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# MINIMAL CYCLIC-4-CONNECTED GRAPHS

BY

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ABSTRACT. A theory of cyclic-connectivity is developed, matroid dual to the standard vertex-connectivity. The cyclic-4-connected graphs minimal under the elementary operations of single-edge deletion or contraction and removal of a trivalent vertex are classified. These turn out to belong to three simple infinite families of indecomposable graphs, or to be decomposable into constituent subgraphs which themselves belong to three simple infinite families. This is modeled after W. T. Tutte's theorem classifying the minimal 3-connected graphs under single-edge deletion or contraction as forming the single infinite family of "wheels." Such theorems serve two main purposes: (1) illustrating the structure of graphs in the class by isolating a type of extremal graph, and (2) by providing a set-up so that induction on $|E(G)|$ can be carried out effectively within the class.

**1. Introduction.** A theory of graph connectivity is developed in Chapter 10 of **[2]**, along with a proof that nondegenerate 3-connected graphs, which are minimal with respect to deletions or contractions of single edges, must be wheels with $k \geqslant 3$ spokes. This gives a method to apply induction on $|E(G)|$ within the class of 3-connected graphs $G$. In this paper an analogous theory is established for cyclic-4-connected graphs which are minimal with respect to deletions or contractions of single edges or removal of single trivalent vertices. Cyclic-connectivity is defined and its elementary properties derived in §3. It is formulated to be as much as possible the matroid dual **[3]** of the well-known vertex-connectivity, given in **[4]**.

The minimal graphs are shown to be indecomposable, or to decompose uniquely into constituent subgraphs. There appear only three simple infinite families of indecomposable graphs, and three simple infinite families of constituents. These graphs resemble ladders and the planar duals to ladders and so this classification is called the *ladder theorem* for cyclic-4-connected graphs. The decomposition theory is similar to the more complete structure theorem in **[1]**.

**2. Terminology.** For notation and theoretical background the reader is referred to **[2]**. Some material is collected here to establish the viewpoint taken in this paper.

Given $X \subseteq V(G)$ define the *edgeless* subgraph $[X] = \mathrm{MIN}\{H \subseteq G\colon V(H) = X\}$, and the *induced* subgraph $G[X] = \mathrm{MAX}\{H \subseteq G\colon V(H) = X\}$. Similarly, when

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$S \subseteq E(G)$ define the *reduced* subgraph $G \cdot S = \mathrm{MIN}\{H \subseteq G\colon E(H) = S\}$, and the *spanning* subgraph $G\colon S = \mathrm{MAX}\{H \subseteq G\colon E(H) = S\}$. Let $J \subseteq G$ be a fixed subgraph of $G$. Then any $x \in V(J)$ incident in $G$ with some edge $A \in E(G) - E(J)$ is a *vertex of attachment* of $J$ in $G$. The set of such vertices of attachment is denoted $W(G, J)$. Define the *complement* $\bar{J} = \mathrm{MIN}\{K \subseteq G\colon J \cup K = G\}$. Clearly, $W(G, J) = V(J \cap \bar{J})$. Following **[2]** we generalize the notion of component of $G$ to that of $J$-component of $G$, and obtain a decomposition of the complement $\bar{J}$.

An *inner J-component* is a loop-graph or link-graph of $G$ not in $J$ whose endvertices are in $V(J)$. An *outer J-component* is the union of a component of $G[V(G) - V(J)]$ with all link-graphs of $G$ joining a vertex of the component to a vertex of $J$. The set $C_J(G)$ of *J-components* in $G$ is the union of its sets of inner and outer $J$-components. If $J$ is the null graph $\Omega$ then $C_J(G) = C(G)$ is the set of components of $G$. One easily sees that the $J$-components of $G$ are subgraphs of $\bar{J}$, intersections of distinct $J$-components are contained in $[W(G, J)]$, and the union of all $J$-components is $\bar{J}$.

A *contraction* $K$ of $G$, written $K \leqslant G$, is a graph such that $E(K) \subseteq E(G)$, $V(K) \subseteq C(G\colon (E(G) - E(K)))$, and each edge $A \in E(K)$ has endvertices $m, n$ in $G$ and $M, N$ in $K$, with $m \in V(M)$ and $n \in V(N)$. For a fixed $S \subseteq E(G)$ define the *reduced contraction* $G \times S = \mathrm{MIN}\{K \leqslant G\colon E(K) = S\}$, and the *spanning contraction* $G \operatorname{ctr} S = \mathrm{MAX}\{K \leqslant G\colon E(K) = S\}$. These graphs differ only on the set $C(G) \cap C(G\colon (E(G) - E(K)))$ of isolated vertices in $G \operatorname{ctr} S$. Let $C'(K)$ denote the set of those components $H'$ of $G$ such that $H \leqslant H'$ for some $H \in C(K)$. The contractions $J, K$ of $G$ are partially ordered by the contraction relation $J \leqslant K$ when $E(J) \subseteq E(K)$ and $C'(J) \subseteq C'(K)$. Note that the first condition implies the second except when $J$ contains an isolated vertex of $G \operatorname{ctr} E(K)$ not present in $K$.

Let $P(G)$ be the set of polygon subgraphs of $G$ and $P_k(G)$ be its subset of polygons with $k$ edges. Define a *bond* to be a connected loopless graph with exactly 2 vertices. Let $B(G)$ be the set of bond contractions of $G$ and $B_k(G)$ be its subset of bonds with $k$ edges (i.e. *k-bonds*). Define the *polygon-girth* of $G$ by

$$\gamma_P(G) = \mathrm{MIN}(\{k\colon P_k(G) \neq \varnothing\} \cup \{\infty\})$$

and the *bond-girth of* $G$ by

$$\gamma_B(G) = \mathrm{MIN}(\{k\colon B_k(G) \neq \varnothing\} \cup \{\infty\})$$

where $\infty$ is a symbol larger than any integer. We see that in connected graphs, bond-girth is the same as edge-connectivity.

Let $D_B(G)$ be the set of *maximal forest* subgraphs of $G$, and $D_P(G)$ be the set of *maximal coforest* contractions of $G$. A *coforest* is a graph whose edges are all loops. The mapping $F \to F'$, defined for $F \in D_B(G)$ and $F' \in D_P(G)$ by $E(F') = E(G) - E(F)$, is a bijection. The edge sets of these maximal graphs have constant cardinality, called the *bond-rank* $\rho_B(G) = |V(G)| - |C(G)|$ in $D_B(G)$, and the *polygon-rank* $\rho_P(G) = |E(G)| - |V(G)| + |C(G)|$ in $D_P(G)$.

Graph connectivity is introduced in Chapter 10 of **[2]**. Take $Q(G) = \{H\colon \Omega \neq H \subsetneqq G$ and $|W(G, H)| \leqslant \mathrm{MIN}\{|E(H)|, |E(G) - E(H)|\}\}$ and $Q_k(G) = \{H \in Q(G)\colon |W(G, H)| = k\}$, for nonnegative integer $k$. Then $G$ is *k-separated* when $Q_k(G) \neq \varnothing$, is *k-connected* when $k$ is positive and $Q_j(G) = \varnothing$ for all $j < k$, and has connectivity $\kappa(G) = \mathrm{MIN}(\{k\colon Q_k(G) \neq \varnothing\} \cup \{\infty\})$. It is readily seen that the following elementary remarks apply.

REMARK 2.1. *$G$ is 1-connected when connected.*

REMARK 2.2. *$G$ is 2-connected when nonseparable.*

REMARK 2.3. *$\kappa(H) = \infty$ exactly for the seven graphs $G$ with $\kappa(G) \geqslant 2$ and $|E(G)| \leqslant 3$.*

REMARK 2.4. *If $\kappa(G) < \infty$ then $\kappa(G) \leqslant \mathrm{MIN}\{\gamma_B(G), \gamma_P(G)\}$.*

**3. Cyclic-connectivity.** Let $Q(P(G)) = \{H \in Q(G)\colon P(H) \neq \varnothing \text{ and } P(\overline{H}) \neq \varnothing\}$, and $Q_k(P(G)) = Q(P(G)) \cap Q_k(G)$ for nonnegative integer $k$. Define the *cyclic-connectivity* of $G$ as follows:

$\kappa_P(G) = 0$ if $G$ is not connected,

$\kappa_P(G) = \infty$ if $G$ is a tree, and otherwise

$\kappa_P(G) = \mathrm{MIN}(\{k\colon Q_k(P(G)) \neq \varnothing\} \cup \{\rho_P(G)\})$.

Then $G$ is *cyclic-k-connected* for any positive integer $k \leqslant \kappa_P(G)$. This definition is formulated to apply to the polygon matroid $P(G)$ as vertex-connectivity applies to the bond matroid $B(G)$. It also assigns a connectivity to all graphs, where Whitney [**4**] defines vertex-connectivity for connected loopless graphs with two or more vertices. The vertex-connectivity of a $k$-clique for $k \geqslant 2$ is defined as $\kappa_B(G) = \rho_B(G) = k - 1$. The cyclomatic number $\rho_P(G)$ serves a similar purpose here.

We now state some direct consequences of the definition. A *block* of a graph is a maximal nonseparable nonnull subgraph.

PROPOSITION 3.1. *In general $\kappa_P(G) \geqslant 0$ with:*

(A) *$\kappa_P(G) \geqslant 1$ if and only if $G$ is connected,*

(B) *$\infty > \kappa_P(G) \geqslant 2$ if and only if $\rho_P(G) \geqslant 2$, $G$ is connected, and at most one block of $G$ contains polygons, and*

(C) *$\kappa_P(G) \geqslant 2$ and $\gamma_B(G) \geqslant 2$ if and only if $\kappa(G) \geqslant 2$ and $\rho_P(G) \geqslant 2$.*

The graph $G$ is *cyclic-k-separated* when $Q_k(P(G)) \neq \varnothing$. The effect of this on $\kappa_P(G)$ can be made more evident.

PROPOSITION 3.2. *We can write $\kappa_P(G) = \mathrm{MIN}\{k\colon Q_k(P(G)) \neq \varnothing\}$ except in two cases:*

(A) *$G$ is not connected and at most one component contains polygons, or*

(B) *$G$ is connected but does not contain two edge-disjoint polygons.*

PROOF. This is obvious if $\kappa_P(G) = 0$ or $\kappa_P(G) = \infty$. Assume that $1 \leqslant \kappa_P(G) < \infty$. Then 3.2(B) means requiring that $Q(P(G)) = \varnothing$. If $Q(P(G)) = \varnothing$ then $\kappa_P(G) = \mathrm{MIN}\{k\colon Q_k(P(G)) \neq \varnothing\}$ cannot apply. Suppose $Q(P(G)) \neq \varnothing$. Then $Q_k(P(G)) \neq \varnothing$, for $k$ minimum, and complementary connected $H, K \in Q_k(P(G))$ exist. Then

$$\begin{aligned}\rho_P(G) &= |E(G)| - |V(G)| + 1\\ &= \big(|E(H)| - |V(H)| + 1\big) + \big(|E(K)| - |V(K)| + 1\big) + k - 1\\ &= \rho_P(H) + \rho_P(K) + k - 1 \geqslant k + 1,\end{aligned}$$

using $\rho_P(H) \geqslant 1$ and $\rho_P(K) \geqslant 1$. Now $\kappa_P(G) = k < \rho_P(G)$, and so $\kappa_P(G) = \mathrm{MIN}\{k\colon Q_k(P(G)) \neq \varnothing\}$ must apply.

The next proposition shows that condition (B) of Proposition 3.2 takes effect in essentially only five cases.

PROPOSITION 3.3. *Suppose $G_1$ is the union of all polygons in a graph $G$ and that no two of these polygons are edge-disjoint. Then $G_1$ is the null graph, or a subdivision of a loop, 3-bond, $K_4$, or $K_{3,3}$.*

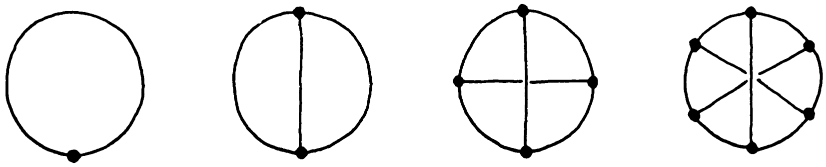


FIGURE 3A. *Graphs without edge-disjoint polygons*

PROOF. If $G_1 \neq \Omega$ there exists $P \in P(G)$. Because $G_1$ is nonseparable, a tower

$$P = H_1 \subsetneqq H_2 \subsetneqq H_3 \subsetneqq \cdots \subsetneqq H_k = G_1$$

exists, where $H_{i+1} = H_i \cup L_i$ for some arc $L_i \subseteq G$ avoiding $H_i$ but with its end-vertices in $V(H_i)$. It is routine to see that $H_i$ is a subdivision of the $i$th graphs in Figure 3A for $i = 1, 2, 3, 4$ and that $k \leqslant 4$ in any such tower.

