Lecture 7

Multi-view geometry

- The SFM problem
- Affine SFM
- Perspective SFM
- Self-calibration
- Applications

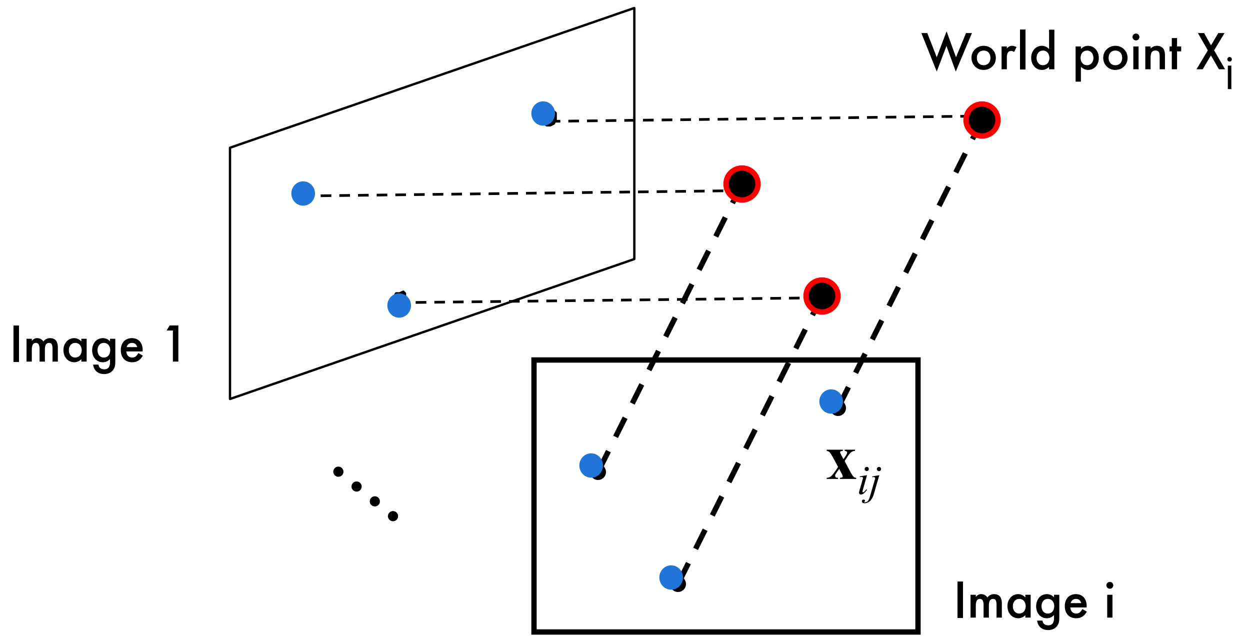
Reading:

[HZ] Chapter 10 “3D reconstruction of cameras and structure”
Chapter 18 “N-view computational methods”
Chapter 19 “Auto-calibration”

[FP] Chapter 13 “projective structure from motion”

[Szelisky] Chapter 7 “Structure from motion”

Affine structure from motion
(simpler problem)

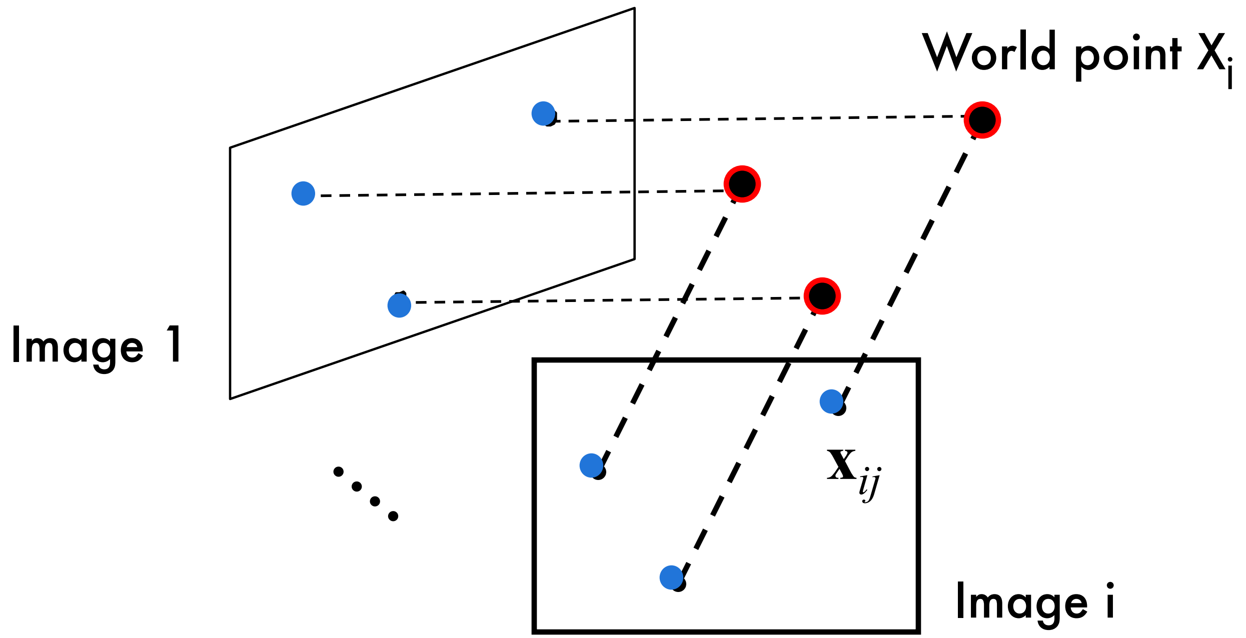


From the m×n observations $\mathbf{x}_{ij}$, estimate:

- *m* projection matrices $\mathbf{M}_i$ (affine cameras)
- *n* 3D points $\mathbf{X}_j$

Affine structure from motion

(simpler problem)



For the affine case (in Euclidean space)

$$\underbrace{\mathbf{x}_{ij}}_{2\times1} = \underbrace{\mathbf{A}_i}_{2\times3}\underbrace{\mathbf{X}_j}_{3\times1} + \underbrace{\mathbf{b}_i}_{2\times1}$$ [Eq. 4]

The Affine Structure-from-Motion Problem

Two approaches:

- Algebraic approach (affine epipolar geometry; estimate F; cameras; points)

- Factorization method

A factorization method – Tomasi & Kanade algorithm

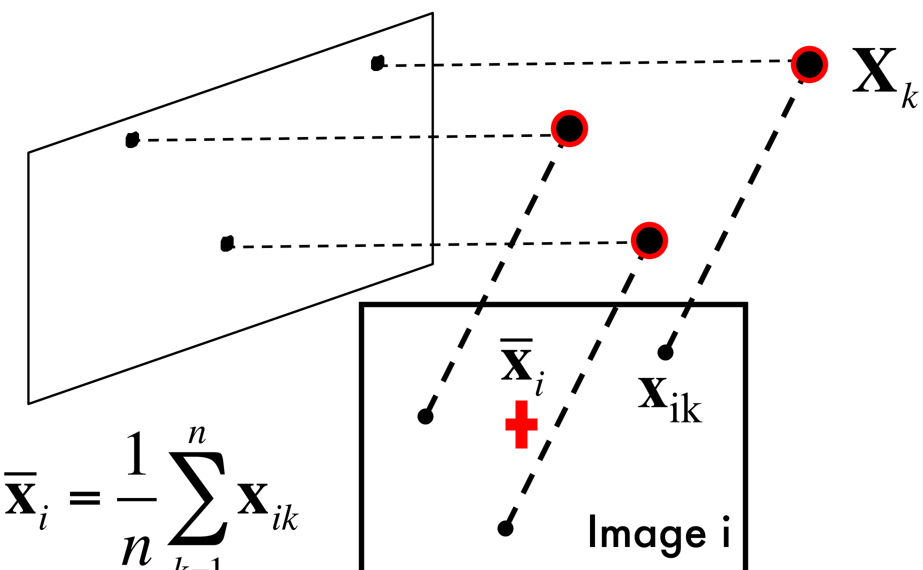
C. Tomasi and T. KanadeShape and motion from image streams under orthography: A factorization method. *IJCV*, 9(2):137-154, November 1992.

- Data centering
- Factorization

A factorization method - Centering the data

Centering: subtract the centroid of the image points

[Eq. 6] $$\hat{\mathbf{x}}_{ij} = \mathbf{x}_{ij} - \underbrace{\boxed{\frac{1}{n}\sum_{k=1}^{n}\mathbf{x}_{ik}}}_{\bar{\mathbf{x}}_i}$$



[Eq. 5] $$\bar{\mathbf{x}}_i = \frac{1}{n}\sum_{k=1}^{n}\mathbf{x}_{ik}$$

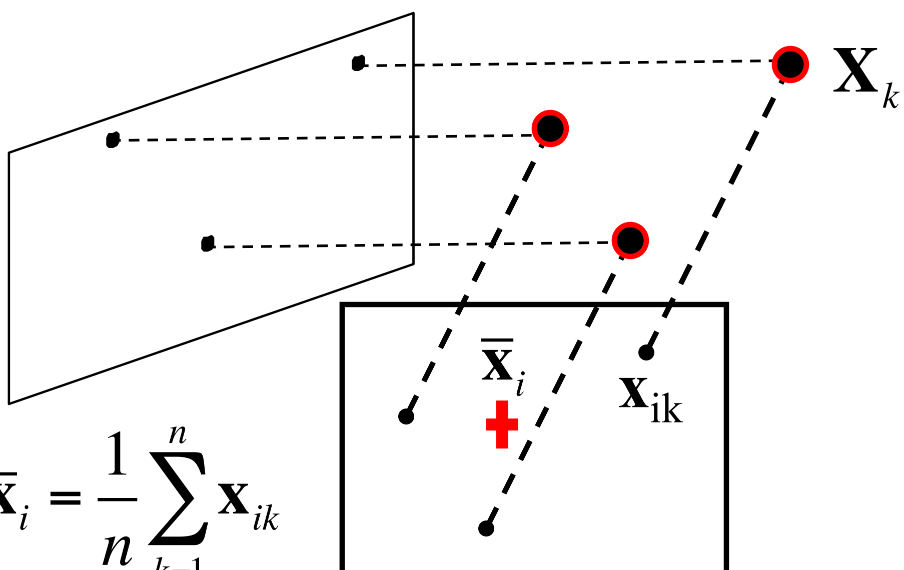
A factorization method - Centering the data

Centering: subtract the centroid of the image points

[Eq. 6] $$\hat{\mathbf{x}}_{ij} = \mathbf{x}_{ij} - \frac{1}{n}\sum_{k=1}^{n}\mathbf{x}_{ik} = \mathbf{A}_i\mathbf{X}_j + \mathbf{b}_i - \frac{1}{n}\sum_{k=1}^{n}\mathbf{A}_i\mathbf{X}_k - \frac{1}{n}\sum_{k=1}^{n}\mathbf{b}_i$$

$$\mathbf{x}_{ik} = \mathbf{A}_i\mathbf{X}_k + \mathbf{b}_i$$

[Eq. 4]



[Eq. 5] $$\bar{\mathbf{x}}_i = \frac{1}{n}\sum_{k=1}^{n}\mathbf{x}_{ik}$$

A factorization method - Centering the data

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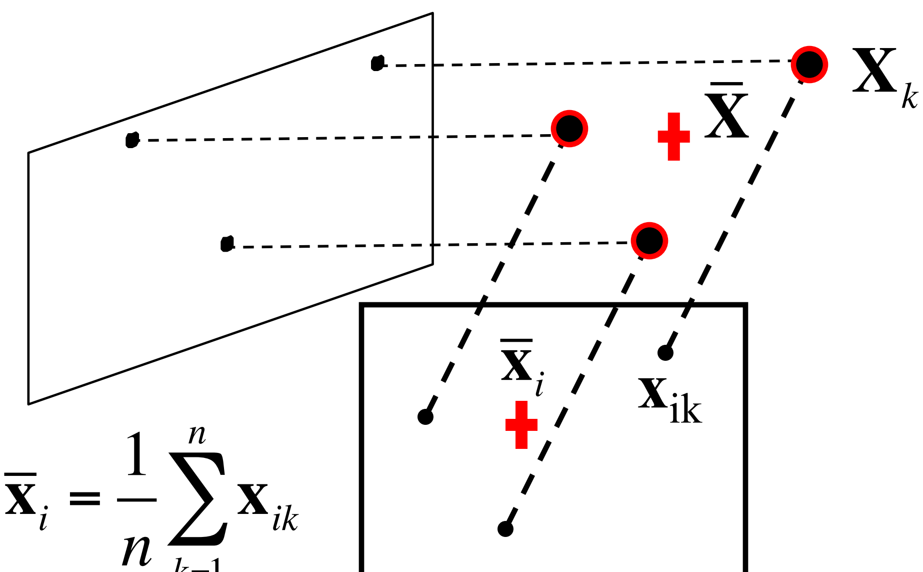
[Eq. 6] $$\hat{\mathbf{x}}_{ij} = \mathbf{x}_{ij} - \frac{1}{n}\sum_{k=1}^{n}\mathbf{x}_{ik} = \mathbf{A}_i\mathbf{X}_j + \mathbf{b}_i - \frac{1}{n}\sum_{k=1}^{n}\mathbf{A}_i\mathbf{X}_k - \frac{1}{n}\sum_{k=1}^{n}\mathbf{b}_i$$

$$= \mathbf{A}_i\left(\mathbf{X}_j - \frac{1}{n}\sum_{k=1}^{n}\mathbf{X}_k\right) = \mathbf{A}_i\left(\mathbf{X}_j - \bar{\mathbf{X}}\right)$$

$$= \mathbf{A}_i\hat{\mathbf{X}}_j$$ [Eq. 8]

$$\mathbf{x}_{ik} = \mathbf{A}_i\mathbf{X}_k + \mathbf{b}_i$$
[Eq. 4]



$$\bar{\mathbf{x}}_i = \frac{1}{n}\sum_{k=1}^{n}\mathbf{x}_{ik}$$

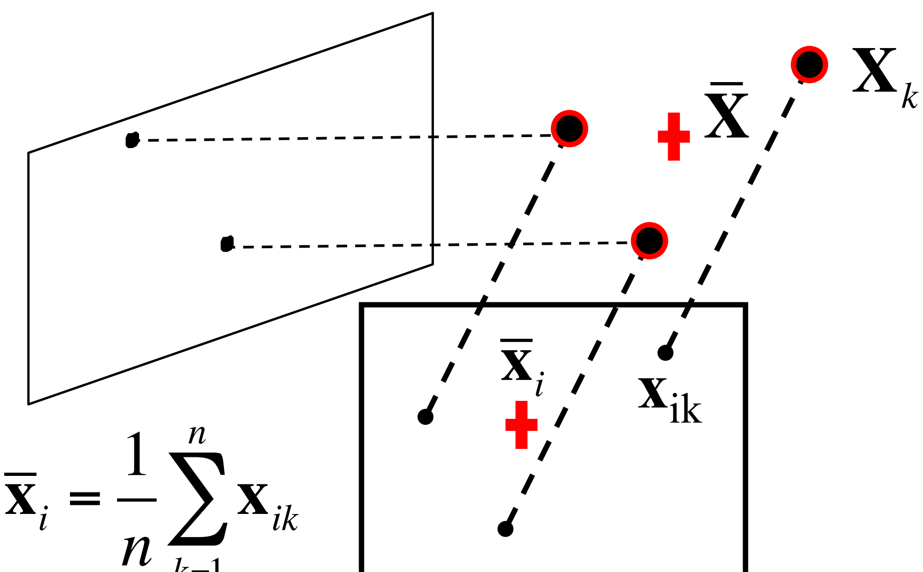
$$\bar{\mathbf{X}} = \frac{1}{n}\sum_{k=1}^{n}\mathbf{X}_k$$ [Eq. 7]

