Algorithms – I (CS29003/203)

Autumn 2022, IIT Kharagpur

Priority Queues and Heaps

Breakthrough during the Vacation

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@demishassabis

Since 1969 Strassen's algorithm has famously stood as the fastest way to multiply 2 matrices - but with #AlphaTensor we've found a new algorithm that's faster, with potential to improve efficiency by 10-20% across trillions of calculations per day!

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In our paper, published today in Nature, we introduce AlphaTensor, the first artificial intelligence (AI) system for discovering novel, efficient, and provably ...

10:30 PM · Oct 5, 2022 · Twitter Web App

Resources

- Apart from the book
- UC Davis ECS 36C Course by Prof. Joël Porquet-Lupine

Introduction

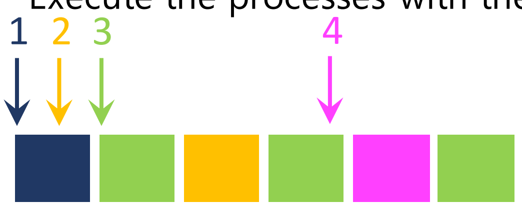
- Let us think about a scenario where you are performing the following tasks simultaneously
  - Start a long code compilation
  - Refresh moodle page
  - Send Gmail chat message
  - Receive email notification
- Objective: Execute multiple processes until completion, but keep the system responsive

- Are processes executed until completion?
- Will the system feel responsive?

Process Scheduling

- Round-robin scheduling
  - Add processes to a (typically FIFO) queue
  - Execute them for equal chunks of time

  - Will the system feel more responsive? Can we do better?
- Priority scheduling
  - Associate priority with each process, e.g., short processes get higher priority
  - Execute the processes with the highest priority first

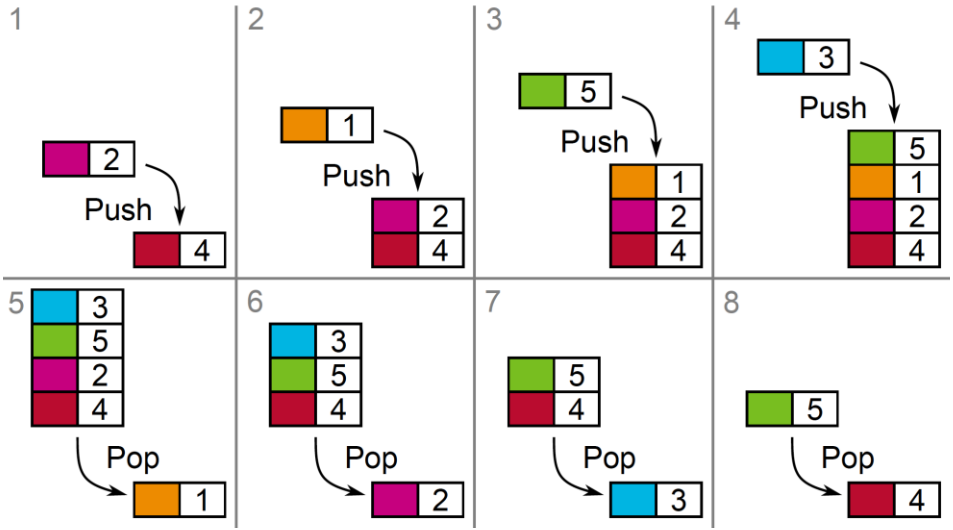


A very good resource on process scheduling:
[Prof. Krzyzanowski's lecture notes]

Source: UC Davis, ECS 36C course, Spring 2020

Priority Queue

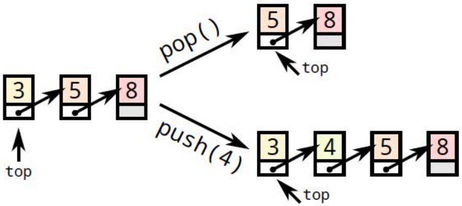
- Definition: **Queue** where each item is associated to a priority (i.e., a comparable key)
  - **Push/Insert**: add item and associated priority
  - **Pop/Pull**: remove item with highest priority



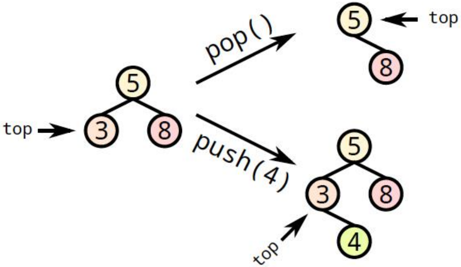
Source: UC Davis, ECS 36C course, Spring 2020

Naïve Implementations

- Sorted list
  - Keep sorted by priority when pushing
  - Pop highest priority item from front



- Binary Search Tree (self-balanced)
  - Restructure tree by priority when inserting
  - Return highest priority item from minimum of tree



Source: UC Davis, ECS 36C course, Spring 2020

List Implementations

- Struct node

```
// Node
struct node {
  // Lower values indicate higher priority
  int priority;
  struct node* next;
};
```

- Function to create a new node

```
// Function to Create A New Node
struct node* newNode(int p)
{
  struct node* temp = (struct node*)malloc(sizeof(struct node));
  temp->priority = p;
  temp->next = NULL;
  return temp;
}
```

- Function to check if queue is empty

```
bool isEmpty(struct node* head)
{
  return head == NULL;
}
```

List Implementations

- Get value of the maximum priority (minimum element)

```
int getMin(struct node* head)
{
  if(isEmpty(head))
  {
    printf("Empty priority queue!\n");
    exit(0);
  }
  return head->priority;
}
```

- Extract the element with maximum priority (minimum element)

```
struct node* pop(struct node* head){
  if(isEmpty(head))
  {
    printf("Empty priority queue!\n");
    exit(0);
  }
  struct node* temp = head;
  head = head->next;
  free(temp);
  // Return new head
  return head;
}
```

