Math 5632 Book

Brad Waller

August 3, 2020

Contents

0.1 Rates of Growth . . . 3
0.2 Stock Purchases . . . 6
0.3 Stocks and Dividends . . . 8
0.4 Measuring Returns . . . 11
0.5 Forward Contracts . . . 14
0.6 Short Selling . . . 20
0.7 Futures . . . 25
0.8 Calls and Puts . . . 28
0.9 The Long and the Short . . . 34
0.10 Or-Nothing Options . . . 35
0.11 Parity . . . 39
0.12 Inequalities . . . 45
0.13 Asian Options . . . 53
0.14 Duality . . . 57
0.15 Single Period . . . 61
0.16 Futures . . . 68
0.17 Multiple Periods . . . 71
0.18 Movement Models . . . 76
0.19 Different Styles . . . 82
0.20 Volatility . . . 89
0.21 Lognormality . . . 92
0.22 Favorite Expectation . . . 95
0.23 The Black-Scholes Formula . . . 100
0.24 Examples . . . 105
0.25 Examples (Advanced) . . . 110
0.26 Exchange Options . . . 117
0.27 Portfolio Changes . . . 124
0.28 Delta, Gamma, and Theta . . . 129
0.29 Elasticity . . . 138
0.30 Approximation . . . 142

0.31 Elasticity 147

1 Introduction

We now begin!

List outcomes if you like:

- blah

1.1 Rates of Growth

To be able to discuss how financial instruments change over time, it is necessary to know how they grow. This section gives you the tools needed to discuss growth.

In this book, it will be necessary to measure how things grow. The things we are usually measuring are the quantities of shares of some asset or dollars. To measure growth, we will need some terminology and notation. Many of the letters used in this section for quantities are ad hoc and will be replaced with other conventions.

Definition 1. *The* **periodic rate of growth** *will be denoted by g. The growth of q units over one period is the quantity given by the product*

$$q \cdot (1 + g).$$

Example 1. *You are given a monthly periodic rate of growth of 2% on your deposit to some account. You deposit \$100, and you leave it there for three months. How much will you have after three months?*

Solution: After three months, you will have

$$\$100(1 + 0.02)^3 = \$106.12$$

In the example, we rounded the dollar value to two decimal places. This is a convention we will work with throughout most of the book. The rounding is for display purposes only, so when trying to execute the computations in an example make sure to save decimal places!

Periodic rates of growth are wonderful for computation; however, they don't give people a good intuition for what happens over the course of one year. That is why annual effective rates are used.

Definition 2. *Given m equal periods per year, each with a periodic rate of g, we have that the* **annual effective rate of growth** *is the value i such that*

$$(1 + g)^m = 1 + i$$

holds.

Example 2. *In the earlier example, our annual effective rate of growth comes from solving*

$$(1 + 0.02)^{12} = 1 + i.$$

Solution: The result is $i = 0.2682$. The growth rate is often expressed as a percent; in such a situation, we would write $i = 26.82\%$.

Notice that we rounded the annual effective rate to four decimal places in this example. This choice was arbitrary. It did not follow the earlier convention since the final result was not a dollar amount.

Oftentimes, periodic rates of growth are translated into something called a nominal rate of growth. This is useful for having an intuitive grasp for what an investment will do over the course of one year; however, this intuitive benefit fails when dealing with extreme values.

Definition 3. *Given m equal periods per year, each with a periodic rate of g, we say that the* **nominal rate of growth**, *denoted $i^{(m)}$, is the value*

$$i^{(m)} = mg.$$

Example 3. *The nominal rate of growth in our previous examples was $i^{(12)} = 12 \cdot 0.02 = 0.24$. This is "close" to the annual effective rate of growth, and it is much easier to compute.*

The last type of rate of growth we need will be derived from taking limits of nominal rates of growth. If we allow the nominal rate of growth to be fixed but let the periods increase to infinity over one period, we have the notion of **compounding continuously**. In mathematical terms, we let $i^{(m)} = \kappa$. Then the following holds:

$$\lim_{m \to \infty} \left(1 + \frac{i^{(m)}}{m}\right)^m = e^{\kappa}$$

To compute growth using this notion is very simple!

Example 4. *You invest \$100 in an account that grows at 8%, compounded continuously.How much will you have after 3.2 years?*

Solution: After 3.2 years, you will have

$$100e^{0.08 \cdot 3.2} = 129.18.$$

Remark 1. *When dealing with the growth of a cash investment, we will use the word interest instead of growth.*

Assumption 1. *Unless otherwise stated, our interest rates will be compounded continuously.*

Question 1 *You deposit \$100 into some savings account that pays an interest rate of 8%. How many dollars do you have after $\sqrt{2}$ years?*

Multiple Choice:

(a) *100.00*

(b) *111.50*

(c) *111.98* ✓

(d) *117.35*

(e) *None of the above are correct.*

Solution: The calculation is straight-forward: you will have $100e^{0.08\cdot\sqrt{2}} = 111.97$ dollars after $\sqrt{2}$ years. Notice that we are already using continuous compounding in our work!

1.2 Stock Purchases

Much of our discussions will revolve about how a stock is used. Here we will learn how stock trades will typically viewed in the real world.

This short section is self contained. The main purposes of this section are to make the reader familiar with terms used in the market and to make the reader more comfortable with computations of payoff, profit, and rate of return. Let's start with a definition related to buying and selling stock.

Definition 4. *When buying or selling stocks there are two values that will be listed by a broker: the* **bid price** *and the* **ask price**. *The bid price is the amount you would receive when selling the stock, and the ask price is the amount you would pay when buying the stock.*

Remark 2. *The ask price should always greater than or equal to the bid price. Otherwise, there would be an arbitrage opportunity. The* **bid-ask spread** *is the difference of the ask price and the bid price.*

$$\textit{bid-ask spread} = \textit{ask price} - \textit{bid price}$$

Example 5. *Suppose that you purchased 30 shares of XYZ six months ago. Today you wish to sell all of your shares. The necessary bid and ask prices are given below.*

Time	*Bid*	*Ask*
6 Months Ago	*50*	*51*
Today	*54*	*56*

The risk free rate is 6% and the dividend rate is 1%. Additionally, there is a 0.5% transaction cost for all purchases and sales. Determine the payoff, profit, and rate of return on your investment in XYZ.

Solution: The payoff is the money we walk away with. Today, we can sell the shares for

$$30e^{0.01\cdot 0.5}\cdot 54 = 1628.12;$$

however, we must pay the transaction fee. Our payoff is $1628.12 \cdot 0.995 = 1619.98$.

To compute the profit, we need the initial investment.

$$\begin{aligned}
\text{initial investment} &= 30\cdot 51 + \text{transaction cost}\\
&= 30\cdot 51\cdot 1.005\\
&= 1537.65\\
\text{profit} &= 1619.98 - 1537.65e^{0.06\cdot 0.5}\\
&= 35.50.
\end{aligned}$$

The last part is the rate of return. to compute the rate of return, we solve the equation below.

$$\begin{aligned} 1537.65e^{\alpha \cdot 0.5} &= 1619.98 \\ \alpha &= 0.104 \end{aligned}$$

Fees do not need to be percentages of transactions. In fact, they are often flat fees. Examples of this will be seen in the exercises.

1.3 Stocks and Dividends

Dividends are a way that companies pay back investors. They are very stable over time, so we can make some continuity assumptions in the future that make dividends easy to deal with.

In much of our future discussions, we will be investing in stocks of some company. One share of a company's stock represents fractional ownership in that company. If a company has 100 outstanding shares of their stock, and you own one share then you own 1% of that company. As an owner of a company's stock, you are entitled to a share of the company's profits. These profits are often paid back to investors as dividends or stock buybacks. Such payments are usually periodic without too much variability.

There are several different models that can be used to determine the price of one share of a company's stock. Such models are not of interest to us; however, one approach is given here. It is the dividend discount model. The principle behind this pricing scheme is relatively simple: you compute a present value of all future dividends that the stock will pay in its (possibly infinite) life.

Example 6. *Suppose you own one share of stock in XYZ. You know that the company will pay quarterly dividends of \$3 starting in three months for the next two years. No more dividends will be paid after the eights dividend. The nominal rate of interest is 8%, compounded quarterly. How much is one share under the dividend discount model?*

The value of one share is

$$\sum_{j=1}^{8} 3\left(1+\frac{0.08}{4}\right)^{-j} = 21.98.$$

I could have used an actuarial formula for this as follows:

$$3 \cdot \frac{1-(1.02)^{-8}}{0.02} = 21.98.$$

■

For the first computation, it is probably good to remember how to apply the geometric sum formula. If you don't remember it, here it is:

$$\sum_{j=1}^{n} ar^j = ar\frac{1-r^n}{1-r}$$

It will give you the actuarial formula after a manipulation or two. Also, when $|r| < 1$, you can allow $n \to \infty$.

Question 2 *Suppose you one one share of stock in XYZ. You know that the company will pay quarterly dividends of \$3 starting in three months forever. The nominal rate of interest is 8%, compounded quarterly. How much is one share under the dividend discount model?* The share's value is = $\boxed{150}$

Solution: We apply the sum formula with $a = 3$ and $r = 1/1.02$.

$$3 \cdot 1/1.02 \cdot \frac{1-0}{1-1/1.02} = 150$$

The alternative is to recall the perpetuity formula for an annuity immediate, and the result is faster.

$$3 \cdot \frac{1}{0.02} = 150$$

It is possible to use dividends to simply buy more shares of a company's stock. The computations involved are not too complicated, and they are not the topic of this book. As such, we won't be going into that material here. Whenever we speak about dividend reinvestment it will be from the simplifying assumption given below. The assumption will hold unless otherwise stated.

Assumption 2. *All stocks will pay dividends continuously, and those dividends will be used to purchase more shares of stock. The dividend rate will be denoted by* δ *or* $\delta_{some\ subscript}$.

What does this mean? It sounds complicated, but it's really easy. Let's see in an example.

Example 7. *You purchase one stock for \$100 today. The dividend rate is 8%. In* $\sqrt{2}$ *years, one share of the company's stock is still worth \$100. What is the value of your investment in* $\sqrt{2}$ *years?*

The value of your investment is equal to the number of shares multiplied by the value of each share.

$$1 \cdot e^{0.08 \cdot \sqrt{2}} \cdot 100 = 111.98$$

■

The result should look familiar. The numbers are identical to the ones we used in a question from an earlier section. The growth in the question from the previous section was applied to the number of dollars. Here the growth from the example was applied to the number of shares.

For this reason, investment in stock can experience two types of growth: share value growth and quantity growth.

Question 3 *You purchase one stock for \$100 today. The dividend rate is 8%. In* 1.5 *years, one share of the company's stock is worth \$106. What is the value of your investment in* 1.5 *years?* *The value of the investment is* = $\boxed{119.51}$

Solution: The computation is similar to the previous example.

$$1 \cdot e^{0.08 \cdot 1.5} \cdot 105 = 119.51$$

Food for thought: You would like to have one share of a company's stock in one year. Think of (at least) two different ways you could do that. Did one of the ways require you to use dividends? If not, keep thinking!

1.4 Measuring Returns

Returns are the measuring stick of the investment world. It is importand to be able to compare two assets, and the raw value of the assets is not always preferred.

People invest in financially risky assets so that they can yield the reward of higher returns. The objective is to exceed the return that would be received by investing money in a risk-free government bond. To ensure that you exceed the risk-free rate, it is necessary to measure returns. There are three ways that we will measure returns, and they are all beneficial in some way. They are

(a) payoff,

(b) profit,

(c) and rate of return.

The first is perhaps the least useful for measuring a return; however, it is extremely useful going forward. We will almost exclusively use payoffs to determine the price of a class of investments called **derivatives**. These will be seen in the upcoming sections with a formal definition.

Profit and rate of return are very useful in measuring return. It may inspire curiousity as to why there are two. Profit indexes your payoff against the opportunity cost of your investment. Rate of return answers the question, "what rate of return would my initial investment have needed to arrive at a specified payoff?" More formally, the definitions for each are provided in below.

Definition 5. *The* **payoff** *of a financial position at time T is the amount of money you could walk away with if you liquidated your position. The* **profit** *of a financial position at time T is the difference of your payoff and your initial investment grown at the risk-free rate. The* **rate of return** *of a financial position at time T is the continuously compounded rate of return that would grow your initial investment to your payoff.*

Remark 3. *In the event that the financial position is worth nothing, one might say that the rate of return is $-\infty$. Also, we will usually compute rate of return when dealing with a positive investment.*

Let's see our definitions in action!

Example 8. *You purchased 100 shares of stock in a company two years ago for 25/share. The dividend rate is 2%, and the risk-free rate is 6%. Today, the price/share of your investment is 28/share. Determine the payoff, profit, and rate of return on your investment.*

Solution: We'll proceed in order. The payoff is the vaue of the investment after 2 years,

$$100e^{0.02\cdot 2}\cdot 28 = 2914.27.$$

The profit is the difference of the payoff and the opportunity cost of the initial investment,

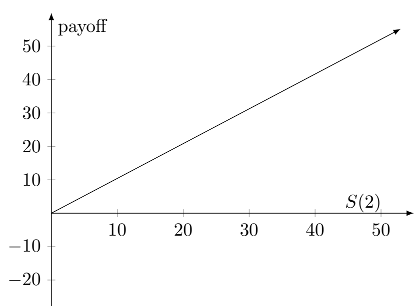
$$2914.27 - 25\cdot 100e^{0.06\cdot 2} = 95.53$$

The rate of return is the rate required to grow the initial investment to the payoff,

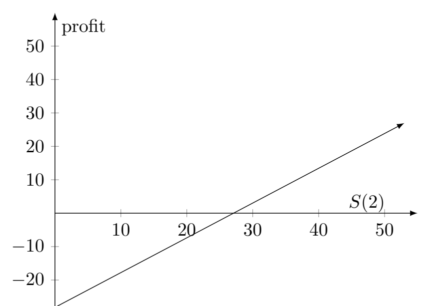
$$\begin{aligned} 25\cdot 100e^{\alpha\cdot 2} &= 2914.27 \\ \alpha &= 0.077 \end{aligned}$$

Remark 4. *Observe that the quantity of shares in the previous example influences the payoff and the profit; however, it has no effect on the rate of return. If you aren't convinced, double the quantity of shares involved and recompute everything in the example.*

Payoffs will be **the most important aspect of a derivative** when determining the price of the derivative. A great tool in building our intuition will be payoff and profit diagrams. Suppose that in our previous example, we only purchased one share initially. The 25 we paid initially would be at time 0, and the payoff is at time 2. When we are at time 0, our payoff at time 2 is unknown; the payoff is a variable dependent on the value $S(2)$. In fact, we can write it as a function: $S(2)e^{0.02\cdot 2}$. The profit is similar; it is the payoff shifted down. $S(2)e^{0.02\cdot 2} - 25e^{0.06\cdot 2}$. The difference comes from the opportunity cost consideration. The graphs for payoff and profit are given below.



Payoff Diagram



Profit Diagram

The picture for profit is the motivation for the term **break-even value**. The break-even value of an investment is the value(s) of the underlying asset that would yield a profit of 0. For our diagram, the break-even value for $S(2)$ is approximately 27.08.

Question 4 *You invest in ZYX. The value today is \$120, the dividend rate is $r = 0.07$, and the risk-free rate is $\delta = 0.02$. What value would ZYX need to take in one year so that you broke even on your investment?* The share's value is = $\boxed{126.15}$

Solution: The solution is short, but it takes some thought to get to the correct result. The quantity invested does not matter, so we may as well assume that one share was purchased initially. After one year, you will have dividend growth in the quantity of shares, so your investment will look like

$$S(1)e^{0.02}$$

You also know that you could have invested your \$120 at the risk-free rate. That would give you

$$120e^{0.07}$$

The result follows when you equate the two.

$$\begin{aligned} S(1)e^{0.02} &= 120e^{0.07} \\ S(1) &= 120e^{0.07-0.02} = 126.15 \end{aligned}$$

The value in the previous question has a special name: it is the forward price of ZYX. Let's make the following definition.

Definition 6. *The time T* **forward price** *of an asset S at time t is denoted*

$$F_{t,T}(S).$$

The value is given by

$$S(t)e^{(r-\delta)(T-t)}.$$

Remark 5. *In our normal context, we will be talking about an investment today and at some particular point in the future. When these values are understood, we will just say the forward price. That is how we used it after the question, and it is the least awkward when we speak about such things in the future. The value would look different if we permitted discrete dividends.*

1.5 Forward Contracts

Forward contracts are our first example of a derivative. The beauty of them is that they happen constantly in business!

A **forward contract** is one of the most natural examples of a **derivative** that we will see.

Definition 7. *A* **forward contract** *is an agreement between two parties, a buyer and a seller. The buyer agrees to pay a specified price for an asset at a specified time. The seller agrees to provide the asset at the specified time.*

Remark 6. *Other things can be specified in a forward contract. They are private arrangements between individuals, so they are highly unregulated.*

People are constantly entering into forward contract. Almost all businesses enter into forward contract. A restaurant owner will have to arrange for foods to be delivered to their restaurant. Such arrangements can be made via forward contract.

You could offer to buy your friend's textbook for a class at the end of the semester, as you plan to take the course next semester. It is likely that your friend would get more by selling to you than they would by selling to the university book store. It is also likely that you would pay less to your friend than you would to the university book store. Both you and your friend are entering into a mutually beneficial contract.

As it turns out, the contract's dependence on another asset is what makes it a derivative.

Definition 8. *A* **derivative** *is a financial instrument that derives its value from another financial asset. The other financial asset is called the* **underlying asset**.

To determine the price of a derivative, we will need to know the payoff of the derivative. First, we must discuss notation that will frequently be used.

We will usually have many variables to consider when computing the price of our derivatives. In addition to our underlying asset, we will have the asset's dividend rate, and some risk-free interest rate. Previously, we have seen that the dividend rate would be denoted δ and the risk-free interest rate is denoted r. We will denote the time t price of our underlying asset as $S(t)$. This is also called the time t spot price of the asset.

The risk-free rate usually comes from a government issued, zero coupon bond. Bonds are issued by governments and businesses to borrow money from the public. If you buy bonds, you are lending money to the issuer. If you sell bonds, you are borrowing money. This distinction will be important whenever we are trying to explain our transactions. The explanations that we provide are just as important as the computations we make.

In addition to the asset, dividend and risk-free rate, there are time variables to consider. Many of the derivatives we deal with are contracts that expire at a particular point in time in the future. We will use T to denote that time of expiration. For variable time, we will use t. s will be be used when t has been taken. Let's hope to never need any more time variables.

There is another variable that we deal with frequently; it is called volatility. We will discuss it in more depth when it becomes necessary.

Earlier, the importance of computing payoffs of derivatives was stressed. Let's compute some payoffs of a forward contract.

Example 9. *You wish to become a cattle herder. You will need some land, cows, and corn to feed them (the land you plan to purchase has no grass). You've already arranged your purchase of cows to be in three months. You would like to arrange to buy 10,000 bushels of corn at the same time as you have the cows (you don't want the corn to go bad by buying it today). The price today for corn is \$4/bushel. You agree to pay \$4.15/bushel for the corn in three months to a corn farmer that lives nearby. What is your payoff under the following three scenarios three months from now:*

(a) *the price of corn drops to \$3.85/bushel,*

(b) *the price of corn rises to \$4.15/bushel,*

(c) *and the price of corn rises to \$4.70/bushel.*

Solution: The results differ, but the setup is always the same. Your payoffs are as follows in order.

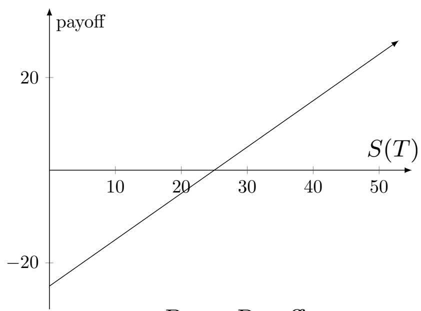
$$10,000(3.85 - 4.15) = -3000$$
$$10,000(4.15 - 4.15) = 0$$
$$10,000(4.70 - 4.15) = 5500$$

The computations seem straight-forward, and that is because they are. In general, the buyer in a forward contract will have a payoff that is equal to

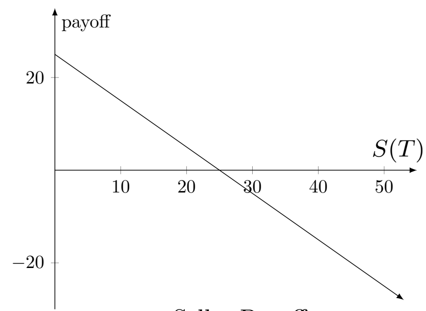
$$q[S(T) - K]$$

where q is the agreed quantity, $S(T)$ is the price of the underlying asset at expiration, and K is the agreed upon price at expiration.

The statement "the buyer" in the previous paragraph is important. The seller in the forward contract will always have the opposite payoff diagram. Let's compare them below.



Buyer Payoff



Seller Payoff

Now that we have the payoffs, we can determine the price of a forward contract. We will need two assumptions to proceed, but first we will need a definition.

Definition 9. **Arbitrage** *is a type of investment opportunity that carries with it no risk and has a guarantee that the* **arbitrageur** *earns a profit.*

Arbitrage opportunities are usually constructed by matching two payoffs in an advantageous way so that they cancel out.

Example 10. *Suppose that you have access to two betting websites. Both give odds on the winner of a game between the Cavaliers and the Lakers. The first site gives the teams equal odds of victory. The second site gives the Cavaliers a 2/3 chance of victory and the Lakers a 1/3 chance of victory. You have $200 to bet. Determine an arrangement that maximizes your return. The first site will give you a 100% return on your bet if you choose the correct winner. The second site will pay you a 50% return if you chose the Cavaliers (and they win) and a 200% return if you chose the Lakers (and they win). Determine an arbitrage arrangement.*

Solution: One possibility is to bet 110 on the Cavaliers at the first betting site and 90 on the Lakers at the second betting site. The results will be either 220 or 270 (depending on who wins). Since you started with 200, you have made an arbitrage profit!

Question 5 *Suppose that you have $200 as in the previous example. How much must you bet with each site so that you always end up with the same amount? What is that amount?*

Who do you bet on at the first site? $\boxed{\textit{Cavaliers}}$

Who do you bet on at the second site? $\boxed{\textit{Lakers}}$

What is the value of your bet at the first site? $\boxed{120}$

What is the value of your bet at the second site? $\boxed{80}$

How much money will you walk away with? $\boxed{240}$

Solution: You should pick the best odds for each team. This will give you should bet on the Cavaliers at the first site and the Lakers at the second site. Now, the calculation relies on a system of two equations in two unknowns. The first is

$$x + y = 200$$

where x is your bet on the Cavaliers and y is your bet on the Lakers. Using that information, we compute the payoffs from either bet. They would be $2x$ if the Cavaliers win and $3y$ if the Lakers win. We equate these values.

$$2x = 3y$$

Solving this system yields that $x = 120$ and $y = 80$. This will result in a constant payoff of $240.

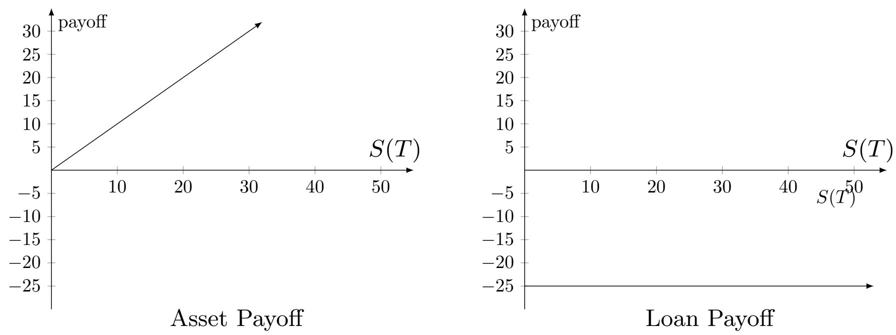
Now that we have seen how arbitrage works, let's see those two assumptions.

