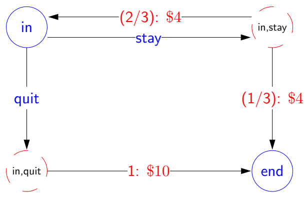
Lecture 8.1: MDPs II

Review: MDPs



Definition: Markov decision process

States: the set of states

$s_{\text{start}} \in$ States: starting state

Actions(s): possible actions from state s

$T(s, a, s')$: probability of s' if take action a in state s

Reward(s, a, s'): reward for the transition (s, a, s')

IsEnd(s): whether at end of game

$0 \le \gamma \le 1$: discount factor (default: 1)

- Last time, we talked about MDPs, which we can think of as graphs, where each node is either a state s or a chance node (s, a). Actions take us from states to chance nodes (which we choose), and transitions take us from chance nodes to states (which nature chooses according to the transition probabilities).

Review: MDPs

- Following a **policy** π produces a path (**episode**)

$$s_0; a_1, r_1, s_1; a_2, r_2, s_2; a_3, r_3, s_3; \dots; a_n, r_n, s_n$$

- **Value** function $V_\pi(s)$: expected utility if follow π from state s

$$V_\pi(s) = \begin{cases} 0 & \text{if IsEnd}(s) \\ Q_\pi(s, \pi(s)) & \text{otherwise.} \end{cases}$$

- **Q-value** function $Q_\pi(s, a)$: expected utility if first take action a from state s and then follow π

$$Q_\pi(s, a) = \sum_{s'} T(s, a, s')[\text{Reward}(s, a, s') + \gamma V_\pi(s')]$$

- Given a policy π and an MDP, we can run the policy on the MDP yielding a sequence of states, action, rewards $s_0; a_1, r_1, s_1; a_2, r_2, s_2; \dots$. Formally, for each time step t, $a_t = \pi(s_{t-1})$, and s_t is sampled with probability $T(s_{t-1}, a_t, s_t)$. We call such a sequence an **episode** (a path in the MDP graph). This will be a central notion in this lecture.
- Each episode (path) is associated with a utility, which is the discounted sum of rewards: $u_1 = r_1 + \gamma r_2 + \gamma^2 r_3 + \cdots$. It's important to remember that the utility u_1 is a **random variable** which depends on how the transitions were sampled.
- The value of the policy (from state s_0) is $V_\pi(s_0) = \mathbb{E}[u_1]$, the expected utility. In the last lecture, we worked with the values directly without worrying about the underlying random variables (but that will soon no longer be the case). In particular, we defined recurrences relating the value $V_\pi(s)$ and Q-value $Q_\pi(s, a)$, which represents the expected utility from starting at the corresponding nodes in the MDP graph.
- Given these mathematical recurrences, we produced algorithms: policy evaluation computes the value of a policy, and value iteration computes the optimal policy.

Roadmap

Tram problem

Value iteration

Transportation example

Example: transportation

Street with blocks numbered 1 to n.
Walking from s to $s+1$ takes 1 minute.
Taking a magic tram from s to $2s$ takes 2 minutes.
How to travel from 1 to n in the least time?
Tram fails with probability 0.5.

[live solution]

- Let us revisit the transportation example. As we all know, magic trams aren't the most reliable forms of transportation, so let us asume that with probability $\frac{1}{2}$, it actually does as advertised, and with probability $\frac{1}{2}$ it just leaves you in the same state.

Evaluating a policy

Definition: utility

Following a policy yields a **random path**.
The **utility** of a policy is the (discounted) sum of the rewards on the path (this is a random quantity).

Path	Utility
[in; stay, 4, end]	4
[in; stay, 4, in; stay, 4, end]	8
[in; stay, 4, in; stay, 4, in; stay, 4, in; stay, 4, end]	16
...	...

Definition: value (expected utility)

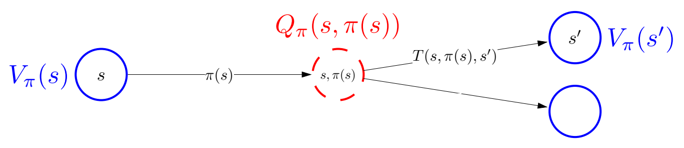
The **value** of a policy is the **expected** utility.

[live solution]

- Recall that we measure how good a policy is by computing the value, or expected utility of the policy. First, I'll show a simple way to estimate this approximately: just simulate lots of episodes and take the average utility.

