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CSC420 Homework 1

Question 1: Nearest Neighbours and the Curse of Dimensionality

Question 1a: Expectation and Variance of V

Final Answer (work below): $E(Z) = \frac{1}{6}, V(Z) = \frac{7}{180}$

We know:

a) $Z = (X - Y)^2$
b) $X \sim Uniform(0, 1)$
c) $Y \sim Uniform(0, 1)$
d) X, Y are independent

Using additive property of expectation $E(A + B) = E(A) + E(B)$ to obtain $E(Z)$

$$E(Z) = E((X - Y)^2)$$
$$\Rightarrow E(Z) = E(X^2 + Y^2 - 2XY)$$
$$\Rightarrow E(Z) = E(X^2) + E(Y^2) - 2E(XY)$$

Since X, Y are independent: $E(XY) = E(X)E(Y)$

$$\Rightarrow E(Z) = E(X^2) + E(Y^2) - 2E(X)E(Y)$$

Since X, Y are sampled from the identical distributions, all their moments must be the same

$$\forall n \in \mathbb{R}, E(X^n) = E(Y^n)$$

$$\Rightarrow E(Z) = 2E(X^2) - 2E(X)^2$$

$$\Rightarrow E(Z) = 2(E(X^2) - E(X)^2)$$

Recall properties of uniform distributions: $X \sim Uniform(a, b)$

$$f(x) = \frac{1}{b - a}$$

Thus for $X \sim Uniform(0, 1)$

$$f(x) = 1$$

Creating an equation for X's n-th moment about 0 ($E(X^n)$)

$$E(X^n) = \int_0^1 x^n f(x) dx = \int_0^1 x^n dx = \frac{1}{n+1} x^{n+1} \Big|_0^1 = \frac{1}{n+1}(1 - 0) = \frac{1}{n+1}$$

Using the above $E(X) = \frac{1}{2}, E(X^2) = \frac{1}{3}$

$$E(Z) = 2(E(X^2) - E(X)^2)$$

$$\Rightarrow 2\left(\frac{1}{3} - \left(\frac{1}{2}\right)^2\right) = 2\left(\frac{1}{3} - \frac{1}{4}\right) = \frac{2}{12}$$

$$\Rightarrow E(Z) = \frac{1}{6}$$

Question 1a, continued…

Definition of variance in terms of expectations

$$V(Z) = E(Z^2) - E(Z)^2$$

Using the additive property of expectation to obtain $E(Z^2)$

$$E(Z^2) = E(((X - Y)^2)^2)$$

$$\Rightarrow E(Z^2) = E((X - Y)^4)$$

$$\Rightarrow E(Z^2) = E(X^4 - 4X^3Y + 6X^2Y^2 - 4XY^3 + Y^4)$$

$$\Rightarrow E(Z^2) = E(X^4) - 4E(X^3Y) + 6E(X^2Y^2) - 4E(XY^3) + E(Y^4)$$

Since X, Y are independently sampled: $E(X^nY^m) = E(X^n)E(Y^m)$

$$\Rightarrow E(Z^2) = E(X^4) - 4E(X^3)E(Y) + 6E(X^2)E(Y^2) - 4E(X)E(Y^3) + E(Y^4)$$

Since X, Y are sampled from identical distributions $\forall n \in \mathbb{R}, E(X^n) = E(Y^n)$

$$\Rightarrow E(Z^2) = 2E(X^4) - 8E(X^3)E(X) + 6E(X^2)^2$$

Using the previous moment equation $E(X^n) = \frac{1}{n+1} \Rightarrow E(X^3) = \frac{1}{4}, E(X^4) = \frac{1}{5}$

$$\Rightarrow E(Z^2) = \frac{2}{5} - \left(\frac{8}{4} * \frac{1}{2}\right) + 6\left(\frac{1}{3}^2\right)$$

$$\Rightarrow E(Z^2) = \frac{2}{5} - 1 + \frac{6}{9}$$

$$\Rightarrow E(Z^2) = \frac{1}{15}$$

Using all prior results to compute $V(Z)$

$$\left(E(Z^2) = \frac{1}{15}\right) \wedge \left(E(Z) = \frac{1}{6}\right) \wedge (V(Z) = E(Z^2) - E(Z)^2)$$

$$\Rightarrow V(Z) = \frac{1}{15} - \frac{1}{36}$$

$$\Rightarrow V(Z) = \frac{7}{180}$$

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Question 1b: Expectation and Variance of R

Final Answer (work below):

$$E(R_d) = \frac{d}{6}, V(R_d) = \frac{7d}{180}$$

Define R_d to be the squared Euclidean distance between two unit-uniformly distributed d-dimensional points

$$R_d = \sum_{i=1}^{d} Z_i$$

Expectation of R_d (simplify using the additive property of expectation)

$$E(R_d) = E\left(\sum_{i=1}^{d} Z_i\right) = \sum_{i=1}^{d} E(Z_i)$$

Since all X_i, Y_i are drawn from identical distributions, and all Z_i have the same relationship with X_i and Y_i, all Z_i must also be drawn from identical distributions. Therefore, all Z_i have identical expectations:

$$\forall(i,j): i \neq j \in \{1 \dots d\} \times \{1 \dots d\} : E(Z_i) = E(Z_j)$$

