Lecture 5
Epipolar Geometry

Professor Silvio Savarese

Computational Vision and Geometry Lab

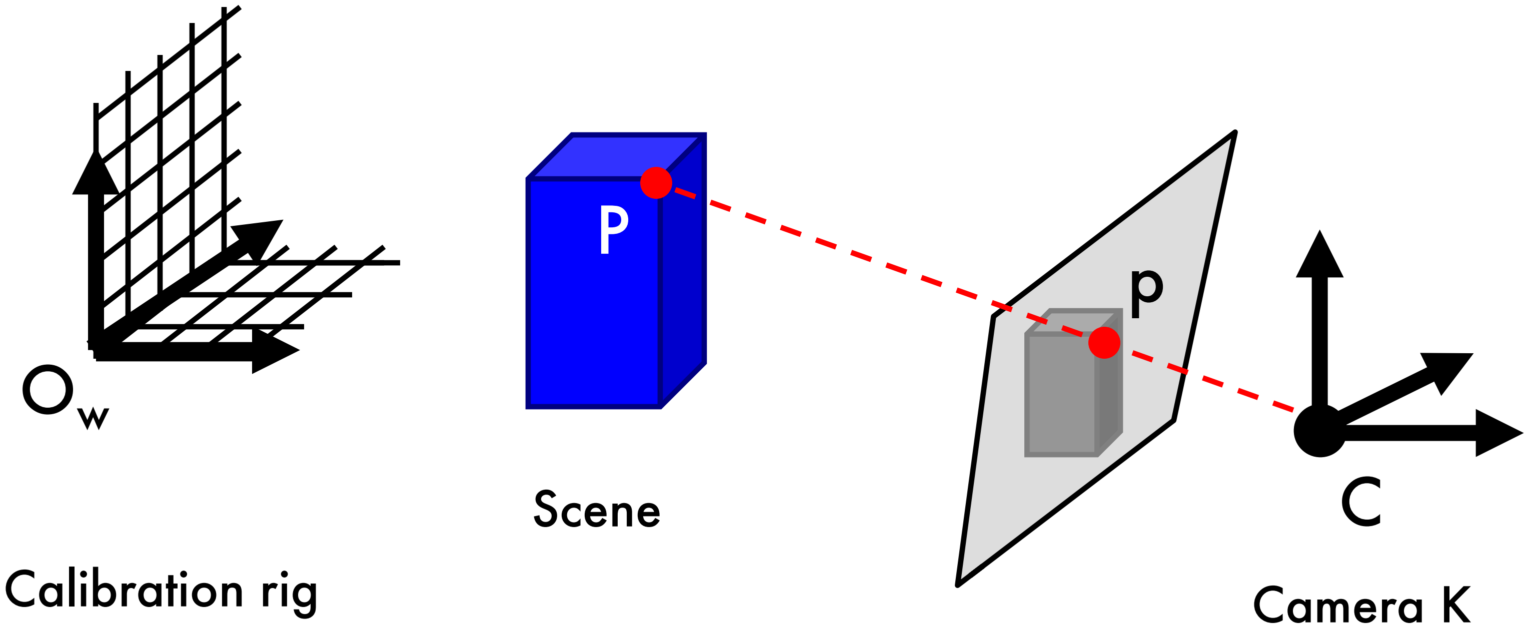
Lecture 5

Epipolar Geometry

- Why is stereo useful?
- Epipolar constraints
- Essential and fundamental matrix
- Estimating F
- Examples

Reading: [AZ] Chapter: 4 “Estimation – 2D perspective transformations
Chapter: 9 “Epipolar Geometry and the Fundamental Matrix Transformation”
Chapter: 11 “Computation of the Fundamental Matrix F”
[FP] Chapter: 7 “Stereopsis”
Chapter: 8 “Structure from Motion”

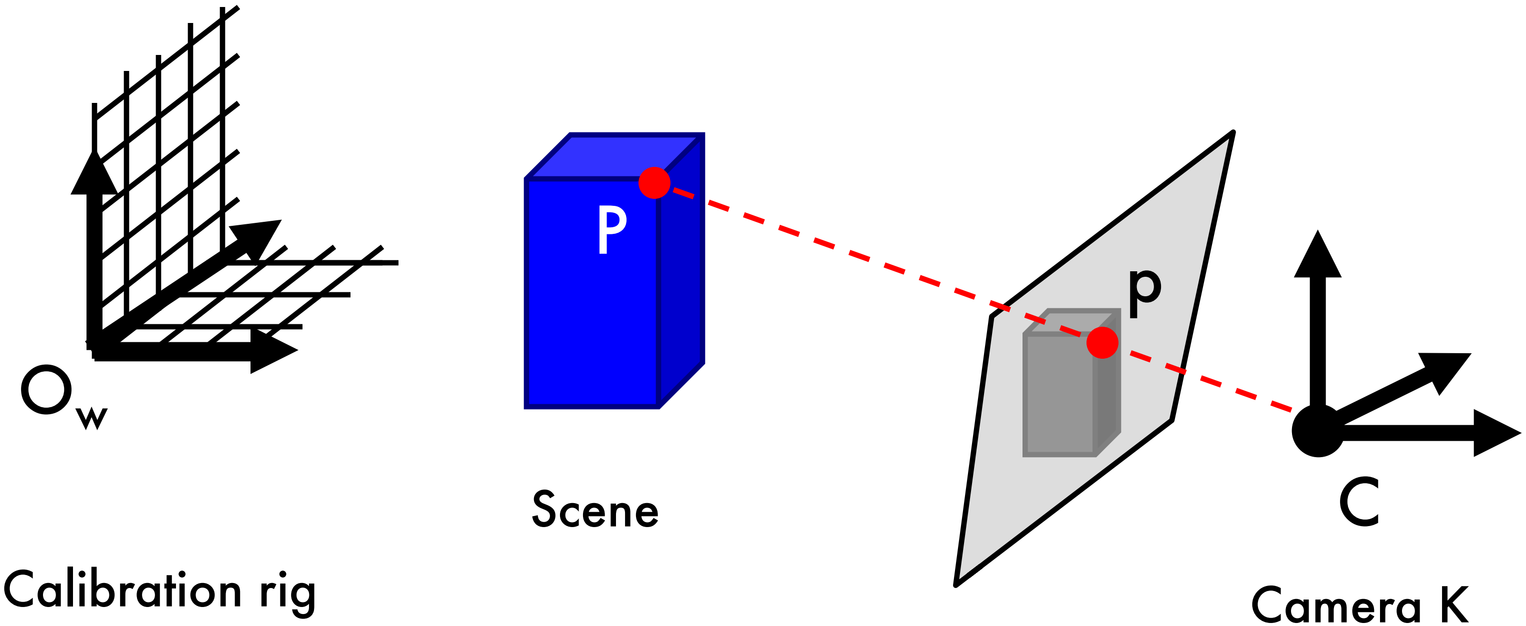
Recovering structure from a single view



From calibration rig	$\rightarrow$	location/pose of the rig, K
From points and lines at infinity + orthogonal lines and planes	$\rightarrow$	structure of the scene, K

Knowledge about scene (point correspondences, geometry of lines & planes, etc…

Recovering structure from a single view



Why is it so difficult?

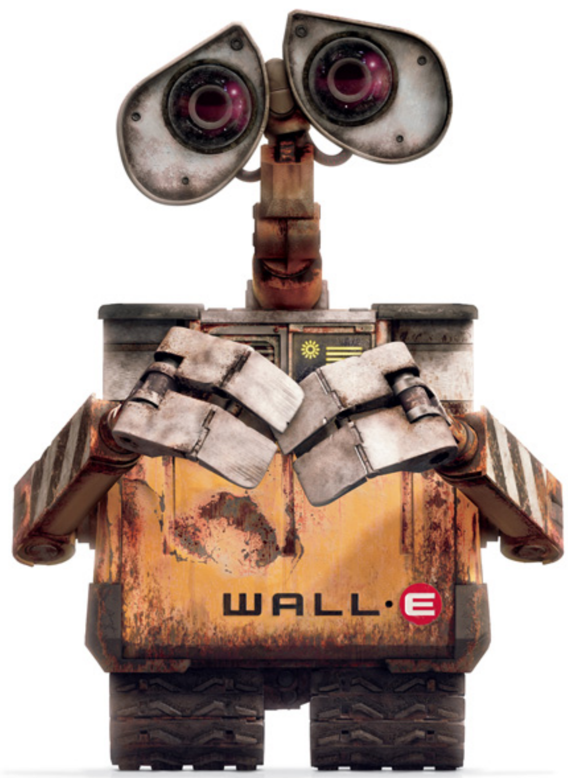
Intrinsic ambiguity of the mapping from 3D to image (2D)

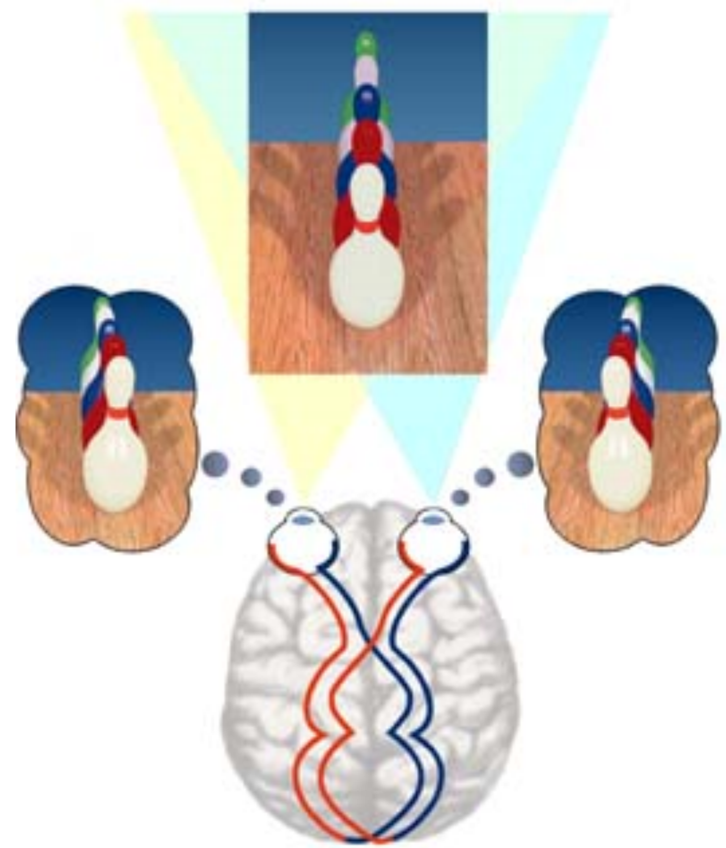
Recovering structure from a single view

Intrinsic ambiguity of the mapping from 3D to image (2D)

Courtesy slide S. Lazebnik

Two eyes help!

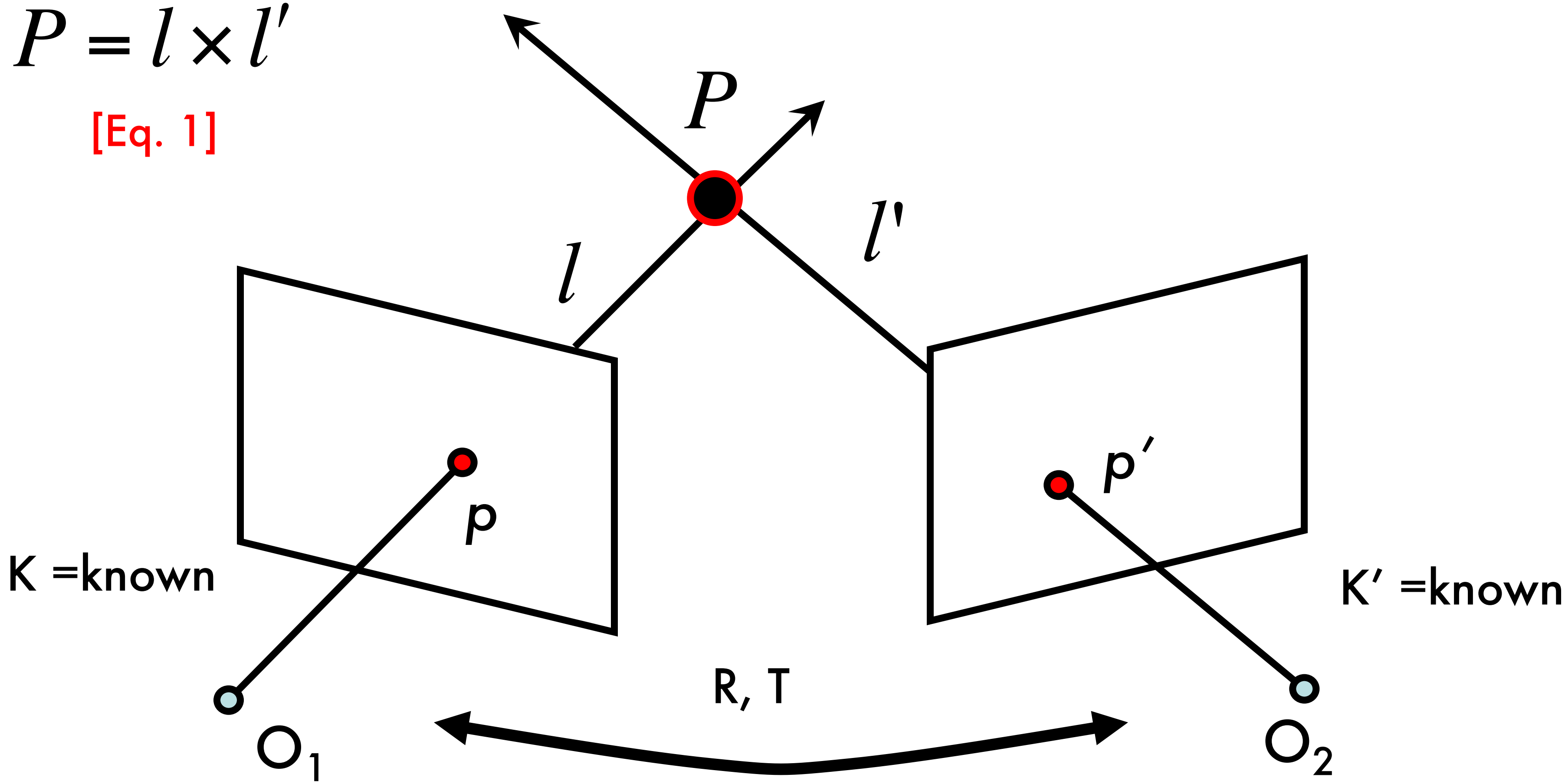




Two eyes help!

$$P = l \times l'$$

[Eq. 1]

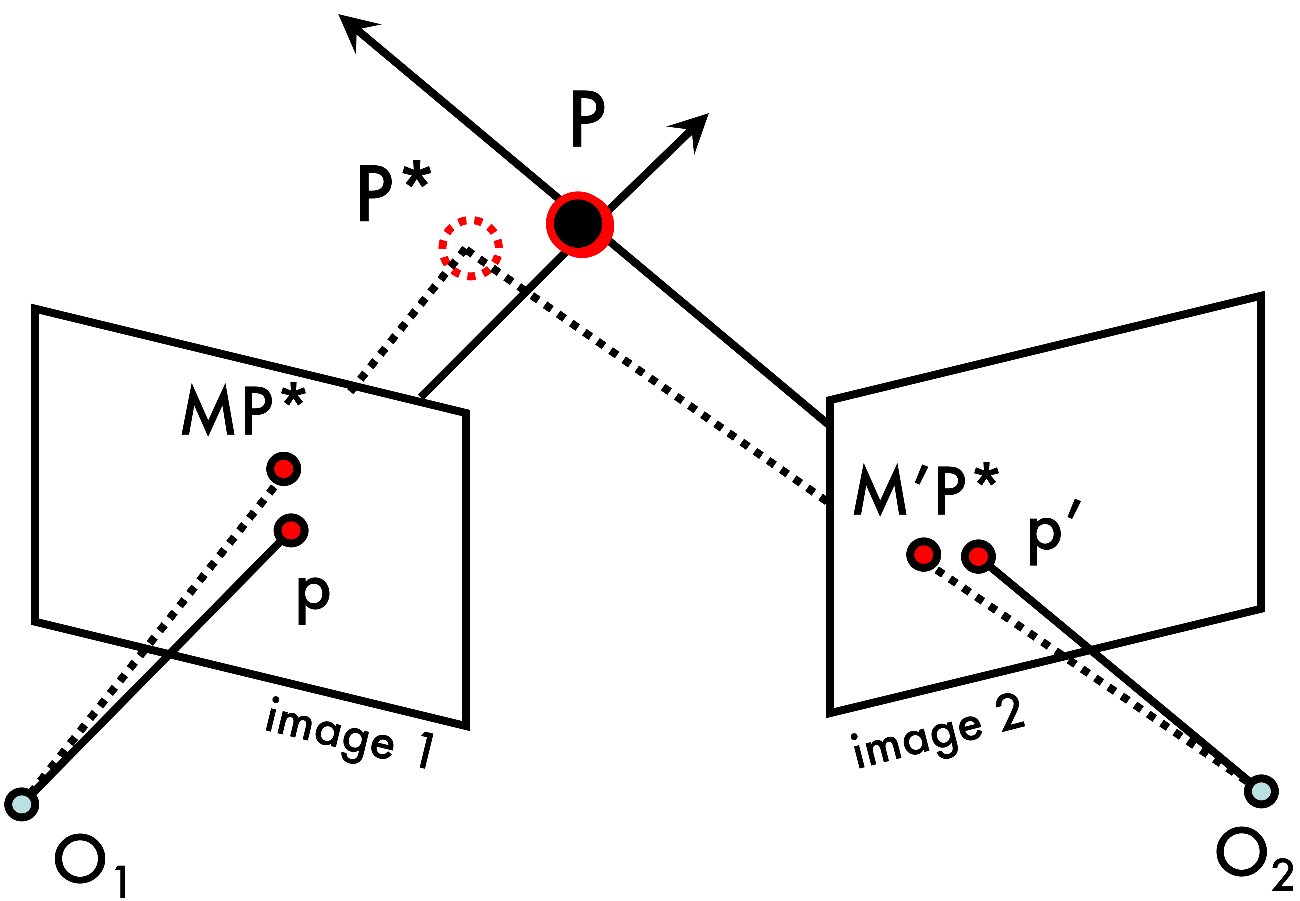


This is called triangulation

Triangulation

- Find P* that minimizes

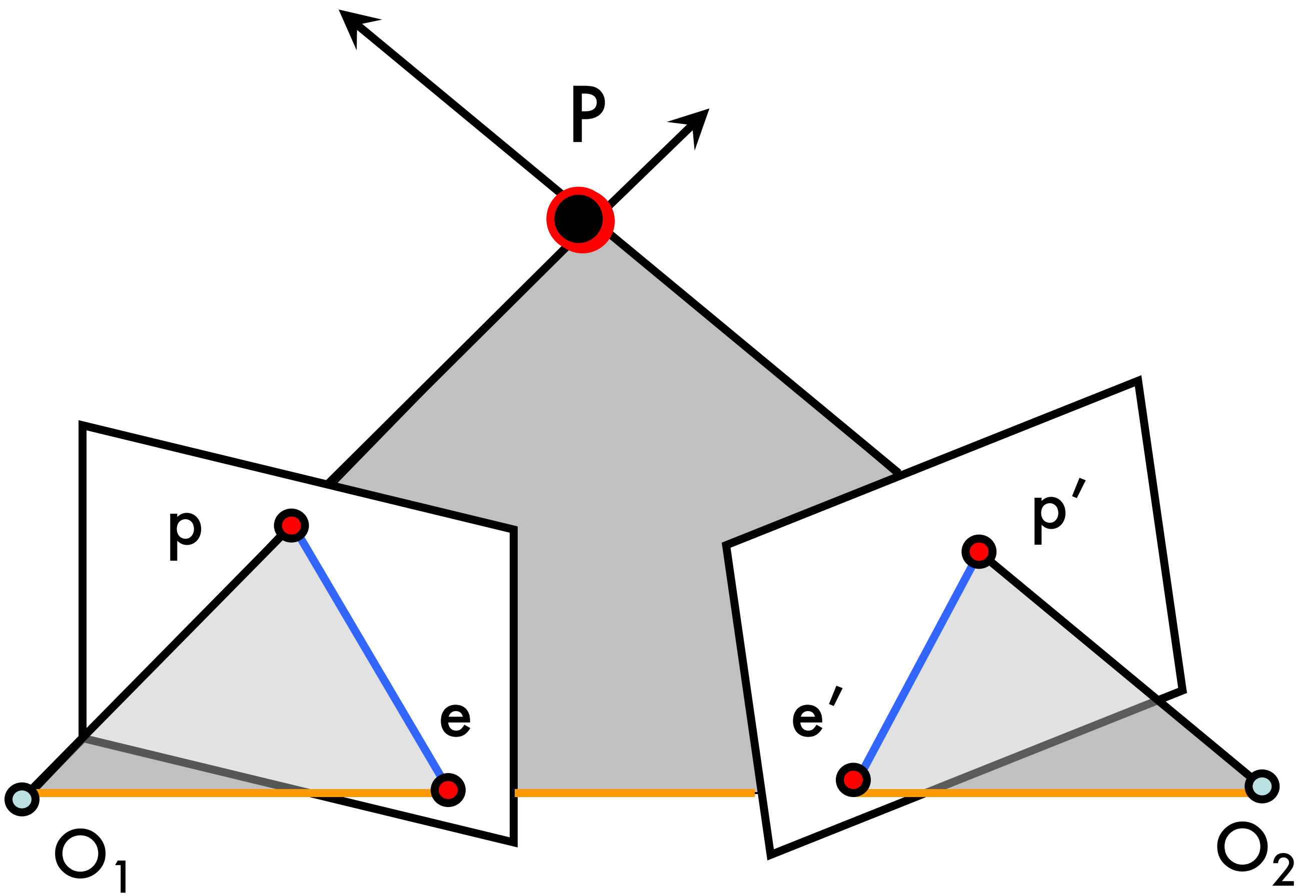
$$d(p, M\ P^*) + d(p', M' P^*) \quad \text{[Eq. 2]}$$



Multi (stereo)-view geometry

- **Camera geometry:** Given corresponding points in two images, find camera matrices, position and pose.
- **Scene geometry:** Find coordinates of 3D point from its projection into 2 or multiple images.
- **Correspondence:** Given a point **p** in one image, how can I find the corresponding point **p′** in another one?