Assumption 3. *Unless a problem is in search of arbitrage, problems will have no arbitrage opportunities.*

Assumption 4. *There will be no fees or taxes unless specified in a problem.*

In the argument to determine the price of a forward contract, we will assume that there are no transaction costs associated with purchases or sales. The idea is to think of a way to end up in the identical payoff position as the buyer in a forward contract. This is not too difficult to do. When you look at the graph of the payoff, you see that there is an axis corresponding to the asset price. In Figure **??**, we had a line with slope 1. In fact, the graph is of the function $S(T) - 25$ (for the buyer). In general, this will look like $S(T) - K$. The goal is to come up with a portfolio (collection of assets) that has payoff $S(T) - K$.

The first position is to enter into a situation where we will end up with one share of the underlying asset at time T. The next position is to end up with $-K$ at time T. If we borrow money today that is due at time T, we will have a negative cash position at time T. These two together give us the desired portfolio. Let's use graphs to visualize.



The loan payoff in the figure is negative since we borrowed money at the beginning. That means money will be leaving us at expiration, so its flow is negative. If these two positions are combined, we get the buyer's payoff from earlier. Since there are no other fees or taxes to worry about, we can state our theorem.

Theorem 1 (Forward Contract Price)**.** *The time t price of a forward contract to purchase q shares of S for K/share that expires at time T is given by the formula*

$$q[S(t)e^{-\delta(T-t)} - Ke^{-r(T-t)}].$$

The time t portfolio that replicates the time T payoff of the forward contract is

(a) *the purchase of $qe^{-\delta(T-t)}$ shares of the underlying asset and*

(b) *selling risk-free bonds valued at $qKe^{-r(T-t)}$ today.*

Alternatively, this could be stated

(a) *the purchase of $qe^{-\delta(T-t)}$ shares of the underlying asset and*

(b) *borrowing $qKe^{-r(T-t)}$.*

Remark 7. *It is important to understand that this price is what the buyer in the contract would have to pay. The situation is reversed for the seller.*

Question 6 *You enter into a forward contract to purchase 100 shares of ABC in three months for \$75/share. The current price of one share of ABC is \$74. The dividend rate is $\delta = 0.03$ and the risk-free rate is $r = 0.09$. How much do you pay for the forward contract?*

You would pay $\boxed{11.57}$

Solution: This is a direct calculation using the formula provided by Theorem 1.

$$100[74e^{-0.03/4} - 75e^{-0.09/4}] = 11.57$$

We can modify the question slightly to see the benefit of having such a formula.

Example 11. *In one month you decide that you don't want to be involved with purchasing 100 shares of ABC, so you decide to sell your forward contract. The only new condition is that the spot price of ABC has changed to 76. What should you receive for the forward contract?*

Solution: You should receive

$$100[76e^{-0.03 \cdot 2/12} - 75e^{-0.09 \cdot 2/12}] = 173.76.$$

Remark 8. *If we were to enter into a forward contract where the agreed upon price per share is the forward price, then the contract would have no value. We would not need to pay anything for that contract. We could save that quantity today by making a risk-free loan of* $qS(t)e^{-\delta(T-t)}$.

Definition 10. *The time* T **prepaid forward price** *of an asset* S *at time* t *is denoted*

$$F^P_{t,T}(S).$$

The value is given by

$$S(t)e^{-\delta(T-t)}.$$

Now that we have a way to compute the theoretical price of a forward contract, we would like to use it for practical reasons. This will rely on something called **short selling**, which is the topic of the next section.

1.6 Short Selling

This section delves into the unusual trading activity called short selling. In this approach, the investor borrows an asset to immediately sell it. Just imagine if you took such an approach when borrowing a friend's car!

In the last section, we ended with a statement regarding **short selling**. The concept of short selling is fairly straight-forward, and a lot of common sense is involved.

Definition 11. *A* **short sale** *is a position where you borrow an asset from another party and immediately sell it. To gain access to the asset, you must offer collateral, which is typically a percent of the asset value. In some sources, you will see that this collateral payment is called a* **haircut**. *The collateral is placed into an account that may earn interest. This is a* **margin account**. *The proceeds from the short sale are typically invested into an account that earns the risk-free rate. When a short sale is terminated, the asset is returned to the original owner plus any dividends that were paid while the short sale was open.*

Remark 9. *Short selling might seem a little odd. Why would anybody want to let you borrow something of value just so you could sell it off? The original owner of the asset might want some liquidity today, but they might also want to be in a position to have their assets back in the future. Entering into this agreement will allow them to do this.*

Also, it is natural to ask where one can short sell. Many online investment brokerages will allow you to do this, provided you have the collateral to do so.

Example 12 (Original Owner's Perspective)**.** *You wish to generate some more revenue from your investments, so you decide to let some short sellers borrow your investment. You ask for collateral of \$1000. You will pay the short sellers 2% on their collateral to attract borrowersl, and you will invest the collateral in a risk-free account earning 6%. The short seller wishes to close out the sale after six months. What is your benefit from the collateral arrangement?*

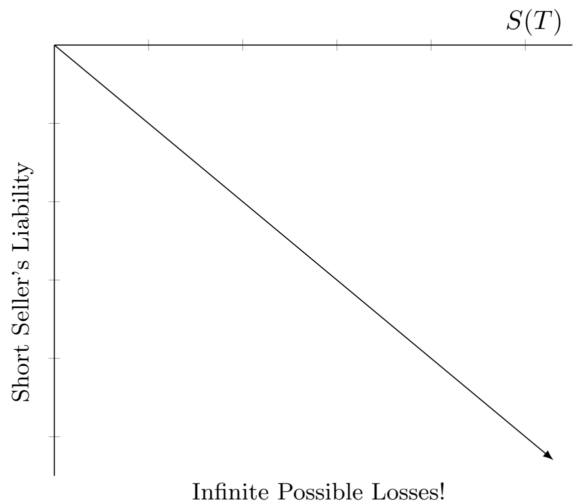
Solution: You would gain

$$1000(e^{0.06\cdot 0.5} - e^{0.02\cdot 0.5}) = 20.40$$

from the collateral in the arrangement. In addition, you would receive your asset plus dividends at the end of the short sale.

In the previous example you enhance your initial, passive position of investment; however, this did come at a price. You assumed default risk when you entered into this arrangement. The borrower may not be able to pay you back.

In almost everything we do going forward, we wlil view a short sale from theshort seller's position. Let's take a look at a diagram that demonstrates the risk of short selling.



In the event that the asset keeps gaining value, the short seller may find themselves bankrupt! Let's compute the payoff of a short sale under a variety of values $S(T)$.

Example 13. *Suppose that you short sell 100 shares of ZYX. You have the following information:*

- $S(0) = 50$,
- $\delta = 0.03$,
- $p = 0.8$
- $m = 0.01$, *and*
- $r = 0.07$.

Here, p represents the proportion of the short sale that must be brought in collateral, and m represents the rate that the margin account will pay.

What is the payoff of the short sale for the following values of $S(0.5)$: 35, 50, and 65.

Solution: The computations have the same structure; the only difference is the value used for $S(T)$. They all start with the initial investment of $qmS(0) = 100 \cdot 0.8 \cdot 50 = 4000$. This is the money that will sit in the margin account for six months. The amount you receive from the short sale will be $qS(0) = 5000$. The computation for the payoff will follow the format given in the following equation.

$$q[S(0)e^{rT} + pS(0)e^{mT} - S(T)e^{\delta T}]$$

The first term represents the amount you have from the sale itself, the second term represents the amount you had in margin, and the third value is the cost of repurchasing the asset and returning it to the counterparty. For our three values, this will yield the following three payoffs:

$$100[50e^{0.07\cdot 1/2} + 0.8 \cdot 50e^{0.01\cdot 1/2} - 35e^{0.03\cdot 1/2}] = 5645.25$$
$$100[50e^{0.07\cdot 1/2} + 0.8 \cdot 50e^{0.01\cdot 1/2} - 50e^{0.03\cdot 1/2}] = 4122.58$$
$$100[50e^{0.07\cdot 1/2} + 0.8 \cdot 50e^{0.01\cdot 1/2} - 65e^{0.03\cdot 1/2}] = 2599.91$$

As can be seen, the higher the price of the asset at expiration, the lower the payoff. It would be useful to know what the profits are.

Question 7 *Compute the profit for the short sale in each of the three scenarios given above.*

When $S(0.5) = 35$ *the profit is*= $\boxed{1502.77}$

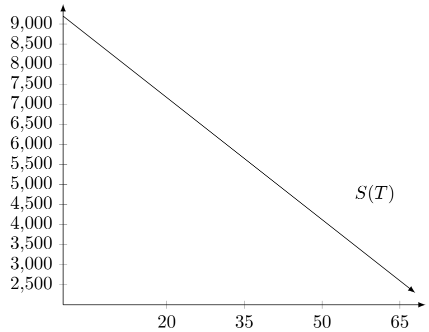
When $S(0.5) = 50$ *the profit is*= $\boxed{-19.90}$

When $S(0.5) = 65$ *the profit is*= $\boxed{-1542.57}$

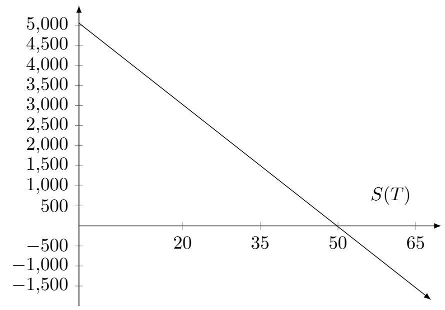
Solution: The computation used above piggybacks off of the work we did in the earlier example. Simply take the payoffs in each of the cases and subtract the opportunity cost $4000e^{0.07\cdot 1/2} = 4142.48$ from each of the three payoffs. Alternatively, use the equation

$$q[pS(0)e^{mT} + S(0)e^{rT} - S(T)e^{\delta T} - pS(0)e^{rT}]$$

We can also view both the payoff and profit graphically. See the figure below.



Short Sale Payoff



Short Sale Profit

These diagrams are meant to illustrate the dangers of short-selling. This is not an activity to enter into lightly. If you find yourself short-selling, you will want to be very knowledgable of the assets you are trading. Alternatively, you may want to try to hedge your position to reduce your loss liability. Derivatives that will be introduced in the next chapter will allow us to protect ourselves from such losses.

Let's return to the discussion of arbitrage. We have a way to determine the price of forward contracts, and we also have ways of replicating that payoff with investments in the underlying asset and cash. Since the payoffs are identical, we can enter into opposite positions that will offset to give us a payoff os zero. This may seem useless; however, when the theoretical price of forward contracts don't match reality we have arbitrage opportunities.

Example 14 (Forward Contract Arbitrage)**.** *You are in a position to buy a forward contract to purchase 100 shares of XYZ in nine months. The contract costs \$425. The underlying asset price is \$90, and the contract's agreed upon price is \$88. In addition, the dividend rate of the asset is 4%, and the risk-free rate is 8%. Determine your arbitrage opportunity.*

Solution: We must start with a computation of the theoretical price of the forward contract:

$$q[S(0)e^{-\delta T} - Ke^{-rT}] = 100[90e^{-0.03} - 88e^{-0.06}] = 446.48$$

From that, we know that the derivative is underpriced. Since it is underpriced, we should buy it! To ensure that we have no liability in the future, we offset the position by (short) selling $100e^{0.03}$ shares of XYZ today and buying $8800e^{-0.06}$ in risk-free bonds today. The resulting arbitrage opportunity is as follows:

- The forward contract,
- a position short of $100e^{-0.03}$ shares in XYZ,
- the investment in risk-free bonda of $8800e^{-0.06}$,

- and an arbitrage gain of \$21.48

The formal computation for the arbitrage gain would be

$$100e^{-0.03} \cdot 90 - 8800e^{-0.06} - 425 = 21.48$$

The reason I didn't do it before displaying the arbitrage gain is that you should know the gain before you write down the portfolio explicitly. It will always be the absolute difference of your two positions. In this case, that is $446.48 - 425 = 21.48$.

1.7 Futures

We explore some of the properties of futures such as margin accounts, marking to market, and margin calls.

This section deals with derivatives that are called **futures contracts**. Futures have a lot in common with forward contracts; however, there are some distinct differences as well. Some similarities that futures have with forward contracts are that they both have

(a) an underlying asset,

(b) a time to expiration,

(c) a buyer and a seller,

(d) and an agreed upon price.

The agreed upon price is called the **futures price** in a futures contract. It is the spot price of the futures contract.

Some of the differences between futures and forward contracts are that futures are

(a) regulated by the market (forward contracts are tailored to the buyer and seller),

(b) traded on exchanges (forward contracts don't have exchanges),

(c) and marked to market.

One of the regulations that markets require of futures exchanges is that buyers must make a margin deposit for to enter into a position. This is to protect the sellers in the exchange. **Marking to market** is the process where gains and losses to the futures position are credited (debited) to the margin account. The credits and debits are absolute; they are not some proportion of the total value of the contract. Both the buyer and seller must open a margin account to participate in the exchange. Typically, the margin required is small relative to the size of the contract.

A margin account must maintain a certain value to keep all participants in the exchange. This value is called **maintenance margin**. The maintenance margin will typically be given in terms of a percentage of the initial margin account value; however, it is possible to be given the value as a dollar amount. In the event that one of the parties' margin account falls in value below the mintenance margin, a **margin call** is initiated. A margin call requires the party that received the call to deposit an amount of collatera so that the account is back to its initial value.

A lot has been said, but little has been done in terms of numerics. Let's try an example.

Example 15. *You would like to invest in the S&P 500 index. To do so, you can enter into a futures contract on the same exchange. A typical S&P 500 futures contract costs 250 times the value of the S&P 500 index. The initial margin is 25% of the total contract value. The margin account pays 2%, and the risk-free rate is 6%. The index value today is \$3400. On the following days, you observe your investment.*

- *After 1 day, the index rose to \$3340*
- *After 2 days, the index dropped to \$3210*
- *After 3 days, the index rose to \$3410*

You decide to exit your position after 3 days. What is your payoff, profit, and rate of return?

Solution: We begin by determining the margin deposit: $250 \cdot 3400 \cdot 0.25 = 212,500$. Now we must compute the chages that occur from marking to market.

$$\begin{aligned} 212,500e^{0.02 \cdot 1/365} + 250(3340 - 3400) &= 197,511.64 \\ 197,511.64e^{0.02 \cdot 1/365} + 250(3210 - 3340) &= 165,022.47 \\ 165,022.47e^{0.02 \cdot 1/365} + 250(3410 - 3210) &= 215,031.51 \end{aligned}$$

The payoff is the last value listed in the computations, $\$215,031.51$. We arrive at the profit by taking the difference

$$215,031.51 - 212,500e^{0.06 \cdot 3/365} = 2426.69.$$

We can also compute the rate of return on the futures investment.

$$\begin{aligned} 212,500e^{\alpha \cdot 3/365} &= 215,031.51 \\ \alpha &= 1.441 \end{aligned}$$

When dealing with futures contract in such a short term, it is almost useless to discuss the rate of return of the investment. It will almost always be exaggerated. Time frames over a couple of weeks will probably be more meaningful. The result of the above should be clear, futures contracts can be a wild ride! We saw our investment almost evaporate after one day of poor performance in the market. The only thing this example didn't cover was a maintenance margin.

Question 8 *Suppose you have the same futures arrangement as in the previous example. In addition, there is a maintenance margin of 80% of the initial margin value. Determine the payoff and profit under this scenario.*

The payoff is $\boxed{262511.64}$

The profit is $\boxed{2421.49}$

Solution: Many of the computations are similar. The difference comes after the second day. You must bring a $212,500 - 165,022.47 = 47,477.53$ in collateral to your account. The calculation after the third day would be

$$212,500e^{0.02 \cdot 1/365} + 250(3410 - 3210) = 262,511.64$$

It follows that your payoff is \$262,511.64. Your profit calculation is a bit messier.

$$262,511.64 - 47,477.53e^{0.06 \cdot 1/365} - 212,500e^{0.06 \cdot 3/365} = 2421.49$$

As it should be expected, the margin call will result in reduced profits.

1.8 Calls and Puts

We learn about our fundamental options. These will be revisited throughout much of the remainder of the book. Payoffs are of huge importance!

In our introductory chapter, we learned about some basic types of derivatives, namely forward contracts and futures arrangements. This chapter deals with a special type of derivative called an option. Options are contracts that are used to make trades based on some underlying asset.

Definition 12. *A* **call option** *gives the option owner the option to buy the underlying asset from the writer (seller) of the option. There are three things specified in the call option contract: the underlying asset, the time to expiration, and the* **strike price**. *The strike price is the value that the owner would pay the writer in the event that the buyer exercises the option. A* **put option** *gives the owner the option to sell the underlying asset to the writer of the option. That is the only difference between call options and put options. An options contract has no value after the expiration or the contract's exercise.*

There are two parties to options contracts: they buyer and the writer. The buyer is the position that we will usually consider, and it is the buyer that chooses to exercise the option. The writer is the individual that is willing to take a **premium** at the time of sale to assume the liability of the options contract.

In addition to the characteristics in the definition, there is another component to an option. It is called **style**. The style of the option dictates timing of exercise. In addition, it may influence the computation of the payoffs. The two styles we will focus on for now will be American and European. An American option that expires at time T may be exercised at any time in the interval $(0, T]$. A European option with the same expiration may only be exercised at expiration. The additional opportunity to exercise provided by an American option gives it greater value than its European counterpart. In symbols, this would be

$$\begin{aligned} C_{am} \geq & C_{eu} \\ P_{am} \geq & P_{eu} \end{aligned}$$

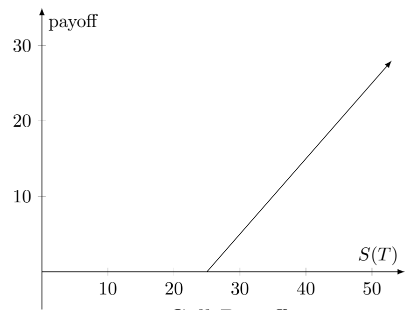
where all other terms are assumed to be the same.

In the above inequalities, I am using the symbols to denote values or prices of the options. These same symbols may be used to describe a specific portfolio. When variables between options are the same, they will be supressed. For example

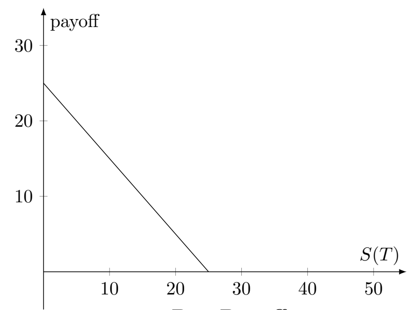
$$10c(30) + 5c(40) - 15c(50)$$

is the number that represents the value of the portfolio consisting of 10 calls with strike 30, 5 calls with strike 40, and 15 **written** calls with strike 50. All of the options have the same underlying asset, time to expiration, and by default we will assume that they are European.

Let's examine the payoffs of some of our European options.



Call Payoff



Put Payoff

Let's work through the reasoning attached to the diagram for the European call option with strike 25 and expiration T. If you are the option holder, you have the choice to buy the underlying asset for 25. If the market value is less than the strike value, would you buy at the strike price? Your answer should be no. If you wanted the asset, and the asset was valued less than the strike you would purchase it on the market. In this case, you allow the option to expire without exercise. In the event that the market price was greater than (or equal to) the strike price, you would exercise the option and purchase the asset for the strike price. You could then immediately sell the asset at the market price to receive the payoff of $S(T) - 25$. The payoffs of our European options are

$$\text{Time } T \text{ Call Payoff} = \max\{S(T) - K, 0\} = \begin{cases} S(T) - K & \text{if } S(T) > K \\ 0 & \text{otherwise,} \end{cases}$$

$$\text{Time } T \text{ Put Payoff} = \max\{0, K - S(T)\} = \begin{cases} 0 & \text{if } S(T) > K \\ K - S(T) & \text{otherwise.} \end{cases}$$

Please work through reasoning used to determine the call's payoff in your determination of the put's payoff.

Example 16. *Suppose that I buy 30 European call options on XYZ. The options all have strike* 50*, and the options expire in one week. In one week, the price of one share of XYZ is 52.37. What is the payoff of my position in one week?*

Solution: Since the price at expiration is greater than the strike, I would exercise the options. The payoff is

$$30 \cdot (52.37 - 50) = 71.10$$

Why don't you try something more challenging. In the following, you will be purchasing and writing options. A written option has payoff that is -1 times the payoffs given above. We did something similar when we discussed the seller of a forward contract.

Question 9 *You enter into several options contracts on the asset ZYX. They are all European, and they expire in one month. Your position follows.*

- *50 calls with strike 40*
- *20 calls with strike 45*
- *70 written calls with strike 50*
- *10 puts with strike 60*

At the end of the month, the share price of ZYX is 55*. What is your payoff?*

$$\text{Your payoff is} = \boxed{650}$$

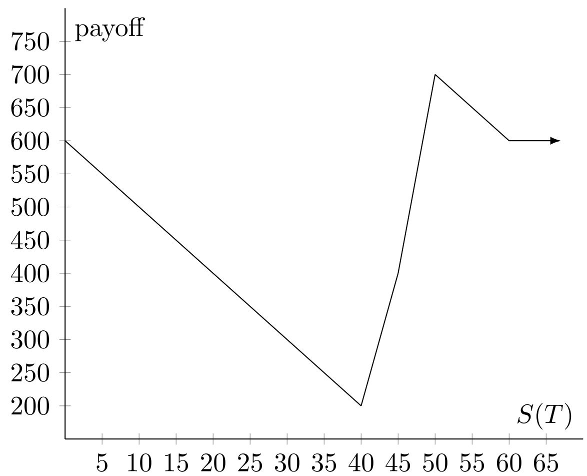
Solution: Strangely enough, each of the four options has a payoff! It is probably easiest to compute each as its own entity.

$$\begin{aligned} 50 \cdot \max\{55-40,0\} &= 750 \\ 20 \cdot \max\{55-45,0\} &= 200 \\ -70 \cdot \max\{55-50,0\} &= -350 \\ 10 \cdot \max\{0,60-55\} &= 50 \end{aligned}$$

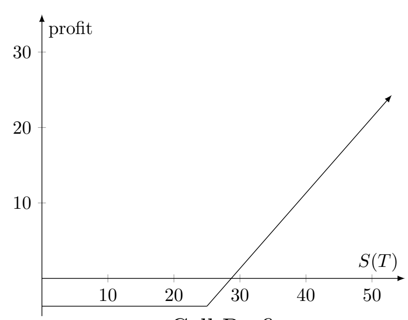
Summing the results yields 650.

As a "fun" exercise, let's try and graph the payoff of your position over all values! This may seem like it is a very difficult task, but there is something that should guide you in all of your work: all of the individual payoffs are piecewise linear (and continuous). The result of summing piecewise linear (and continuous) functions is also piecewise linear (and continuous). To determine how to draw the diagram, you only need to determine where the result will change slope.