The three remaining propositions compare $\gamma_P(G)$, $\gamma_B(G)$, $\kappa_P(G)$, and $\kappa(G)$.

PROPOSITION 3.4. *In general $\kappa_P(G) \leqslant \gamma_P(G)$.*

PROOF. If Proposition 3.4 is false then

$$1 \leqslant \gamma_P(G) < \kappa_P(G) \leqslant \rho_P(G)$$

obtains, and a polygon $P \subseteq G$ exists with $|E(P)| = \gamma_P(G)$. Now $\overline{P}$ is a forest so that

$$|E(G)| - |E(P)| \leqslant |V(G)| - 1, \quad \text{and}$$
$$\rho_P(G) = |E(G)| - |V(G)| + 1 \leqslant |E(P)| = \gamma_P(G),$$

contrary to assumption.

PROPOSITION 3.5. *If $\kappa_P(G) < \kappa(G)$ then $G$ is either a 3-bond or a polygon.*

PROOF. By hypothesis, and the consequent connectedness of $G$, $1 \leqslant \kappa_P(G) < \kappa(G)$, whence $\kappa_P(G) = \rho_P(G)$ and $G$ is a graph for which Proposition 3.3 applies with $G_1 \neq \Omega$. If $\rho_P(G) = 1$ then $\kappa(G) \geqslant 2$ and $G$ must be a polygon. If $\rho_P(G) = 2$ then $\kappa(G) \geqslant 3$ and $G$ is a 3-bond. Finally, if $\rho_P(G) \geqslant 3$ then $\kappa(G) \geqslant 4$, contrary to $\kappa(G) \leqslant 3$ in Proposition 3.3.

Two vertices in $G$ are *adjacent* when distinct and joined by an edge. The *degree* $d_G(x)$ of a vertex $x$ in $G$ is its number of adjacent vertices. This differs from the *valency* $v_G(x)$ of the vertex $x$, which is its number of incident edges, each loop counted twice incident. Call the connected $H \subseteq G$ with $x \in V(H)$ and $E(H) = \{A \in E(G)\colon A$ is incident with $x\}$ the *vertex-star* with *centre* $x$ in $G$. The *degree* of a vertex-star is the degree of its centre.

LEMMA 3.6. *Suppose $G$ is connected, $H \in Q_k(G)$, and $Q_j(G) = \varnothing$ for all $j < k$. When $k = 1$ or $k = 2$ choose $H$ minimal in $Q_k(G)$. Now either*

(A) *$H$ is a $k$-gon,*

(B) *$H$ is a simple vertex-star of degree $k$, or*

(C) *$H$ contains a polygon and $W(G, H) \subsetneqq V(H)$.*

PROOF. When $x \in V(H) - W(G, H)$ exists let $K$ be its vertex-star in $G$. Then $K \subseteq H$, and $\kappa(G) = k$ implies $K$ has degree at least $k$. If (C) fails then $H$ is a tree and $K$ is simple. Also $H$ has at least $d_G(x) \geqslant k$ monovalent vertices distinct from $x$. If $k = 1$ then $K = H$ is a link-graph, by the minimality of $H$. When $k \geqslant 2$ the set of monovalent vertices in $H$ is $W(G, H)$. This implies $d_G(x) = k$ and $d_H(y) \leqslant 2$ for all $y \in V(H)$ distinct from $x$. If $k = 2$ then $K = H$ is a vertex-star of degree 2, by the minimality of $H$. When $k \geqslant 3$ there are no divalent vertices in $H$ and hence $K = H$ and again (B) applies.

The alternative $W(G, H) = V(H)$ remains. A polygon $P \subseteq H$ exists, because $|E(H)| \geqslant |V(H)|$, and may be chosen with smallest possible girth $j$. Now $k$ is a minimum with $Q_k(G) \neq \varnothing$, and $j \leqslant k$, whence $j = k$. When $k \leqslant 2$ the minimal condition on $H$ and the minimum condition on $k$ imply $P = H$. If $k \geqslant 3$ then all the edges of $H$ are in $P$, because $j$ is minimal and hence $P = H$. The proposition is valid.

PROPOSITION 3.7. *If $G$ is neither an $h$-bond for $h \leqslant 3$ nor a polygon, then $\kappa(G) = \mathrm{MIN}\{\kappa_P(G), \gamma_B(G)\}$.*

PROOF. Under these hypotheses $\kappa(G) = \infty$ only for the null graph and vertex graphs, and then $\kappa_P(G) = \gamma_B(G) = \infty$. Otherwise $H \in Q_k(G)$ exists, for $k = \kappa(G)$, and $\overline{H} \in Q_k(G)$. Remark 2.4 gives $\kappa(G) \leqslant \gamma_B(G)$ when $Q(G) \neq \varnothing$, and Proposition 3.5 gives $\kappa(G) \leqslant \kappa_P(G)$ under our hypothesis. Thus $k < \mathrm{MIN}\{\kappa_P(G), \gamma_B(G)\}$ when Proposition 3.7 is false. No member of $Q_k(G)$ is a $k$-gon, by Proposition 3.4, or a simple vertex-star of degree $k$, by $k < \gamma_B(G)$. Applying Lemma 3.6 to $H$ and $\overline{H}$, or minimal members of $Q_k(G)$ contained in these graphs when $k \leqslant 2$, we see that both $H$ and $\overline{H}$ contain polygons, contrary to $k < \kappa_P(G)$. This completes the proof.

Note that when $\kappa(G) = \gamma_B(G) < \kappa_P(G)$ there is a simple vertex-star of degree $\kappa(G)$ in $G$. Apart from the tie-in with the connectivity $\kappa(G)$ of Tutte **[2]**, this development parallels that of Whitney for vertex-connectivity in **[1]**.

Define $G'_A = G{:}\ (E(G) - \{A\}), G''_A = G\,\mathrm{ctr}(E(G) - \{A\}), G_t = G[V(G) - \{t\}]$ when $A \in E(G)$ and $t \in V(G)$. Then set $L = \{G{:}\ 4 \leqslant \kappa_P(G)$ and $3 \leqslant \gamma_B(G)\}$ and $M = \{G \in L{:}\ G'_A \notin L$ and $G''_A \notin L$ for all $A \in E(G)$, and $G_t \notin L$ for all trivalent $t \in V(G)\}$. The members of $M$ are called *minimal* cyclic-4-connected graphs. By 3.7 members of $L$ and $M$ are 3-connected and admit only *triads* (simple vertex-stars of degree 3) and their complements in $Q_3(G)$. This paper aims to effectively describe these minimal graphs, and thus to provide an inductive theory of cyclic-4-connectivity.

**4. Lemmas.** Some lemmas useful for the next sections will now be established.

LEMMA 4.1. *Suppose $3 \leqslant \kappa(G) = k < \kappa_P(G)$ and $S \subseteq E(G)$. Then $G \times S$ is a bond of girth $k$ if and only if $G \cdot S$ is a vertex-star of degree $k$.*

PROOF. When $G \cdot S$ is a vertex-star $G \times S$ is a union of bonds joining $[x]$ to the components of $G_x$ containing vertices adjacent to $x$. However, $\kappa(G) \geqslant 3$ implies $G$ is simple and nonseparable, hence $|S| = k$ and $G \times S$ is a single bond. Conversely, assume $G \times S$ is a bond of girth $k$ with vertex set $\{H, H_1\}$. Then $\{H, H_1\} \not\subseteq \bigcup_{j \leqslant k} Q_j(P(G))$ and so $H$ may be assumed to be a tree. If $H$ has two or more monovalent vertices then each is incident with at least $k-1$ edges in $S$, and so $k = |S| \geqslant 2(k-1)$ or $2 \geqslant k$, contrary to hypotheses. Thus $H$ is a vertex-graph, and $G \cdot S$ is a vertex-star of degree $k$.

LEMMA 4.2. *If $\kappa(G) = k < \kappa_P(G)$ and $T$ is a vertex-star of degree $k$ in $G$ with centre $t$ and no endvertex $x$ of valency $v_G(x) = k$, then $\kappa(G_t) \geqslant k$.*

PROOF. Suppose $j = \kappa(G_t) < k$ and choose complementary $H, H_1 \in Q_j(G_t)$. Then $H, H_1 \notin Q_j(G)$ implies vertices $x \notin V(H_1)$, $y \notin V(H)$ adjacent to $t$ in $G$ exist, so that $k \geqslant 2$. Now $G$ is simple because $\kappa_P(G) \geqslant 3$, thus $v_H(x), v_{H_1}(y) \geqslant k$, which forces $H, H_1 \in Q_j(P(G_t))$. We can write $T = T_1 \cup T_2$ where $(T_1)_t \subseteq H$, $(T_2)_t \subseteq H_1$, and $T_1 \cap T_2 = [t]$. Define $N = H \cup T_1$ and $N_1 = H_1 \cup T_2$. Then $N, N_1 \in Q_{j+1}(P(G))$, contrary to $k < \kappa_P(G)$.

LEMMA 4.3. *Suppose $\kappa(G) \geqslant 3$, $\gamma_p P(G) > 3$, and that $|W(G, H)| \leqslant 3$ for some $H \subseteq G$. Then $\overline{H}$ is connected and exactly one of the following applies*:

(A) *$H$ is a forest with $V(H) = W(G, H)$.*

(B) *$H \cong D_i$ for $i \in \{1, 2, 3\}$, with $D_i$ as in Figure* 4A, *under an isomorphism sending $W(G, H)$ onto $\{x, y, z\}$, or*

(C) $|E(H)| \geqslant 8$.

PROOF. Because $\overline{H}$ is a subgraph of a 3-connected graph $G$, which has no isolated vertices and at most three vertices of attachment, it must be connected. Assume (A) and (C) do not apply. Because (A) does not hold $|W(G, H)| \leqslant 3$ and $\gamma_P(G) > 3$ imply $G$ has a vertex $x \notin V(\overline{H})$. Then $v_G(x) \geqslant \kappa(G) \geqslant 3$ and so $|E(H)| \geqslant 3$. By hypotheses $|E(G)| \geqslant 9$. Because (C) also does not hold we see that $|W(G, H)| = 3$, and either $H \in Q_3(G)$ or $\overline{H}$ is a 2-arc with $V(H) = W(G, H)$.

If each edge of $H$ has an endvertex in $W(G, H)$ then $H \cong D_1$ or $H \cong D_2$ as in (B). Otherwise, an edge of $H$ exists with endvertices $a, b \notin W(G, H)$. The vertex-stars with centres $a$ and $b$ form a tree with at least four monovalent vertices. Thus a 2-arc $N \subseteq H$ disjoint from $\overline{H}$ exists, and $H$ contains at least five edges not in $N$. Then $|E(H)| \leqslant 7$ implies $H$ is the union of $N$ and five link-graphs, each having one endvertex in $V(N)$ and the other in $W(G, H)$. This leads finally to $H \cong D_3$ as in (B).

Lemma 4.3 is usually applied in the form of the following remark.

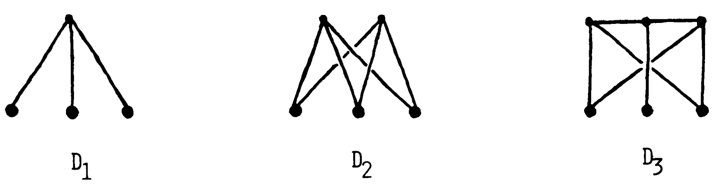



FIGURE 4A. *Some small subgraphs of $G \in L$*

REMARK 4.4. *Suppose* $G \in L$ *and* $H \subseteq G$ *is such that* $\overline{H}$ *contains a polygon. We have*:

(A) *If* $|W(G, H)| \leqslant 3$ *then* $H$ *is a forest and either* $V(H) = W(G, H)$, *or* $H \cong D_1$ *is a triad of* $G$. *Moreover, either* $|E(\overline{H})| \geqslant 8$ *or* $G \cong K_{3,3}$, *and* $\overline{H} \cong D_2$ *or* $\overline{H} \cong D_3$, *as in* (B) *of Lemma* 4.3.

(B) *If* $|W(G, H)| = 4$ *then either* $H$ *is connected or the connected components of* $H$ *are vertex-graphs, link-graphs*, 2-*arcs, or triads of* $G$.

PROOF. In Remark 4.4(A), as $Q_j(P(G)) = \varnothing$ for $j \leqslant 3$ it follows that $H$ is a forest. Either 4.3(A) obtains, or 4.3(B) with $H \cong D_1$. Now $\overline{H}$ contains a polygon, so that 4.3(C) applies to it, or else $\overline{H} \cong D_2$ or $\overline{H} \cong D_3$ as in 4.3(B). We readily see that $G \cong K_{3,3}$ when 4.3(B) holds. In Remark 4.4(B) either $H$ is connected or its connected components $C$ satisfy $|W(G, C)| \leqslant 3$. Applying 4.4(A) each $C$ must be a vertex-graph, link-graph, 2-arc, or a triad of $C$. This completes the proof.