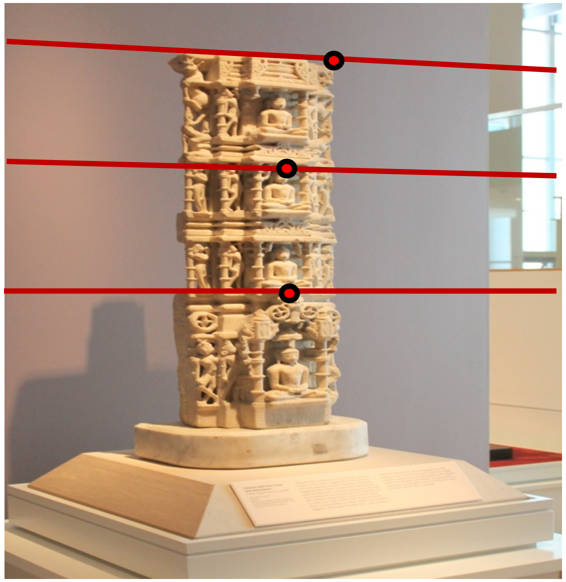
Epipolar geometry

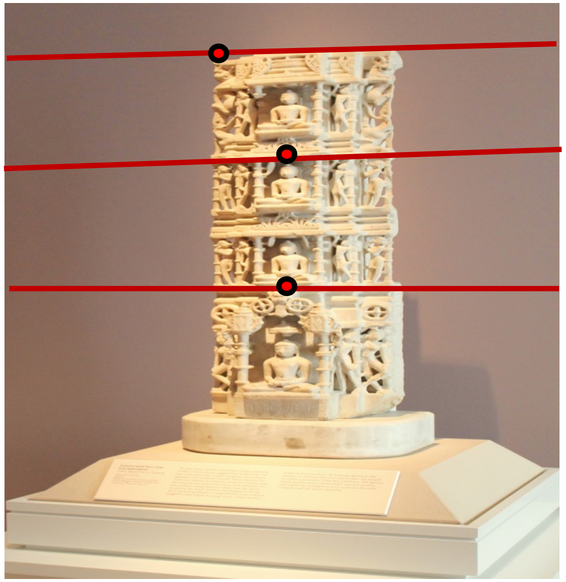


- Epipolar Plane
- Baseline
- Epipolar Lines
- Epipoles e, e′

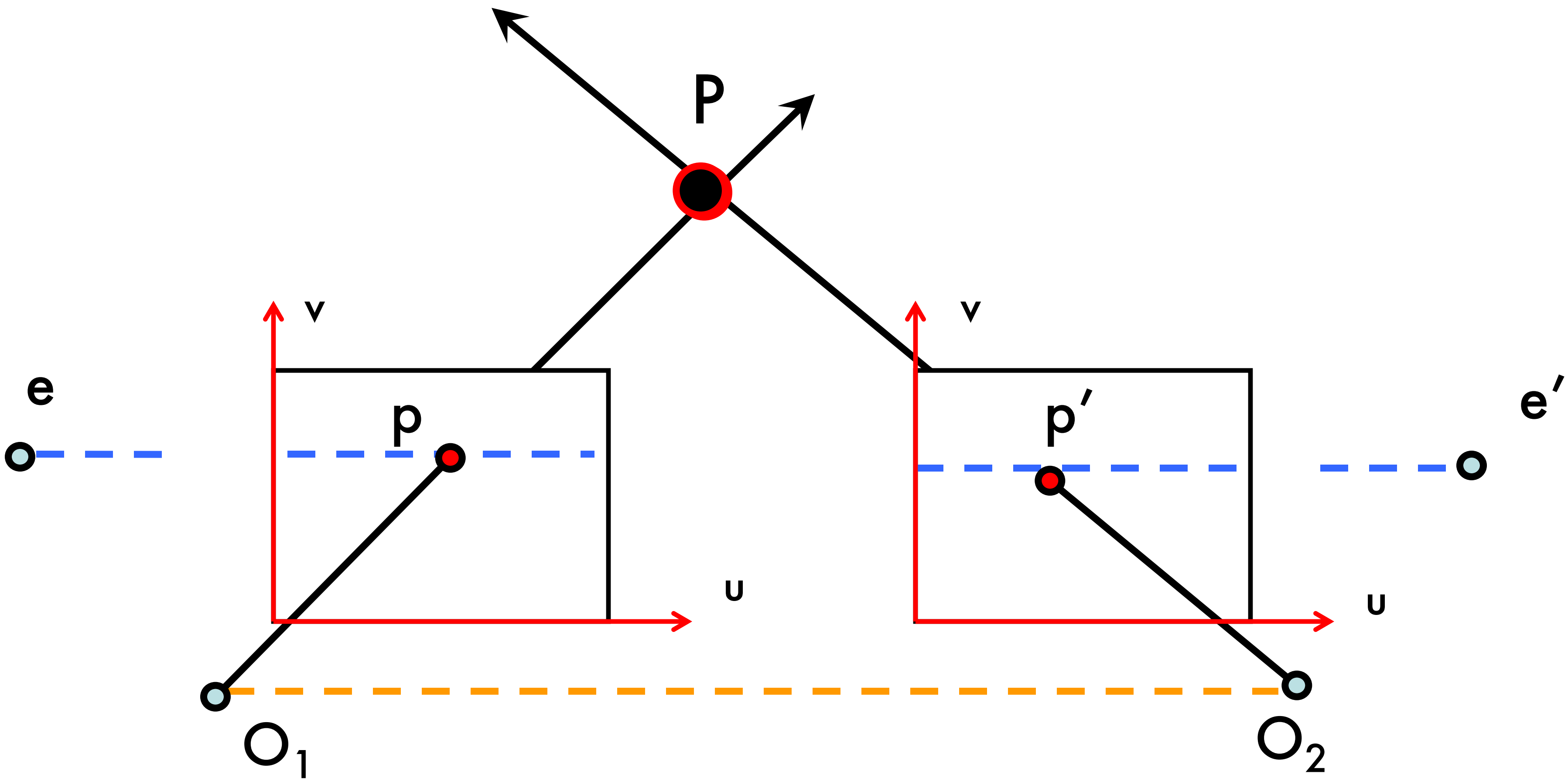
  = intersections of baseline with image planes
  = projections of the other camera center

Example of epipolar lines





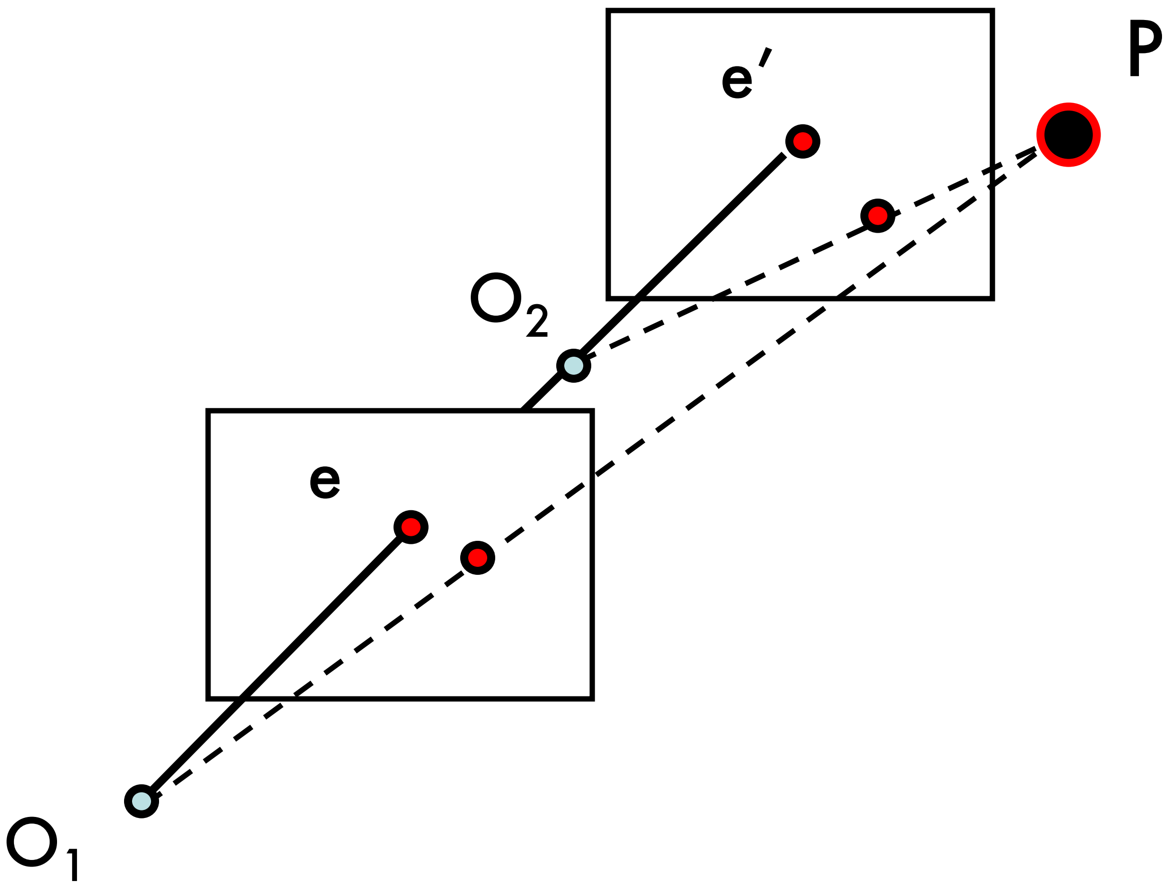
Example: Parallel image planes

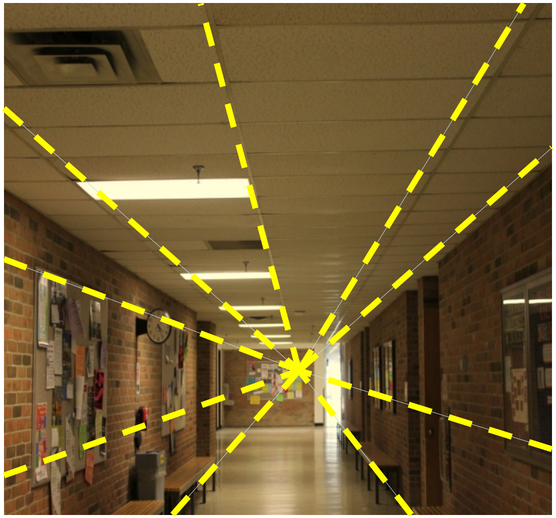


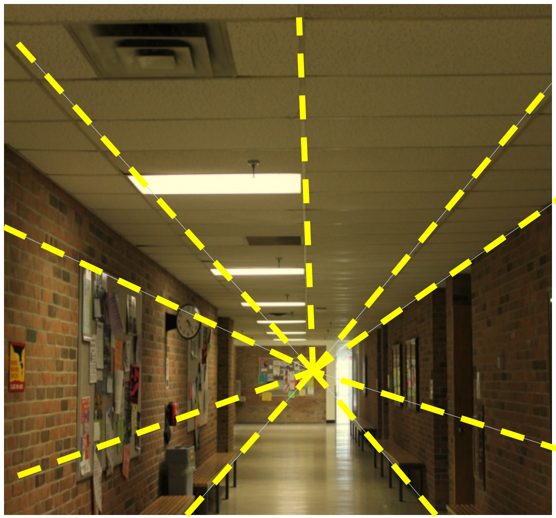
- Baseline intersects the image plane at infinity
- Epipoles are at infinity
- Epipolar lines are parallel to u axis

Example: Parallel Image Planes

Example: Forward translation





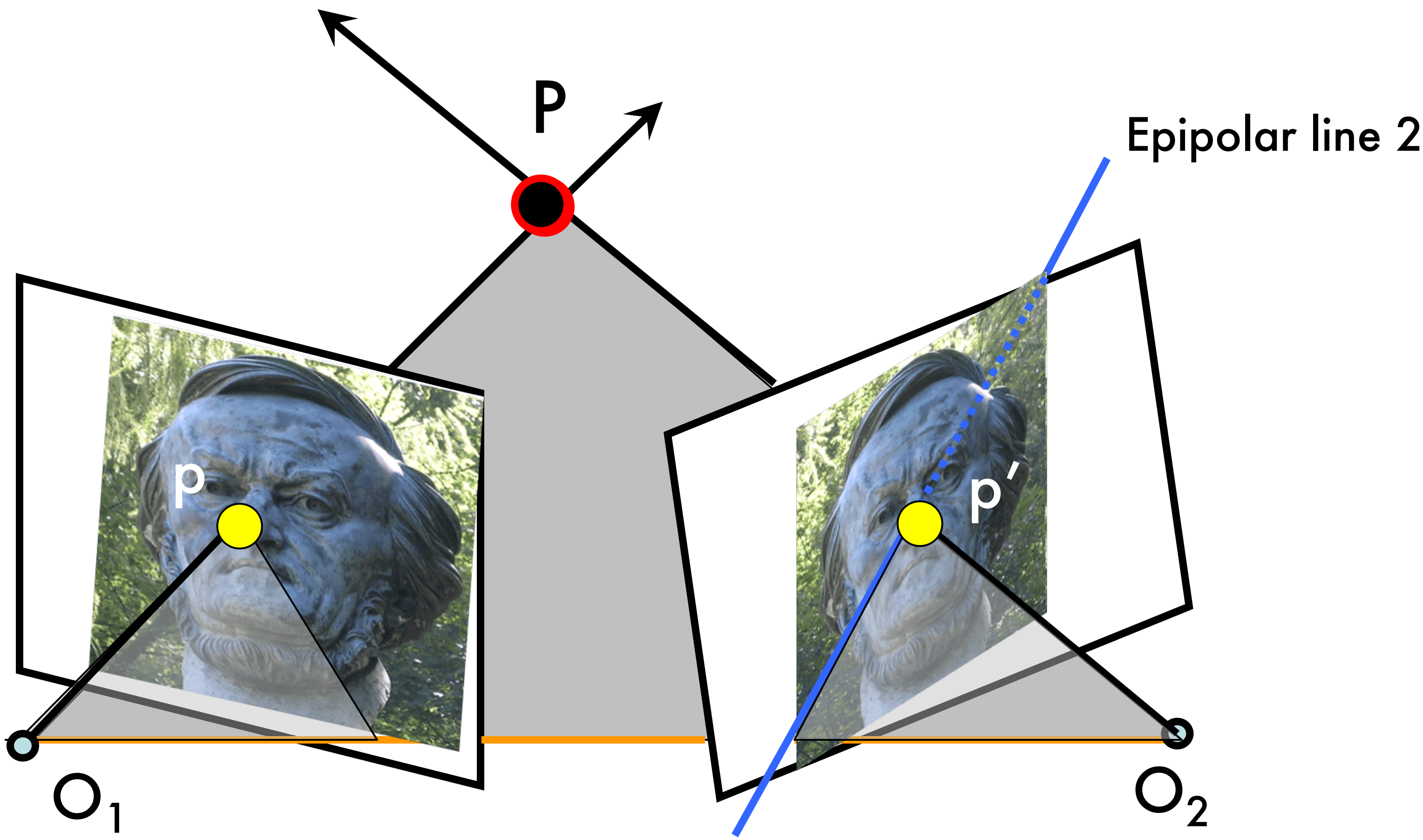


- The epipoles have same position in both images
- Epipole called FOE (focus of expansion)

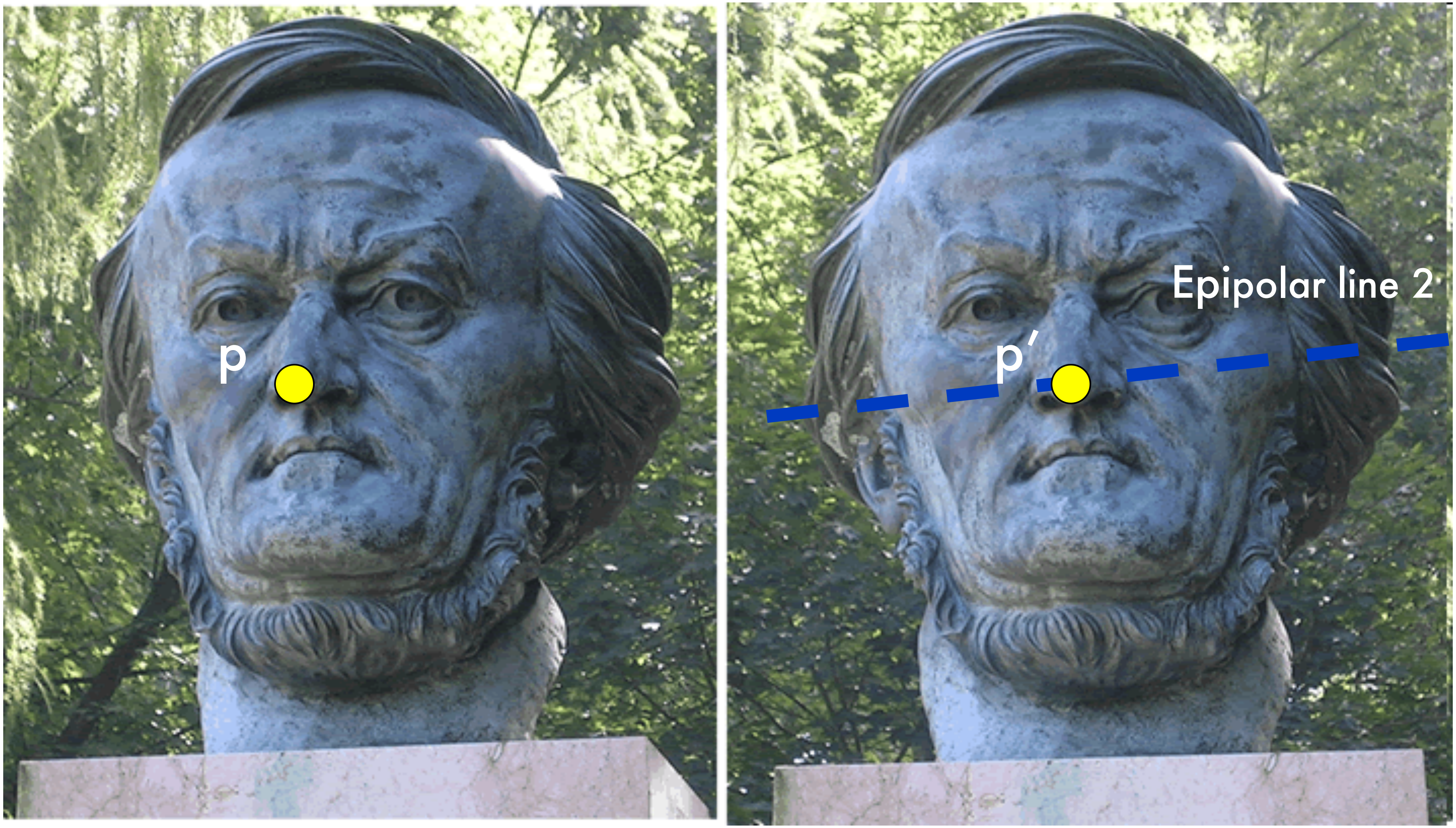
Epipolar Constraint

- Two views of the same object
- Given a point on left image, how can I find the corresponding point on right image?