Centroid of 3D points

A factorization method - Centering the data

Thus, after centering, each **normalize**d observed point is related to the 3D point by

$$\hat{\mathbf{x}}_{ij} = \mathbf{A}_i \hat{\mathbf{X}}_j \quad \text{[Eq. 8]}$$



$$\bar{\mathbf{x}}_i = \frac{1}{n}\sum_{k=1}^{n}\mathbf{x}_{ik}$$

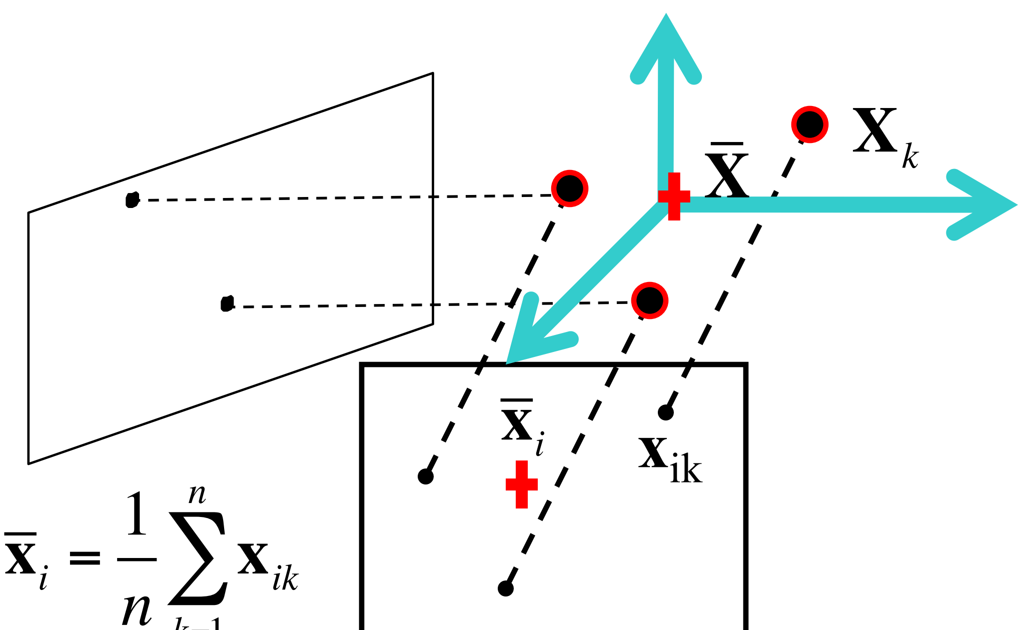
$$\bar{\mathbf{X}} = \frac{1}{n}\sum_{k=1}^{n}\mathbf{X}_k \quad \text{[Eq. 7]}$$

Centroid of 3D points

A factorization method - Centering the data

If the centroid of points in 3D = center of the world reference system

$$\hat{\mathbf{x}}_{ij} = \mathbf{A}_i \hat{\mathbf{X}}_j = \mathbf{A}_i \mathbf{X}_j \quad \text{[Eq. 9]}$$



$$\bar{\mathbf{x}}_i = \frac{1}{n}\sum_{k=1}^{n} \mathbf{x}_{ik}$$

$$\bar{\mathbf{X}} = \frac{1}{n}\sum_{k=1}^{n} \mathbf{X}_k \quad \text{[Eq. 7]}$$

Centroid of 3D points

A factorization method - factorization

Let's create a $2m \times n$ data (measurement) matrix:

$$\mathbf{D} = \begin{bmatrix} \hat{\mathbf{x}}_{11} & \hat{\mathbf{x}}_{12} & \cdots & \hat{\mathbf{x}}_{1n} \\ \hat{\mathbf{x}}_{21} & \hat{\mathbf{x}}_{22} & \cdots & \hat{\mathbf{x}}_{2n} \\ & & \ddots & \\ \hat{\mathbf{x}}_{m1} & \hat{\mathbf{x}}_{m2} & \cdots & \hat{\mathbf{x}}_{mn} \end{bmatrix}$$

cameras ($2m$)

points (n)

Each $\hat{\mathbf{x}}_{ij}$ entry is a 2x1 vector!

A factorization method - factorization

Let's create a $2m \times n$ data (measurement) matrix:

$$\mathbf{D} = \underbrace{\begin{bmatrix} \hat{\mathbf{x}}_{11} & \hat{\mathbf{x}}_{12} & \cdots & \hat{\mathbf{x}}_{1n} \\ \hat{\mathbf{x}}_{21} & \hat{\mathbf{x}}_{22} & \cdots & \hat{\mathbf{x}}_{2n} \\ & & \ddots & \\ \hat{\mathbf{x}}_{m1} & \hat{\mathbf{x}}_{m2} & \cdots & \hat{\mathbf{x}}_{mn} \end{bmatrix}}_{(2m \times n)} = \underbrace{\boxed{\begin{bmatrix} \mathbf{A}_1 \\ \mathbf{A}_2 \\ \vdots \\ \mathbf{A}_m \end{bmatrix}}}_{\text{cameras } (2m \times 3)\ \ M} \underbrace{\boxed{\begin{bmatrix} \mathbf{X}_1 & \mathbf{X}_2 & \cdots & \mathbf{X}_n \end{bmatrix}}}_{\text{points } (3 \times n)\ \ S}$$

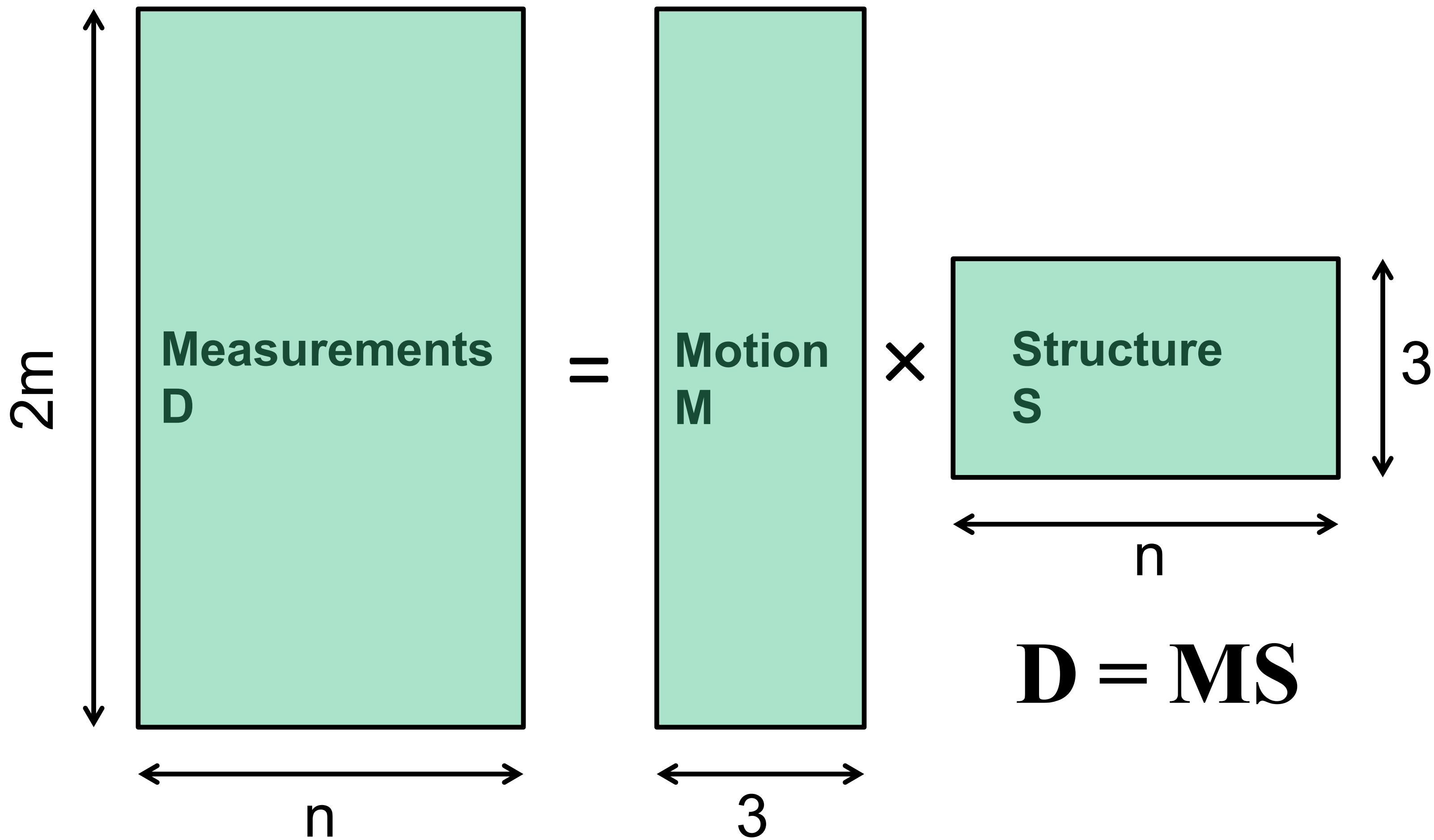
[Eq. 10]

Each $\hat{\mathbf{x}}_{ij}$ entry is a 2x1 vector!
$\mathbf{A}_i$ is 2x3 and $\mathbf{X}_j$ is 3x1

The measurement matrix **D = M S** has rank 3
(it's a product of a 2mx3 matrix and 3xn matrix)

Factorizing the Measurement Matrix

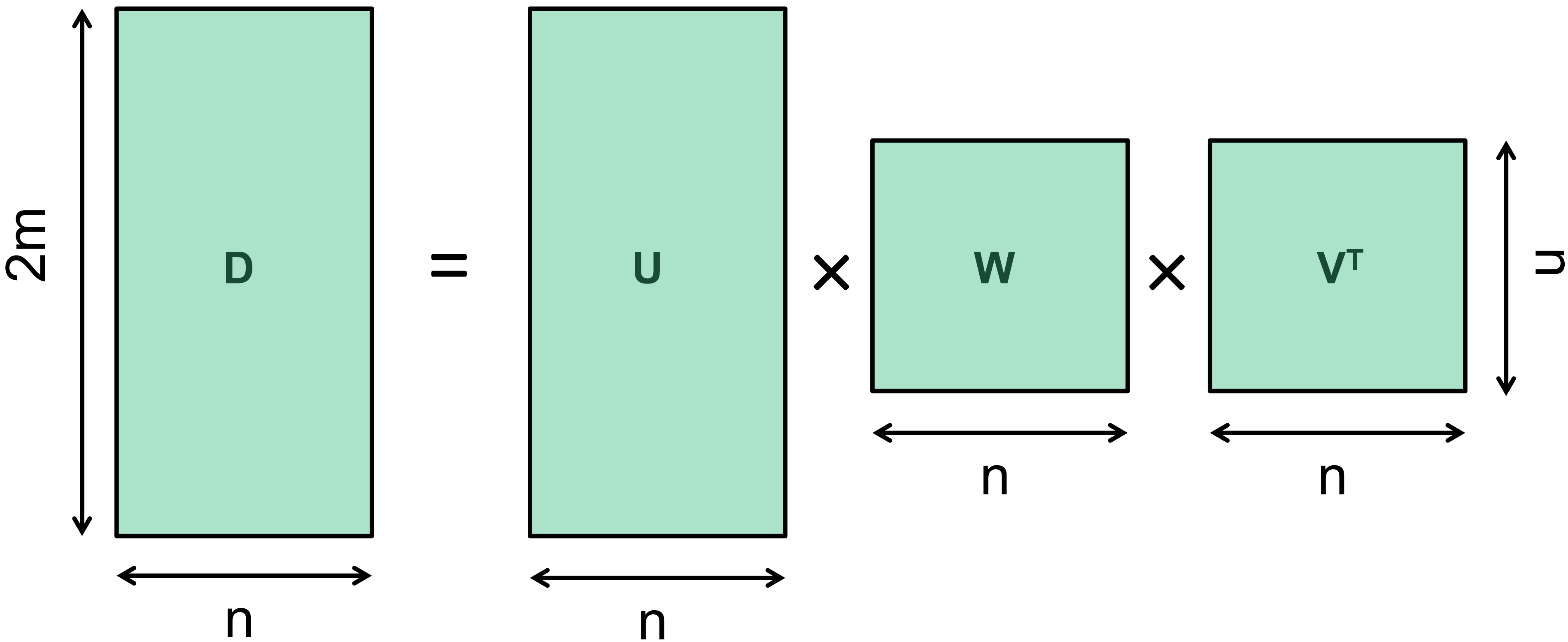
How to factorize D?



$$\mathbf{D} = \mathbf{MS}$$

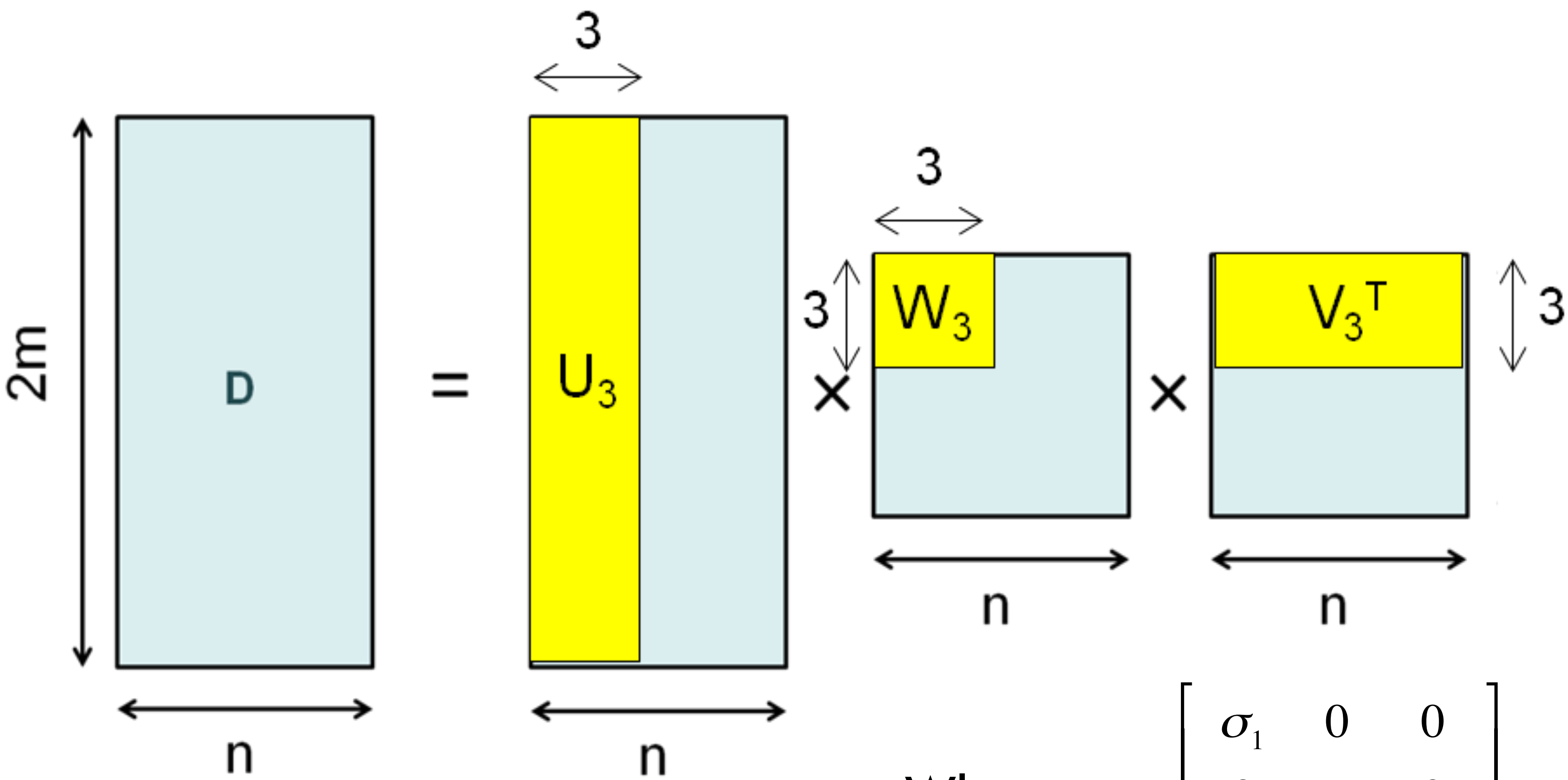
Factorizing the Measurement Matrix

- By computing the Singular value decomposition of D!



