijective, we therefore can deduce that each U_α looks locally like Euclidean space. Since the collection $\{U_\alpha\}_{\alpha\in I}$ form a cover of M, this means everywhere on M looks locally like Euclidean space, i.e. M is locally Euclidean. This allows us to think of each U_α as an open subset of M and an open subset of $\mathbb{R}^n$. Furthermore, thinking of two corresponding coordinate grids on M and $\mathbb{R}^n$ allows us to think of any point on the manifold as its corresponding point in Euclidean space. Figure 1.6, below, shows an example of this diagrammatically.

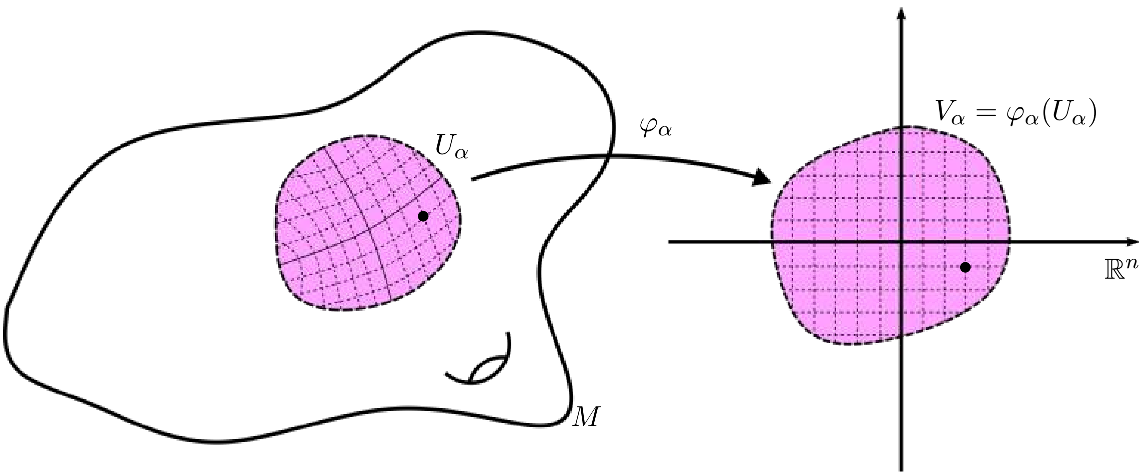

Figure 1.6: Corresponding coordinate grids on $U_\alpha \subset M$ and $V_\alpha \subset \mathbb{R}^n$

We will often represent a point $p \in U_\alpha \subset M$ by its corresponding coordinates in $\mathbb{R}^n$, $(x_1, \ldots, x_n) = \varphi_\alpha(p) \in V_\alpha \subset \mathbb{R}^n$, by saying "$(x_1, \ldots, x_n)$ is the local coordinate representation of p". This will be particularly useful when dealing with functions and curves on manifolds, which we will discuss in the latter sections of chapter 1.

This correspondence between smooth manifolds and Euclidean space is also useful as smooth manifolds inherit many properties of $\mathbb{R}^n$ and we know more about Euclidean space than we do about a general manifold. The properties that smooth manifolds inherit include local compactness, local connectedness and first countability.

This also, alongside the smoothness of the manifold, allows us to do calculus which we will see in sections 1.4 to 1.6.

1.4 Differentiable Maps

By equipping a manifold with a smooth structure (maximal differentiable atlas), we can take the notions of smooth functions and maps in Euclidean space and transfer them to smooth manifolds. As all the coordinate changes/transition maps within an atlas are smooth, the charts are all "smoothly compatible". This allows the smoothness of a map/function to be well-defined because it is independent with respect to the choice of chart.

As John Lee points out on [13, pg 32], when studying smooth manifolds we often make a slight distinction between *functions* and *maps*. We will generally use the term *function* to mean a map with a codomain of $\mathbb{R}^n$, for any $n \geq 1$. Whereas, a *map/mapping* can be any type of map, including those between smooth manifolds.

We will begin by defining a smooth map between manifolds. Note, $\mathbb{R}^n$ is a smooth manifold for all $n \geq 1$, so definition 1.20 defines both a smooth function and a general smooth map/mapping.

Definition 1.20 (Smooth map between manifolds)**:** Let M and N be smooth manifolds of dimension m and n, respectively. Then, a map $f : M \to N$ is **smooth at $p \in M$** if, given a chart $\widetilde{\varphi} : \widetilde{U} \subset N \to \widetilde{V} \subset \mathbb{R}^n$ around $f(p) \in N$, there exists a chart $\varphi : U \subset M \to V \subset \mathbb{R}^m$

around $p \in M$ such that $f(U) \subset \widetilde{U}$ and the map

$$\widetilde{\varphi} \circ f \circ \varphi^{-1} : V \subset \mathbb{R}^m \to \widetilde{V} \subset \mathbb{R}^n \tag{1.1}$$

is differentiable at $\varphi(p) \in V \subset \mathbb{R}^m$. We say that f is a **smooth map** if it is smooth at every $p \in M$.

We call eq. (1.1) the **coordinate representation of f**, and is often denoted by $\hat{f}$ or $\widehat{f}$.

Remark 1.21: As mentioned at the beginning of this section, the definition does not depend on the choice of charts, as they are smooth functions. Therefore, f is smooth if Equation 1.1, above, is smooth.

Example 1.22: Let $M = M^- \cup M^+$ (as in example 1.17) and $N = S^1 := \{x = (x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 = 1\}$ be two smooth manifolds, equipped with the identity atlas (from example 1.4) and stereographic projection atlas (from example 1.19), respectively. Let $(U_\alpha, \varphi_\alpha)$ denote the relevant charts on M and $(\widetilde{U}_\beta, \widetilde{\varphi}_\beta)$ denote the relevant charts on N.

Then, the map $f : M \to N$ defined by

$$f(x) = \left(\frac{x_1}{\sqrt{x_1^2 + x_2^2}}, \frac{x_2}{\sqrt{x_1^2 + x_2^2}} \right), \text{ for } x \in M$$

is a smooth map between smooth manifolds M and N.

Indeed, $(\frac{x_1}{\sqrt{x_1^2+x_2^2}}, \frac{x_2}{\sqrt{x_1^2+x_2^2}})$ is a smooth map between Euclidean spaces, excluding $(0, 0)$. Now, all that remains to check is that $\widetilde{\varphi}_\beta \circ f \circ \varphi_\alpha^{-1} = \widetilde{\varphi}_\beta \circ f$ is smooth, for $\beta = N, S$ (as $\varphi_\alpha = \varphi_\alpha^{-1} = Id$).

Recall,

$$\widetilde{\varphi}_N(x) = \frac{1}{1 - x_2}(x_1) \in \mathbb{R} \text{ and } \widetilde{\varphi}_S(x) = \frac{1}{1 + x_2}(x_1) \in \mathbb{R}.$$

Then,

$$\widetilde{\varphi}_N \circ f(x) = \frac{1}{1 - \frac{x_2}{\sqrt{x_1^2+x_2^2}}} \left(\frac{x_1}{\sqrt{x_1^2 + x_2^2}} \right),$$

and

$$\widetilde{\varphi}_S \circ f(x) = \frac{1}{1 + \frac{x_2}{\sqrt{x_1^2+x_2^2}}} \left(\frac{x_1}{\sqrt{x_1^2 + x_2^2}} \right).$$

It can be verified that these functions are smooth on the domain M. Hence, f is a smooth map between smooth manifolds. $\triangle$

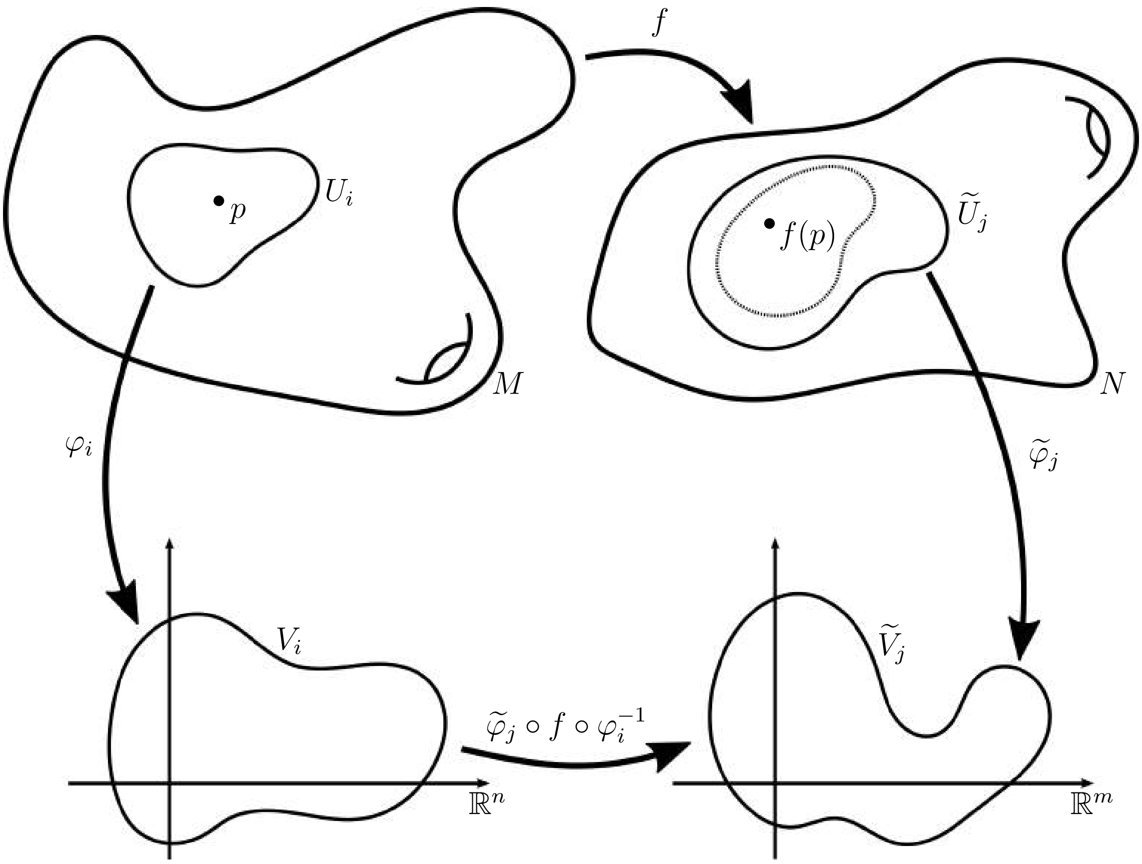

Figure 1.7: A smooth map f between smooth manifolds M and N

It is much easier to visualise these smooth maps using a diagram. Let us take two smooth manifolds, M^m and N^n, equipped with the smooth structures $\{(U_\alpha, \varphi_\alpha)\}_{\alpha\in I}$ and $\{(\widetilde{U}_\alpha, \widetilde{\varphi}_\alpha)\}_{\alpha\in J}$, respectively. Then, let $f : M \to N$ be a map between the manifolds. Figure 1.7 shows this map, along with the map between Euclidean spaces on which we define smoothness.

So, for $f : M \to N$ to be smooth, we must have for every $p \in M$:

- There exists a $(U_i, \varphi_i) \subset M$ which contains p and a $(\widetilde{U}_j, \widetilde{\varphi}_j) \subset N$ which contains $f(p)$ such that $f(U_i) \subset \widetilde{U}_j$,
- The map $\widetilde{\varphi}_j \circ f \circ \varphi_i^{-1}$ is a smooth map between Euclidean spaces.

Note: The dotted subset shown in fig. 1.7 represents $f(U_i) \subset \widetilde{U}_j$.

Definition 1.23 (Diffeomorphism)**:** Let M and N be two smooth manifolds. A smooth map $f : M \to N$ is called a **diffeomorphism** if it is bijective and if its inverse $f^{-1} : N \to M$ is also smooth.

If there exists a diffeomorphism between two manifolds M and N, we say that M and N are **diffeomorphic**.

Diffeomorphisms are incredibly strong relations between smooth manifolds; they are the smooth manifold version of an isomorphism. Any two smooth manifolds that are diffeomorphic are often said to be “essentially the same” or “indistinguishable”.

Example 1.24: For any smooth manifold M^n, any coordinate chart

$$\varphi_\alpha : U_\alpha \subset M \to V_\alpha \subset \mathbb{R}^n$$

is a diffeomorphism. $\triangle$

Definition 1.25 (Differentiable curve)**:** Let $J \subset \mathbb{R}$ be an interval[5] and let M be a smooth manifold. A smooth function $\gamma : J \to M$ is a **differentiable curve** in M.

Definition 1.26: Let M be a smooth manifold and let $p \in M$. Define

$$D(M,p) := \{f : M \to \mathbb{R} \mid f \text{ is differentiable at } p\}.$$

In other words, $D(M,p)$ is the set of real-valued functions on M that are differentiable at the point $p \in M$.

We define the set $D(M,p)$ as it is useful to use this shorthand notation in future definitions and proofs, rather than writing out the full set definition each time.