Suppose $G \in L$ and $H \in Q_3(G)$ is such that $\overline{H}$ contains a polygon. Then $H$ is a triad, by Lemma 4.3, and $G \times E(H)$ is the only type of bond in $G$ with girth 3, by Lemma 4.1. Indeed, such $H$ are what remains of 3-connectivity in $G$. Lemma 4.2 states that if no endvertex of $H$ is trivalent then $H$ can be removed from $G$ leaving the 3-connected graph $\overline{H}$. Thus it is natural to allow the removal of certain such trivalent vertices in defining the set $M$ of minimal members of $L$.

The next three lemmas deal with $G \in L$ under the following three separate conditions:

(A) when $G_t \notin L$ for some trivalent $t \in V(G)$,

(B) when $G'_A \notin L$ for some $A \in E(G)$, and

(C) when $G''_A \notin L$ for some $A \in E(G)$.

LEMMA 4.5. *Let* $G \in L$ *and suppose* $G_t \notin L$ *for some* $t \in V(G)$ *which is the centre of a triad with no trivalent endvertices. Then* $\kappa(G_t) = 3$, *and complementary* $J, J_1 \in Q_3(P(G_t))$ *exist with* $|E(J)| \geqslant 6$ *and* $|E(J_1)| \geqslant 6$, *and* $t$ *is adjacent to some* $x \notin V(J_1)$ *and some* $y \notin V(J)$.

PROOF. By Lemma 4.2 it follows that $\kappa(G_t) \geqslant 3$. Then $\gamma_B(G_t) \geqslant 3$, and $\kappa_P(G_t) \geqslant 3$. By Proposition 3.7, we have $\kappa(G_t) = 3$ and by Propositions 3.2 and 3.3 either $G_t \cong K_4$, $G_t \cong K_{3,3}$ or $Q_3(P(G_t)) \neq \varnothing$. Then $\gamma_P(G) \geqslant 4$, by Proposition 3.4, hence $G_t \ncong K_4$. If $G_t \cong K_{3,3}$ then $\gamma_P(G) \geqslant 4$ forces $G \cong K_{3,4}$, contrary to $\kappa_P(G) \geqslant 4$. Thus complementary $J, J_1 \in Q_3(P(G_t))$ exist, and Lemma 4.3 ensures $|E(J)| \geqslant 6$ and $|E(J_1)| \geqslant 6$. Finally, the condition $\kappa_P(G) \geqslant 4$ implies there exist vertices $x \notin V(J_1)$ and $y \notin V(J)$ adjacent to $t$ in $G$.

LEMMA 4.6. *Suppose* $G \in L$ *and that* $G'_A \notin L$ *for some edge* $A \in E(G)$ *not contained in a triad of* $G$. *Then* $\kappa(G'_A) = 3$ *and complementary* $H, H_1 \in Q_3(P(G'_A))$ *exist, where necessarily* $|E(H)| \geqslant 6$, $|E(H_1)| \geqslant 6$, *and* $A$ *has endvertices* $x \notin V(H_1)$ *and* $y \notin V(H)$.

PROOF. When $G'_A \notin L$ either $\gamma_B(G'_A) < 3$ or $\kappa_P(G'_A) < 4$. Suppose $\gamma_B(G'_A) < 3$ and derive a contradiction. By the definition of a bond and the hypothesis $\gamma_B(G) \geqslant 3$, there exists $K \in B(G)$ with $A \in E(K)$ such that $K'_A \in B_j(G'_A)$ for $j = \gamma_B(G'_A)$. Then

$3 \leqslant \gamma_B(G) \leqslant j+1 \leqslant 3$ implies $\gamma_B(G) = 3$. Using Proposition 3.7 we have $\kappa(G) = 3$. The hypotheses of Lemma 4.1 are satisfied with $S = E(B)$. Thus $G \cdot S$ is a triad, contrary to the hypotheses of this lemma.

We conclude that $\gamma_B(G'_A) \geqslant 3$ and $\kappa_P(G'_A) < 4$. Proposition 3.4 ensures $\gamma_P(G'_A) \geqslant \gamma_P(G) \geqslant 4$. Then Propositions 3.2 and 3.3 imply that $\kappa_P(G'_A) = \mathrm{MIN}\{k\colon Q_k(P(G'_A)) \neq \varnothing\}$. There thus exist complementary $H, H_1 \in Q_j(P(G'_A))$ for $j = \kappa_P(G'_A)$. Because $H \notin Q_j(P(G))$ and $H_1 \notin Q_j(P(G))$ the edge $A$ has endvertices $x \notin V(H_1)$ and $y \notin V(H)$. Then $H, H_1 \in Q_{j+1}(P(G))$ so that $4 \geqslant j+1 \geqslant 4$ and hence $j = \kappa_P(G'_A) = 3$ and $\kappa_P(G) = 4$. Now Proposition 3.7 implies $\kappa(G'_A) = 3$ and Lemma 4.3 implies $|E(H)| \geqslant 6$ and $|E(H_1)| \geqslant 6$.

LEMMA 4.7. *Suppose $G \in L$ and that $G''_A \notin L$ for some edge $A \in E(G)$ not contained in a quadrilateral of $G$. Then $\kappa(G''_A) = 3$ and complementary $K, K_1 \in Q_4(P(G'_A))$ exist, where $A$ has endvertices $x, y \in V(K \cap K_1)$ and necessarily $|E(K)| \geqslant 6, |E(K_1)| \geqslant 6$.*

PROOF. If $A$ is in no quadrilaterial then $\gamma_P(G''_A) \geqslant 4$. Now $\gamma_B(G''_A) \geqslant \gamma_B(G) \geqslant 3$, so that $\kappa_P(G''_A) \leqslant 3$ when $G''_A \notin L$. There exists, by Propositions 3.2 and 3.3, complementary $J, J_1 \in Q_j(P(G''_A))$ for $j = \kappa_P(G''_A)$. Let $f\colon G \to G''_A$ be the induced contractive mapping and $z = fA$. If $z \notin V(J \cap J_1)$ then either $J \in Q_j(P(G))$ or $J_1 \in Q_j(P(G))$, contrary to $\kappa_P(G) \geqslant 4$, and so $z \in V(J \cap J_1)$. Let $K = (f^{-1}J)'_A$ and $K_1 = (f^{-1}J_1)'_A$. Using $\gamma_P(G''_A) \geqslant 4$, we have $|E(J)| = |E(K)| \geqslant 4$ and $|E(J_1)| = |E(K_1)| \geqslant 4$. Then $4 \leqslant \kappa_P(G) \leqslant j+1 \leqslant 4$ forces $\kappa_P(G) = 4$ and $3 = j = \kappa_P(G''_A)$. Proposition 3.7 implies $\kappa(G''_A) = 3$ and Lemma 4.3 applied to $G''_A$ gives $|E(K)| \geqslant 6$, $|E(K_1)| \geqslant 6$. Now $K \notin Q_3(P(G))$ and $K_1 \notin Q_3(P(G))$ imply $A$ has endvertices $x, y \in V(K \cap K_1)$, completing the proof.

Figure 4B illustrates Lemmas 4.6 and 4.7. We make two remarks for reference.

REMARK 4.8. *When $H$ is minimal under the conditions of Lemma* 4.6, *no two of $u, v, w$ are adjacent in $H$, and $v_H(u), v_H(v), v_H(w) \geqslant 2$.*

REMARK 4.9. *When $K$ is minimal under the conditions of Lemma* 4.7, *no two of $x, y, s, t$ are adjacent in $K$, and $v_K(s), v_K(t) \geqslant 2$.*

LEMMA 4.10. *If $H, K \subseteq G$ then*

$$W(G, H \cap K) = V(H \cap K) \cap (W(G, H) \cup W(G, K)).$$

PROOF. This follows because $x \in W(G, H \cap K)$ if and only if $x \in V(H \cap K)$ and $x$ in incident with some $A \in E(G)$ not contained in both $H$ and $K$.

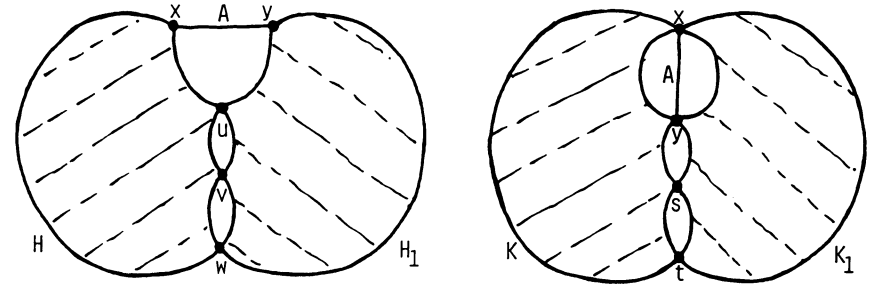



FIGURE 4B. *Diagrams for Lemmas* 4.6 *and* 4.7

When using Lemma 4.10, we often write $W(G, H \cap K) \subseteq \{a_1, a_2, \ldots, a_n\}$ where $a_1, a_2, \ldots, a_n$ need not be distinct. However, when we write $W(G, H \cap K) = \{a_1, a_2, \ldots, a_n\}$ the elements are understood to be distinct. Lemma 4.10 is recognizable enough to be used without being repeatedly identified.

**5. Local minimum structure.** Three crucial propositions are proved here. When applied to a graph $G \in M$ they show that:

(A) *each edge of $G$ is in a triad or a quadrilaterial,*

(B) *at most one edge in a triad of $G$ is not also in a quadilateral, and*

(C) *at most one edge in a quadrilateral of $G$ is not also in a triad.*

The general decomposition theory for the $G \in M$ given in §6 is based on these facts.

PROPOSITION 5.1. *Suppose $G \in L$ and $G_t \notin L$ for all trivalent $t \in V(G)$. If $G'_A \notin L$ and $G''_A \notin L$ for $A \in E(G)$, then a triad or quadrilateral containing $A$ exists in $G$.*

PROOF. Let $G \in L$ and suppose $G'_A \notin L$ and $G''_A \notin L$ for some $A \in E(G)$ which is contained in no triad or quadrilateral of $G$. Then Lemma 4.6 and Lemma 4.7 apply in the notation of Figure 4B. Without loss of generality $H$ can be assumed minimal in $Q_3(P(G'_A))$ and the notation of Figure 4B taken so that $u, v \in V(K_1)$ and $u \notin V(K)$. Now $K$ can be assumed minimal in $\{J \in Q_4(P(G))\colon \{x, y\} \subseteq W(G, J)\}$ without altering any notation. Applying Lemma 4.10 we obtain $W(G, H \cap K) \subseteq \{x, w, s, t\}$, $W(G, H_1 \cap K) \subseteq \{y, w, s, t\}$, and $W(G, H_1 \cap K_1) \subseteq \{y, u, v, w, s, t\}$. Here $u \notin V(K)$, while $v \in V(K)$ implies $v \in \{s, t\}$.

Suppose first that $s \notin V(H)$. In general $v_{H_1 \cap K}(y) \geqslant 1$ and, because $K$ is minimal, $v_{H_1 \cap K}(s) \geqslant 2$ and the vertices $s$, $y$ are not adjacent in $K$. Applying Remark 4.4(A) to $H_1 \cap K$ these facts ensure $|W(G, H_1 \cap K)| \geqslant 4$ and hence that $W(G, H_1 \cap K) = \{y, w, s, t\}$. Now $H$ is minimal, and $w \notin V(K_1)$ because $w \in V(K)$ and $w \notin W(G, K)$ so that Remark 4.8 ensures $v_{H \cap K}(w) \geqslant 2$. Then $W(G, H \cap K) \subseteq \{x, w, t\}$ and 4.4(A) force $H \cap K$ to be a 2-arc with internal vertex $w$ and endvertices $x$, $t$. However, then $t \in V(H \cap H_1)$ and $t$, $w$ are adjacent in $H$, contrary to $H$ being minimal.