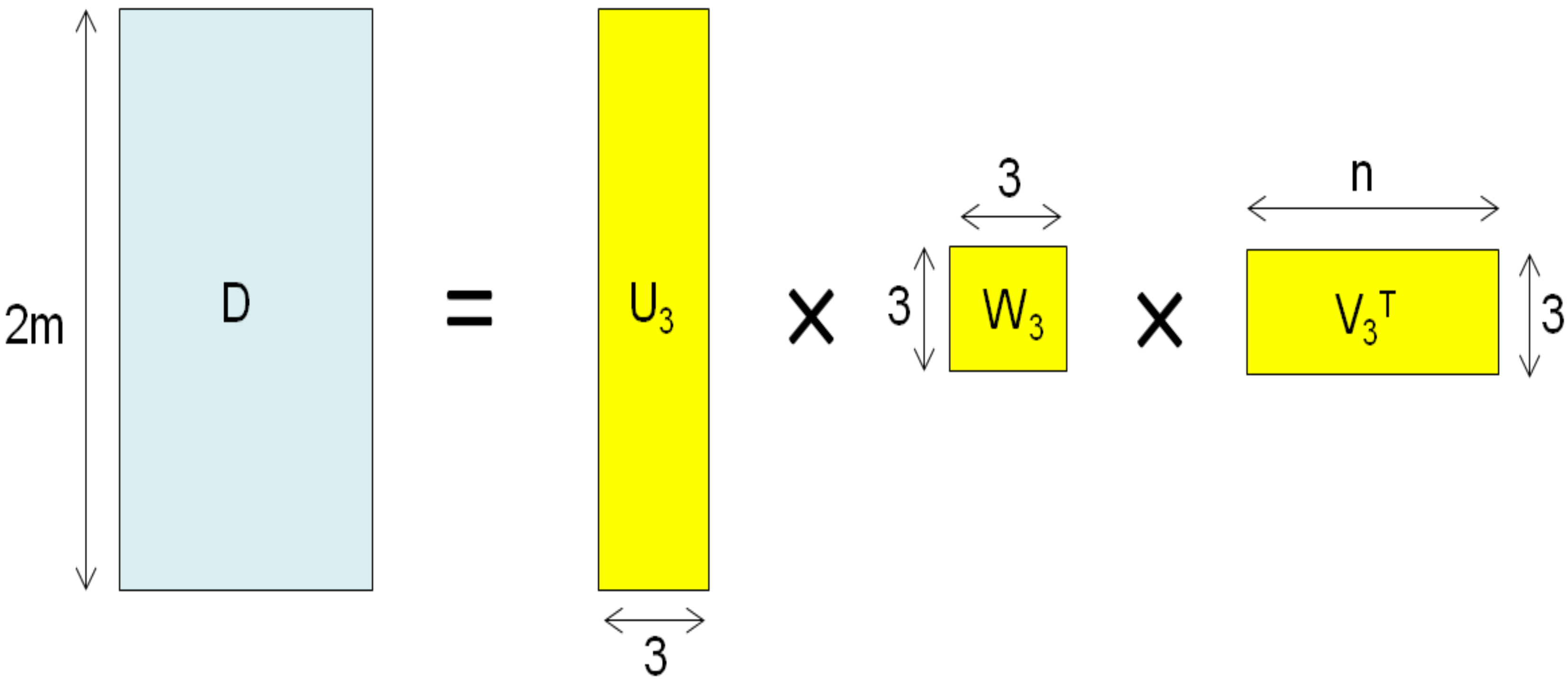
Factorizing the Measurement Matrix

Since rank (D)=3, there are only 3 non-zero singular values σ_1 , σ_2 and σ_3

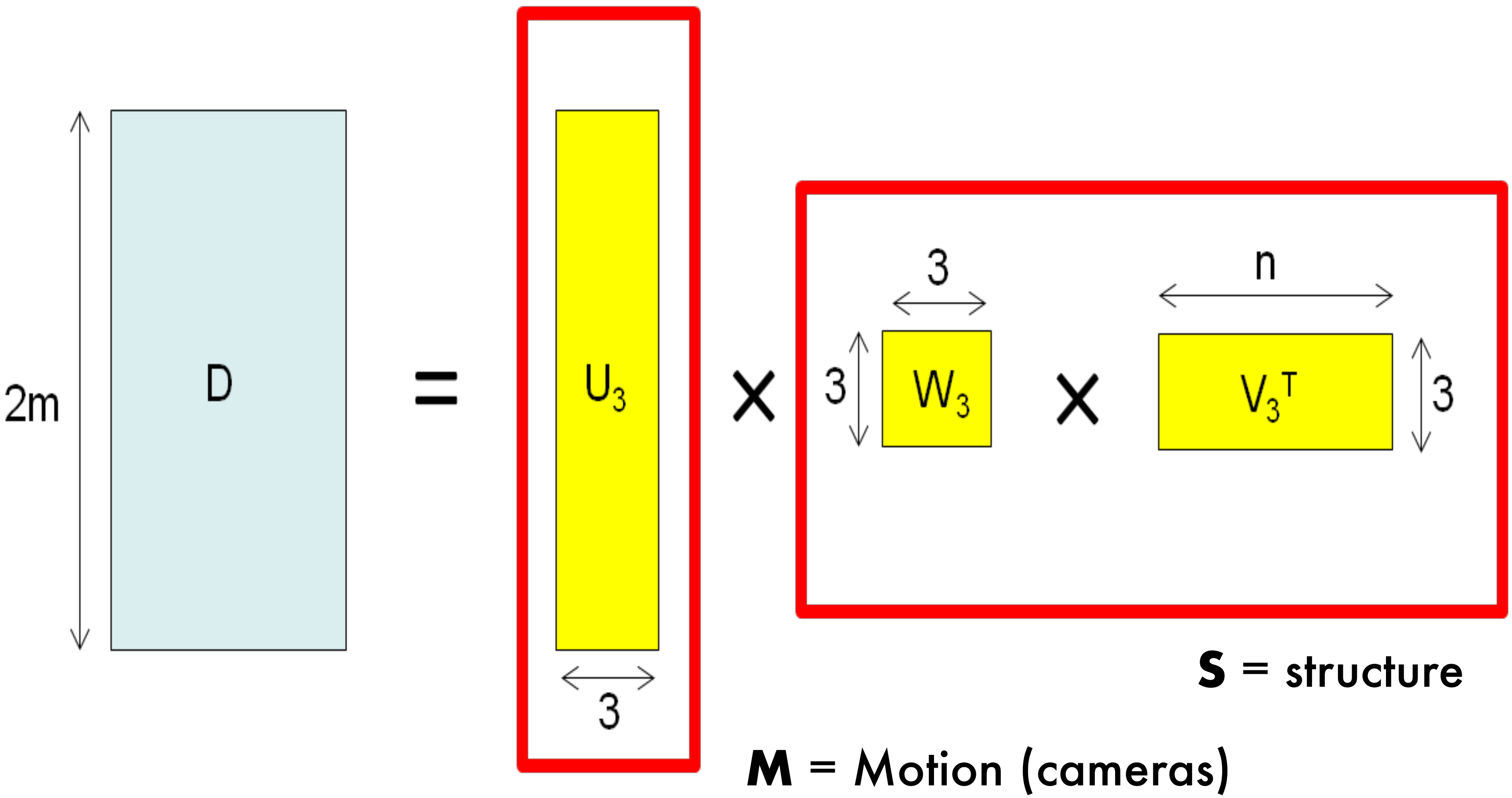


Where $W_3 = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix}$ [Eq. 11]

Factorizing the Measurement Matrix



Factorizing the Measurement Matrix



$$\mathbf{D} = \mathbf{U}_3\ \mathbf{W}_3\ \mathbf{V}_3^{T} = \mathbf{U}_3\ (\mathbf{W}_3\ \mathbf{V}_3^{T}) = \mathbf{M}\ \mathbf{S}$$ [Eq. 12]

Factorizing the Measurement Matrix

$$\mathbf{D} = \mathbf{U}_3\, \mathbf{W}_3\, \mathbf{V}_3^{T} = \mathbf{U}_3\, (\mathbf{W}_3\, \mathbf{V}_3^{T}) = \mathbf{M}\, \mathbf{S} \quad \text{[Eq. 12]}$$

What is the issue here?

D has rank>3 because of:

- measurement noise
- affine approximation

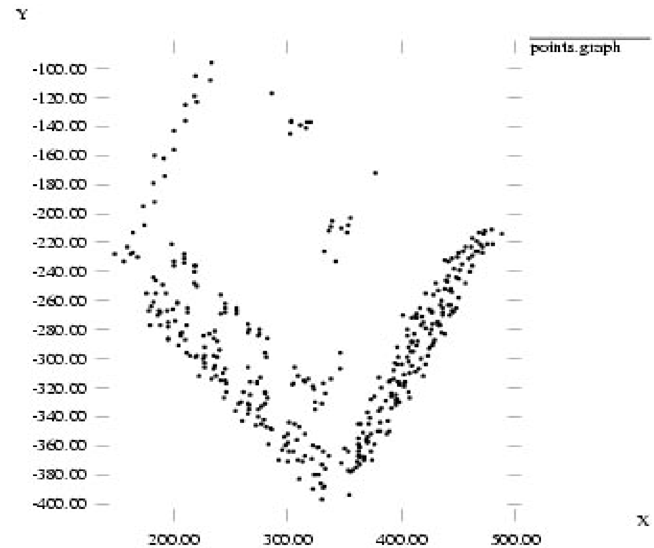
Theorem: When $\mathbf{D}$ has a rank greater than 3, $\mathbf{U}_3\mathbf{W}_3\mathbf{V}_3^T$ is the best possible rank- 3 approximation of **D** in the sense of the Frobenius norm.

$$\mathbf{D} = \mathbf{U}_3\mathbf{W}_3\mathbf{V}_3^T \qquad \begin{cases} \mathbf{M} \approx \mathbf{U}_3 \\ \mathbf{S} \approx \mathbf{W}_3\mathbf{V}_3^T \end{cases}$$

$$\boxed{\|A\|_F = \sqrt{\sum_{i=1}^{m}\sum_{j=1}^{n} |a_{ij}|^2} = \sqrt{\sum_{i=1}^{\min\{m,n\}} \sigma_i^2}}$$

Reconstruction results

1



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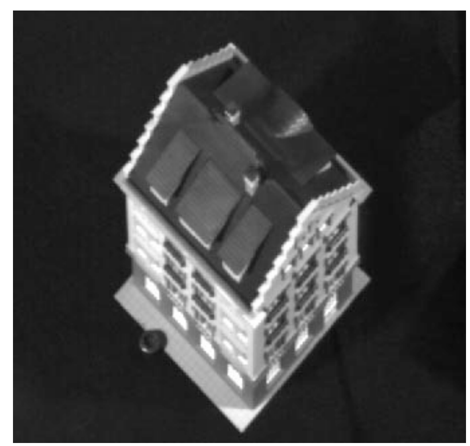
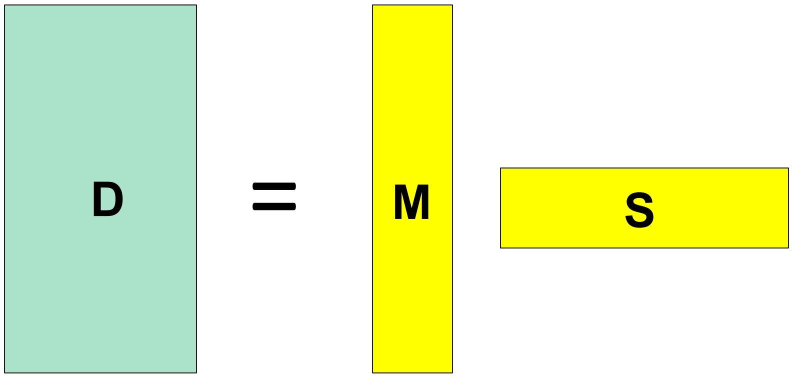
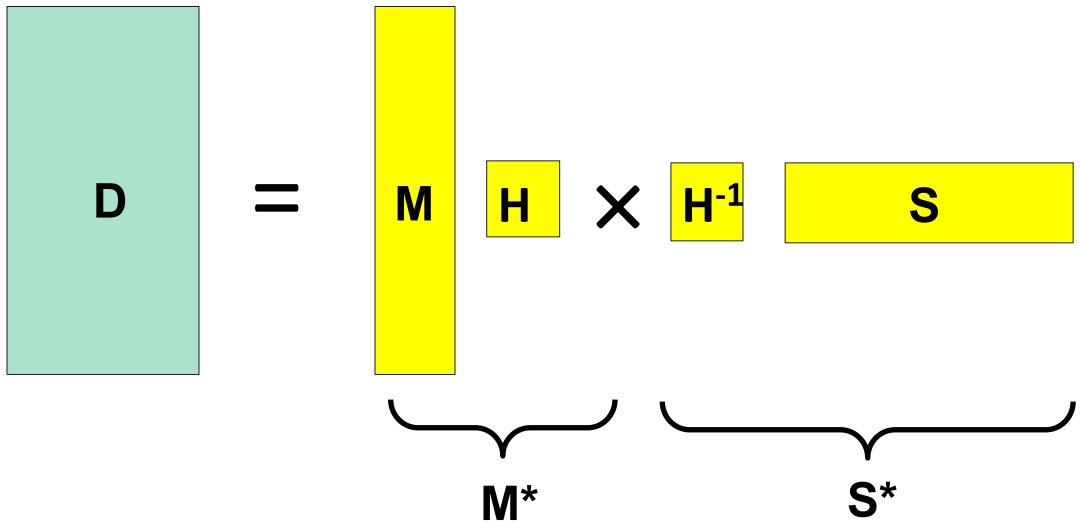


C. Tomasi and T. Kanade. [Shape and motion from image streams under orthography: A factorization method.](#) *IJCV*, 9(2):137-154, November 1992.

Affine Ambiguity



Affine Ambiguity



- The decomposition is not unique. We get the same **D** by applying the transformations:

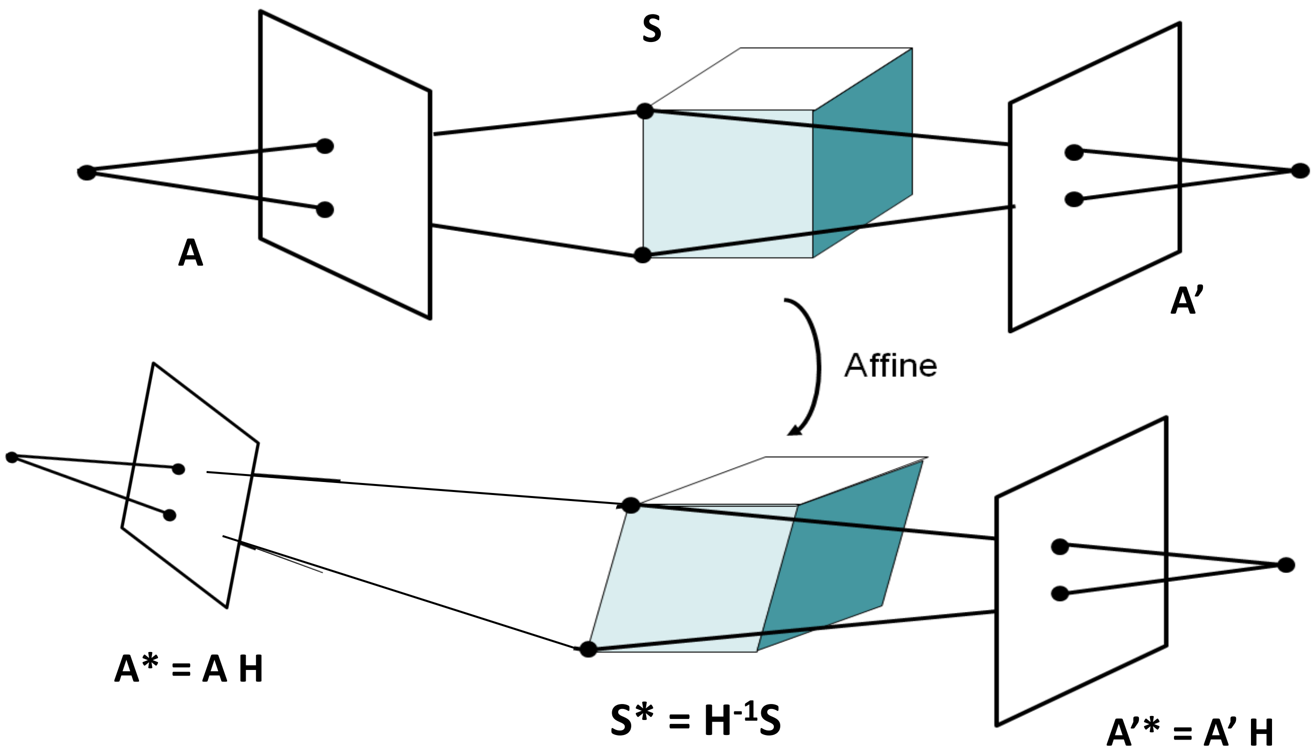
$$\mathbf{M^* = M\,H}$$

$$\mathbf{S^* = H^{-1}S}$$

where **H** is an arbitrary 3x3 matrix describing an affine transformation

- Additional constraints must be enforced to resolve this ambiguity

Affine Ambiguity



The Affine Structure-from-Motion Problem

Given m images of n fixed points $\mathbf{X}_j$ we can write

$$\mathbf{x}_{ij} = \mathbf{A}_i \mathbf{X}_j + \mathbf{b}_i \qquad \text{for } i = 1, \ldots, \boxed{m} \text{ and } j = 1, \ldots, \boxed{n}$$

N. of cameras N. of points

Problem: estimate m matrices A_i, m matrices b_i and the n positions $\mathbf{X}_j$ from the $m \times n$ observations $\mathbf{x}_{ij}$.