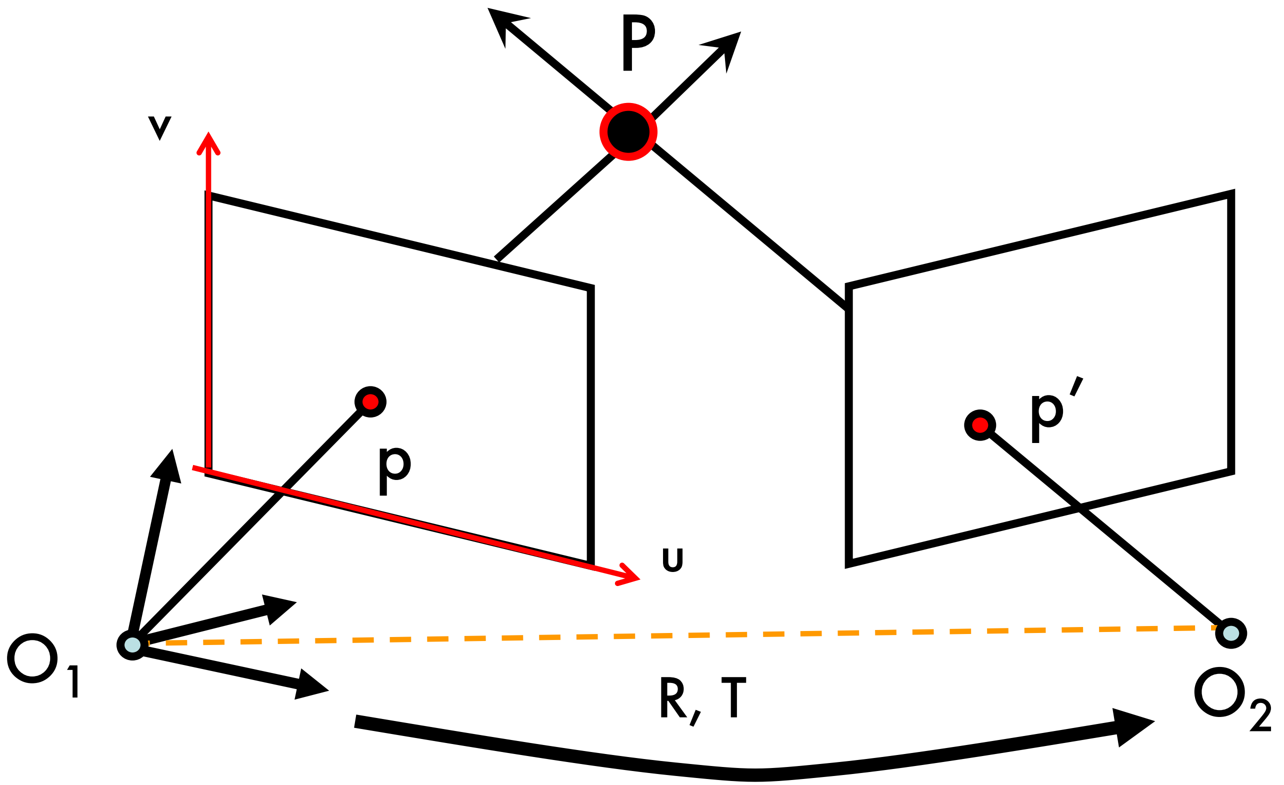
Epipolar geometry



Epipolar Constraint



Epipolar Constraint

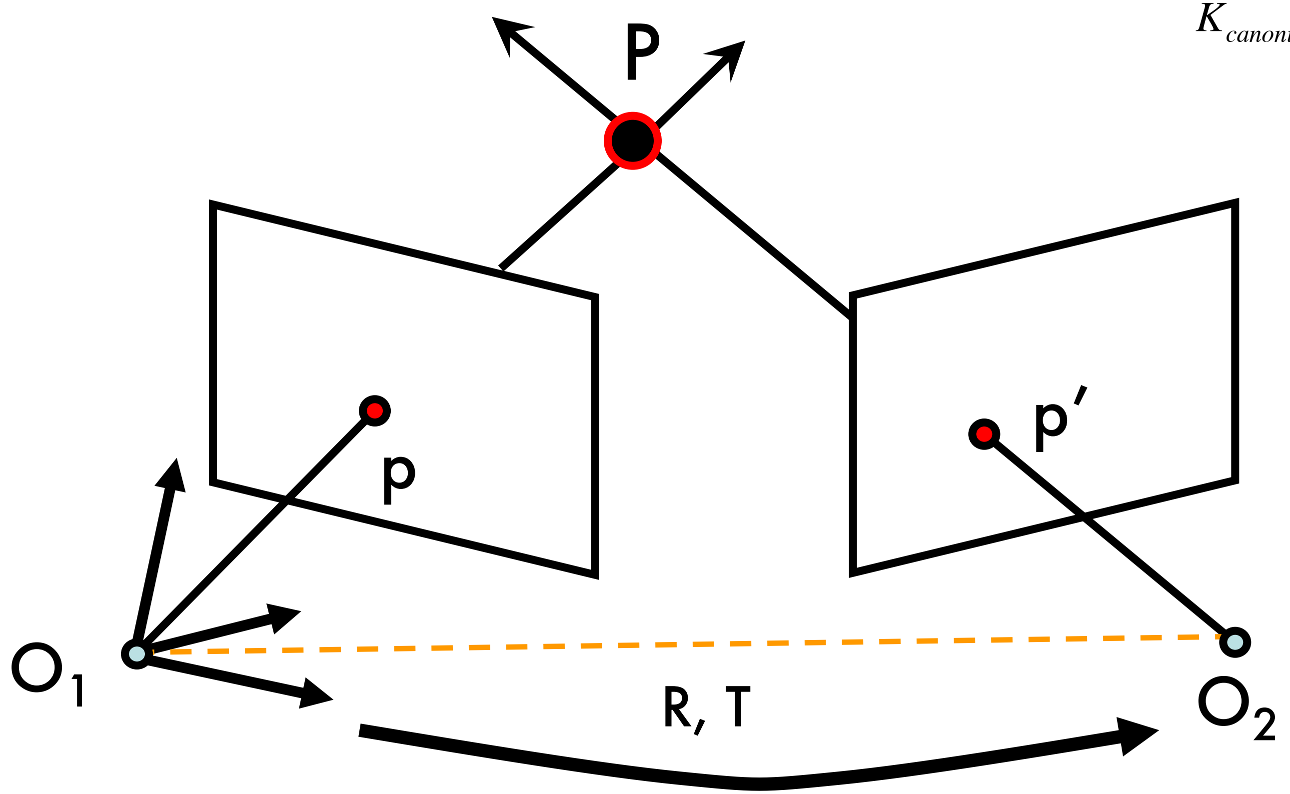


$$M = K\begin{bmatrix} I & 0 \end{bmatrix} \qquad M' = K'\begin{bmatrix} R^T & -R^T T \end{bmatrix}$$

$$M\ P = \begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = p \quad \text{[Eq. 3]} \qquad M'\ P = \begin{bmatrix} u' \\ v' \\ 1 \end{bmatrix} = p' \quad \text{[Eq. 4]}$$

Epipolar Constraint

$$K_{canonical} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



$$M = K\begin{bmatrix} I & 0 \end{bmatrix}$$

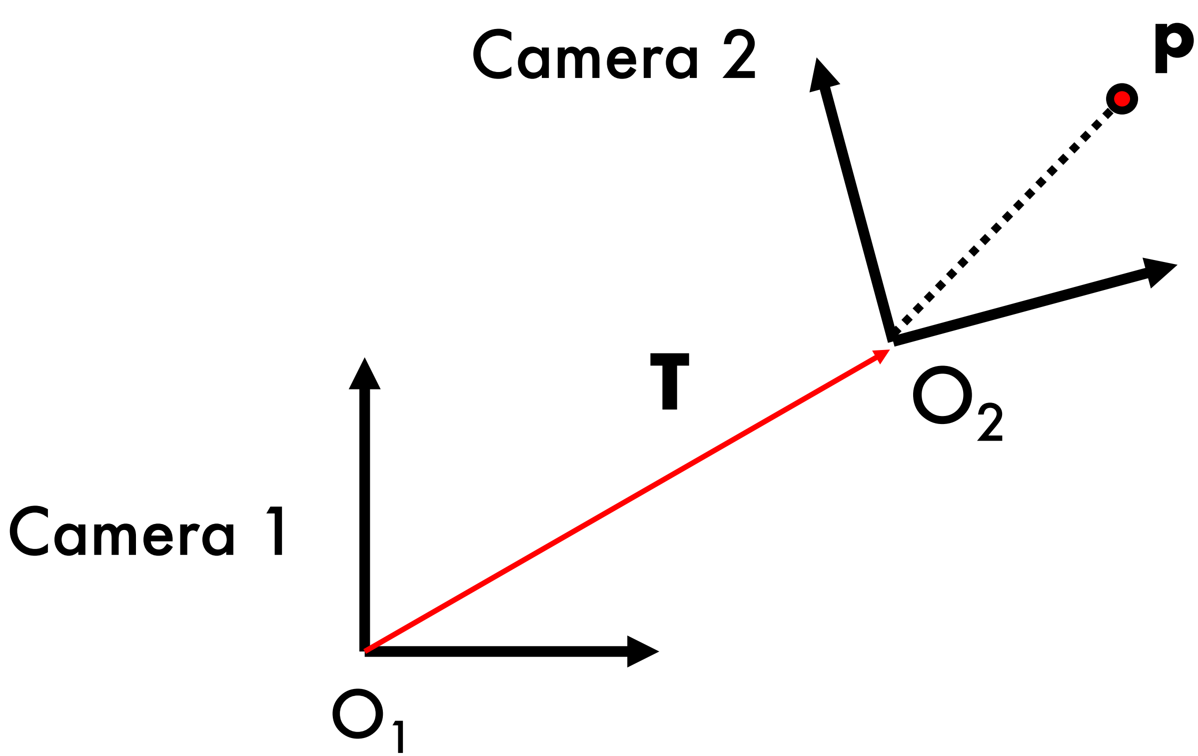
K = K′ are known
(canonical cameras)

$$M' = K'\begin{bmatrix} R^T & -R^T T \end{bmatrix}$$

$$M = \begin{bmatrix} I & 0 \end{bmatrix}$$ [Eq. 5]

$$M' = \begin{bmatrix} R^T & -R^T T \end{bmatrix}$$ [Eq. 6]

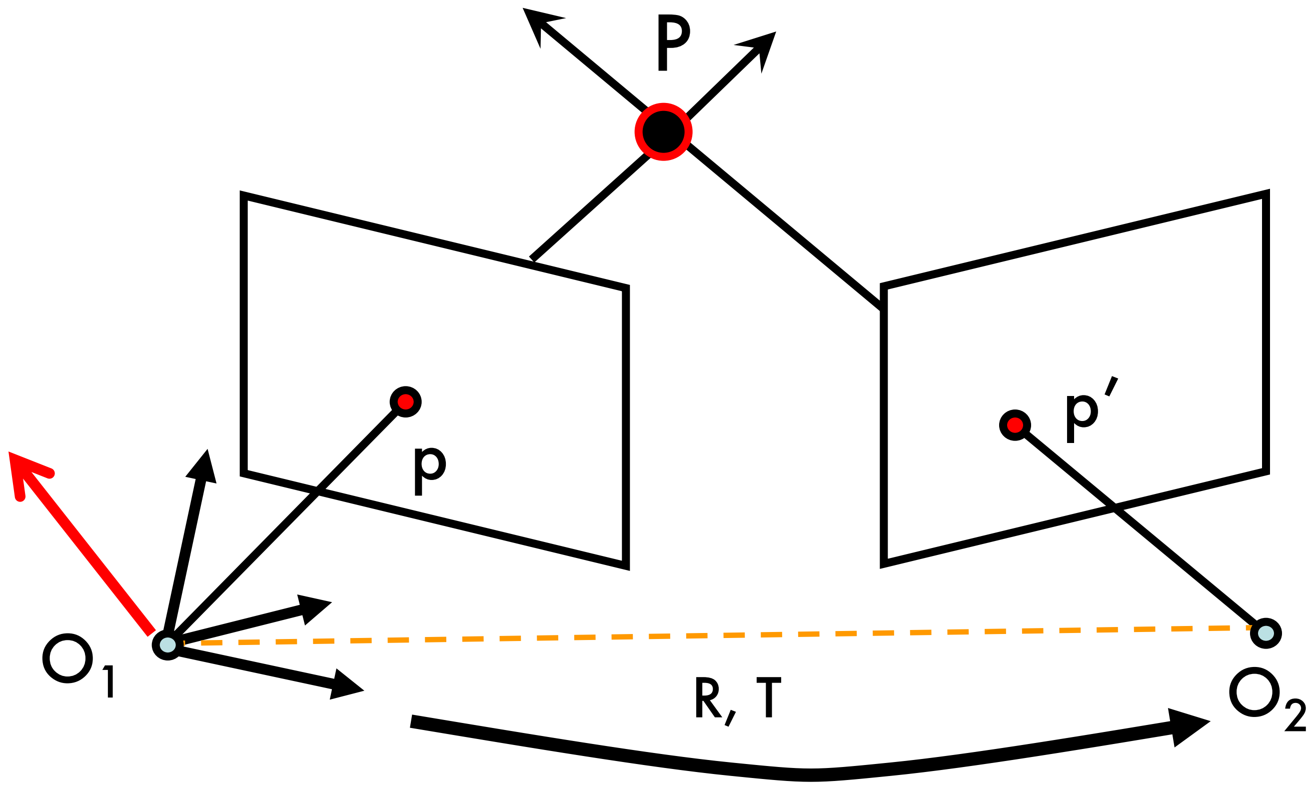
The cameras are related by R, T



T = $\mathbf{O_2}$ in the camera 1 reference system

R is the rotation matrix such that a vector **p’** in the camera 2 is equal to **R p’** in camera 1.

Epipolar Constraint



p′ in first camera reference system is $= R\,p' + T$

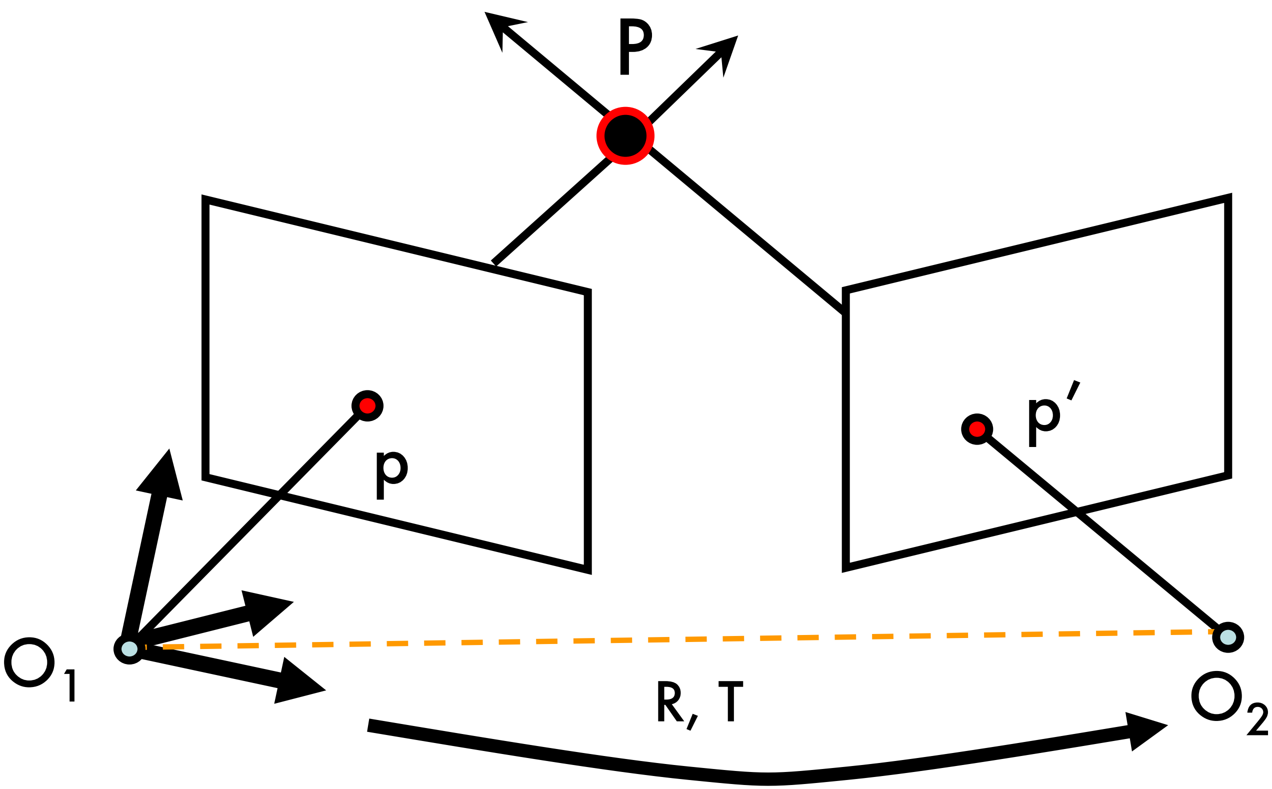
$T \times ((R\,p') + T) = T \times (R\,p')$ is perpendicular to epipolar plane

$$\rightarrow p^T \cdot \left[T \times (R\,p')\right] = 0$$ [Eq. 7]

Cross product as matrix multiplication

$$\mathbf{a} \times \mathbf{b} = \begin{bmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{bmatrix} \begin{bmatrix} b_x \\ b_y \\ b_z \end{bmatrix} = [\mathbf{a}_\times]\mathbf{b}$$

Epipolar Constraint



$$p^T \cdot [T \times (R\, p')] = 0 \rightarrow p^T \cdot \boxed{[T_\times] \cdot R}\, p' = 0$$

[Eq. 8]

[Eq. 9]