The case $t \notin V(H)$ is similar to $s \notin V(H)$. Suppose alternatively that $s, t \in V(H)$, and assume $s = v$ or $t = w$ when $s \in V(H_1)$, or $t \in V(H_1)$, respectively. Then $W(G, H \cap K) \subseteq \{x, w, s, t\}$, $W(G, H_1 \cap K) \subseteq \{y, w, s\}$, and $W(G, H_1 \cap K_1) \subseteq \{y, u, v, w\}$. Remark 4.9 applies, because $K$ is minimal in $Q_4(P(G))$ subject to $x, y \in W(G, K)$. There exist $m, n \notin V(K_1)$ adjacent to $x$, $y$, respectively, in $G$, for otherwise a contradiction can be derived by applying 4.4(A) to $K_x$ or $K_y$ and noting that $|E(K)| \geqslant 6$. Then $m \neq n$, by $\gamma_P(G) \geqslant 4$. Using $|E(H_1)| \geqslant 6$ and 4.4(A) there similarly can be seen to exist a vertex $u' \notin V(\overline{H}_1)$ adjacent to $u$ in $G$. Now $y$, $s$ are not adjacent in $K$, and $y$ is adjacent to $n$ in $H_1 \cap K$, so that 4.4(A) implies $v_{H_1 \cap K}(y) = 1$. Then $v_G(y) \geqslant 4$ forces $v_{H_1 \cap K_1}(y) \geqslant 2$. Now $u \notin V(K)$ and $u' \notin \{s, t\} \subseteq V(H)$, hence $u' \notin V(H \cup K)$, and so 4.4(A) ensures $W(G, H_1 \cap K_1) = \{y, u, v, w\}$. Then $n \in V(H_1)$ and $n \notin \{u, v, w\} \subseteq V(K_1)$ imply $n \notin V(H \cup K_1)$. Applying 4.4(A) again $H_1 \cap K$ is a triad with center $n$ and endvertices $y, w, s$. But now $w \in V(K \cap K_1)$ and $s \in V(H \cap H_1)$, so that $s = v$ and $t = w$. Then

$W(G, H \cap K) \subseteq \{x, w, s\}$ and $\{u, v, w\} \subseteq V(K_1)$. Now $m \in V(H), W(G_A', H) = \{u, v, w\}$ and $m \notin V(K_1)$ imply $m \notin V(H_1 \cup K_1)$. But then $H \cap K$ is a triad with centre $m$ and endvertices $x, w, s$, by 4.4(A). This determines $K = (H \cap K) \cup (H_1 \cap K)$.

Let $I = H_m$ and $I_1 = (H_1)_n$. Then $|E(H)| \geqslant 6$, $|E(H_1)| \geqslant 6$, $v_G(x) \geqslant 4$ and $v_G(y) \geqslant 4$ imply $|E(I)| \geqslant 3$, $|E(I_1)| \geqslant 3$, $v_I(x) \geqslant 2$ and $v_{I_1}(y) \geqslant 2$. Thus $W(G, I) = \{x, u, v, w\}$ and $W(G, I_1) = \{y, u, v, w\}$, by 4.4(A), while $I$ and $I_1$ are connected, by 4.4(B). Thus $v_G(v) \geqslant 4$ and $v_G(w) \geqslant 4$. By hypothesis $G_m \notin L$, and so Lemma 4.5 implies complementary $J, J_1 \in Q_3(P(G_m))$ exist. Because $J, J_1 \notin Q_3(P(G))$, and both $J, J_1$ and $v, w$ are interchangeable, we may assume without loss of generality $w \notin V(J)$ where $J$ is chosen minimal in $Q_3(P(G_m))$. If $n \in V(J)$ then $n \in W(G_m, J)$, because $n$ is adjacent to $w$, and $v, y \notin V(J_1)$, by the minimality of $J$. Now $y \notin V(J_1)$ implies $x \in V(J)$. But then $(J_1)_n \in Q_3(P(G))$, contrary to $\kappa_P(G) \geqslant 4$. Thus $n \notin V(J)$, which implies $v, y \in V(J_1)$. Then $x \notin V(J_1)$ and $y \in V(J \cap J_1)$. Write $V(J \cap J_1) = \{y, p, q\}$ with $v = p$ when $v \in V(J)$, as in Figure 5A.

We now have $W(G, I \cap J) \subseteq \{x, u, p, q\}$ and $W(G, I_1 \cap J) \subseteq \{y, u, p, q\}$ and $v_{I \cap J}(x) \geqslant 2$. Lemma 4.5 states that $|E(J)| \geqslant 6$. Lemma 4.3 and the hypothesis that $A$ is in no quadrilateral force $|E(J)| \geqslant 7$. Assume $u \in V(J_1)$ so that $W(G, I \cap J) \subseteq \{x, p, q\}$ and $W(G, I_1 \cap J) \subseteq \{y, p, q\}$. Then $J_A' = (I \cap J) \cup (I_1 \cap J)$ and 4.4(A) imply $I \cap J$ and $I_1 \cap J$ are triads of $G$ with endvertices $x, p, q$ and $y, p, q$, respectively. This contradicts $v_{I \cap J}(x) \geqslant 2$. We may assume that $u \notin V(J_1)$. Then $v_{I \cap J}(x) \geqslant 2$, $v_{I \cap J}(u) \geqslant 2$, and so 4.4(A) implies $W(G, I \cap J) = \{x, u, p, q\}$.

Now $u, w$ are distinct from $p, q$. If $q \in V(I_1)$ then $q = v$. But then $v \in V(J)$ and $v = p$, which is impossible. It follows that $q \notin V(I_1)$. Note that $u' \notin V(I)$ because $u' \notin V(H)$, and $u' \notin V(J_1)$ because $u \notin V(J_1)$ and $u' \neq y, p, q$. This implies $I_1 \cap J$ is a triad with centre $u'$ and endvertices $y, u, p$, by 4.4(A). No other such triad can exist when $\kappa_P(G) \geqslant 4$ and $G \not\cong K_{3,3}$, hence $u'$ is the unique vertex of $G$ adjacent to $u$, and not in $V(\overline{H}_1)$. Then $p \in V(I \cap I_1)$ and $p \neq u, w$ imply $p = v$. Now let $I_2 = ((I_1)_u)_{u'}$. Then $W(G, I_2) \subseteq \{y, v, w\}$ and, because $v_G(y) \geqslant 4$ and $\gamma_P(G) \geqslant 4$, there exists a vertex $k \notin V(\bar{I}_2)$ adjacent to $y$ in $I_2$. Now 4.4(A) implies $I_2$ is a triad of $G$ with centre $k$ and endvertices $y, v, w$. But this is impossible because $H_1 \cap K$ is a triad of $G$ with centre $n \neq k$ and endvertices $y, v, w$ while $G \not\cong K_{3,3}$. All cases lead to contradictions, hence the theorem is valid.

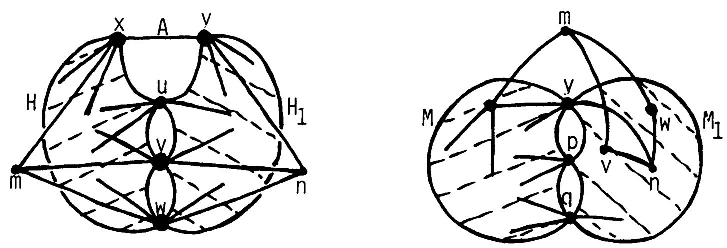



FIGURE 5A. *A case to be eliminated*

PROPOSITION 5.2. *If $G \in L$ and $T$ is a triad of $G$ with centre $t$ such that $G_A'' \notin L$ for all $A \in E(T)$, then $t \in V(Q)$ for some quadrilateral $Q \in P_4(G)$.*

PROOF. Assume the hypotheses and suppose $t \notin V(Q)$ for any $Q \in P_4(G)$. Let $T$ have endvertices $x_1, x_2, x_3$. Then Lemma 4.7 applies in cases $i = 1, 2, 3$, with $t = x$ and $x_i = y$. Using Remark 4.4 and $|E(K)| \geqslant 6$, $|E(K_1)| \geqslant 6$ we see that $x$ is adjacent to vertices $a \notin V(K_1)$ and $b \notin V(K)$ in Figure 4B. Using $v_G(t) = 3$, and switching notation between $K$ and $K_1$ if necessary, $x_{i+1} = a$ and $x_{i-1} = b$ can be assumed, subscripts reduced mod 3. Denote $K_x$ and $(K_1)_x$ from Figure 4B by $H_i$ and $K_i$, respectively, throughout this proof. Then $H_i \cup K_i = G_t$ and $H_i \cap K_i = [x_i, y_i, z_i]$ for appropriate $y_i, z_i \in V(G)$, while $W(G, H_i) = \{x_{i+1}, x_i, y_i, z_i\}$, $W(G, K_i) = \{x_{i-1}, x_i, y_i, z_i\}$, $|E(H_i)| \geqslant 5$ and $|E(K_i)| \geqslant 5$. These decompositions are illustrated in Figure 5B.

Lemma 4.10 implies $W(G, H_1 \cap H_2) \subseteq \{x_2, y_1, z_1, y_2, z_2\}$, $W(G, H_1 \cap K_2) \subseteq \{x_1, x_2, y_1, z_1, y_2, z_2\}$, $W(G, K_1 \cap H_2) \subseteq \{x_3, y_1, z_1, y_2, z_2\}$, and $W(G, K_1 \cap K_2) \subseteq \{x_1, y_1, z_1, y_2, z_2\}$. Furthermore $v_{H_1 \cap H_2}(x_2) \geqslant 1$, $v_{H_1 \cap K_2}(x_1) \geqslant 1$, $v_{H_1 \cap K_2}(x_2) \geqslant 1$, $v_{K_1 \cap H_2}(x_3) \geqslant 2$, and $v_{K_1 \cap K_2}(x_1) \geqslant 1$.

Assume first that $y_1, z_1 \in V(K_2)$. Then $W(G, H_1 \cap H_2) \subseteq \{x_2, y_2, z_2\}$ and $W(G, K_1 \cap H_2) \subseteq \{x_3, y_2, z_2\}$. Now $v_{K_1 \cap H_2}(x_3) \geqslant 2$, $|E(H_2)| \geqslant 5$, and $H_2 = (H_1 \cap H_2) \cup (K_1 \cap H_2)$ imply by Remark 4.4(A) that $K_1 \cap H_2$ is a 2-arc with internal vertex $x_3$ and endvertices $y_2, z_2$, while $H_1 \cap H_2$ is a triad of $G$ with centre some vertex $r \notin V(K_1 \cup K_2)$ and endvertices $x_2, y_2, z_2$. Thus $y_2, z_2 \in V(H_1 \cap K_1)$, and so without loss of generality we can write $y_1 = y_2$ and $z_1 = z_2$, and see that $W(G, K_1 \cap K_2) \subseteq \{x_1, y_2, z_2\}$. Again, $K_1 = (K_1 \cap H_2) \cup (K_1 \cap K_2)$ and $|E(K_1)| \geqslant 5$ imply $K_1 \cap K_2$ is a triad of $G$ with centre $s \notin V(H_1 \cup H_2)$ and endvertices $x_1, y_2, z_2$. Now $|E(H_3)| \geqslant 5$, $|E(K_3)| \geqslant 5$, $v_G(x_3) = 3$, and $H_3$, $K_3$ are connected, hence 4.4(A) ensures $x_3$ is adjacent to a vertex not in $\overline{H}_3$ and one not in $\overline{K}_3$. Without loss of generality $y_2 \notin V(\overline{H}_3)$ and $z_2 \notin V(\overline{K}_3)$ can be written. But then $V(H_3 \cap K_3) = \{x_3, r, s\}$, and the monovalent $x_3$ and $s$ can be removed from $K_3$ to form $K_4$. This gives $W(G, K_4) \subseteq \{x_2, z_2, r\}$, $|E(K_4)| \geqslant 3$, and $v_{K_4}(r) = 2$, contrary to 4.4(A).

Alternatively, suppose that $y_1 \notin V(\overline{H}_2)$. If $y_2, z_2 \in V(H_1)$ this is essentially the preceding case, and so $y_2 \notin V(\overline{K}_1)$ may also be assumed. Then $W(G, H_1 \cap K_2) \subseteq \{x_1, x_2, z_1, z_2\}$, $W(G, K_1 \cap H_2) \subseteq \{x_3, y_1, y_2, z_1, z_2\}$ and $W(G, K_1 \cap K_2) \subseteq \{x_1, y_2, z_1, z_2\}$. In $H_1 \cap K_2$ vertices $x_1, x_2$ are incident with disjoint edges since $t$ is not in any quadrilateral, hence 4.4(A) is contradicted if $|W(G, H_1 \cap K_2) \leqslant 3$;

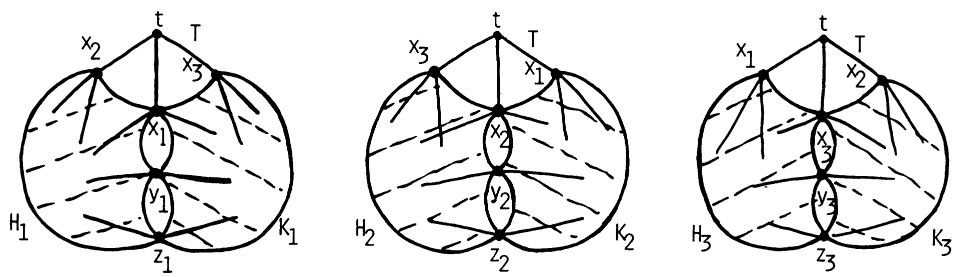



FIGURE 5B. *Decompositions of G with respect to T* therefore $W(G, H_1 \cap K_2) = \{x_1, x_2, z_1, z_2\}$. Then $z_1 \in V(K_2)$ and $z_1 \neq z_2$. Also $z_1 \neq y_2$, because $y_2 \in V(\overline{K}_1)$, so that $z_1 \notin (\overline{K}_2)$. Similarly $z_2 \notin V(\overline{H}_1)$. Now $W(G, K_1 \cap H_2) \subseteq \{x_3, y_1, y_2\}$ and $v_{K_1 \cap H_2}(x_3) \geqslant 2$, which implies $K_1 \cap H_2$ is a 2-arc with internal vertex $x_3$ and endvertices $y_1, y_2$. Using $|E(K_1)| \geqslant 5$,

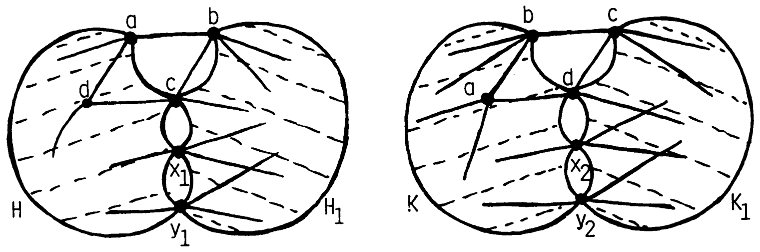
$$W(G, K_1 \cap K_2) \subseteq \{x_1, y_2, z_1\} \quad \text{and} \quad K_1 = (K_1 \cap H_2) \cup (K_1 \cap K_2),$$

4.4(A) implies $K_1 \cap K_2$ is a triad with endvertices $x_1, y_2, z_1$. With $y_2 \notin V(\overline{K}_1)$, this implies that $v_G(y_2) = 2$, contrary to $\gamma_B(G) \geqslant 3$. Neither alternative obtains and so the proposition is valid.