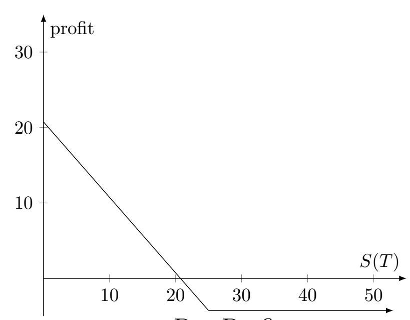
In our example, this occurs at the values $S(T) = 40, 45, 50,$ and 60. The payoffs are $200, 450, 700,$ and 600, respectively. This gives us four points on our graph. We can connect the dots with a line, and we have the desired diagram; however, we still need to continue the diagram beyond $S(T) = 60$ and before $S(T) = 40$. For the former, we note that the slope must be 0. This is because the put does not contribute anything to the payoff for large values of $S(T)$. For the latter, the put is the only contributing factor. Since there are 10 puts, the slope will be -10.



In everything we've done so far in this section there has been no consideration to prices of the options. We will consider various models to determine prices of the options later in the course; however, it should be clear that they option buyer must pay something for the options. This is because they always have a positive payoff, and they never suffer any liability. It follows that the profit diagrams for the two types of European options will look like the figure below.



Call Profit



Put Profit

It is natural to wonder why anybody would want an options contract. There are many reasons.

- If you currently own shares of a company's stock, and you would like to protect your investment from losses in its value, you could purchase put options. This allows you to pick a price today to sell your asset to another party in the future.
- If you are currently the short seller in some asset, you could purchase calls using some of the proceeds of your short sale. These call options would lock in a maximum price that you could buy back the asset when you close out your short sale.

- You could write calls on an asset that you own. This can be viewed as a way to enhance an investment's rate of return, provided you are comfortable with selling the asset to the buyer of the option at expiration if the asset price is greater than the strike price. This strategy is not advised if you are unwilling to part with the underlying asset.
- You could enter into the options market as a speculator, wishing only to trade the options themselves with no interest in the underlying asset. This can be very lucrative since many options are quite elastic! More on elasticity will be seen in later chapters.

The first two items would be considered hedging strategies because they protect you from negative moves in the asset's value.

Remark 10. *Many types of insurance may be viewed as options contracts. For example, if you buy a homeowner's policy that covers damages on your home in excess of \$5000 and your home is destroyed, then the policy will pay you the value of your home minus \$5000. This would be similar to an American put option on your home with strike \$5000 because they payoffs to you are identical. It is for this reason that actuaries are interested in options.*

Let's see an example of one of the hedging strategies.

Example 17. *Suppose that you short sell 100 shares of ZYX. You have the following information:*

- $S(0) = 50$,
- $\delta = 0.03$,
- $p = 0.8$
- $m = 0.01$, *and*
- $r = 0.07$.

In addition, you hedge your position by purchasing 101.51 European call options with strike 50 that all expire in six months. What is the profit of your porition if $S(0.5) = 65$ *if the price of the calls was \$469.52?*

Solution: We have seen this example previously, and the profit without the calls was $-\$1542.57$. The computation is altered now. We will proceed in two ways:

(a) we will view the portfolio as its parts, or

(b) we will view the portfolio as a whole.

Both should yield the same result.

The profit of the call options is a straight-forward computation, as the payoff is obviously
$101.51(65 - 50) = 1522.65$.

$$1522.65 - 469.52e^{0.07\cdot 0.5} = 1036.41$$

It follows that the profit of the whole position is the sum of the parts:
$-1542.57 + 1036.41 = -\$506.16$

We won't always have the benefit of knowing the short sale's profit. Let's see the second approach. Recall that the short sale requires 80% collateral. Our setup would look something like this where the payoff is in the first set of parenthesis and the profit is the negative at the end of the equation.

$$100\left(S(0)e^{0.07\cdot 0.5} + 0.8S(0)e^{0.01\cdot 0.5} - S(0.5)e^{0.03\cdot 0.5} + [S(0.5) - 50]e^{0.03\cdot 0.5}\right) - 4469.52e^{0.07\cdot 0.5}$$

Now, before doing too much, notice that we could change the equation as follows:

$$100\left(S(0)e^{0.07\cdot 0.5} + 0.8S(0)e^{0.01\cdot 0.5} - 50e^{0.03\cdot 0.5}\right) - 4469.52e^{0.07\cdot 0.5}$$

It's as though we altered the short-sale payoff by changing $S(0.5)$ to 50. That is the benfit of having the call options. The reason this is permitted is because the quantity of call options is the same as the quantity of shares at the time of the buyback. The value of the last equation is $-\$506.14$. We can attribute the difference to some rounding error. The benefit of the second approach is that it allows you to view the portfolio as a modified short sale. That can save you time when dealing with complex portfolios.

It should be clear that our poortfolio benefited from the purchase of the call options. Try graphing the profit of this position for various values of the underlying asset at expiration. After graphing the profit, it should be clear that there is a trade off taking place. Think about what that might be.

1.9 The Long and the Short

Calls and puts are financial assets on their own. Since they are derivatives, their price is determined by their underlying asset. Terms regarding this relationship are given in this section.

We have seen the term "Short Sale" in the past, but we did not go into the etymology at all. With the study of derivatives and, more specifically, options, it is good to know how your derivatives will react to changes in the price of the underlying asset.

Definition 13. *You are said to be* **long** *in an asset if you benefit by its increase in value, and you are said to be* **short** *in an asset if you benefit by its decrease in value.*

The short sale uses this terminology. Every short sale has an underlying asset. When you are the short seller, you are short in the underlying asset.

Things are a little more interesting/convoluted when you are dealing with derivatives. This is because derivatives are also assets! We have only really dealt with three derivatives: forward contracts (futures), calls, and puts. Let's discuss each with respect to their (respective) underlying assets.

	Forward Contract	Call Option	Put Option
Buyer	Long in underlying asset	Long in underlying asset	Short in underlying asset
Seller	Short in underlying asset	Short in underlying asset	Long in underlying asset

In addition to these positions, the buyer is always long in the derivative while the seller is short. There is not much to ask with regard to these concepts, but they do creep up in the launguage of our subject with some regularity. For example, it might be said that you enter into a long position on a call. Thet just means you are buying a call. However, I will never say something like you are short in a put. I feel that there is ambiguity in this, and it should be avoided. I would prefer to say you write a put.

1.10 Or-Nothing Options

The cash-or-nothing and asset-or-nothing options can be used as building blocks for almost all of our derivatives. With a little creativity, they can be used to approximate almost any derivative!

In our section introducing calls and put, we saw that they had payoffs that are piecewise functions. We graphed the payoff diagram of a collection of these options and saw what a mess it could become, and it wasn't the easiest thing to produce. The benefit of "Or-Nothing" options is that they make graphs much easier to produce. In addition, they readily approximate payoffs of the most exotic of derivatives.

Definition 14. *An* **asset-or-nothing option** *pays the underlying asset at expiration depending on whether it is a call or a put. A* **cash-or-nothing option** *pays* \$1 *at expiration depending on whether it is a call or a put.*

The payoffs for each of these four derivatives should be written explicitly. We will assume that these options are European throughout this text.

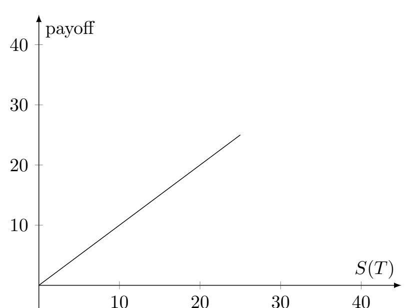
$$c_{\mathrm{A/N}} \text{ has payoff } \begin{cases} S(T) & \text{if } S(T) > K \\ 0 & \text{otherwise,} \end{cases}$$

$$p_{\mathrm{A/N}} \text{ has payoff } \begin{cases} 0 & \text{if } S(T) > K \\ S(T) & \text{otherwise,} \end{cases}$$

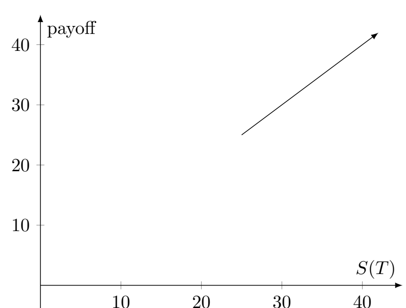
$$c_{\mathrm{C/N}} \text{ has payoff } \begin{cases} 1 & \text{if } S(T) > K \\ 0 & \text{otherwise,} \end{cases}$$

$$p_{\mathrm{C/N}} \text{ has payoff } \begin{cases} 0 & \text{if } S(T) > K \\ 1 & \text{otherwise.} \end{cases}$$

Graphically, this would be

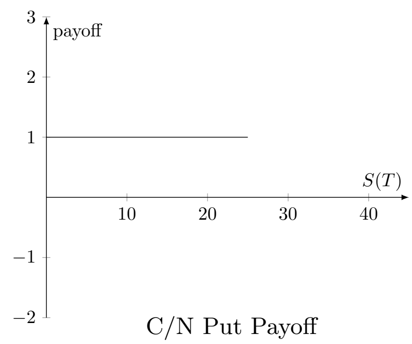


A/N Put Payoff

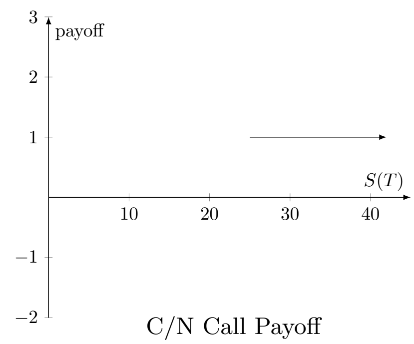


A/N Call Payoff

and



C/N Put Payoff



C/N Call Payoff

The payoffs for these derivatives are quite simple, and it is for that reason that we use these options to approximations to complicated payoffs. There are three rules of thumb when dealing with these options:

(a) To approximate nonzero slopes, use asset-or-nothing options.

(b) To approximate zero slopes, use cash-or-nothing options.

(c) To shift a diagram up or down, use cash-or-nothing options.

Let's try an example!

Example 18. *Suppose you have a derivative that has time T payoff given by the function*

$$f(S) = \begin{cases} 5 & \text{if } S(T) \leq 20, \\ 3S - 55 & \text{if } 20 < S(T) \leq 25 \\ 20 & \text{if } 25 < S(T). \end{cases}$$

Construct a portfolio that will give you the exact payoff described by the function.

Solution: It should be clear where to use the various types of options based on our rules of thumb. The first and third pieces follow directly from the rules. For the first piece, it is probably best to use 5 cash-or-nothing puts with strike 20. For the third piece, it is probably best to use 20 cash-or-nothing calls with strike 25.

The middle piece requires the most thought. The rule of thumb tells us that when slope is involved, we should use asset-or-nothing options. Here, the slope is 3. It follows that we should buy 3 asset-or-nothing calls with strike 20. What is not clear, is that we need to cancel out the payoff of these three calls after the value 25 (otherwise we would interfere with the third piece). To make the appropriate cancellation, we write three asset-or-nothing calls with strike 25.

We are close to the desired payoff. The slope is right, but the location is not. We must shift the diagram down!

We use the last rule of thumb. If we use the value $S(T) = 20$ with our asset-or-nothing options as an anchor point, we see that the payoff value is 60. We need to shift our diagram down by 55!. To do this, write 55 cash-or-nothing calls with strike 20 and purchase 55 cash-or-nothing calls with strike 25.

The required portfolio is as follows:

- 5 cash-or-nothing puts with $K = 20$,
- 20 cash-or-nothing calls with $K = 25$,
- 3 asset-or-nothing calls with $K = 20$,
- 3 written asset-or-nothing calls with $K = 25$,
- 55 written cash-or-nothing calls with $K = 20$,
- and 55 cash-or-nothing calls with $K = 25$.

We could go further by combining like terms, but I'll leave that to you in your spare time!

Why don't you try to work through a similar problem.

Question 10 *Select all of the following that will produce the payoff described by the function*

$$f(S) = \begin{cases} 2(S-20) & \text{if } 20 < S(T) \leq 30 \\ 0 & \text{otherwise.} \end{cases}$$

For a greater challenge, try this problem before seeing the options below. Your answer may be correct, but it might look different than the options given.

Select All Correct Answers:

(a) *Two asset-or-nothing calls with strike 20*

(b) *Two asset-or-nothing puts with strike 30* ✓

(c) *Forty cash-or-nothing calls with strike 30* ✓

(d) *Two written asset-or-nothing calls with strike 25*

(e) *Two written asset-or-nothing puts with strike 20* ✓

(f) *Forty cash-or-nothing puts with strike 30*

(g) *Forty written cash-or-nothing calls with strike 20* ✓

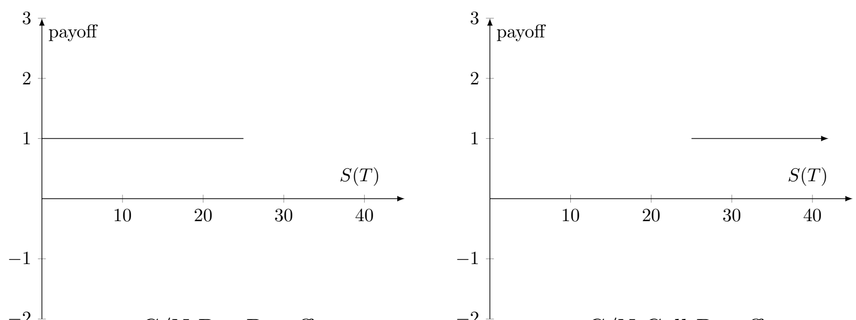
Solution: It is clear that there should be two asset-or-nothing options from our rules. The only pair that have the appropriate values are the asset-or-nothing puts. If you try to graph the position of these two types of options, you will see that the function is forty units too high! We must shift down by 40 using the cash-or-nothing options. Writing 40 cash-or-nothing calls with strike 20 and purchasing 40 cash-or-nothing calls with strike 30 accomplishes this.

Remark 11. *In our definitions of the or-nothing options, it was specified that $S(T) > K$ in the conditions. This makes these functions left-continuous. I won't expect you to remember this specific of a detail. Because of this, you won't need to worry if the endpoints in your graphs need hollow dots or filled in dots (unless you choose to). The reason I don't require you to remember this detail is that when we deal with continuous probabilities in the future, the value of these functions at one point will not change the price of the derivatives in question.*

1.11 Parity

Our no arbitrage and frictionless transaction assumptions have some hefty implications, many of which are explored here.

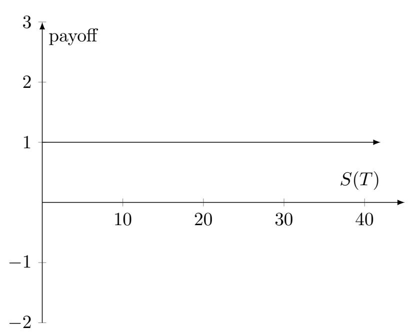
In the last few sections, we have developed the notion of options, a special type of derivative. These are the derivatives that much of this text will work with. With these derivatives come some special relationships that follow from our no arbitrage assumption and the further assumption of frictionless markets (that is, no fees on transactions). The simplest relationship comes from comparing our cash-or-nothing options. Recall our diagram for these options.



C/N Put Payoff

C/N Call Payoff

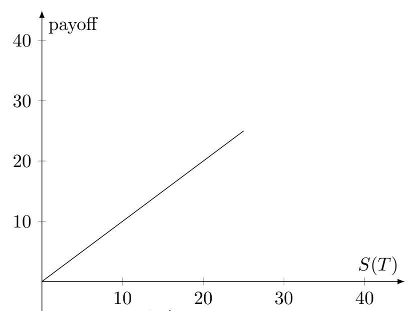
If we were lazy and drew these both on one graph, we would have the following:



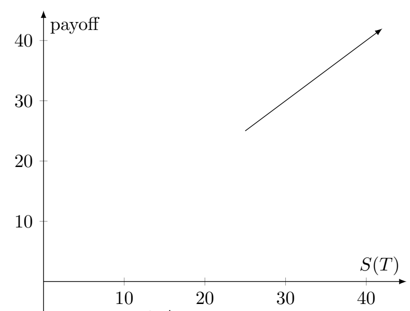
That's actually quite revealing. What this says is that the position of owning a cash-or-nothing put and cash-or-nothing call is equivalent to \$1 at time T! We should all know that the price of the position of \$1 at time T is simply its present value. Without knowing anything more, we have

$$c_{C/N} + p_{C/N} = e^{rT}. \tag{1}$$

In words, the price of the portfolio consisting of a cash-or-nothing put and cash-or-nothing call is the present value of one dollar. This is all under the assumption that the put and call have the same underlying asset, same strike, expiration at time T, and are European. This is particularly wonderful since we have no idea how much our options cost on their own! Let's see if we can do something similar with asset-or-nothing options.

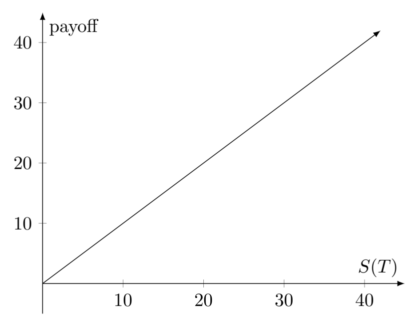


A/N Put Payoff



A/N Call Payoff

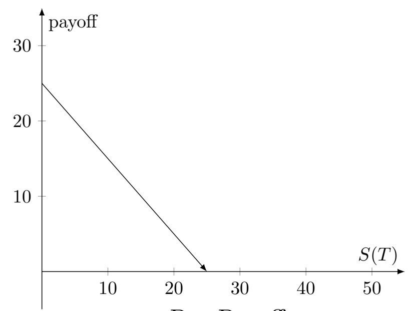
Let's be lazy once again!



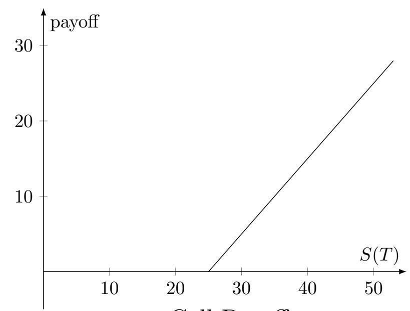
This laziness is really working out! The picture here is quite revealing as well. It shows that the portfolio consisting of an asset-or-nothing put and an asset-or-nothing call will pay just like the underlying asset itself at time T. The bad news is that it's a little bit more difficult to price this position; however, if we recall there is a special value we can pay today to guarantee this payoff. Try to think what it is before you read the next sentence. It is the prepaid forward price for the underlying asset. We have

$$c_{A/N} + p_{A/N} = S(0)e^{-\delta T}. \tag{2}$$

In words, the price of the portfolio consisting of a asset-or-nothing put and asset-or-nothing call is the prepaid forward price of the underlying asset. This is all under the assumption that the put and call have the same underlying asset, same strike, expiration at time T, and are European. Everything is going rather well. Let's try something similar to our regular calls and puts.

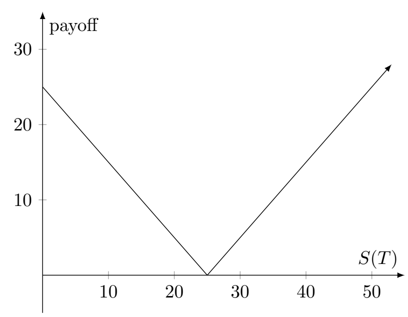


Put Payoff

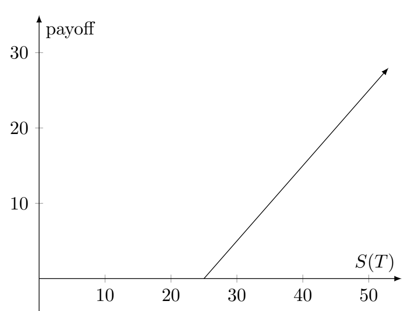


Call Payoff

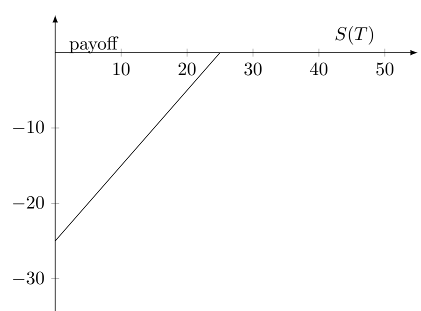
Letting laziness strike, we have



Hmmm... that didn't do what I had hoped. Perhaps if I write the put instead.

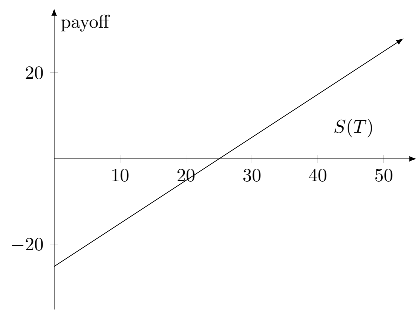


Call Payoff



Written Put Payoff

Unleash the lazy!



That certainly looks better. There's a few ways to think of this. I think the most direct is that the position of ownership in a call and a written put is equivalent to a position that pays $S(T) - K$ at time T. Fortunately, we have a derivative that does just that: a forward contract. This yields

$$c - p = S(0)e^{-\delta T} - Ke^{-rT}. \tag{3}$$

In words, since a call and a written put have the same payoff of the forward contract, it must be the case that the portfolio consisting of the call and written put has equivalent value to the forward contract. All derivatives in this arrangement have the same expiration and same underlying asset. In addition, the strikes of the options are the same as the agreed upon price in the forward contract. Finally, the options must all be European.

We can combine these equations into the following theorem.

Theorem 2 (Parity)**.** *The equations numbered in this section are given special names due to their importance. In order they are* **cash-or-nothing parity**

$$c_{C/N} + p_{C/N} = e^{-rT},$$

asset-or-nothing parity

$$c_{A/N} + p_{A/N} = S(0)e^{-\delta T},$$

and **put-call-parity**

$$c - p = S(0)e^{-\delta T} - Ke^{-rT}.$$

In each equation, the derivatives must all have the same terms.

Remark 12. *These are not all the possibilities for parity. This is simply a glimpse into how parity relationships work. For example, an alternative to put call parity would be to say that the position consisting of a call and a written put is equivalent to purchasing $e^{-\delta T}$ shares of the underlying asset and selling (or issuing) Ke^{-rT} in bonds today.*

Since the cash-or-nothing options don't have the most exciting arbitrage opportunities, let's work with an asset-or-nothing option.

Example 19. *You are given the price of two asset-or-nothing options:* $c_{A/N} = 31.97$ *and* $p_{A/N} = 34.08$. *Both options have the same underlying asset, the same strike and the same time to expiration (three months from today). The asset costs* 70 *today and has a dividend rate of* 2%. *Determine if there is an arbitrage opportunity.*

Solution: We only have one test for this condition: asset-or-nothing parity. We must verify that the equality holds. If it doesn't, then we have the recipe for arbitrage.

$$c_{A/N} + p_{A/N} = 31.97 + 34.08 = 66.05$$
$$S(0)e^{-\delta T} = 70e^{-0.005} = 69.65$$

Since those numbers are different, we have an arbitrage opportunity. We must buy the low cost side of the equation and write or sell the high cost side of the equation. This will result in an arbitrage gain of 3.60. The portfolio is as follows:

- Buy the asset-or-nothing call

- Buy the asset-or-nothing put

- (Short) sell $e^{-0.005}$ shares of the underlying asset.

The wonderful thing about this relationship is that we would have no liability in the future. The position we attain from the asset-or-nothing options will be used to give back to the counterparty in the short sale. In the real world, we would make this transaction as many times as we could afford. The only limitation would be the collateral that we can provide for the short sale.

Let's try a different type of problem.

