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ON THE FREQUENCY OF 3-CONNECTED SUBGRAPHS OF PLANAR GRAPHS

NICHOLAS C. WORMALD

The concept of dependence of subgraphs of a plane graph is defined, as a measure of how much they overlap. It is shown that if M is a 3-connected plane graph, then the number of copies of M in a plane graph which are dependent on a given copy is bounded above by a constant $c(M)$. The number of copies of M in any n-vertex plane graph is at most $nc(M)$.

1. Introduction.

Hakimi and Schmeichel [2] gave bounds on the number of k-cycles in a maximal planar graph with n vertices. Let $f(n,G)$ denote the maximum number of copies of a graph G (that is, subgraphs isomorphic to G) occuring in any planar graph on n vertices. Alon and Caro [1] studied $f(n,G)$ for various G, determining it precisely if G is complete bipartite or a maximal planar graph with no non-facial triangles, and obtaining bounds for other triangulations G. From their results it follows that $f(n,G)$ is less than $12n$ for any triangulation G. The main object of this article is to prove the conjecture, attributed in [1] to Perles, that $f(n,G) \le c_G n$ for all 3-connected planar graphs G

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and all n. Here and in what follows, c_G denotes a positive constant depending only on the graph G, perhaps different at different occurrences of the notation, even within the same formula. It is interesting to compare this with the results of [4], which state that if F is one of four sets of triangulations (2-connected or 3-connected or 2-face-colourable 2-or 3-connected), then almost all n-vertex elements of F contain at least $c_G n$ copies of G for any particular $G \in F$.

On the way to deriving the main result in Section 3, we find that the number of copies of G which overlap a given copy (in a certain sense) is at most c_G. In Section 4, bounds on this constant are given if G is a wheel, and a general bound is conjectured.

2. Notation.

Unless otherwise specified, G denotes a 3-connected planar graph and $\nu(G)$ is the cardinality of its vertex set. Our graphs have no loops or multiple edges. A plane graph is as usual a proper embedding of a planar graph in the plane, and its faces are the connected components of the plane remaining when the vertices and edges are removed. In a 2-connected plane graph, the cycle of vertices and edges around the boundary of a face is called the bounding cycle of that face; the 2-connectedness ensures that no vertex appears at two different places in the bounding cycle. The bounding cycle of the unbounded face of a plane graph M is called the bounding cycle of M. The vertices of M not on its bounding cycle are called interior vertices. Also, M is internally 3-connected if it is 2-connected and removal of any two or fewer vertices (together with incident edges) leaves every remaining vertex of M connected (by a path along vertices and edges) to at least one vertex in the bounding cycle of M.

By an embedding of a plane graph M in a plane graph N, we mean an isomorphism from the underlying graph of M to a subgraph of the underlying graph of N, which is induced by a homeomorphism of the plane from M to N. We may regard the embedding as a superposition of M onto N, and accordingly we may speak of the vertices and faces of the embedding. Given two embeddings M_1 and M_2 of a plane graph in a plane graph, the distribution of the interior vertices of M_2 into the faces and vertices of M_1 determines a function between these two sets. We denote this function by $D(M_1, M_2)$.

3. The main results.

First a lemma is given which is the key to the main result. Then in Theorem 1 the number of copies of a plane graph which can overlap a given copy is bounded. Theorem 2 relates this bound to the maximum number of copies of G is a plane graph.

LEMMA. *Let M be an internally 3-connected plane graph, and suppose an embedding of the bounding cycle of M in a plane graph N is specified. Then there are at most c_M different ways to complete this embedding to an embedding of M of N.*

Proof. This is by induction on the number, say m , of interior vertices of M . If $m = 0$ then the result is immediate, since it only remains to map certain edges of M to edges of N , and since our graphs have no multiple edges this can be done in at most one way. So suppose $m \geq 1$.