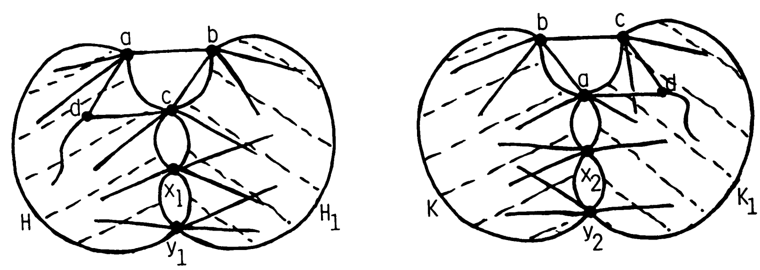
Proposition 5.3. *Suppose $G \in L$ and $G_t \notin L$ for all trivalent $t \in V(G)$. If $Q \in P_4(G)$ is a quadrilateral such that $G'_A \notin L$ for all $A \in E(Q)$, then $Q$ has at least two vertices that are trivalent in $G$.*

Proof. Assume the hypotheses, that $(a, A, b, B, c, C, d, D)$ is the circular sequence of vertices and edges in $Q$, and that $a, b, c$ are not trivalent in $G$. By Lemma 4.6 complementary $H, H_1 \in Q_3(P(G'_A))$ and $K, K_1 \in Q_3(P(G'_B))$ exist, each having at least six edges, with $a \notin V(H_1)$, $b \notin V(H \cup K_1)$, and $c \notin V(K)$. Choose $H_1$ minimal in $Q_3(P(G'_A))$ .

Suppose $c \notin V(H)$. Then $d \in V(H \cap H_1)$ and $v_{H_1}(d) \geqslant 2$ by the minimality of $H_1$. Now $v_G(d) \geqslant 4$ because $H_d \notin Q_3(P(G))$, and we change notation $H, H_1, a, b, c, d$ to $H_1, H, b, a, d, c$, respectively, and choose new $K, K_1 \in Q_3(P(G'_A))$ with regard to the new $B \in E(Q)$. This done, $c \in V(H)$ and $H_1$ can be replaced by a minimal member of $Q_3(P(G'_A))$ it contains. We may thus assume $c \in V(H)$ without loss of generality.




*Case* (1). *Here $c \in V(H)$, $d \in V(K)$, and $H_1$, $K_1$ are chosen minimal.*




*Case* (2). *Here $d \notin V(H_1 \cup K)$, $v_H(c) \geqslant 2$, $v_{K_1}(a) \geqslant 2$, and $H_1$, $K$ are chosen minimal.*

Figure 5C. *Notation for Cases* (1) *and* (2)

The argument divides into two cases, as shown in Figure 5C. If $d \in V(K)$, or $d \notin V(K)$ and $v_{K_1}(a) = 1$, replace $K_1$ by a minimal member of $Q_3(P(G'_B))$ it contains. Then $d \in V(K)$ and Case (1) applies with possibly $d \in V(H_1)$ or $a \in V(K_1)$. Alternatively $d \notin V(K)$ and $v_{K_1}(a) \geqslant 2$. Then change $H, H_1, K, K_1, a, b, c, d$ to $K_1, K, H_1, H, c, b, a, d$ respectively, so that $d \notin V(H_1)$ and $v_H(c) \geqslant 2$. Replace $H_1$ by a minimal member of $Q_3(P(G'_A))$ it contains. This does not alter the relationships of the previous paragraph. Now either Case (1) can again be arranged, or $d \notin V(K)$, $v_{K_1}(a) \geqslant 2$ and $K$ can also be chosen minimal. This is Case (2).

It is convenient to treat these two cases with respect to another pair of alternatives. Using elementary properties of the $H \cup Q$-components of $G$, with $|E(H)| \geqslant 6$ and Remark 4.4, we see that either:

(A) *$H_1$ is the union of two triads in $G'_A$ with distinct centres $b, t \notin V(H)$ and common endvertices $c, x, y$, where $x = x_1$ and $y = y_1$, or*

(B) *$H_1$ contains an arc of length at least three, having only its endvertices $b, c$ in $\overline{H}_1$.*

These alternatives may hold in either of the above cases. Suppose first that (A) obtains. To eliminate various possibilities Remark 4.4 and Lemma 4.10 will often be used.

In both Cases (1) and (2), if $t \in V(K \cap K_1)$ then $(K_1)_t \notin Q_3(P(G))$ implies a vertex $z \notin V(\overline{K}_1)$ exists adjacent to $t$ in $G$. However $\{x, y, c\} \subseteq V(\overline{K}_1)$ because $b \notin V(K_1)$. But then $v_G(t) \geqslant 4$, contrary to assumption. Thus $t \notin V(K)$ and $x = x_2$, $y = y_2$ may be assumed. Now $\gamma_P(G) \geqslant 4$ implies $d \notin V(\overline{H})$, and in Case (1) $a \notin V(K_1)$ because $a \notin \{x, y\}$. In Case (1) take $I = (H \cap K)'_D$ and $I_1 = (H \cap K_1)'_C$. Then $W(G, I) \subseteq \{a, x, y, d\}$, $W(G, I_1) \subseteq \{c, x, y, d\}$, $v_I(a) \geqslant 2$, $v_{I_1}(c) \geqslant 1$, and $v_{I_1}(d) \geqslant 1$. Remark 4.4, using edges $C$ and $D$, implies $I$ and $I_1$ are both connected and have all four possible vertices of attachment. Thus $d, x, y$ are not trivalent in $G$. The second diagram in Figure 5D shows this situation. In Case (2) we see the graph $K$ is the union of two triads of $G'_B$ with distinct centres $b, u \notin V(K_1)$ and common endvertices $a, x, y$, by applying 4.4(A) to $K_b$. Then $W(G, (K_1)_t) \subseteq \{a, c, x, y\}$ and $|E((K_1)_t)| \geqslant 6$ imply $(K_1)_t$ is connected, by 4.4, hence $x$ and $y$ are not trivalent in $G$. The third diagram in Figure 5D pertains here.

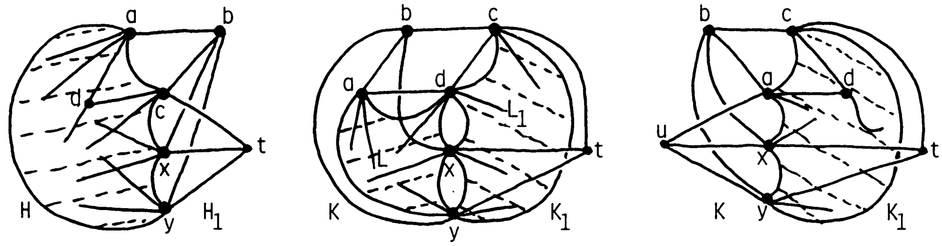



FIGURE 5D. *Cases* (1) *and* (2) *under* (A)

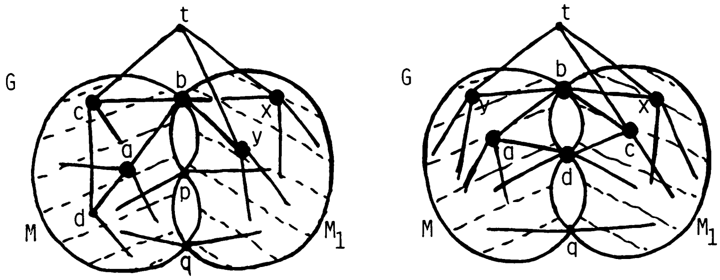



FIGURE 5E. *Alternatives for* $G_t \notin L$ *in* (A)

Both these possibilities can be eliminated using $G_t \notin L$. By Lemma 4.5 complementary $J$, $J_1 \in Q_3(P(G_t))$ exist. Now $J$, $J_1 \notin Q_3(P(G))$ implies $\{c, x, y\} \not\subseteq V(J)$ and $\{c, x, y\} \not\subseteq V(J_1)$, so that $b \in V(J \cap J_1)$. Then $J_b, (J_1)_b \notin Q_3(P(G))$ and $v_G(b) = 4$ imply $v_J(b) = v_{J_1}(b) = 2$ and that $b$ is adjacent to no vertex in $V(J \cap J_1)$. Notation is easily arranged so that one of the following alternatives obtains,

(A1) $a, c \notin V(J_1)$ *and* $x, y \notin V(J)$, *or*

(A2) $a, y \notin V(J_1)$, $c, x \notin V(J)$, *and* $d \in V(J \cap J_1)$, as in Figure 5E.

Assume Case (1) applies. In (A1) we have $W(G, I \cap J) \subseteq \{a, d, p, q\}$ and $W(G, I_1 \cap J) \subseteq \{c, d, p, q\}$. Then $v_{I \cap J}(a) \geqslant 2$, $v_{I_1 \cap J}(c) \geqslant 1$, and the fact that the vertex $d$ is adjacent to $a$ and $c$ in $G$ imply, using 4.4(A), that $W(G, I \cap J) = \{a, d, p, q\}$ and $|W(G, I_1 \cap J)| \geqslant 3$, which is contrary to $p, q \notin V(I \cap I_1)$. In (A2) we have $W(G, I \cap J) \subseteq \{a, d, y, q\}$ and $W(G, I_1 \cap J) \subseteq \{d, y, q\}$. Then $v_{I \cap J}(a) \geqslant 2$, $v_{I \cap J}(y) \geqslant 1$ and $y$ is not adjacent to $a$ in $G$. Thus $W(G, I \cap J) = \{a, d, y, q\}$, by 4.4(A). Also $v_{I_1 \cap J}(y) \geqslant 1$ and $y$ is adjacent to some vertex $y' \notin V(\bar{I}_1)$, by the minimality of $K_1$. Thus $y' \in V(I_1 \cap J)$, so that $q \in W(G, I_1 \cap J)$, by 4.4(A). This contradicts the fact that $q \notin V(I \cap I_1)$. Case (1) is ruled out. In Case (2), for both (A1) and (A2), we can assume $u = q$. Then $|W(G, (J_b)_u)| = 3$ and $v_{(J_b)_u}(a) \geqslant 2$. Now 4.4(A) implies $(J_b)_u$ is an arc of length 2 with internal vertex $a$. In both (A1) and (A2) this contradicts $\gamma_P(G) \geqslant 4$.

This leads us back to alternative (B). The arc in $H_1$ contains $x_2$ or $y_2$. In Case (1) assume, without loss of generality, that $x_2 \notin V(H)$. Then $W(G, H \cap K) \subseteq \{a, d, x_1, y_1, y_2\}$, $W(G, H \cap K_1) \subseteq \{c, d, x_1, y_1, y_2\}$, $W(G, H_1 \cap K) \subseteq \{b, x_1, y_1, x_2, y_2\}$, and $W(G, H_1 \cap K_1) \subseteq \{c, x_1, y_1, x_2, y_2\}$. Then $v_{H_1 \cap K}(b) \geqslant 2$, $v_{H_1 \cap K_1}(c) \geqslant 1$, $v_{H_1 \cap K_1}(x_2) \geqslant 2$, and $d$ is adjacent to $a$ and $c$ in $G$. Suppose that $y_2 \in V(H_1)$. Then $a \notin V(K_1)$, $v_{H \cap K}(a) \geqslant 3$, and so $W(G, H \cap K) = \{a, d, x_1, y_1\}$, by Remark 4.4(A). Now $W(G, H_1 \cap K_1) \subseteq \{c, x_2, y_2\}$ and, using $v_{H_1 \cap K_1}(x_2) \geqslant 2$ with 4.4(A), this implies $H_1 \cap K_1$ is a 2-arc with centre $x_2$ and endvertices $c$, $y_2$. But $x_2$ and $y_2$ are not adjacent, by the minimality of $K_1$. It follows that $y_2 \notin V(H_1)$ and $x_1, y_1, x_2, y_2$ are distinct. Now $v_{H_1 \cap K}(b) \geqslant 2$ and $W(G, H_1 \cap K) \subseteq \{b, x_1, y_1, x_2\}$ so that $x_1 \in V(K)$ can be assumed. It follows that $x_1 \notin V(\bar{K})$, $v_{H_1 \cap K}(x_1) \geqslant 2$, and $W(G, H_1 \cap K) = \{b, x_1, y_1, x_2\}$. But then $y_1 \notin V(\bar{K})$ also, which implies that $W(G, H_1 \cap K_1) \subseteq \{c, x_2\}$, contrary to $v_{H_1 \cap K_1}(x_2) \geqslant 2$.