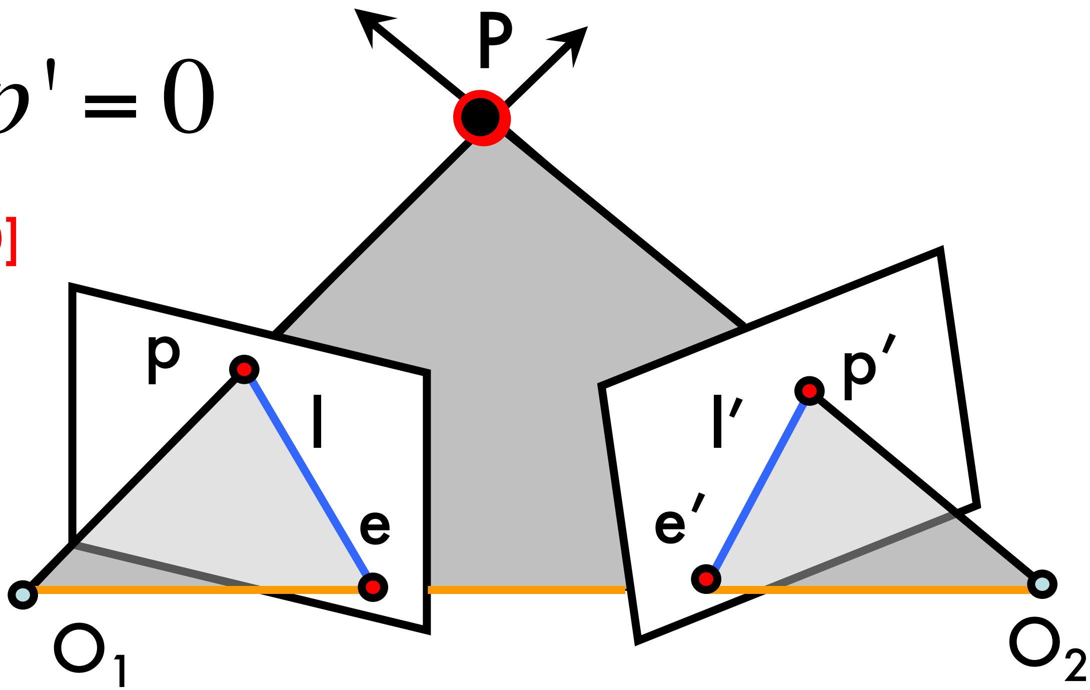
E = Essential matrix

(Longuet-Higgins, 1981)

Epipolar Constraint

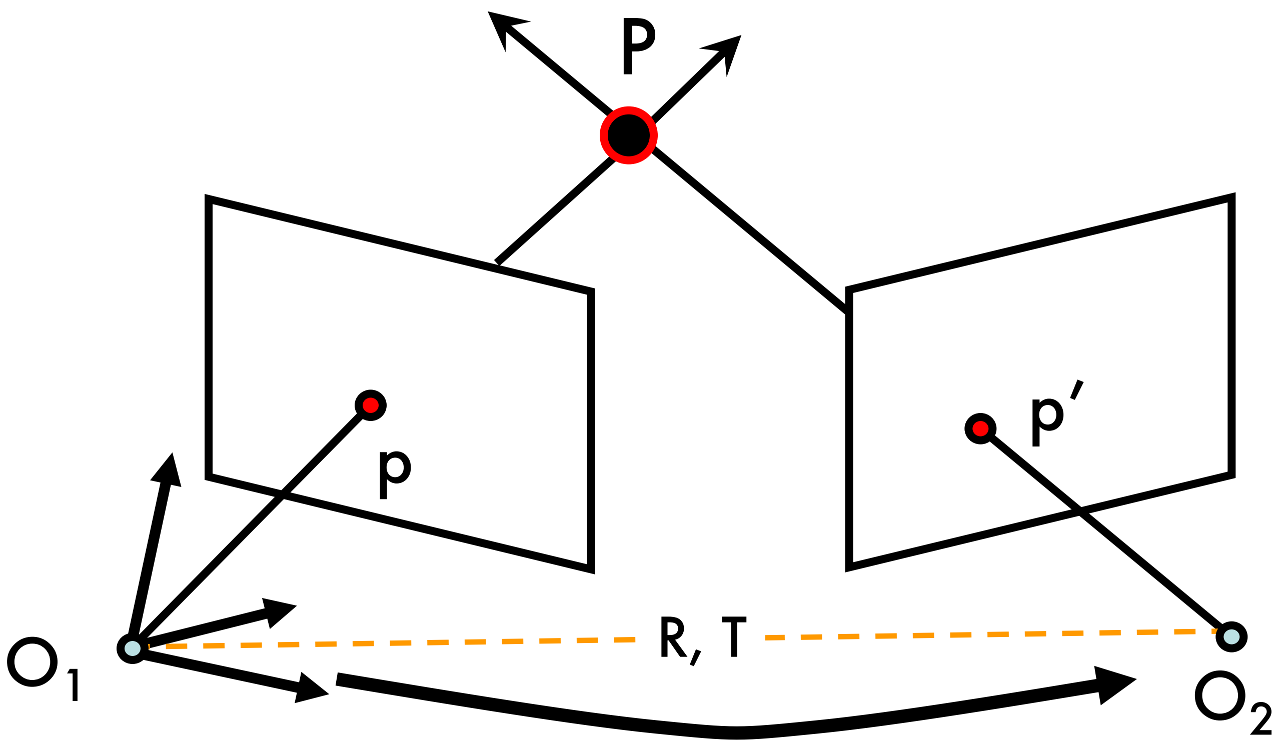
$$p^T \cdot E\ p' = 0$$

[Eq. 10]



- l = E p′ is the epipolar line associated with p′
- l′ = E^T p is the epipolar line associated with p
- E e′ = 0   and E^T e = 0
- E is 3x3 matrix; 5 DOF
- E is singular (rank two)

Epipolar Constraint



$$M = K\begin{bmatrix} I & 0 \end{bmatrix} \qquad M' = K'\begin{bmatrix} R^T & -R^T T \end{bmatrix}$$

$$p_c = K^{-1}\, p$$ [Eq. 11]

$$p'_c = K'^{-1}\, p'$$ [Eq. 12]

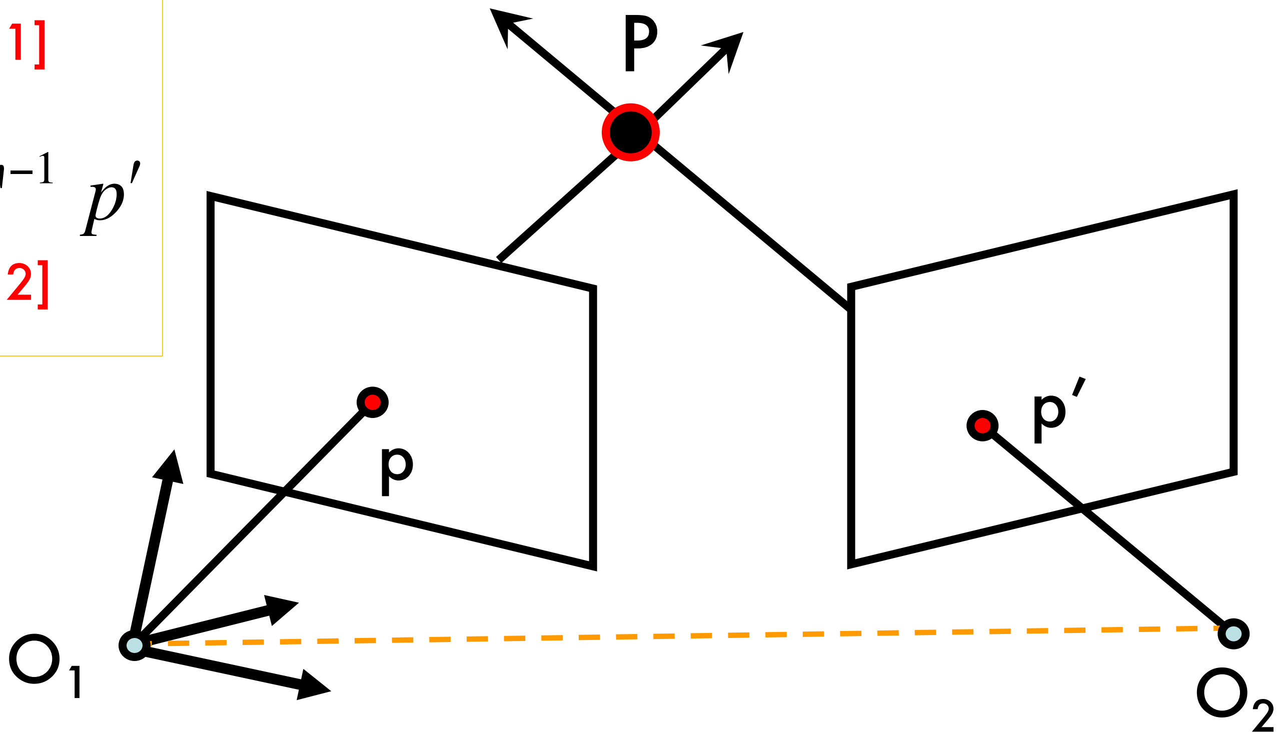
Epipolar Constraint

$$p_c = K^{-1}\ p$$

[Eq. 11]

$$p'_c = K'^{-1}\ p'$$

[Eq. 12]



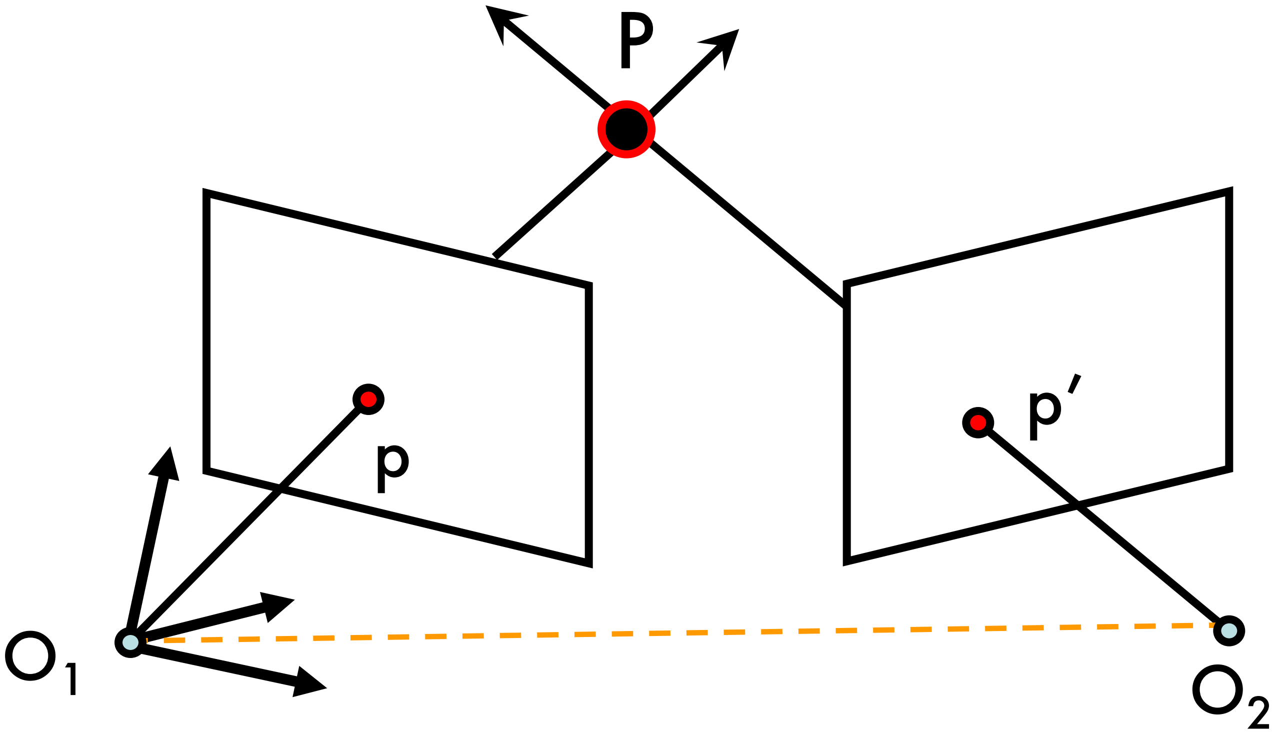
[Eq.9]

$$p_c^T \cdot \left[T_\times\right] \cdot R\ p'_c = 0 \rightarrow (\mathrm{K}^{-1}\ \mathrm{p})^{\mathrm{T}} \cdot \left[\mathrm{T}_\times\right] \cdot \mathrm{R}\ \mathrm{K}'^{-1}\ \mathrm{p}' = 0$$

$$\mathrm{p}^{\mathrm{T}} \boxed{\mathrm{K}^{-\mathrm{T}} \cdot \left[\mathrm{T}_\times\right] \cdot \mathrm{R}\ \mathrm{K}'^{-1}}\ \mathrm{p}' = 0 \rightarrow \mathrm{p}^{\mathrm{T}} \boxed{\mathrm{F}}\, \mathrm{p}' = 0$$

[Eq. 13]

Epipolar Constraint



[Eq. 13] $\boxed{p^T F p' = 0}$

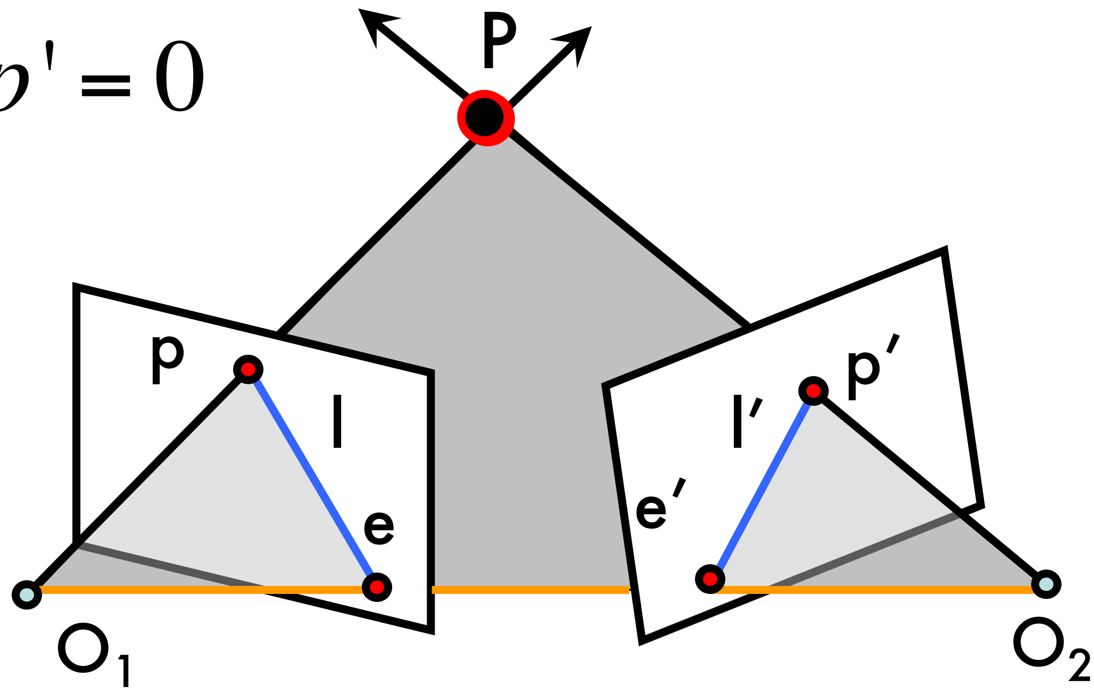
$$F = K^{-T} \cdot [T_\times] \cdot R\, K'^{-1}$$

[Eq. 14]

F = Fundamental Matrix
(Faugeras and Luong, 1992)

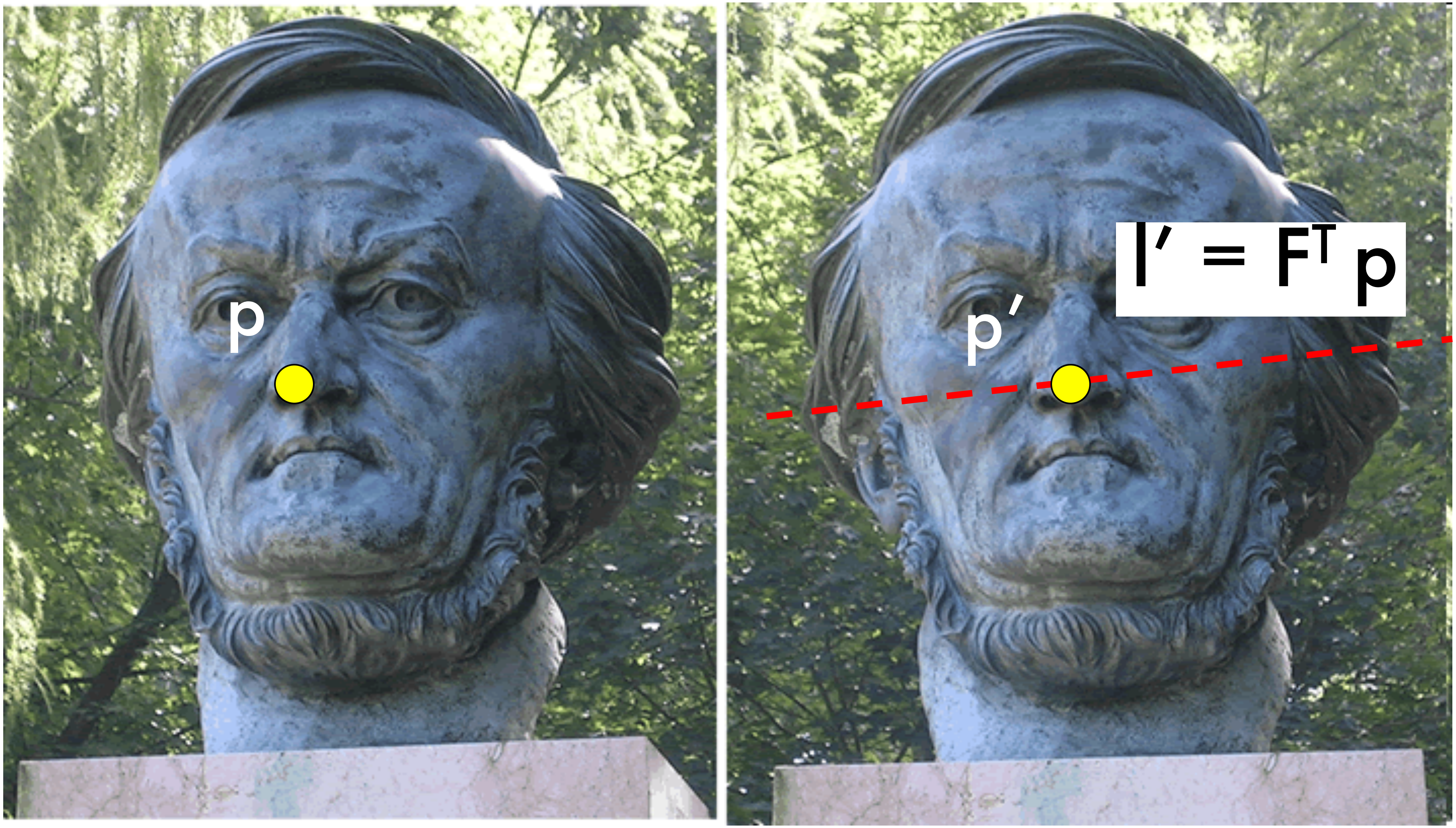
Epipolar Constraint

$$p^T \cdot F\ p' = 0$$



- l = F p′ is the epipolar line associated with p′
- l′= F^T p is the epipolar line associated with p
- F e′ = 0 and F^T e = 0
- F is 3x3 matrix; 7 DOF
- F is singular (rank two)

Why F is useful?



- Suppose F is known
- No additional information about the scene and camera is given
- Given a point on left image, we can compute the corresponding epipolar line in the second imag

Why F is useful?

- F captures information about the epipolar geometry of 2 views + camera parameters

- MORE IMPORTANTLY: F gives constraints on how the scene changes under view point transformation (without reconstructing the scene!)

