Assignment 3

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Problem 1

Answer

We will use the **Theorem 3.3** and the **Definition 3.7** from [1] to check global stability

Theorem 3.3 (Global Stability) *Assume that there exists a scalar function V of the state* $\mathbf{x}$, *with continuous first order derivatives such that*

- $V(\mathbf{x})$ *is positive definite*
- $\dot{V}(\mathbf{x})$ *is negative definite*
- $V(\mathbf{x}) \to \infty$ *as* $\|\mathbf{x}\| \to \infty$

then the equilibrium at the origin is globally asymptotically stable.

Definition 3.7 *A scalar continuous function* $V(\mathbf{x})$ *is said to be locally positive definite if* $V(\mathbf{0}) = 0$ *and, in a ball* $\mathbf{B}_{R_o}$

$$\mathbf{x} \neq \mathbf{0} \quad => \quad V(\mathbf{x}) > 0$$

If $V(\mathbf{0}) = 0$ *and the above property holds over the whole state space, then* $V(\mathbf{x})$ *is said to be globally positive definite.*

- $V(\boldsymbol{x})$ is **globally positive definite**
  $V(\boldsymbol{x})$ is defined for all $x_1, x_2 \in \mathbb{R}$ and is positive unless $\boldsymbol{x} = \mathbf{0}$
- $\dot{V}(\boldsymbol{x})$ is **not negative definite** unless there is some appropriate connection between x_1 and x_2:
  $$\dot{V}(\boldsymbol{x}) = \left(\frac{x_1^2}{(1+x_1^2)^2} + x_2^2\right)'_t = 2\,x_2\dot{x}_2 + \frac{2\,x_1\dot{x}_1}{(x_1^2+1)^2} - \frac{4\,x_1^3\,\dot{x}_1}{(x_1^2+1)^3}$$

Problem 2

Answer 1

We will use the **Definition 3.7**, **Definition 3.8**, **Theorem 3.2** from [1] to check local stability.

Definition 3.8 *If, in a ball* $\mathbf{B}_{R_o}$, *the function* $V(\mathbf{x})$ *is positive definite and has continuous partial derivatives, and if its time derivative along any state trajectory of system (3.2) is negative semi-definite, i.e.,*

$$\dot{V}(\mathbf{x}) \leq 0$$

then $V(\mathbf{x})$ *is said to be a Lyapunov function for the system (3.2).*

Theorem 3.2 (Local Stability) *If, in a ball* $\mathbf{B}_{R_o}$, *there exists a scalar function* $V(\mathbf{x})$ *with continuous first partial derivatives such that*

- $V(\mathbf{x})$ *is positive definite (locally in* $\mathbf{B}_{R_o}$*)*
- $\dot{V}(\mathbf{x})$ *is negative semi-definite (locally in* $\mathbf{B}_{R_o}$*)*

then the equilibrium point $\mathbf{0}$ *is stable. If, actually, the derivative* $\dot{V}(\mathbf{x})$ *is locally negative definite in* $\mathbf{B}_{R_o}$, *then the stability is asymptotic.*

We select as a Lyapunov candidate function which is **globally positive definite**:

$$V(\boldsymbol{x}) = x_1^2 + x_2^2$$

Its derivative

$$\dot{V}(\boldsymbol{x}) = -2\,(x_1^2 + x_2^2)\,(c - x_1^2 - x_2^2)$$

is locally negative definite for $x_1^2 + x_2^2 < c$. Therefore, the given system is **locally asymptotically stable** near the origin.

Answer 2

Let $\boldsymbol{\Omega}_c$ be the region where $V(\boldsymbol{x}) = x_1^2 + x_2^2 < c$. Then only $\mathbf{0} \in \boldsymbol{R}$, where $\boldsymbol{R}$ is the set of all points of $\boldsymbol{\Omega}_c$ where $\dot{V}(\boldsymbol{x}) = 0$. Hence, the largest invariant set in $\boldsymbol{R}$ is $\boldsymbol{M} = \{\mathbf{0}\}$, the **region of attraction**.

Problem 3

Answer 1

$V = \frac{1}{2}\tilde{H}^2 \succ 0$

Let $u = \ddot{\theta} + \sin(\theta) - \tanh(k\theta)\sqrt{2H_d}$. We will use $\tanh(k\theta)$ with some large k to have a differentiable, $\operatorname{sign}(\theta)$-like function.While in theory $\tanh(0) = 0$, it may not be noticeable with finite-precision numbers.

Then $\dot{\theta} = -\tanh(k\theta)\sqrt{2H_d}$ and $H = \dfrac{1}{2}\dot{\theta}^2 + 1 - \cos(\theta) = H_d \cdot \tanh^2(k\theta) + 1 - \cos(\theta)$.We want that $\tilde{H} = H_d - H = H_d(1 - \tanh^2(k\theta)) + \cos(\theta) - 1 \simeq \cos(\theta) - 1$ converges to zero. In fact, we want that θ converges to 0.