In Case (2) of (B) both $x_2 \notin V(H)$ and $x_1 \notin V(K_1)$ can be assumed without loss of generality, for otherwise Case (2) of (A) applies. Then $W(G, H \cap K_1) \subseteq \{a, c, y_1, y_2\}$ and $W(G, H_1 \cap K) \subseteq \{b, x_1, y_1, x_2, y_2\}$, while $v_{H \cap K_1}(a)$, $v_{H \cap K_1}(c)$, $v_{H_1 \cap K}(b)$, $v_{H_1 \cap K}(x_1)$, $v_{H_1 \cap K}(x_2) \geqslant 2$, and $d \notin V(H \cap K_1)$. This implies that $W(G, H \cap K_1) = \{a, c, y_1, y_2\}$ and $|W(G, H_1 \cap K)| \geqslant 4$. The former conclusion implies $y_1 \in V(K_1)$, $y_2 \in V(H)$ and $y_1 \neq y_2$. By the assumptions of this case both $y_1 \neq x_2$ and $y_2 \neq x_1$. Thus $y_1 \notin V(K)$ and $y_2 \notin V(H_1)$, so that $W(G, H_1 \cap K) \subseteq \{b, x_1, x_2\}$, contrary to the latter conclusion above. This eliminates the last alternative and proves the theorem.

**6. The ladder theorem.** A decomposition theory for the $G \in M$ is presented here. It is shown that $G$ is either indecomposable or is decomposable and decomposes into certain fragments. Amongst the possible kinds of fragments we shall distinguish some which will be called "degenerate". Figure 6A depicts the indecomposable $G$, Figure 6C the nondegenerate fragments, and Figure 6D the decomposable $G$ with degenerate fragments.

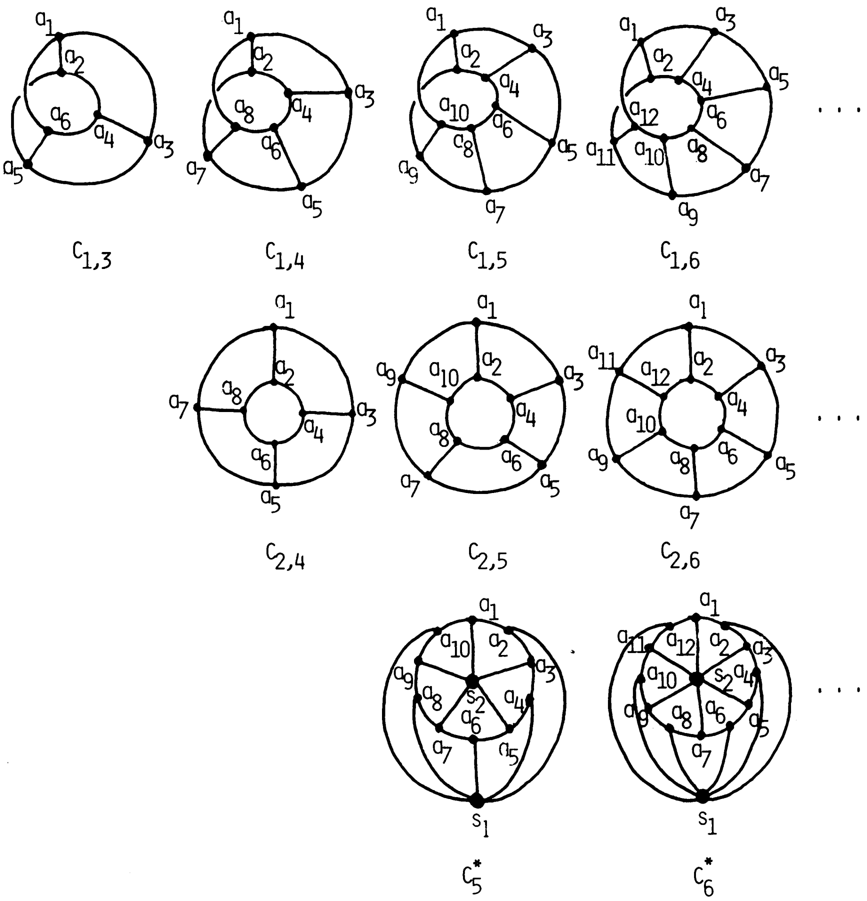



FIGURE 6A. *The indecomposable $G \in M$*

Suppose $G \in M$. Call $x \in V(G)$ a *node* when $v_G(x) \geqslant 4$. Then an edge $A \in E(G)$ is *nodal* when it is incident with a node, and a triad $T \subseteq G$ is *nodal* if it contains a nodal edge. For any $Q, Q' \in P_4(G)$ write $Q \sim Q'$ when a sequence $(Q_0, Q_1, \ldots, Q_n)$ drawn from $P_4(G)$ exists such that

(A) $Q = Q_0$, $Q = Q_n$, *and*

(B) $|E(Q_{j-1} \cap Q_j)| \geqslant 1$ *and* $E(Q_{j-1} \cap Q_j) \neq \{A\}$ *where $A$ is a nodal edge, for* $1 \leqslant j \leqslant n$.

Then $\sim$ is an equivalence relation on $P_4(G)$. Define a *constituent* of $G$ to be the union of all quadrilaterals in an equivalence class of $P_4(G)$. Then $G$ is *indecomposable* or *decomposable* according as it has one or more than one constituent, respectively. A *fragment* of $G$ is the union of a constituent of $G$ with the triads whose centres it contains.

There are three classes of indecomposable graphs, the *Möbius ladders* $C_{1,j}$ for $j \geqslant 3$, the *cylindrical ladders* $C_{2,j}$ for $j \geqslant 4$, and the *circular coladders* $C_j^*$ for $j \geqslant 5$. There are also three classes of constitutents of decomposable graphs, the *ladders* $L_j$ for $j \geqslant 1$, the (2, $j$)-*bicliques* $K_{2,j}$ for $j \geqslant 3$, and the *coladders* $L_j^*$ for $j \geqslant 3$. These appear in Figures 6A and 6B. Corollary 6.6 will show that constituents are induced subgraphs of $G$.

The constitutents of $G \in M$ do not always contain every edge of $G$. Isomorphic constituents may be imbedded differently in $G$, especially $L_1$, $L_2$, and $K_{2,3}$. Fragments better illustrate the structure of a decomposition. Figure 6C gives the *ladder*

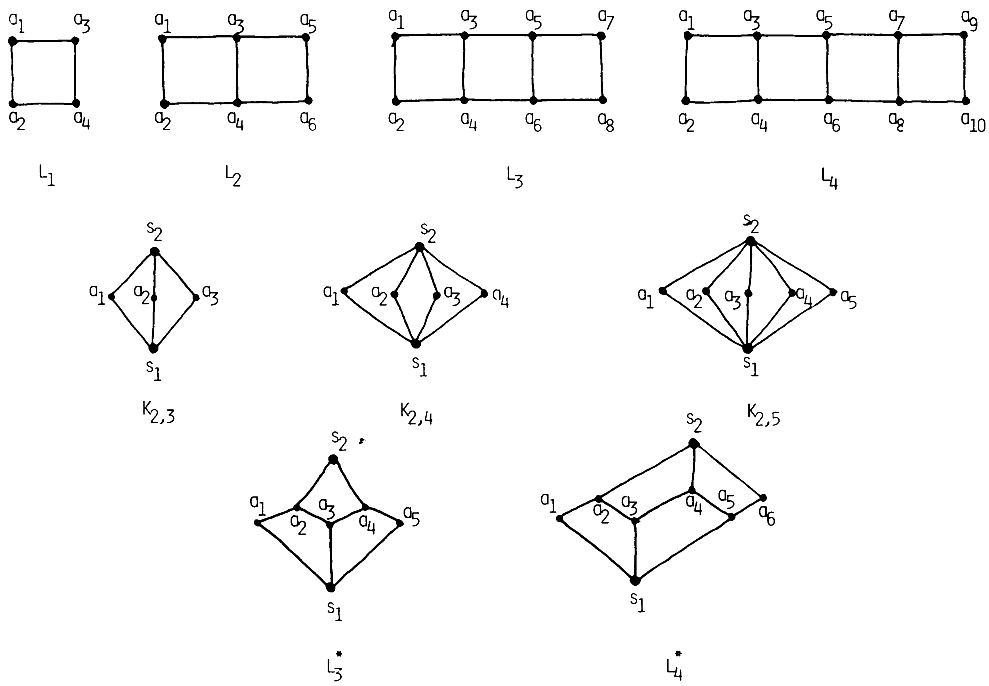



FIGURE 6B. *Possible constitutents of $G \in M$*

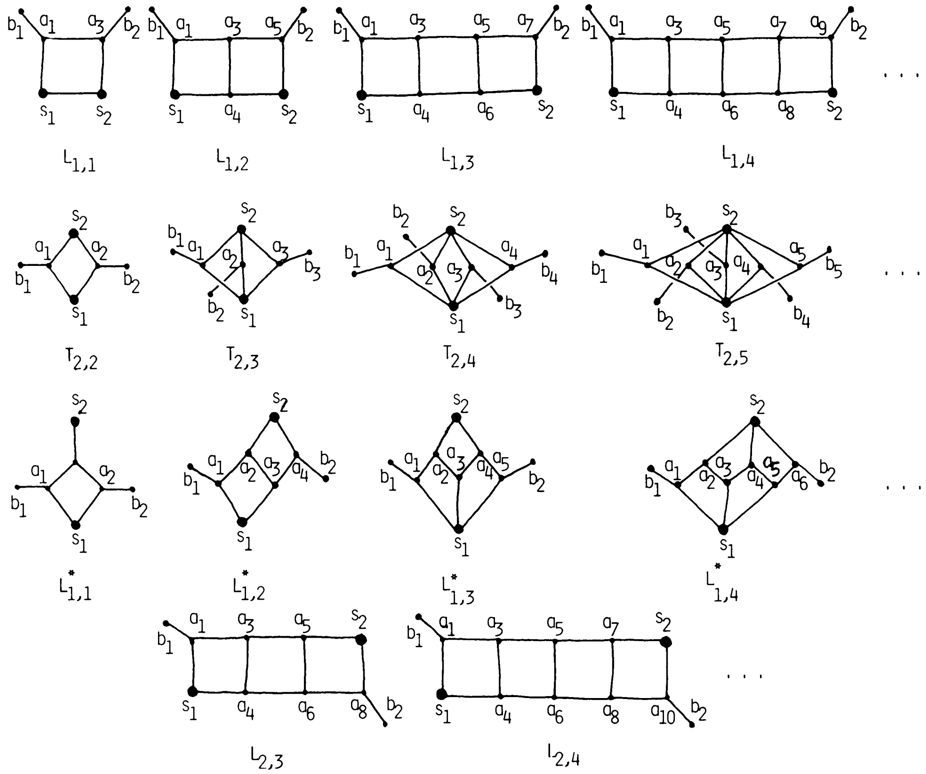



FIGURE 6C. *Fragments for decomposable* $G \in M$

fragments $L_{1,j}$ for $j \geqslant 1$ and $L_{2,j}$ for $j \geqslant 3$, the nondegenerate *coladder* fragments $L^*_{1,j}$ for $j \geqslant 1$, and the *triad clusters* $T_{2,j}$ for $j \geqslant 2$. The large vertices $s_1$ and $s_2$ in these diagrams represent the nodes of $G$ contained in the corresponding constituent, except possibly for $s_2$ in either $L^*_{1,1}$ or $T_{2,3}$. In these diagrams the vertices labelled $a_i$ are trivalent in $G$ and hence cannot be vertices of attachment for the fragments. Degenerate coladder fragments $L^*_{2,j}$ for even $j \geqslant 4$ arise from the $L^*_{1,j}$ by identifying $b_1$ and $b_2$ as $b$. If $G \in M$ has a fragment $F \cong L^*_{2,j}$ for even $j \geqslant 4$ then $\bar{F}$ is a triad, by 4.4(A), with centre $t$ and endvertices $b, s_1, s_2$. Then $G$ is determined by $j$ up to isomorphism, as shown in Figure 6D. In Figures 6D and 6E edges in two constituents are specially marked to make identification of constituents easier.