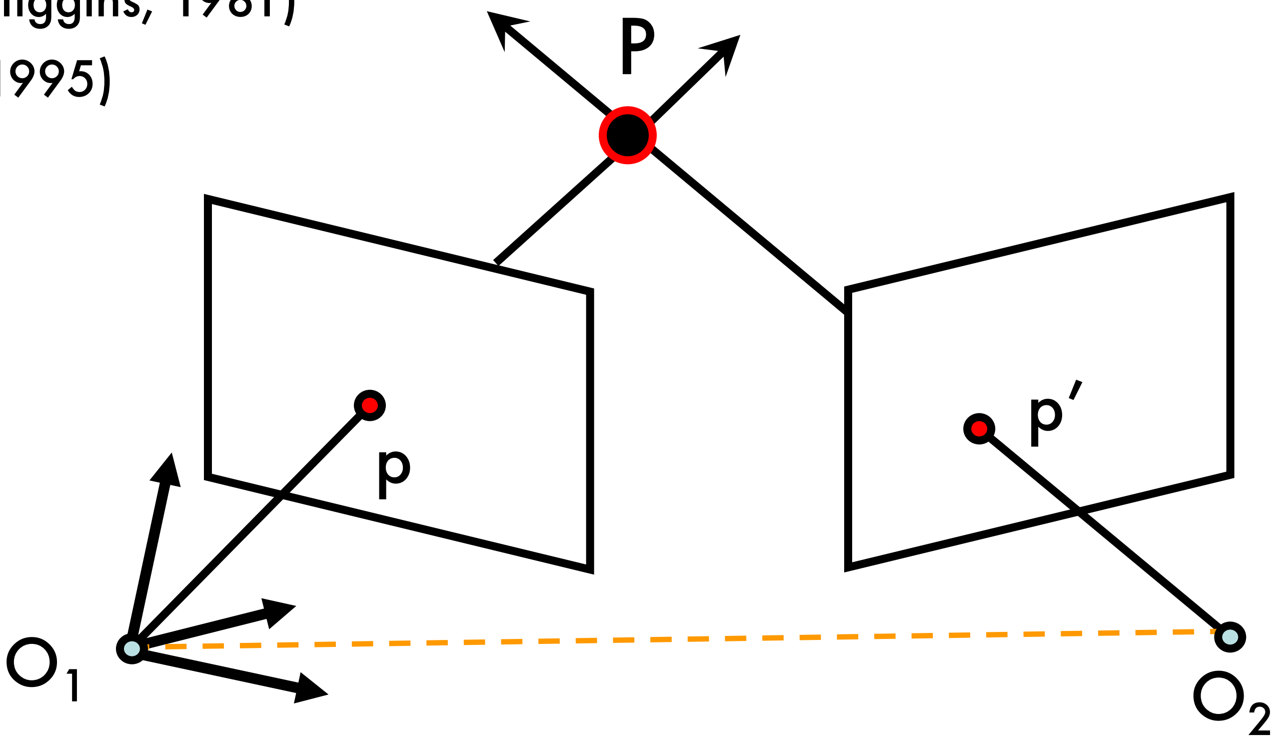
- Powerful tool in:
  - 3D reconstruction
  - Multi-view object/scene matching

Estimating F

The Eight-Point Algorithm

(Longuet-Higgins, 1981)

(Hartley, 1995)



$$p^T F p' = 0$$

Estimating F

[Eq. 13] $$\mathrm{p}^{\mathrm{T}}\ \mathrm{F}\ \mathrm{p}' = 0 \quad \Rightarrow$$

$$p = \begin{bmatrix} u \\ v \\ 1 \end{bmatrix} \qquad p' = \begin{bmatrix} u' \\ v' \\ 1 \end{bmatrix}$$

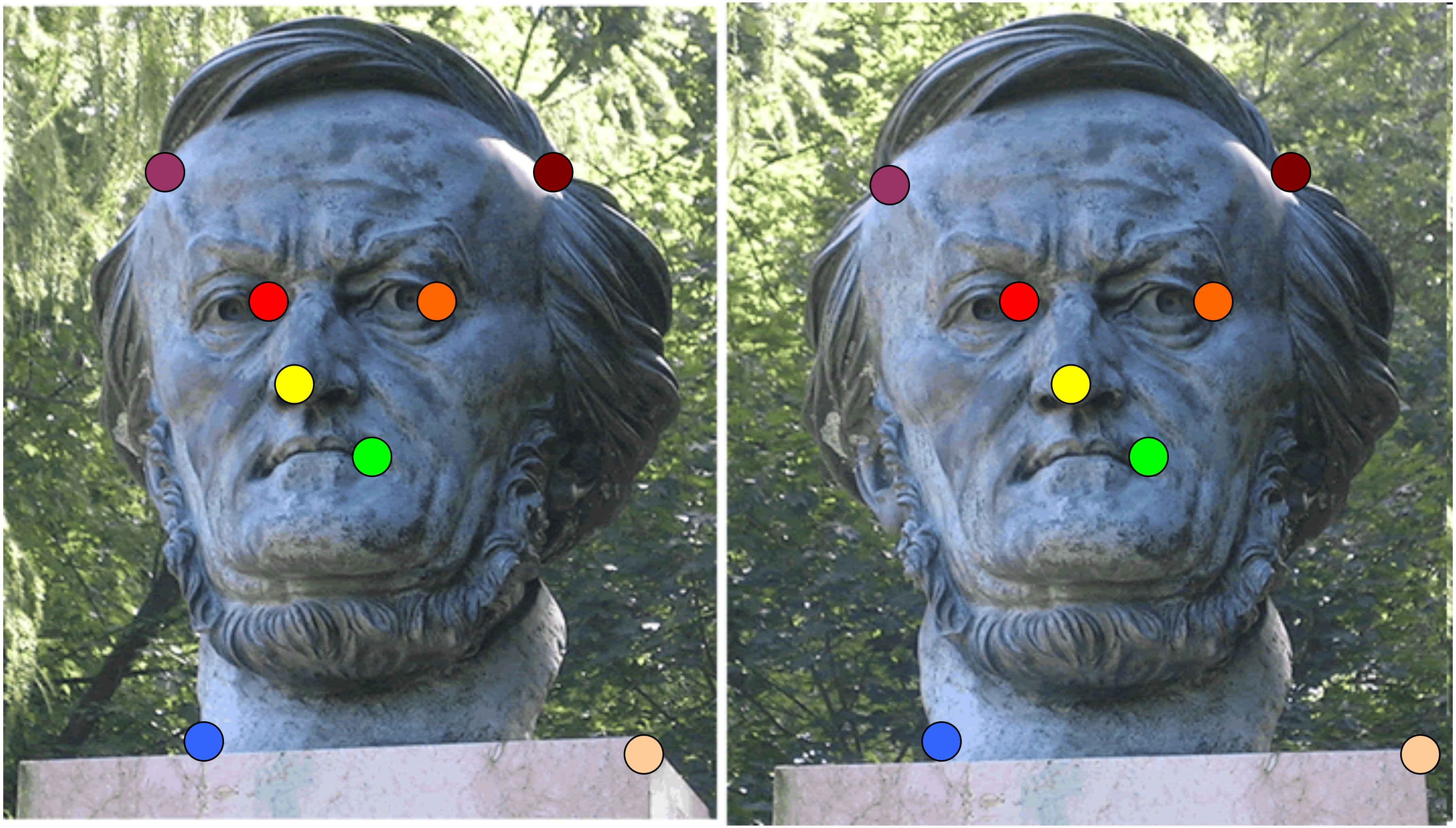
$$(u, v, 1)\begin{pmatrix} F_{11} & F_{12} & F_{13} \\ F_{21} & F_{22} & F_{23} \\ F_{31} & F_{32} & F_{33} \end{pmatrix}\begin{pmatrix} u' \\ v' \\ 1 \end{pmatrix} = 0$$

$$\Rightarrow \quad (uu', uv', u, vu', vv', v, u', v', 1)\begin{pmatrix} F_{11} \\ F_{12} \\ F_{13} \\ F_{21} \\ F_{22} \\ F_{23} \\ F_{31} \\ F_{32} \\ F_{33} \end{pmatrix} = 0$$

[Eq. 14]

Let's take 8 corresponding points

Estimating F



Estimating F

$$\left(u_i u'_i, u_i v'_i, u_i, v_i u'_i, v_i v'_i, v_i, u'_i, v_i', 1\right)\begin{pmatrix} F_{11} \\ F_{12} \\ F_{13} \\ F_{21} \\ F_{22} \\ F_{23} \\ F_{31} \\ F_{32} \\ F_{33} \end{pmatrix} = 0 \qquad \text{[Eq. 14]}$$

Estimating F

$$\mathbf{W}\begin{pmatrix} u_1u'_1 & u_1v'_1 & u_1 & v_1u'_1 & v_1v'_1 & v_1 & u'_1 & v'_1 & 1 \\ u_2u'_2 & u_2v'_2 & u_2 & v_2u'_2 & v_2v'_2 & v_2 & u'_2 & v'_2 & 1 \\ u_3u'_3 & u_3v'_3 & u_3 & v_3u'_3 & v_3v'_3 & v_3 & u'_3 & v'_3 & 1 \\ u_4u'_4 & u_4v'_4 & u_4 & v_4u'_4 & v_4v'_4 & v_4 & u'_4 & v'_4 & 1 \\ u_5u'_5 & u_5v'_5 & u_5 & v_5u'_5 & v_5v'_5 & v_5 & u'_5 & v'_5 & 1 \\ u_6u'_6 & u_6v'_6 & u_6 & v_6u'_6 & v_6v'_6 & v_6 & u'_6 & v'_6 & 1 \\ u_7u'_7 & u_7v'_7 & u_7 & v_7u'_7 & v_7v'_7 & v_7 & u'_7 & v'_7 & 1 \\ u_8u'_8 & u_8v'_8 & u_8 & v_8u'_8 & v_8v'_8 & v_8 & u'_8 & v'_8 & 1 \end{pmatrix} \underbrace{\begin{pmatrix} F_{11} \\ F_{12} \\ F_{13} \\ F_{21} \\ F_{22} \\ F_{23} \\ F_{31} \\ F_{32} \\ F_{33} \end{pmatrix}}_{\mathbf{f}} = 0$$

[Eqs. 15]

- Homogeneous system $\mathbf{W}\,\mathbf{f} = 0$
- Rank 8 $\longrightarrow$ A non-zero solution exists (unique)
- If N>8 $\longrightarrow$ Lsq. solution by SVD! $\longrightarrow$ $\hat{\mathrm{F}}$

$$\|\mathbf{f}\| = 1$$

$\hat{F}$ satisfies: $p^T \hat{F} p' = 0$

and estimated $\hat{F}$ may have full rank ($\det(\hat{F}) \neq 0$)

But remember: fundamental matrix is Rank2

Find F that minimizes $\|F - \hat{F}\| = 0$

Frobenius norm (*)

Subject to det(F)=0

SVD (again!) can be used to solve this problem

(*) Sq. root of the sum of squares of all entries

Find F that minimizes $\left\| \mathrm{F} - \hat{\mathrm{F}} \right\| = 0$

Frobenius norm (*)

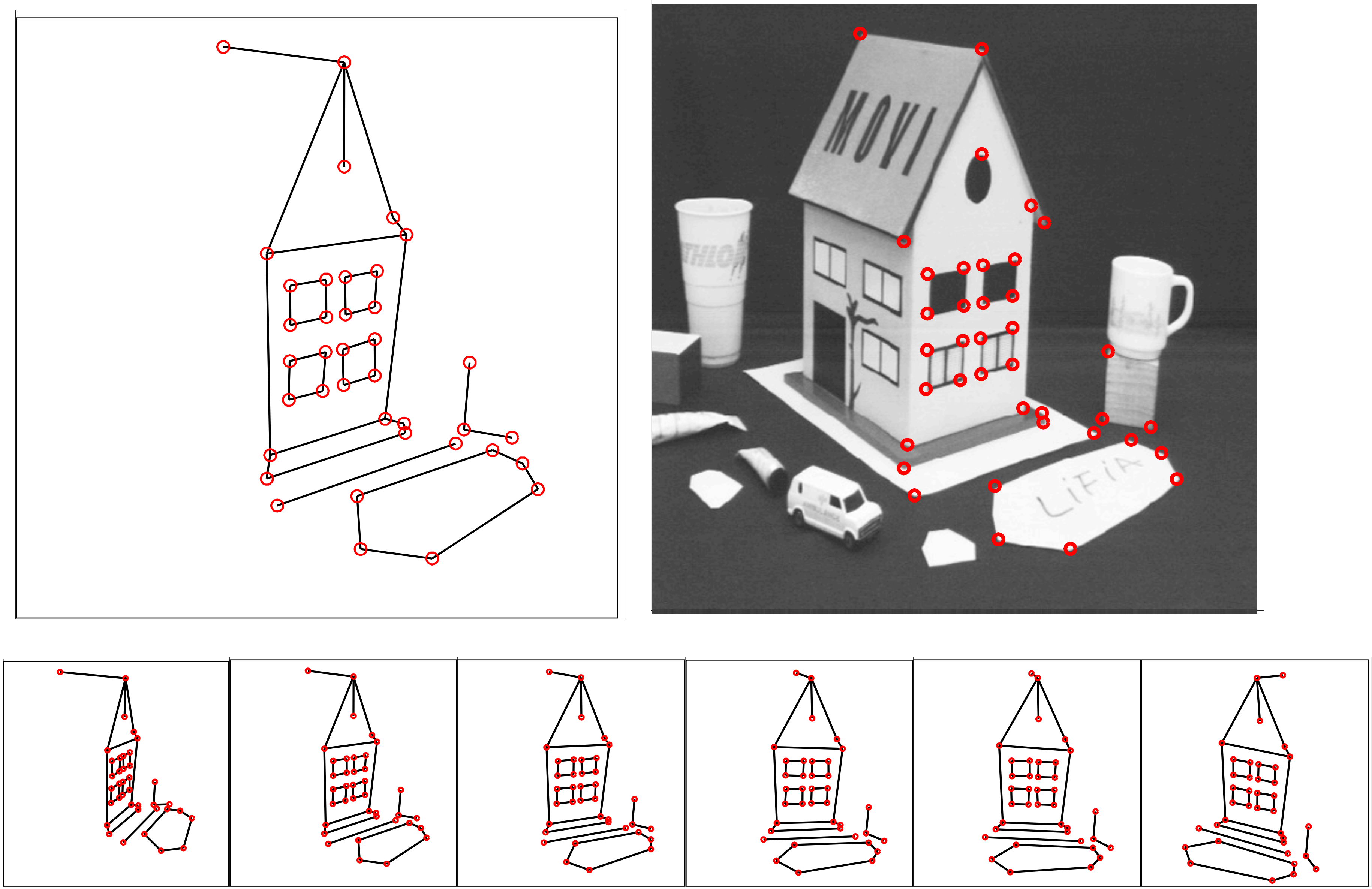
Subject to det(F)=0

$$F = U \begin{bmatrix} s_1 & 0 & 0 \\ 0 & s_2 & 0 \\ 0 & 0 & 0 \end{bmatrix} V^T$$

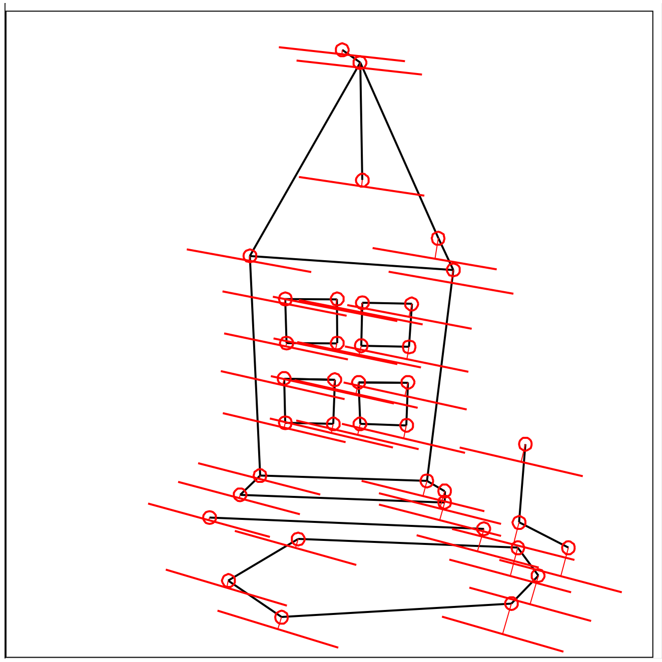
Where:

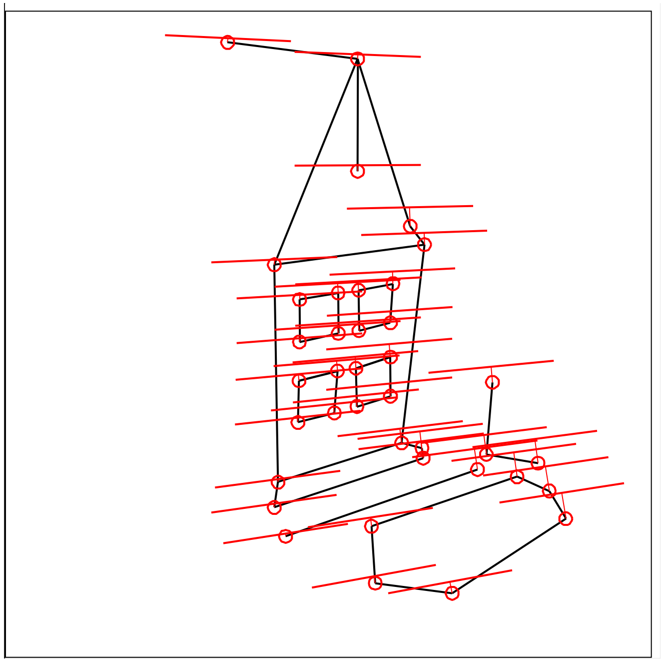
$$U \begin{bmatrix} s_1 & 0 & 0 \\ 0 & s_2 & 0 \\ 0 & 0 & s_3 \end{bmatrix} V^T = SVD(\hat{F})$$

[HZ] pag 281, chapter 11, “Computation of F”



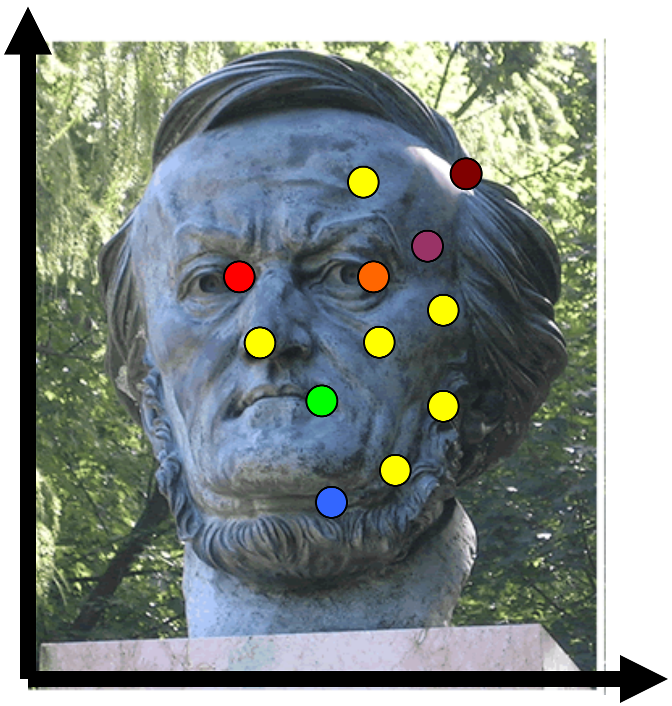
Data courtesy of R. Mohr and B. Boufama.





Mean errors:
10.0pixel
9.1pixel

Problems with the 8-Point Algorithm



$$\mathbf{W}\,\mathbf{f} = 0,\quad \|\mathbf{f}\| = 1 \xrightarrow{\text{Lsq solution by SVD}} F$$

- Recall the structure of W:
  - do we see any potential (numerical) issue?