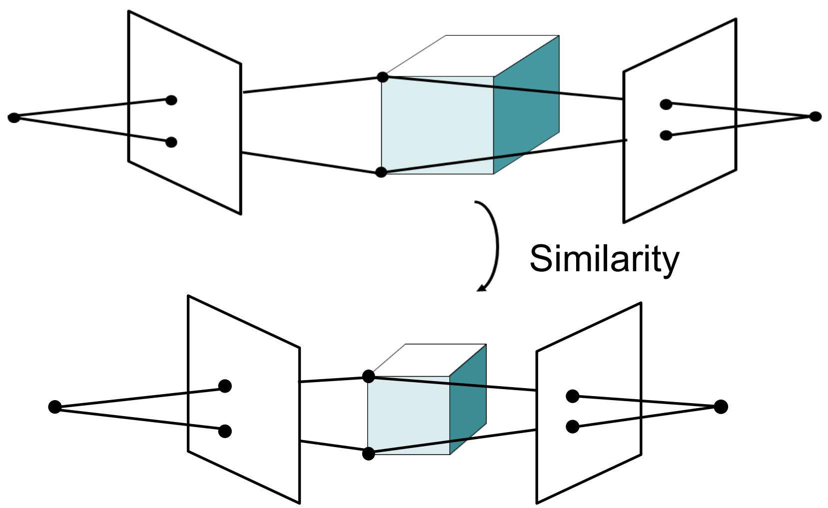
How many equations and how many unknown?

$2m \times n$ equations in $8m + 3n - 8$ unknowns

Similarity Ambiguity

- The scene is determined by the images only up a similarity transformation (rotation, translation and scaling)

- This is called **metric reconstruction**



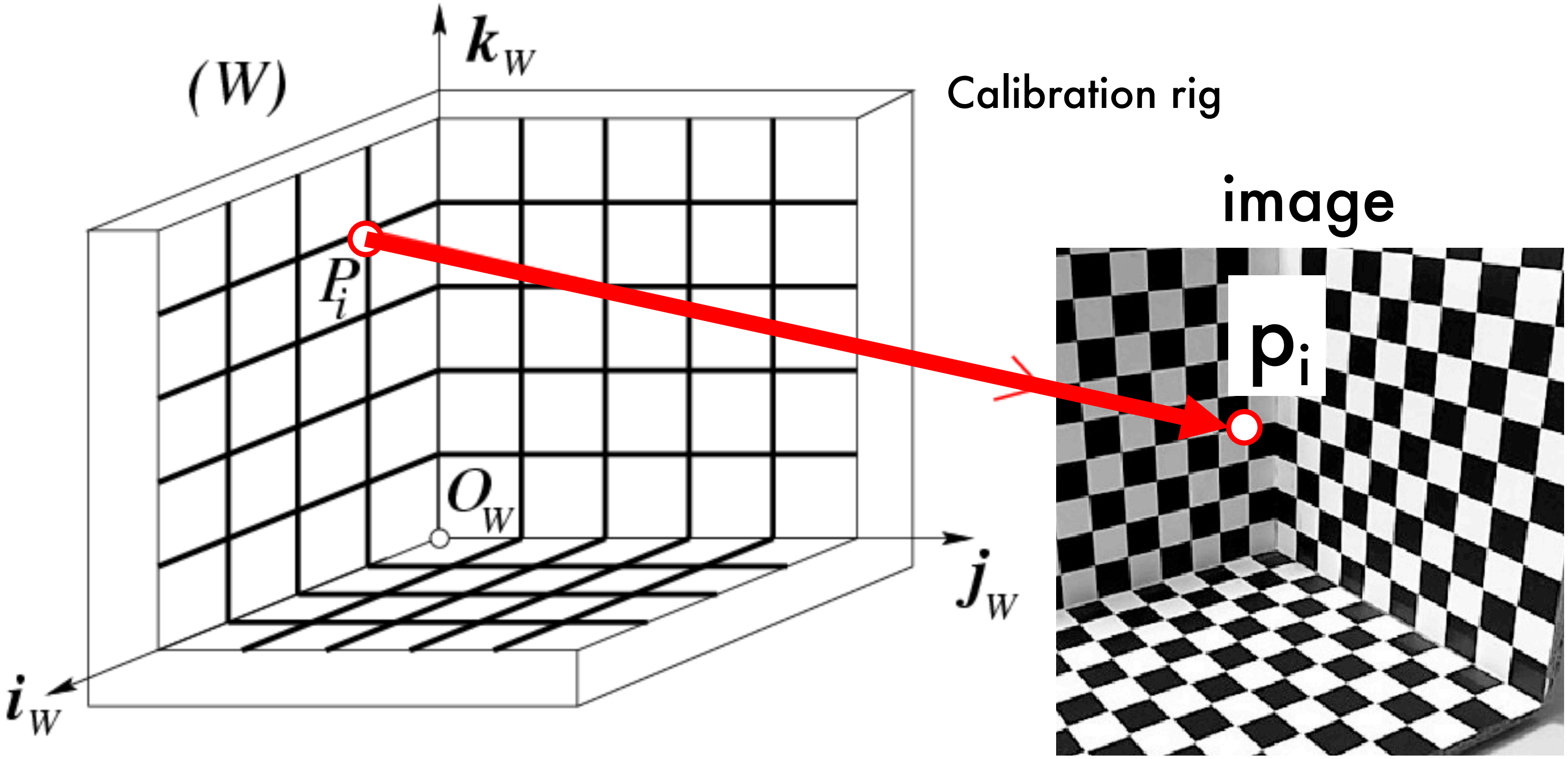
- The ambiguity exists even for (intrinsically) calibrated cameras
- For calibrated cameras, the similarity ambiguity is the only ambiguity

[Longuet-Higgins '81]

Similarity Ambiguity

- It is impossible, based on the images alone, to estimate the absolute scale of the scene

Resolving the similarity ambiguity



While calibrating a camera, we make assumptions about the geometry of the world

Lecture 7

Multi-view geometry

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- Affine SFM
- Perspective SFM
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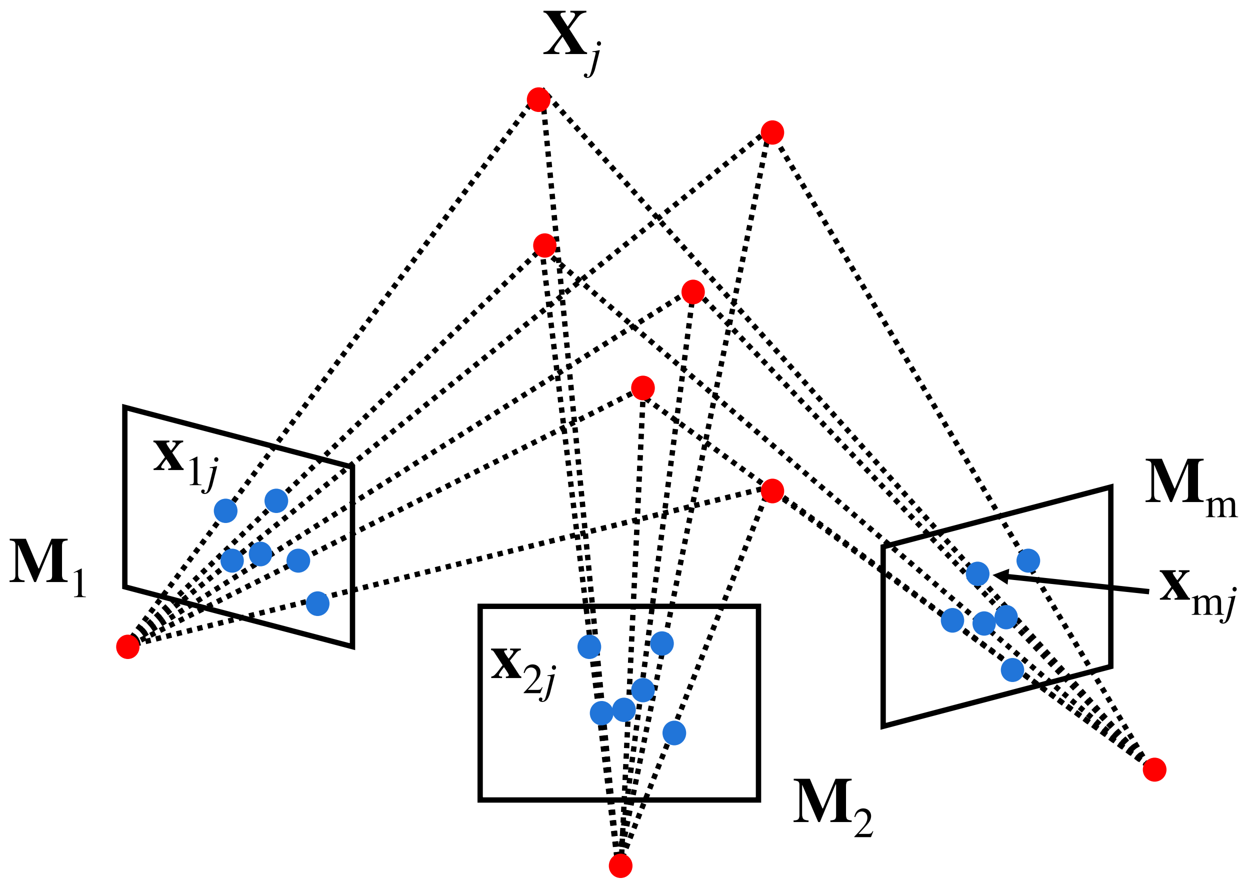
- Applications

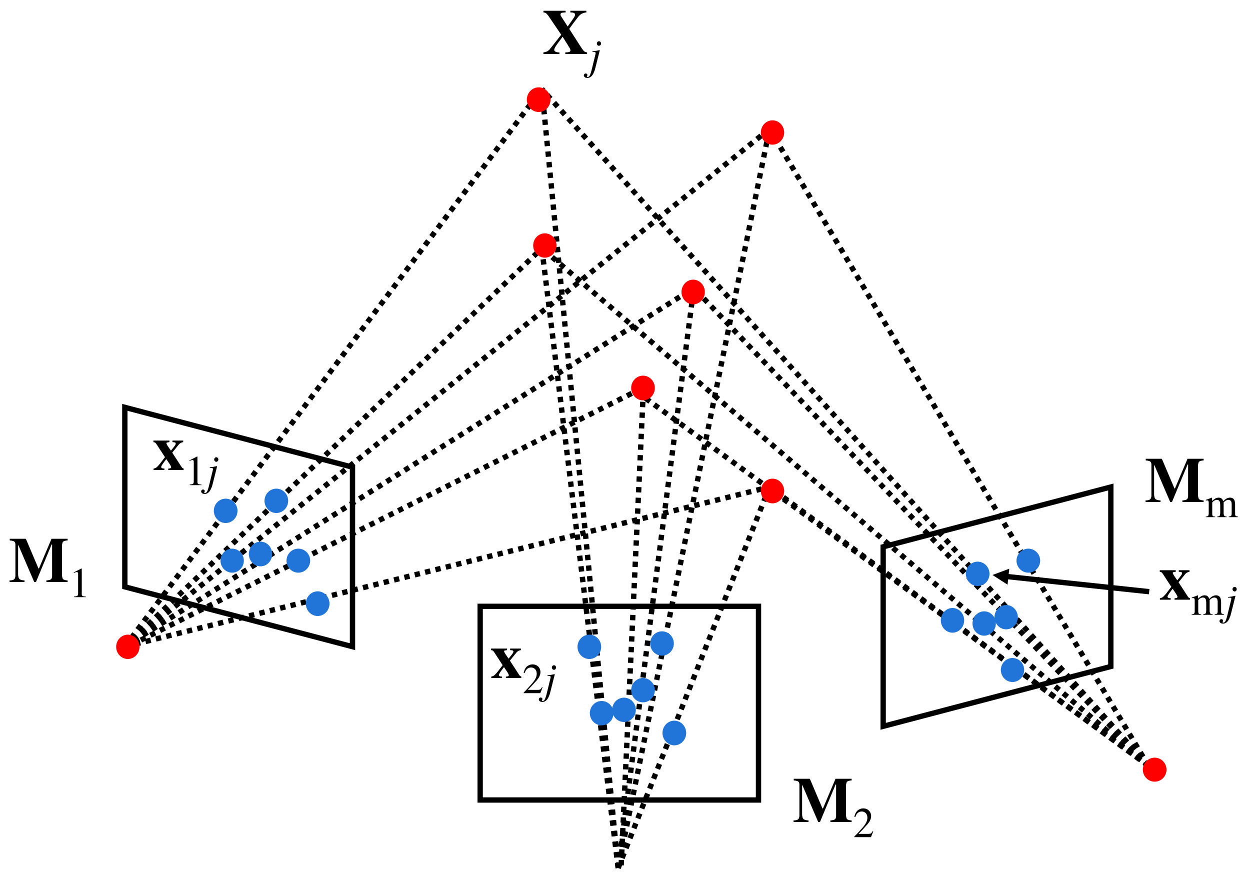
Structure from motion problem



From the m×n observations $\mathbf{x}_{ij}$, estimate:

- *m* projection matrices $\mathbf{M}_i$ = motion
- *n* 3D points $\mathbf{X}_j$ = structure

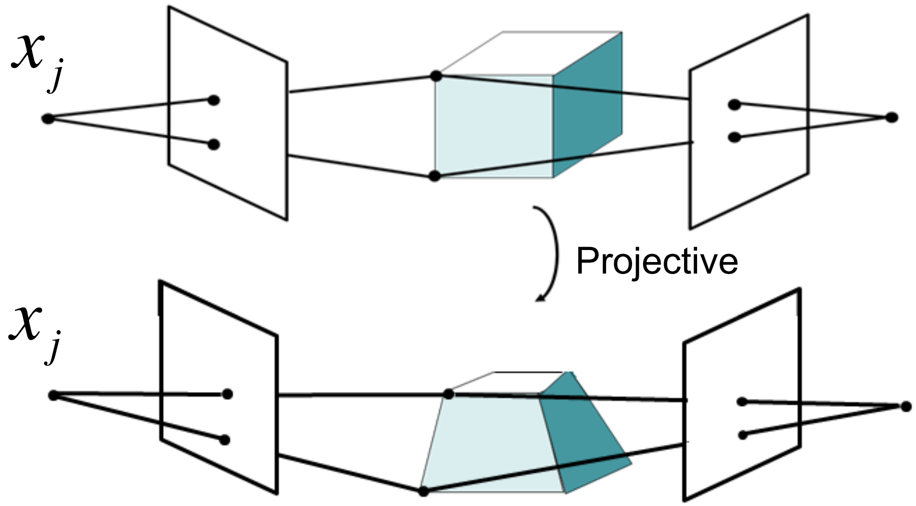
Structure from motion problem



m cameras $M_1 \ldots M_m$

$$M_i = \begin{bmatrix} a_{11} & a_{12} & a_{13} & b_1 \\ a_{21} & a_{22} & a_{23} & b_2 \\ a_{31} & a_{32} & a_{33} & 1 \end{bmatrix}$$

Structure from Motion Ambiguities



- In the general case (nothing is known) the ambiguity is expressed by an arbitrary 4X4 projective transformation

$$x_j = M_i \boxed{X_j} \qquad \boxed{M_i} = K_i \begin{bmatrix} R_i & T_i \end{bmatrix}$$

$$\boxed{H X_j} \qquad \boxed{M_j H^{-1}}$$

$$x_j = M_i X_j = \left(M_i H^{-1}\right)\left(H X_j\right)$$

The Structure-from-Motion Problem

Given *m* images of *n* fixed points X_j we can write

$$x_{ij} = M_i \, X_j$$

for i = 1, ..., $\boxed{m}$ and j = 1, ... $\boxed{, n}$

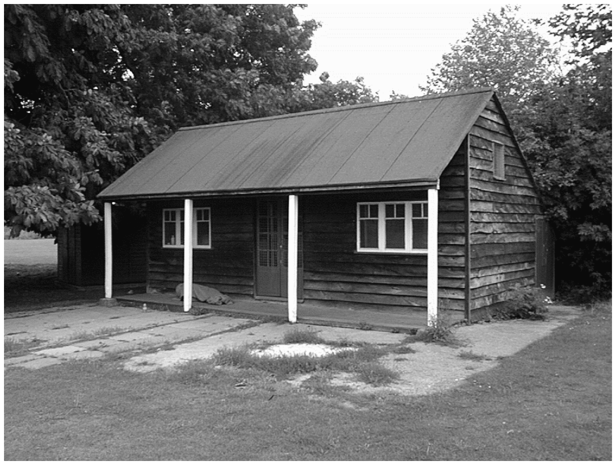
N. of cameras N. of points

Problem: estimate m 3×4 matrices M_i and n positions X_j from m×n obvesrvations x_{ij} .