1.5 Tangent Vectors & the Tangent Space

Definition 1.27 (Tangent vectors)**:** Let M be a smooth manifold. Let $\gamma : (-\varepsilon, \varepsilon) \to M$ be a differentiable curve in M with $\gamma(0) = p \in M$. The **tangent vector to the curve $\boldsymbol{\gamma}$ at $\boldsymbol{t = 0}$** is the linear derivation $\gamma'(0) : D(M,p) \to \mathbb{R}$ given by

$$\gamma'(0)(f) \;=\; \frac{d(f \circ \gamma)}{dt}\bigg|_{t=0}, \quad f \in D(M,p).$$

A **tangent vector at $\boldsymbol{p}$** is the tangent vector of some curve $\gamma : (-\varepsilon, \varepsilon) \to M$ with $\gamma(0) = p$.

Definition 1.28 (Tangent space)**:** The set of all tangent vectors at $p \in M$ is the **tangent space of $\boldsymbol{M}$ at $\boldsymbol{p}$**, denoted by T_pM.

We will take the following proposition without proof. This is because the proof involves some concepts that are not directly relevant to this report, and hence have not been defined. A proof by Lee can be found in [13, pgs 60-61].

Proposition 1.29: *Let M be a smooth manifold of dimension n and take $p \in M$. Then, the tangent space T_pM is an n-dimensional real vector space. Therefore, T_pM is isomorphic to $\mathbb{R}^n$.*

[5] We will mostly be using open intervals but it is useful to allow J to include one or both endpoints for some purposes.

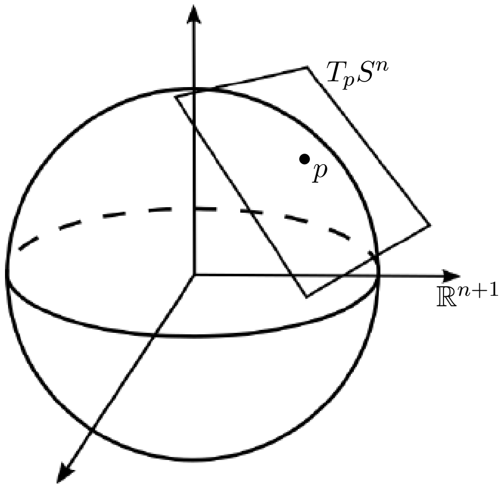

Figure 1.8: T_pS^n at $p \in S^n$

Example 1.30 (Tangent space of the n-sphere)**:** Let us take $S^n = \{x \in \mathbb{R}^{n+1} : ||x|| = 1\}$, the unit n-sphere. By example 1.19, we know this is a smooth manifold. Then, T_pS^n is the tangent space to the unit n-sphere at a point $p \in S^n$. This can be seen in Figure 1.8. As S^n is embedded in $\mathbb{R}^{n+1}$, the tangent space at a point is, simply, the affine hyperplane which is orthogonal to p, with respect to the standard Euclidean inner product, that also passes through p. As S^n is a smooth manifold of dimension n, and a hyperplane within $\mathbb{R}^{n+1}$ is, by definition, a space of dimension n, this example therefore agrees with the above proposition 1.29. △

Definition 1.31 (Tangent bundle)**:** Let M^n be a smooth manifold. The **tangent bundle of M**, denoted TM, is defined as follows,

$$\begin{aligned} TM &= \bigsqcup_{p \in M} T_pM \\ &= \{(p, v) : p \in M, v \in T_pM\}, \end{aligned}$$

where "$\bigsqcup$" is the disjoint union ([24]).

Remark 1.32: The tangent bundle TM is a $2n$-dimensional smooth manifold, where $n = \dim M$.

Definition 1.33 (Coordinate tangent vectors)**:** Let M^n be a smooth manifold with $\varphi : U \subset M \to V \subset \mathbb{R}^n$ be a coordinate chart. Similarly to what we presented in section 1.3, we can write $\varphi = (x_1, \ldots, x_n)$ because φ takes values in $\mathbb{R}^n$. This is equivalent to saying that $x_i : U \to \mathbb{R}$ is the i-th coordinate function of φ. Let $\{e_i\}_{i=1}^n$ be the standard basis for $\mathbb{R}^n$ and let $\varepsilon > 0$. Then, the ***i*-th coordinate curve through p** $\in U$ is given by

$$\begin{aligned} c_i : (-\varepsilon, \varepsilon) \subset \mathbb{R} &\to U \subset M \\ t &\mapsto \varphi^{-1}(\varphi(p) + te_i). \end{aligned}$$

Note that $c_i(0) = p$ for each $i = 1, \ldots, n$. The ***i*-th coordinate tangent vector at p** is the tangent vector $c_i'(0)$, denoted by

$$c_i'(0) =: \left.\frac{\partial}{\partial x_i}\right|_p .$$

Note: Coordinate tangent vectors are particular tangent vectors associated to the coordinate directions of the chart that a point is within.

The following proposition will, again, be taken without proof. It is fairly simple to verify this holds true by taking any tangent vector and showing it can be decomposed into a linear combination of coordinate tangent vectors. A proof can be found in [13, pgs 60-61].

Proposition 1.34: *Let M be an n-dimensional smooth manifold and $p \in M$. From proposition 1.29, we know that T_pM is an n-dimensional real vector space. The coordinate tangent vectors defined in definition 1.33, $\left\{ \frac{\partial}{\partial x_i}\Big|_p \right\}_{i=1}^{n}$, form a basis for T_pM.*

Example 1.35: For Euclidean space $\mathbb{R}^n$, its tangent space $T_p\mathbb{R}^n$ at $p \in \mathbb{R}^n$ has basis vectors $\frac{\partial}{\partial x_i}\Big|_p = e_i$, with e_i being the standard basis vectors for Euclidean space. $\triangle$

1.6 The Differential

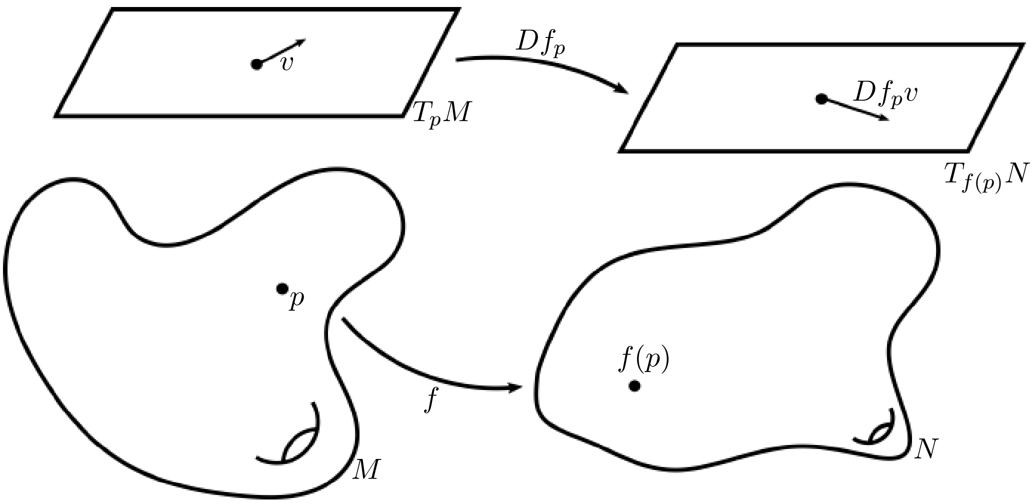

Figure 1.9: The differential $Df_p : T_pM \to T_{f(p)}N$

Definition 1.36 (The differential)**:** Let M and N be smooth manifolds of dimension m and n, respectively. Let $f : M \to N$ be a smooth map. For each $p \in M$, we define a map

$$Df_p : T_pM \to T_{f(p)}N,$$

called **the differential of f at p** as follows:

$$Df_p(v)(h) = v(h \circ f), \quad v \in T_pM,\ h \in D(N, f(p)).$$

In other words, given a differentiable curve $\alpha : (-\varepsilon, \varepsilon) \to M$ with $\alpha(0) = p$, $\alpha'(0) = v$, and letting $\beta = f \circ \alpha$, Df_p is given by $Df_p(v) = \beta'(0)$.

To put this in simpler terms, the differential of a smooth map between smooth manifolds is a linear map between their tangent spaces. If the smooth manifolds are Euclidean spaces, the differential is simply the usual derivative given by the Jacobian matrix (see example 1.37). The differential can be seen diagrammatically in Figure 1.9.

Example 1.37 (Differential of a map between Euclidean spaces)**:** Recall, from example 1.11, that Euclidean spaces are smooth manifolds.

Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a smooth function between Euclidean spaces. Let $p \in \mathbb{R}^n$. Let $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_m)$ represent the coordinates in $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively.

We know, from proposition 1.34, that the coordinate tangent vectors

$$\left\{ \frac{\partial}{\partial x_i}\bigg|_p \right\}_{i=1}^n \quad \text{and} \quad \left\{ \frac{\partial}{\partial y_j}\bigg|_{f(p)} \right\}_{j=1}^m$$

form bases for $T_p\mathbb{R}^n$ and $T_{f(p)}\mathbb{R}^m$, respectively. Recall that these bases correspond to the canonical bases of each Euclidean space. Now, the differential of f at p, $Df_p : T_p\mathbb{R}^n \to T_{f(p)}\mathbb{R}^m$, is a linear map between finite dimensional vector spaces. This means that it can be represented as a matrix. If we denote the matrix that represents Df_p by $A = (a_{ij})$, $a_{ij} \in \mathbb{R}$, then we can see that:

$$Df_p\left(\frac{\partial}{\partial x_i}\bigg|_p \right) = \sum_{j=1}^m a_{ij} \frac{\partial}{\partial y_j}\bigg|_{f(p)}. \tag{1.2}$$

We can compare coefficients of eq. (1.2) to determine the values of the entries of A. Indeed, let $f_k = y_k \circ f$ be the k-th component of f. Then,

$$\begin{aligned} Df_p\left(\frac{\partial}{\partial x_i}\bigg|_p \right) y_k &= \frac{\partial}{\partial x_i}\bigg|_p (y_k \circ f) = \frac{\partial f_k}{\partial x_i}(p), \\ \sum_{j=1}^m a_{ij} \frac{\partial}{\partial y_j}\bigg|_{f(p)} y_k &= \sum_{j=1}^m a_{ij}\delta_{jk} = a_{ik}, \end{aligned}$$

where δ_{ij} is the Kronecker delta. Therefore, the matrix representation of the differential Df_p is given as:

$$\begin{bmatrix} \frac{\partial f_1}{\partial x_1}(p) & \cdots & \frac{\partial f_1}{\partial x_n}(p) \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1}(p) & \cdots & \frac{\partial f_m}{\partial x_n}(p) \end{bmatrix}. \tag{1.3}$$

Indeed, this is the Jacobian matrix $\mathbf{J}$ of f at the point p. This shows that the differential between smooth manifolds is simply the generalisation of the derivative in normal Euclidean space. △

1.7 Partitions of Unity

Definition 1.38 (Locally finite, partition of unity)**:** Let M be a smooth manifold. A family of open sets $U_\alpha \subset M$ with $\bigcup_\alpha U_\alpha = M$ is **locally finite** if every point $p \in M$ has a neighbourhood W such that $W \cap U_\alpha \neq \emptyset$ for only a finite number of indices α.

Recall the **support** of a function $f : M \to \mathbb{R}$ is defined as

$$\operatorname{supp}(f) = \overline{\{x \in M : f(x) \neq 0\}}.$$

A family $\{f_\alpha\}$ of smooth functions $f_\alpha : M \to \mathbb{R}$ is a **smooth partition of unity** (subordinate to the open cover $\{U_\alpha\}$) if the following three conditions are satisfied:

(i) For each α, $f_\alpha \geq 0$ and $\text{supp}(f_\alpha) \subset U_\alpha$ for some coordinate chart U_α of a smooth atlas $\{(U_\alpha, \varphi_\alpha)\}$ of M,

(ii) The family $\{U_\alpha\}$ is locally finite,

(iii) The following equality holds – $\sum_\alpha f_\alpha(p) = 1$, for all $p \in M$.

Since the collection of U_α-s are locally finite, and $\text{supp}(f_\alpha) \subset U_\alpha$, every point $p \in M$ lies in only finitely many of the sets $\text{supp}(f_\alpha)$. Hence, $f_\alpha(p) \neq 0$ for only finitely many α. Therefore, the sum in (iii) is a finite sum that is well-defined.

Partitions of unity are useful generally as they can be used to patch together objects that are only defined locally into something that is defined on a global scale. We will find them particularly useful when exploring Riemannian metrics and their existence, in section 3.1.

Chapter 2

Lie Groups

2.1 Definitions

Definition 2.1 (Lie group)**:** A **Lie group** is a smooth manifold G (without boundary) that is also a group in the algebraic sense, with the condition that the multiplication map $(g, h) \mapsto gh$ and the inverse map $g \mapsto g^{-1}$, for $g, h \in G$, are both differentiable.

Definition 2.2 (Left & right-translation)**:** Let G be a Lie group and $g \in G$. Then, the maps $L_g : G \to G$ and $R_g : G \to G$ defined by

$$L_g(h) = gh \text{ and } R_g(h) = hg$$

are called **left** and **right-translation** by g, respectively. These maps are also often known as left and right-multiplication.

It is simple to see that L_g and R_g are both diffeomorphisms. One can observe that the inverse to L_g is given by $L_g^{-1} = L_{g^{-1}}$. Similarly, $R_g^{-1} = R_{g^{-1}}$. Then, by definition, the multiplication map $(g, h) \mapsto gh$ is a smooth map for all $g, h \in G$. Therefore, L_g and R_g, and their inverses $L_g^{-1} = L_{g^{-1}}$ and $R_g^{-1} = R_{g^{-1}}$, are all smooth. Hence, both left and right-translation maps are diffeomorphisms.