Question 11 *You are given that the price of some three-month European call and put are* 2.83 *and* 4.28, *respectively. The strike of the options is* 72. *In addition, the underlying asset costs* 70 *and has dividend rate* 2%. *What is the implied risk-free rate?*

$$\textit{The risk-free rate is } r = \boxed{0.05}$$

Solution: We must use put-call parity.

$$\begin{aligned} c - p &= S(0)e^{-\delta T} - Ke^{-rT} \\ 2.83 - 4.28 &= 70e^{-0.05} - 72e^{-0.25r} \\ 72e^{-0.25r} &= 71.10 \\ r &= 0.05 \end{aligned}$$

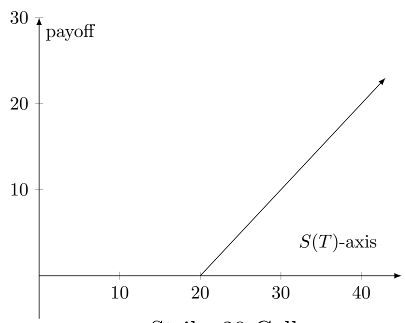
The main point of this section was to show that you can compare two seemingly different portfolios to get implied information regarding values of assets. This can lead to a variety of arbitrage opportunities. In the next section, we will extend this kind of reasoning to inequalities. The principle that will guide us is this: If a portfolio has only non-negative payoffs, then the position today must also be non-negative.

1.12 Inequalities

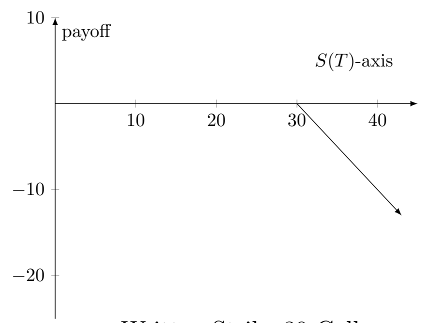
Several fundamental inequalites are explored. They give us some powerful tools to analyze the price of calls and puts.

Last section, we worked with positions that had liabilities that perfectly matched. This match allowed us to use the positions to offset one another. This allowed us to infer values associated to other portfolios. Here, we will deal with portfolios that are simply compared to others to try to derive aggregate portfolios that have non-negative value (as opposed to zero value). There are three different portfolios that we will consider here; however, in the finance world, there are many more (many of which have bizarre names.

The first portfolio we will consider is the bull spread. The idea of a bull spread is to buy a call and to write another call with a higher strike. The diagram below depicts a call with strike 20 and a written call with strike 30.



Strike 20 Call



Written Strike 30 Call

The lazy approach from last section doesn't work as well since there are pieces of these functions that are both nonzero. Perhaps if we thought about these as piecewise functions first and added them together we would have better luck. The functions for each are

$$f_1(S) = \max\{S - 20, 0\} = \begin{cases} S - 20 & \text{if } S > 20 \\ 0 & \text{otherwise,} \end{cases}$$

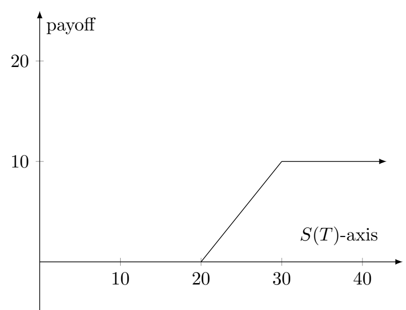
$$f_2(S) = -\max\{S - 30, 0\} = \begin{cases} 30 - S & \text{if } S > 30 \\ 0 & \text{otherwise.} \end{cases}$$

We can add these functions. We just have to remember to break this into three pieces. The only part that will be "tricky" wil be the values of S larger than 30.

$$f_1(S) + f_2(S) = \begin{cases} S - 20 & \text{if } 20 < S \leq 30 \\ 10 & \text{if } 30 < S \\ 0 & \text{otherwise.} \end{cases}$$

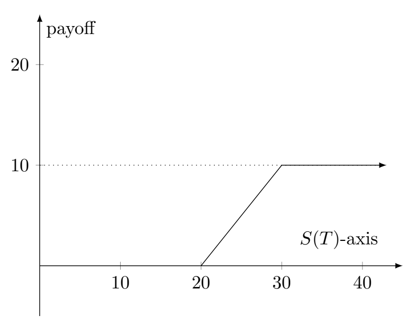
I would probably never do this in practice. My view is that I only need to note the slopes of each function. Once I have that information, these functions become a lot easier to graph. In this problem, I would start with slope 0 from 0 to 20. The slope would be 1 from 20 to 30. Then the written call would contribute a negative one to the slope, bringing the sum's slope back to 0 from 30 to infinity.

Let's graph the resulting function.



Bull Spread

Now, we can add one little dotted line to the picture that will reveal how this inequality will be realized.



We see that our payoff diagram is always wedged between the values 0 and 10. This gives us the bull-spread inequality.

$$0 \le c(20) - c(30) \le \mathrm{PV}(10) = 10e^{-rT}$$

In greater generality, we have that for strikes $K_1 < K_2$

$$0 \le c(K_1) - c(K_2) \le (K_2 - K_2)e^{-rT}$$

We could go further and say the following:

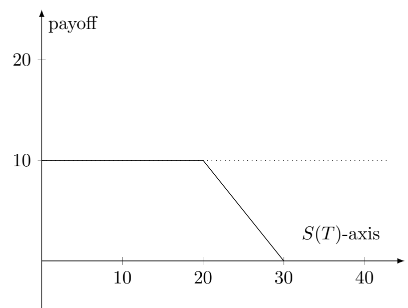
$$0 \le c(K_1) - c(K_2) \le \min\{c(K_1), (K_2 - K_1)e^{-rT}\},$$

but it only makes things messier. One useful piece of information can be gleaned.

$$c(K_1) \geq c(K_2)$$

That is to say, calls are alwasy non-increasing in value as a function of strike. This information is useful.

We can apply everything we have done to puts. The difference is that we will need to buy the put with the higher strike and write the one with the lower strike. Please work out any of the details if they don't make sense.



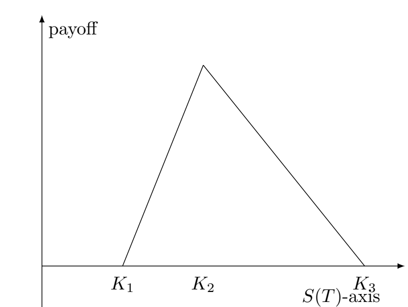
Bear Spread

This picture give almost the same information as the one used for the bull spread.

$$0 \leq p(K_2) - p(K_1) \leq (K_2 - K_1)e^{-rT}$$
$$p(K_1) \leq p(K_2)$$

The second says that puts are always non-decreasing in value as a function of strike. We will use this fact later.

The last inequality we will develop comes from the butterfly spread. This opportunity is a little different than the previous ones. It uses only calls (or puts). Let's start with the diagram below.



Butterfly Spread

The payoff depicted here is never negative, and it is sometimes positive. This position should probably cost something. In addition, I have included three values. These should indicate strikes of some kind of option. The lack of labels gives me some freedom in my choices. I will begin by assuming that the first option is a call with strike K_1. This will match my diagram from 0 to K_2. At K_2, an adjustment needs to be made. We need to determine that adjustment.

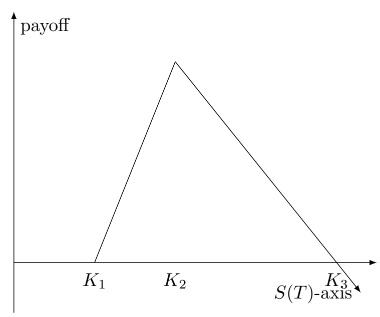
Since we purchased one call with strike K_1, the slope of our line connecting the points at K_1 and K_2 is 1. That means the value at the apex is (K_2, K_1). We need to write a number of options so that the resulting line will connect the points (K_2, K_1) and $(K_3, 0)$. The slope of the line must be

$$-\frac{K_2 - K_1}{K_3 - K_2}.$$

To move from a line with slope one to the line with slope we just mentions, we will need to write a number of options with strike K_2. That number is

$$1 + \frac{K_2 - K_1}{K_3 - K_2} = \frac{K_3 - K_1}{K_3 - K_2}.$$

If we stopped there, the diagram would be



We need to purchase some calls with strike K_3 to level things out. Fortunately, we know the slope of the line we are dealing with. We need to buy

$$\frac{K_2 - K_1}{K_3 - K_2}$$

calls with strike K_3 to level everything out. This gives us what we wanted.

$$\begin{aligned}
c(K_1) - \frac{K_3 - K_1}{K_3 - K_2}c(K_2) + \frac{K_2 - K_1}{K_3 - K_2}c(K_3) &\geq 0 \\
c(K_1) + \frac{K_2 - K_1}{K_3 - K_2}c(K_3) &\geq \frac{K_3 - K_1}{K_3 - K_2}c(K_2) \\
\frac{K_3 - K_2}{K_3 - K_1}c(K_1) + \frac{K_2 - K_1}{K_3 - K_1}c(K_3) &\geq c(K_2) \\
\alpha c(K_1) + (1 - \alpha)c(K_3) &\geq c(K_2)
\end{aligned}$$

where $\alpha = \dfrac{K_3 - K_2}{K_3 - K_1}$. This relationships is different from the bull and bear spreads. It does not rely on time. This is simply a relationship regarding calls with various strikes. This also says that the price of a call is **convex** or **concave up** with respect to the strike. If we combine this with the bull spread information, we have an idea for what the price of a call should look like as we vary the strike.

In addition, we can use the same analysis to show that the result will also hold for put options. The value of α is the same as well.

$$\alpha p(K_1) + (1-\alpha)p(K_3) \geq p(K_2)$$

Since this information is all important, let's group it all together into a theorem.

Theorem 3 (Inequalities)**.** *When our European call and put options have the same underlying asset and time to expiration, T, we have the following inequalities.*

Name of Relationship	Inequality
Bull Spread	$0 \leq c(K_1) - c(K_2) \leq (K_2 - K_1)e^{-rT}$
Bear Spread	$0 \leq p(K_2) - p(K_1) \leq (K_2 - K_1)e^{-rT}$
Butterfly Spread	$c(K_2) \leq \dfrac{K_3 - K_2}{K_3 - K_1} \cdot c(K_1) + \dfrac{K_2 - K_1}{K_3 - K_1} \cdot c(K_3)$
Butterfly Spread	$p(K_2) \leq \dfrac{K_3 - K_2}{K_3 - K_1} \cdot p(K_1) + \dfrac{K_2 - K_1}{K_3 - K_1} \cdot p(K_3)$

We would like to graph the calls and put with respect to their strikes, but we need a little more information. Let's answer the following questions:

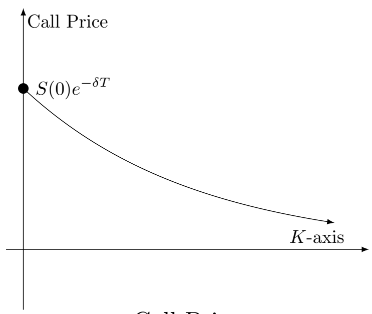
- What should the price of a strike 0 call be?
- What should the price of a strike 0 put be?

This isn't too difficult if you think of the payoffs. The call's will always be the underlying asset at expiration. That means the price of a strike 0 call should be the prepaid forward price of the underlying asset. In contrast, the put's will always be 0. That means the price of a strike 0 put should be 0.

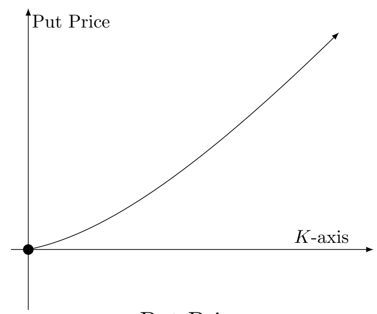
Now, we need one more piece of information for the graphs of call and put prices. We can call this an assumption for now. It may be something we can prove later under specific pricing models for calls and puts.

$$\lim_{K \to \infty} c(K) = 0$$

The figures below are cartoons of the reality, but they are certainly illustrative.



Call Prices



Put Prices

Another observation that we could make is that the put has a slant asymptote with slope e^{-rT}. This can be derived using put-call parity.

We've developed a lot of theory. Let's put it to practice!

Example 20. *You are given the table below with various values of European calls and puts on the same underlying asset. The time to expiration is six months, and the risk-free rate is 7%. Determine if there is an arbitrage opportunity present.*

Strike	45	50	55
Call	7.48	4.63	2.67
Put	1.67	3.65	6.52

Solution: There is a lot that we could do here: there are three pairs of bull spreads to check, three pairs of bear spreads to check, and two butterfly spreads to check. Let's just check one of each, and I will leave it to you to do the remainder. We'll check them in order.

Let's start with a bull spread consisting of one purchased call with strike 45 and one written call with strike 55.

$$\begin{aligned} c(45) - c(55) &\leq \mathrm{PV}(10) \\ 7.48 - 2.67 &\leq 10e^{-0.07 \cdot 0.5} \\ 4.81 &\leq 9.66 \checkmark \end{aligned}$$

There is no opportunity with that pair. Now let's try one of the bear spreads.

$$\begin{aligned} p(55) - p(50) &\leq \mathrm{PV}(5) \\ 6.52 - 3.65 &\leq 5e^{-0.07 \cdot 0.5} \\ 2.87 &\leq 4.83 \checkmark \end{aligned}$$

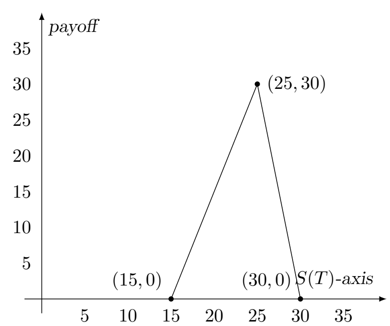
There is no opportunity with that pair. Finally, we will try a butterfly spread.

$$\begin{aligned} \alpha p(45) + (1-\alpha)p(55) &\geq p(50) \\ 0.5 \cdot 1.67 + 0.5 \cdot 6.52 &\geq 3.65 \\ 4.095 &\geq 3.65 \checkmark \end{aligned}$$

There is no opportunity with that butterfly spread. If you go and check the remaining 5 groupings of options, you will find that no opportunity exists!

Now let's try a different kind of problem. This one will make you combine butterfly options with portfolio construction.

Question 12 *You plan to use three different calls to construct the payoff diagram given below.*



I should buy $\boxed{3}$ *calls with strike* 15.

I should write $\boxed{9}$ *calls with strike* 25.

I should buy $\boxed{6}$ *calls with strike* 30.

Solution: It may seem that the numbers from the prompted response are arbitrary, but they can be deduced in a very clear way. The key here is to observe the slopes of your line segments.

The first line segment connects the points $(15, 0)$ and $(25, 30)$. This line segment has slope 3. To arrive at this position, it is necessary to purchase 3 call options! The slope give you exactly what you need!

The second line segment connects the points $(25, 30)$ and $(30, 0)$. This line segment has slope -6. To go from a line with slope 3 to one with -6 requires you to write $(3 - (-6)) = 9$ calls with strike 25.

The last part comes from eliminating the slope of -6. This requires you to buy 6 calls with strike 30!

If any of this seems unclear, you can always write out the piecewise functions. This will never fail; however, it can be tedious.

In addition, we could have done this exact same portfolio using put options! It would have consisted of 6 puts with strike 30, 9 written puts with strike 25, and 3 puts with strike 15.

1.13 Asian Options

Here we introduce Asian options. Averages will be discussed, and payoffs will be computed.

This section diverges a bit from our earlier discussion regarding payoff diagrams. The reason this section is so different is that the options we discuss have payoffs that are dependent on the path that the underlying asset takes over time. This averaging can serve as a tool to smooth out the randomness of stock prices. Before we can define the Asian options, we first need to formalize the notions of averaging that we will use.

Definition 15. *Let x_1, x_2, ..., x_n be nonnegative numbers and $n \geq 1$. The* **arithmetic average** *is the quantity*

$$\bar{x} = AA = \frac{1}{n}\sum_{j=1}^{n} x_j,$$

and the **geometric average (mean)** *is the quantity*

$$GA = \left(\prod_{j=1}^{n} x_j\right)^{1/n}.$$

It is very likely that you have seen the arithmetic average before. It is a little less likely that you have seen the geometric average. The restriction that these numbers be nonnegative is required so we don't end up with undefined or imaginary geometric averages. This assumption makes sense when we are dealing with stock prices, as they are never negative. Let's try an example of some averaging.

Example 21. *Let $x_1 = 3$, $x_2 = 8$, and $x_3 = 9$. Compute both averages for these three numbers.*

Solution: The averages are

$$\begin{aligned}\text{AA} &= \frac{3+8+9}{3} = 6.\bar{6}\\ \text{GA} &= (3 \cdot 8 \cdot 9)^{1/3} = 6\end{aligned}$$

It's a bit of a stretch to generalize after only one example, but it just so happens that the geometric average is always less than or equaly to the arithmetic average. This is known as the AM-GM inequality, or using our letters the AA-GA inequality. It will not be proven here, but we will use it when comparing our Asian options.

Definition 16. *An Asian option is an option that pays based on averages. There are 8 varieties that are determined by the three following choices:*

- *call or put,*
- *average applied to the strike value or the stock price,*
- *and the average is arithmetic or geometric.*

To be explicit, let's give a table describing the payoff of each of the eight Asian options.

Type of Option	Payoff
Arithmetic average strike call	$\max\{S(T) - AA, 0\}$
Arithmetic average price call	$\max\{AA - K, 0\}$
Geometric average strike call	$\max\{S(T) - GA, 0\}$
Geometric average price call	$\max\{GA - K, 0\}$
Arithmetic average strike put	$\max\{AA - S(T), 0\}$
Arithmetic average price put	$\max\{K - AA, 0\}$
Geometric average strike put	$\max\{GA - S(T), 0\}$
Geometric average price put	$\max\{K - GA, 0\}$

An Asian option will not always be explicitly stated as such. The indication for these options is usually the phrases "average strike" or "average price." Personally, I would not bother memorizing a table like this. It seems like a waste of resources. I would remember that there are three "switches" for the Asian options: c/p, K/S, and AA/GA. Let me give an example.

Example 22. *Describe the payoff of a one year, geometric average strike put option.*

Solution: Without looking at the table, I might write something like this:

$$\begin{aligned}\max\{K - S(T)\} &= \max\{A - S(T), 0\} \\ &= \max\{GA - S(T), 0\}.\end{aligned}$$

The left hand side of the first line is recognizing that this is a put option. Put options have payoff $\max\{K - S(T), 0\}$. The right hand side of the first line indicates that the average is going in place of the strike. The right hand side of the second line recognizes that the average is geometric.

When we are computing payoffs of Asian options, we will use all available information when computing averages except for the underlying asset's initial value.

Example 23. *You would like to determine the payoff of a six month, geometric average price call option with strike 45. You are given*

Time (months)	*0*	*1*	*2*	*3*	*4*	*5*	*6*
$S(T)$	*45*	*48*	*50*	*49*	*52*	*49*	*48*

Solution: We first compute the geometric average, and then we will compute the payoff.

$$\begin{aligned} GA &= (48 \cdot 50 \cdot 49 \cdot 52 \cdot 49 \cdot 48)^{1/6} \\ &= 49.31 \\ \max\{GA - K, 0\} &= \max\{49.31 - 45, 0\} \\ &= 4.31 \end{aligned}$$

Let's try something a little different.

Problem 13 *Choose the more valuable more option:*

Multiple Choice:

(a) *Arithmetic average strike put option* ✓

(b) *Geometric average strike put option*

Solution: We stated earlier that the arithmetic average is greater than or equal to the geometric average. This is instrumental.

$$\begin{aligned} AA &\geq GA \\ AA - S(T) &\geq GA - S(T) \\ \max\{AA - S(T), 0\} &\geq \max\{GA - S(T), 0\} \end{aligned}$$

Since the payoffs satisfy this inequality so do the prices.

Let's try one more concept before finishing this section.

Example 24 (Parity?)**.** *Suppose that I know the payoff of an arithmetic average strike call option. What can I say about the payoff of an arithmetic average strike put option, provided the underlying asset and date of expiration are the same?*

Solution: Without more information, all I can say is the following:

$$\begin{aligned} \max\{S(T) - AA, 0\} - \max\{AA - S(T), 0\} &= S(T) - AA \\ \max\{S(T) - AA, 0\} - S(T) + AA &= \max\{AA - S(T)\}. \end{aligned}$$

This suggests a parity relationship; however, I will never know the arithmetic average at time zero. This makes Asian options more difficult to price than European options. Fortunately, the first pricing models we wil discuss are well equipped to handling these options.

1.14 Duality

Parity is not the only relationship between calls and puts. Sometimes, a barter between two different assets can be established. There lies the relationship known as duality.

This section is another divergence from the meat of this chapter; however, it does relate call and put options in an entirely new way. The difficulty here lies in the lack of payoff diagrams to relate the calls and puts in question. In a standard scenario, duality is a reference to two positions in currencies.

Let's say that the price to buy one of currency 2 today using currency 1 as a payment is $S(0)$. Some authors use $x(0)$, but I think that makes things seem needlessly complicated and different from what we are used to. Each currency has its own risk-free rate, say r_1 and r_2. Since we are using currency 1 to purchase currency 2, the risk-free rate, r_1, will go in position r in any of our formulae. Since currency 2 is acting as an asset, r_2 will go in position δ in any of our formulae.

We wish to examine the payoffs of two different options at expiration. We will assign values so everything makes more sense.

Example 25. *Suppose that the price to purchase one €1 was $1.10. Similarly, the price to purchase $1 is €1/1.1. There are calls and puts available to users of each currency to purchase or sell the other currency. We are going to compare their payoffs under the right circumstances.*

Since I live in Ohio, it makes sense for me to be a speculator that spends dollars on their investments. That is to say, if I wanted to get into the currency market, I could buy calls and puts for euros. Let's calculate a couple of payoffs for two options: a call and a put with strike $K = 1.05$. *The options have the euro as an underlying asset, and they have the same time to expiration.*

Price at Expiration	*0.9*	*1.1*
Call Payoff	*$0*	*$0.05*
Put Payoff	*$0.15*	*$0*

Now we must ask the question: what would this all look like for a speculator in Europe that wishes to buy calls and puts based on the dollar? Well, since the price is inverted, perhaps we should also invert the strike. We could assign a new letter for this strike or simply write $1/K = 1/1.05 = 0.95238$. *Let's see a similar table.*

Price at Expiration	*1.11111*	*0.90909*
Call Payoff	*€0.15873*	*€0*
Put Payoff	*€0*	*€0.04329*

This doesn't seem like it illustrates anything... Wait! The currencies are different. I can convert one to the other. Once again, I live in Ohio, so I think in dollars. I will convert the second table to dollars. I will also multiply by K. That second part doesn't make any sense, but the proof is in the pudding!

Price at Expiration	*1.11111*	*0.90909*
Call Payoff	$0.15873 \cdot 1/1.11111 \cdot 1.05 = 0.15$	*0*
Put Payoff	*0*	$\frac{0.04329}{0.90909} \cdot 1.05 = 0.05$

This is really strange. It seems that there is a relationship between the call to buy euros and the put to sell dollars. Similarly, there appears to be a relationship between the put to sell Euros and the call to buy dollars. This is illuastrated via the following equations:

Theorem 4 (Put-Call Duality)**.** *The price to buy currency 2 using currency 1 is S, and a strike price of K is written into currency 1-denominated option contracts to buy/sell currency 2. Then*

$$\begin{aligned} c(S,K) &= SKp(1/S, 1/K) \\ p(S,K) &= SKc(1/S, 1/K), \end{aligned}$$

where the right hand side consists of currency 2 denominated options to buy/sell currency 1.

The really strange part is that these prices are given in values today! That is because I would have no idea of the valus of S at time T.