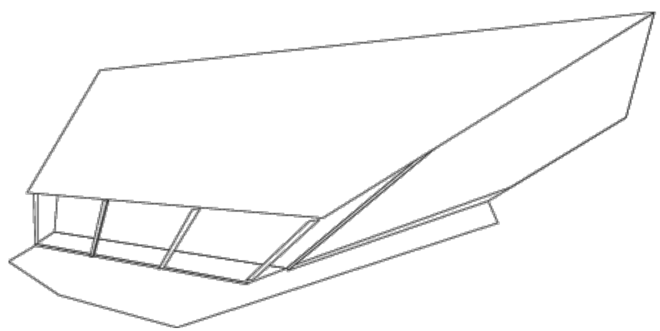
- If the cameras are not calibrated, cameras and points can only be recovered up to a 4x4 projective (where the 4x4 projective is defined up to scale)
- Given two cameras, how many points are needed?
- How many equations and how many unknown?

2m × n equations in 11m+3n – 15 unknowns

Projective Ambiguity



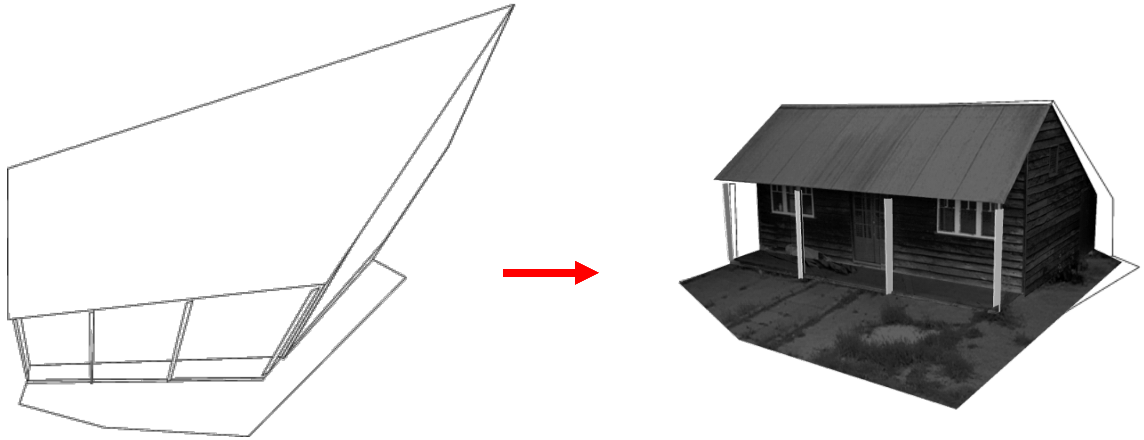
S =



R. Hartley and A. Zisserman, Multiple View Geometry in Computer Vision, 2nd edition, 2003

Metric reconstruction (upgrade)

- The problem of recovering the metric reconstruction from the perspective one is called self-calibration



Structure-from-Motion methods

1. Recovering structure and motion up to perspective ambiguity

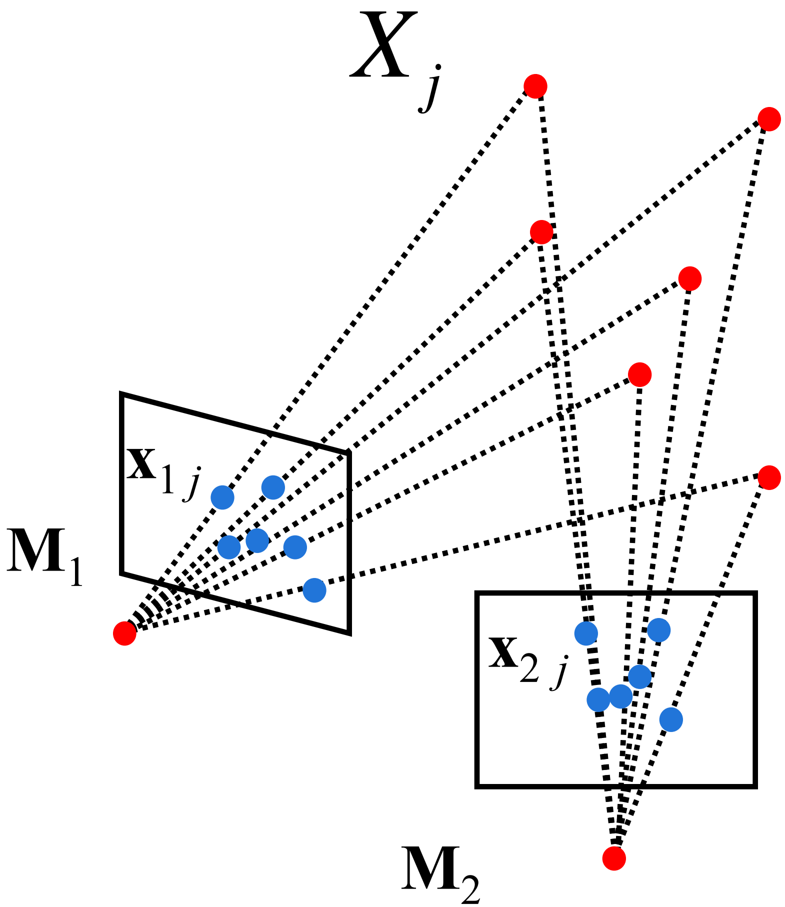
- Algebraic approach (by fundamental matrix)
- Factorization method (by SVD)
- Bundle adjustment

2. Resolving the perspective ambiguity

Algebraic approach (2-view case)

1. Compute the fundamental matrix F from two views
2. Use F to estimate projective cameras
3. Use these cameras to triangulate and estimate points in 3D

Algebraic approach (2-view case)



$$x_{1j} = M_1 X_j$$

$$x_{2j} = M_2 X_j$$

For j = 1, … ,n

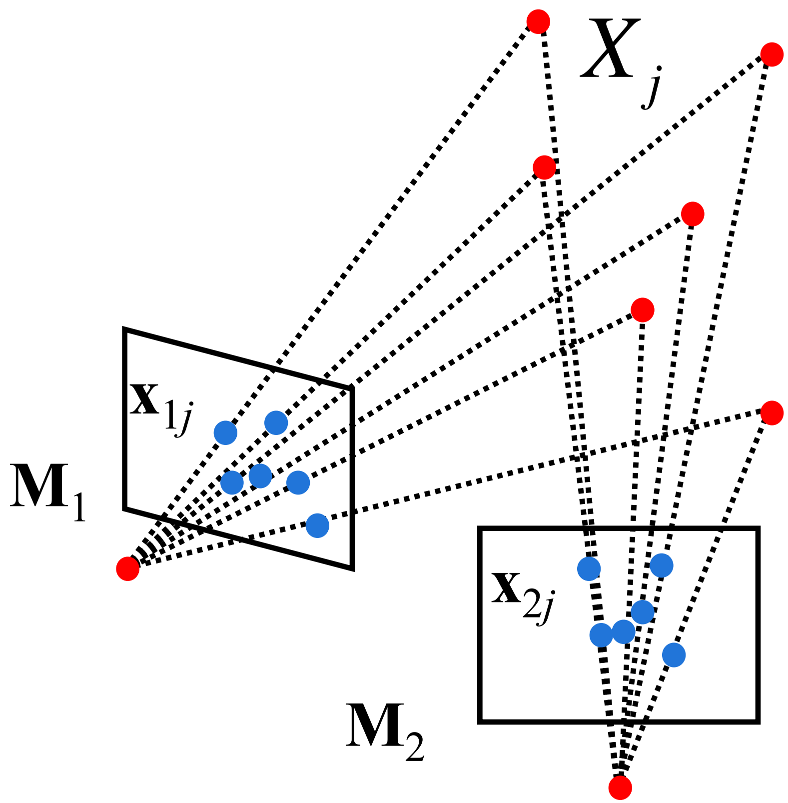
N. of points

From at least 8 point correspondences, compute F associated to camera 1 and 2

Algebraic approach (2-view case)

1. Compute the fundamental matrix F from two views (eg. 8 point algorithm)
2. Use F to estimate projective cameras
3. Use these cameras to triangulate and estimate points in 3D

Algebraic approach (2-view case)



$$x_{1j} = M_1 X_j$$

$$x_{2j} = M_2 X_j$$

For $j = 1, \ldots, \boxed{n}$

N. of points

Because of the projective ambiguity, we can always apply a projective transformation H such that:

$$\mathrm{M}_1 \mathrm{H}^{-1} = \begin{bmatrix} \mathrm{I} & 0 \end{bmatrix}$$

[Eq. 3]

Canonical perspective camera

$$\mathrm{M}_2 \mathrm{H}^{-1} = \begin{bmatrix} \mathrm{A} & \mathrm{b} \end{bmatrix}$$

[Eq. 4]

Algebraic approach (2-view case)

- Call $\mathbf{X}$ a generic 3D point $\mathbf{X}_{ij}$
- Call $\mathbf{x}$ and $\mathbf{x}'$ the corresponding observations to camera 1 and respectively

[Eqs. 5]

$$\begin{cases} \tilde{M}_1 = M_1 H^{-1} = \begin{bmatrix} \mathbf{I} & \mathbf{0} \end{bmatrix} \\ \tilde{M}_2 = M_2 H^{-1} = \begin{bmatrix} \mathbf{A} & \mathbf{b} \end{bmatrix} \\ \tilde{\mathbf{X}} = \mathrm{H}\,\mathbf{X} \end{cases}$$

$$\mathbf{x} = M_1 \mathbf{X} = M_1 H^{-1} H \mathbf{X} = [\mathbf{I} \,|\, \mathbf{0}]\tilde{\mathbf{X}}$$ [Eq. 6]

$$\mathbf{x}' = M_2 \mathbf{X} = M_2 H^{-1} H \mathbf{X} = [\mathbf{A} | \mathbf{b}]\tilde{\mathbf{X}}$$

?

$$\mathbf{x}' = [\mathbf{A} | \mathbf{b}]\tilde{\mathbf{X}} = [\mathbf{A} \,|\, \mathbf{b}]\begin{bmatrix} \tilde{X}_1 \\ \tilde{X}_2 \\ \tilde{X}_3 \\ 1 \end{bmatrix} = \mathbf{A}[\mathbf{I} | \mathbf{0}]\begin{bmatrix} \tilde{X}_1 \\ \tilde{X}_2 \\ \tilde{X}_3 \\ 1 \end{bmatrix} + \mathbf{b} = \mathbf{A}\boxed{[\mathbf{I} | \mathbf{0}]\,\tilde{\mathbf{X}}} + \mathbf{b} = \mathbf{A}\mathbf{x} + \mathbf{b}$$

[Eq. 7]

$$\mathbf{x}' \times \mathbf{b} = (\mathbf{A}\mathbf{x} + \mathbf{b}) \times \mathbf{b} = \mathbf{A}\mathbf{x} \times \mathbf{b}$$ [Eq. 8]

$$\mathbf{x}'^T \cdot (\mathbf{x}' \times \mathbf{b}) = \mathbf{x}'^T \cdot (\mathbf{A}\mathbf{x} \times \mathbf{b}) = 0$$ [Eq. 9]

$$\mathbf{x}'^T (\mathbf{b} \times \mathbf{A}\mathbf{x}) = 0$$ [Eq. 10]

Cross product as matrix multiplication

$$\mathbf{a} \times \mathbf{b} = \begin{bmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{bmatrix} \begin{bmatrix} b_x \\ b_y \\ b_z \end{bmatrix} = [\mathbf{a}_\times]\mathbf{b}$$

Algebraic approach (2-view case)

[Eqs. 5]

$$\begin{cases} \tilde{M}_1 = M_1 H^{-1} = \begin{bmatrix} I & 0 \end{bmatrix} \\ \tilde{M}_2 = M_2 H^{-1} = \begin{bmatrix} A & b \end{bmatrix} \\ \widetilde{\mathbf{X}} = \mathrm{H}\,\mathbf{X} \end{cases}$$

$$\mathbf{x} = M_1 H^{-1} H \mathbf{X} = [\mathbf{I} \,|\, \mathbf{0}]\widetilde{\mathbf{X}}$$

$$\mathbf{x}' = M_2 H^{-1} H \mathbf{X} = [\mathbf{A} \,|\, \mathbf{b}]\widetilde{\mathbf{X}}$$

[Eq. 6]

⋮

$$\mathbf{x}'^{T}(\mathbf{b} \times \mathbf{A}\mathbf{x}) = 0$$ [Eq. 10]

$$\mathbf{x}'^{\mathrm{T}}\boxed{[\mathbf{b}_{\times}]\mathbf{A}}\,\mathbf{x} = 0$$

is this familiar?

$$\boxed{\mathbf{F} = [\mathbf{b}_{\times}]\mathbf{A}}$$

fundamental matrix!

$$\boxed{\mathbf{x}'^{\mathrm{T}}\mathrm{F}\,\mathbf{x} = 0}$$

Compute cameras

$$\mathbf{x}'^{\mathrm{T}} \mathrm{F}\, \mathbf{x} = 0 \qquad \mathbf{F} = [\mathbf{b}_{\times}]\mathbf{A} = \mathbf{b} \times \mathbf{A} \quad \text{[Eq. 11]}$$

Compute **b**:

- Let's consider the product **F b**

$$\mathbf{F} \cdot \mathbf{b} = [\mathbf{b}_{\times}]\mathbf{A} \cdot \mathbf{b} = \mathbf{b} \times \mathbf{A} \cdot \mathbf{b} = 0 \quad \text{[Eq. 12]}$$

- Since **F** is singular, we can compute **b** as least sq. solution of **F b** = 0, with |**b**|=1 using SVD

- Using a similar derivation, we have that $\mathbf{b}^{\mathrm{T}}\, \mathbf{F} = 0$ [Eq. 12-bis]

Compute cameras

$$\mathbf{x}'^{\mathrm{T}} \mathrm{F}\, \mathbf{x} = 0 \qquad \mathbf{F} = [\mathbf{b}_{\times}]\mathbf{A} \qquad \begin{cases} \mathbf{F}\,\mathbf{b} = 0 & \text{[Eq. 12]} \\ \mathbf{b}^{\mathrm{T}}\,\mathbf{F} = 0 & \text{[Eq. 12-bis]} \end{cases}$$

[Eq. 11]

Compute $\mathbf{A}$:

- Define: $\mathbf{A}' = -[\mathbf{b}_{\times}]\,\mathbf{F}$
- Let's verify that $[\mathbf{b}_{\times}]\mathbf{A}'$ is equal to $\mathbf{F}$:

Indeed: $[\mathbf{b}_{\times}]\mathbf{A}' = -[\mathbf{b}_{\times}][\mathbf{b}_{\times}]\,\mathbf{F} = -(\mathbf{b}\,\mathbf{b}^{T} - |\mathbf{b}|^{2}\,\mathbf{I})\,\mathbf{F} = -\mathbf{b}\,\mathbf{b}^{T}\mathbf{F} + |\mathbf{b}|^{2}\,\mathbf{F} = 0 + 1 \cdot \mathbf{F} = \mathbf{F}$

[Eq. 13]