Remark 2.3: Left-translation and right-translation commute with each other: $L_g R_h = R_h L_g$.

2.2 Examples of Lie groups

In this section, we will first give some basic examples of Lie groups in example 2.4. We will then follow that with a more in depth example that shows how to prove a set is a Lie group in example 2.5.

Example 2.4: The following are all examples of Lie groups:

- $\mathbb{R}^n$, with the group operation being addition
- $\mathbb{R}^+$, with the group operation being multiplication
- $GL(n, \mathbb{R})$ and $GL(n, \mathbb{C})$
- The orthogonal group, $O(n)$

- The special orthogonal group, $SO(n)$

$\triangle$

Example 2.5 (The orthogonal group is a Lie group)**:** Recall that the orthogonal group $O(n)$ in Euclidean geometry is defined as follows:

$$O(n) = \{A \in M_n(\mathbb{R}) \mid AA^T = I_n\}$$

where $M_n(\mathbb{R})$ is the set of all n-dimensional square matrices and I_n is the n-dimensional identity matrix. It is simple to show that $O(n)$ is a group in the algebraic sense, so we will omit this. The multiplication map (matrix multiplication) is smooth because matrix multiplication is a smooth map. The inverse map (matrix inversion) is smooth by Cramer's rule [23]. So, we simply need to show that $O(n)$ is a smooth manifold. To do this, we will use the preimage theorem[1,2] by finding a regular value of a smooth function that we will construct.

First, define S_n as the space of real, symmetric, square matrices of dimension n. Then, define a map $f : M_n(\mathbb{R}) \to S_n$ by

$$f(A) = AA^T \text{ , for } A \in M_n(\mathbb{R}). \tag{2.1}$$

By using linear algebra, it is clear that AA^T is a symmetric matrix. Hence, $O(n) = f^{-1}(I_n)$. As f is a smooth function, we simply need to show that I_n is a regular value of f to show that $f^{-1}(I_n) = O(n)$ is a differentiable manifold. To do this, we need to show that the differential Df_A is a surjective map for all $A \in O(n)$.

First, let's compute the differential. In this example, we are only working within Euclidean spaces, so we can compute the differential as we would in normal Euclidean geometry.

$$\begin{aligned}
Df_A(B) &= \lim_{h\to 0} \frac{f(A+hB) - f(A)}{h} \\
&= \lim_{h\to 0} \frac{(A+hB)(A+hB)^T - AA^T}{h} \\
&= \lim_{h\to 0} \frac{AA^T + hAB^T + hBA^T + h^2BB^T - AA^T}{h} \\
&= \lim_{h\to 0} (AB^T + BA^T - hBB^T) \\
&= AB^T + BA^T
\end{aligned}$$

Hence, we need to find a matrix B for every matrix C with $Df_A(B) = C$. We know that C is symmetric because Df_A maps to S_n, just as f does. Therefore, $AB^T + BA^T$ is symmetric, so $AB^T + BA^T = 2BA^T = C$. This means that $B = \frac{1}{2}CA$. Then,

$$Df_A(B) = Df_A(\frac{1}{2}CA) = \frac{1}{2}[A(CA)^T + (CA)A^T] = \frac{1}{2}[AA^TC^T + CAA^T] = \frac{1}{2}(C^T + C) = C$$

using the symmetry of C and the orthogonality of A.

Therefore, we have shown that Df_A is surjective. Hence, I_n is a regular value of f and $f^{-1} = O(n)$ is a smooth manifold.

Finally, this shows that $O(n)$ is a Lie group. $\triangle$

[1] Also often called the regular value theorem for manifolds or inverse value theorem.
[2] See [13, Corollary 5.14].

Example 2.6 (Simple left-translation example)**:** Let's take $\mathbb{R}^n$ with the group operation being addition. This is a Lie group as stated in example 2.4. Let $x \in \mathbb{R}^n$. The left-translation map $L_x : \mathbb{R}^n \to \mathbb{R}^n$ is defined by $L_x(y) = x + y$, for all $y \in \mathbb{R}^n$. Right-translation $R_x : \mathbb{R}^n \to \mathbb{R}^n$ is defined in a similar manner. As Euclidean space equipped with addition is an abelian group, $L_x(y) = x + y = y + x = R_x(y)$, hence $L_x \equiv R_x$ in $\mathbb{R}^n$. It is a trivial exercise to verify that L_x and R_x are diffeomorphisms. $\triangle$

Chapter 3

Riemannian Manifolds

In standard Euclidean geometry, we can use inner products and metrics to define and measure the lengths of tangent vectors, the angle between tangent vectors, and many other "metric" ideas that are used in geometry. To do this on smooth manifolds, we need a tool to help us. This tool will be Riemannian metrics.

3.1 Riemannian Metrics

Definition 3.1 (Riemannian metric)**:** A **Riemannian metric** on a smooth manifold M is a correspondence which assigns each point $p \in M$ an inner product

$$g_p : T_pM \times T_pM \to \mathbb{R}$$

that varies smoothly on p, i.e., for any two smooth vector fields X, Y on M, the function $p \mapsto g_p(X(p), Y(p))$ is differentiable.

Note: We may also use the notation

$$g_p(v, w) = \langle v, w \rangle_p \text{ for any } v, w \in T_pM$$

for the Riemannian metric. We may often omit the index p if it is clear and obvious which Riemannian metric we are working with.

Note: The Riemannian metric, as defined above, is actually *not* a "metric" in the classical sense of metric spaces[1]. However, we will see in chapter 4 that they are closely related to one another. In this report, we will explicitly state whether we are talking about a Riemannian metric or metric in the sense of metric spaces, unless it is obvious in the context.

Definition 3.2 (Length of a tangent vector)**:** Let (M, g) be a Riemannian manifold with $p \in M$. Then, the **length of a tangent vector** $v \in T_pM$ is defined as

$$|v|_p = ||v||_p = \sqrt{\langle v, v \rangle_p}.$$

Many sources use both of these notations to represent the same thing, we define them both here to avoid any possible confusion.

[1] See appendix A.3.

If the specific p is obvious or not important, then we will use the notation $|v|_g = ||v||_g = \sqrt{g(v,v)}$.

Remark 3.3: The length of a tangent vector, $||\cdot||_p$, is a norm. More specifically, it is the norm induced by the inner product $\langle \cdot\,,\cdot\rangle_p$.

Proposition 3.4 (Existence of Riemannian metrics)**:** *Every smooth manifold M admits a Riemannian metric g.*

Proof: Let $M = \bigcup_\alpha U_\alpha$ be a cover of M by a locally finite family of coordinate charts $\{(U_\alpha, \varphi_\alpha)\}$. For each α, take g_α in U_α to be the Riemannian metric with its local expression $(g_\alpha)_{i,j}$ being the identity matrix. Let $\{f_\alpha : M \to \mathbb{R}\}$ be a smooth partition of unity subordinate to the open cover $\{U_\alpha\}$. Then, define

$$g = \sum_\alpha f_\alpha g_\alpha.$$

We know that the family of supports of the f_α-s is locally finite, therefore the above sum is a finite sum. Hence, g is well-defined and differentiable. Clearly, g is also symmetric and bilinear (linear in each argument). We also have, by definition of partitions of unity, that $f_\alpha \geq 0$ for each α, and $\sum_\alpha f_\alpha = 1$. Hence, g is also positive-definite. Therefore, g satisfies the conditions of inner products and defines a Riemannian metric on the smooth manifold M. $\square$

3.2 Riemannian Manifolds & Examples

Definition 3.5 (Riemannian manifold)**:** A **Riemannian manifold (M, g)** is a smooth manifold M paired with a Riemannian metric g.

Note: If the choice of Riemannian metric g on M is clear, we may simply write M to refer to the Riemannian manifold (M, g).

Example 3.6 (Euclidean space as a Riemannian manifold)**:** Let us take Euclidean space $\mathbb{R}^n$ as our smooth manifold. Then, for any $p \in \mathbb{R}^n$, its tangent space $T_p\mathbb{R}^n$ has basis vectors $\{e_i\}_{i=1}^n$, where e_i are the standard basis vectors in Euclidean space (see example 1.35). Therefore, we can let the Riemannian metric on $\mathbb{R}^n$ be the standard inner product ("dot product") on $\mathbb{R}^n$. Indeed, let $\bar{g}_p : T_p\mathbb{R}^n \times T_p\mathbb{R}^n \to \mathbb{R}$ be the Riemannian metric at p. This Riemannian metric is defined by

$$\bar{g}_p(v, w) = \langle v, w\rangle_p = \sum_{i=1}^n v_i \cdot w_i,$$

for $v = \sum_{i=1}^n v_i e_i \in T_p\mathbb{R}^n$ and $w = \sum_{i=1}^n w_i e_i \in T_p\mathbb{R}^n$. Therefore, the geometry of this Riemannian manifold is that of standard Euclidean space. $\triangle$

Example 3.7 (The n-sphere as a Riemannian manifold)**:** As the n-sphere, S^n, is embedded (contained) in Euclidean space $\mathbb{R}^{n+1}$, it inherits the metric of $\mathbb{R}^{n+1}$ (see example 3.6) restricted to S^n. We call this the round metric on S^n, and denote it by g_o. Therefore, (S^n, g_o) is a Riemannian manifold. $\triangle$

3.3 Metric Space Structure of Riemannian Manifolds

We saw that the concept of a Riemannian metric is not a metric in the classical metric space sense. From here though, one may wonder whether there is a distance preserving metric on Riemannian manifolds. In this section, we will see how we construct the distance function on manifolds, and see that while they are not equal to the Riemannian metric, they are heavily linked.

3.3.1 Length of Curves

Definition 3.8 (Piecewise differentiable curve)**:** Let M be a smooth manifold. A curve $\gamma : [a, b] \to M$ is called **piecewise differentiable** if it is continuous and there exists a partition $a = t_0 < \cdots < t_n = b$ such that every $\gamma\big|_{[t_i, t_{i+1}]}$ is differentiable.

Definition 3.9 (Reparametrisation)**:** Let M be a smooth manifold. Let $\gamma : [a, b] \to M$ and $\widetilde{\gamma} : [c, d] \to M$ be piecewise differentiable curves. We say that $\widetilde{\gamma}$ is a **reparametrisation of γ** if $\widetilde{\gamma} = \gamma \circ \varphi$ for some diffeomorphism $\varphi : [c, d] \to [a, b]$. A diagram of a reparametrisation can be seen in fig. 3.1, below.

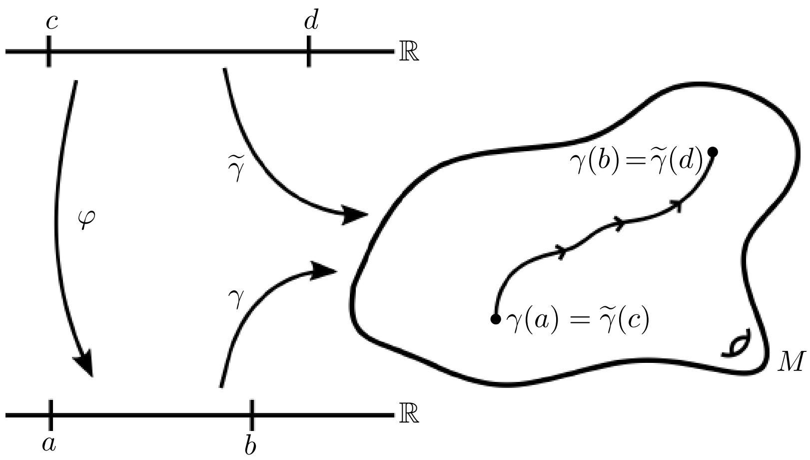

Figure 3.1: A curve γ reparametrised as $\widetilde{\gamma} = \gamma \circ \varphi$

Definition 3.10 (Length of a curve)**:** Let (M, g) be a Riemannian manifold and $\gamma : [a, b] \to M$ be a differentiable curve. The **length of γ**, denoted by $L(\gamma)$, is defined as

$$L(\gamma) = \int_a^b ||\gamma'(t)||_{\gamma(t)} dt,$$

where $||v||_p$ is defined as in definition 3.2. If $\gamma : [a, b] \to M$ is a *piecewise* differentiable curve, then

$$L(\gamma) = \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} ||\gamma'(t)||_{\gamma(t)} dt.$$

Example 3.11: Let $S^3 \subset \mathbb{R}^4$ be the 3-dimensional unit sphere, equipped with its standard round metric induced by the Euclidean metric from $\mathbb{R}^4$. Let $\gamma : [0, \pi] \to S^3$ be the smooth curve defined by $\gamma(t) = \frac{1}{\sqrt{2}}(\cos(2t), \sin(t), \sin(2t), \cos(t))$.

Then, for t in our domain, we have

$$\gamma'(t) = \frac{1}{\sqrt{2}}(-2\sin(2t), \cos(t), 2\cos(2t), -\sin(t)).$$

From this, we get

$$\begin{aligned} ||\gamma'(t)||_{\gamma(t)} &= \sqrt{\langle \gamma'(t), \gamma'(t) \rangle} \\ &= \sqrt{\frac{1}{2}(4\sin^2(2t) + \cos^2(t) + 4\cos^2(2t) + \sin^2(t))} \\ &= \sqrt{5/2}. \end{aligned}$$

Finally, we can calculate the length of γ:

$$\begin{aligned} L(\gamma) &= \int_0^\pi ||\gamma'(t)||_{\gamma(t)} dt \\ &= \sqrt{\frac{5}{2}} \int_0^\pi dt \\ &= \sqrt{\frac{5}{2}} \cdot \pi. \end{aligned}$$

△

Proposition 3.12: *Let (M, g) be a Riemannian manifold and let $\gamma : [a, b] \to M$ be a piecewise smooth curve. If $\widetilde{\gamma}$ is a reparametrisation of γ, then $L(\gamma) = L(\widetilde{\gamma})$.*

The proof of this proposition has been adapted from John Lee [13].