Let M_1 and M_2 denote two distinct completions of the embedding of the bounding cycle to an embedding of M . Then the number of possibilities for $D(M_1, M_2)$, for a given M_1 , is at most c_M . Also, for any face F of M_1 , the portion of M_2 lying within F , together with the bounding cycle of F , determine a plane graph $P(F)$ which is clearly internally 3-connected. Given M_1 and $D(M_1, M_2)$, the isomorphism type of $P(F)$ and the embedding of its bounding cycle in N are determined. Since M is internally 3-connected, not all of its interior vertices can lie in F . Hence $P(F)$ has fewer than m internal vertices, and so by induction there are at most $c_{P(F)}$ distinct ways to complete the embedding of the bounding cycle of $P(F)$ to an embedding of $P(F)$ in N. Since this holds for each face F , it follows that there are at most c_M ways to complete the embedding of M_2 , given $D(M_1,M_2)$. Hence, given M_1 , the number of possibilities for M_2 is at most c_M , and the lemma follows. □

We say that two embeddings M_1 and M_2 of plane graphs in a plane graph are <u>independent</u> (of each other) if the vertices of one are contained entirely in the unbounded face and bounding cycle of the other. Otherwise they are <u>dependent</u> (on each other).

THEOREM 1. *For each 3-connected plane graph* M *, there is a constant* $r(M)$ *with the following property: For any plane graph* N *and any embedding* M_1 *of* M *in* N *, there are at most* $r(M)$ *distinct embeddings of* M *in* N *which are dependent on* M_1 *.*

Proof. Let M_2 be an embedding of M in N dependent on M_1 . Then by the 3-connectedness of M , for each face F of M_1 , the portion of M_2 lying within F , together with the bounding cycle of F , can be assumed to determine a plane graph $P(F)$ which is internally 3-connected. (This is not immediate when F is the unbounded face of M_1 . In this case, we can either re-define "internally 3-connected" appropriately or argue that with regard to the examination of F , M_1 can be projected onto the sphere and then re-projected onto the plane so that F becomes bounded and the assumption holds.) Hence, it now follows by the Lemma that, given M_1 and $D(M_1,M_2)$, there are at most $c_{P(F)}$ ways to complete the embedding of $P(F)$. Since this holds for each face F of M_1 , there are at most c_M ways to choose M_2 . As the number of potential $D(M_1,M_2)$ is at most c_M , the theorem follows. □

If $\nu(G) = p$ then G can be embedded in the plane in at most $4(p-2)$ ways. This is because it has at most $2(p-2)$ faces, and each face can be the unbounded face in at most two distinct ways in view of Whitney's theorem [6] that G is uniquely embeddable in the sphere. (For our present purposes, we regard the plane as possessing an orientation.) Let $r(G)$ denote the sum of $r(M)$ over all the different embeddings of G in the plane. An upper bound on $f(n,G)$ can now be given as follows.

THEOREM 2. *If* $\nu(G) = p$ *then* $f(n,G) \le (n-p+1)r(G)$ *.*

Proof. Let H be a planar graph with $\nu(H) = n$ and let N be any proper planar embedding of H. Then each copy of G in H corresponds to an embedding of some plane graph M in N, where the underlying graph of M is G. We bound the maximum number, say $g(n,M)$, of embeddings of M in <u>any</u> plane graph with n vertices, and then sum over all plane graphs M with underlying graph G to bound $f(n,G)$.

Let M_1 denote an embedding of M in N, and denote the faces of M_1 by $F_1,\ldots,F_f$. For $i = 1,\ldots, f$, let k_i denote the number of vertices of N lying in F_i or on its bounding cycle. Then the number of embeddings of M in N which are independent of M_1 is at most $\sum g(k_i,M)$. This summation and the following ones are for $i = 1,\ldots, f$ unless otherwise indicated.