List Implementations

- Push new element at the right place

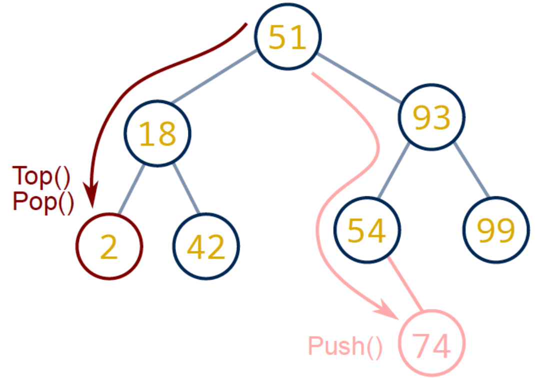
```
struct node* push(struct node* head, int newNum){
  // Create new Node
  struct node* nNode = newNode(newNum);
  // The head of list has lesser priority than new node
  if(head->priority>newNum)
  {
    nNode->next = head;
    // Return new head
    return nNode;
  }
  else
  {
    // Traverse the list and find the position
    // to insert the new node
    struct node* start = head;
    struct node* startPrev = head;
    while(newNum>start->priority)
    {
      if(start->next!=NULL)
      {
        startPrev = start;
        start = start->next;
      }
      else
      {
        // It has reached the end of list. Insert the new node
        start->next = nNode;
        // Return head
        return head;
      }
    }
    // Store what is now pointed by 'next' field of the node whose
    // priority is just above it (previous node of current start)
    nNode->next = startPrev->next;
    // Connect 'next' field of previous node to newNode's next
    startPrev->next = nNode;

    // Return head
    return head;
  }
}
```

List Implementations

- Complexities?
  - getMin() - $O(1)$
  - pop() - $O(1)$
  - push() - $O(n)$

Binary Search Tree Implementations

- A priority queue can also be entirely mapped onto a BST implementation
- getMin()
  - Use getMin() - $O(\log n)$
- pop()
  - Use delete(getMin()) - $O(\log n)$
- push()
  - Use insert() method of BST - $O(\log n)$

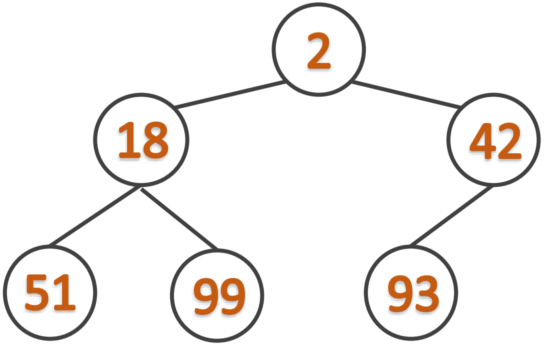


- Conclusion on naïve implementations
  - A list implementation gives getMin()/pop() in $O(1)$
    - But push() in $O(n)$
  - A BST implementation gives push() in $O(\log n)$
    - But getMin()/pop() in $O(\log n)$ as well
  - How to get the best of both worlds?

Source: UC Davis, ECS 36C course, Spring 2020

Binary Heap

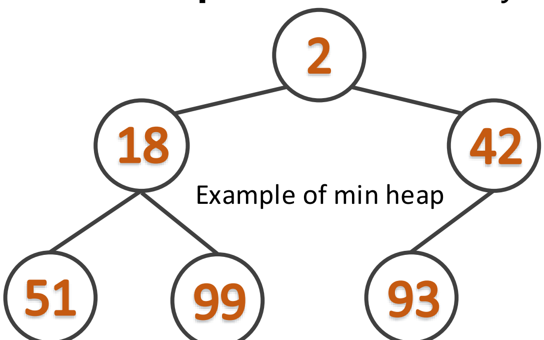
- Most common data structure for implementing a priority queue
- So ubiquitous in implementing priority queues that the word '*heap*' is used without any qualifier in this context
- A heap is a binary tree with two properties
  - Heap-order property
  - Structure property
- Operations on heaps can destroy one or more of these properties
- So a heap operation must not terminate until all heap properties are in order



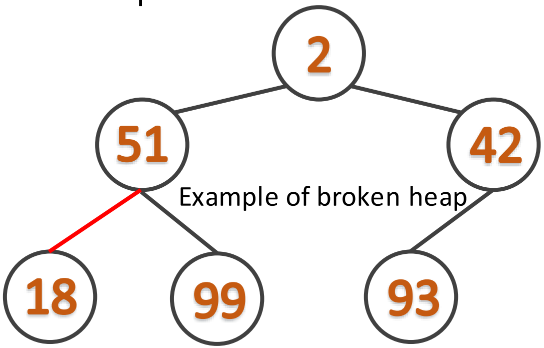
Source: UC Davis, ECS 36C course, Spring 2020

Heap-Order Property

- Since we want to find the minimum quickly, it makes sense that the smallest element should be at the root.
- Continuing, any node should be smaller than all of its descendants
- Giving two types of Heaps
  - **min-heaps**: each node's key is less than or equal to each of its children
  - **max-heaps**: each node's key is greater than or equal to each of its children



Example of min heap

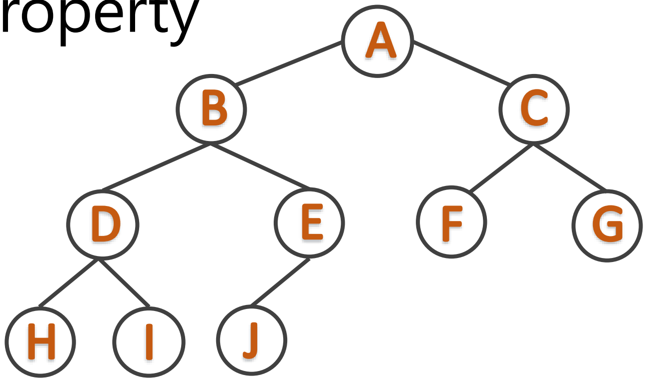


Example of broken heap

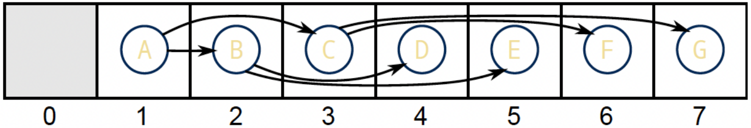
- By design, item with highest priority is always the root
  - getMin() is $O(1)$