Problems with the 8-Point Algorithm

$$\mathbf{Wf} = 0$$

$$\begin{pmatrix} u_1u'_1 & u_1v'_1 & u_1 & v_1u'_1 & v_1v'_1 & v_1 & u'_1 & v'_1 & 1 \\ u_2u'_2 & u_2v'_2 & u_2 & v_2u'_2 & v_2v'_2 & v_2 & u'_2 & v'_2 & 1 \\ u_3u'_3 & u_3v'_3 & u_3 & v_3u'_3 & v_3v'_3 & v_3 & u'_3 & v'_3 & 1 \\ u_4u'_4 & u_4v'_4 & u_4 & v_4u'_4 & v_4v'_4 & v_4 & u'_4 & v'_4 & 1 \\ u_5u'_5 & u_5v'_5 & u_5 & v_5u'_5 & v_5v'_5 & v_5 & u'_5 & v'_5 & 1 \\ u_6u'_6 & u_6v'_6 & u_6 & v_6u'_6 & v_6v'_6 & v_6 & u'_6 & v'_6 & 1 \\ u_7u'_7 & u_7v'_7 & u_7 & v_7u'_7 & v_7v'_7 & v_7 & u'_7 & v'_7 & 1 \\ u_8u'_8 & u_8v'_8 & u_8 & v_8u'_8 & v_8v'_8 & v_8 & u'_8 & v'_8 & 1 \end{pmatrix} \begin{pmatrix} F_{11} \\ F_{12} \\ F_{13} \\ F_{21} \\ F_{22} \\ F_{23} \\ F_{31} \\ F_{32} \\ F_{33} \end{pmatrix} = 0$$

- Highly un-balanced (not well conditioned)
- Values of W must have similar magnitude
- This creates problems during the SVD decomposition

HZ pag 108

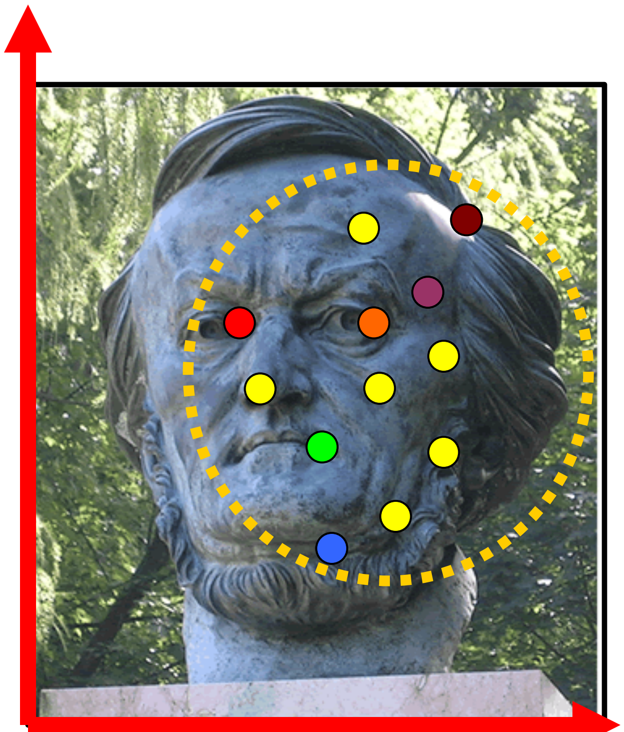
Normalization

IDEA: Transform image coordinates such that the matrix **W** becomes better conditioned (**pre-conditioning**)

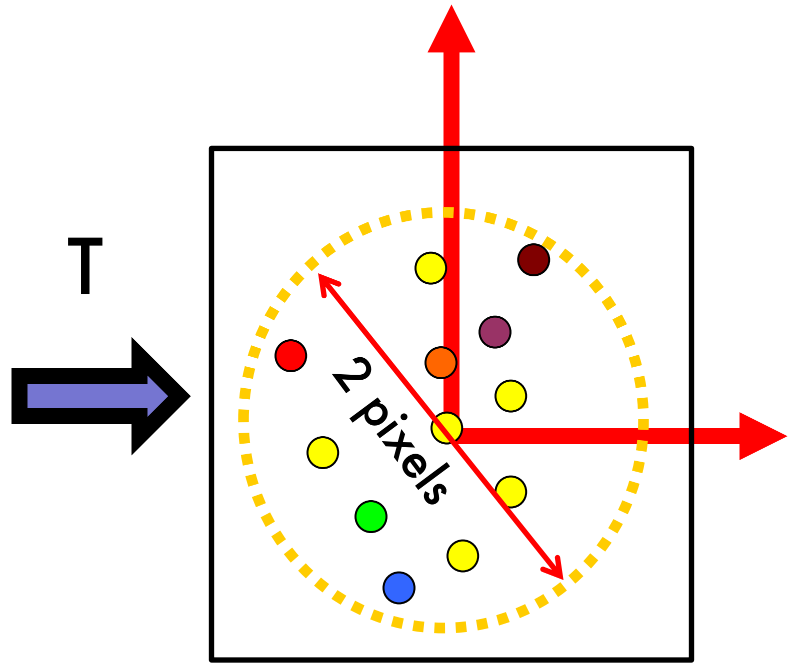
For each image, apply a transformation T (translation and scaling) acting on image coordinates such that:

- Origin = centroid of image points
- Mean square distance of the image points from origin is ~2 pixels

Example of normalization



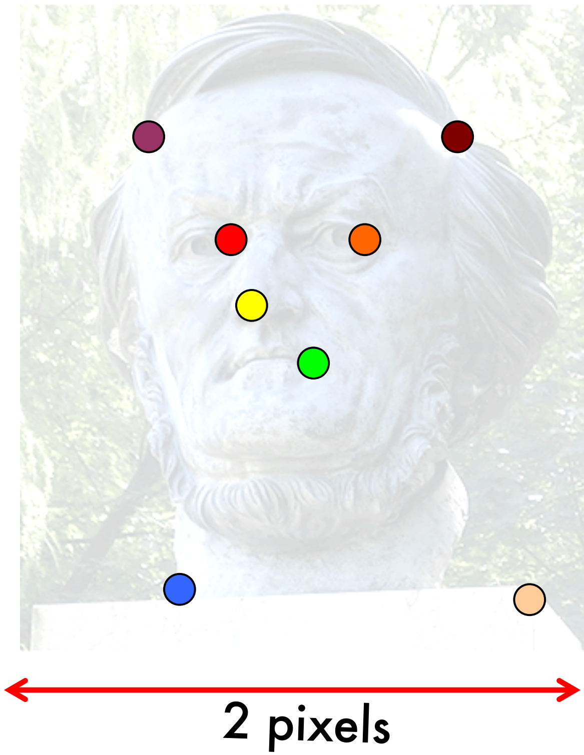
Coordinate system of the image before applying T

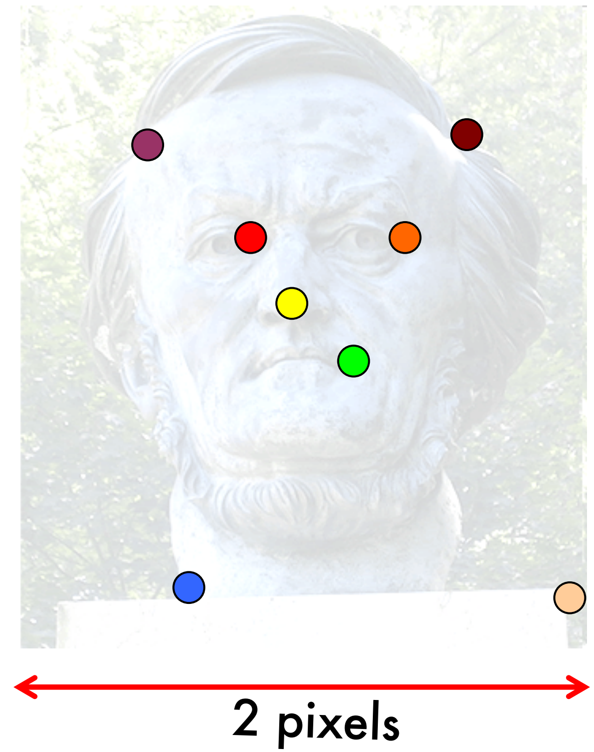


Coordinate system of the image after applying T

- Origin = centroid of image points
- Mean square distance of the image points from origin is ~2 pixels

Normalization





$$q_i = T \; p_i$$

$$q_i' = T' \; p_i'$$

The Normalized Eight-Point Algorithm

0. Compute T and T′ for image 1 and 2, respectively

1. Normalize coordinates in images 1 and 2:

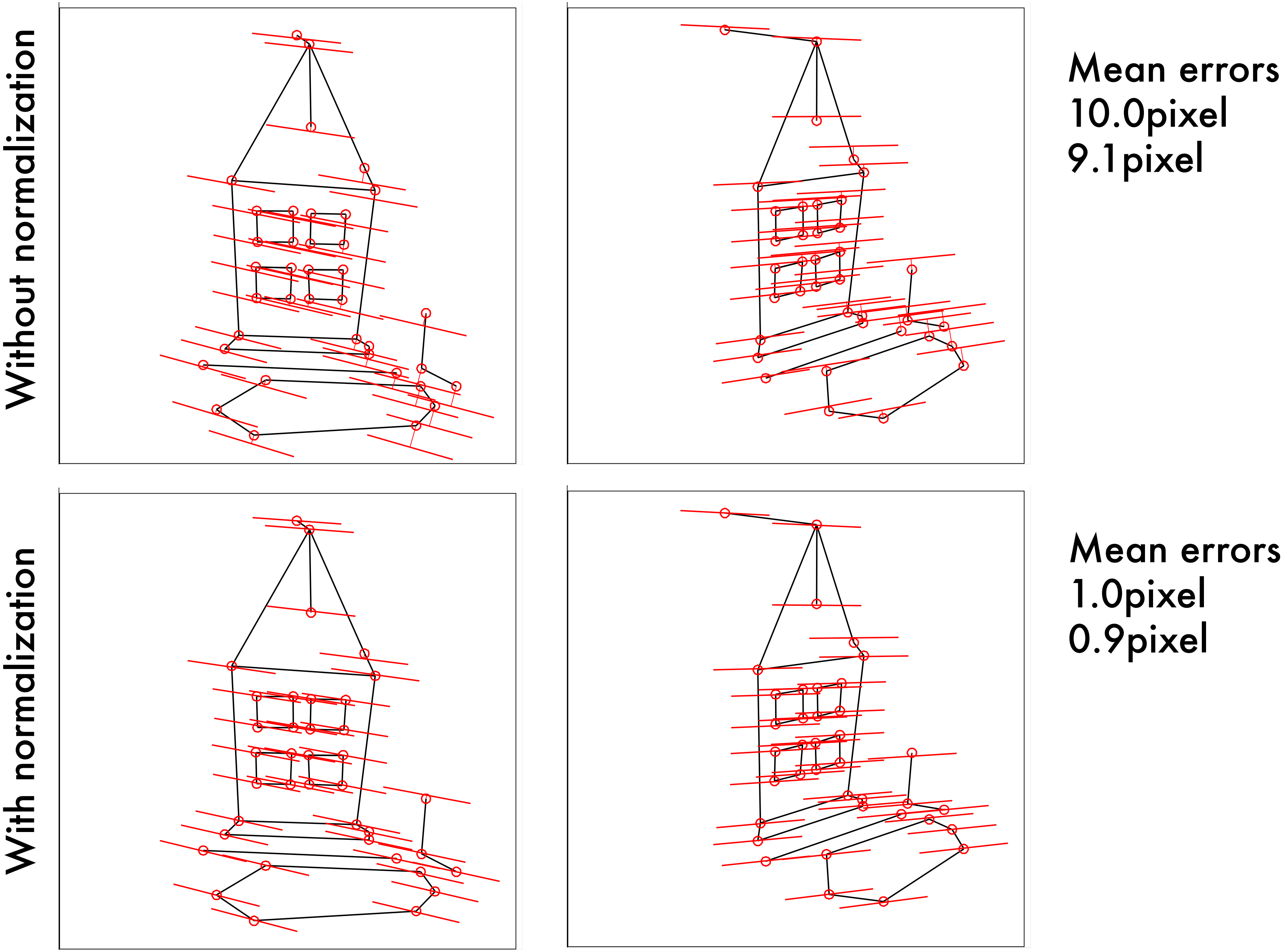
$$q_i = T\ p_i \qquad q_i' = T'\ p_i'$$

2. Use the eight-point algorithm to compute $\hat{F}_q$ from the corresponding points q_i and q_i' .

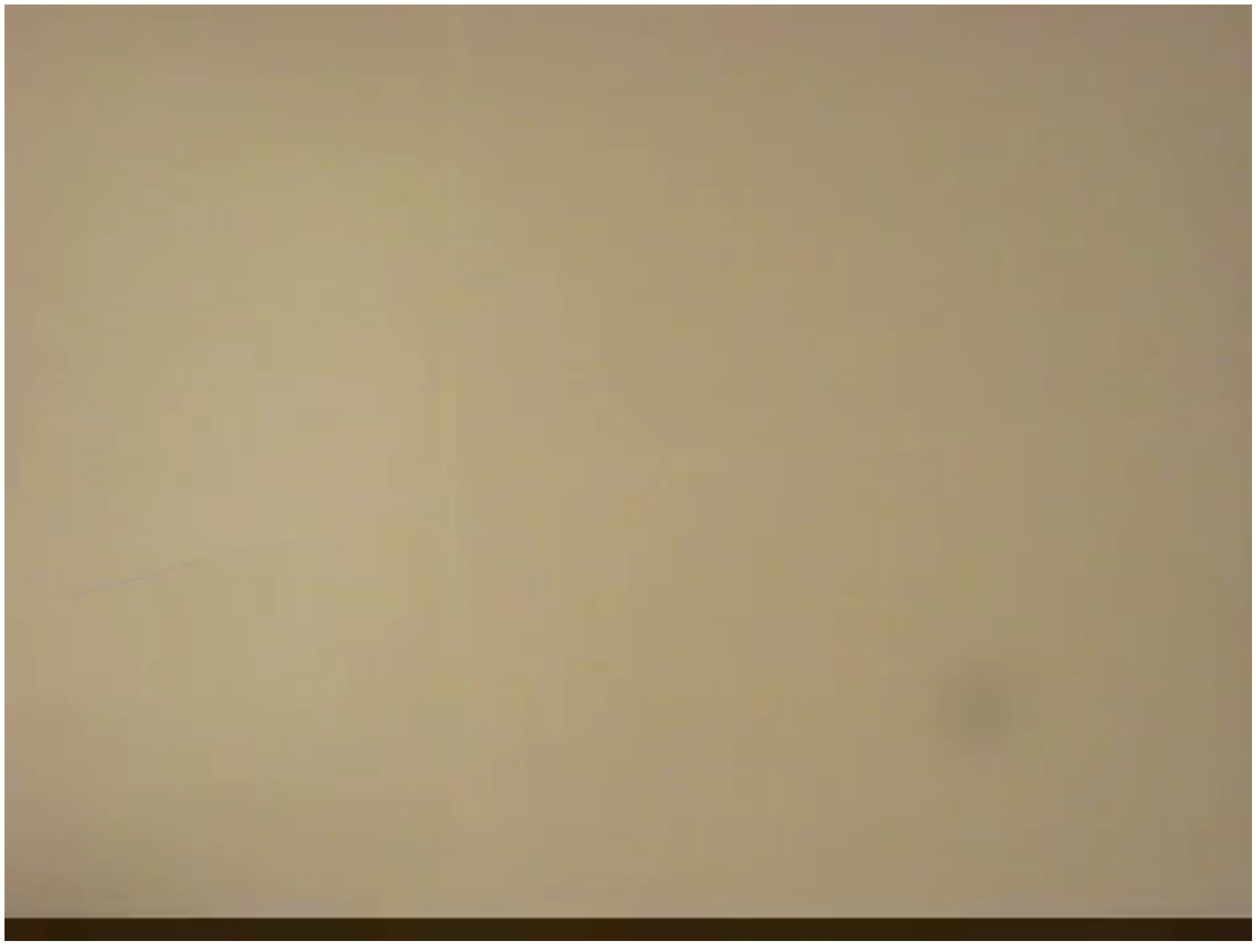
1. Enforce the rank-2 constraint: $\rightarrow \mathrm{F_q}$ such that:

$$\begin{cases} \mathrm{q^T\ F_q\ q' = 0} \\ \mathrm{det(\ F_q\ ) = 0} \end{cases}$$

2. De-normalize F_q: $F = T^T F_q\ T'$

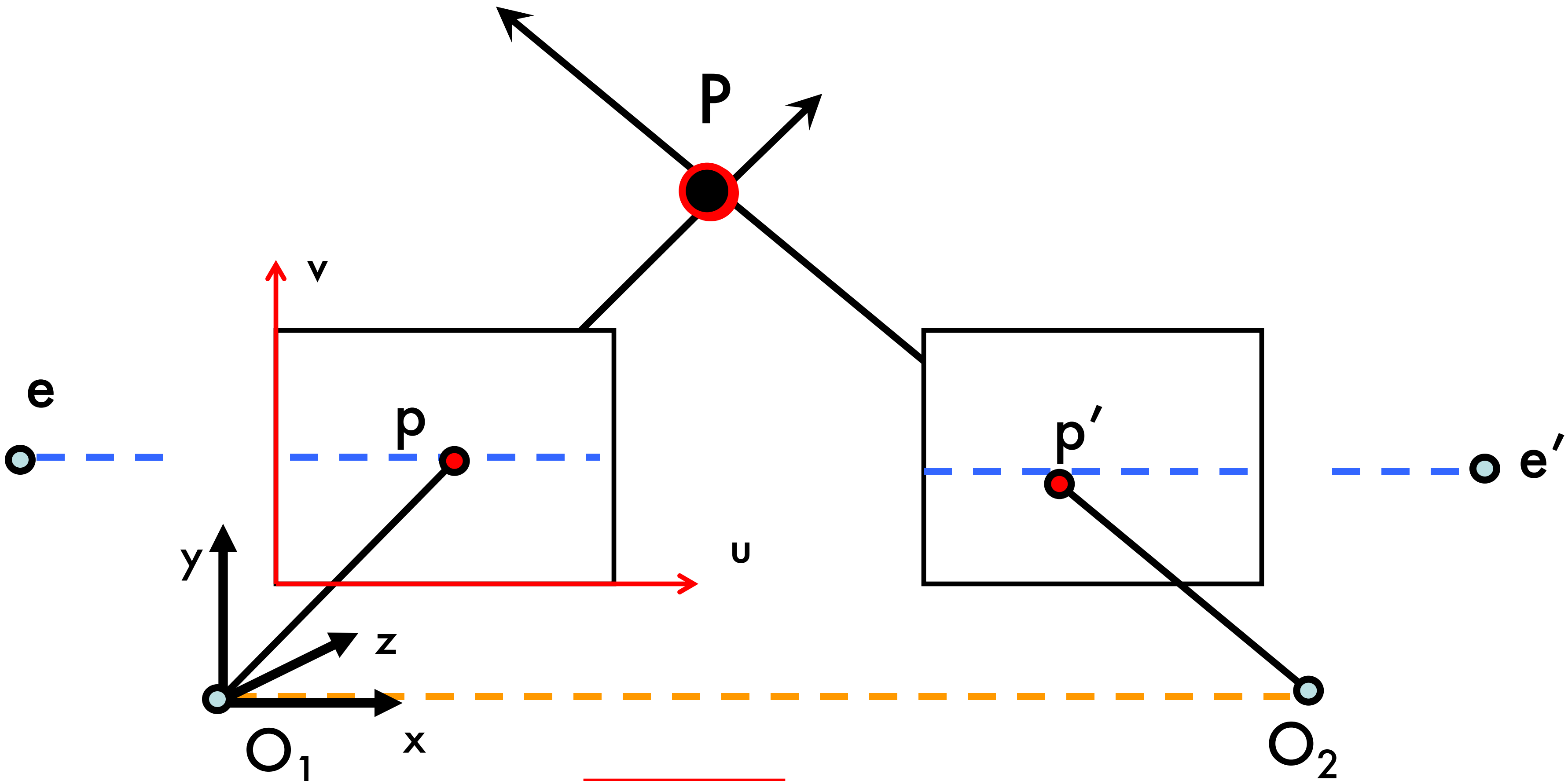

Without normalization
Mean errors:
10.0pixel
9.1pixel
With normalization
Mean errors:
1.0pixel
0.9pixel