- Thus, $\mathbf{A} = \mathbf{A}' = -[\mathbf{b}_{\times}]\,\mathbf{F}$

[Eqs. 14] $$\boxed{\tilde{M}_1 = \begin{bmatrix} I & 0 \end{bmatrix} \qquad \tilde{M}_2 = \begin{bmatrix} -[\mathbf{b}_x]\mathbf{F} & \mathbf{b} \end{bmatrix}}$$

Interpretation of **b**

$$\mathbf{x}'^{\mathrm{T}} \mathrm{F}\, \mathbf{x} = 0$$

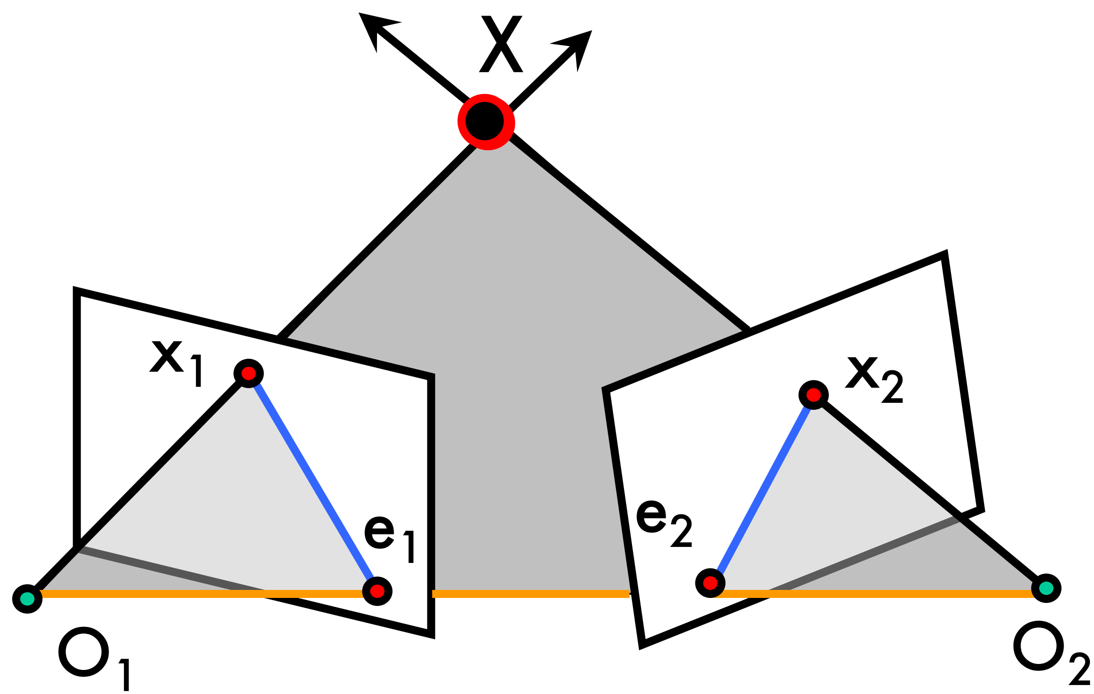
$$\mathbf{F} = [\mathbf{b}_{\times}]\mathbf{A}$$

[Eq. 11]

$$\begin{cases} \mathbf{F}\,\mathbf{b} = 0 & \text{[Eq. 12]} \\ \mathbf{b}^{\mathrm{T}}\,\mathbf{F} = 0 & \text{[Eq. 12-bis]} \end{cases}$$

What's **b**??

Epipolar Constraint [lecture 5]



$F x_2$ is the epipolar line associated with x_2 ($l_1 = F x_2$)

$F^T x_1$ is the epipolar line associated with x_1 ($l_2 = F^T x_1$)

F is singular (rank two)

$F e_2 = 0$ and $F^T e_1 = 0$

F is 3x3 matrix; 7 DOF

Interpretation of **b**

$$\mathbf{x}'^{\mathrm{T}} \mathrm{F}\, \mathbf{x} = 0$$

$$\mathbf{F} = [\mathbf{b}_{\times}]\mathbf{A}$$

[Eq. 11]

$$\begin{cases} \mathbf{F}\,\mathbf{b} = 0 \\ \mathbf{b}^{\mathrm{T}}\,\mathbf{F} = 0 \end{cases}$$

$\boxed{\textbf{b} \text{ is an epipole!}}$

$$\tilde{M}_1 = \begin{bmatrix} I & 0 \end{bmatrix} \qquad \tilde{M}_2 = \begin{bmatrix} -[\mathbf{b}_x]\mathbf{F} & \mathbf{b} \end{bmatrix}$$

$$\Downarrow \qquad\qquad \Downarrow$$

$$\tilde{M}_1 = \begin{bmatrix} I & 0 \end{bmatrix} \qquad \tilde{M}_2 = \begin{bmatrix} -[\mathbf{e}_x]\mathbf{F} & \mathbf{e} \end{bmatrix}$$

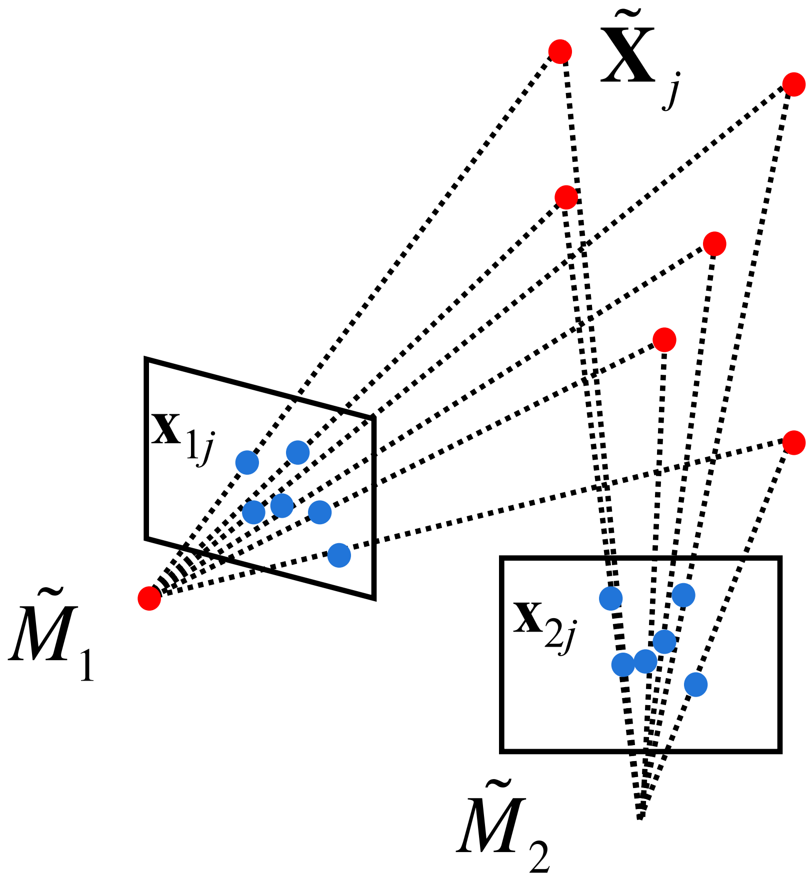
[Eq. 15] [Eq. 16]

HZ, page 254
PF, page 288

Algebraic approach (2-view case)

1. Compute the fundamental matrix F from two views (eg. 8 point algorithm)
2. Use F to estimate projective cameras
3. Use these cameras to triangulate and estimate points in 3D

Triangulation



$$x_{1j} = \tilde{M}_2 \tilde{\mathbf{X}}_j$$

$$x_{2j} = \tilde{M}_1 \tilde{\mathbf{X}}_j$$

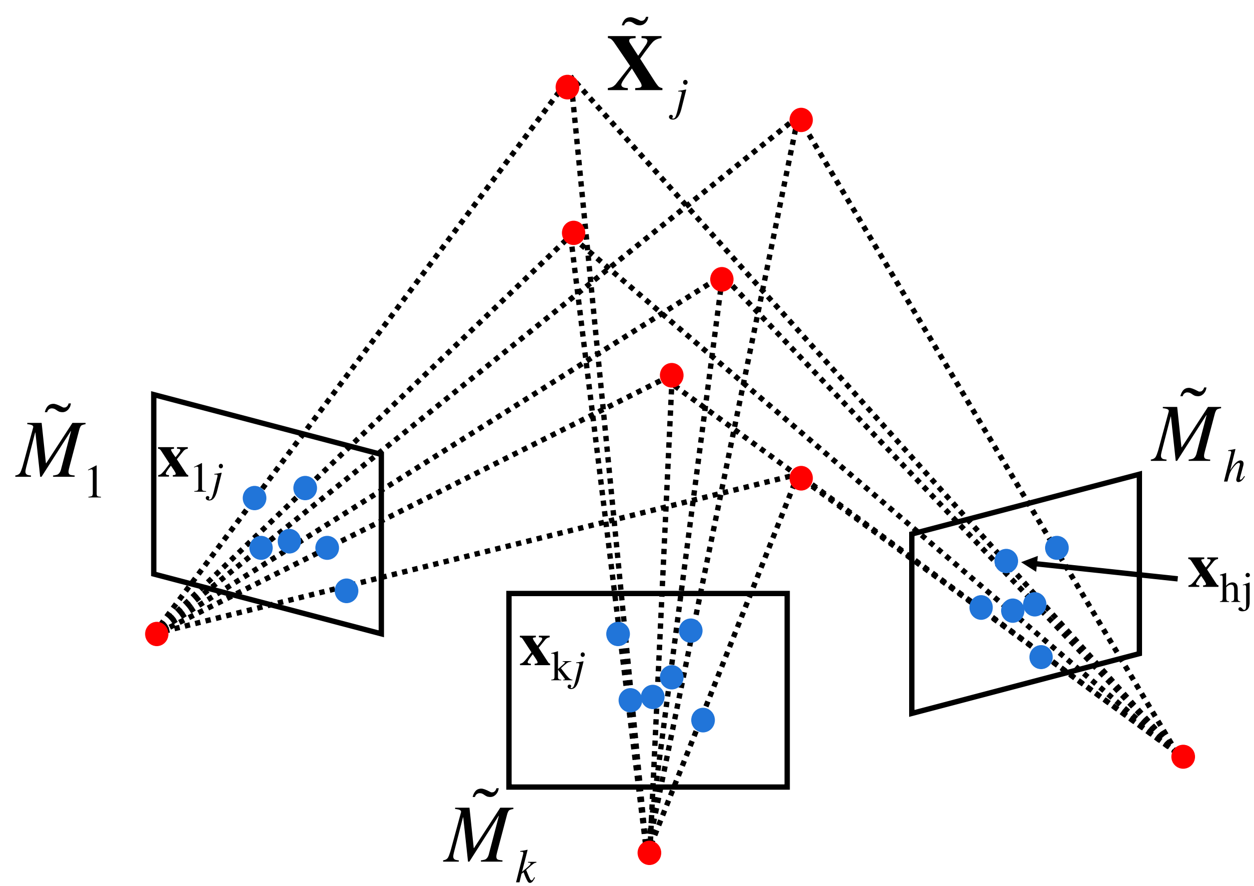
$$\tilde{M}_1 = \begin{bmatrix} I & 0 \end{bmatrix}$$

$$\tilde{M}_2 = \begin{bmatrix} -[\mathbf{e}_x]\mathbf{F} & \mathbf{e} \end{bmatrix}$$

$\rightarrow$ $\tilde{\mathbf{X}}_j$ For j = 1, … ,n

3D points can be computed from camera matrices via SVD (see page 312 of HZ for details)

Algebraic approach: the N-views case



- From I_k and I_h $\rightarrow \tilde{M}_k, \tilde{M}_h, \tilde{X}_{[k,h]}$

3D points associated to point correspondences available between I_k and I_h

- Pairwise solutions may be combined together using *bundle adjustment*

Structure-from-Motion Algorithms

- Algebraic approach (by fundamental matrix)
- Factorization method (by SVD)
- Bundle adjustment

Limitations of the approaches so far

- Factorization methods assume all points are visible.

This not true if:

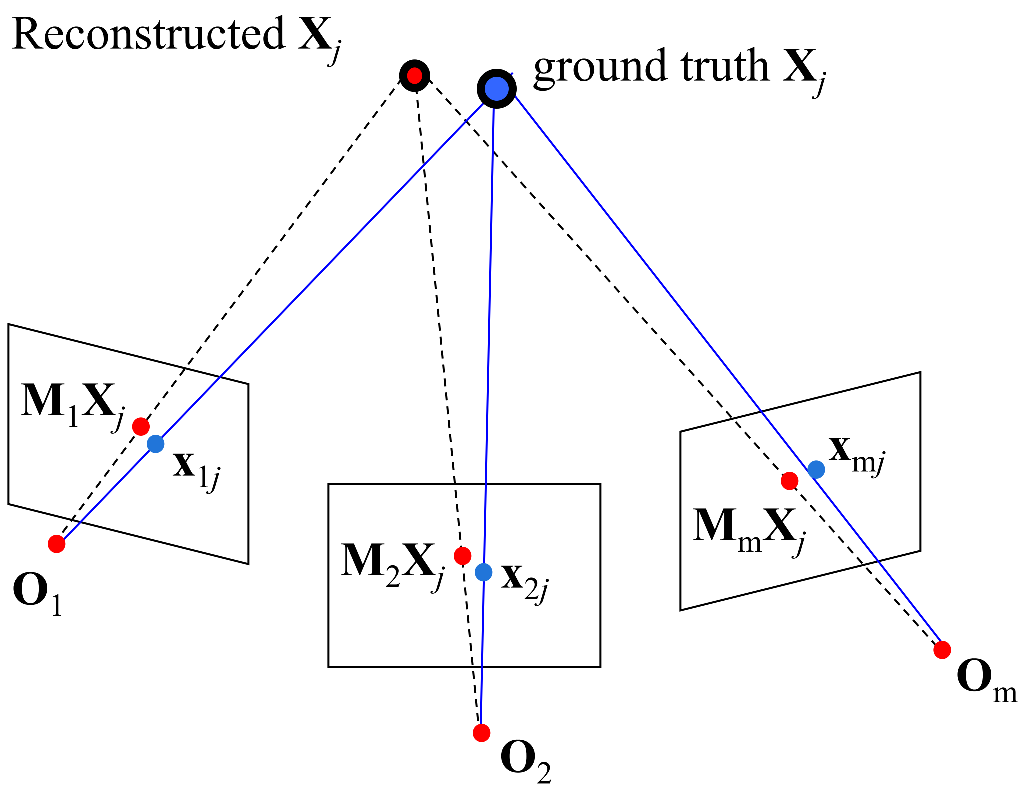
  - occlusions occur
  - failure in establishing correspondences

- Algebraic methods work with 2 views

Bundle adjustment

- Non-linear method for refining structure and motion
- Minimizes re-projection error

$$E(\mathrm{M}, \mathbf{X}) = \sum_{i=1}^{m} \sum_{j=1}^{n} D\left(\mathbf{x}_{ij}, \mathrm{M}_i \mathbf{X}_j\right)^2$$



General Calibration Problem

$$E(M, \mathbf{X}) = \sum_{i=1}^{m} \sum_{j=1}^{n} D\left(\mathbf{x}_{ij}, M_i \mathbf{X}_j\right)^2$$

parameters

measurements

D is the nonlinear mapping

- Newton Method
- Levenberg-Marquardt Algorithm
  - Iterative, starts from initial solution
  - May be slow if initial solution far from real solution
  - Estimated solution may be function of the initial solution
  - Newton requires the computation of J, H
  - Levenberg-Marquardt doesn't require the computation of H

Bundle adjustment

- Advantages
  - Handle large number of views
  - Handle missing data
- Limitations
  - Large minimization problem (parameters grow with number of views)
  - Requires good initial condition

- Used as the final step of SFM (i.e., after the factorization or algebraic approach)
- Factorization or algebraic approaches provide a initial solution for optimization problem