Source: UC Davis, ECS 36C course, Spring 2020

Structure Property



- A heap is a complete binary tree
  - All levels are completely filled, apart from possibly the last
  - The last level is packed to the left
- Guarantee of height in $O(\log n)$
- Easy representation
  - Complete binary tree is so regular, it can be represented in an array
  - No need for complicated link management, and fast traversal



- Root is always at index 1

For node at index i*:
- Left child at index $2i$
- Right child at index $2i+1$
- Parent (if not root) at index $\lfloor \frac{i}{2} \rfloor$

*Heap could start at array index 0, but node indexing would become a little more complex

Source: UC Davis, ECS 36C course, Spring 2020

Implementation

- Min-heap version
- Simple array (capacity incremented by 1 for the unused first element)

```
#define CAPACITY 10

// Global variable to track the size of Heap
int size = 0;
```

```
// Remember we are starting at index 1
int H[CAPACITY+1];
```

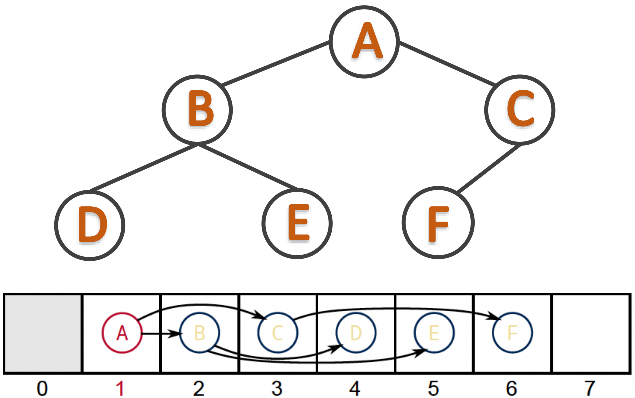
```
// Helper methods for indices
int Root(){
return 1;
}
int Parent(int n){
return n/2;
}
int LeftChild(int n){
return 2*n;
}
int RightChild(int n){
return 2*n + 1;
}
```

```
// Helper methods for node testing
bool HasParent(int n){
  return n!=Root();
}
bool IsNode(int n){
  return n<=size;
}
```

Source: UC Davis, ECS 36C course, Spring 2020

Implementation

- getMin() – Returns the node with highest priority (minimum value in min-heap)
- It is as simple as returning the root only



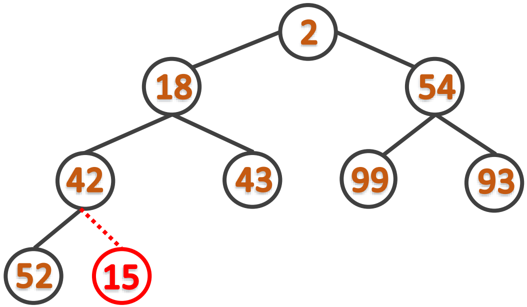

```c
// Function to get the node with highest
// priority minimum element (root)
int getMin(int H[])
{
    if(size == 0)
    {
      printf("Empty priority queue!\n");
      exit(0);
    }
    return H[Root()];
}
```

- Next, we will insert or push() an element into the heap
- What is the most natural position for the new element? (Don't worry about the heap order property, just keep the structure property intact)

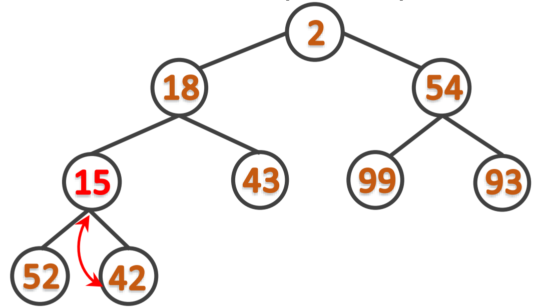
Source: UC Davis, ECS 36C course, Spring 2020

Implementation

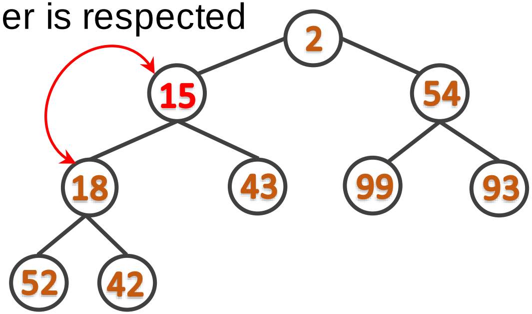
- push() – Insert only at locations that keeps the tree complete



- If heap-order not broken, stop!
- Otherwise, swap with parent



- Continue going up tree until heap-order is respected



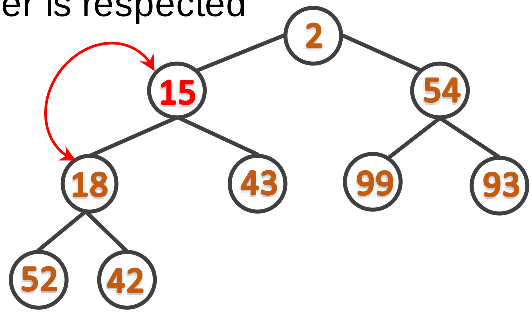
- Strategy known as shift up
- Also called bubble up, heapify-up

```
// Function to shift up the node in order
// to maintain the heap property
void shiftUp(int H[], int n){
  while (HasParent(n) && (H[Parent(n)] > H[n]))
  {
    swap(&H[Parent(n)], &H[n]);
    // Inside the while loop change
    // n to go to its parent
    n = Parent(n);
  }
}
```