Proof: Assume that γ is smooth. If γ is only piecewise smooth, we can apply the following arguments to each segment which is smooth separately and come to the same conclusion.

Let $\varphi : [c, d] \to [a, b]$ be a diffeomorphism with $\widetilde{\gamma} = \gamma \circ \varphi$. As φ is a diffeomorphism, its derivative φ' is never zero. We can see this by looking at the formula for the derivative of the inverse: $(\varphi^{-1})' = \frac{1}{\varphi'(\varphi^{-1})}$. We know the derivative of the inverse exists, by definition, so $\varphi'(\varphi^{-1}) \neq 0 \Rightarrow \varphi' \neq 0$. As φ is smooth and $\varphi' \neq 0$, we conclude that either $\varphi' > 0$ or $\varphi' < 0$. First, assume $\varphi' > 0$. Then,

$$\begin{aligned} L(\widetilde{\gamma}) &= \int_c^d ||\widetilde{\gamma}'(t)||_g \, dt = \int_c^d \left|\left| \frac{d}{dt}(\gamma \circ \varphi)(t) \right|\right|_g dt \\ &\stackrel{*}{=} \int_c^d ||\varphi'(t)\gamma'(\varphi(t))||_g \, dt = \int_c^d ||\gamma'(\varphi(t))||_g \varphi'(t) \, dt \\ &\stackrel{\dagger}{=} \int_a^b ||\gamma'(u)||_g \, du = L(\gamma), \end{aligned}$$

where "*" denotes the use of the chain rule and "†" denotes the use of the change of variables formula $u = \varphi(t)$.

For $\varphi' < 0$, the calculation is very similar, with sign changes in the fourth and fifth equalities which cancel each other out. Hence, $L(\gamma) = L(\widetilde{\gamma})$. □

Example 3.13: Let us carry on from example 3.11. Define $\varphi : [0, 2\pi] \to [0, \pi]$ by $\varphi(t) = t/2$. Then $\widetilde{\gamma} = \gamma \circ \varphi$ is a reparametrisation of γ. We know that $L(\gamma) = \sqrt{5/2} \cdot \pi$.

Now, $||\widetilde{\gamma}'(t)||_{\widetilde{\gamma}(t)} = \sqrt{5/8}$. Then,

$$\begin{aligned} L(\widetilde{\gamma}) &= \int_0^{2\pi} ||\widetilde{\gamma}'(t)||_{\widetilde{\gamma}(t)} dt \\ &= \sqrt{\frac{5}{8}} \cdot 2\pi \\ &= \sqrt{\frac{5}{2}} \cdot \pi = L(\gamma). \end{aligned}$$

Indeed, the length is invariant under reparametrisation. △

3.3.2 Riemannian Distance Function

By being able to measure the length of smooth curves, we can introduce a *distance function* on a Riemannian manifold. We will see that the manifold together with this distance function defines a metric space structure.

Definition 3.14 (Distance function induced by a Riemannian metric)**:** Let (M, g) be a (path-) connected Riemannian manifold. Then, by definition, for $p, q \in M$ there exists a curve $\gamma : [a, b] \to M$ with $\gamma(a) = p$ and $\gamma(b) = q$. We define the **distance function** $d_g : M \times M \to [0, \infty)$ induced by the Riemannian metric g by

$$d_g(p, q) = \inf \{L(\gamma) \mid \gamma : [a, b] \to M \text{ piecewise smooth}, \gamma(a) = p, \gamma(b) = q\}.$$

The distance d_g can be referred to as the **Riemannian distance** (function).

Example 3.15: Let $\bar{g}$ be the Euclidean inner product on $\mathbb{R}^n$. Then $d_{\bar{g}}(p, q) = ||p - q||_{\bar{g}}$, which is the usual Euclidean metric. △

Before we prove that d_g is a metric, we first need a result that shows $|| \cdot ||_g$ is locally equivalent to the Euclidean inner product.

Lemma 3.16: *Let $U \subset \mathbb{R}^n$ be an open subset, and let g be a Riemannian metric on U. Let $\bar{g}$ be the standard Euclidean inner product on $\mathbb{R}^n$, as in example 3.15. If $K \subset U$ is a compact subset, then there exists constants $c, C > 0$ such that, for all $x \in K$ and $v \in T_xU$,*

$$c||v||_{\bar{g}} \leq ||v||_g \leq C||v||_{\bar{g}}. \tag{3.1}$$

Proof: By proposition 1.29, we know that $T_xU \cong \mathbb{R}^n$. Therefore, we can also see that $TU \cong U \times \mathbb{R}^n$. Now, let us first consider $K \times S^{n-1} \subset U \times \mathbb{R}^n \subset TU$. This is a compact set because K is compact by definition, and the $(n-1)$-sphere is a closed and bounded subset of Euclidean space, which is equivalent to compactness.

Then, for $(x, v) \in K \times S^{n-1}$, $||v||_{\bar{g}} = 1$ by definition of S^{n-1}. Therefore, the function $(x, v) \mapsto ||v||_{\bar{g}}$ is a continuous, strictly positive function. Hence, by compactness, there exists constants $c, C > 0$ such that $c \leq ||v||_g \leq C$ for $(x, v) \in K \times S^{n-1}$.

Now, let us take $(x, v) \in K \times \mathbb{R}^n$. Here, we either have $v = 0$ or $v \neq 0$. For the trivial case $v = 0$, we get that $||v||_{\bar{g}} \equiv ||v||_g \equiv 0$ for all x. For $v \neq 0$, there must exist a unique $\lambda > 0$ such that $\lambda^{-1}v \in S^{n-1}$. Clearly, $\lambda = ||v||_{\bar{g}}$. Therefore, using the homogeneity* of norms, for $(x, v) \in K \times \mathbb{R}^n$,

$$||v||_g = ||\lambda\lambda^{-1}v||_g \overset{*}{=} |\lambda| \cdot ||\lambda^{-1}v||_g = \lambda||\lambda^{-1}v||_g \leq \lambda C = C||v||_{\bar{g}},$$

$$\Rightarrow ||v||_g \leq C||v||_{\bar{g}}, \text{ for some } C > 0.$$

This completes the right hand side of the inequality. A similar computation shows the other half to be also true. □

Theorem 3.17: *Let (M, g) be a connected Riemannian manifold. Then, (M, d_g) forms a metric space, where d_g is the Riemannian distance function defined in definition 3.14.*

This proof is adapted from John Lee's [13, Theorem 13.29].

Proof: Let $p, q, r \in M$. To prove (M, d_g) is a metric space, we must show that d_g satisfies all three properties from definition A.3.1 along with non-negativity, namely positivity, symmetry, and the triangle inequality.

Non-negativity, i.e. $d_g(p, q) \geq 0$, follows immediately.

Symmetry follows simply from the definition of d_g as any curve from p to q can be reparametrised to instead travel from q to p. For example, if α goes from p to q, then $(-\alpha)$ will run from q to p. Hence, using proposition 3.12, $d_g(p, q) = d_g(q, p)$.

The triangle inequality follows directly from the definition also. By using properties of inf, we see that, indeed, $d_g(p, r) \leq d_g(p, q) + d_g(q, r)$.

All that remains to show is positivity, or $d_g(p, q) > 0$ for $p \neq q$. Let $p, q \in M$ be distinct points. Let (U, φ) be a coordinate chart containing p, but not q. Let $\bar{g}$ denote the Euclidean metric in $\varphi(U) \subset \mathbb{R}^n$. Let V be a regular coordinate ball of radius ε centered around p such that $\overline{V} \subset U$. Using these, lemma 3.16 states that there exists $c, C > 0$ such that eq. (3.1) is satisfied for $x \in \overline{V}$ and $v \in T_xM$. Then, for any piecewise smooth curve segment γ which lies entirely in $\overline{V}$, we can integrate eq. (3.1) to get the following inequality:

$$cL_{\bar{g}}(\gamma) \leq L_g(\gamma) \leq CL_{\bar{g}}(\gamma), \tag{3.2}$$

where L_g is the length of a curve with respect to the Riemannian metric g.

Now, suppose we have a piecewise smooth curve segment $\gamma : [a, b] \to M$ travelling from p to q. Define $t_0 = \inf\{t \in [a, b] : \gamma(t) \notin \overline{V}\}$. By continuity, $\gamma(t_0)$ lies on the boundary of $\overline{V}$. Note that we used inf, so even though $\gamma(t_0) \in \overline{V}$, it is the correct result as $\overline{V}^c$ is an open set. Also note that $\gamma(t) \in \overline{V}$ for $a \leq t \leq t_0$. Therefore,

$$L_g(\gamma) \geq L_g(\gamma|_{[a,t_0]}) \overset{*}{\geq} cL_{\bar{g}}(\gamma|_{[a,t_0]}) \overset{\dagger}{\geq} \bar{g}(\gamma(a) = p, \gamma(t_0)) \overset{\ddagger}{=} c\varepsilon,$$

where we used eq. (3.2) in "*", the definition of $d_{\bar{g}}$ in "†", and the definition of $\overline{V}$ in "‡". Taking the infimum over all such γ gives us

$$d_g(p, q) = \inf L_g(\gamma) \geq c\varepsilon > 0.$$

Hence, all necessary conditions are satisfied. Therefore, (M, d_g) is a metric space. □

3.3.3 Geodesics

The study of geodesics is much deeper than we need for this report. Here, we will outline some features and useful properties of them, but we will not go into much detail to save time. To gain a further understanding of geodesics, one can look to [14, Chapters 5/6], [3, Chapter 3], or most other textbooks on Riemannian geometry.

Definition 3.18 ((Piecewise) regular curve/Admissible curve)**:** A **(piecewise) regular curve** is a (piecewise) smooth curve γ where $\gamma' \neq 0$. A piecewise regular curve on a closed interval is also known as an **admissible curve**.

Definition 3.19 (Minimizing curve)**:** An admissible curve $\gamma : [a, b] \to M$ is **minimizing** if its length is equal to the distance between its endpoints, i.e., $L(\gamma) = d_g(\gamma(a), \gamma(b))$.

Definition 3.20 (Locally minimizing curve)**:** A curve $\gamma : I \to M$ is **locally minimizing** if any $t \in I$ has a neighbourhood $U \subset I$ such that $\gamma|_U$ is minimizing between each pair of its points.

Note: Every minimizing curve is locally minimizing.

Definition 3.21 (Geodesic)**:** Let (M, g) be a Riemannian manifold and I be an interval. A smooth curve $\gamma : I \to M$ is a **geodesic** if its acceleration, $\ddot{\gamma} = \frac{d^2\gamma}{dt^2}$ is zero.

Remark 3.22: A geodesic $\gamma : I \to M$ has constant speed, i.e., $||\gamma'||_g$ is constant.

Geodesics are very useful because every minimizing curve is a geodesic, at least when it is reparametrised to have unit speed ($\gamma' = 1$). There is a reverse to this fact which states that every Riemannian geodesic is locally minimizing.

Proposition 3.23: *Let (M, g) be a Riemannian manifold. For any $p \in M$ and $v \in T_pM$, there is a unique maximal geodesic (one which cannot be extended to a larger interval) $\gamma_v : I \to M$ with $\gamma(0) = p$ and $\gamma'(0) = v$.*

3.3.4 The Exponential Map

Similar to our small subsection on geodesics, we only briefly define and state some properties of the exponential map as most of the theory is beyond the scope of this report. It is useful to be familiar with the exponential though, and we will see it in action in section 4.2. For a more in depth study, see [14, Chapter 5].

Definition 3.24 (Exponential)**:** Let (M, g) be a Riemannian manifold, $p \in M$, $v \in T_pM$, and γ_v be defined as in proposition 3.23. Define the **domain of the exponential map** by

$$\mathcal{E} := \{v \in TM : \gamma_v \text{ is define on an interval containing } [0, 1]\}.$$

Then define the **exponential map** $\exp : \mathcal{E} \to M$ by

$$\exp(v) = \gamma_v(1).$$

For each $p \in M$, the **restricted exponential map** $\exp_p$ is the restriction of exp to the set $\mathcal{E}_p := \mathcal{E} \cap T_pM$.

The exponential, therefore, is a map from a tangent space T_pM back onto the manifold M itself. By applying $\exp_p$ to different tangent vectors v, we get different points in M that are all within a certain distance from the point p.

The following definition is taken from Peter Petersen's [20].

Definition 3.25 (Injectivity radius)**:** The largest radius ε for which $\exp_p : B(0, \varepsilon) \to B(p, \varepsilon)$ is a diffeomorphism is called the **injectivity radius** $\text{inj}(p)$ at p.

The following proposition follows from the fact that the exponential map is a radial isometry and, hence, preserves radial distances (see John Lee's [14, Proposition 6.10]).