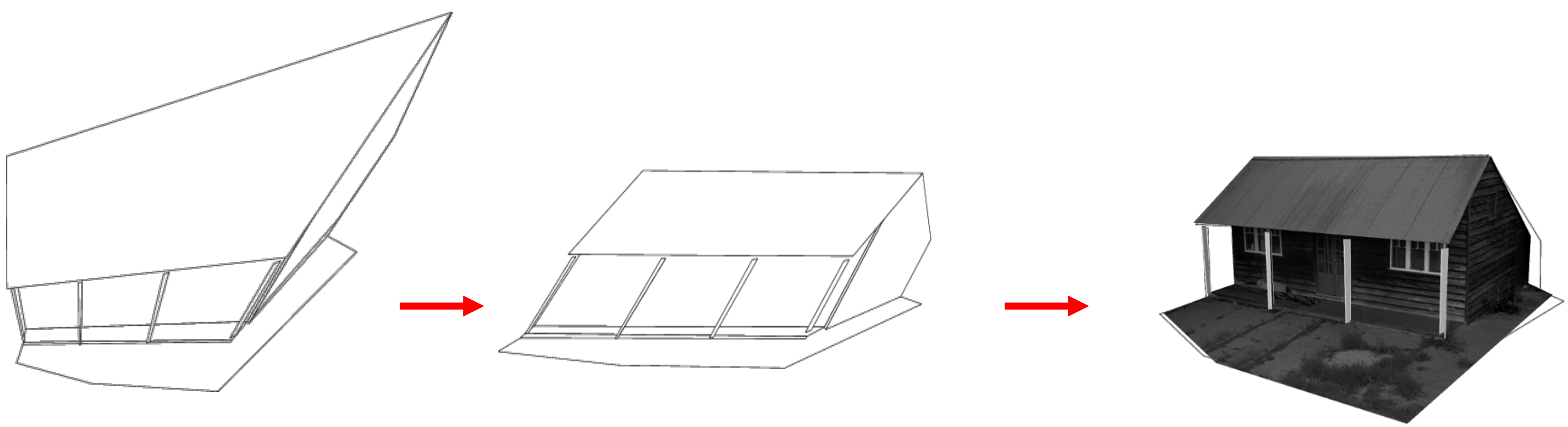
Lecture 7

Multi-view geometry

- The SFM problem
- Affine SFM
- Perspective SFM
- Self-calibration
- Applications

Self-calibration

- Self-calibration is the problem of recovering the metric reconstruction from the perspective (or affine) reconstruction
- We can self-calibrate the camera by making some assumptions about the cameras



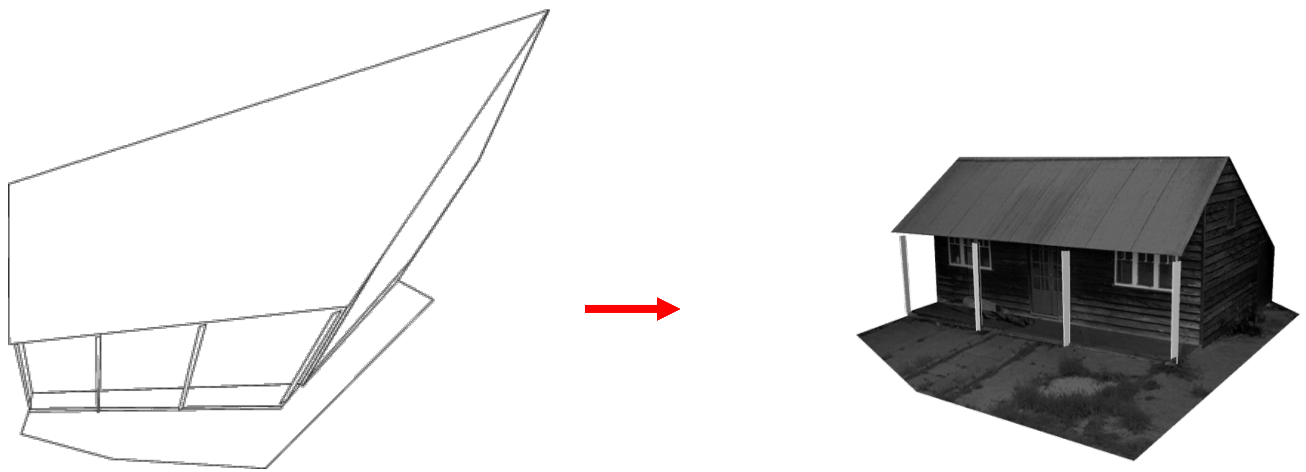
Self-calibration

[HZ] Chapters 19 "Auto-calibration"

Several approaches:

- Use single-view metrology constraints (lecture 4)
- Direct approach (Kruppa Eqs) for 2 views
- Algebraic approach
- Stratified approach

Inject information about the camera during the bundle adjustment optimization

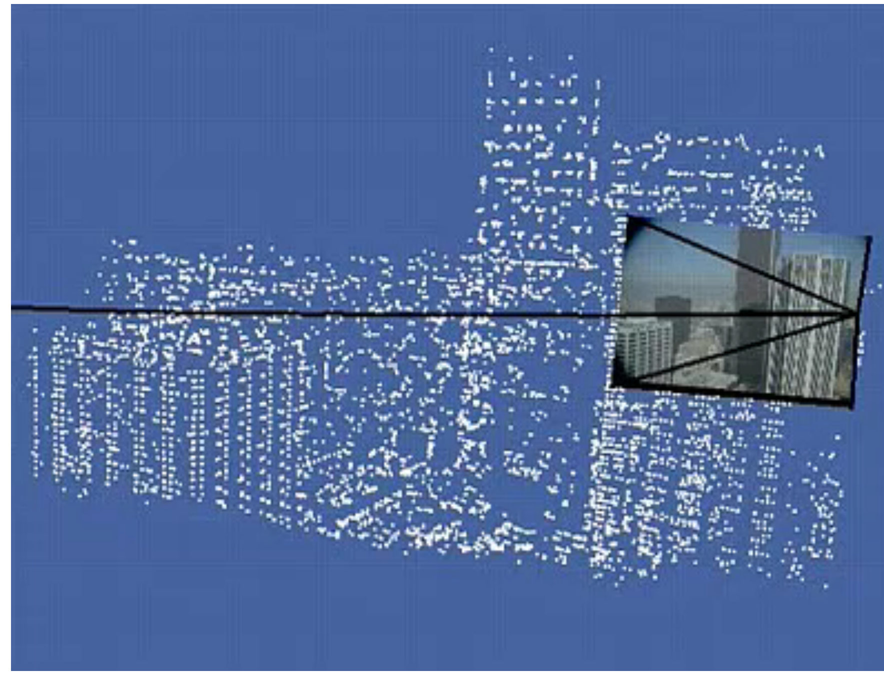


For calibrated cameras, the similarity ambiguity is the only ambiguity [Longuet-Higgins '81]

Lecture 7
Multi-view geometry

- The SFM problem
- Affine SFM
- Perspective SFM
- Self-calibration
- Applications

Structure from motion problem



Courtesy of Oxford **Visual Geometry Group**

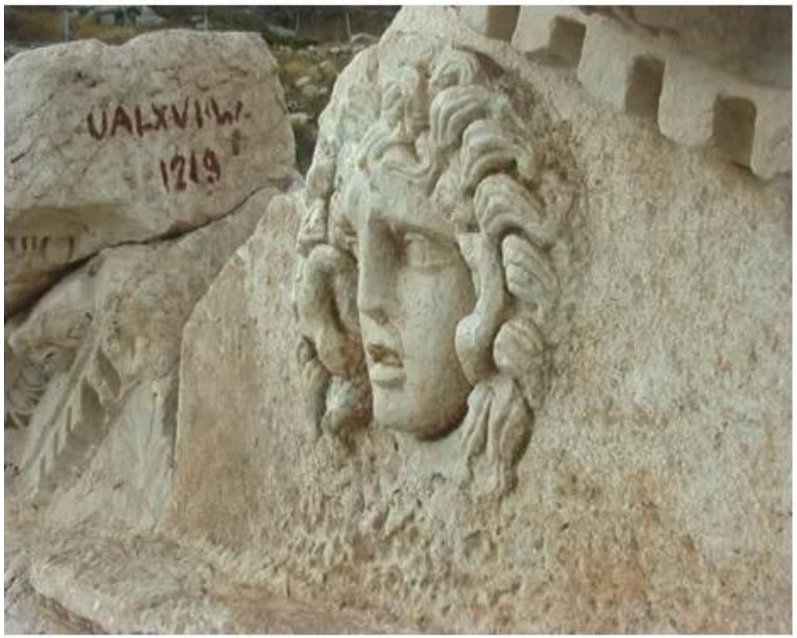
Lucas & Kanade, 81
Chen & Medioni, 92
Debevec et al., 96
Levoy & Hanrahan, 96
Fitzgibbon & Zisserman, 98
Triggs et al., 99
Pollefeys et al., 99
Kutulakos & Seitz, 99

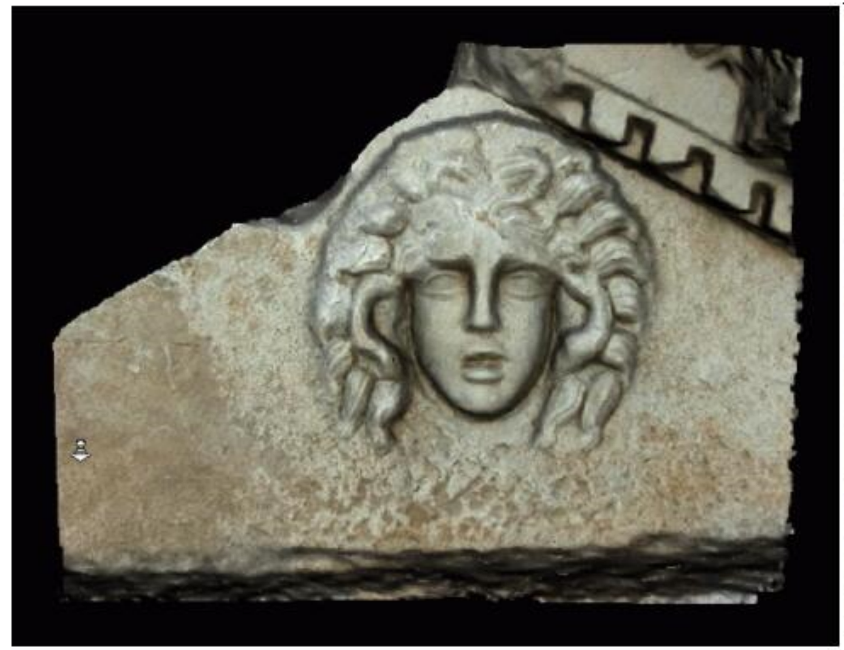
Levoy et al., 00
Hartley & Zisserman, 00
Dellaert et al., 00
Rusinkiewic et al., 02
Nistér, 04
Brown & Lowe, 04
Schindler et al, 04
Lourakis & Argyros, 04
Colombo et al. 05

Golparvar-Fard, et al. JAEI 10
Pandey et al. IFAC , 2010
Pandey et al. ICRA 2011
Microsoft's PhotoSynth
Snavely et al., 06-08
Schindler et al., 08
Agarwal et al., 09
Frahm et al., 10

Reconstruction and texture mapping

M. Pollefeys et al 98–



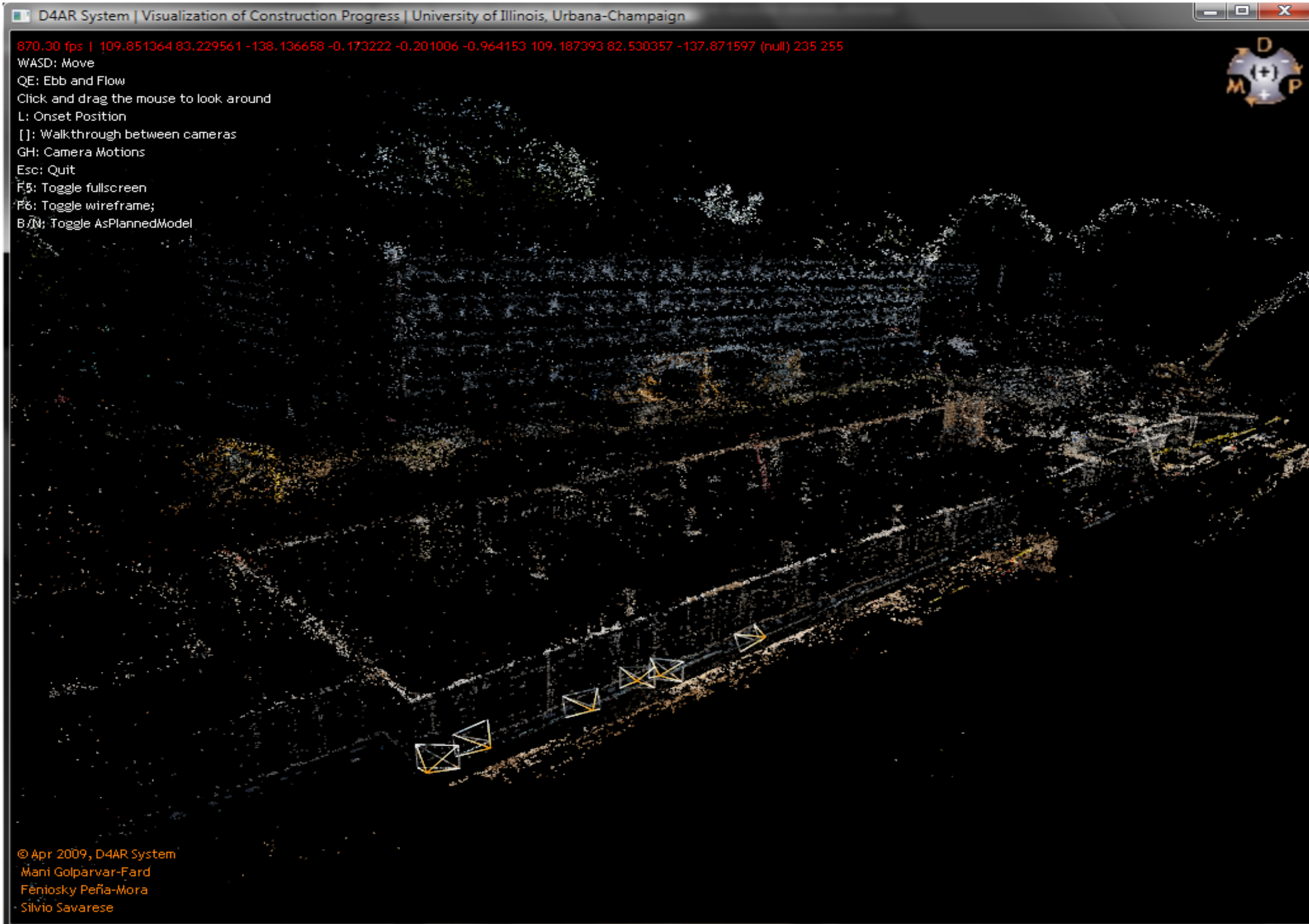


Incremental reconstruction of construction sites

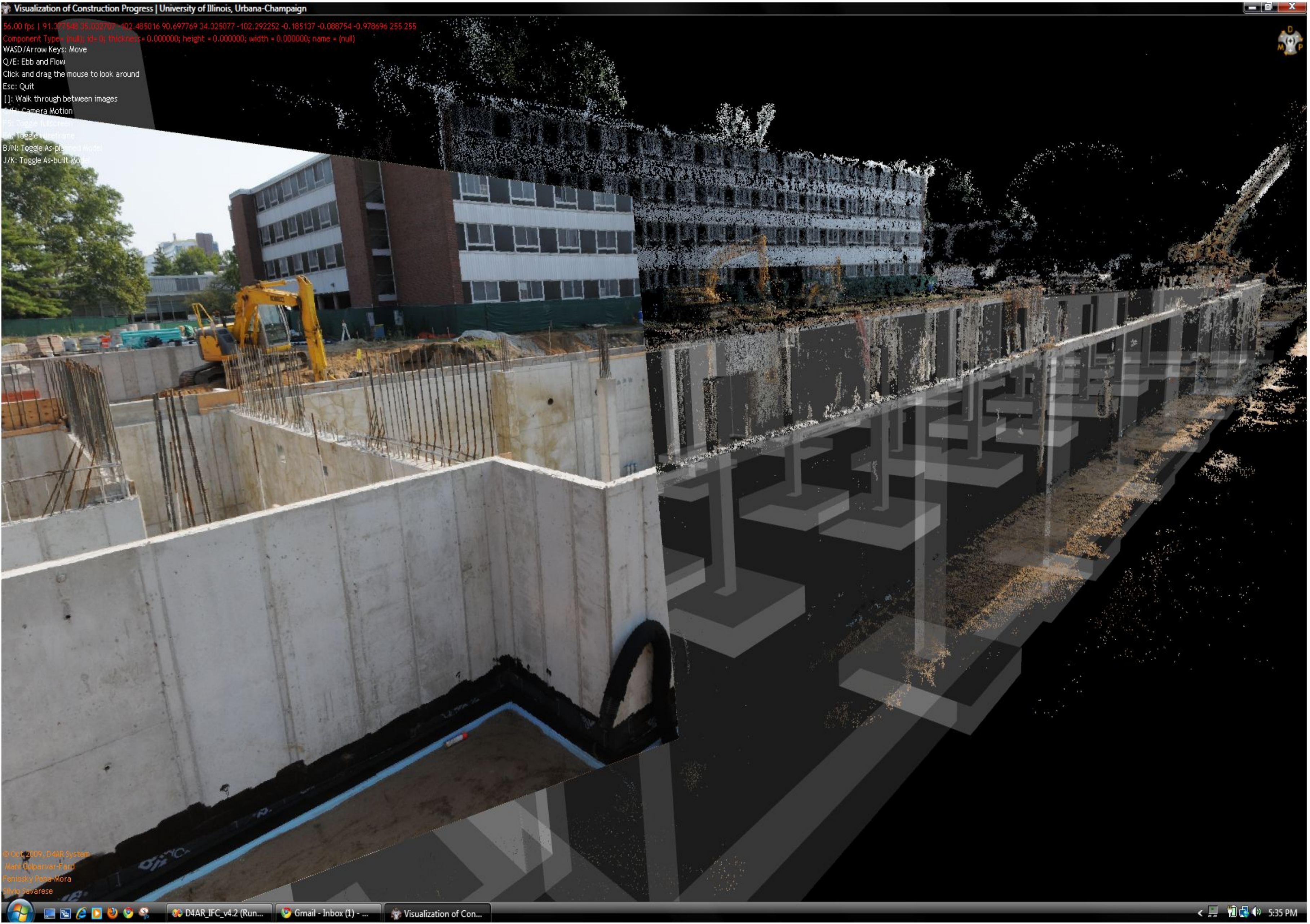
Initial pair – 2168 & Complete Set 62,323 points, 160 images