Source: UC Davis, ECS 36C course, Spring 2020

Implementation

- Continue going up tree until heap-order is respected



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  {
    swap(&H[Parent(n)], &H[n]);
    // Inside the while loop change
    // n to go to its parent
    n = Parent(n);
  }
}
```

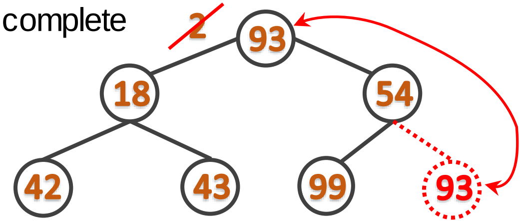
- Insertion of new item

```
void push(int H[], int newNum){
  // Check if heap is full
  if(size == CAPACITY)
  {
    printf("Priority queue full!\n");
    exit(0);
  }
  // Insert at the end
  H[size+1] = newNum;
  size++;
  // Shift up
  shiftUp(H, size);
}
```

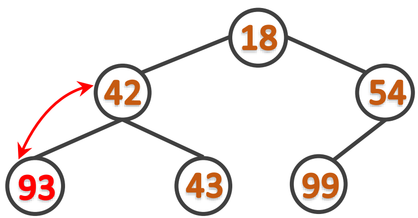
Source: UC Davis, ECS 36C course, Spring 2020

Implementation

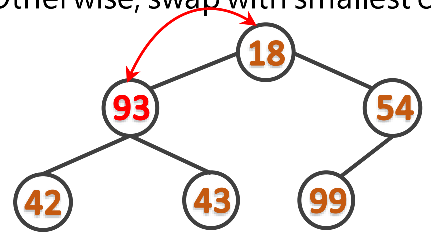
- pop() – Removing min item leaves hole at root. Move last item to root to keep the tree complete



- Continue going down tree until heap-order is respected again



- If heap-order not broken, stop!
- Otherwise, swap with smallest child



- Strategy known as shift down
- Also called bubble down, heapify-down

Source: UC Davis, ECS 36C course, Spring 2020

Implementation

- Strategy known as shift down
- Also called bubble down, heapify-down

```
// Function to shift down the node in order
// to maintain the heap property
void shiftDown(int H[], int n){
  // While node has at least one child
  // (if one, necessarily on the left)
  while(IsNode(LeftChild(n)))
  {
    // Consider left child by default
    int child = LeftChild(n);
    // If right child exists and smaller than
    // left child, then consider right child
    if(IsNode(RightChild(n)) && (H[RightChild(n)] < H[LeftChild(n)]))
      child = RightChild(n);
    // Exchange smallest child with node to
    // restore heap-order if necessary
    if(H[n]>H[child])
      swap(&H[n], &H[child]);
    else
      break;

    // Inside the while loop change
    // n to go one level down
    n = child;
  }
}
```