Proposition 3.26: *In a Riemannian manifold* (M, g) *with* $v \in T_pM$*, for sufficiently small* $0 < s, t < \varepsilon$,

$$d_g(\exp_p(tv), \exp_p(sv)) = |s - t| \cdot ||v||_g. \tag{3.3}$$

Part II

Isometries

Chapter 4

Isometries on Manifolds

We have shown that Riemannian manifolds are metric spaces. A key concept in metric spaces is that of an isometry. We will study the familiar notion of a metric isometry, while also studying a new type of isometry, namely a Riemannian isometry.

When working with isometries, we will explicitly state which type of isometry we are dealing with (metric/Riemannian), unless it is clearly obvious.

4.1 Metric Isometries

Definition 4.1 (Metric isometry)**:** Let (X, d_X) and (Y, d_Y) be metric spaces.[1] A map $f : X \to Y$ is called a **metric isometry** or **distance-preserving** if for any $x_1, x_2 \in X$, we have

$$d_X(x_1, x_2) = d_Y(f(x_1), f(x_2)).$$

Two metric spaces are called **isometric** if there exists a bijective isometry between them.

4.2 Riemannian Isometries

Definition 4.2 (Riemannian isometry)**:** Let (M_1, g_1) and (M_2, g_2) be two Riemannian manifolds. A diffeomorphism $f : M_1 \to M_2$ is a **Riemannian isometry** if, for each $p \in M_1$, the differential

$$Df_p : T_pM_1 \to T_{f(p)}M_2$$

is a linear isometry that preserves the Riemannian metric, i.e.,

$$\langle v, w\rangle_p = \langle Df_p(v), Df_p(w)\rangle_{f(p)} \quad \text{for all } v, w \in T_pM_1. \tag{4.1}$$

Example 4.3 (Simple isometry between Euclidean spaces)**:** Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be the reflection function given by

$$p = (x_1, \ldots, x_{n-1}, x_n) \mapsto (x_1, \ldots, x_{n-1}, -x_n),$$

for $p \in \mathbb{R}^n$. By example 3.6, we know the Riemannian metric $\bar{g}_p$ on Euclidean spaces is given by the standard Euclidean inner product. By example 1.37, we know that the

[1]See definition A.3.1 for definition of a metric space.

differential of a map between Euclidean spaces is given by the Jacobian matrix $\mathbf{J}$ of the map. The equation for the Jacobian is repeated below:

$$\mathbf{J} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(p) & \cdots & \frac{\partial f_1}{\partial x_n}(p) \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1}(p) & \cdots & \frac{\partial f_m}{\partial x_n}(p) \end{bmatrix}. \qquad (1.3 \text{ revisited})$$

Hence, for our map f, we have

$$\mathbf{J} = \begin{bmatrix} 1 & 0 & \cdots & \cdots & 0 \\ 0 & 1 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & 1 & 0 \\ 0 & \cdots & \cdots & 0 & -1 \end{bmatrix}.$$

So, let $v, w \in T_p\mathbb{R}^n$, where $v = \sum_{i=1}^n v_i e_i$ and $w = \sum_{j=1}^n w_j e_j$. Then,

$$\bar{g}_p(v, w) = \langle v, w \rangle = v \cdot w.$$

Applying the differential to v and w, we get

$$\begin{aligned} Df_p(v) &= \mathbf{J}(v_1, \dots, v_{n-1}, v_n) = (v_1, \dots, v_{n-1}, -v_n), \\ Df_p(w) &= \mathbf{J}(w_1, \dots, w_{n-1}, w_n) = (w_1, \dots, w_{n-1}, -w_n). \end{aligned}$$

Therefore, this gives us

$$\begin{aligned} \bar{g}_{f(p)}(Df_p(v), Df_p(w)) &= \langle Df_p(v), Df_p(w) \rangle \\ &= \langle (v_1, \dots, v_{n-1}, -v_n), (w_1, \dots, w_{n-1}, -w_n) \rangle \\ &= \langle v, w \rangle. \end{aligned}$$

Hence, $\bar{g}_p(v, w) = \bar{g}_{f(p)}(Df_p(v), Df_p(w))$, which shows that f is a Riemannian isometry between Euclidean spaces. $\triangle$

From here, a natural question arises: what do metric isometries and Riemannian isometries have in common? We know that the Riemannian distance is induced by the Riemannian metric, but does this mean that all distance preserving maps are Riemannian isometries? We will see that, in fact, this is the case. The converse is also true, but we do not prove that in this report. That result is, however, a much easier result to prove. This is because a map that preserves the Riemannian metric will indeed preserve distance which is defined using the Riemannian metric.

Before we go any further, we must first define some helpful tools which will aid us in the proof of the upcoming theorem. This definition is taken from [20].

Definition 4.4 (Distance coordinates)**:** Let (M^n, g) be a Riemannian manifold. For each $p \in M$, fix a neighbourhood U around p such that, for each $q \in U$, we have that $U \subset B_{\mathrm{inj}(q)}(q)$, where $B_c(x)$ is the regular coordinate ball around x of radius c. Therefore, for each $q \in U$, $r_q(x) = d_g(x, q)$ is smooth on $U \setminus \{q\}$. Now, choose $q_1, \dots, q_n \in U \setminus \{p\}$ such that their derivatives at p form a basis in T_pM. Then, $\varphi = (r_{q_1}, \dots, r_{q_n})$ can be used as coordinates on some neighbourhood of p.

The proof to the following theorem (Theorem 4.5) will be based upon the work by Peter

Petersen in his book, "Riemannian Geometry" [20]. The original theorem and proof can be found in the paper by Myers & Steenrod [19].

Theorem 4.5: *Let (M, g) and (N, h) be Riemannian manifolds with $f : M \to N$ a bijection. Then, f is a Riemannian isometry if f is a metric isometry, i.e., $d_g(p, q) = d_h(f(p), f(q))$ for all $p, q \in M$.*

Proof: We have that f is a bijection and preserves distance. Firstly, we will show that f is a smooth map.

Take a fixed $p \in M$ with $q = f(p)$. Near $q \in N$, introduce distance coordinates $\varphi = (r_{q_1}, \dots, r_{q_n})$. Find p_i-s in M such that $f(p_i) = q_i$. Then,

$$\begin{aligned} (r_{q_i} \circ f)(x) &= d_g(f(x), q_i) \\ &= d_g(f(x), f(p_i)) \quad \text{, by definition of } q_i \\ &= d_g(x, p_i) \quad \text{, as } f \text{ is a distance preserving.} \end{aligned}$$

Since $d_g(q, q_i) = d_g(f(p), f(p_i)) = d_g(p, p_i)$, we can assume that each p_i and q_i are chosen such that the distance coordinates near $p \in M$ are smooth at p. We already have that the distance coordinates near $q \in N$ are smooth, by definition. Therefore, $(r_{p_1}, \dots, r_{p_n}) = \varphi \circ f$ is smooth at p. As we have that both sets of distance coordinates are smooth at p, f must also be smooth at p.

Now, all that is left to show is that f is indeed a Riemannian isometry. It suffices to show that $||v||_g = ||Df(v)||_h$ for all tangent vectors $v \in TM$. This is because these norms are induced by the inner products from the Riemannian metrics, and hence, using the polarization identity, the equivalence of norms implies the equivalence of inner products. The equivalence of inner products clearly means that f is a Riemannian isometry.

So, take a fixed $v \in T_pM$ and let $\gamma(t) = \exp_p(tv)$. If we take t, s small enough, we have

$$L(\gamma|_{[s,t]}) \stackrel{\dagger}{=} |t - s| \cdot ||v||_g = d_g(\gamma(s), \gamma(t)) \stackrel{*}{=} d_h(f \circ \gamma(s), f \circ \gamma(t)),$$

where we used the fact that the exponential function preserves radial distances (proposition 3.26) in "†", and the fact that f is distance preserving in "*".

Therefore, we have

$$\begin{aligned} ||Df(v)||_h &= \left| \frac{d(f \circ \gamma)}{dt} \right|_{t=0} \\ &= \lim_{t \to 0} \frac{d_h(f \circ \gamma(t), f \circ \gamma(0))}{|t|}, \text{ by definition} \\ &\stackrel{*}{=} \lim_{t \to 0} \frac{d_g(\gamma(t), \gamma(0))}{|t|} \\ &= ||\gamma(0)||_g, \text{ by definition} \\ &= ||v||_g, \text{ by definition of } \exp. \end{aligned}$$

Hence, $||v||_g = ||Df(v)||_h$ as required. This completes the proof. □

Remark 4.6: By taking $(M, g) = (N, h)$ in the previous theorem, we see that, in particular, every mapping f of M onto itself which preserves d_g is a Riemannian isometry that preserves g.

Chapter 5

The Isometry Group of a Riemannian Manifold

The study of the groups of isometries of Riemannian manifolds is an important and popular area within Riemannian geometry. In this section, we will define what isometry groups are and prove two vital theorems about their properties.

5.1 Definition

Definition 5.1 (Isometry group)**:** Let (M, g) be a Riemannian manifold. A Riemannian isometry $f : (M, g) \to (M, g)$ is called a **Riemannian isometry of *M***. The set of all Riemannian isometries of M is called the **isometry group of *M***; it is denoted $\text{Iso}(M, g)$.

Remark 5.2: The isometry group of a Riemannian manifold is a group in the algebraic sense because composition of isometries and inverses of isometries are isometries themselves. In particular, $\text{Iso}(M, g)$ is a **transformation group**.[1] Moreover, it is clear that $\text{Iso}(M, g)$ is an **effective/faithful** transformation group.[2]

Example 5.3 (The isometry group of Euclidean space)**:** Take $(\mathbb{R}^n, \bar{g})$ to be our manifold with the standard Euclidean metric. An isometry of $\mathbb{R}^n$ is either a reflection, rotation, translation, or an arbitrary combination of the three. This is simple to see, because these are the only transformations of Euclidean space which preserve distance (which we know is equivalent to being a Riemannian isometry by theorem 4.5). Therefore, for any $x \in \mathbb{R}^n$, we can see that any isometry of Euclidean space is of the form $x \mapsto Ax + b$, where $A \in O(n)$, and $b \in \mathbb{R}^n$ is a constant translation vector. The isometry group is denoted by $\text{Iso}(\mathbb{R}^n, \bar{g}) = E(n)$; this group is known as the Euclidean group. $\triangle$

5.2 Local Compactness

Our goal in this section is to prove a result about general metric spaces and their isometry groups. We can then apply this result to Riemannian manifolds because we know that a Riemannian metric induces a metric space structure on the manifold. To prove the theorem, we will need multiple smaller results which we will present as lemmas. All of the

[1] A group of transformations that acts on a set, i.e., maps from the set into itself.

[2] The only element that sends $x \in M$ to x is the identity element.

results in this section have been adapted from and inspired by the work of Kobayashi and Nomizu in [12, Chapter I-4], except for lemma 5.6 which has been taken from Steenrod's [21]. Additions and changes have been made to proofs where I feel necessary, however some of the proofs could not be changed significantly enough to be classed as anything more than paraphrasing.

Throughout this section, unless stated otherwise, let (M, d) be a locally compact[3], connected metric space with G its group of (metric) isometries. In this section, "isometry" will refer to metric isometries, unless stated otherwise.

Before we delve into local compactness however, it would be useful to remind ourselves of some of the equivalent definitions of compactness.

Definition 5.4 (Compactness)**:** For a metric space (M, d), the following are equivalent:

(i) (M, d) is compact,

(ii) Every countable open cover of (M, d) has a finite subcover,

(iii) (M, d) is sequentially compact, i.e., every sequence in M has a convergent subsequence whose limit lies in X.

Definition 5.5 (Compact-open topology of G)**:** [4] Let $K_i \subset M$ be compact subsets of M and let $U_i \subset M$ be open subsets of M. Let $W = W(K_1, \dots, K_s; U_1, \dots, U_s) := \{f \in G : f(K_i) \subset U_i$ for each $i = 1, \dots, s\}$. Then the sets W of this form are taken as a basis for the open (sub)sets of G. The collection of these sets W is the **compact-open topology** of G.

We are working towards proving that G is locally compact with respect to the compact-open topology defined above. We will use sequences of isometries to eventually show this. By proposition A.3.3, we know every connected, locally compact metric space is second-countable[5]. As M is therefore second-countable and assumed to be locally compact, G is also second-countable. This was one of the first results to be published on the topic of the compact-open topology; a proof can be found in [2, Theorem 5]. Therefore, using sequences to prove local compactness is a valid method due to Kelley's [10, Theorem 5.5] which states that in second-countable spaces, compactness is equivalent to having a subsequence that converges, for each sequence in the space.

Lemma 5.6: *If G has the compact-open topology, and M is regular*[6] *and locally compact, then the group multiplication ($G \times G \to G$) and the group action ($G \times M \to M$) are continuous.*

The proof of this lemma has been taken directly from Steenrod's [21].

Proof: Let $W(K; U)$ be defined as in definition 5.5. Suppose $g_1 g_2 \in W(K; U)$, then either $g_1 g_2 \cdot K \subset U$, or $g_2 \cdot K \subset g_1^{-1} \cdot U$, and the latter set is open. Since M is regular and locally compact, there exists an open subset $V \subset M$ such that $g_2 \cdot K \subset V$, $\overline{V} \subset g_1^{-1} \cdot U$, and $\overline{V}$ is compact. If $g_1' \in W(\overline{V}; U)$ and $g_2' \in W(K; V)$, then it is clear that $g_1' g_2' \in W(K; U)$. Therefore, $W(\overline{V}; U)$ and $W(K; V)$ are neighbourhoods of g_1 and g_2 whose product lies in $W(K; U)$. This implies that group multiplication is continuous.