There is nothing obvious about this relationship. The only information that would even suggest it is the payoff chart we wrote in the example. Fortunately, duality holds under the models we will discuss in this course. A proof of this theorem will be given in the section covering the Black-Scholes model.

Sometimes, it helps me to keep in mind the conversions I am doing at each step along the way. In the first formula, the call is a payment of currency 1 for currency 2, or $\$_1/\$_2$. S and K are similar. The put on the right hand side is a payment in currency 2 for currency 1, or $\$_2/\$_1$. If you let the units cancel as you would in any science class, you will end up with the appropriate units

Let's apply put-call duality in an example.

Example 26. *Let's use the values from our example before. In addition, we will take the position of the American traveler going to visit Europe in six months. The current price of €1 is \$1.1. As a traveler, you would like to ensure that you get a decent conversion rate of dollars to euros. On your trip, you would like to have €2000, and you would like to buy each euro for at most \$1.05. You use a call to achieve this. The six month call costs you \$255.94.*

How much is a six month euro-denominated put option to sell \$500 with strike $1/1.05$*?*

Solution: We must use put-call duality; however, we should convert the values to units first. The original call would be

$$c(S, K) = \frac{1}{2000} \cdot 255.94 = 0.12797$$

Now we can use duality.

$$\begin{aligned} c(S, K) &= SKp(1/S, 1/K) \\ 0.12797 &= 1.1 \cdot 1.05 \cdot p(1/S, 1/K) \\ 0.11080 &= p(1/S, 1/K) \end{aligned}$$

The last step is to scale this number up by 500.

$$500p(1/S, 1/K) = 55.40$$

That is, the euro-denominated put option costs €55.40

That seemed pretty straight forward; however, the question could have been rephrased a little bit. It could have said that a dollar- denominated European call option to buy €2000 in six months for \$2100 costs \$255.94. How much is a euro-denominated European put option to sell \$500 in six months for €476.19? The current currency conversion rate would be the same.

This seems to be missing the strike information, but it is given in the statements "buy €2000 in six months for \$2100" and "sell \$500 in six months for €476.19." You would need to compute the fractions

$$\begin{aligned} K &= \frac{2100}{2000} = 1.05 \\ 1/K &= \frac{476.19}{500} = 1/1.05. \end{aligned}$$

From there, you would still know the correct quantities. With the call, you are buying 2000 of the underlying asset. With the put, you are selling 500 of the underlying asset.

Let's try something related to duality.

Problem 14 *In the framework of the previous problem, determine the euro-denominated price of a six month European call option to buy \$500 for €476.19. The risk-free rate in the US is 6%, and the euro risk-free rate is 3%.*

The price is=€ $\boxed{27.41}$

Solution: For this problem, we actually need put-call parity! That's why the respective risk-free rates were given. Since the quantites of dollars is the same for the call and the put, we have

$$\begin{aligned} c - p &= S(0)e^{-\delta T} - Ke^{-rT} \\ c - 55.4 &= 500\left(\frac{1}{1.1}e^{-0.06\cdot 0.5} - \frac{1}{1.05}e^{-0.03\cdot 0.5}\right) \\ c &= 27.41. \end{aligned}$$

The difficult part here comes in the correct substitutions. Don't marry yourself to the letters themselves. Remember the meaning. In put-call parity, S refers to the underlying asset, and δ refers to the asset's "dividend" rate. In this problem, the asset was dollars. It follows that S was the value of one dollar. That value was $1/1.1$. Similarly, K refers to the agreed upon price of the asset. Here, that was $1/1.05$.

1.15 Single Period

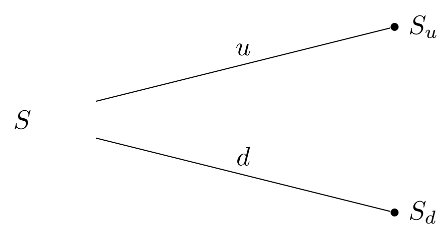
A single period binomial tree is our first example of a model. The models we use for payoffs will be used to help in the construction of payoffs of derivatives. These payoffs lead to the prices of our derivatives.

Now that we have explored a variety of derivatives, we can finally see the benefit of constructing models to estimate the prices of various derivatives. This chapter's goal is to explore the binomial models. Binomial models are wonderful because they can shed light on price variability in all of the derivatives discussed thus far.

When dealing with a totally new concept, it is good to start with something simple. That is the situation we find ourselves in here. Let's begin with defining a one period binomial model.

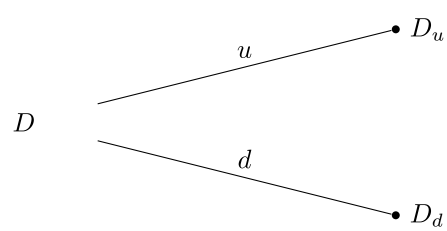
Definition 17. *A* **one-period binomial model** *is used to model the price of one asset over a fixed time period. The model assumes that there are two future positions for the asset: one favorable and one unfavorable. These outcomes are denoted S_u and S_d, respectively (alternatively, $S_u(T)$ and $S_d(T)$).*

This model is very simple. It says that there are only two asset prices available in the future! This structure is easy to draw, as seen below.



Asset Binomial Tree

It is useful to note that such a model will automatically suggest payments for any derivative on the asset. Such a diagram would look like:

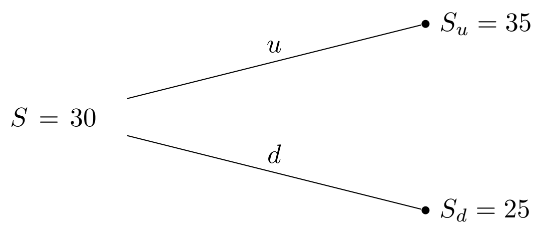


Derivative Binomial Tree

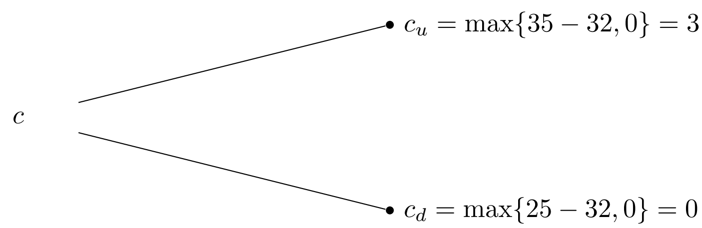
Let's try an example!

Example 27. *You wish to model a strike 32, three month, European call option on some asset S. Your model has the price of the asset today, and it assumes that the stock could either increase by 5 or decrease by 5 in value over the three month period. Model the call's payoff.*

Solution: The stock model is as follows:



This will give a derivative payoff diagram of



Our model is complete!

This is all very nice, but it doesn't model the price of the derivative. We really would like to have a price under the assumed model, or this is all fruitless. The approach is similar to what we did when constructing arbitrage opportunities: we are going to build portfolios that replicate the payoff of the derivative under the model. Since the payoffs will be identical to the derivative's (under the model), the prices of each must be the same.

Such a construction is called a **replicating portfolio**. We will need four components: the underlying asset, its binomial model, the asset's dividend rate, and the risk free rate. We will try to build the payoff of the derivative by combining Δ shares of the underlying asset with an investment of B in bonds. Let's just piggy-back on the previous example.

Example 28. *In addition to our earlier binomial model, we are given that the dividend rate of the derivative is $\delta = 0.03$, and the risk-free rate is $r = 0.07$. What is the price of the derivative using a replicating portfolio?*

Solution: The replicating portfolio is equated to the derivative's payoff diagram. Adjustments must be made for the portfolio in three months since the stock will pay dividends, and the bond will grow in value.

$$
\begin{array}{lll}
 & & 3 = \Delta S_u e^{\delta \cdot 1/4} + B e^{r \cdot 1/4} \\
 & \nearrow & \\
c = \Delta S(0) + B & & \\
 & \searrow & \\
 & & 0 = \Delta S_d e^{\delta \cdot 1/4} + B e^{r \cdot 1/4}
\end{array}
$$

We solve the two equations in two unknowns. I'll use elimination and subtract the second equation from the first to give the third.

$$
\begin{aligned}
3 &= \Delta 35 e^{0.03/4} + B e^{0.07/4} \\
0 &= \Delta 25 e^{0.03/4} + B e^{0.07/4} \\
3 &= \Delta 10 e^{0.03/4} \\
0.3 e^{-0.0075} &= \Delta
\end{aligned}
$$

From there, we can solve for B.

$$
\begin{aligned}
0 &= \Delta 25 e^{0.03/4} + B e^{0.07/4} \\
-7.5 &= B e^{0.07/4} \\
-7.5 e^{-0.0175} &= B
\end{aligned}
$$

Now that we have Δ and B, we can compute the modeled price of the derivative. The value is simply the sum of the price of Δ shares of the underlying asset and the bond B.

$$
c = \Delta S(0) + B = 1.56
$$

This solution makes some intuitive sense. The value of the call should be less than 3. My reasonging is that your maximum payment will be 3, no matter the position. It follows, by our no arbitrage assumptions, that the maximum price will be less than or equal to the present value of 3. 1.56 certainly satisfies that condition.

Replicating portfolios always work like this. You only need the four pieces of information ai listed earlier. The only tricky part could be detrmining the payoff at expiration!

Problem 15 *Use a replicating portfolio to model the price of a strike 32, European put option that expires in three months on the same underlying asset as in our examples.*

$$p = \boxed{3.23}$$

Solution: Our equations are slightly different from what we did in the call example.

$$\begin{array}{lll} & & \bullet\; 0 = \Delta S_u e^{\delta \cdot 1/4} + B e^{r \cdot 1/4} \\ p = \Delta S(0) + B & & \\ & & \bullet\; 7 = \Delta S_d e^{\delta \cdot 1/4} + B e^{r \cdot 1/4} \end{array}$$

Thus, we are solving the following equations.

$$\begin{aligned} 0 &= \Delta 35 e^{0.0075} + B e^{0.0175} \\ 7 &= \Delta 25 e^{0.0075} + B e^{0.0175} \\ -7 &= \Delta 10 e^{0.0075} \\ -0.7 e^{-0.0075} &= \Delta \\ 0 &= -24.5 + B e^{0.0175} \\ 24.5 e^{-0.0175} &= B \end{aligned}$$

It follows that the modeled price of the put is

$$p = \Delta S(0) + B = 3.23,$$

as desired.

If you put some effort into it, you could come up with a formula for Δ and B. I will give them here for completeness, but I do not think these are things you should memorize. It is my belief that solving two equations in two unknowns should be very natural to you at this time. In fact, I think execution of the following formulae is more difficult than solving two equations in two unknowns.

$$\Delta = \frac{D_u - D_d}{S_u - S_d} e^{-\delta h}$$
$$B = \frac{S_d D_u - S_u D_d}{S_d - S_u} e^{-rh}$$

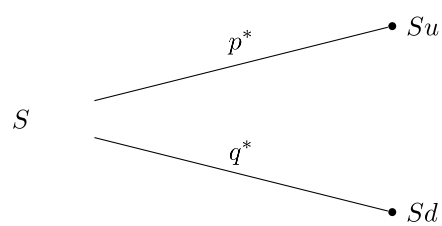
In the event that we treat u and d as factors (which will be usefull very soon), we could cancel the S's out of the formula for B as follows:

$$B = \frac{dD_u - uD_d}{d - u} e^{-rh}.$$

If you are wondering what I meant by treating u and d as factors, check our example. In it, we would have had $u = 7/6$ and $d = 5/6$.

Replicating portfolios are very nice, but they don't generalize in any great way to multiple periods. Multiple period binomial models are superior in that they allow for a greater variety of outcomes. Fortunately, there is an alternative. We can use probability! The idea is to assign a probability of an asset to move up and its complememt to the asset moving down, say p and q. We could make arbitrary choices, but that's not so great.

If we make the assumption that money spent on the underlying asset is just as good as money spent on a risk-free investment, we can arrive at something called the **risk-free probability of an up move**. It is denoted p^*. It's complement is denoted q^*. Let's set this up! Recall our earlier diagram, where I have replaced u and d with p^* and q^*, respectively. The diagram only considers stock prices. We must make sure to remember that there may be dividends in play.



By spending S today, we arrive at some possible future. Alternatively, we could spend S today and arrive at Se^{rh}. This is best treated through expectations. Everything in what follows is a time h value. P denotes the portfolio that is an investment in the stock S and any dividends that may entail.

$$\begin{aligned}
Se^{rh} &= \mathbb{E}[P(h)] \\
Se^{rh} &= Sue^{\delta h}p^* + Sde^{\delta h}q^* \\
e^{rh} &= ue^{\delta h}p^* + de^{\delta h}(1-p^*) \\
e^{(r-\delta)h} &= up^* + d(1-p^*) \\
e^{(r-\delta)h} &= (u-d)p^* + d \\
e^{(r-\delta)h} - d &= (u-d)p^* \\
\frac{e^{(r-\delta)h} - d}{u-d} &= p^*
\end{aligned}$$

In case this is not satisfactory, you could always use replicating portfolios to determine p^*. I encourage you to see if you can set up the equations correctly! We will try something like this in the next section when we determine p^* under a futures arrangement. For now, let's revisit our first example.

Example 29. *We want to determine the price of a three-month, strike 32 European call on S where the following information holds:*

- $S(0) = 30$
- $Su = 35$
- $Sd = 25$
- $\delta = 0.03$
- $r = 0.07$

Use risk-free probabilities to compute the price of the call.

Solution: We must start with computing the risk-free probability:

$$p^* = \frac{e^{(0.07-0.03)/4} - 5/6}{7/6 - 5/6} = 0.53015$$

It follows that the price must be

$$\begin{aligned}
c &= [3p^* + 0q^*]e^{-0.07/4} \\
c &= 1.56
\end{aligned}$$

The reason I am discounting the values is to bring them to today's currency values. All of the payoffs happened in the future, so the currencies would not be compatible to currencies today without such a conversion.

The big boon to using p^* in modeling the prices of derivatives is that they generalize to multiple periods, and it can be used for some very strange derivatives; however, there is a shortcoming. We cannot use risk-free probabilities to construct arbitrage opportunities. Let's see how that might work

Example 30. *You are a speculator wishing to make some money on mispriced assets. You look up the market price on the call discussed in this section. The market price of the call is \$2. Under your binomial model, you determined that the price of the call should be \$1.56. If you model is correct, then there is arbitrage present. The market value of the call is overpriced. Describe your arbitrage opportunity!*

Solution: It should be obvious that the opportunity will net you $2 - 1.56 = 0.44$. This should always be clear once the opportunity is found. Since the market price is higher than the model price, we will write the market call. We will buy the replicating portfolio. From our work earlier, we know that our arbitrage portfolio is

- Buy $0.3e^{-0.0075}$ shares of S.
- Sell 7.37 in bonds.
- Write the call.
- Receive the arbitrage gain of 0.44.

I hope it is obvious that this is only true if our model is correct? This begs the question of what we might do if the market price of the call were \$1. The result is not much different.

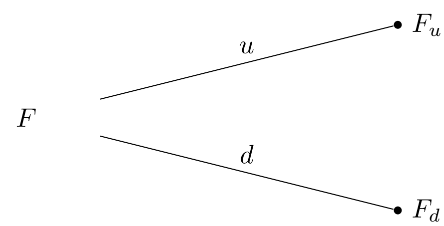
- Sell $0.3e^{-0.0075}$ shares of S.
- Buy 7.37 in bonds.
- Buy the call.
- Receive the arbitrage gain of 0.56.

That concludes our discussion of single period binomial models when dealing with stocks and currencies. In the next section, we will explore what happens when applying these portfolios to futures arrangements. Following that, we will deal with multiple periods.

1.16 Futures

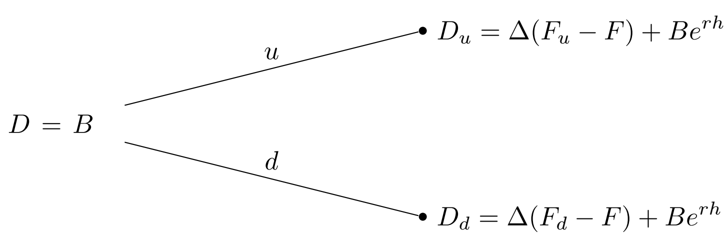
Under a futures arrangement, you simply have an agreement to buy something at the expiration date. This changes how binomial trees are constructed for futures.

It may seem strange that we need to dedicate a section to futures, but there is a reason to do so. First off, do futures even have dividends? No, they do not. Does that mean that $\delta = 0$ in our pricing formula. Strangely, the answer to that is also no! Let's see why later in this section. First, we need to work on our model.



Futures Binomial Tree

This doesn't look that different from our asset model; however, the cartoon is misleading. The F today represents the futures price. The model assumes that at settlement, you will have a payoff of $F_u - F$ or $F_d - F$. Margin accounts will play no role in these models. It follows that at inception, we are not paying anything! We are just making an agreement to pay F at the end of the period. This will affect our replicating portfolios. Let's see what our derivative would look like under such a model.



Derivative Binomial Tree

The derivative being modeled will be valued B because at inception, the futures part of the arrangement costs nothing. It isn't too fruitful to do many example of portfolios under this arrangement, but there is one implication that is counterintuitive. That will be explored now. What value should we give to p^*, the risk-free probability of an up move? Our approach from last section won't work at all because we aren't paying anything for our futures arrangement, so we can't measure an investment's growth in two different ways. The alternative is to measure the up move by using an indicator derivative. That is, use a derivative that pays 1 in the up position and 0 in the down position.

$$
D = B \begin{array}{l} \xrightarrow{u} \; 1 = \Delta(F_u - F) + Be^{rh} \\ \xrightarrow{d} \; 0 = \Delta(F_d - F) + Be^{rh} \end{array}
$$

We solve for the derivative's price in two different ways. First, we use the portfolio from the diagram.

$$
\begin{aligned}
1 &= \Delta(F_u - F) + Be^{rh} \\
0 &= \Delta(F_d - F) + Be^{rh} \\
1 &= \Delta(F_u - F_d) \\
\frac{1}{F_u - F_d} &= \Delta \\
0 &= \frac{F_d - F}{F_u - F_d} + Be^{rh} \\
\frac{F - F_d}{F_u - F_d} &= Be^{rh} \\
\frac{1 - d}{u - d}e^{-rh} &= B
\end{aligned}
$$

The first two lines come from the diagram. The third is the difference of the first two. The fourth should be clear from the third. The fifth comes from substitution of Δ into the second line. The last coulple come from solving for B and treating u and d as multiples.

Now we can compute the value of this derivative using risk-free probabilities. This is much more direct.

$$
\begin{aligned}
D &= [1p^* + 0q^*]e^{-rh} \\
D &= p^* e^{-rh}
\end{aligned}
$$

Remember, $D = B$, so we can equate the last lines of each line of reasoning.

$$\frac{1-d}{u-d}e^{-rh} = p^* e^{-rh}$$
$$\frac{1-d}{u-d} = p^*$$

Remember for our asset models, we had that

$$p^* = \frac{e^{(r-\delta)h} - d}{u-d}$$

This means that for our futures models, we will use the relationship

$$\boxed{\delta = r}$$

in all our computations! This will be very important when we start our multiple period trees in the next section. Commit it to memory!

When dealing with arbitrage opportunities, you will be asked to describe the arbitrage portfolio. If the modeled price is lower than the market price of the derivative, then you will purchase the replicating portfolio and sell the derivative. This will look like

- buy the bond for B,
- agree to buy Δ of F at expiration,
- sell the derivative,
- and receive the arbitrage gain!

If the situation is reversed, then you would be selling the replicating portfolio and buying the derivative. That is,

- Sell the bond for B,
- agree to sell Δ of F at expiration,
- buy the derivative,
- and receive the arbitrage gain!

The key difference for futures is that you are making an agreement. When dealing with assets, you were actually purchasing or selling the underlying asset at time 0.

1.17 Multiple Periods

Our one-period model was nice and simple; however, it was obviously lacking in reality. The multiple period model makes up for these shortcomings in many ways.

It should have been clear that single period binomial trees are overly simple. The framework, however, is useful. We will make a few assumptions when constructing multiperiod binomial trees.

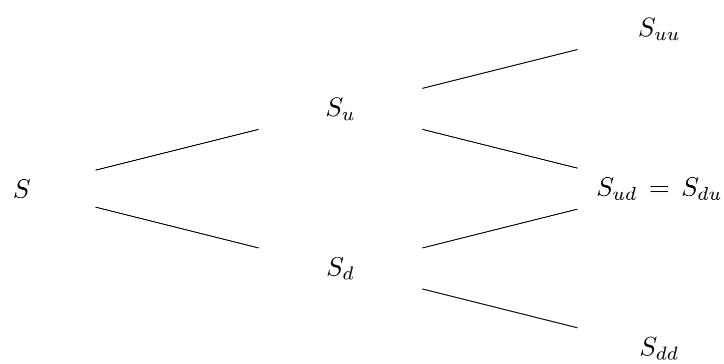
(a) When the asset moves up, it will always be by a common factor of u.

(b) When the asset moves down, it will always be by a common factor of d.

(c) Each step will always take the same amount of time h.

(d) The risk-free rate and dividend rate will remain constant.

These rules ensure that p^* does not vary by position. It eases all computations that we will do with these models. Below, you will see a two period binomial tree. By our first and second rules, we have that $S_{ud} = S_{du}$. In other words, the tree recombines.



Two Period Binomial Tree

It is really easy to construct payoff diagrams for European derivatives under this model, and it is slightly more complex than what we did previously. Our guiding principle for pricing European derivatives is

$$\text{Derivative Price} = \mathbb{E}^*[\text{Payoff}]e^{-rT}$$

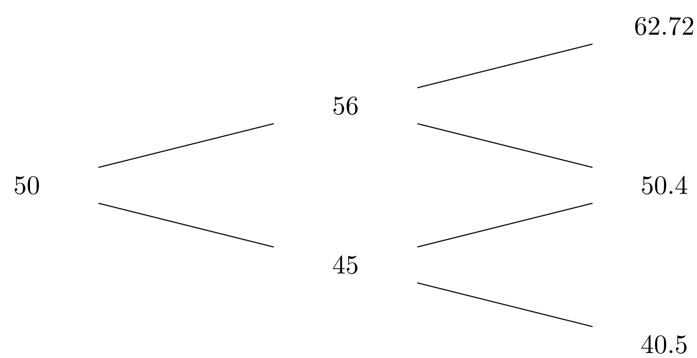
The star indicates that we are using the risk-free rate in our computation of p^*. p^* will be used to compute the expectation. This concept will need to be modified when we work with American options, as their payoffs can be at a variable point in time.

Example 31. *You are given a two period binomial model with the following information: $S(0) = 50$, $r = 0.08$, $\delta = 0.02$. In addition, $u = 1.12$ and $d = 0.9$. Each step in the model covers three months. Use the model to determine the price of a European call option with strike 50 that expires in six months.*

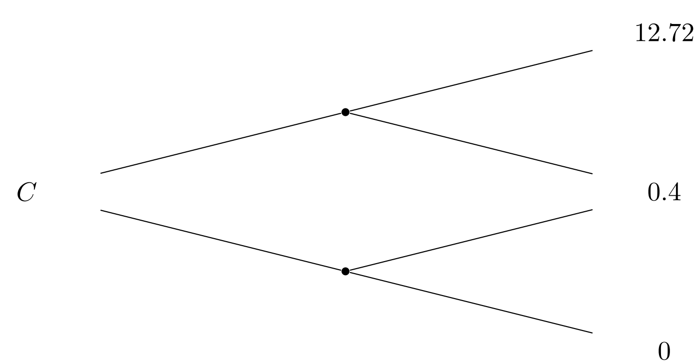
Solution: We begin by calculating p^*. Then we will construct the asset model diagram and the payoff diagram.

$$p^* = \frac{e^{(0.08-0.02)\cdot 0.25} - 0.9}{1.12 - 0.9} = 0.52324$$

Now we will construct the diagrams.