The Fundamental Matrix Song



Next lecture:
Stereo systems

Example: Parallel image planes



$K_1 = K_2$ = known

x parallel to O_1O_2

$\boxed{E = ?}$

Hint :

$\boldsymbol{R} = I \qquad \boldsymbol{T} = (T, 0, 0)$

Essential matrix for parallel images

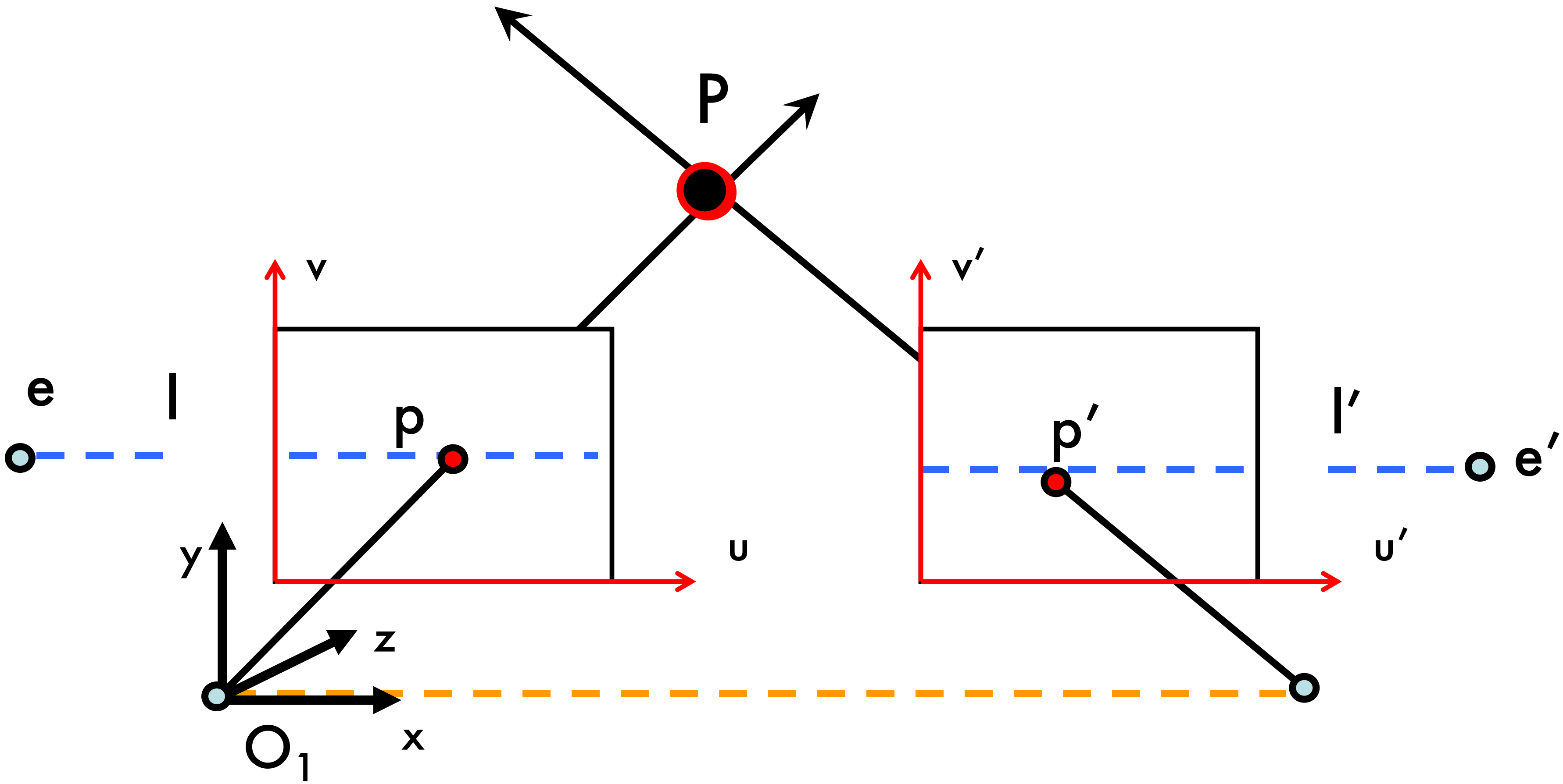
$$\mathbf{E} = [\mathbf{T}_{\times}] \cdot \mathbf{R}$$

$$\mathbf{E} = \begin{bmatrix} 0 & -T_z & T_y \\ T_z & 0 & -T_x \\ -T_y & T_x & 0 \end{bmatrix} \mathbf{R} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -T \\ 0 & T & 0 \end{bmatrix}$$

[Eq. 20]

$$\mathbf{T} = [\ \mathrm{T}\ \ 0\ \ 0\ ]$$
$$\mathbf{R} = \mathbf{I}$$

Example: Parallel image planes

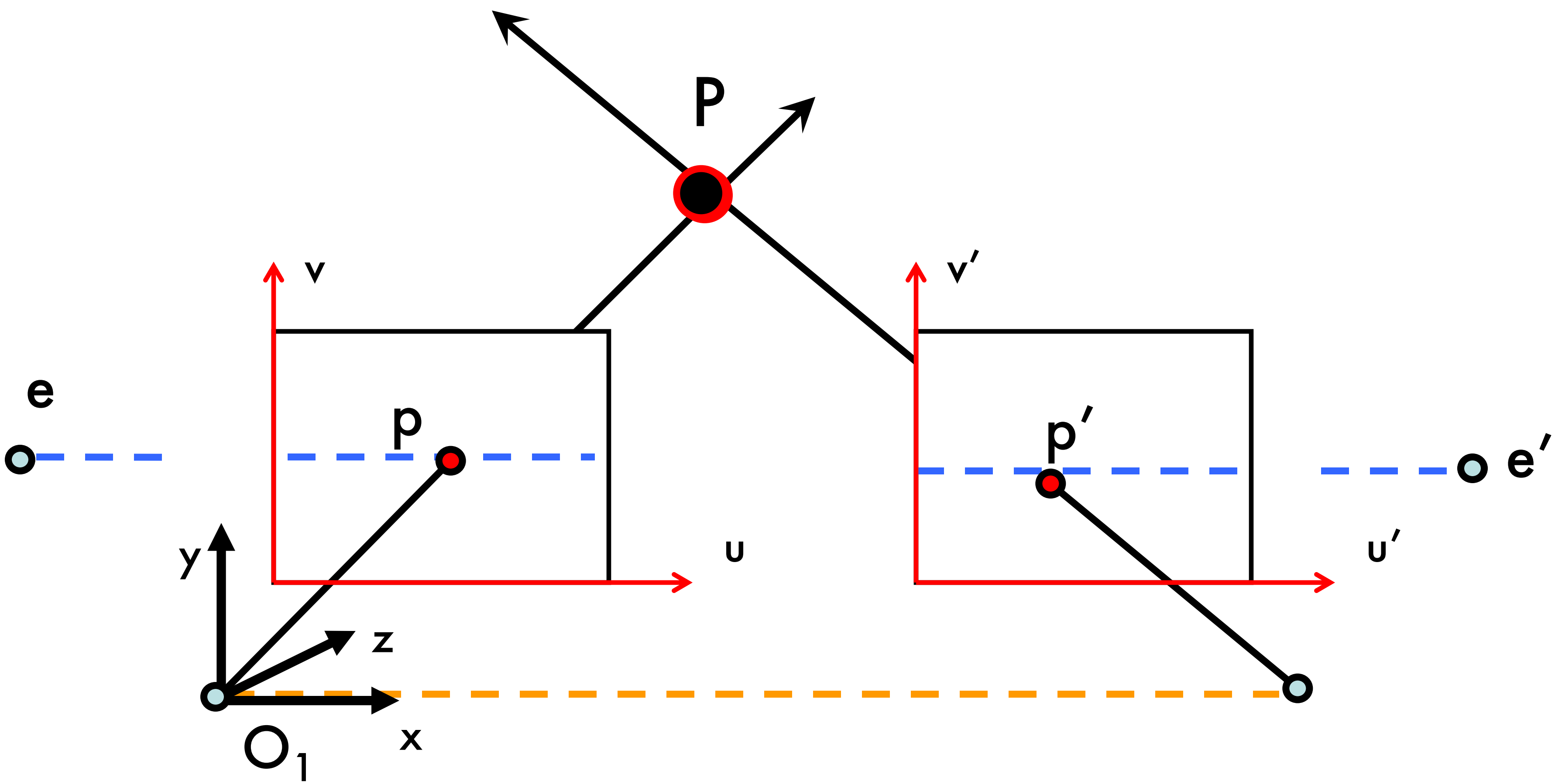


What are the directions of epipolar lines?

$$l = \mathrm{E}\, p' = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -T \\ 0 & T & 0 \end{bmatrix} \begin{bmatrix} u' \\ v' \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -T \\ T v' \end{bmatrix}$$

horizontal!

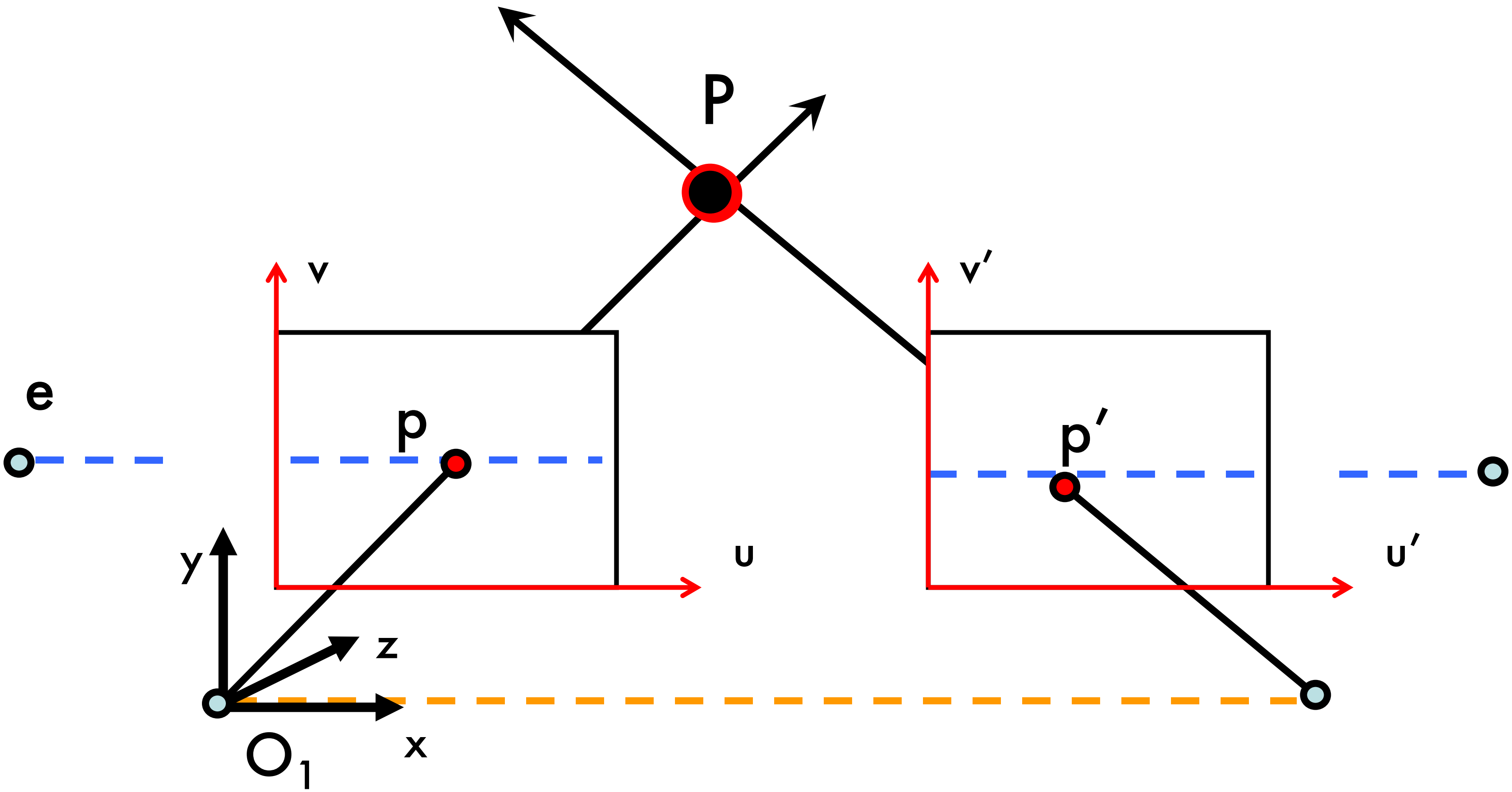
Example: Parallel image planes



How are p and p′ related?

$$p^T \cdot E \, p' = 0$$

Example: Parallel image planes

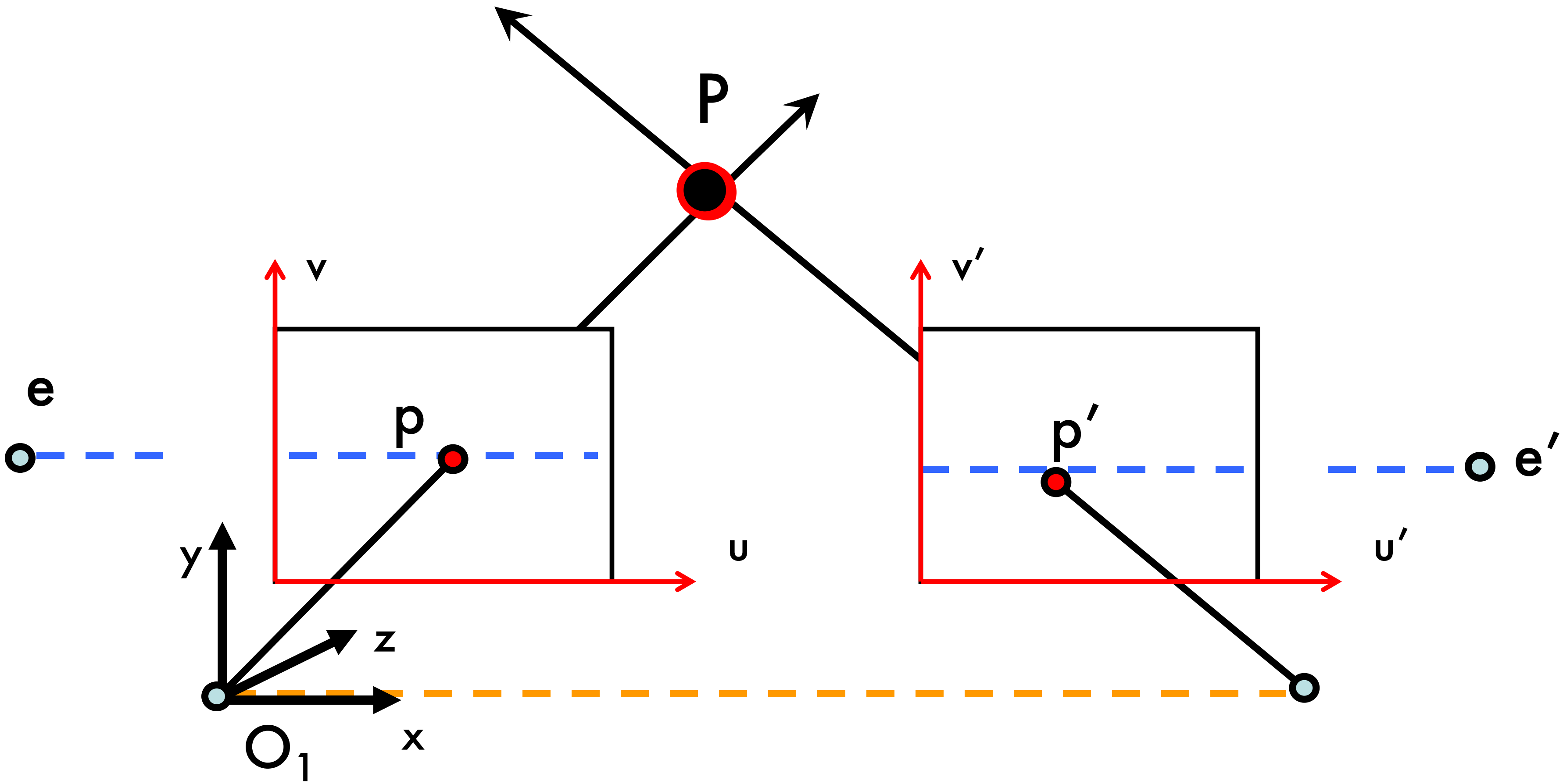


How are p and p′ related?

$$\Rightarrow \begin{pmatrix} u & v & 1 \end{pmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -T \\ 0 & T & 0 \end{bmatrix} \begin{pmatrix} u' \\ v' \\ 1 \end{pmatrix} = 0 \Rightarrow \begin{pmatrix} u & v & 1 \end{pmatrix} \begin{pmatrix} 0 \\ -T \\ Tv' \end{pmatrix} = 0 \Rightarrow Tv = Tv'$$

$$\Rightarrow v = v'$$

Example: Parallel image planes



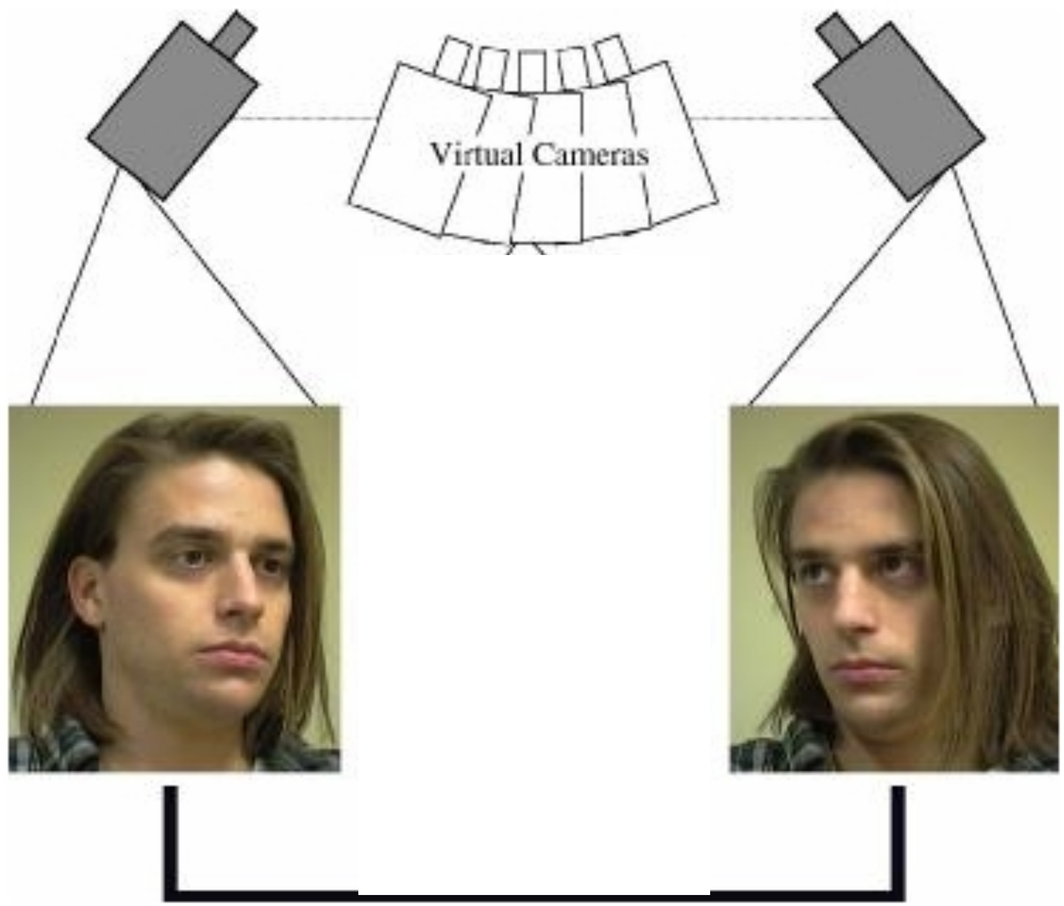
Rectification: making two images "parallel"

Why it is useful?