Golparvar-Fard. Pena-Mora, Savarese 2008

Reconstructed scene + Site photos



Reconstructed scene + Site photos



Results and applications

Noah Snavely, Steven M. Seitz, Richard Szeliski, "Photo tourism: Exploring photo collections in 3D," ACM Transactions on Graphics (SIGGRAPH Proceedings),2006,

Next lecture

- Fitting and Matching

Appendix

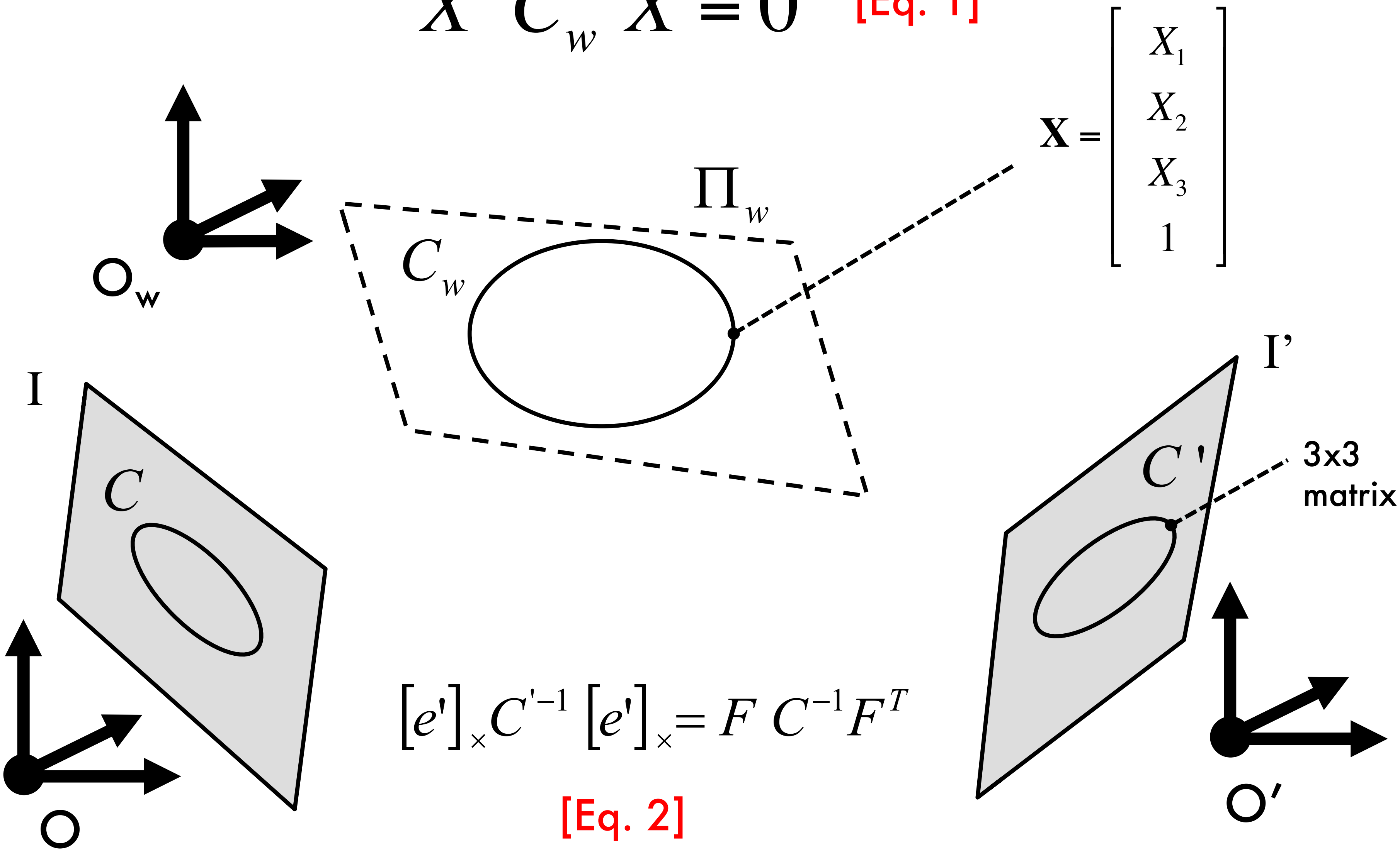
Direct approach

We use the following results:

1. A relationship that maps conics across views
2. Concept of absolute conic and its relationship to K
3. The Kruppa equations

Projections of conics across views

$$X^T C_w \; X = 0$$ [Eq. 1]

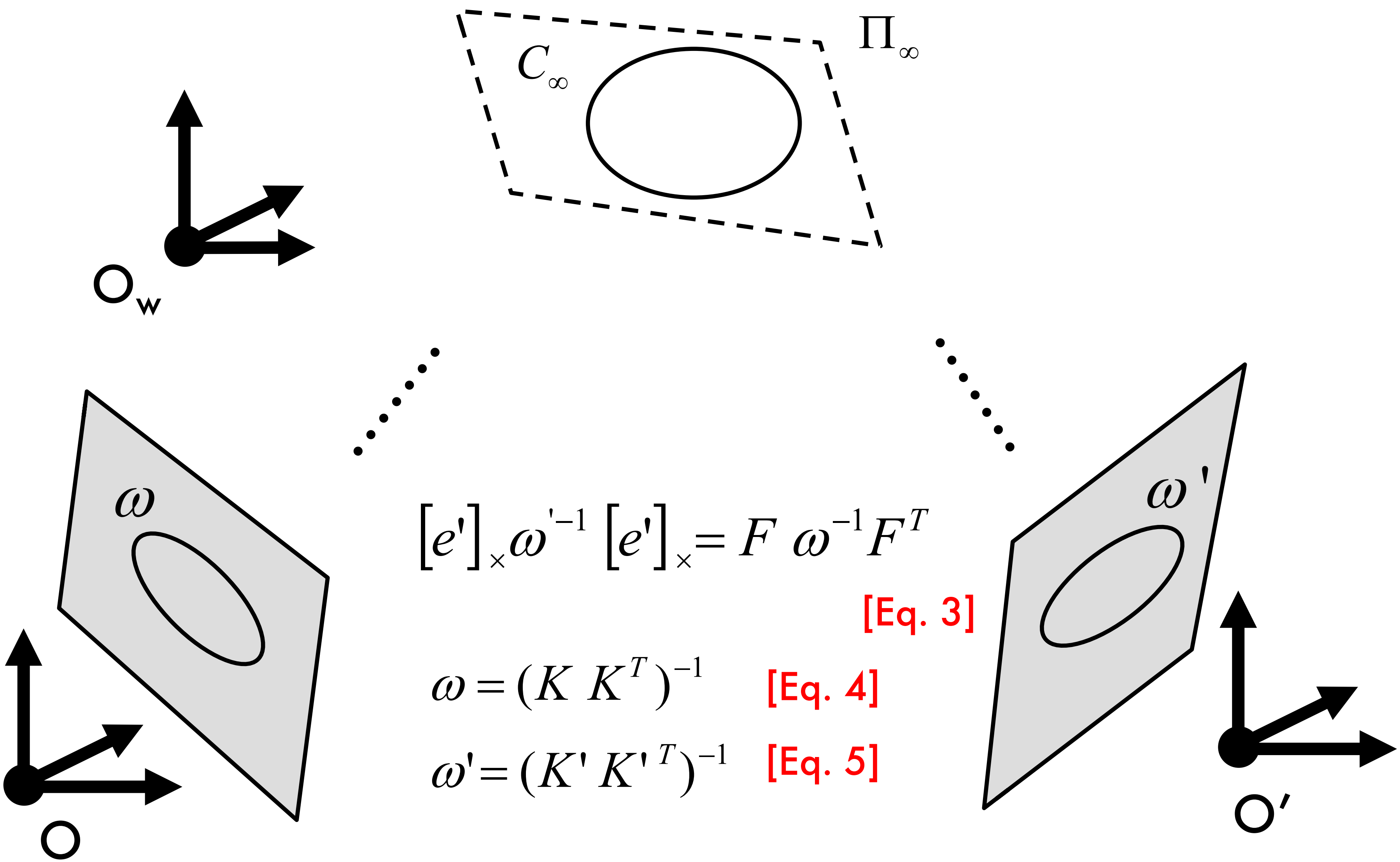


$$[e']_\times C'^{-1} \, [e']_\times = F \; C^{-1} F^T$$

[Eq. 2]

Projection of absolute conics across views

From lecture 4, [HZ] page 210, sec. 8.5.1



$$[e']_\times \omega'^{-1} [e']_\times = F \, \omega^{-1} F^T$$

[Eq. 3]

$$\omega = (K K^T)^{-1}$$ [Eq. 4]

$$\omega' = (K' K'^T)^{-1}$$ [Eq. 5]

Kruppa equations

[Faugeras et al. 92] **From [HZ] page 471**

$$\begin{pmatrix} u_2^T K' K'^T u_2 \\ -u_1^T K' K'^T u_2 \\ u_1^T K' K'^T u_1 \end{pmatrix} \times \begin{pmatrix} \sigma_1^2 \, v_1^T K \, K^T v_1 \\ \sigma_1 \sigma_2 \, v_1^T K \, K^T v_2 \\ \sigma_2^2 \, v_2^T K \, K^T v_2 \end{pmatrix} = 0$$ **[Eq. 6]**

where u_i , v_i and σ_i are the columns and singular values of SVD of F

These give us two independent constraints in the elements of K and K'

Kruppa equations

[Faugeras et al. 92]

$$\begin{pmatrix} u_2^T K' K'^T u_2 \\ -u_1^T K' K'^T u_2 \\ u_1^T K' K'^T u_1 \end{pmatrix} \times \begin{pmatrix} \sigma_1^2 v_1^T K K^T v_1 \\ \sigma_1 \sigma_2 v_1^T K K^T v_2 \\ \sigma_2^2 v_2^T K K^T v_2 \end{pmatrix} = 0$$

$$\frac{u_2^T K K^T u_2}{\sigma_1^2 v_1^T K K^T v_1} = \frac{-u_1^T K K^T u_2}{\sigma_1 \sigma_2 v_1^T K K^T v_2} = \frac{u_1^T K K^T u_1}{\sigma_2^2 v_2^T K K^T v_2}$$ **[Eq. 7]**

- Let's make the following assumption: $K' = K = \begin{pmatrix} f & 0 & 0 \\ 0 & f & 0 \\ 0 & 0 & 1 \end{pmatrix}$ **[Eq. 8]**

[Eq. 9] $\alpha f^2 + \beta f + \gamma = 0 \longrightarrow f$

Kruppa equations

[Faugeras et al. 92]

- Powerful if we want to self-calibrate 2 cameras with unknown focal length
- Limitations:
  - Work on a camera pair
  - Don’t work if R=0

[Eq. 10] $[e']_\times \omega^{-1} [e']_\times = F\ \omega^{-1} F^T$ becomes trivial

Since: $F = [e']_\times$

Self-calibration

[HZ] Chapters 19 "Auto-calibration"

Several approaches:

- Use single-view metrology constraints (lecture 4)
- Direct approach (Kruppa Eqs) for 2 views
- Algebraic approach
- Stratified approach

Algebraic approach Multi-view approach

Suppose we have a projective reconstruction $\{\tilde{M}_i, \tilde{X}_j\}$

Let H be a homography such that:

$$\begin{cases} \text{First perspective camera is canonical: } \tilde{M}_1 = [\ I \quad 0\ ] & \text{[Eq. 11]} \\ \text{i}^{\text{th}} \text{ perspective reconstruction of the camera (known): } \tilde{\mathrm{M}}_{\mathrm{i}} = [\ \mathrm{A}_{\mathrm{i}} \quad \mathrm{b}_{\mathrm{i}}\ ] & \text{[Eq. 12]} \end{cases}$$

[Eq. 13] $$\left(A_i - b_i p^T\right) K_1\, K_1^T \left(A_i - b_i p^T\right)^T = K_i\ K_i^T \qquad \text{i=2...m}$$

[Eq. 14] $$H = \begin{bmatrix} K_1 & 0 \\ -p^T\ K_1 & 1 \end{bmatrix}$$

p is an unknown 3x1 vector

$K_1 \ldots K_m$ are unknown

Algebraic approach Multi-view approach

Suppose we have a projective reconstruction

Let H be a homography such that:

$$\begin{cases} \text{First perspective camera is canonical: } \tilde{M}_1 = [\; I \quad 0 \;] & \text{[Eq. 11]} \\ \text{i}^{\text{th}} \text{ perspective reconstruction of the camera (known): } \tilde{\mathrm{M}}_{\mathrm{i}} = [\; \mathrm{A}_{\mathrm{i}} \quad \mathrm{b}_{\mathrm{i}} \;] & \text{[Eq. 12]} \end{cases}$$

[Eq. 13] $$\left(A_i - b_i p^T\right) K_1\, K_1^T \left(A_i - b_i p^T\right)^T = K_i\; K_i^T \qquad \text{i=2...m}$$

How many unknowns?

- 3 from p
- 5 m from $K_1 \ldots K_m$

How many equations? 5 independent equations [per view]

Algebraic approach Multi-view approach

Suppose we have a projective reconstruction

Let H be a homography such that:

First perspective camera is canonical: $\tilde{M}_1 = [\ I \quad 0\ ]$ [Eq. 11]

i^{th} perspective reconstruction of the camera (known): $\tilde{\mathrm{M}}_\mathrm{i} = [\ \mathrm{A}_\mathrm{i} \quad \mathrm{b}_\mathrm{i}\ ]$ [Eq. 12]

Assume all camera matrices are identical: $\mathrm{K}_1 = \mathrm{K}_2 \ldots = \mathrm{K}_\mathrm{m}$

[Eq. 15] $$\left(A_i - b_i p^T\right) K\ K^T \left(A_i - b_i p^T\right)^T = K\ \ K^T \qquad \text{i=2…m}$$

How many unknowns?
- 3 from p
- 5 from K

How many equations? 5 independent equations [per view]

We need at least 3 views to solve the self-calibration problem

Algebraic approach

Art of self-calibration:

Use assumptions on Ks to generate enough equations on the unknowns

Condition	*N. Views*
• Constant internal parameters	3
• Aspect ratio and skew known • Focal length and offset vary	4
• Skew =0, all other parameters vary	8

Issue: the larger is the number of view,
the harder is the correspondence problem

Bundle adjustment helps!

SFM problem - summary

1. Estimate structure and motion up perspective transformation
    1. Algebraic
    2. factorization method
    3. bundle adjustment
2. Convert from perspective to metric (self-calibration)
3. Bundle adjustment

** or **

1. Bundle adjustment with self-calibration constraints