Policy evaluation

Plan: define recurrences relating value and Q-value



$$V_\pi(s) = \begin{cases} 0 & \text{if IsEnd}(s) \\ Q_\pi(s,\pi(s)) & \text{otherwise.} \end{cases}$$

$$Q_\pi(s,a) = \sum_{s'} T(s,a,s')[\text{Reward}(s,a,s') + \gamma V_\pi(s')]$$

Policy evaluation

Key idea: iterative algorithm

Start with arbitrary policy values and repeatedly apply recurrences to converge to true values.

Algorithm: policy evaluation

```
Initialize $V_\pi^{(0)}(s) \leftarrow 0$ for all states $s$.
For iteration $t = 1, \dots, t_{\text{PE}}$:
    For each state $s$:
        $V_\pi^{(t)}(s) \leftarrow \underbrace{\sum_{s'} T(s,\pi(s),s')[\text{Reward}(s,\pi(s),s') + \gamma V_\pi^{(t-1)}(s')]}_{Q^{(t-1)}(s,\pi(s))}$
```

[live solution]

- Now we'll see how to implement policy evaluation, which takes advantages of recurrences to converge much faster to the value.

Roadmap

Tram problem

Value iteration

- If we are given a policy π, we now know how to compute its value $V_\pi(s_{\text{start}})$. So now, we could just enumerate all the policies, compute the value of each one, and take the best policy, but the number of policies is exponential in the number of states (A^S to be exact), so we need something a bit more clever.
- We will now introduce value iteration, which is an algorithm for finding the best policy. While evaluating a given policy and finding the best policy might seem very different, it turns out that value iteration will look a lot like policy evaluation.

Optimal value and policy

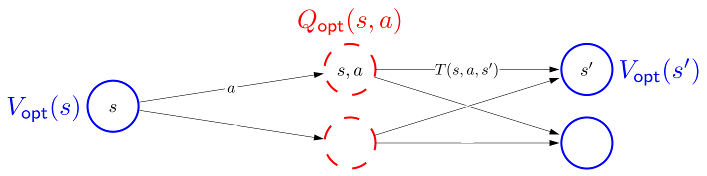
Goal: try to get directly at maximum expected utility

Definition: optimal value

The **optimal value** $V_{\text{opt}}(s)$ is the maximum value attained by any policy.

- We will write down a bunch of recurrences which look exactly like policy evaluation, but instead of having V_π and Q_π with respect to a fixed policy π, we will have V_{opt} and Q_{opt}, which are with respect to the optimal policy.

Optimal values and Q-values



Optimal value if take action a in state s:

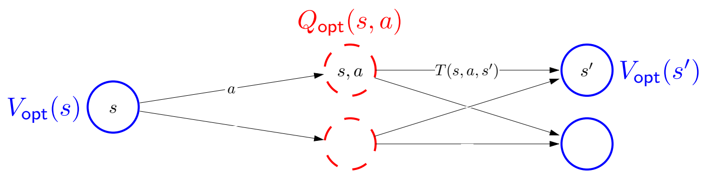
$$Q_{\text{opt}}(s,a) = \sum_{s'} T(s,a,s')[\text{Reward}(s,a,s') + \gamma V_{\text{opt}}(s')].$$

Optimal value from state s:

$$V_{\text{opt}}(s) = \begin{cases} 0 & \text{if IsEnd}(s) \\ \max_{a \in \text{Actions}(s)} Q_{\text{opt}}(s,a) & \text{otherwise.} \end{cases}$$

- The recurrences for V_{opt} and Q_{opt} are identical to the ones for policy evaluation with one difference: in computing V_{opt}, instead of taking the action from the fixed policy π, we take the best action, the one that results in the largest $Q_{\text{opt}}(s,a)$.

Optimal policies



Given Q_{opt}, read off the optimal policy:

$$\pi_{\text{opt}}(s) = \arg\max_{a \in \text{Actions}(s)} Q_{\text{opt}}(s,a)$$

- So far, we have focused on computing the value of the optimal policy, but what is the actual policy? It turns out that this is pretty easy to compute.
- Suppose you're at a state s. $Q_{\text{opt}}(s,a)$ tells you the value of taking action a from state s. So the optimal action is simply to take the action a with the largest value of $Q_{\text{opt}}(s,a)$.

Value iteration

Algorithm: value iteration [Bellman, 1957]

```
Initialize $V_{\text{opt}}^{(0)}(s) \leftarrow 0$ for all states $s$.
For iteration $t = 1, \dots, t_{\text{VI}}$:
    For each state $s$:
        $V_{\text{opt}}^{(t)}(s) \leftarrow \max_{a \in \text{Actions}(s)} \underbrace{\sum_{s'} T(s,a,s')[\text{Reward}(s,a,s') + \gamma V_{\text{opt}}^{(t-1)}(s')]}_{Q_{\text{opt}}^{(t-1)}(s,a)}$
```

Time: $O(t_{\text{VI}} SAS')$

[live solution]

- By now, you should be able to go from recurrences to algorithms easily. Following the recipe, we simply iterate some number of iterations, go through each state s and then replace the equality in the recurrence with the assignment operator.
- Value iteration is also guaranteed to converge to the optimal value.
- What about the optimal policy? We get it as a byproduct. The optimal value $V_{\text{opt}}(s)$ is computed by taking a max over actions. If we take the argmax, then we get the optimal policy $\pi_{\text{opt}}(s)$.