As M is 3-connected, we have $k_i \le n-1$ for each i. Let $t_1,\ldots, t_f$ denote the valencies of $F_1,\ldots, F_f$ respectively. Then $k_i \ge t_i$ for each i, and as M is 3-connected, $p > t_i$ for each i and so $g(k_i,M) = 0$ if $k_i = t_i$. Also $\sum (k_i - t_i) = n-p$. Thus by Theorem 1,

$$g(n,M) \le r(M) + \max \sum g(k_i,M)$$

where the maximum is over all sequences $k_1,\ldots,k_f$ for which $\sum k_i = n-p+ \sum t_i$, and $k_i \ge t_i$ for each i. Let $t = \max t_i$, so $t \le p-1$, and say $t_w = t$. Define a function d on the non-negative integers by $d(j) = 0$ for $j < p$ and

$$d(j) = r(M) + \max_j \sum d(k_i) \tag{3.1}$$

for $j \ge p$, where the maximum of the empty set is taken as 0, and $\max_j$ denotes the maximum over all sequences $k_1,\ldots,k_f$ with $k_i \ge t_i$ for each i and $\sum k_i \le j-t-1 + \sum t_i$.

Then

$$d(j) \ge g(j,M) . \tag{3.2}$$

For $j \ge p$ we have

$$\begin{aligned} d(j) - d(j-1) &\geq \max_j \sum d(k_i) - \max_{j-1} \sum d(k_i) \\ &\geq \max_{j-1}(d(k_w+1) + \sum_{i \neq w} d(k_i) - \sum d(k_i)) \\ &= \max_{j-1}(d(k_w+1) - d(k_w)) \\ &\geq d(j-1) - d(j-2) \end{aligned}$$

since if $k_i = t_i$ $(i \neq w)$ and $k_w = j-2$, then $\sum k_i = j-t-2+\sum t_i$. Hence d is convex, and it follows that the maximum in (3.2) is achieved when $k_w = j-1$ and $k_i = t_i$ otherwise, and is thus equal to $d(j-1)$. So $d(n) = (n-p+1)\ r(M)$. The theorem now follows from (3.2), as $f(n,G)$ is at most the sum of $g(n,M)$ over all M whose underlying graph is G. □

4. Refinements.

Although the proof of Theorem 2 can give a slightly better result than that stated, it seems unlikely that the result can be improved considerably without substantial modifications of the proof. Clearly $f(n,G) \geq \lfloor n/p \rfloor$, and so our bounds on $f(n,G)$ are perhaps best summarised as $c_G n \leq f(n,G) \leq c_G n$.

It would be interesting to know the value of $c(G) \equiv \sup\{f(n,G)/n : n \in \mathbb{N}\}$ for various G . As pointed out in [1] , it is easy to see that for the k-spoked wheel W_k (defined as in Tutte [5,p.78]), we have $c(W_k) > c^k$ for some constant $c > 1$ (and perhaps k large enough). Hence $c(G)$ is not bounded independently of G even though it is less than 12 if G is any triangulation. Even if G has all faces of valency 4 , $c(G)$ is arbitrarily large. For instance, let G be the 1-skeleton of a stack of k cubes (that is the graph $P_{k+1} \times C_4$) . Then if H is a suitable triangulation of G (for example with all diagonals of the faces in the sides of the stack sloping in the same direction around the stack), we can find 2^k copies of G in H . Thus $c(G) \geq 2^k/4(k+1)$.

All upper bounds on $r(G)$ are upper bounds on $c(G)$ by Theorem 2, but tight upper bounds seem to be difficult to obtain. However, in view of the comment above, the next result shows that $r(W_k)$ grows essentially exponentially.

THEOREM 3. *For* $k \geq 2$, $r(W_k) \leq 3.2^{k-1}$.

Proof. Denote the hub of W_k $(k \geq 3)$ by u , and let $v_1,\ldots,v_k$ denote the rim vertices in cyclic order. Let U_k denote the graph obtained by removing the edge v_1v_k from W_k , and let M denote the embedding of U_k in the plane with all vertices in the bounding cycle. Let $h(k)$ denote the maximum number of ways of completing an embedding of M in a planar graph, given an embedding of u, v_1 and v_k , and given the orientation of M (which determines whether $v_1v_2\ldots$ are embedded in clockwise or anticlockwise order). We find an upper bound on $h(k)$, thereby proving its finiteness, and use this to obtain the upper bound on $r(W_k)$.