```
// Function to extract the element with
// maximum priority (in min-heap, this is
// node with the min value)
void Pop(int H[]){
  if(size == 0)
  {
    printf("Empty priority queue!\n");
    exit(0);
  }
  // Move last item back to root and reduce
  // heap's size
  H[Root()] = H[size];
  size--;
  shiftDown(H, Root());
}
```

Conclusion

- Running time complexities

Push	Pop	Top
$O(\log n)$	$O(\log n)$	$O(1)$

- Other operations:
  - In-place item modification, e.g., when heap/priority queue is used for process scheduling, change priority of a process
  - DecreaseItem(Item, DeltaPriority)
  - IncreaseItem(Item, DeltaPriority)
- Heap variants:
  - 2-3 heap, Binomial heaps, d-ary heaps, Fibonacci heaps, Leftist heaps, Skew heaps etc.

Type	Push	Pop	Top
Sorted list	$O(n)$	$O(1)$	$O(1)$
BST	$O(\log n)$	$O(\log n)$	$O(\log n)$
Binary heap	$O(\log n)$	$O(\log n)$	$O(1)$

Source: UC Davis, ECS 36C course, Spring 2020

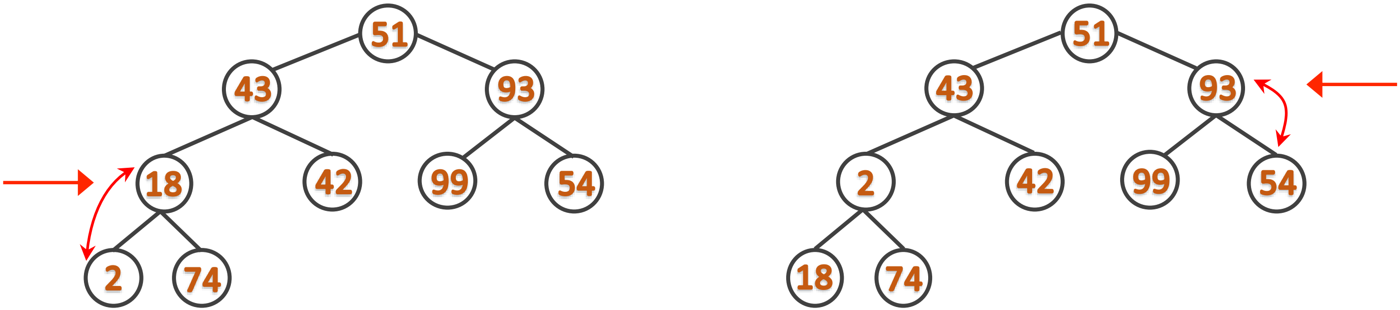
Heapify/BuildHeap

- Sometimes, build an entire heap directly out of an initial collection of items
- One use-case is Heapsort – an efficient way to sort an array
- Naïve approach (also known as Williams' method)
  - For all N items, push() N times
- Each push() operation takes $O(1)$ average time and $O(\log N)$ worst-case time
- For N items, this algorithm will run in $O(N)$ average time and $O(\mathrm{N}\log N)$ worst-case time
- Is there a better way?

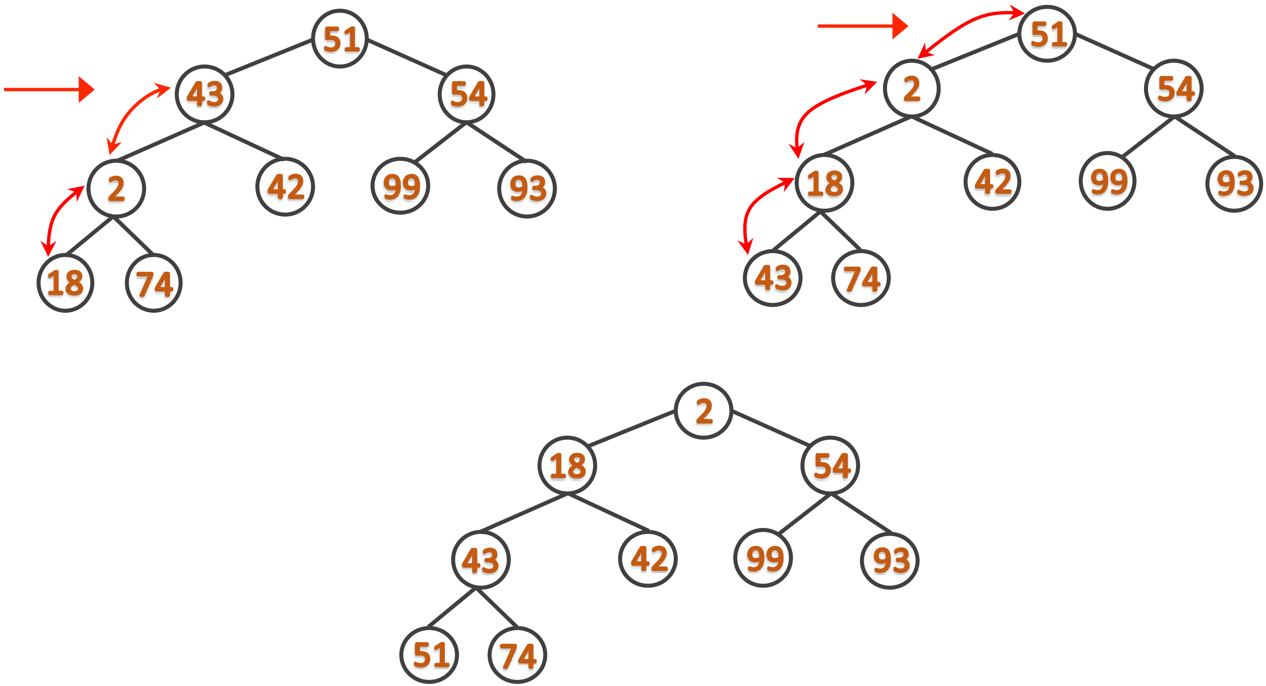
Heapify/BuildHeap

- Linear approach (Floyd's method)
  - Put all the elements on binary tree, only respecting structure property
  - Shift down all the elements starting from the first node who has at least one child, and up to the root

```
// Array
int arr[] = {51, 43, 93, 18, 42, 99, 54, 2, 74};
```



Heapify/BuildHeap



Heapify/BuildHeap Implementations

```
// Main code
int main()
{
  // Array
  int arr[] = {51, 43, 93, 18, 42, 99, 54, 2, 74};
  // Size of array
  int n = sizeof(arr)/sizeof(arr[0]);
  // Print the array
  printf("Array: ");
  for (int i = 0; i < n; i++)
    printf("%d ", arr[i]);
  printf("\n");

  // Remember we are starting at index 1
  int H[CAPACITY+1];
  // Building heap
  buildheap(arr, H, n);
  // Print after heapifying
  printf("Min-Heap: ");
  printHeap(H, size);

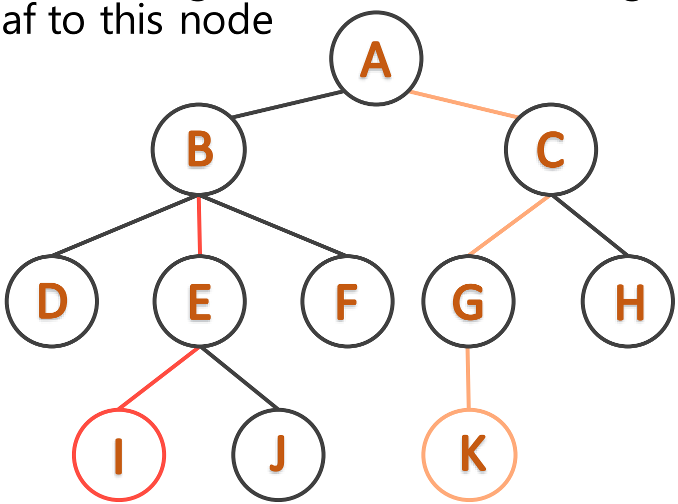
  return 0;
}
```

```
void buildheap(int arr[], int H[], int n){
  // Only enforce structure property
  for(int i = 0; i < n; i++)
    H[i+1] = arr[i];
  // Set the global variable size's value
  size = n;