Using the equivalence relation $\sim$ and statements (A), (B) and (C) at the beginning to §5, we have obtained a unique decomposition of any $G \in M$ into fragments. Denote by $W$ the set of connected graphs defined by Figures 6A and 6C. Figure 6E provides a graph $G \in M$ sufficiently general to include all the types of fragments in Figure 6C. The main theorem in this paper asserts that no other fragments except the degenerate coladder fragments are possible.

THEOREM 6.1. *If* $G \in M$ *and* $F$ *is a fragment of* $G$ *then* $F \cong L^*_{2,k}$ *for even* $k \geqslant 4$ *or there exists some* $H \in W$ *such that* $F \cong H$.

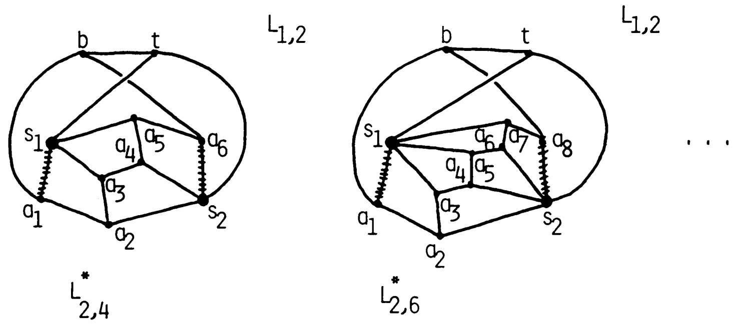



FIGURE 6D. *The $G \in M$ with a degenerate fragment*

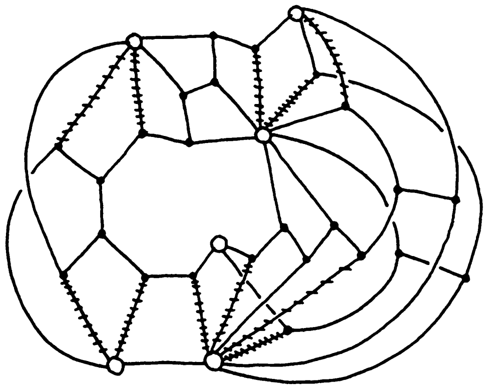


FIGURE 6E. *A typical decomposable $G \in M$*

An edge of $G \in M$ not in both a triad and a quadrilateral is called *singular*. *Singular fragments* are those containing singular edges. Before proving Theorem 6.1 we examine the singular fragments of $G$ more closely.

PROPOSITION 6.2. *Suppose $G \in M$, that $Q \in P_4(G)$, and $A \in E(Q)$ has nodes of $G$ as endvertices. If $F$ is the fragment of $G$ containing $Q$ then $F \cong L_{1,1}$, and connected edge-disjoint subgraphs $J$, $J_1$ of $\bar{F}$ and vertices $c, e \in V(G) - V(F)$ exist such that $G = J \cup F \cup J_1$ with $W(G, F) = \{b_1, b_2, s_1, s_2\}$, $W(G, J) = \{b_1, s_1, c, e\}$, and $W(G, J_1) = \{b_2, s_2, c, e\}$ in the notation of $L_{1,1}$.*

PROOF. By Lemma 4.6 complementary $H, H_1 \in Q_3(P(G'_A))$ exist. Moreover, these can be chosen so that $6 \leqslant |E(H)| \leqslant |E(H_1)|$ and $H$ is minimal in $Q_3(P(G'_A))$. Denote the endvertices of $A$ by $s_1, s_2$ so that $s_1 \notin V(H_1)$ and $s_2 \notin V(H)$, and write $V(H \cap H_1) = \{a, c, e\}$. The hypotheses of Lemma 4.3 apply to $G'_A$ and so $|E(H_1)| = 6$ forces conclusion (B) with $H \cong D_2$ and $H_1 \cong D_2$. But then both $H$ and $H_1$ contain triads of $G$ with endvertices $a, c, e$; a contradiction because $G \not\cong K_{3,3}$. It follows that $|E(H_1)| \geqslant 7$. By the minimality of $H$, $v_H(a) \geqslant 2$, $v_H(c) \geqslant 2$, and $v_H(e) \geqslant 2$, while $a, c, e$ are pairwise nonadjacent in $H$. Applying Remark 4.4 and $|E(H_1)| \geqslant 7$ to $(H_1)_a$, $(H_1)_b$, and $(H_1)_c$ gives the existence of vertices $b, d, f \in V(\bar{H}_1)$, adjacent in $G$ to $a, c, e$, respectively. By Proposition 5.3 two vertices of $Q$ are trivalent in $G$, and so it can be assumed without loss of generality that $V(Q) = \{s_1, s_2, a, b\}$. Set $a = a_1$, $b = a_3$, and let $b_1$, $b_2$, respectively, be the vertices not in $Q$ adjacent to $a_1, a_3$. Set $J = H_a$ and $J_1 = ((H_1)_a)_b$, noting $|E(J)| \geqslant 4$ and $|E(J_1)| \geqslant 4$, so that $J$ and $J_1$ are connected, $W(G, J) = \{b_1, s_1, c, e\}$ and $W(G, J_1) = \{b_2, s_2, c, e\}$, by Remark 4.4(B). Any quadrilateral intersecting $Q$ in one or more edges intersects in exactly a pivot edge, thus $F \cong L_{1,1}$. In general $W(G, F) = \{s_1, s_2, b_2, b_2\}$. Clearly $G = J \cup F \cup J_1$ is an edge-disjoint decomposition. This completes the proof.

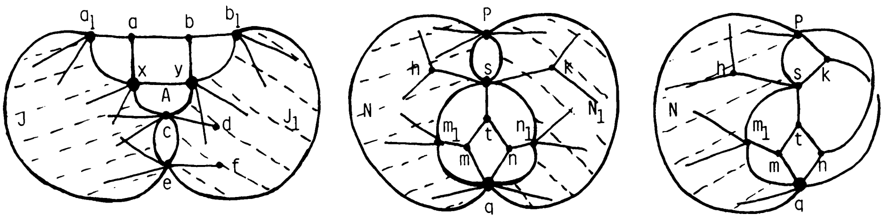



Diagram 1 Diagram 2 Diagram 3

FIGURE 6F. *The singular fragments of a* $G \in M$

REMARK 6.3. Suppose $A \in E(G)$ has nodes of $G$ as endvertices. Then some $Q \in P_4(G)$ exists with $A \in E(Q)$, by Proposition 5.1. By the decomposition of Proposition 6.2, at most one such $Q$ can contain any of $a, c, e$. Thus at most three such quadrilaterals can exist. When $J$ is minimal with respect to these decompositions we see that $v_J(c) \geqslant 2$, $v_J(e) \geqslant 2$. It follows from Proposition 6.2 that any quadilateral in $G \in M$ with two adjacent nodes of $G$ is in a fragment $F \cong L_{1,1}$, which is an induced subgraph of $G$. Using this fact, it is not difficult to see that $c, e$, respectively, are adjacent to $d, f \notin V(\bar{J}_1)$, and that we also may assume $d \neq f$. Diagram 1 of Figure 6F illustrates this situation.

PROPOSITION 6.4. *Suppose* $G \in M$ *and* $F$ *is a fragment of* $G$ *with* $A \in E(F)$ *such that* $\gamma_P(G''_A) \geqslant 4$. *Then* $F \cong L^*_{1,1}$ *with* $W(G, F) = \{b_1, b_2, s_1, s_2\}$ *in the notation used in* $L^*_{1,1}$. *Moreover, there exist edge-disjoint connected subgraphs* $N_i$ *of* $\bar{F}$ *with* $G = N_1 \cup F \cup N_2$ *and vertices* $n_i \notin V(\bar{N}_i)$, *for* $i \in \{1, 2\}$, *adjacent to* $s_2$, *such that either*:

(A) $N_i$ *is a triad with centre* $n_i$ *and endvertices* $b_i, s_1, s_2$; *or*

(B) $N_i \in Q_4(P(G))$ *and* $W(G, N_i) = \{p, s_1, s_2, b_i\}$.

PROOF. By Lemma 4.7, complementary $K, K_1 \in Q_4(P(G'_A))$ exist, with $|E(K)| \geqslant 6$, $|E(K_1)| \geqslant 6$, and $V(K \cap K_1) = \{p, s_1, s_2, a_2\}$, where $s_2$, $a_2$ are the endvertices of $A$. By Proposition 5.1 we may assume $a_2$ is trivalent. Using Remark 4.4 and $\gamma_P(G) \geqslant 4$ it follows that distinct vertices $n_1, a_1 \notin V(K_1)$ and $n_2, a_3 \notin V(K)$ exist, with $n_1, n_2$ adjacent to $s_2$ and $a_1, a_3$ adjacent to $a_2$. By Proposition 5.2 there is a quadrilateral $Q$ with $a_2 \in V(Q)$. Then $A \notin E(Q)$ and without loss of generality we can write $V(Q) = \{s_1, a_1, a_2, a_3\}$. Remark 4.4 and $\gamma_P(G''_A) \geqslant 4$ imply $v_K(s_1) \geqslant 2$ and $v_{K_1}(s_1) \geqslant 2$, so that $s_1$ must be a node of $G$. There exist vertices $b_1, b_2 \notin V(Q)$ adjacent to $a_1, a_3$, respectively. By Proposition 5.3, and the notational interchangeability of $K, K_1$, it may be assumed that $v_G(a_3) = 3$. Write $N_2 = ((K_1)_{a_2})_{a_3}$. By

$\gamma_P(G) \geqslant 4$, $n_2 \neq b_2$, and so $n_2 \notin V(\overline{N}_2)$. If $p = b_2$ then $N_2 = T_2$ is a triad with centre $n_2$ and endvertices $b_2, s_1, s_2$, by Remark 4.4(A). We may assume $p \neq b_2$. Then Remark 4.4(B), with $|E(N_2)| \geqslant 3$ and $v_{N_2}(b_2) \geqslant 2$, implies $N_2 \in Q_4(P(G))$ is connected, with $W(G, N_2) = \{p, s_1, s_2, b_2\}$. Suppose that also $v_G(a_1) = 3$, and let $N_1 = (K_{a_1})_{b_1}$. Just as above, if $p = b$, then $N_1 = T_1$ is a triad with centre $n_1$ and endvertices $b_1, s_1, s_2$ and if $p \neq b_1$ then $N_1 \in Q_4(P(G))$ is connected with $W(G, N_1) = \{p, s_1, s_2, b_1\}$ and $n_1 \notin V(\overline{N}_1)$. If $b_1 = b_2$ then $N_1$ and $N_2$ are triads with the same endvertices, contrary to $G \in L$ and $G \not\cong K_{3,3}$. Thus $b_1 \neq b_2$ and so $F \cong L^*_{1,1}$, with $W(G, F) = \{b_1, b_2, s_1, s_2\}$.

It remains to suppose $v_G(a_1) \geqslant 4$ and look to Proposition 6.2 for a contradiction. Keep the present notation, with $a_2, s_2, a_3, b_2, a_1, s_1$ replacing $a_1, b_1, a_3, b_2, s_1, s_2$, respectively, in diagram 1 of Figure 6F. Then $W(G, J \cap K) \subseteq \{a_1, s_2, c, e, p\}$, $W(G, J_1 \cap K_1) \subseteq \{s_1, p, b_2, c, e\}$, $v_{J \cap K}(a_1) \geqslant 2$, $v_{J \cap K}(s_2) \geqslant 1$, and $v_{J_1 \cap K_1}(s_1) \geqslant 1$. From 6.2 it is clear that $N_2$ satisfies conclusion (B) of this proposition. Now $v_{J_1 \cap K_1}(b_2) \geqslant 2$, and so Remark 4.4 with $\gamma_P(G) \geqslant 4$ imply $|W(G, J \cap K)| \geqslant 4$ and $|W(G, J_1 \cap K_1)| \geqslant 4$. We may assume, without loss of generality, that $c \in V(K \cap K_1)$. Then $c = p$ and so $e \in V(K \cap K_1)$. But now $e = p$, the required contradiction. It follows that $v_G(a_1) = 3$.

Remark 6.5. Some of the possibilities allowed by Proposition 6.4 are listed here for clarity.

(1) If $b_1 \neq p$ and $b_2 \neq p$ then 6.4(B) holds for $i = 1$ and $i = 2$, and diagram 2 of Figure 6F applies.

(2) If $b_2 = p$ then 6.4(A) holds for $i = 2$, and diagram 3 of Figure 6F applies. A similar decomposition occurs when $b_1 = p$.