The payoffs are computed by computing $\max\{S(0.5) - 50, 0\}$ at the three end-points of the diagram.



Now that we have the payoffs, we can compute the price under the model!

$$\begin{aligned}\text{Call Price} &= \mathbb{E}^*[\text{Payoff}]e^{-rT}\\ &= [12.72(p^*)^2 + 0.4p^*q^* + 0.4q^*p^* + 0(q^*)^2]e^{-0.08\cdot 0.5}\\ &= [3.48 + 0.10 + 0.10 + 0]e^{-0.04}\\ &= [3.68]e^{-0.04}\\ &= 3.54\end{aligned}$$

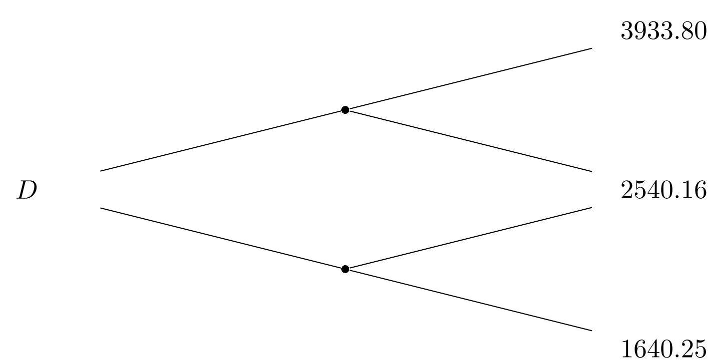
We conclude that the modeled price is $c = 3.54$.

It is really important to remember that p^* is computed using the step length, and the discounting is computed using the entire term of the contract in question. Also, we could use this model to determine many other things! We could model the price of a variety of calls and puts. If the put had strike 50, we could also use put-call parity. The result would be the same. We could get creative and compute the price of a derivative that pays the square of the stock price at expiration, for example!

Problem 16 *Determine the price of the derivative that pays the square of the asset's price in six months under the previous model.*

$$\textit{Derivative Price} = \boxed{2610.62}$$

Solution: The model is constructed in a similar way to the call diagram we constructed earlier. The only difference is the payoff!



Now we compute the price.

$$\begin{aligned}
\text{Price} &= [3933.80(p^*)^2 + 2540.16p^*q^* + 2540.16q^*p^* + 1640.25(q^*)^2]e^{-0.08\cdot 0.5} \\
&= [1077.00 + 633.67 + 633.67 + 372.83]e^{-0.04} \\
&= 2610.62
\end{aligned}$$

There was a little rounding that I did in each calculation, but since it was to the nearest penny I can rest easy knowing that the answer will only be off by at most one penny. If I were multiplying the result by 1,000,000, I would have been a bit more cautious. Also, you might have noticed that we don't need to add the middle terms. We could simply double them since they always appear twice. Our European price formula can now be rewritten as you can see below.

$$\begin{aligned}
\text{European Derivative Price} &= [D_{uu}(p^*)^2 + 2D_{ud}p^*q^* + D_{dd}]e^{-rT} \\
&= \left[\sum_{j=0}^{2} \binom{2}{j} Du^j d^{2-j} (p^*)^j (q^*)^{2-j}\right] e^{-rT}
\end{aligned}$$

The second formula gets a little abusive with notation. It uses u and d as though they are factors; however, they are only there to keep track of position. Also, term before the derivative payoff is called a binomial coefficient. It is a way to count paths in our binomial tree. The value of such a term is given in the equation below.

$$\binom{n}{m} = \frac{n!}{m!(n-m)!}$$

In the sum we wrote earlier, we had $\binom{2}{0}$, $\binom{2}{1}$, and $\binom{2}{2}$. The values are 1, 2, and 1, respectively.Now that we have our binomial coefficients, we can write a compact formula for larger trees. Let's say that we are dealing with a tree that has n steps. Then the following formula will hold.

$$\text{European Derivative Price} = \left[\sum_{j=0}^{n} \binom{n}{j} Du^j d^{n-j} (p^*)^j (q^*)^{n-j}\right] e^{-rT}$$

It is likely that you will never need to use this formula beyond three or four steps, so you will become familiar with those particular binomial coefficients. You can also get these coefficients by drawing Pascal's triangle.

We will have plenty of time to practice this formula in later sections, so don't fret! For now, we need to focus our attention on particular binomial models. You may have noticed in our example that the choice of u and d seemed arbitrary. It was! I just made them up as I was writing. This is probably not a good way to construct models. Our next section will focus on three particular model choices.

1.18 Movement Models

In our previous work the values u and d were arbitrary. Here, we give three named models and apply them to several examples.

As stated in the last section, the choice of u and d was arbitrary. Making arbitrary choices is usually not advisable in any modeling situation. In this section, we will cover three different models for the factors u and d. They are very similar, so they won't require much in terms of memory. In all the models, we have a term σ, called the volatility. This quantity will always be non-negative, and we will discuss its formulation in a later section.

Cox-Ross-Rubenstein

$$u = e^{\sigma\sqrt{h}} \qquad d = e^{-\sigma\sqrt{h}}$$

Forward

$$u = e^{(r-\delta)h+\sigma\sqrt{h}} \qquad d = e^{(r-\delta)h-\sigma\sqrt{h}}$$

Jarrow-Rudd

$$u = e^{(r-\delta-\sigma^2/2)h+\sigma\sqrt{h}} \qquad d = e^{(r-\delta-\sigma^2/2)h-\sigma\sqrt{h}}$$

Before we use the models, let's make some observations regarding some of the values. First of all, if we are modeling futures prices then there are only two models. That is because we assume that $r = \delta$ when modeling futures, so the Cox-Ross-Rubenstein and Forward models collapse into one. Let's compute p^* for the forward model.

$$\begin{aligned}
p^* &= \frac{e^{(r-\delta)h} - d}{u - d} \\
&= \frac{e^{(r-\delta)h} - e^{(r-\delta)h-\sigma\sqrt{h}}}{e^{(r-\delta)h+\sigma\sqrt{h}} - e^{(r-\delta)h-\sigma\sqrt{h}}} \\
&= \frac{1 - e^{-\sigma\sqrt{h}}}{e^{\sigma\sqrt{h}} - e^{-\sigma\sqrt{h}}} \\
&= \frac{1 - e^{-\sigma\sqrt{h}}}{e^{\sigma\sqrt{h}}(1 - e^{-2\sigma\sqrt{h}})} \\
&= \frac{1 - e^{-\sigma\sqrt{h}}}{e^{\sigma\sqrt{h}}(1 - e^{-\sigma\sqrt{h}})(1 + e^{-\sigma\sqrt{h}})} \\
&= \frac{1}{e^{\sigma\sqrt{h}} + 1} \\
&\leq \frac{1}{2}
\end{aligned}$$

This may seem like some useless pushing around of letters, but this gives some useful intuition. When calculating the risk-free probability of an up move for a forward tree, you must have that the value is less than one half! This is from the computation above and the fact that you will never run into a problem that gives you $\sigma = 0$. We can try something similar for the other models, but it won't be as fruitful. Let's work through a problem!

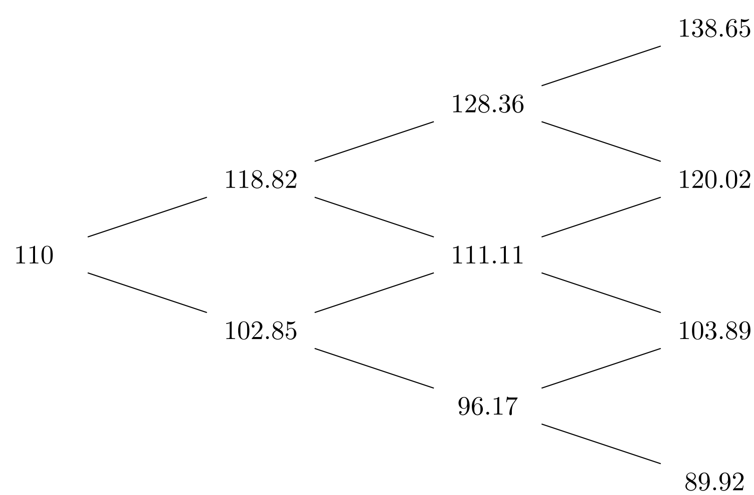
Example 32. *Use a three period forward tree to model the price of a strike* 112*, three month European put option on an asset S under the following conditions.*

- $r = 0.07$
- $\delta = 0.01$
- $\sigma = 0.25$
- $S(0) = 110$

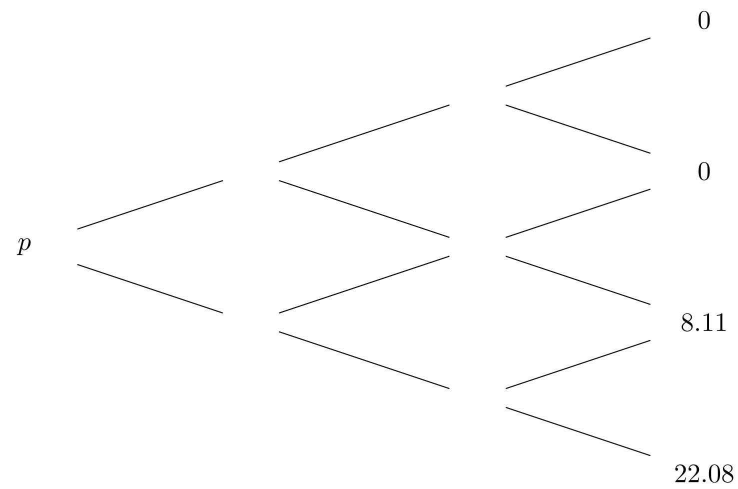
Solution: We start by determining the values u, d, and p^*.

$$
\begin{aligned}
u &= e^{(r-\delta)h+\sigma\sqrt{h}} \\
&= e^{(0.07-0.01)\cdot 1/12+0.25\sqrt{1/12}} \\
&= 1.08022 \\
d &= e^{(r-\delta)h-\sigma\sqrt{h}} \\
&= e^{(0.07-0.01)\cdot 1/12-0.25\sqrt{1/12}} \\
&= 0.93504 \\
p^* &= \frac{e^{(r-\delta)h} - d}{u-d} \\
&= \frac{e^{(0.07-0.01)\cdot 1/12} - 0.93504}{1.08022 - 0.93504} \\
&= 0.48197
\end{aligned}
$$

We did p^* a little early, but it was an opportune time since we needed u and d for that value. Now, we can construct the asset model.



The hard work has been done. Now we can build the model for the put's payoff. All of the payoffs are computed at the endpoint of each path using $\max\{112 - S(0.25), 0\}$.



Now we can execute our pricing formula.

$$
\begin{aligned}
p &= [0(p^*)^3 + 3 \cdot 0(p^*)^2 q^* + 3 \cdot 8.11 p^*(q^*)^2 + 22.08(q^*)^3]e^{-0.07 \cdot 0.25} \\
&= [0 + 0 + 3.15 + 3.07]e^{-0.0175} \\
&= 6.11
\end{aligned}
$$

Now you should test your knowledge! Try the following problem.

Question 17 *Determine the price of a six month, strike 41 European asset-or-nothing call option on some futures contract F. Use a three period, Jarrow-Rudd model with the following assumptions:*

- $F(0) = 40$,
- $r = 0.06$, *and*
- $\sigma = 0.35$.

Multiple Choice:

(a) 22.72

(b) 23.12

(c) 23.52 ✓

(d) 23.92

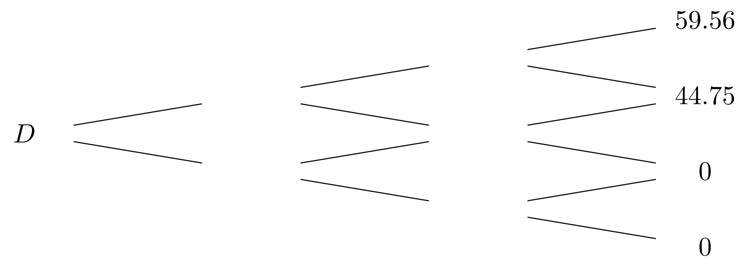
Solution: We must always construct the model, so first we determine u, d, and p^*. It may seem like we don't have enough information, but we must remember that for futures $r = \delta$.

$$
\begin{aligned}
u &= e^{(r-\delta-\sigma^2/2)h+\sigma\sqrt{h}} \\
&= e^{(-0.35^2/2)\cdot 1/6+0.35\sqrt{1/6}} \\
&= 1.14188 \\
d &= e^{(r-\delta-\sigma^2/2)h-\sigma\sqrt{h}} \\
&= e^{(-0.35^2/2)\cdot 1/6-0.35\sqrt{1/6}} \\
&= 0.85805 \\
p^* &= \frac{e^{(r-\delta)h}-d}{u-d} \\
&= \frac{1-0.85805}{1.14188-0.85805} \\
&= 0.50012
\end{aligned}
$$

Now we can construct our Jarrow-Rudd binomial tree using u and d.

$$
\begin{tikzcd}
 & & & 59.56 \\
 & & 52.16 \arrow[ur, dash] \arrow[dr, dash] & \\
 & 45.68 \arrow[ur, dash] \arrow[dr, dash] & & 44.75 \\
40 \arrow[ur, dash] \arrow[dr, dash] & & 39.19 \arrow[ur, dash] \arrow[dr, dash] & \\
 & 34.32 \arrow[ur, dash] \arrow[dr, dash] & & 33.63 \\
 & & 29.45 \arrow[ur, dash] \arrow[dr, dash] & \\
 & & & 25.27
\end{tikzcd}
$$

Since our derivative is a European, asset-or-nothing call with strike 41, we can simply replace all values less than 41 with a 0. We also don't need to list any of the values before expiration. This is because they don't contribute to the payoff of the derivative. When we begin dealing with American options, these intermediary values will become important.



Now we apply our European derivative formula.

$$\begin{aligned}
D &= [59.56(p^*)^3 + 3 \cdot 44.75(p^*)^2 q^* + 3 \cdot 0 p^*(q^*)^2 + 0(q^*)^3]e^{-0.5 \cdot 0.06} \\
&= [7.45 + 16.79]e^{-0.03} \\
&= 23.52
\end{aligned}$$

This concludes our discussion of our movement models. They will be used for our binomial models going forward. In our next section, we will apply them to American and Asian options.

1.19 Different Styles

Most of what we have done has dealt with determining the price of a European derivative. Here we explore some other possibilities.

It should be evident that binomial models easily apply to European derivatives. In fact, it is not too difficult to program any binomial model into Excel. The real benefit to binomial modela is that they are easy to apply to other styles of derivatives as well! This will not be the case for our more sophisticated Black-Scholes model that we will see in the next chapter. Let's see how our binomial model applies to American and Asian options.

Before our American option example, it is important to think about how an American option works. Remember, an American option is like a European option. The only difference is that the option can be exercised at any time before expiration. The option can only be exercised once. After exercise, the contract is gone. That is of huge importance when determining the price of the option. At every position of a payoff diagram, you must ask the question, "is it advantageous to exercise here?" The following example will demonstrate how we answer this question.

Example 33. *Use a three period Jarrow-Rudd model to construct some stock's price over a nine month period under the following assumptions:*

- $S(0) = 60$
- $\delta = 0.02$
- $\sigma = 0.3$
- $r = 0.08$

Use the model to determine the price of a strike 72 American put option that expires in nine months.

Solution: We must begin with our model. You can verify the following values.

$$\begin{aligned} u &= 1.16620 \\ d &= 0.86394 \\ p^* &= 0.50014 \end{aligned}$$

This gives our asset diagram.

$$\begin{tikzcd}[column sep=small, row sep=tiny]
 & & & 95.16 \\
 & & 81.60 \arrow[ur, dash] \arrow[dr, dash] & \\
 & 69.97 \arrow[ur, dash] \arrow[dr, dash] & & 70.50 \\
60 \arrow[ur, dash] \arrow[dr, dash] & & 60.45 \arrow[ur, dash] \arrow[dr, dash] & \\
 & 51.84 \arrow[ur, dash] \arrow[dr, dash] & & 52.23 \\
 & & 44.78 \arrow[ur, dash] \arrow[dr, dash] & \\
 & & & 38.69
\end{tikzcd}$$

Now, we can construct the payoff diagram of the option. The diagram will have payoffs at each point in time. This is only part of the work, as we will need to make comparisons to establish where it is advantageous to exercise.

$$\begin{tikzcd}[column sep=small, row sep=tiny]
 & & & 0.00 \\
 & & 0.00 \arrow[ur, dash] \arrow[dr, dash] & \\
 & 2.03 \arrow[ur, dash] \arrow[dr, dash] & & 1.50 \\
p \arrow[ur, dash] \arrow[dr, dash] & & 11.55 \arrow[ur, dash] \arrow[dr, dash] & \\
 & 20.16 \arrow[ur, dash] \arrow[dr, dash] & & 19.77 \\
 & & 27.22 \arrow[ur, dash] \arrow[dr, dash] & \\
 & & & 33.31
\end{tikzcd}$$

Now we can begin our comparisons. We must work from right to left. We begin from the up-up node to the up-up-up and up-up-down nodes.

$$\begin{tikzcd}[column sep=small, row sep=tiny]
 & 0.00 \\
0.00 \arrow[ur, dash] \arrow[dr, dash] & \\
 & 1.50
\end{tikzcd}$$

It should be obvious that waiting is advantageous, but let's make the necessary computation anyway. Since the values on the right are further into the future, they must be brought back three months to account for the time value of money.

$$0 < [0p^* + 1.50q^*]e^{-0.08\cdot 0.25} = 0.73$$

Now we can go to the next position, up-down (or down-up, if you prefer).

$$\begin{tikzcd}[column sep=small, row sep=tiny]
 & 1.50 \\
11.55 \arrow[ur, dash] \arrow[dr, dash] & \\
 & 19.77
\end{tikzcd}$$

Nothing is obvious here. Let's make the necessary computation.

$$11.55 > [1.50p^* + 19.77q^*]e^{-0.08 \cdot 0.25} = 10.42$$

It is advantageous to exercise early! Make note of this. Now let's work with the down-down node.

$$\begin{tikzcd}
 & 19.77 \\
27.22 \arrow[ur, dash] \arrow[dr, dash] & \\
 & 33.31
\end{tikzcd}$$

The comparison is

$$27.22 > [19.77p^* + 33.31q^*]e^{-0.08 \cdot 0.25} = 26.01.$$

Once again, it is advantageous to exercise early. Now, it will be useful for us to see what we have done in a picture.

$$\begin{tikzcd}
 & & 0.73 \\
 & 2.03 \arrow[ur, dash] \arrow[dr, dash] & \\
p \arrow[ur, dash] \arrow[dr, dash] & & 11.55 \\
 & 20.16 \arrow[ur, dash] \arrow[dr, dash] & \\
 & & 27.22
\end{tikzcd}$$

We repeat the process at the up and down nodes.

$$\begin{tikzcd}
 & 0.73 & & & 11.55 \\
2.03 \arrow[ur, dash] \arrow[dr, dash] & & & 20.16 \arrow[ur, dash] \arrow[dr, dash] & \\
 & 11.55 & & & 27.22
\end{tikzcd}$$

We make the necessary computations.

$$2.03 < [0.73p^* + 11.55q^*]e^{-0.08 \cdot 0.25} = 6.02$$
$$20.16 > [11.55p^* + 27.22q^*]e^{-0.08 \cdot 0.25} = 19.00$$

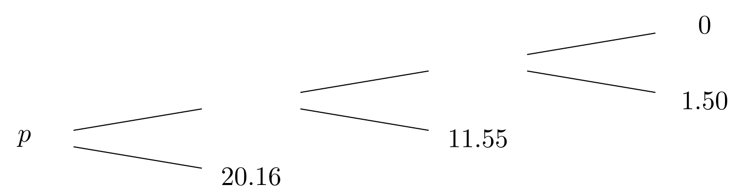
It is advantageous to wait in the up position and to exercise early in the down position. This gives us our last diagram to analyze.

$$\begin{tikzcd}
 & 6.02 \\
p \arrow[ur, dash] \arrow[dr, dash] & \\
 & 20.16
\end{tikzcd}$$

The price of the American put option is

$$p = [6.02p^* + 20.16q^*]e^{-0.08 \cdot 0.25} = 12.83$$

We could go further and draw the diagram showing only the nodes where exercise is advantageous. That would look like this:



Using this diagram, we could also determine the price of the American put. The difficulty here is that you must count all paths to each particular payoff. Fortunately, that is not so difficult with this problem!

$$p = 20.16q^*e^{-0.08 \cdot 0.25} + 11.55p^*q^*e^{-0.08 \cdot 0.5} + 1.50(p^*)^2q^*e^{-0.08 \cdot 0.75} = 12.83$$

Notice that there were actually two ways of computing the price of the option. The first was very algorithmic. You could program it into a computer with ease, provided you have maximum functions. Unfortunately, you could lose site of where the positions of early exercise are. The second shows you exactly where early exercise takes place; however, it makes a new problem of counting paths.

Since you haven't done too much with American options, you likely lack intuition regarding their prices. It is worthwhile to compute the price of the European option with similar terms. In this case, you should find that the price is 11.43. This is less than the value we computed in the example. That's always a good check for an American option.

The ability to price American options without sophisticated mathematical techniques is one of the benefits of having binomial models. The approach to Asian options is fairly direct: compute an average for each path based on the option you are dealing with. The average computation will **not** include the initial asset value.

Fortunately, the computations necessary for the Asian options are easy. They are just averages. Unfortunately, there is a unique computation for each path. In a three period binomial tree, there are 8 unique paths. That means we must make 8 average computations. In contrast, the American option only required 6 computations.

The disparity becomes even more stark when going to larger trees. A tree with $n = T/h$ periods will require $n(n+1)/2+1$ computaions for an American option while a similar length Asian option will require $2^n + 1$ computations! The $+1$ in each comes from the price of the derivative computation that comes after computing all of the payoffs.

Example 34. *Construct a three period Cox-Ross-Rubenstein model for some asset S with each step equal to two months using the following assumptions:*

- $S(0) = 90$
- $\sigma = 0.22$
- $\delta = 0.03$
- $r = 0.09$

Use your model to estimate the price of a six month geometric average strike call option.

Solution: Once again, we compute u, d, and p^*. The results are below.

$$
\begin{aligned}
u &= e^{0.22\sqrt{1/6}} = 1.09397 \\
d &= e^{-0.22\sqrt{1/6}} = 0.91410 \\
p^* &= \frac{e^{(0.09-0.03)/6} - d}{u - d} = 0.53344
\end{aligned}
$$

Now that we have u and d, we can construct our asset model.

$$
\begin{tikzcd}
 & & & 117.83 \\
 & & 107.71 \arrow[ur, dash] \arrow[dr, dash] & \\
 & 98.46 \arrow[ur, dash] \arrow[dr, dash] & & 98.46 \\
90.00 \arrow[ur, dash] \arrow[dr, dash] & & 90.00 \arrow[ur, dash] \arrow[dr, dash] & \\
 & 82.27 \arrow[ur, dash] \arrow[dr, dash] & & 82.27 \\
 & & 75.20 \arrow[ur, dash] \arrow[dr, dash] & \\
 & & & 68.74
\end{tikzcd}
$$

The payoff computations may now begin! We must do eight of them since this is a three period binomial tree.

- Position up-up-up (98.46-107.71-117.83): GA $= \sqrt[3]{98.46 \cdot 107.71 \cdot 117.83} = 107.71$.