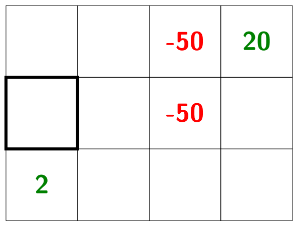
Value iteration: dice game

s	end	in	
$V_{\text{opt}}^{(t)}$	0.00	12.00	($t = 100$ iterations)
$\pi_{\text{opt}}(s)$	-	stay	

- Let us demonstrate value iteration on the dice game. Initially, the optimal policy is "quit", but as we run value iteration longer, it switches to "stay".

Value iteration: volcano crossing

Run (or press ctrl-enter)



- As another example, consider the volcano crossing. Initially, the optimal policy and value correspond to going to the safe and boring 2. But as you increase `numIters`, notice how the value of the far away 20 propagates across the grid to the starting point.
- To see this propagation even more clearly, set `slipProb` to 0.

Convergence

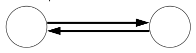
Theorem: convergence

Suppose either
- discount $\gamma < 1$, or
- MDP graph is acyclic.

Then value iteration converges to the correct answer.

Example: non-convergence

discount $\gamma = 1$, zero rewards



- Let us state more formally the conditions under which any of these algorithms that we talked about will work. A sufficient condition is that either the discount γ must be strictly less than 1 or the MDP graph is acyclic.
- We can reinterpret the discount $\gamma < 1$ condition as introducing a new transition from each state to a special end state with probability $(1-\gamma)$, multiplying all the other transition probabilities by γ, and setting the discount to 1. The interpretation is that with probability $1-\gamma$, the MDP terminates at any state.
- In this view, we just need that a sampled path be finite with probability 1.
- We won't prove this theorem, but will instead give a counterexample to show that things can go badly if we have a cyclic graph and $\gamma = 1$. In the graph, whatever we initialize value iteration, value iteration will terminate immediately with the same value. In some sense, this isn't really the fault of value iteration, but it's because all paths are of infinite length. In some sense, if you were to simulate from this MDP, you would never terminate, so we would never find out what your utility was at the end.

Summary of algorithms

- Policy evaluation: (MDP, π) $\rightarrow V_\pi$

- Value iteration: MDP $\rightarrow (V_{\text{opt}}, \pi_{\text{opt}})$

Unifying idea

Algorithms:

- Search DP computes FutureCost(s)
- Policy evaluation computes policy value $V_\pi(s)$
- Value iteration computes optimal value $V_{\text{opt}}(s)$

Recipe:

- Write down recurrence (e.g., $V_\pi(s) = \cdots V_\pi(s') \cdots$)
- Turn into iterative algorithm (replace mathematical equality with assignment operator)

- There are two key ideas in this lecture. First, the policy π, value V_π, and Q-value Q_π are the three key quantities of MDPs, and they are related via a number of recurrences which can be easily gotten by just thinking about their interpretations.
- Second, given recurrences that depend on each other for the values you're trying to compute, it's easy to turn these recurrences into algorithms that iterate between those recurrences until convergence.

Summary

- **Markov decision processes** (MDPs) cope with uncertainty
- Solutions are **policies** rather than paths
- **Policy evaluation** computes policy value (expected utility)
- **Value iteration** computes optimal value (maximum expected utility) and optimal policy
- Main technique: write recurrences $\rightarrow$ algorithm
- Next time: reinforcement learning — when we don't know rewards, transition probabilities

Value iteration

Algorithm: value iteration [Bellman, 1957]

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Initialize $V_{\text{opt}}^{(0)}(s) \leftarrow 0$ for all states $s$.
For iteration $t = 1, \ldots, t_{\text{VI}}$:
    For each state $s$:
        $V_{\text{opt}}^{(t)}(s) \leftarrow \max_{a \in \text{Actions}(s)} \underbrace{\sum_{s'} T(s,a,s')[\text{Reward}(s,a,s') + \gamma V_{\text{opt}}^{(t-1)}(s')]}_{Q_{\text{opt}}^{(t-1)}(s,a)}$
```

Time: $O(t_{\text{VI}} SAS')$

[live solution]