(3) It is possible that 6.4(A) holds for $i = 1$, yielding $T_1$ and $N_2$ with $p = b_1$, and separately for $i = 2$, yielding $T_2'$ and $N_1'$ with $p' = b_2$. Let $H = \overline{T_1 \cup F \cup T_2'}$ in this case. Then $W(G, H) \subseteq \{b_1, b_2, s_1, s_2\}$ and in particular we may have:

(a) $W(G, H) = \{b_1, b_2\}$, $H$ is a link-graph and $G$ has fragments $L^*_{1,1}$, $L^*_{1,1}$, $L^*_{1,1}$, $L^*_{1,1}$;

(b) $W(G, H) = \{b_1, b_2, s_1\}$, $H$ is a triad and $G$ has fragments $L^*_{1,1}$, $L^*_{1,1}$, $T_{2,3}$, $T_{2,3}$;

(c) $W(G, H) = \{b_1, b_2, s_2\}$, $H$ is a triad and $G$ has fragments $L^*_{1,1}$, $L^*_{1,2}$, $L^*_{1,2}$, $T_{2,2}$; and

(d) $W(G, H) = \{b_1, b_2, s_1, s_2\}$, and the number of fragments in $H$ is unrestricted.

(4) In case 3(d), Propositions 6.2 and 6.3 can be used to show that $b_1$, $b_2$ are not adjacent in $H$. Thus, except for the single graph in case 3(a), any edge of $G$ in no quadrilateral is in a fragment $F \cong L^*_{1,1}$ which is an induced subgraph of $G$.

(5) An edge of $G$ in no quadrilateral may be in one or two fragments $F \cong L^*_{1,1}$ only. This depends on whether the edge is in one or two triads of $G$, respectively.

Proof of Theorem 6.1. Suppose $S \subseteq P_4(G)$ is an equivalence class of the relation $\sim$ and $F$ is the fragment of $G$ it determines. If $F$ contains a singular edge of $G$ then $F \cong L_{1,1}$ or $F \cong L^*_{1,1}$, by Propositions 6.2 and 6.4. Assume henceforth that $F$ contains no singular edge. Then any $Q \in S$ has $V(Q) = \{a_1, a_2, c_1, c_2\}$ with vertices $a_1, a_2$ trivalent and nonadjacent in $G$. For some maximum $k \geqslant 2$ there exist triads $T_1, T_2, \ldots, T_k$ in $G$ with distinct centres $a_1, a_2, \ldots, a_k$, and endvertices $c_1$, $c_2$, and

$b_1, b_2, \dots, b_k$, respectively. If $b_i = b_j$ for unequal $i$, $j$ then $T_i \cup T_j \cong D_2$ as in Lemma 4.3(b), in which case $G \cong K_{3,3}$ by Remark 4.4, and $F = G \cong C_{1,3}$. Otherwise $b_1, b_2, \dots, b_k$ are distinct. If either $k \geqslant 3$ (when $k = 3$ one of $c_1$, $c_2$ may be trivalent), or $k = 2$ and both $c_1$, $c_2$ are nodes of $G$ then it is easily verified that $F = \bigcup_{i=1}^{k} T_i \cong T_{2,k}$. Assume these cases do not hold. Then $|S| \geqslant 2$, any $Q \in S$ contains at most one node of $G$, and quadrilaterals $Q'$ of $G$ intersect those of $S$ in either the null graph, a vertex-graph or a link-graph of $G$. When $Q' \notin S$ such a vertex-graph or link-graph is nodal.

Adopt the notation of Figure 6B and show quadrilaterals $Q_1, Q_2, \dots, Q_n \in S$ exist for $n \geqslant 2$ such that either statement (A) or (B) which follow apply. The quadrilaterals are specified by their vertices written in circular order, and edges by their endvertices.

(A) *We can write* $Q_i = (a_{2i-1}, a_{2i}, a_{2i+2}, a_{2i+1})$ *for* $1 \leqslant i \leqslant n$, *where the edges* $a_{2i-1}a_{2i+1}$, $a_{2i}a_{2i+2}$ *are in no other quadrilateral of* $G$, *the vertices* $a_1, a_2, \dots, a_{2n+2}$ *are distinct, and at most* $a_1$ *and one of* $a_{2n+1}$, $a_{2n+2}$ *are nodes of* $G$.

(B) *We can write* $Q_i = (s_j, a_i, a_{i+1}, a_{i+2})$ *for* $1 \leqslant i \leqslant n$ *and* $j \in \{1, 2\}$ *with* $j \equiv i \pmod 2$, *where the vertices* $s_1, s_2, a_1, \dots, a_{n+2}$ *are distinct and, when* $n \geqslant 3$, *at most* $s_1$ *and* $s_2$ *are nodes of* $G$.

Set $H_n = Q_1 \cup Q_2 \cup \cdots \cup Q_n$. Because $|S| \geqslant 2$ there exist $Q_1, Q_2 \in S$, adjacent under $\sim$, such that $Q_1 \cap Q_2$ is a link-graph with endvertices trivalent in $G$. Statement (B) with $n = 2$ applies trivially. Now $H_2$ is an induced subgraph of $G$, by $\gamma_P(G) \geqslant 4$, and the assumption that quadrilaterals of $G$ meet those of $S$ in at most one edge. Either (A) with $n = 2$ also applies or, with some simple adjustments in notation, a quadrilateral $Q_3' = (s_1, a_3, a_4, a_5)$ can be assumed to exist. Suppose the latter case applies. If $s_1$ and $a_4$ are nodes of $G$, then Proposition 6.2 ensures $a_1$ and $s_2$ are centres of triads $T_1$ and $T_2$, respectively. Let $b_1'$ and $b_2'$ be the endvertices of these triads not in $H_2$. If $b_1' = b_2'$ then Remark 4.4 applied to $T_1 \cup H_2 \cup T_2$ with $s_1$ and $a_4$ nodes yields a contradiction, so that $b_1' \neq b_2'$. It is now routine to check $F = T_1 \cup H_2 \cup T_2 \cong L_{1,2}$. If $s_1$ and $a_4$ are not both nodes of $G$ assume $a_4$ is trivalent, without loss of generality. Then $Q_3' \in S$ and we may drop the prime and consider $H_3$ in the notation of (B). If $s_1$ is trivalent then Remark 4.4 applies to yield $\overline{H}_3$ is a triad and that $G \cong C_{2,4}$ is the indecomposable graph of a cube. We may assume $s_1$ is a node of $G$. Then $a_1$ and $a_5$ are trivalent, by Proposition 6.2, and case (B) applies with $n \geqslant 3$.

Suppose (A) holds and, inductively, that $n$ is maximum. Then $n \geqslant 2$ and $W(G, H_n) = \{a_1, a_2, a_{2n+1}, a_{2n+2}\}$. If $H_n$ contains nodes of $G$ we can assume without loss of generality that $a_1$ is a node. By Proposition 6.2 it follows that $a_2$ is trivalent. If one of $a_{2n+1}$ or $a_{2n+2}$ is also a node of $G$ then let $T_1$ and $T_2$ be the triads of $G$ with respective centres $a_2$ and either $a_{2n+2}$ or $a_{2n+1}$. Let $b_1$ and $b_2$ be their endvertices not in $H_n$. If $b_1 = b_2$ then Remark 4.4 with $|E(\overline{H}_n)| \geqslant 4$ contradicts $G \in L$. Thus $b_1 \neq b_2$ and we readily see that $F = T_1 \cup H_n \cup T_2$, and $F \cong L_{1,n}$ or $F \cong L_{2,n}$ for $n \geqslant 2$. Here $L_{2,2} \cong L_{2,2}^*$. Alternatively $a_{2n+1}$ and $a_{2n+2}$ are trivalent. As $F$ has no singular edge there is a quadrilateral $Q_{n+1}' = (a_{2n+1}, a_{2n+2}, a_{2n+4}', a_{2n+3}') \in S$. By the maximality of $n$ and Remark 4.4 for $G \in L$ it follows that $Q_1 \cap Q_{n+1}'$ is a link-graph with endvertices $a_1$ and $a_2$, which are trivalent. Thus $F = H_n \cup Q_{n+1}' = G$, and either $G \cong C_{1,n+1}$ for $n \geqslant 3$ or $G \cong C_{2,n+1}$ for $n \geqslant 4$.

In the cases remaining (B) applies, with $n$ maximum. Then $n \geqslant 3$, and $s_1$ is a node of $G$, by earlier remarks. As quadrilaterals of $G$ intersect those of $S$ in at most one edge it follows that $s_1$ and $s_2$ are not adjacent. By inductive assumption $a_1, a_2, \ldots, a_{n+2}$ are trivalent, whence $W(G, H_n) = \{s_1, s_2, a_1, a_{n+2}\}$. If $a_1$ and $a_{n+2}$ are adjacent in $G$ then $n = 2k$ for $k \geqslant 3$, and $G \cong C^*_{k+1}$. The case $k = 2$ was treated earlier and gave the cube. It is excluded here because $s_1$ is a node. At this point all the indecomposable $G \in M$ shown in Figure 6B have been constructed. In what remains assume $a_1$ and $a_{n+2}$ are not adjacent, so that $H_n$ is an induced subgraph of $G$.

Applying Remark 4.4 and $\gamma_P(G) \geqslant 4$ we see that $s_2 \in W(G, H_n)$. If $s_1 \notin W(G, H_n)$ then $\overline{H}_n$ is a triad of $G$ with centre $x \in V(H_n)$ and endvertices $a_1, a_{n+2}, s_2$. When $n$ is odd this contradicts the maximality of $n$, and when $n$ is even this contradicts $\gamma_P(G) \geqslant 4$. It remains to consider the case where $W(G, H_n) = \{a_1, a_{n+2}, s_1, s_2\}$. Then $s_1, s_2$ are nodes, except possibly for $s_2$ when $n = 3$, and $a_1, a_{n+2}$ are centres of triads $T_1$, $T_2$ of $G$ with endvertices $b_1, b_2 \in V(H_n)$, respectively. If $s_2$ is trivalent when $n = 3$ let $b_3$ be the vertex not in $H_3$ adjacent to $s_2$. If $b_1 = b_3$ or $b_2 = b_3$ then Remark 4.4 implies $\kappa_P(G) \leqslant 3$. Thus $s_2$ is incident with a singular edge of $G$, contrary to Proposition 6.4. If $b_1 = b_2 = b$ then the complement $T_1 \cup H_n \cup T_2$ is a triad of $G$ with centre $x$ and endvertices $b, s_1, s_2$. When $n$ is odd this contradicts Proposition 6.4, because $xs_2$ is a singular edge of $G$ and $s_1$ is a node. When $n$ is even this produces the degenerate fragments of Figure 6D. Finally, when $b_1 \neq b_2$ the maximality of $n$ implies $F = T_1 \cup H_n \cup T_2 \cong L^*_{1,n}$, for $n \geqslant 3$. This completes the proof.

Corollary 6.6. *The constituents of any $G \in M$ are induced subgraphs of $G$.*

Proof. Let $F$ be a fragment of $G$ and $F_1$ be its corresponding constituent. By Theorem 6.1 the fragment $F$ can be expressed as in Figure 4C or 4D. Then $F_1$ is induced provided $s_1, s_2$ are nonadjacent in $\overline{F}_1$. When $F \cong L_{1,1}$, $F \cong L_{1,1}$, $F \cong T_{2,k}$ for $k \geqslant 2$ or $F \cong L^*_{2,k}$ for even $k \geqslant 4$, then $F$ is induced, by $\gamma_P(G) \geqslant 4$. If $F \cong L^*_{1,1}$ then $F$ is singular and $s_1, s_2$ are not adjacent. Otherwise, $s_1$ and $s_2$ are nodes and any edge joining them is singular. Assume such an edge exists. If $F \cong L_{1,3}$ or $F \cong L^*_{1,k}$ for $k \geqslant 3$, then there are quadrilaterals not in $F$ equivalent under $\sim$ to those in $F$, which is impossible. When $F \cong L_{1,k}$ for $k \geqslant 4$ or $F \cong L_{2,k}$ for $k \geqslant 3$, Remarks 6.3 implies $s_1, s_2$ are separated in $J \cup J_1$ by the edges $cd$, $ef$. This is contrary to the minimality of $J$ because these edges are in a common quadrilateral. It follows that constituents are induced subgraphs.

The above theory shows how the decomposable $G \in M$ break up in a reasonably simple way into constituents. A full characterization should include how the constituents recombine to produce decomposable $G \in M$. Reference [**1**] provides an adequate theory of this type in a similar context. Three ways to proceed seem feasible: a direct description of how fragments combine at triads and nodes to give $G \in M$; a theory of how these graphs can be built up within $M$; and a study of further simple operations within $L$ to obtain a less complicated minimal class. For example, in the second approach one would show how to remove singular fragments and singular triads (those whose endvertices are nodes) and then show how to delete and contract constituents (possibly producing more of these singularities). Combined, these operations should lead to the indecomposable $G \in M$ and perhaps to a few decomposable $G \in M$. At present this looks like a rather technical and repetitive task. It is left open until the context of similar problems expands to better motivate the approach to be taken and provide stronger lemmas to cut down on repetitive work.

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