  // Now enforce heap-property
  for (int i = size/2; i >= 1; i--)
    shiftDown(H, i);
}
```

Heapify/BuildHeap Complexity Analysis

- The running time of heapify operation is $\Theta(n)$ where n is the number of elements in the heap
- RECAP - <u>Height</u>: The height of a node is the length of the longest path from a leaf to this node



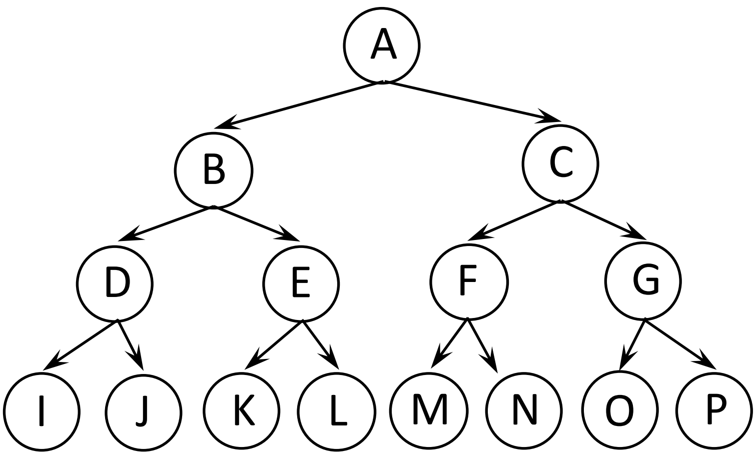
- All leaves are at height 0
- The height of a tree is equal to the height of the root from the deepest leaf (which is always equal to the depth of the tree)

Heapify/BuildHeap Complexity Analysis

- The running time of heapify operation is $\Theta(n)$ where n is the number of elements in the heap
- Some intuitions first [Great resource on this topic - https://stackoverflow.com/a/18742428]
- The basic idea is after creating a complete binary tree - move an offending node until it satisfies the heap property
  - shiftUp - swaps a node that is less than its parent (thereby moving it up) until it is no smaller than the node above it
  - shiftDown - swaps a node that is more than its smallest child (thereby moving it down) until it is no larger than both nodes below it
- The number of operations required for shiftUp and shiftDown is proportional to the distance the node may have to move
- For shiftUp, it is the distance to the top of the tree, so shiftUp is expensive for nodes at the bottom of the tree
- For shiftDown, it is the distance to the bottom of the tree, so shiftDown is expensive for nodes at the top of the tree