- Position up-up-down (98.46-107.71-98.46): GA $= \sqrt[3]{98.46 \cdot 107.71 \cdot 98.46} =$ 101.45.
- Position up-down-up (98.46-90.00-98.46): GA $= \sqrt[3]{98.46 \cdot 90 \cdot 98.46} =$ 95.56.
- Position up-down-down (98.46-90.00-82.27): GA $= \sqrt[3]{98.46 \cdot 90 \cdot 82.27} =$ 90.00.
- Position down-up-up (82.27-90.00-98.46): GA $= \sqrt[3]{82.27 \cdot 90 \cdot 98.46} =$ 90.00.
- Position down-up-down (82.27-90.00-82.27): GA $= \sqrt[3]{82.27 \cdot 90 \cdot 82.27} =$ 84.77.
- Position down-down-up (82.27-75.20-82.27): GA $= \sqrt[3]{82.27 \cdot 75.20 \cdot 82.27} =$ 79.84.
- Position down-down-down (82.27-75.20-68.74): GA $= \sqrt[3]{82.27 \cdot 75.20 \cdot 68.74} =$ 75.20.

The eight payoff computations will come from replacing K in the payoff of a European call option payoff with each of the eight averages given above. They are done in the same order.

$$
\begin{aligned}
\text{payoff}_1 &= \max\{117.83 - 107.71, 0\} = 10.12 \\
\text{payoff}_2 &= \max\{98.46 - 101.45, 0\} = 0 \\
\text{payoff}_3 &= \max\{98.46 - 95.56, 0\} = 2.90 \\
\text{payoff}_4 &= \max\{82.27 - 90.00, 0\} = 0 \\
\text{payoff}_5 &= \max\{98.46 - 90.00, 0\} = 8.46 \\
\text{payoff}_6 &= \max\{82.27 - 84.77, 0\} = 0 \\
\text{payoff}_7 &= \max\{82.27 - 79.84, 0\} = 2.43 \\
\text{payoff}_8 &= \max\{68.74 - 75.20, 0\} = 0
\end{aligned}
$$

The modeled price of the option is

$$
\begin{aligned}
c &= [10.12(p^*)^3 + 2.90p^*q^*p^* + 8.46q^*(p^*)^2 + 2.43(q^*)^2p^*]e^{-0.09 \cdot 0.5} \\
&= 3.18
\end{aligned}
$$

In this last computation, I tried to illustrate the path in the ordering of the terms p^* and q^*. For example, $p^*q^*p^*$ represents the position up-down-up. Before this example, I said that there were $2^n + 1$ computations necessary for this computation. That would give 9 computations for this problem. If you go through and count them, you will see that I actually did $8 + 8 + 1 = 17$ (not including the binomial model). That's definitely larger than $2^3 + 1 = 9$.

Once you get good at this, you can go immediately from the average computation to the payoff computation. Combining these would cut out 8 of the lines, bringing the total down to $2^3 + 1$.

Now that we have covered different styles of options, we are really finished with our application of our binomial models. The last bit of information we need to shore up is where our volatility comes from. Our next section will cover that topic.

1.20 Volatility

We have used volatility throughout this chapter without really knowing what it is. Here is where we (hopefully) answer any of those lingering volatility questions.

We have seen our binomial models in many different applications now. That whole time, we took volatility as a given. That will usually be the case in an academic setting; however, in the real world you can only assume a particular model and estimate the value of the volatility. The risk-free rate would likely be based on some government instrument while the dividend rate could be based off of a company's dividend payout history.

The key to getting our hands on a notion of volatility is to try to extract factors over envenly spaced time intervals. In this discussion, we will assume that the stock follows a Jarrow-Rudd model for movement. To get our hands on the u and d values, we will need to take quotients of successive observations. To determine the values in the exponents of u and d, we will need logarithms.

Suppose that we make $n+1$ evenly spaced observations of some stock's price over a T year period. We will let h denote the value T/n. For our purposes, we can call the values $S(0)$, $S(h)$, $S(2h)$, ... $S((n-1)h)$, and $S(nh) = S(T)$. We take quotients of successive terms.

$$\frac{S(h)}{S(0)} \qquad \frac{S(2h)}{S(h)} \qquad \frac{S(3h)}{S(2h)} \qquad \cdots \qquad \frac{S(nh)}{S((n-1)h)}$$

There are n such quotients. Each one should resemble one of our factors u or d. Let me stress the word resemble. They will not give only two values! We knew the binomial model had limitations, and one of them rears its head here. Our final step is to apply logarithms to each quotient.

$$\ln\left[\frac{S(h)}{S(0)}\right] \qquad \ln\left[\frac{S(2h)}{S(h)}\right] \qquad \ln\left[\frac{S(3h)}{S(2h)}\right] \qquad \cdots \qquad \ln\left[\frac{S(nh)}{S((n-1)h)}\right]$$

Each of these terms should resemble the logarithm of u or d. Specifically, they should be either

$$\ln u = (\alpha - \delta - \sigma^2/2)h + \sigma\sqrt{h} \qquad \text{or} \qquad \ln d = (\alpha - \delta - \sigma^2/2)h - \sigma\sqrt{h}.$$

The term $(\alpha - \delta - \sigma^2/2)h$ is called the deterministic part. It is always dragging the asset's price in one direction. It is not random. I am using α in place of r since we can actually observe an asset's rate of return when we look into the past. The other part, $\pm\sigma\sqrt{h}$ is called the random part. It has variance $\sigma^2 h$. The estimation of the deterministic and random parts will be done using a sample average and a sample variance, respectively. Remember that a sample variance is computed as follows:

$$s_x^2 = \frac{1}{n-1}\sum_{j=1}^{n}(x_j - \bar{x})^2$$

Example 35. *You observe a stock's price over a 4 month period, making observations twice every month at even intervals. You record the following:*

Time (in half months)	*0*	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*
$S(h)$	90	91	94	92	89	85	92	90	95

Estimate the volatility using the method described above.

Solution: We must compute 8 terms. From there, we can compute our sample variance and derive the volatility.

$$\begin{aligned}
x_1 &= \ln\frac{91}{90} = 0.01105 \\
x_2 &= \ln\frac{94}{91} = 0.03244 \\
x_3 &= \ln\frac{92}{94} = -0.02151 \\
x_4 &= \ln\frac{89}{92} = -0.03315 \\
x_5 &= \ln\frac{85}{89} = -0.04599 \\
x_6 &= \ln\frac{92}{85} = 0.07914 \\
x_7 &= \ln\frac{90}{92} = -0.02198 \\
x_8 &= \ln\frac{95}{90} = 0.05407 \\
\bar{x} &= 0.00676 \\
s_x^2 &= 0.00202 \\
\hat{\sigma}^2 h &= 0.00202 \\
\hat{\sigma} &= 0.22
\end{aligned}$$

I rounded all values to five decimal places except the result. You will usually be given volatility to two places, so it makes sense to give the answer to two places as well. The value of h used in the computation was $h = 1/24$. That is because there are 24 half months in every year.

Now you should try part of the problem!

Question 18 *In the previous problem, assume that the stock in question pays no dividends. Use the above values to estimate the asset's rate of return over the same four month period. Select the answer that is closest to the answer.*

Multiple Choice:

(a) 15%

(b) 17%

(c) 19% ✓

(d) 21%

Solution: Much of the work has been done in the example. We just need to extract the appropriate values.

$$\begin{aligned}(\hat{\alpha} - \hat{\sigma}^2/2)/24 &= 0.00676 \\ (\hat{\alpha} - 0.22^2/2) &= 0.16224 \\ \hat{\alpha} &= 0.18644\end{aligned}$$

The answer rounds to 0.19 or 19%.

This process illustrates how we can come up with the value σ. Volatility will still be important in our future models, but we seldom will estimate it.

1.21 Lognormality

Our binomial models were effective at handling almost all derivatives; however, their discrete nature was unrealistic. Here, we will learn about a continuous model that is related to those binomial models we discussed.

In our last chapter, we saw the binomial model in many different situations. The model was very versatile; however, the computations could grow cumbersome when there are many periods. This difficulty is diminished with the growing power of computers. If we flex some of our probability muscles, we can show that our binomial models converge to a lognormal distribution. This term may be unfamiliar, but it isn't soo hard to analyze it when we write it down:

$$S(T) = S(0)e^{(r-\delta-\sigma^2/2)T+\sigma\sqrt{T}\mathcal{N}(0,1)}$$

That is to say, the time T price of an asset given the price today is given by a random variable with a standard normal distribution in the exponent. There is nothing obvious about this Prakash Balachandran gives some of the background here. There is a slight typo at the end, but that has been corrected in the formula given above. Fortunately (or not), the proof of the convergence is not within the scope of this course.

The formula given above has been simplified a bit. A more general form is given below. It assumes you know the value of the asset at time t.

$$S(T) = S(t)e^{(r-\delta-\sigma^2/2)(T-t)+\sigma\sqrt{T-t}\mathcal{N}(0,1)}$$

This will be our **Black-Scholes model** for stock prices. Futures prices are defined similarly with the condition $r = \delta$ used as usual. Now that we have a model, it is useful to answer questions regarding that model. This section will cover how to compute probabilities and percentiles.

Example 36. *Determine the risk-free probability that an asset will be greater than 75 in three months under the following assumptions:*

- $S(0) = 70$
- $\delta = 0.01$
- $\sigma = 0.41$
- $r = 0.11$

Solution: The probability is given by the computations below.

$$\begin{aligned}\mathbb{P}(S(1/4) > 75) &= \mathbb{P}(70e^{(0.11-0.01-0.41^2/2)/4+0.41\sqrt{1/4}\mathcal{N}(0,1)} > 75)\\ &= \mathbb{P}(e^{(0.01595)/4+0.205\mathcal{N}(0,1)} > 75/70)\\ &= \mathbb{P}(0.004 + 0.205\mathcal{N}(0,1) > \ln 75/70)\\ &= \mathbb{P}(0.205\mathcal{N}(0,1) > 0.065)\\ &= \mathbb{P}(\mathcal{N}(0,1) > 0.317)\\ &= \mathbb{P}(\mathcal{N}(0,1) < -0.317)\\ &= 0.37558\end{aligned}$$

This solution really hinged on knowledge of standard normal distributions. In the second to last line I used the symmetry of the standard normal distribution, and in the final line I used an online calculator. If you are using a table, you may arrive at a slightly different answer. That is fine. I will provide you with tables when you need one!

It is natural to ask, "Why would I ever care about this probability?" One possible answer would be that it answers the investor's question, "What is the probability that a three month, strike 75 European call is exercised?" It also answers the question regarding the similar put (just apply the complement rule from probability).

Another thing to note is that this is a risk-free probability. When there is more than one possible probability measure in question (i.e. a rate of return is given), the risk-free measure will always have an asterisk (like this: $\mathbb{P}^*$).

Every probability question will be similar, in principle, so let's turn our attention to percentiles.

Definition 18. *The* $100p$ **percentile** *of a distribution* X *is the value* $\pi_p(X)$ *such that*

$$\mathbb{P}(X < \pi_p(X)) = p.$$

The **median** *is the* 50^{th} *percentile.*

Oftentimes, it is necessary to give a more involved definition of a percentile. We will only care about percentiles for continuous distributions, so this is unnecessary. Our percentiles are always defined and unique when we use them.

Example 37. *Compute the median for the* $S(1/4)$ *in the previous example.*

Solution: This sounds more involved than it really is. A wonderful simplification occurs when you observe that the median will occur for $S(1/4)$ exactly when the median for the standard normal random variable happens. That is when $\mathcal{N}(0,1) = 0$.

$$\begin{aligned}\pi_{0.5}(S(1/4)) &= 70e^{(0.11-0.01-0.41^2/2)/4+0.41\cdot 1/2\cdot 0}\\ &= 70e^{0.004}\\ &= 70.28\end{aligned}$$

Now you can practice!

Question 19 *What is the probability that the same stock will exceed the three month median value in six months? (Intuitively, the result should be larger than 1/2 since $(r-\delta-\sigma^2/2)>0$.)*

$$\mathbb{P}(S(1/2) > \pi_{0.5}(S(1/4)) = \boxed{0.51}$$

Solution: There would be computations if the second example of this section didn't have the first piece.

$$\begin{aligned}\mathbb{P}(S(1/3) > 70.28) &= \mathbb{P}(70e^{(0.11-0.01-0.41^2/2)/2+0.41\sqrt{1/2}\mathcal{N}(0,1)} > 70.28)\\ &= \mathbb{P}(0.41\sqrt{1/2}\mathcal{N}(0,1) > -0.004)\\ &= \mathbb{P}(\mathcal{N}(0,1) > -0.0137)\\ &= \mathbb{P}(\mathcal{N}(01) < 0.0137)\\ &= 0.50547\end{aligned}$$

This rounds to 0.51, as the answer stated.

Notice that our more general model did not apply since we weren't given any information about the asset at time 1/4. We were only comparing two numbers that could be computed using the time zero value.

That concludes our work for this section. In the next section, we will expand our analysis of our model to expectations!

1.22 Favorite Expectation

Since our European derivatives formula relies on an expectation, we will establish that expectation for a lognormal random variable. This will pay dividends in the future.

The Black-Scholes model easily prices European derivatives using the same principle that we used with the binomial models. The price of a derivative should be

$$\text{Derivative Price} = \mathbb{E}^*[\text{payoff}]e^{-rT}.$$

Since our asset follows a lognormal model, we will need to determine how to compute this expectation for calls and puts. We must start with our lognormal (Black-Scholes) model for our asset.

$$\mathbb{E}^*[S(T)] = \mathbb{E}^*[S(0)e^{(r-\delta-\sigma^2/2)T+\sigma\sqrt{T}\mathcal{N}(0,1)}]$$

In our discussion here, we will deal with something a little more general. Let $Y = Me^{a+b\mathcal{N}(0,1)}$. We will compute the expectation of Y. To compute this expectation, we need the density function for the standard normal random variable.

$$f(z) = \frac{1}{\sqrt{2\pi}}e^{-(z^2)/2}$$

The expectation of Y is computed by integrating Y against this density function.

$$
\begin{aligned}
\mathbb{E}[Y] &= \int_{-\infty}^{\infty} Me^{a+bz} f(z)\mathrm{d}z \\
&= \int_{-\infty}^{\infty} Me^{a+bz} \frac{1}{\sqrt{2\pi}} e^{-(z^2)/2}\mathrm{d}z \\
&= \frac{M}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-z^2/2+bz+a}\mathrm{d}z \\
&= \frac{M}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(z^2-2bz-2a)/2}\mathrm{d}z \\
&= \frac{M}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-[(z-b)^2-b^2-2a]/2}\mathrm{d}z \\
&= \frac{M}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(z-b)^2/2} e^{a+b^2/2}\mathrm{d}z \\
&= Me^{a+b^2/2} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-(z-b)^2/2}\mathrm{d}z \\
\mathbb{E}[Y] &= Me^{a+\frac{b^2}{2}} \\
&= M\exp\left(a+\frac{b^2}{2}\right)
\end{aligned}
$$

You may be wondering what happened to the integral. The reason the integral disappeared is that the argument represents a normal density function. It is shifted right by b. Also, I wrote the favorite expectation in two ways. The first is the way that I would usually write it. The second is to really emphasise that the argument $a + b^2/2$ lives in the exponent of the exponential.

How does this help with our asset question? Well, under the Black-Scholes model we have that $a = (r - \delta - \sigma^2/2)T$ and $b = \sigma\sqrt{T}$. Let's see what this gives us.

$$
\begin{aligned}
\mathbb{E}^*[S(T)] &= S(0)e^{a+b^2/2} \\
&= S(0)e^{(r-\delta-\sigma^2/2)T+\sigma^2T/2} \\
&= S(0)e^{(r-\delta)T}
\end{aligned}
$$

This is wonderful! Under the Black-Scholes model, we have that the asset's expectation at time T is the forward price of the asset. This is all worthy of a theorem.

Theorem 5 (Favorite Expectation)**.** *Let $Y = Me^{a+b\mathcal{N}(0,1)}$. Then we have the expectation below.*

$$\mathbb{E}[Y] = Me^{a+b^2/2}$$

Furthermore, when we have S under the Black-Scholes model, then the expectation is the forward price.

$$\mathbb{E}^*[S(T)] = S(0)e^{(r-\delta)T}$$

This theorem seems like it isn't saying much, but it is quite powerful. Much of what we do for the remainder of this book will rely on this fact. Let's see how we can apply this theorem to some derivatives pricing.

Example 38. *Suppose that we have an asset with the following parameters:*

- $S(0) = 5$
- $\delta = 0.02$
- $\sigma = 0.28$

In addition, the risk-free rate is $r = 0.08$. *Compute the price of a derivative that pays* $S(3/4)^2$ *in nine months.*

Solution: The price is given by the formula for European derivatives. We must compute the risk-free expectation of the payoff and discount at the risk-free rate.

$$\begin{aligned}
S(3/4)^2 &= [5e^{(0.08-0.02-0.28^2/2)\cdot 3/4+0.28\sqrt{3/4}\mathcal{N}(0,1)}]^2 \\
&= [5e^{0.0156+0.28\sqrt{3/4}\mathcal{N}(0,1)}]^2 \\
&= 25e^{0.0312+0.56\sqrt{3/4}\mathcal{N}(0,1)} \\
\mathbb{E}^*[S(3/4)^2] &= \mathbb{E}^*[25e^{0.0312+0.56\sqrt{3/4}\mathcal{N}(0,1)}] \\
&= 25e^{0.0312+(0.56\sqrt{3/4})^2/2} \\
&= 25e^{0.1488} \\
&= 29.01
\end{aligned}$$

The price of the derivative is

$$29.01e^{-0.08\cdot 3/4} = 27.32$$

You will notice that in the solution, I squared the stock price before computing the expectation. This is because I didn't want to falsely state that $\mathbb{E}[X^2] = \mathbb{E}[X]^2$. If this were the case, all variances would be zero!

Question 20 *Compute the variance of $S(3/4)$, where S satisfies all the conditions given in the previous example.*

$$Var(S(3/4)) = \boxed{1.66}$$

Solution: The variance is given by the formula $\mathbb{E}[S(3/4)^2]-\mathbb{E}[S(3/4)]^2$. Since no rate of return is given, we must use the risk-free values. The example computed the first term, and we can use the theorem for the second term.

$$\mathbb{E}[S(3/4)] = 5e^{(0.08-0.02)\cdot 3/4} = 5.23$$

The variance is computed below.

$$\begin{aligned}
\text{Var}(S(3/4)) &= \mathbb{E}[S(3/4)^2] - \mathbb{E}[S(3/4)]^2 \\
&= 25e^{0.1488} - [5e^{0.045}]^2 \\
&= 1.66
\end{aligned}$$

There are some people that love to study and memorize formulae. I am not one of them; however, the following is to cater to those types. The variance of an asset price can be determined by the parameters of the asset and the risk-free rate.

$$\begin{aligned}
\mathbb{E}^*[S(T)^2] &= \mathbb{E}[S(0)^2 e^{(r-\delta-\sigma^2/2)2T+2\sigma\sqrt{T}\mathcal{N}(0,1)}] \\
&= S(0)^2 e^{(r-\delta-\sigma^2/2)2T+2\sigma^2 T} \\
&= S(0)^2 e^{[2(r-\delta)+\sigma^2]T} \\
\mathbb{E}^*[S(T)]^2 &= S(0)^2 e^{2(r-\delta)T} \\
\text{Var}(S(T)) &= S(0)^2 e^{2(r-\delta)T}[e^{\sigma^2 T} - 1] \\
&= [F_{0,T}]^2 [e^{\sigma^2 T} - 1]
\end{aligned}$$

We have a lot we can do now, and we are finally equipped to handle harder questions regarding derivatives. It may seem like this new model is making things more complicated. That is not the case. Imagine trying to model the price of the derivative in this section using a 10 step binomial tree. That would require an immense amount of labor. The Black-Scholes model gets around all of that!

If you really want a challenge, try to determine the price of a derivative that pays the square of the difference of the asset price and the median price in nine months. The asset is the only one we discussed in this section. For clarity, I give the payoff below.

$$[S(3/4) - \pi_{0.5}(S(3/4))]^2$$

Solution: First, compute the median.

$$\pi_{0.5}(S(3/4)) = 5e^{(0.08-0.02-0.28^2/2)\cdot 3/4} = 5.08$$

Now we can compute the derivative's value by expanding the payoff and computing expectations term-by-term.

$$[S(3/4) - \pi_{0.5}(S(3/4))]^2 = S(3/4)^2 - 10.16S(3/4) + 5.08^2$$
$$\mathbb{E}^*[(S(3/4) - \pi_{0.5}(S(3/4)))^2] = \mathbb{E}^*[S(3/4)] - 10.16\mathbb{E}^*[S(3/4)] + 5.08^2$$

The two expectations have already been computed. Substitute them in and make sure to discount your result at the risk-free rate.

That concludes our discussion of our favorite expectation. Remember, it is one of the most important formulae in this course. It will be used in the next section when we derive the famous Black-Scholes formula (sometimes called the Black-Scholes-Merton formula).

1.23 The Black-Scholes Formula

Now that we have our favorite expectation, we can derive the Black-Scholes formula. It doesn't require anything more than some elementary probability theory; however, the computations are involved.

In the last section, we computed the prices of several derivatives. We were fortunate in that the derivatives always paid a power of the underlying asset's price. Unfortunately, that is not how call and put options operate. These derivatives have payoffs that are computed using piecewise functions. Therefore, any computation of expectation will require a piecewise function. We will proceed as we did in the last section by using a generic lognormal random variable. Then we will use the special cases given under the Black-Scholes model to derive our formula.

Before we begin the mathematical argument for the Black-Scholes formula, we will make some (strange) definitions.

Definition 19. *Two important quantities in our arguments will be useful.*

$$\begin{aligned} D_1 &= \frac{\ln \frac{M}{K} + a}{b} + b \\ D_2 &= \frac{\ln \frac{M}{K} + a}{b} \end{aligned}$$

Also, recall our generic lognormal model $Y = Me^{a+b\mathcal{N}(0,1)}$ We are going to compute

$$\mathbb{E}[\max\{Y - K, 0\}].$$

This won't take too much effort beyond what we did in the last section. First, we will need to know when the lognormal variable Y is greater than K.

$$\begin{aligned} \mathbb{P}(Y > K) &= \mathbb{P}(Me^{a+b\mathcal{N}(0,1)} > K) \\ &= \mathbb{P}\left(e^{a+b\mathcal{N}(0,1)} > \frac{K}{M}\right) \\ &= \mathbb{P}\left(a + b\mathcal{N}(0,1) > \ln \frac{K}{M}\right) \\ &= \mathbb{P}\left(\mathcal{N}(0,1) > \frac{\ln \frac{K}{M} - a}{b}\right) \\ &= \mathbb{P}(\mathcal{N}(0,1) > -D_2) \end{aligned}$$

This may not seem too important, but it answers the question, "When does a call option have a non-zero payoff?" The answer is, "When the normal random variable in the exponent is larger than $-D_2$." This will give us the lower bound on our integral when we are computing the expectation that is the target of this section.