[3] See definition A.3.2.

[4] Taken from [12, pg. 46].

[5] See definition A.2.2.

[6] See definition A.2.10.

Now, suppose that $g_0 \cdot y_0 \in U$. Since M is regular and locally compact, there is a neighbourhood V of y_0 such that $\overline{V}$ is compact and $\overline{V} \subset g_0^{-1} \cdot U$. Hence, $g_0 \in W(\overline{V}; U)$. If $g \in W(\overline{V}; U)$ and $y \in V$, it then follows that $g \cdot y \in U$. This implies that the group action is continuous. □

Lemma 5.7: *Let $x \in M$ and let $\varepsilon > 0$ be such that $B_\varepsilon(x) = \{a \in M : d(x, a) < \varepsilon\}$ has compact closure (i.e., $\overline{B_\varepsilon(x)}$ is compact). Denote the open neighbourhood $B_{\varepsilon/4}(x)$ of x by V_x. Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of isometries such that $f_n(y)$ converges for some point $y \in V_x$. Then, there exists a compact set K and $N \in \mathbb{N}$ such that $f_n(V_x) \subset K$ for all $n > N$.*

Proof: As (f_n) converges for some point $y \in V_x$, we can choose an N such that, for all $n > N$: $d(\, f_n(y), f_N(y)\,) < \varepsilon/4$. Then, for $z \in V_x$ and $n > N$,

$$\begin{aligned} d(f_n(z), f_N(x)) &\overset{*}{\leq} d(f_n(z), f_n(y)) + d(f_n(y), f_N(x)) \\ &\overset{*}{\leq} d(f_n(z), f_n(y)) + d(f_n(y), f_N(y)) + d(f_N(y), f_N(x)) \\ &\overset{\dagger}{=} d(z, y) + d(f_n(y), f_N(y)) + d(y, x) \\ &< 2 \cdot \varepsilon/4 + \varepsilon/4 + \varepsilon/4 \\ &= \varepsilon, \end{aligned}$$

where "*" denotes use of the triangle inequality, and "†" denotes use of the fact that f_n and f_N are isometries. In the second to last line, we used the fact that $x, y, z \in V_x$ and $d(f_n(y), f_N(y)) < \varepsilon/4$. Therefore, $f_n(V_x) \subset B_\varepsilon(f_N(x))$. Since f_N is an isometry,

$$\begin{aligned} f_N(B_\varepsilon(x)) &= \{f_N(a) \in M : d(x, a) < \varepsilon\} \\ &= \{f_N(a) \in M : d(f_N(x), f_N(a)) < \varepsilon\} \\ &= \{a \in M : d(f_N(x), a) < \varepsilon\} \\ &= B_\varepsilon(f_N(x)). \end{aligned}$$

Then, define $K := \overline{f_N(B_\varepsilon(x))} = \overline{B_\varepsilon(f_N(x))}$. By the assumption in the statement of the lemma, K is compact.

Clearly, $f_n(V_x) = f_n(B_{\varepsilon/4}(x)) \subset K$, for every $n > N$. This completes the proof. □

Lemma 5.8: *As in lemma 5.7, assume that $f_n(y)$ converges for some $y \in V_x$. Then, there is a subsequence $(f_{n_k})_{k\in\mathbb{N}}$ of (f_n) such that $f_{n_k}(z)$ converges for every $z \in V_x$.*

Proof: For metric spaces, separability is equivalent to second-countability. Therefore, we know that M is separable. This means that, by definition of separable, we have a countable set $\{y_n\}_{n\in\mathbb{N}}$ that is dense in V_x. By lemma 5.7, we know that there exists an $N \in \mathbb{N}$ such that $f_n(V_x) \subset K$, for all $n > N$. In particular, we have that $f_n(y_1) \in K$.

Now, choose a subsequence $(f_{1,k})_{k\in\mathbb{N}}$ such that $f_{1,k}(y_1)$ converges. From this subsequence, choose another subsequence $f_{2,k}$ such that $f_{2,k}(y_2)$ converges, etc. The diagonal sequence $f_{k,k}(y_n)$ then converges for every $n \in \mathbb{N}$. We want to show that $f_{k,k}(z)$ converges for any $z \in V_x$. We can change our notation without consequence, i.e., assume that $f_n(y_i)$ converges for each $i \in \mathbb{N}$. Let $z \in V_x$ and $\delta > 0$. Choose a y_i such that $d(z, y_i) < \delta/4$.

There is an N' such that $d(f_n(y_i), f_m(y_i)) < \delta/4$ for any $n, m > N'$. Then,

$$\begin{aligned} d(f_n(z), f_m(z)) &\leq d(f_n(z), f_n(y_i)) + d(f_n(y_i), f_m(z)) \\ &\leq d(f_n(z), f_n(y_i)) + d(f_n(y_i), f_m(y_i)) + d(f_m(y_i), f_m(z)) \\ &= d(z, y_i) + d(f_n(y_i), f_m(y_i)) + d(z, y_i) \\ &< 3 \cdot \delta/4 \\ &< \delta. \end{aligned}$$

Therefore, $f_n(z)$ is a Cauchy sequence. We know, by lemma 5.7, that $f_n(z)$ is in a compact set for all $n > N$. By the properties of compact sets (see definition 5.4 (iii)), we therefore conclude that $f_n(z)$ converges for each $z \in V_x$. Note, we changed notation part way through, so even though it may not appear to be, we did indeed prove that a subsequence converges. $\square$

Lemma 5.9: *Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of isometries such that $f_n(a)$ converges for some point $a \in M$. Then, there is a subsequence $(f_{n_k})_{k\in\mathbb{N}}$ such that $f_{n_k}(x)$ converges for each $x \in M$.*

Proof: For each $x \in M$, define $V_x = B_{\varepsilon/4}(x)$ such that $\overline{B_\varepsilon(x)}$ is compact. Note, ε may depend on x. We define a chain as a finite sequence of open sets V_i that have the following properties:

1. For each i, $V_i = V_x$, for some x,
2. $a \in V_1$,
3. V_i and V_{i+1} have a shared point.

As we have a connected manifold M, the chains of open sets can reach any point $z \in M$. Thus, the set of points z coincides with the manifold M.

From here, let $\{y_i\}_{i\in\mathbb{N}}$ be a countable set which is dense in M. Let $V_1, \ldots, V_s$ be a chain with $y_1 \in V_s$. We know, therefore, that $a \in V_1$ and $f_n(a)$ converges. Using lemma 5.8, we can find a subsequence that converges for all points in V_1. By the final property of the sets V_i we defined, we know that $V_1 \cap V_2 \neq \emptyset$. So by taking a point $x \in V_1 \cap V_2$, we just showed that $f_n(x)$ because $x \in V_1$ (changing the notation of the subsequence). But, applying lemma 5.8 again allows us to find a subsequence that converges for all $x \in V_2$. We can carry this on for all open sets V_i in the chain. As in the proof of the lemma above, we can get a diagonal subsequence $f_{k,k}(y_n)$ that converges for all n, as we had that $y_1 \in V_1$. Again, change the notation of this diagonal subsequence to obtain that $f_n(y_i)$ converges for every i.

Now, we have that some $y_i \in V_x$. So, using lemma 5.7, there exists an $N \in \mathbb{N}$ and a compact set K such that $f_n(V_x) \subset K$ for $n > N$. As we did in the previous proof, we can see that $f_n(x)$ is Cauchy. As we also have that $f_n(x) \in K$ for all $n > N$, we can therefore see that $f_n(x)$ converges for all $x \in M$ due to the compactness of K. $\square$

Lemma 5.10: *Assume that $(f_n)_{n\in\mathbb{N}}$ is a sequence of isometries such that $f_n(x)$ converges for each $x \in M$. Define $f(x) := \lim_{n\to\infty} f_n(x)$ for each x. Then, f is an isometry.*

Proof: We have $d(f(x), f(y)) = d(\lim_{n\to\infty} f_n(x), \lim_{n\to\infty} f_n(y)) = d(x, y)$ because every f_n is distance preserving. Now, for all $a \in M$, define $a' := f(a)$. Then,

$$d(f_n^{-1}(a'), a) = d(f_n(f_n^{-1}(a')), f_n(a)) = d(a', f_n(a)) = d(f(a), f_n(a)) \to 0, \text{ as } n \to \infty.$$

Hence, $f_n^{-1}(a') \to a$. Using lemma 5.9, there exists a subsequence f_{n_k} such that $f_{n_k}^{-1}(y)$ converges for all $y \in M$. Define a map φ by $\varphi(y) = \lim_{k\to\infty} f_{n_k}^{-1}(y)$. It is simple to see that φ preserves distance. Then,

$$\begin{aligned} d(\varphi(f(x)), x) &= d(\lim_{k\to\infty} f_{n_k}^{-1}(f(x)), x) \\ &= \lim_{k\to\infty} d(f_{n_k}^{-1}(f(x)), x) \\ &= \lim_{k\to\infty} d((f(x)), f_{n_k}(x)) \\ &= d(f(x), f(x)) = 0. \end{aligned}$$

From this, we see that $\varphi(f(x)) = x$, $\forall x \in M$. This shows that f maps M onto itself. Since we have that φ preserves distance and also maps M onto itself, φ^{-1} exists and is equal to f. Therefore, f is an isometry. $\square$

Lemma 5.11: *Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of isometries and let f be an isometry. If $f_n(x) \to f(x)$ for each $x \in M$, then the convergence is uniform on every compact subset $K \subset M$.*

Proof: Let $\delta > 0$ be given. For each $a \in K$, choose an $N_a \in \mathbb{N}$ such that for all $n > N_a$, $d(f_n(a), f(a)) < \delta/4$. Let $W_a = B_{\delta/4}(a)$. Then, for any $x \in W_a$, $n > N_a$, we have

$$\begin{aligned} d(f_n(x), f(x)) &\le d(f_n(x), f_n(a)) + d(f_n(a), f(x)) \\ &\le d(f_n(x), f_n(a)) + d(f_n(a), f(a)) + d(f(a), f(x)) \\ &= 2d(a, x) + d(f_n(a), f(a)) \\ &< 2 \cdot \delta/4 + \delta/4 \\ &< \delta. \end{aligned}$$

As K is compact, it can be covered by a finite number of W_a-s (see definition 5.4 (ii)). Say, $K \subseteq \bigcup_{i=1,\dots,s} W_{a_i}$. Therefore, any $x \in K$ will also be an element of some W_{a_i}. Let $N := max_i \{N_{a_i}\}$. So, if $n > N$, we have

$$d(f_n(x), f(x)) < \delta \quad \text{, for any } x \in K.$$

$\square$

Lemma 5.12: *If $f_n(x)$ converges to $f(x)$ as in lemma 5.11, then $f_n^{-1}(x)$ converges to $f^{-1}(x)$, for all $x \in M$.*

Proof: For any $x \in M$, let $y = f^{-1}(x)$. Then,

$$\begin{aligned} d(f_n^{-1}(x), f^{-1}(x)) &= d(f^{-1}(f(y), y) \quad \text{, by definition of } y \\ &= d(f(y), f_n(y)) \quad \text{, because } f_n \text{ is an isometry.} \end{aligned}$$

Then, $d(f(y), f_n(y)) \to 0$ as $n \to \infty$, by definition. Hence, $f_n^{-1} \to f^{-1}$. $\square$

These six lemmas form the bulk of the proof of the following important theorem.

Theorem 5.13: *The group G of isometries of a connected, locally compact metric space M is itself locally compact with respect to the compact-open topology.*

Proof: By lemma 5.6, we know that the group multiplication $G \times G \to G$ and the group action $G \times M \to M$ are both continuous.

Now, $f_n \to f$ with respect to the compact-open topology is equivalent to the uniform convergence of f_n to f on every compact subset of M. This is simply by the definition of the compact open topology. If $f_n \to f$ in G, then lemma 5.12 implies that $f_n^{-1}(x) \to f^{-1}(x)$ for all $x \in M$. Theorem 5.11 shows that the convergence is uniform on every compact subset of M. Therefore, $f_n^{-1} \to f^{-1}$ in G. Hence, the inverse mapping $G \to G$, $f \mapsto f^{-1}$, is continuous.

Finally, we must prove that G is locally compact with respect to the compact-open topology. Let $x \in M$ and let U be an open neighbourhood of x with compact closure. We will show that the neighbourhood $W = W(x; U) = \{f \in G : f(x) \in U\}$ of the identity of G has compact closure (clearly the identity is contained in W). This is because the topology of G is determined by a neighbourhood basis for the identity (indeed, a family $\{U_j\}_{j \in J}$ of arbitrarily small neighbourhoods around the identity element in G determines the family $\{g \cdot U_j\}_{j \in J}$ of arbitrarily small neighbourhoods around another element $g \in G$). In other words, it is sufficient to verify this in any neighbourhood of the identity, as from there it follows for the rest of the group via group translation.