Heapify/BuildHeap Complexity Analysis

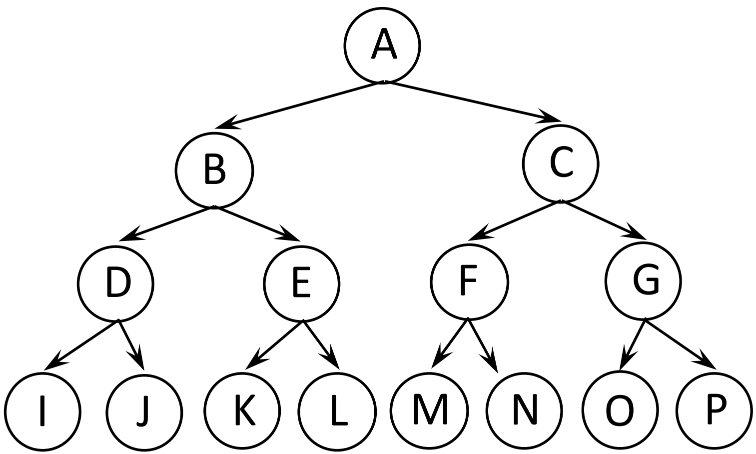
- Although both operations are O(log n) in the worst case, only one node is at the top whereas half the nodes lie in the bottom layer
- So it shouldn't be too surprising that if we have to apply an operation to every node, we would prefer shiftDown over shiftUp



Max # of swaps	Max # of nodes
height h	$1 = 2^0$
height $h - 1$	$2 = 2^1$
height $h - 2$	$4 = 2^2$
height $h - 3$ $= 0$	$8 = 2^3$

Heapify/BuildHeap Complexity Analysis

- Total swaps (max) = $\sum_{i=0}^{h} 2^i(h-i)$
- $A \;=\; h + 2(h-1) + 2^2(h-2) + \cdots + 2^{h-1}1 + 0 \ldots (1)$
- $2A = 2h + 2^2(h-1) + 2^3(h-2) + \cdots + 2^h 1 + 0 \ldots (2)$



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Heapify/BuildHeap Complexity Analysis

- Total swaps (max) = $\sum_{i=0}^{h} 2^i (h-i)$
- $A \ = \ h + 2(h-1) + 2^2(h-2) + \cdots + 2^{h-1}1 + 0 \ldots (1)$
- $2A = 2h + 2^2(h-1) + 2^3(h-2) + \cdots + 2^h 1 + 0 \ldots (2)$

- (2) – (1) $\Rightarrow A = -h + 2 + 2^2 + 2^3 + \cdots + 2^{h-1} + 2^h$

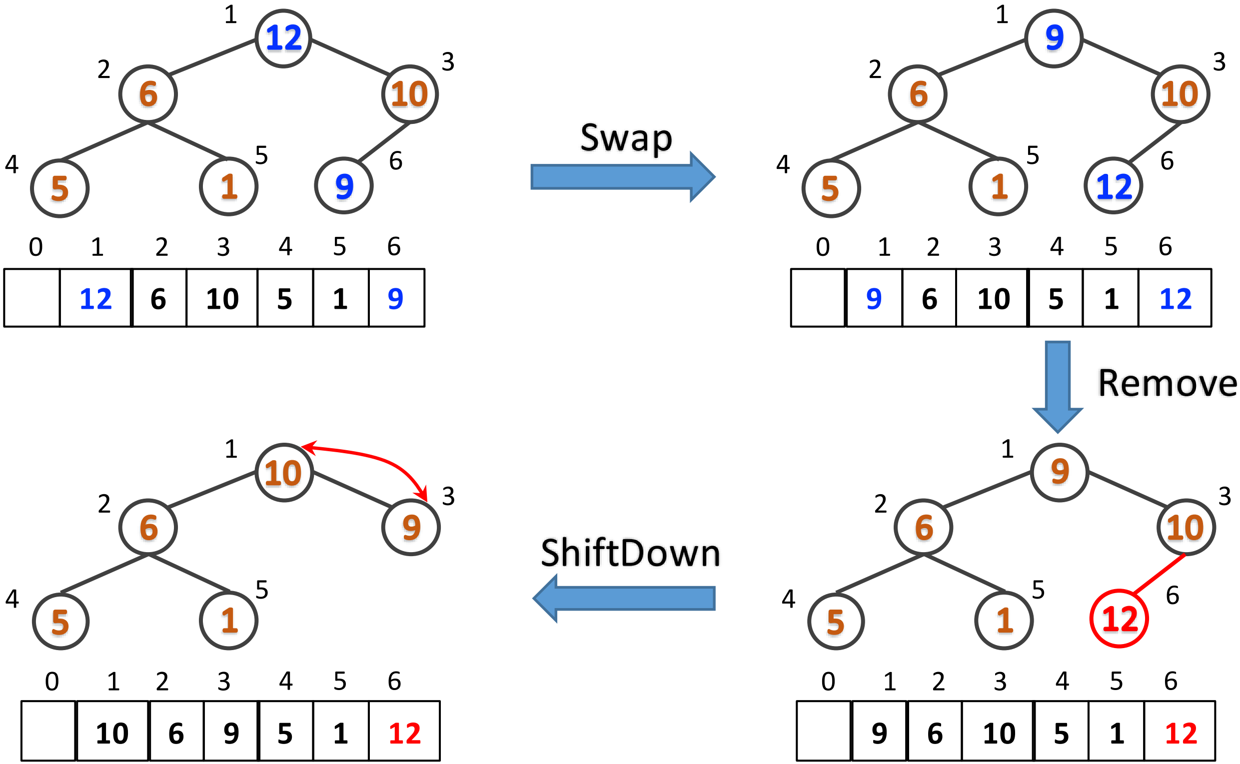
$$= -h + \left(\frac{2^{h+1}-1}{2-1} - 1\right) = (2^{h+1} - 1) - (h+1)$$

$$= (N-1) - (\log N + 1) = O(N)$$

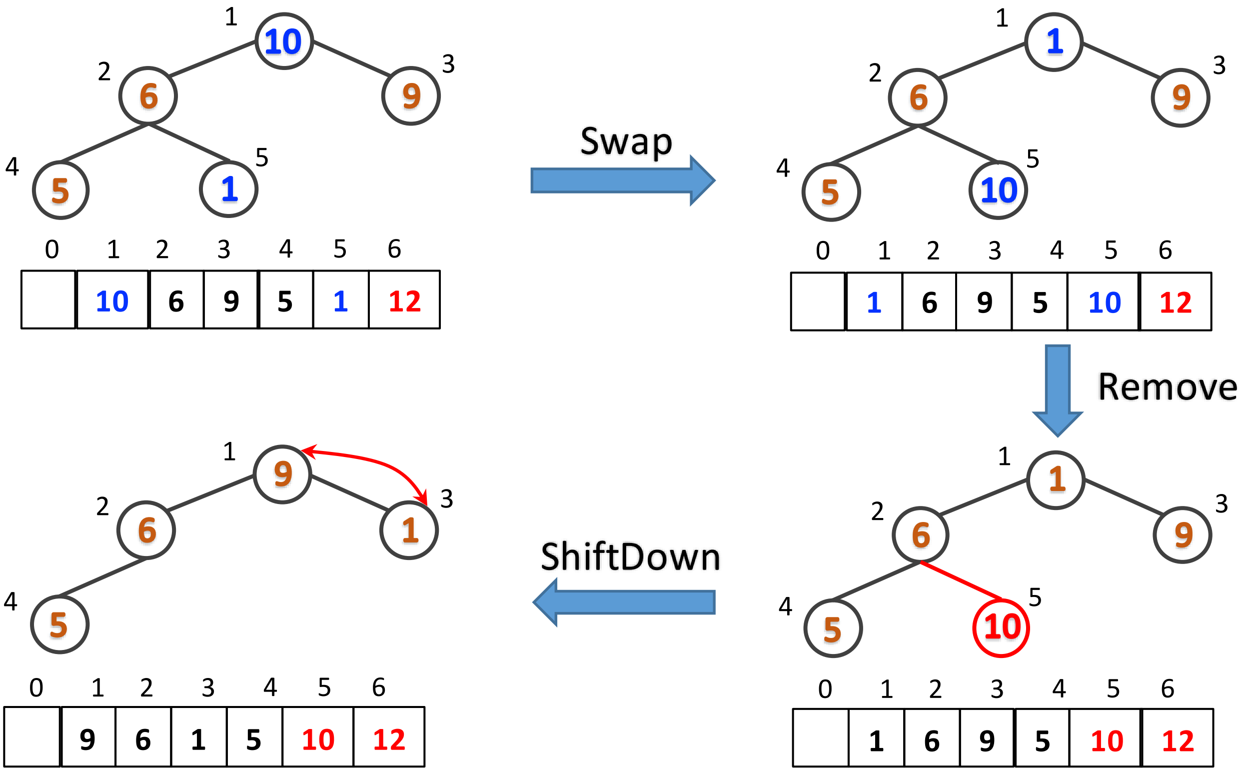
Heapsort

- Heapsort uses heaps to sort in $O(N \log N)$ time
- Basic strategy:
  - Build a min-heap of N elements in $O(N)$ time
  - Perform N pop() operations
  - Store these elements in a second array giving N sorted elements