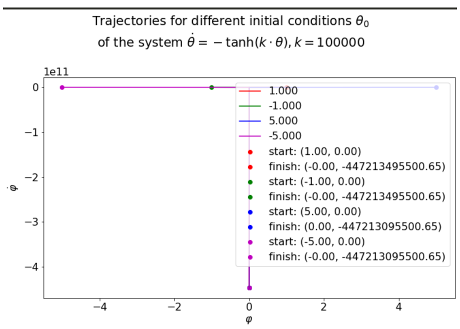
Approximately, $\dot{\theta} = \left\{ \begin{smallmatrix} -H_d & \theta > 0 \\ H_d & \theta < 0 \end{smallmatrix} \right.$, which will drive θ to 0 in case it switches the sign

But I get that

$\dot{V} = \frac{\sqrt{2}\,\sqrt{H_d}\,(2\,H_d\,k\tanh(\theta\,k) + \sin(\theta)\cosh^2(\theta\,k))\,(H_d + \cos(\theta)\cosh^2(\theta\,k) - \cosh^2(\theta\,k))\tanh(\theta\,k)}{\cosh^4(\theta\,k)}$

is always positive definite in the neighborhood of zero. I don't know why:(

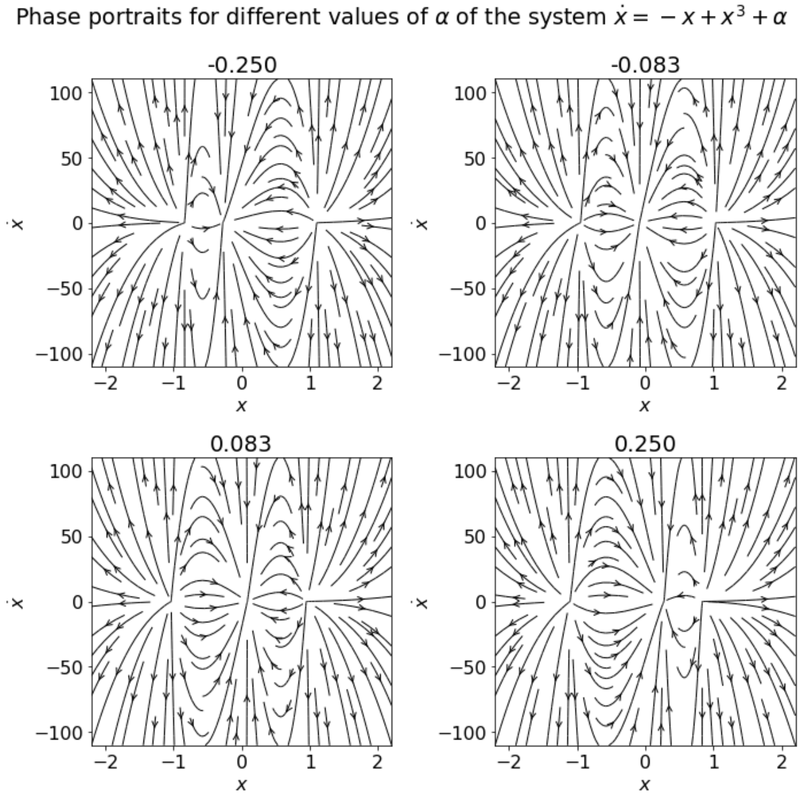
Answer 2



Unfortunately, I was unable to plot the phase plot properly due to some float errors:(

Problem 4

Answer



1 Option

Let $V = \frac{1}{2}x^2$ and $\dot{V} = x\,\dot{x} = x(-x + x^3 + \alpha) = -x^2 + x^4 + \alpha x$.
Suppose $x^2 < 1$ and $|\alpha| < \frac{1}{4}$ (by given region).

Let us inspect $(-x + x^3 + \alpha) = E$:

$$\begin{cases} E < 0 & 0.269594 < x < 0.837565, \alpha = \frac{1}{4} \\ E < 0 & x < 1, \alpha < 0 \\ E < 0 & -0.269594 < x < 0, \alpha = -\frac{1}{4} \\ E > 0 & -1 < x < 0, \alpha > 0 \end{cases}$$

As E is a subexpression of $\dot{V}$, we see that Lyapunov function $\dot{V}$ is not very conclusive and may have several zeros. On the other hand, the plots show that robust invariant set is $|x| < 1$.

2 Option

Let $V = \frac{1}{2}\dot{x}^2$.
Then $\dot{V} = \dot{x}\ddot{x} = \dot{x}(-\dot{x} + 3x^2\dot{x}) = (-x + x^3 + \alpha)^2(-1 + 3x^2)$
This function is negative definite on $(-1 + 3x^2) < 0$, or $x^2 < \frac{1}{3}$. Possibly, it can be extended to include the whole $|x| < 1$ region.

Bibliography

[1] J.J.E. Slotine, J.J.E. Slotine, and W. Li. *Applied Nonlinear Control*. Prentice Hall, 1991.