Let $(f_n)_{n \in \mathbb{N}}$ be a sequence within W. Then, since $f_n(x) \in U \subset \overline{U}$, we can choose a subsequence $(f_{n_k})_{k \in \mathbb{N}}$ such that $f_{n_k}(a)$ converges for each $a \in M$ (by lemma 5.9). Furthermore, by lemma 5.10, the map f defined by $f(a) = \lim_{k \to \infty} f_{n_k}(a)$ is an isometry of M. By lemma 5.11, the subsequence converges uniformly to f on every compact subset of M, that is to say, $f_{n_k} \to f$ in G. This proves that W has compact closure. Therefore, by the definition of local compactness, G is locally compact with respect to the compact-open topology. $\square$

Theorem 5.14: *If M is a locally compact metric space with a finite number of connected components, the group G of isometries of M is locally compact with respect to the compact-open topology.*

Proof: We can split up M into its finite number of connected components, say $M = \bigcup_{i=1}^{s} M_i$. Then, for each M_i, choose a point $x_i \in M_i$ and an open neighbourhood U_i of x_i with compact closure. Then, $W(a_1, \dots, a_s; U_1, \dots, U_s) = \{f \in G : f(a_i) \in U_i \text{ for } i = 1, \dots, s\}$ is a neighbourhood of the identity of G that has compact closure. We discuss neighbourhoods of the identity here for the same reason we did in the previous proof. Hence, G is locally compact with respect to the compact-open topology. $\square$

5.3 Lie Group Structure

We are at the final hurdle in our effort to prove the Myers–Steenrod theorem. All that is left is to show that locally compact transformation groups are actually Lie groups. To do this, we will work towards showing that transformation groups that are locally compact and effective contain no small subgroups. Then, using the solution to Hilbert's fifth problem, we can conclude that these transformation groups are indeed Lie groups.

All results from this section have been taken from the work of Montgomery and Zippin in [18, Section 5.2], unless stated otherwise. Again, I have altered proofs where I feel I can add clarity or improvements, but for some proofs I have done nothing more than paraphrased.

For ease of notation, for a map $f = (f_1, \ldots, f_n)$, denote $\left(\frac{\partial}{\partial x_j}\right) f_i$ by f_{ij}. For two sets U and V, we define $UV := \{uv : u \in U, v \in V\}$.

Lemma 5.15: *Let G be a topological group*[7] *which acts on a manifold M of class C^1 and assume that for each $g \in G$, $g(x) = f(g; x)$ is of class C^1 in the argument x. Let W_1 and W_2 be open admissible convex coordinate neighbourhoods in M with $\overline{W_1} \subset W_2$. Let U be a neighbourhood of the identity element $e \in G$ such that $\overline{U}\,\overline{W_1} \subset W_2$. Let $g, g^2, \ldots, g^q \in \overline{U}$. Then, for all $x \in \overline{W_1}$,*

$$f_i(g^q; x) - x_i = \sum_j [\delta_{ij}(g; x; q)] [q\{f_j(g; x) - x_j\}],$$

where with $y_i(g; x) = f_i(g; x) - x_i$,

$$\delta_{ij}(g; x; q) = \frac{1}{q} \int_0^1 [\delta_{ij} + f_{ij}(g; x + ty) + \ldots + f_{ij}(g^{q-1}; x + ty)] dt.$$

Proof: Firstly, define

$$T_i(g; x) := x_i + f_i(g; x) + \ldots + f_i(g^{q-1}; x).$$

By definition of T_i and y_i, we see that

$$\begin{aligned} T_i(g; x + y) &= T_i(g;\ f(g; x)) \\ &= f_i(g; x) + f_i(g; f(g; x)) + \ldots + f_i(g^{q-1}; f(g; x)) \\ &= f_i(g; x) + f_i(g^2; x) + \ldots + f_i(g^q; x), \quad \text{by definition of } f. \end{aligned} \tag{5.1}$$

This gives us

$$T_i(g; x + y) - T_i(g; x) = f_i(g^q; x) - x_i. \tag{5.2}$$

We also have that, by using properties of summations and integrals that cause cancellations,

$$T_i(g; x + y) - T_i(g; x) = \sum_j y_j \int_0^1 T_{ij}(g;\ x + ty) dt. \tag{5.3}$$

Then, by combining eq. (5.2) and eq. (5.3), we get

$$\begin{aligned} f_i(g^q; x) - x_i &= T_i(g; x + y) - T_i(g; x) \\ &= \sum_j \left[\int_0^1 \delta_{ij} + f_{ij}(g; x + ty) + \ldots + f_{ij}(g^{q-1}; x + ty)\right] [f_j(g; x) - x_j] \\ &= \sum_j \frac{1}{q} \left[\int_0^1 \delta_{ij} + f_{ij}(g; x + ty) + \ldots + f_{ij}(g^{q-1}; x + ty)\right] [q\{f_j(g; x) - x_j\}] \\ &= f_i(g^q; x) - x_i = \sum_j [\delta_{ij}(g; x; q)] [q\{f_j(g; x) - x_j\}], \end{aligned}$$

as required. □

[7] See definition A.2.11.

The following two corollaries follow from lemma 5.15.

Corollary 5.16: *Take the hypothesis of lemma 5.15, along with the further assumption that the functions $f_{ij}(g;x)$ are continuous in both arguments g and x. Then, it follows that for every $\varepsilon > 0$, $\overline{U}$ can be chosen such that if $g, \dots, g^q \in \overline{U}$, then*

$$q[f_j(g;x) - x_j] = \sum_i \alpha_{ij}(g;x;q)\,[f_i(g^q;x) - x_i]\,,$$

where $\alpha_{ij}(g;x;q) = [\delta_{ij}(g;x;q)]^{-1}$ and $|\alpha_{ij}(g;x;q) - \delta_{ij}| < \varepsilon$.

Corollary 5.17: *Take the hypotheses of lemma 5.15 and corollary 5.16. Then, it follows that if all powers of g are in $\overline{U}$, then for any $x \in \overline{W_1}$,*

$$g(x) = f(g;x) = x.$$

Theorem 5.18: *Let G be a compact group of transformations of a manifold M of class C^k $(k \geq 1)$ and let each transformation of G be of class C^k. Then, in the vicinity of a stationary point, admissible coordinates may be chosen such that the transformations are linear.*

Proof: Take $T(g) := f(g;x)$. As G is assumed to be compact, the stationary point is in at least one arbitrarily small neighbourhood which is invariant under any transformation of G. In this (/one of these) neighbourhood, choose a convex admissible coordinate system where the origin is defined to be the stationary point. Using the mean value theorem, we see that

$$T_i(g) = f_i(g;x) = \sum_{j=1}^{n} [a_{ij}(g)x_j + b_{ij}(g;x)x_j]\,, \tag{5.4}$$

where $b_{ij}(g;x) \to 0$ with $\sum x_i^2$.

Let $L(g) = (a_{ij}(g))$ be the linear transformation given by the first term in eq. (5.4). Then, clearly, $L(g)L(h) = L(gh)$. Consider, for each $g \in G$, the transformation $L(g^{-1})T(g)$ given by

$$h_i(g;x) = \sum_{j=1}^{n} a_{ij}(g^{-1})f_j(g;x) \to x_i, \text{ with } \sum x_i^2.$$

Now, consider one final transformation R given by

$$R = \int_G L(g^{-1})T(g)dg,$$

where the integration is normalised to give total volume of one. Then, by definition, $R_i = \int_G h_i(g;x)dg$. It can be verified that R and its inverse are both of class C^k in a neighbourhood of the origin (in our convex admissible coordinate system).

Then, for any element $s \in G$, we have

$$\begin{aligned}
L(s)R &= L(s)\int_G L(g^{-1})T(g)dg \\
&= \int_G L(ag^{-1})T(g)dg \\
&= \int_G L(h^{-1})T(hs)dh, \text{ by changing variables } g = hs \\
&= \int_G L(h^{-1})T(h)dh \cdot T(s) \\
&= RT(s) \quad \Rightarrow \quad L(s) = RT(s)R^{-1}.
\end{aligned}$$

Hence, R transforms $T(g)$ into a linear transformation $L(g)$. □

Theorem 5.19: *Let G be a locally compact effective transformation group of a connected manifold M of class C^k, and let each transformation of G be of class C^k. Then, G does not contain small subgroups and therefore must be a Lie group.*

Proof: By corollary 5.17, we know that a sufficiently small neighbourhood of e in G will leave some maximal open coordinate neighbourhood $W \subset M$ stationary. But, by theorem 5.18, we know that in the vicinity of stationary points, we can choose admissible coordinates such that the transformations are linear. In other words, G is locally linear around any stationary points. As we have a stationary set of points rather than a singular point, the linear transformations allow us to see that our set W never stops, i.e., it includes the whole space. Therefore, any sufficiently small group leaves the whole of M stationary. As G is effective, by definition, this small group has to consist of only the identity element of G. Hence, there are no such small groups. Hence, by the combined effort of Gleason, Montgomery, and Zippin in [6] and [17], the solution to Hilbert's fifth problem indeed tells us that G is a Lie group. □

Chapter 6

Myers–Steenrod Theorem

In 1938, Sumner Myers and Norman Steenrod collaborated on a paper titled *'The Group of Isometries of a Riemannian Manifold'* [19]. In classical theory surrounding isometries of Riemannian manifolds, "neither 'whole' groups nor 'whole' spaces were ever considered, but only 'group germs' and spaces in the neighbourhood of a point". The paper by Myers and Steenrod changed this by exploring "the total group of isometries of a Riemannian manifold in the large". Their paper was hugely important as it proved two very useful theorems: Theorem 4.5 and the following Theorem 6.1, below.

6.1 Statement of Theorem

Theorem 6.1 (Myers–Steenrod)**:** *For a Riemannian manifold* (M, g) *with a finite number of connected components, its isometry group* $\mathrm{Iso}(M, g)$ *is a Lie group of isometries.*

6.2 Proof of Theorem

Proof of the Myers–Steenrod Theorem: In theorem 6.1, we proved that every Riemannian manifold (M, g) is also a metric space (M, d_g), where d_g is the distance function induced by the Riemannian metric g.

Now, every Riemannian manifold is locally Euclidean, and clearly Euclidean space is locally compact. Hence, every Riemannian manifold is locally compact.

Then, theorem 5.14 showed that for a locally compact metric space with a finite number of connected components, its metric isometry group is locally compact with respect to the compact-open topology.

Theorem 4.5 showed that every metric isometry between Riemannian manifolds is also a Riemannian isometry. Then, in particular, remark 4.6 states every metric isometry of M onto itself is, in fact, a Riemannian isometry that preserves g. Hence, for a Riemannian manifold, its metric isometry group is equal to the Riemannian isometry group, $\mathrm{Iso}(M, g)$.

Finally, theorem 5.19 states that a locally compact effective transformation group of a manifold (in our case, its isometry group) is a Lie group. This completes the proof. $\square$

6.3 Example

Example 6.2 (Isometry group of the n-sphere)**:** Let us take the smooth manifold S^n, the n-sphere. We showed that this was a smooth manifold in example 1.19. Furthermore, we showed in example 3.7 that S^n equipped with g_o is a Riemannian manifold.

As (S^n, g_o) is embedded in Euclidean space, we can visualise what transformations applied to the space are isometries. It is simple to see that these transformations are all elements of the orthogonal group, $O(n+1)$. The orthogonal group is the group of distance preserving transformations around a fixed point (the origin in our case), hence these will preserve the structure and size of the sphere. Hence, this group forms the isometry group of $S^n(\subset \mathbb{R}^{n+1})$.

In example 2.5, we proved that $O(n')$ is a Lie group, for any n'. So, by taking $n' = n+1$, we see that $O(n+1)$ is, in fact, a Lie group.

So indeed, in the case of the n-sphere S^n, its isometry group $\text{Iso}(S^n, g_o) = O(n+1)$ is a Lie group! Hence, the Myers–Steenrod theorem (theorem 6.1) holds, as expected. $\triangle$

Chapter 7

Further Results

7.1 Dimension of the Group of Isometries of a Riemannian Manifold

We can build upon the work of Myers and Steenrod, and discover more about the isometry group of a Riemannian manifold. More precisely, these results will revolve around the dimension of the isometry group and classifying Riemannian manifolds with maximal dimension isometry groups.

Both of the following theorems can be found in [11, Theorem 3.1].

7.1.1 Upper Bound of the Dimension

Theorem 7.1: *Let (M, g) be a connected, n-dimensional Riemannian manifold. Then, its isometry group* $\mathrm{Iso}(M, g)$ *satisfies the following inequality:*

$$\dim(\mathrm{Iso}(M, g)) \leq \frac{1}{2}n(n+1). \tag{7.1}$$

The proof of this theorem requires knowledge we haven't yet encountered in this report. So, we will give an outline of the proof (which can be seen in full in [11]) by introducing the necessary concepts. We will then provide a full proof by induction of a similar theorem, if we assume that the isometry group is compact and effective/faithful.

Outline of proof: For every point $p \in M$, the Riemannian metric determines the set of all orthonormal frames (an orthonormal frame is an ordered orthonormal basis of T_pM with respect to g_p). The collection of all orthonormal frames on M is called the orthonormal frame bundle, denoted $O(M)$. The set of orthonormal frames on $\mathbb{R}^n$ is isomorphic to $O(n)$, hence has dimension $\frac{1}{2}n(n-1)$ (this can be seen below in example 7.3). As the orthonormal frame bundle associates each point of a manifold with the orthonormal frames there, the dimension of $O(M)$ is therefore $\dim O(M) = n + \frac{1}{2}n(n-1) = \frac{1}{2}n(n+1)$.

It can then be shown that $\mathrm{Iso}(M, g)$ is a closed submanifold of $O(M)$, hence $\dim \mathrm{Iso}(M, g) \leq \dim O(M)$. □

Proof (assuming compact & effective): Assume that $\mathrm{Iso}(M, g)$ is compact and acts effectively on M.