Let M_1 denote an embedding of M in a plane graph N . We use $v_{i,j}$ to denote the image of v_j in M_i , and (by a slight abuse of notation) v_1 , v_k and u denote the images of v_1 , v_k and u . Let ℓ denote the greatest integer less than k for which some edge of N joins the vertices v_1 and v_ℓ of M_1 . Then $2 \leq \ell \leq k-1$. If M_2 is an embedding of M which coincides with M_1 on u, v_1 and v_k and has the same orientation, then $v_{1,\ell}$ must equal $v_{2,j}$ for some $2 \leq j \leq k-1$. Hence M_2 is determined by an embedding of U_j (with u, v_1 and v_j mapped onto u, $v_{1,1}$ and $v_{1,\ell}$ respectively) and an embedding of U_{k+1-j} (with u, v_1 and v_{k+1-j} mapped onto u, $v_{1,\ell}$ and v_k respectively). If ℓ is 2 or $k-1$, we refer here to U_2 , which we define to be the triangle uv_1v_2 , and $h(2)$ is accordingly defined to be 1 . It is now evident that

$$h(k) \leq \sum_{\ell=2}^{k-1} h(\ell)h(k+1-\ell) . \tag{4.1}$$

Assume for the moment that equality holds here. Then since

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$h(2) = 1$, this recurrence defines the Catalan numbers and we have $h(k) = \frac{1}{k-1}\binom{2k-4}{k-2} = q_k$ say (see Liu [3,p. 75] for example). Therefore q_k is an upper bound on $h(k)$ even without assuming equality in (4.1).

Now suppose that M is the plane graph with underlying graph W_k and with unbounded face of valency k , and M_1 and M_2 are dependent embeddings of M in a plane graph N . Then the hub of M_2 must coincide with a vertex of M_1 : either the hub or one of the rim vertices. We consider these two cases separately.

In the first case, some two adjacent vertices in the rim of M_1 must be vertices (not necessarily adjacent) in the rim of M_2 . Hence the number of possibilities for M_2 , given M_1 , is in this case at most $k\sum_{i=2}^{k} h(i)h(k+2-i) \le kq_{k+1}$. This count includes the event that $M_2 = M_1$.

In the second case, the hub, u , and another vertex, say v , of M_1 are in the rim of M_2 . The edge uv lies in the unbounded face of M_1 or in its bounding cycle. There can be at most $2k-3$ such edges in N . So the number of possibilities for M_2 in this case is at most $(2k-3)q_{k+1}$, by an argument similar to that used in the first case.

Next suppose M is the other plane embedding of W_k , so that the unbounded face is a triangle. Then a similar argument again gives $r(M) \le 3(k-1)q_{k+1}$. The theorem follows. □

Since $r(G) \le 12$ for any triangulation G by the results of [1], and since W_k has a face of largest possible valency amongst those planar graphs on $k+1$ vertices, a plausible conjecture is that $r(G)$ is maximised for p-vertex graphs when $G = W_{p-1}$. This would yield $c(G) \le 2^p$ whenever $\nu(G) = p$, but so far the evidence for this is weak. The following much weaker conjecture would still, if true, give a nice description of the nature of $c(G)$.

Conjecture. There exists an absolute constant $c > 1$ such that if $\nu(G) = p$ then $f(n,G) < c^p n$.

References

[1] N. Alon and Y. Caro, "On the number of subgraphs of prescribed type of planar graphs with a given number of vertices", *Ann. Discrete Math.* 20 (1984), 25-36.

[2] S. L. Hakimi and E. F. Schmeichel, "On the number of cycles of length k in a maximal planar graph", *J. Graph Theory* 3 (1979), 69-86.

[3] C. L. Liu, *Introduction to Combinatorial Mathematics*, (McGraw-Hill, New York, 1968).

[4] L. B. Richmond and N. C. Wormald, "Random triangulations of the plane", (to appear).

[5] W. T. Tutte, *Graph Theory* , (Encyclopedia of Mathematics and its Applications, Vol. 21, Addison-Wesley, Mass., 1984).

[6] H. Whitney, "Congruent graphs and the connectivity of graphs", *Amer. J. Math.* 54 (1932), 150-168.

Department of Mathematics and Statistics
University of Auckland,
Private Bag,
Auckland,
New Zealand.