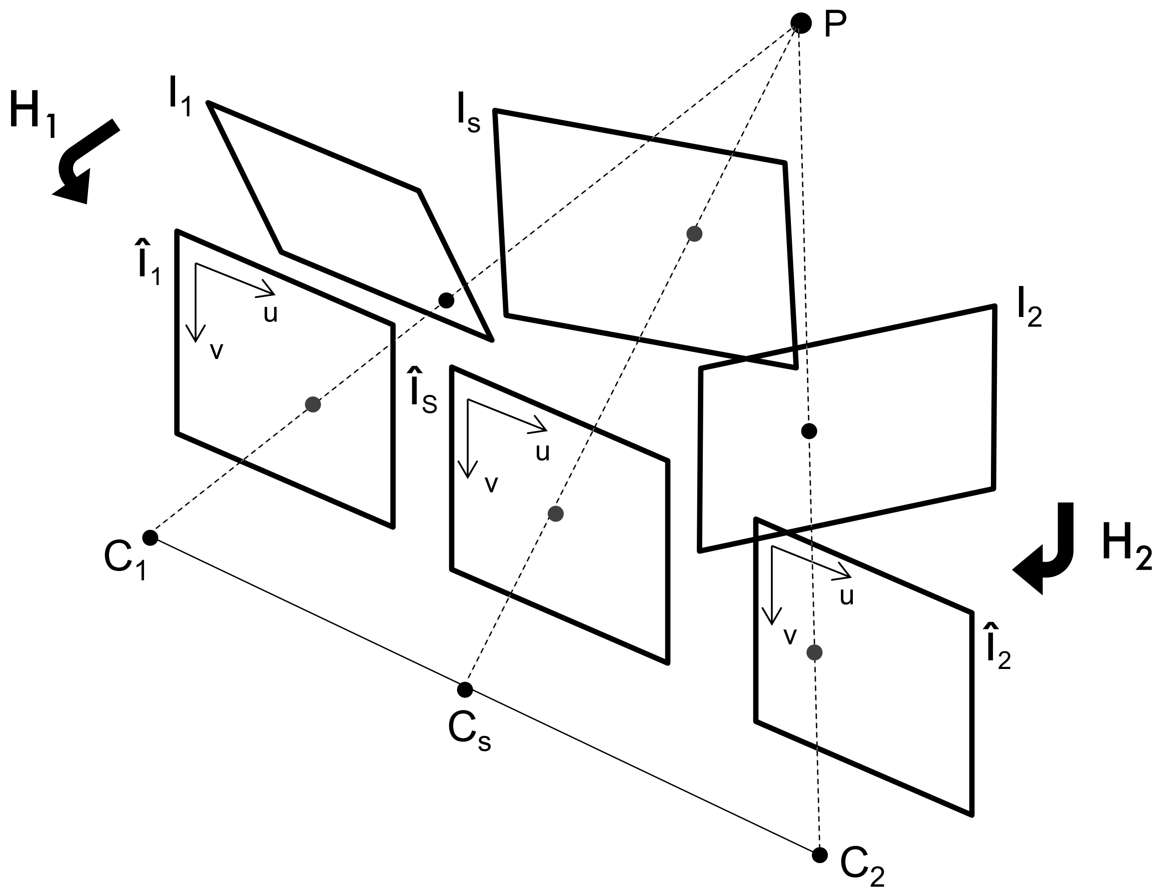
- Epipolar constraint → v = v′
- New views can be synthesized by linear interpolation

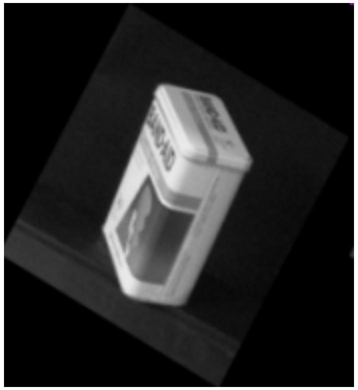
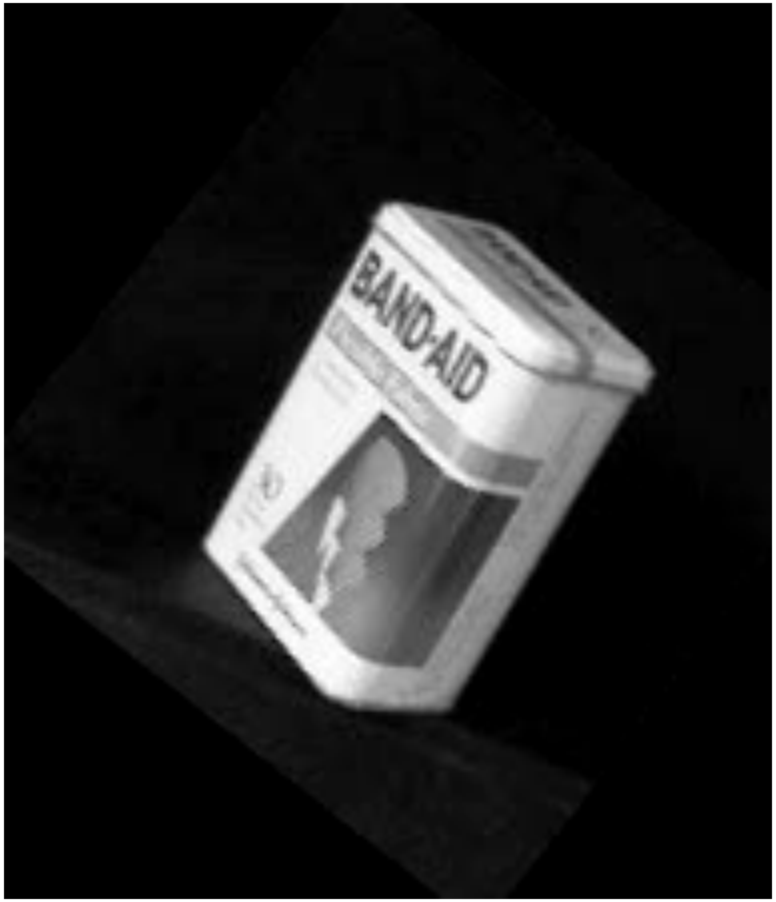
Application: view morphing

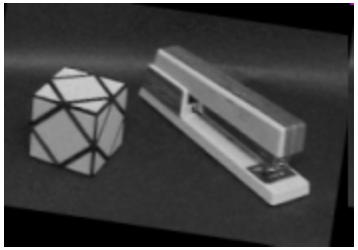
S. M. Seitz and C. R. Dyer, *Proc. SIGGRAPH 96*, 1996, 21-30

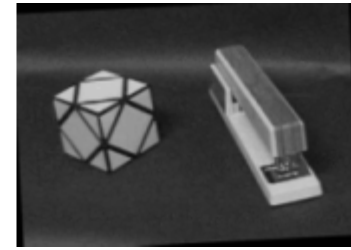


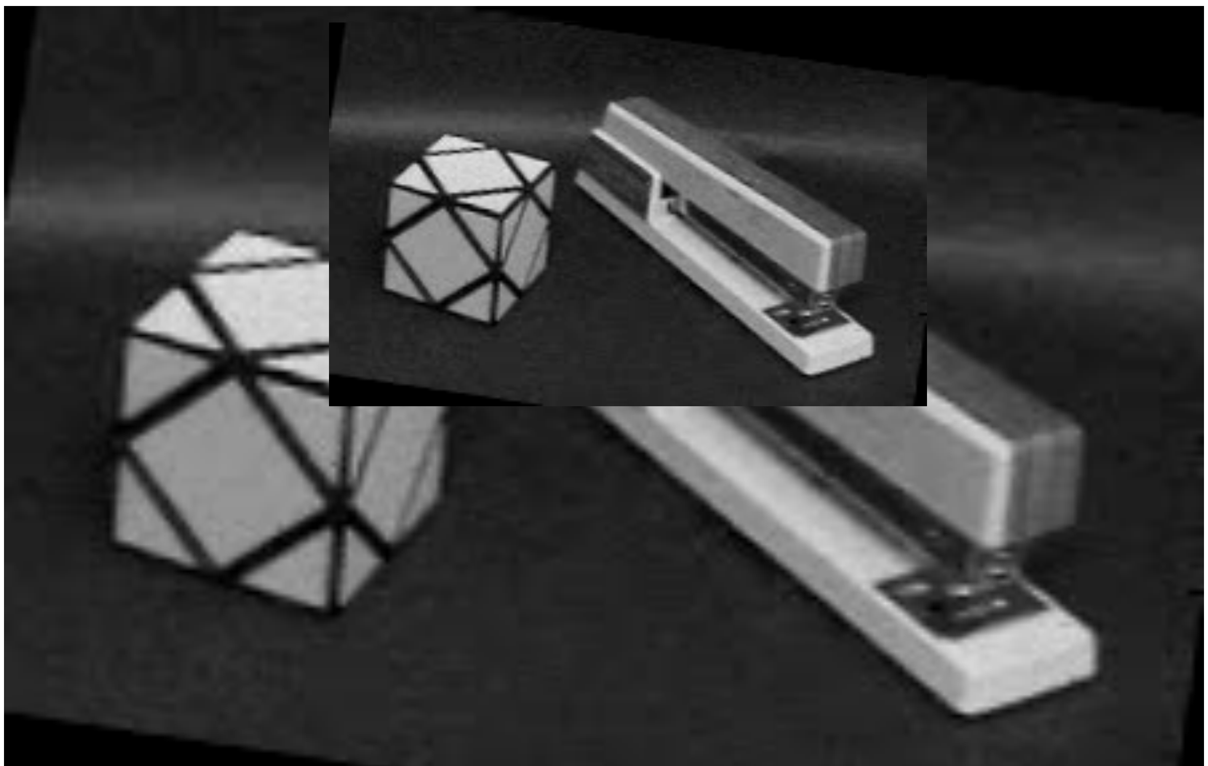
Rectification

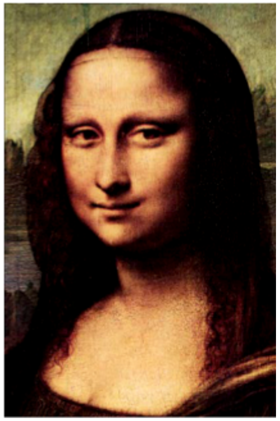


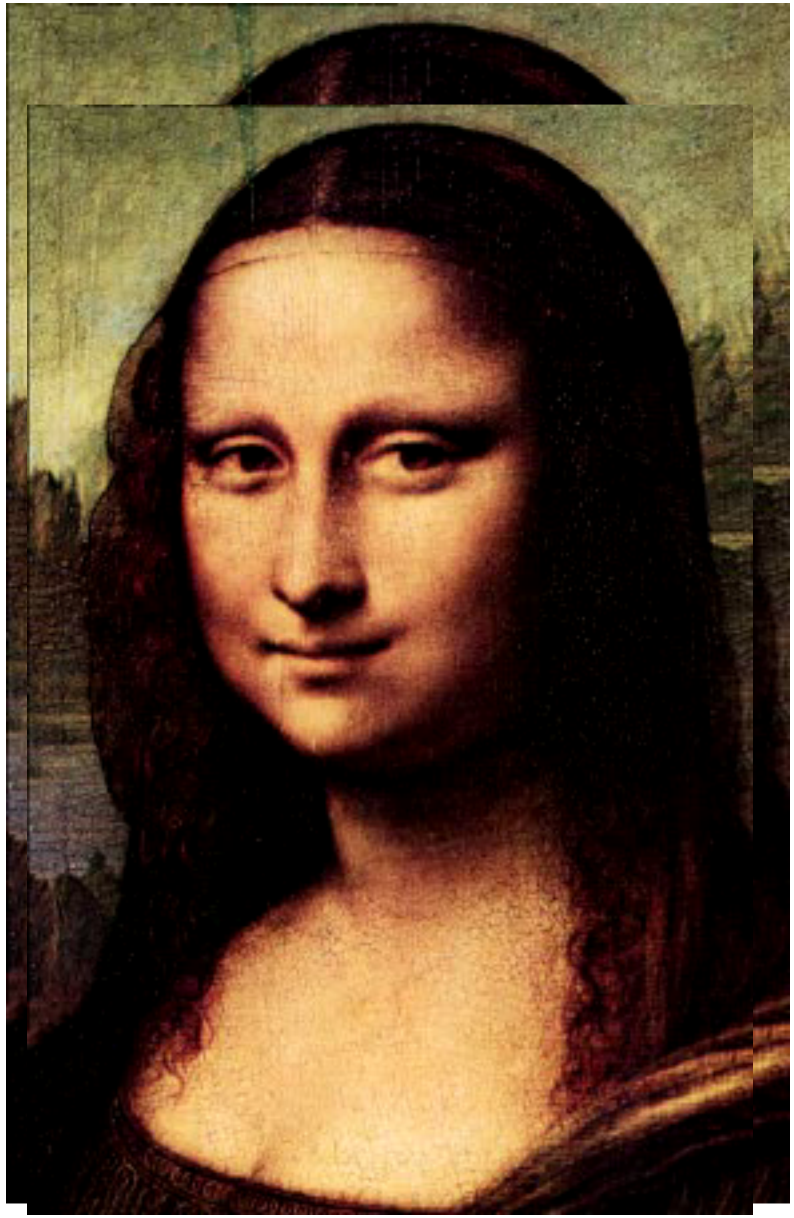

BAND-AID

BAND-AID

BAND-AID

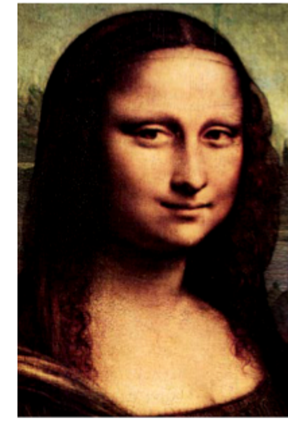




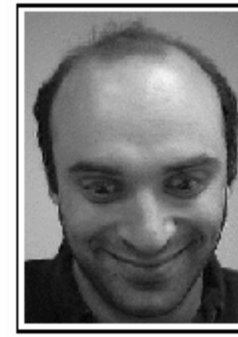






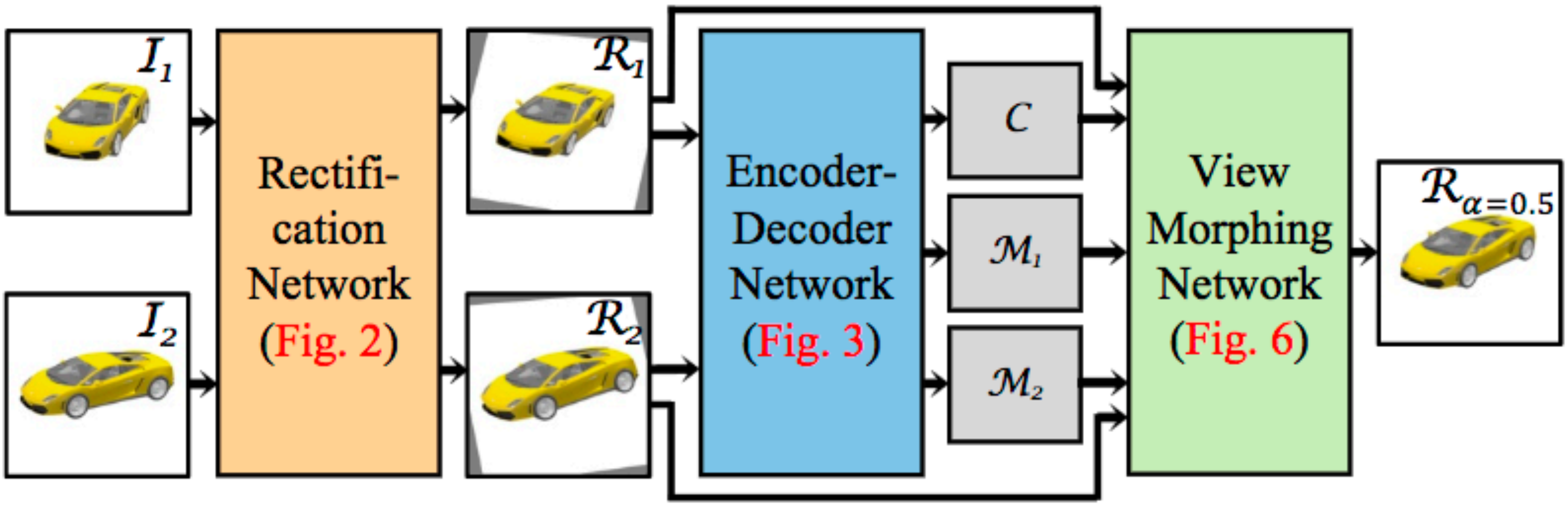


From its reflection!



Deep view morphing

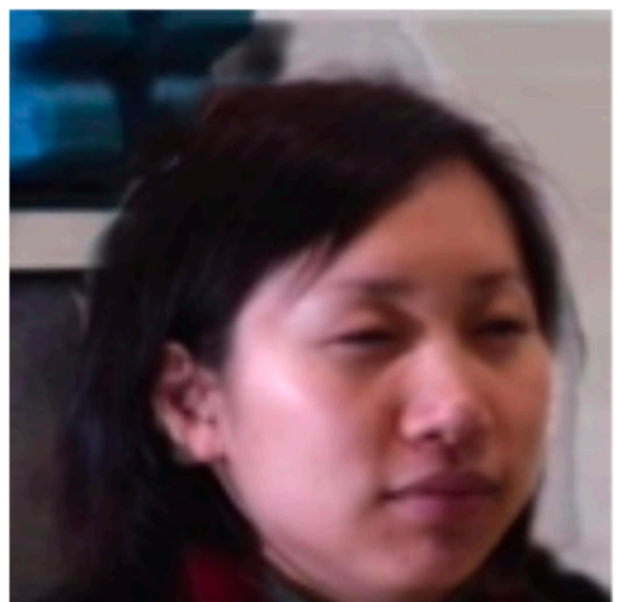
D. Ji, J. Kwon, M. McFarland, S. Savarese, CVPR 2017



Deep view morphing

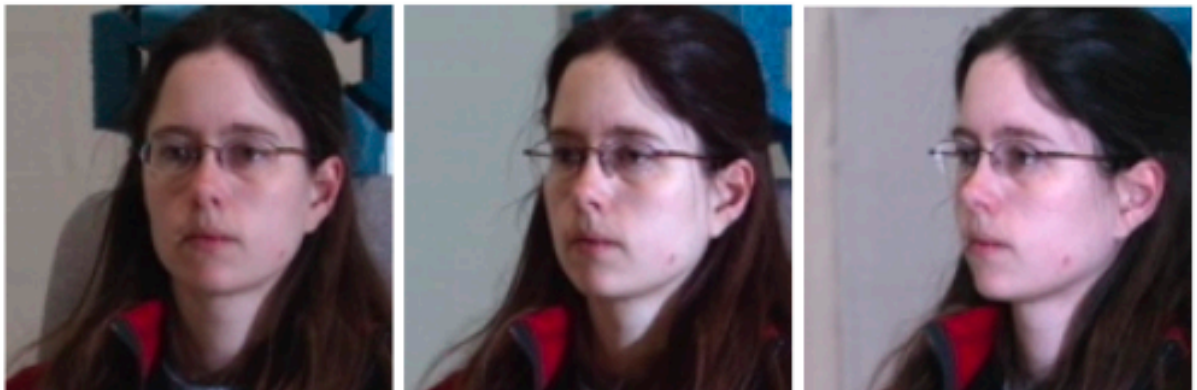
D. Ji, J. Kwon, M. McFarland, S. Savarese, CVPR 2017

I_1 GT I_2



Deep view morphing

D. Ji, J. Kwon, M. McFarland, S. Savarese, CVPR 2017



I_1 GT I_2