Throughout this proof, let $x \in M$ and $f \in \mathrm{Iso}(M, g)$. For simplicity, let $G = \mathrm{Iso}(M, g)$.

We will prove this theorem via induction on the dimension of the manifold, namely n. The result clearly holds for $n = 0$, as there is only one connected smooth manifold of dimension zero, the set including a single point. Now, assume that the inequality 7.1 holds for all $n < n'$, for some $n' \in \mathbb{N}$. Consider a point $x \in M^{n'}$, where $M^{n'}$ is a n'-dimensional Riemannian manifold. Using simple group theory, we know that the mapping $f \mapsto f(x)$ induces a diffeomorphism between the orbit of x, $G(x)$, and G/G_x, where G_x is the stabiliser subgroup of x. Therefore, using the fact that G is compact and effective, $\dim G(x) = \dim G - \dim G_x \leq n'$.

However, we know that G_x acts on the unit sphere of T_xM effectively via isotropy representation. By the induction hypothesis, and example 6.2, we can therefore see that $\dim G_x = \frac{1}{2}n'(n'-1)$. Hence,

$$\dim G \leq \dim G_x + n' \leq \frac{1}{2}n'(n'-1) + n' = \frac{1}{2}n'(n'+1).$$

The inequality holds for $n = n'$ and hence, by induction, holds for all $n \in \mathbb{N}$. □

7.1.2 Structure of the Manifold with Equality

Theorem 7.2: *Let (M, g) be a connected, n-dimensional Riemannian manifold. If*

$$\dim(\mathrm{Iso}(M, g)) = \frac{1}{2}n(n+1), \tag{7.2}$$

then M is isometric to one of the following spaces of constant curvature:

(1) n-dimensional Euclidean space $\mathbb{R}^n$,

(2) n-dimensional sphere S^n,

(3) n-dimensional, simply connected hyperbolic space $\mathbb{H}^n$,

(4) n-dimensional real projective space $\mathbb{R}P^n$ (as defined in example 1.14).

The proof to this theorem is very lengthy and involved, and goes beyond the scope of this report as it contains many concepts that we haven't defined or encountered. We will give a basic outline of the proof below; for a full proof, see Kobayashi's [11, Theorem 3.1].

Basic outline of proof: If $\dim \mathrm{Iso}(M, g) = \dim O(M)$ (as defined in the proof outline of theorem 7.1), then either $\mathrm{Iso}(M, g) = O(M)$, or $\mathrm{Iso}(M, g)$ looks like one of the connected components of $O(M)$. It can be shown that the bundle $O(M)$ has either one or two connected components.

Following this train of thought, one can arrive to the conclusion that M has to be a space of constant curvature. After this, one can come to the conclusion that M is homogeneous and, therefore, complete. From this, if M is simply connected, it must be either (1), (2), or (4).

The rest of the proof needs much more machinery to be accessible. We omit the latter half. □

7.1.3 Example

Example 7.3 (Revisiting the n-sphere)**:** Throughout this report, we have shown that the n-sphere, S^n, is a Riemannian manifold and that its isometry group is the orthogonal group, $O(n+1)$, which is a Lie group.

From eq. (2.1) and the inverse value theorem, we know that $O(n) = f^{-1}(I_n)$, where $f : M_n(\mathbb{R}) \to S_n$ (S_n is the space of real, symmetric $n \times n$ matrices). Therefore,

$$\dim O(n) = \dim f^{-1}(I_n) = \dim M_n(\mathbb{R}) - \dim S_n.$$

Clearly, $\dim M_n(\mathbb{R}) = n^2$. For any matrix in S_n, the elements on the diagonal can be any value, whereas the elements below/above the diagonal determine the elements above/below the diagonal, respectively. So, the dimension of S_n is $\dim S_n = \sum_{k=1}^{n} k = \frac{1}{2}n(n+1)$. Hence,

$$\dim O(n) = n^2 - \frac{1}{2}n(n+1) = \frac{1}{2}n(n-1).$$

Therefore, for $O(n+1)$, its dimension is $\dim O(n+1) = \frac{(n+1)((n+1)-1)}{2} = \frac{n(n+1)}{2}$.

All this shows that theorem 7.2 indeed holds, because $M = S^n$ is clearly isometric to S^n and its isometry group satisfies the equality 7.2. $\triangle$

Part III

Conclusion

Throughout this report, we aimed to introduce the reader to a variety of new content and material. The primary field of interest was Riemannian geometry, with a focus on Riemannian manifolds and their isometry groups. The final goal was to give a proof of a theorem first presented by Myers and Steenrod in their paper, "The group of Isometries of a Riemannian Manifold" [19].

To summarise the material we have covered, we first introduced smooth manifolds, and the smooth structures that they are equipped with. They generalise ideas that the reader is already familiar with, namely curves and surfaces, to arbitrarily many dimensions. We built a strong knowledge base surrounding these, mainly consisting of the tools and concepts necessary to study calculus on smooth manifolds. We then extended this concept further in two directions to study Lie groups and Riemannian manifolds. Riemannian manifolds allowed us to define notions of length and angles on the manifold and its tangent space. We showed how the Riemannian metric, while not a metric itself, can induce a metric on the smooth manifold, transforming it into a metric space. Our first major theorem was encountered here, where we saw that every distance preserving map (i.e. metric isometry) is also a Riemannian isometry. Carrying on from this, we studied the groups of isometries on Riemannian manifolds, coming to two more important theorems. Here, we showed that the group of isometries is locally compact, for a Riemannian manifold with finitely many connected components. Furthermore, we saw that transformation groups that act on Riemannian manifolds are, in fact, Lie groups. Putting it all together, we culminated with a proof of the Myers–Steenrod theorem which states that the isometry group of a Riemannian manifold, with finitely many connected components, is a Lie group of isometries. We then quickly explored some further results regarding the dimension of this isometry group and its upper bound.

However, despite this report studying Riemannian manifolds in great depth, there are still many interesting results we did not discuss and open hypotheses yet to be proved. A lot of "recent" research is into manifolds that are the "nicest" or most "optimal", these manifolds tend to be ones with large isometry groups, as stated by Grove in [7, Section 8]. In this article by Grove, he explores and discusses some of the main tools that have been discovered and worked on over the years. Another area of interest, that carries on from the results we explored in chapter 7, is characterising manifolds that have certain properties. For example, in this article by Grove and Wilkin, [9], they show that a closed, simply connected 4-manifold with non-negative curvature whose isometry group contains a circle must be diffeomorphic to S^4, $\mathbb{R}P^4$, or $\mathbb{C}P^2$.

Many other interesting results that extend beyond this report can be found in [1, 4, 5, 8, 16]. For example, in Mirzaie's [16], he characterises certain Riemannian manifolds with negative curvature under the action of a compact Lie group of isometries. Another, very recent, article [4] explores and "computes the full isometry group of any left-invariant metric on a simply connected, non-unimodular Lie group of dimension three".

These further examples shown barely scratch the surface of the amazing and profoundly interesting work being done surrounding Riemannian manifolds and their Lie groups of isometries.

Part IV

Appendix

Appendix A

Reviews of Selected Necessary Knowledge

Any propositions/results in the appendix is common knowledge which will be taken without proof, unless stated otherwise.

A.1 Groups

Definition A.1.1 (Group)**:** A **group** is a set G together with a binary operation. We will denote the binary operation between $a, b \in G$ as $a \cdot b$ (this may very depending on context). The set G together with the binary operation must satisfy the four following conditions:

(i) Closure: if $a, b \in G$, then $a \cdot b \in G$.

(ii) Associativity: for all $a, b, c \in G$, $(a \cdot b) \cdot c = a \cdot (b \cdot c)$.

(iii) Identity: There exists an $e \in G$ such that $e \cdot a = a \cdot e = a$ for every $a \in G$.

(iv) Inverse: For each $a \in G$, there exists an $a^{-1} \in G$ such that $a \cdot a^{-1} = a^{-1} \cdot a = e$.

A.2 Topological Spaces

Definition A.2.1 (Topological Space)**:** A **topological space** is a pair $(X, \mathcal{T})$ consisting of a set X together with a collection of subsets of X, $\mathcal{T}$, called **open subsets**. They satisfy the following conditions:

(i) X and $\emptyset$ are open.

(ii) The union of any family of open subsets is open.

(iii) The intersection of any finite collection of open subsets is open.

The collection $\mathcal{T}$ is called a **topology**.

We will most often omit any explicit mention of the topology, as long as the choice of topology is clear, and instead simply write X to refer to the topological space.

Definition A.2.2 (Second-countable)**:** A topological space $(X, \mathcal{T})$ is **second-countable** if $\mathcal{T}$ has a countable basis.

Definition A.2.3 (Homeomorphism)**:** Let X, Y be topological spaces. A map $f : X \to Y$ is a **homeomorphism** if it is bijective and both f and f^{-1} are continuous.

Definition A.2.4 (Connectedness)**:** A topological space X is said to be **disconnected** if it has two disjoint non-empty open subsets whose union is X. Otherwise, X is **connected**. Equivalently, X is connected if and only if the only subsets of X that are both open and closed are X and $\emptyset$.

Definition A.2.5 (Connected components)**:** Given a point $x \in X$, X a topological space, the union of any collection of connected subsets that each contain x is also a connected subset. The union of all connected subsets of X containing x is called the **connected component of x**; it is the unique largest connected subset of X that contains x.

The maximal connected subsets of a non-empty topological space are called the **connected components** of the space. They form a partition of the topological space.

Definition A.2.6 (Path)**:** Let X be a topological space and $p, q \in X$. Then, a **path in X from p to q** is a continuous map $f : [0, 1] \to X$ such that $f(0) = p$ and $f(1) = q$.

Definition A.2.7 (Path-connectedness)**:** If, for every pair of points $p, q \in X$ there exists a path in X from p to q, then X is said to be **path-connected**.

Proposition A.2.8: *Every path-connected space is connected.*

Proposition A.2.9: *Let X and Y be topological spaces. Then, a union of connected subspaces of X with a point in common is connected.*

Similarly, a union of path-connected subspaces of X with a point in common is path-connected.

Definition A.2.10 (Regular topological space)**:** A topological space X is regular if, given any closed set $C \in X$ and given a point $x \notin C$, there exists a neighbourhood U of x and a neighbourhood V of C such that $U \cap V = \emptyset$.

Definition A.2.11 (Topological group)**:** We call $(G, \mathcal{T}, \cdot)$ a **topological group** if the following conditions are satisfied:

- $(G, \cdot)$ is a group,
- $(G, \mathcal{T})$ is a topological space,
- the group operation $\cdot : G \times G \to G$ is continuous,
- the inverse map $g \mapsto g^{-1}$ is continuous, for $g \in G$.

If the topology and/or the group operation are clear in the context, we will often omit writing them.

Remark A.2.12: Any group G can be made into a topological group by equipping it with the discrete topology.

A.3 Metric Spaces

Definition A.3.1 (Metrics & metric spaces)**:** Let X be a non-empty set. A function $d : X \times X \to \mathbb{R}_+$ is called a **metric** if it satisfies the following three conditions:

(i) Positivity: $d(x, y) = 0 \Leftrightarrow x = y$, or equivalently, $d(x, y) > 0 \Leftrightarrow x \neq y$,

(ii) Symmetry: $d(x, y) = d(y, x)$ for all $x, y \in X$,

(iii) Triangle inequality: $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

The couple (X, d) is called a **metric space**.

Definition A.3.2 (Local compactness)**:** A metric space (X, d) is said to be **locally compact** if every point $x \in X$ has a compact neighbourhood.

The following proposition is proved in [12, Appendix 2] and [26, Theorem 16.11] combined.

Proposition A.3.3: *A connected, locally compact metric space is separable. As separability is equivalent to second countability for metric spaces, a connected, locally compact metric space is second countable.*

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Summary of Notation

This list serves as a useful reference for notation that is used within this report.

Manifolds

M	(Smooth/Riemannian) Manifold
M^n	(Smooth/Riemannian) Manifold of dimension n
(M, g)	Riemannian manifold
$(U_\alpha, \varphi_\alpha)$	Coordinate chart
$M \cong N$	The two spaces M and N are diffeomorphic
T_pM	Tangent space to manifold M at point p
TM	Tangent bundle to manifold M
Df_p	Differential of f at p
$\frac{\partial}{\partial x_i}\|_p$	The i-th coordinate tangent vector at p
$g_p(\ , \ )$	Riemannian metric at point $p \in M$
$\langle \ , \ \rangle_p$	Riemannian metric at point $p \in M$
d_g	Riemannian distance
exp	Exponential map
$\exp_p$	Restricted exponential map at p
$\mathrm{Iso}(M, g)$	Isometry group of Riemannian manifold (M, g)

Common Spaces

$\mathbb{R}^n$	n-dimensional Euclidean space
S^n	n-dimensional sphere
$\mathbb{H}^n$	n-dimensional hyperbolic space
$\mathbb{R}P^n$	n-dimensional real projective space
$O(n)$	The orthogonal group of dimension n
$M_n(\mathbb{R})$	Space of $n \times n$ square matrices
S_n	Space of $n \times n$ symmetric matrices

Other

C^∞	Infinitely differentiable
e_i	Standard basis vector
Id	Identity map
I_n	Identity matrix in n dimensions
$\mathrm{supp}(f)$	Support of f

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