- Since each pop() takes $O(\log n)$ times, total running time is $O(N \log N)$
- The main problem with this algorithm is that it uses an extra array
- A clever way to avoid this is to use the fact that after each pop() operation, the heap shrinks by 1
- Thus, the previous last cell can be used to store the element that was just popped
- Using this strategy, the array will contain the elements in decreasing sorted order
- If we want in more typical increasing sorted order, we can change the heap to max-heap so that parent has larger element than child

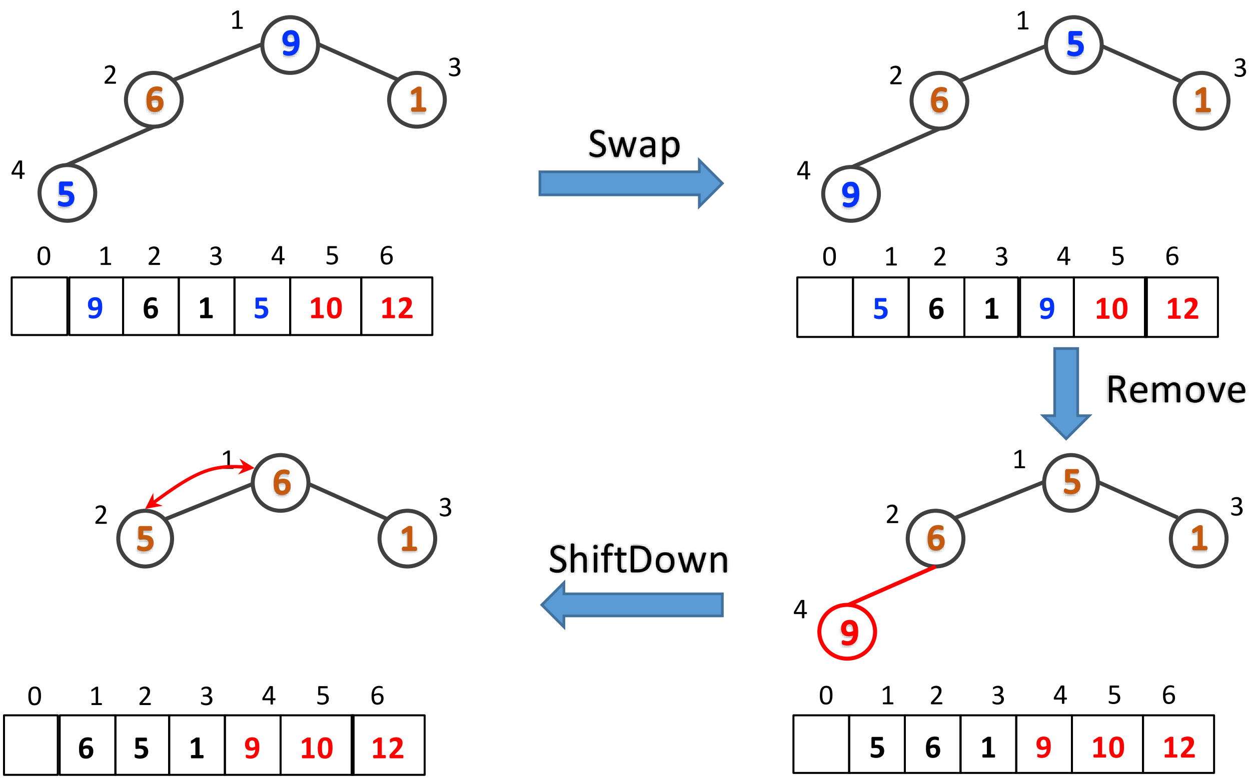
Heapsort



Heapsort


Swap
Remove
ShiftDown
0 1 2 3 4 5 6
10 6 9 5 1 12
1 6 9 5 10 12
1 6 9 5 10 12
9 6 1 5 10 12

Heapsort


Swap
Remove
ShiftDown
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9 6 1 5 10 12
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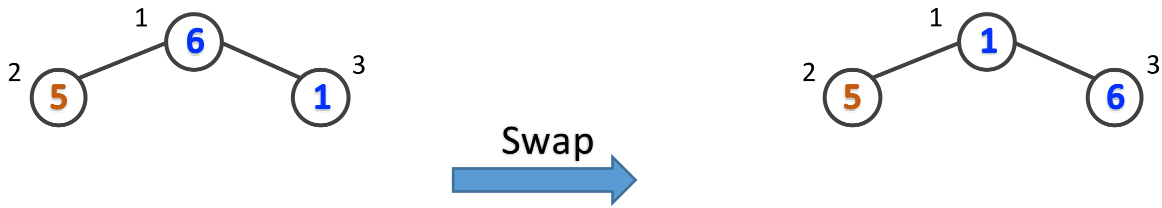
Heapsort



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	6	5	1	9	10	12

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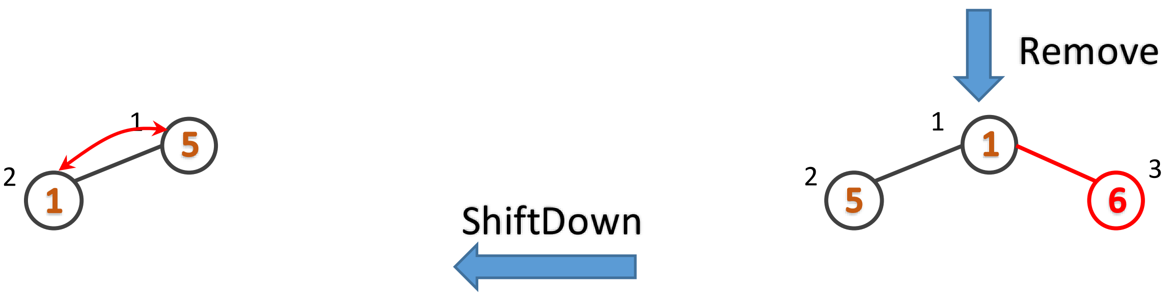

0	1	2	3	4	5	6
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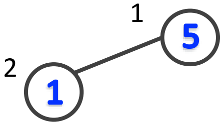
Heapsort



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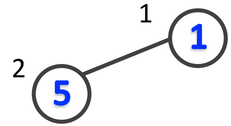

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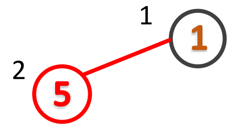
Remove



0	1	2	3	4	5	6
	1	5	6	9	10	12

0	1	2	3	4	5	6
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Thank You