6) (a) $y^t P y = y^t A^T A y$

$= (Ay)^T (Ay)$

$A \Rightarrow (m \times n)$ matrix

$y \Rightarrow (n \times 1)$ vector

So, $Ay \Rightarrow (m \times 1)$ vector

The square of the l-2 norm of a vector (z) is defined as $z^T z$

Here $z = Ay$

$\Rightarrow y^T P y =$ L-2 norm of (Ay)

From the property of L2-norms,

L2-norm ≥ 0

$\Rightarrow (Ay)^T (Ay) \geq 0$.

Similarly, $z^t Q z = \cancel{z^T Q} \; z^T A A^T z$

$= (A^T z)^T (A^T z)$

$=$ L2-norm of $(A^T z)$

≥ 0

Now, consider the eigen values - λ of P

$Pv = \lambda v$

$\Rightarrow v^T P v = \lambda v^T v$

$\Rightarrow \|Av\|_2^2 = \lambda \|v\|_2^2$

as $\|Av\|_2^2 \geq 0$ & $\|v\|_2^2 \geq 0$, $\lambda \geq 0$

$\therefore$ eigen values are non negative

Similarly, consider an eigen value λ of Q,

$$Qv = \lambda v$$

$$\Rightarrow v^T Q v = \lambda (v^T v)$$

$$\Rightarrow \|A^T v\|_2^2 = \lambda \|v\|_2^2$$

By the same argument as before eigen values of Q are non-negative

(b) $Au = \lambda u$

$A^T A u = \lambda u$

$\Rightarrow A(A^T A u) = \lambda (Au)$

using associative property,

$$(A A^T)(Au) = \lambda (Au)$$

$$Q(Au) = \lambda (Au)$$

So, Au is an eigen vector of Q with eigen value λ.

Now,

$$Qv = \mu v$$

$$AA^T v = \mu v$$

$$A^T (AA^T v) = \mu A^T v$$

using associative property,

$$(A^T A)(A^T v) = \mu (A^T v)$$

$$P(A^T v) = \mu (A^T v)$$

So, $A^T v$ is an eigenvector of P with eigenvalue μ.

Now, coming to the no. of elements,

$A \Rightarrow (m \times n)$ matrix

$u \Rightarrow (y \times 1)$ vector

for Au to be defined, $y = n$ is required

So, $u \Rightarrow (n \times 1)$ vector – It has 'n' elements

Similarly $A^T \Rightarrow (n \times m)$ matrix

$v \Rightarrow (x \times 1)$ vector

$x = m$ for $A^T v$ to be defined

So, $v \Rightarrow (m \times 1)$ vector – It has 'm' elements

(c) $Qv_i = \lambda_i v_i$

$\Rightarrow AA^T v_i = \lambda_i v_i$

$\Rightarrow A(A^T v_i) = \lambda_i v_i$

$\Rightarrow A \dfrac{(A^T v_i)}{\|A^T v_i\|_2} = \left(\dfrac{\lambda_i}{\|A^T v_i\|_2}\right) v_i$

$\Rightarrow A u_i = \left(\dfrac{\lambda_i}{\|A^T v_i\|_2}\right) v_i$

Taking $\gamma_i = \dfrac{\lambda_i}{\|A^T v_i\|_2}$, we show there there indeed exists γ_i such that $Au_i = \lambda_i v_i$

(d) ~~$Pu_i = \lambda u_i$~~ & ~~$Qv_i = \lambda_i v_i$~~

$$Au_i = \gamma_i v_i$$

$$\Rightarrow AV = \underbrace{A[u_1 | u_2 | \ldots | u_m]}$$

Block matrix multiplication is valid as sizes of A & u_i match & is applied

$$= [Au_1 | Au_2 | Au_3 | \ldots | Au_m]$$

$$= [\gamma_1 v_1 | \gamma_2 v_2 | \ldots | \gamma_m v_m]$$

$$= \underbrace{[v_1 | v_2 | \ldots | v_m]}_{U} \underbrace{\begin{bmatrix} \gamma_1 & 0 & \cdots & 0 \\ 0 & \gamma_2 & \cdots & 0 \\ 0 & 0 & \ddots & \\ \vdots & \vdots & & \\ 0 & 0 & & \end{bmatrix}}_{\Gamma}$$

$$AV = U\Gamma$$

$$\Rightarrow A(VV^T) = U\Gamma V^T$$

$$VV^T = [u_1 | u_2 | \ldots | u_m] \begin{bmatrix} u_1^T \\ \hline u_2^T \\ \hline \vdots \\ \hline u_m^T \end{bmatrix} = I_n$$ (from lecture slides)

$$\Rightarrow A I_n = U\Gamma V^T$$

~~(using block matrix multiplication)~~

$$\therefore A = U\Gamma V^T$$