According to the above, we can turn the summation in $E(R_d)$ into a multiplication of $E(Z)$

$$\big(E(Z) = E(Z_1) = \cdots = E(Z_d)\big)$$

$$\Rightarrow E(R_d) = \sum_{i=1}^{d} E(Z_i) = \sum_{i=1}^{d} E(Z)$$

$$\Rightarrow E(R_d) = dE(Z)$$

Since Z is distributed the same as in part (a), $E(Z) = \frac{1}{6}$

$$\Rightarrow E(R_d) = \frac{d}{6}$$

Since each Z_i is independent of its peers (i.e. $i \neq j \Rightarrow P\big(Z_i, Z_j\big) = P(Z_i)P(Z_j)$), all variances can simply be summed together

$$V(R_d) = V\left(\sum_{i=1}^{d} Z_i\right) = \sum_{i=1}^{d} V(Z_i)$$

Since Z is distributed identically, we can turn the above summation into a multiplication of $V(Z)$, and use the previous result $V(Z) = \frac{7}{180}$

$$\big(V(Z) = V(Z_1) = \cdots = V(Z_d)\big) \Rightarrow V(R_d) = \sum_{i=1}^{d} V(Z_i) = dV(Z) = \frac{7d}{180}$$

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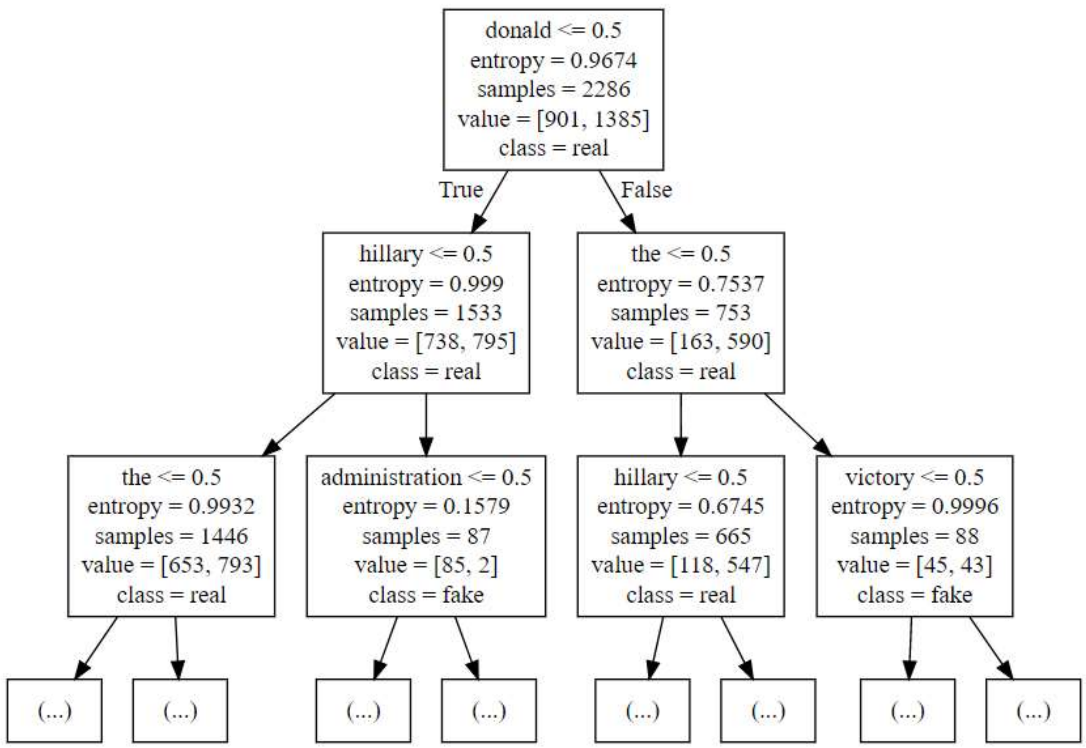
Question 2: Decision Trees

Question 2b: Output of select_model (tested over both criteria, and depths 2-6, inclusive)

```
Model(depth=2, criteria=entropy): score = 0.6346938775510204
Model(depth=3, criteria=entropy): score = 0.6857142857142857
Model(depth=4, criteria=entropy): score = 0.7061224489795919
Model(depth=5, criteria=entropy): score = 0.689795918367347
Model(depth=6, criteria=entropy): score = 0.689795918367347
Model(depth=2, criteria=gini): score = 0.7040816326530612
Model(depth=3, criteria=gini): score = 0.7040816326530612
Model(depth=4, criteria=gini): score = 0.7040816326530612
Model(depth=5, criteria=gini): score = 0.6877551020408164
Model(depth=6, criteria=gini): score = 0.7
```

Question 2c: Visualization of Best Model



Question 2d: Output from compute_information_gain

Bolded is the topmost label ("donald")

Information Gain in Label by splitting on donald: 0.03412305015881989
Information Gain in Label by splitting on hillary: 0.026697700148983317
Information Gain in Label by splitting on clinton: 0.007298694213891288
Information Gain in Label by splitting on korea: 0.011570446505243082
Information Gain in Label by splitting on america: 0.00849756958300818
Information Gain in Label by splitting on putin: 0.0017367934909430227