Now that we know the bounds on our integral, we may compute the desired expectation. Since some of the computations would require completing the square (as we saw in the last section), those steps are suppressed.

$$\begin{aligned}
\mathbb{E}[\max\{Y-K,0\}] &= \int_{-D_2}^{\infty} [Me^{a+bz}-K]\frac{1}{\sqrt{2\pi}}e^{z^2/2}\mathrm{d}z \\
&= \int_{-D_2}^{\infty} Me^{a+bz}\frac{1}{\sqrt{2\pi}}e^{z^2/2}\mathrm{d}z - \int_{-D_2}^{\infty} K\frac{1}{\sqrt{2\pi}}e^{z^2/2}\mathrm{d}z \\
&= \int_{-D_2}^{\infty} Me^{a+b^2/2}\frac{1}{\sqrt{2\pi}}e^{-(z-b)^2/2}\mathrm{d}z - K\mathbb{P}(\mathcal{N}(0,1) > -D_2) \\
&= Me^{a+b^2/2}\int_{-D_2-b}^{\infty} \frac{1}{\sqrt{2\pi}}e^{-u^2/2}\mathrm{d}z - K\mathbb{P}(\mathcal{N}(0,1) > -D_2) \\
&= Me^{a+b^2/2}\mathbb{P}(\mathcal{N}(0,1) > -D_1) - K\mathbb{P}(\mathcal{N}(0,1) > -D_2)
\end{aligned}$$

There is something unappealing about all these negative D_1 and D_2 terms. Observe that all of the probabilities in question are survivals. These are great; however, normal tables are in terms of cumulative distributions functions. We need to write them that way. Fortunately, the standard normal random variable is symmetric. The tails all have equal probabilities. That means that we can drop the negative signs by switching the inequalities as such:

$$\mathbb{E}[\max\{Y-K,0\}] = Me^{a+b^2/2}\mathbb{P}(\mathcal{N}(0,1) < D_1) - K\mathbb{P}(\mathcal{N}(0,1) < D_2)$$

We make one further modification. Instead of writing the probabilities as I have above, we will begin writing them as $\mathcal{N}(D_1)$ and $\mathcal{N}(D_2)$. Our expectation becomes

$$\mathbb{E}[\max\{Y-K,0\}] = Me^{a+b^2/2}\mathcal{N}(D_1) - K\mathcal{N}(D_2)$$

Our general Black-Scholes formula is derived by taking a present value of the above.

$$\text{“call” price} = \left[Me^{a+b^2/2}\mathcal{N}(D_1) - K\mathcal{N}(D_2)\right]e^{-rT}$$

The quotation marks indicate that the payoff in question simply looks like that of a call. When we use our Black-Scholes model, the marks will be dropped.

Under the model, we have that $a = (r - \delta - \sigma^2/2)T$, $b = \sigma\sqrt{T}$, and $M = S(0)$. For this model, we use lowercase letters d_1 and d_2 in place of D_1 and D_2, respectively.

$$
\begin{aligned}
d_1 &= \frac{\ln \frac{S(0)}{K} + (r - \delta - \sigma^2/2)T}{\sigma\sqrt{T}} + \sigma\sqrt{T} \\
&= \frac{\ln \frac{S(0)}{K} + (r - \delta + \sigma^2/2)T}{\sigma\sqrt{T}} \\
d_2 &= \frac{\ln \frac{S(0)}{K} + (r - \delta - \sigma^2/2)T}{\sigma\sqrt{T}} \\
&= d_1 - \sigma\sqrt{T} \\
c &= \left[S(0)e^{(r-\delta)T}\mathcal{N}(d_1) - K\mathcal{N}(d_2)\right] e^{-rT} \\
&= S(0)e^{-\delta T}\mathcal{N}(d_1) - Ke^{-rT}\mathcal{N}(d_2)
\end{aligned}
$$

This may seem like a lot, but if you just remember that d_2 comes from the probability that the call has a non-zero payoff then you can generate the values d_1 and d_2 from scratch. Additionally, the price of a call looks very similar to the right-hand side of put-call parity. In fact, if you cover up $\mathcal{N}(d_1)$ and $\mathcal{N}(d_2)$, then you have the right-hand side!

It would be wonderful if this was everything, but there is actually so much more! We could determine the price of a cash-or-nothing call option.

$$
\begin{aligned}
\mathbb{E}[1_{\{S(T)>K\}}] &= \int_{-d_2}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-z^2/2} \mathrm{d}z \\
&= \mathcal{N}(d_2)
\end{aligned}
$$

We can also compute the price of an asset-or-nothing call option. This was accomplished earlier, and the result is

$$
\text{asset-or-nothing call price} = S(0)e^{-\delta T}\mathcal{N}(d_1).
$$

If you aren't convinced, simply add K cash-or-nothing call options to our call price formula.

We can come up with all sorts of similar formula for our put options, but we want to be a little more efficient. You can use parity relationships to derive all of them. Let's do this with ordinary calls and puts.

$$\begin{aligned}
c - p &= S(0)e^{-\delta T} - Ke^{-rT} \\
S(0)e^{-\delta T}\mathcal{N}(d_1) - Ke^{-rT}\mathcal{N}(d_2) - p &= S(0)e^{-\delta T} - Ke^{-rT} \\
S(0)e^{-\delta T}[\mathcal{N}(d_1) - 1] - Ke^{-rT}[\mathcal{N}(d_2) - 1] &= p \\
Ke^{-rT}[1 - \mathcal{N}(d_2)] - S(0)e^{-\delta T}[1 - \mathcal{N}(d_1)] &= p \\
Ke^{-rT}\mathcal{N}(-d_2) - S(0)e^{-\delta T}\mathcal{N}(-d_1) &= p
\end{aligned}$$

Let's wrap everything up in two theorems: the first will be more general while the second will be more expansive.

Theorem 6 (General Black-Scholes)**.** *Let $Y = Me^{a+b\mathcal{N}(0,1)}$, D_1 and D_2 be as in the definition. The price of a derivative that pays* $\max\{Y - K, 0\}$ *at time T is*

$$\left[Me^{a+b^2/2}\mathcal{N}(D_1) - K\mathcal{N}(D_2)\right] e^{-rT}.$$

The price of a derivative that pays $\max\{K - Y, 0\}$ *at time T is*

$$\left[K\mathcal{N}(-D_2) - Me^{a+b^2/2}\mathcal{N}(-D_1)\right] e^{-rT}.$$

Theorem 7 (Black-Scholes Formulae)**.** *Under the Black-Scholes model, we have the following prices of our options:*

(a) *Our regular options have prices*

$$\begin{aligned}
c &= S(0)e^{-\delta T}\mathcal{N}(d_1) - Ke^{-rT}\mathcal{N}(d_2) \\
p &= Ke^{-rT}\mathcal{N}(-d_2) - S(0)e^{-\delta T}\mathcal{N}(-d_1)
\end{aligned}$$

(b) *Our cash-or-nothing options have prices*

$$\begin{aligned}
c_{C/N} &= e^{-rT}\mathcal{N}(d_2) \\
p_{C/N} &= e^{-rT}\mathcal{N}(-d_2)
\end{aligned}$$

(c) *Our asset-or-nothing options have prices*

$$\begin{aligned}
c_{A/N} &= S(0)e^{-\delta T}\mathcal{N}(d_1) \\
p_{A/N} &= S(0)e^{-\delta T}\mathcal{N}(-d_1)
\end{aligned}$$

In all of these formulae, d_1 and d_2 are the same.

$$\begin{aligned}
d_1 &= \frac{\ln\frac{S(0)}{K} + (r - \delta + \sigma^2/2)T}{\sigma\sqrt{T}} \\
d_2 &= \frac{\ln\frac{S(0)}{K} + (r - \delta - \sigma^2/2)T}{\sigma\sqrt{T}}
\end{aligned}$$

In the event that we are given $S(t)$, we would replace $S(0)$ with that value. In addition, all the the T's that appear in the theorems would be replaced with $T - t$. That will seldom happen, so it isn't worth writing down all six of those formulae. Just know that it's a possibility.

A lot was covered in this section. The theorems are what you will need to know. Our next section will explore several examples of these theorems in action.

1.24 Examples

Our last section derived the Black-Scholes formula. It is here that we will see some applications. They are very straight-forward.

Now that we have the Black-Scholes formula, we should see several examples. We will cover assets, currencies, and futures in this section. They are all really the same; however, it is good to see these to remind you that they exist.

Example 39. *Today's price of an asset is* 140. *The dividend rate is* $\delta = 0.02$, *and the volatility is* $\sigma = 0.43$. *Determine the price of a three month, strike* 150 *European call option under the Black-Scholes model if the risk-free rate is* 9%.

Solution: It is best to use the more expansive theorem of the previous section for this example. We begin with d_1 and d_2 then move on to the call formula.

$$\begin{aligned}
d_1 &= \frac{\ln \frac{S(0)}{K} + (r - \delta + \sigma^2/2)T}{\sigma\sqrt{T}} \\
&= \frac{\ln \frac{140}{150} + (0.09 - 0.02 + 0.43^2/2) \cdot 0.25}{0.43\sqrt{0.25}} \\
&= -0.132 \\
d_2 &= d_1 - \sigma\sqrt{T} \\
&= -0.132 - 0.43/1 \\
&= -0.347 \\
c &= S(0)e^{-\delta T}\mathcal{N}(d_1) - Ke^{-rT}\mathcal{N}(d_2) \\
&= 140e^{-0.005}\mathcal{N}(-0.132) - 150e^{-0.0225}\mathcal{N}(-0.347) \\
&= 139.30 \cdot 0.44749 - 146.66 \cdot 0.36430 \\
&= 8.91
\end{aligned}$$

This is, essentially, how every Black-Scholes formula problem will go. You just need to identify what the problem is asking for! Why don't you try the following question.

Question 21 *What is the risk-free probability that the previous call is profitable?*

The risk-free probability that the call is profitable is $\boxed{0.27}$.

Solution: This might seem like it is in the wrong section; however, we use many of the ideas that are tied in with the Black-Scholes formula. The call will be profitable if the payoff is greater than the future value of the price:

$$8.91e^{0.09\cdot 0.25} = 9.11.$$

The payoff will be greater than this amount precisely when the stock's value is that much larger than the strike at time 1/4. That is to say when

$$S(1/4) \geq 150 + 9.11 = 159.11.$$

To solve this question, we only need to determine $\mathcal{N}(d_2)$ when the strike is 159.11.

$$\begin{aligned} d_2 &= \frac{\ln \frac{140}{159.11} + (0.09 - 0.02 - 0.43^2) \cdot 0.25}{0.43\sqrt{0.25}} \\ &= -0.621 \\ \mathcal{N}(d_2) &= 0.267 \end{aligned}$$

Notice the words "risk-free" probability used in the question. In the event that there is no language like this, you would use α, the asset's rate of return (assuming that you have that information or can derive it).

Let's turn our attention to a futures example.

Example 40. *You wish to make a derivative that has a payoff based on some futures index, F, of* $\max\{100-F(1/12)^2, 0\}$ *at the end of one month. Currently, the futures value is* 7 *and the volatility is* 0.54*. The risk-free rate is* 0.06*. What should the price of your derivative be under the Black-Scholes model?*

Solution: This problem requires the use of the more general Black-Scholes formula. The derivative pays like a put. We need to determine what $F(1/12)^2$ looks like.

$$\begin{aligned} F(1/12) &= 7e^{(-0.54^2/2)/12+0.54\sqrt{1/12}\mathcal{N}(0,1)} \\ F(1/12)^2 &= 49e^{-0.0243+1.08\sqrt{1/12}\mathcal{N}(0,1)} \\ M &= 49 \\ a &= -0.0243 \\ b &= 1.08\sqrt{1/12} \\ K &= 100 \end{aligned}$$

Now that we have made all the necessary identifications, we can proceed with the formula. The option in this example pays like a put option.

$$
\begin{aligned}
D_1 &= \frac{\ln \frac{M}{K} + a}{b} + b \\
&= \frac{\ln \frac{49}{100} - 0.0243}{1.08\sqrt{1/12}} + 1.08\sqrt{1/12} \\
&= -2.054
\end{aligned}
$$

$$
\begin{aligned}
D_2 &= \frac{\ln \frac{M}{K} + a}{b} \\
&= -2.366 \\
p &= \left[K\mathcal{N}(-D_2) - Me^{a+b^2/2}\mathcal{N}(-D_1)\right] e^{-rT} \\
&= \left[100\mathcal{N}(2.366) - 49e^{-0.0243+(1.08\sqrt{1/12})^2/2}\mathcal{N}(2.054)\right] e^{-0.06/12} \\
&= [100 \cdot 0.99109 - 50.21 \cdot 0.98001]\, e^{-0.005} \\
&= 49.66
\end{aligned}
$$

Now that we have seen assets and futures, we can tackle currency exchanges.

Example 41. *You are speculating in the currencies exchange between the United States and Japan. It is your belief that the dollar will fall with respect to the yen. You would like to buy* $1,000,000$ *yen if your belief is correct. The current price of one yen is* \$0.0093. *You would like to buy* $1,000,000$ *asset-or-nothing options to buy the yen at a strike of* \$0.0125 *in two months. You know that the dollar's risk-free rate is* $r_d = 0.03$*, and the yen's risk-free rate is* $r_y = 0.07$*. In addition, you assume that the volatility of the exchange is* $\sigma = 0.38$*. What is the price of the options to buy yen?*

Solution: This is very similar to the first example of this section. In fact, in some ways it is easier.

$$
\begin{aligned}
d_1 &= \frac{\ln \frac{0.0093}{0.0125} + (0.03 - 0.07 + 0.38^2/2)/6}{0.38\sqrt{1/6}} \\
&= -1.872 \\
c_{A/N} &= S(0)e^{-\delta T}\mathcal{N}(d_1) \\
&= 0.0093e^{-0.07/6} \cdot 0.03060 \\
&= 0.00028 \\
1,000,000c_{A/N} &= 281.28
\end{aligned}
$$

I stored many of the values in my calculator when doing the previous problem. This is necessary since I am multiplying the resulting value from the call calculation by $1,000,000$. I did end up rounding d_1 to three decimal places. You can usually get away with two decimal places since normal tables typically go to two places. I am using an online calculator in my computations. The result I get by hand is slightly different than the one I get by using Excel. In Excel, my result is 281.57. You shouldn't worry about such small differences. I know I won't!

Since we just demonstrated how to handle a Black-Scholes problem using currencies, it is only fitting that you should get a chance.

Question 22 *Suppose that you are examining two calls. The first is in the United States, and the second is in the United Kingdom. The conditions on the first asset and call are* $S_1(0) = 50$, $\delta_1 = 0.04$, *and* $K_1 = 55$. *The U.S. risk-free rate is* $r_1 = 0.08$. *The conditions on the second asset and call are* $S_2(0) = 110$, $\delta_2 = 0.02$, *and* $K_2 = 121$. *The U.K. risk-free rate is* $r_2 = 0.06$. *Each option is European and expires in six months. Each asset is assumed to have the same volatility. You know that the dollar-denominated price of the first call is* 2.92. *What must the pound-denominated price of the second option be?*

The pound-denominated price of the second option is $\boxed{6.49}$.

Solution: This is one of those problems where it seems like you aren't given enough information. There are two possible approaches: first try to back out the value for the volatility using some sort of software/tricks, and the second is that there must be some observation that simplifies this problem. We will the first method in the next section when we discuss implied volatility. The second is the approach we take here. Notice that under the assumptions of the problem we have that the d_1 and d_2 values are identical for each option. Let's write down each Black-Scholes formula.

$$
\begin{aligned}
c_1 &= 2.92 \\
&= 50e^{-0.02}\mathcal{N}(d_1) - 55e^{-0.04}\mathcal{N}(d_2) \\
c_2 &= 110e^{-0.01}\mathcal{N}(d_1) - 121e^{-0.03}\mathcal{N}(d_2) \\
\frac{c_2}{c_1} &= \frac{110e^{-0.01}\mathcal{N}(d_1) - 121e^{-0.03}\mathcal{N}(d_2)}{50e^{-0.02}\mathcal{N}(d_1) - 55e^{-0.04}\mathcal{N}(d_2)} \\
&= 2.2e^{0.01} \\
c_2 &= 2.2e^{0.01}c_1 \\
&= 6.49
\end{aligned}
$$

In case you are curious, I did use the Black-Scholes formula to check the values for the previous problem. I used a volatility of $\sigma = 32$ for each option.

That concludes our section covering some basic examples of the Black-Scholes formula. Our next section will give some more involved examples. We will derive the put-call duality formula from our Black-Scholes formula. Please try this before you proceed to that section. Additionally, we will see implied volatility and gap options.

1.25 Examples (Advanced)

Previously, we saw some elementary applications of the Black-Scholes formula. Here, we will see some more advanced applications. This includes the more general Black-Scholes formula.

We have seen several examples of the Black-Scholes formula in action. This section deals with some consequences of that formula, and it covers some new concepts that result from using the formula. Let's begin with what was promised: the proof of put-call duality. Recall the formulae for duality:

$$
\begin{aligned}
c(S,K) &= S\cdot K\cdot p(1/S,1/K)\\
p(S,K) &= S\cdot K\cdot c(1/S,1/K)
\end{aligned}
$$

We will prove that the first formula holds and leave the second to the reader. To prove it, we just apply the Black-Scholes formula to each side. We begin with the left-hand side. δ will represent the risk-free rate for the asset currency in this part.

$$
\begin{aligned}
d_1 &= \frac{\ln\frac{S(0)}{K} + (r-\delta+\sigma^2/2)T}{\sigma\sqrt{T}}\\
d_2 &= \frac{\ln\frac{S(0)}{K} + (r-\delta-\sigma^2/2)T}{\sigma\sqrt{T}}\\
c &= S(0)e^{-\delta T}\mathcal{N}(d_1) - Ke^{-rT}\mathcal{N}(d_2)
\end{aligned}
$$

Now, we apply the formula to the right hand side. Due to the ambiguity in using d_1 and d_2 again, we will use tildes over the d's. Recall that the asset for the right hand side is $1/S$ and the strike is $1/K$. In the computations here, the roles of δ and r are reversed. Finally, the volatility remaines unchanged since it is the same measurement between these two assets. If you aren't convinced of this part, just reverse the order of your fractions from a volatility computation. The sample variance will remain unchanged.

$$
\begin{aligned}
\tilde{d}_1 &= \frac{\ln \frac{1/S(0)}{1/K} + (\delta - r + \sigma^2/2)T}{\sigma\sqrt{T}} \\
&= -\frac{\ln \frac{S(0)}{K} + (r - \delta - \sigma^2/2)T}{\sigma\sqrt{T}} \\
&= -d_2 \\
\tilde{d}_2 &= \frac{\ln \frac{1/S(0)}{1/K} + (\delta - r - \sigma^2/2)T}{\sigma\sqrt{T}} \\
&= -\frac{\ln \frac{S(0)}{K} + (r - \delta + \sigma^2/2)T}{\sigma\sqrt{T}} \\
&= -d_1
\end{aligned}
$$

That's rather convenient! Let's put these values into the Black-Scholes formula for the put option.

$$
\begin{aligned}
p &= 1/Ke^{-\delta T}\mathcal{N}(-d_2) - 1/S(0)e^{-rT}\mathcal{N}(-d_1) \\
&= 1/Ke^{-\delta T}\mathcal{N}(d_1) - 1/S(0)e^{-rT}\mathcal{N}(d_2) \\
S(0) \cdot K \cdot p(1/S, 1/K) &= S(0)e^{-\delta T}\mathcal{N}(d_1) - Ke^{-rT}\mathcal{N}(d_2) \\
&= c(S, K)
\end{aligned}
$$

It's good that we could validate the formula we came up with some time ago. If you want an extra challenge, try to verify the formula using a binomial model. I've never tried it myself, but it would be a fun exercise. My suspicion is that it will work for the Cox-Ross-Rubenstein and the Forward tree models. You may run into some difficulty with the Jarrow-Rudd model.

Let's move our attention on to the subject of gap options. A gap option is simply a modified call or put option.

Definition 20. *A* **gap option** *is an option with strike K_1 and payoff equal to*

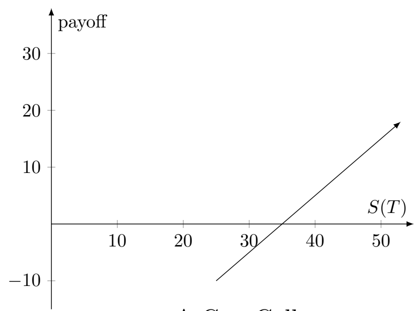
$$\max\{S(t) - K_2, 0\}$$

for a gap call and payoff equal to

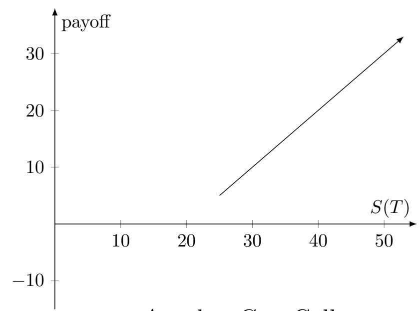
$$\max\{K_2 - S(t), 0\}$$

for a gap put. The payoffs are at time t.

The given definition allows for American gap options; however, we will only consider the case when they are European. That is, we will price them with an expiration time of T. Before we work out some prices, it is useful to visualize these options.



A Gap Call



Another Gap Call

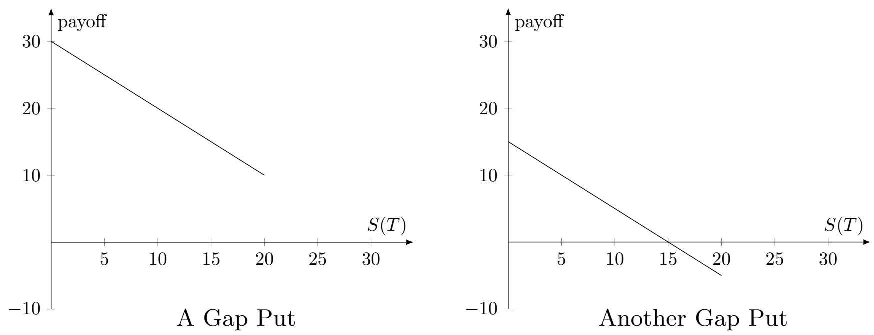
Both of the options in the figure can be viewed as gap options, but it is more useful if you can view them as combinations of asset-or-nothing options combined with cash-or-nothing options where each option has the same strike, $K = 25$. The first gap call should have a price equivalent to a strike 25 asset-or-nothing option with 35 written cash-or-nothing options. The second gap call should have a price equivalent to a strike 25 asset-or-nothing options with 20 written cash-or-nothing options. When this is written, it would look like this:

$$c_{\text{gap},1} = S(0)e^{-\delta T}\mathcal{N}(d_1) - 35e^{-rT}\mathcal{N}(d_2)$$
$$c_{\text{gap},2} = S(0)e^{-\delta T}\mathcal{N}(d_1) - 20e^{-rT}\mathcal{N}(d_2)$$

Do not make the mistake and assume that $K = 35$ or $K = 20$ in your computations of d_1 and d_2! That is a pitfall that many stumble into. In both, the values for d_1 and d_2 are identical.

$$d_1 = \frac{\ln \frac{S(0)}{25} + (r - \delta + \sigma^2/2)T}{\sigma\sqrt{T}}$$
$$d_2 = \frac{\ln \frac{S(0)}{25} + (r - \delta - \sigma^2/2)T}{\sigma\sqrt{T}}$$

To complete our visualization, let's graph the gap puts as well.



Both of the put options have the same strike, $K = 20$. To get the first gap put, you could purchase an asset-or-nothing put and buy 10 cash-or-nothing puts. The get the second gap put, you could purchase an asset-or-nothing put and write 5 cash-or-nothing puts.

We can summarize what we have done in the following theorem.

Theorem 8 (Gap Options)**.** *The Black-Scholes price for gap calls and gap puts are given by the formulae below.*

$$
\begin{aligned}
c_{gap} &= S(0)e^{-\delta T}\mathcal{N}(d_1) - K_2 e^{-rT}\mathcal{N}(d_2) \\
p_{gap} &= K_2 e^{-rT}\mathcal{N}(-d_2) - S(0)e^{-\delta T}\mathcal{N}(-d_1) \\
d_1 &= \frac{\ln \frac{S(0)}{K_1} + (r - \delta + \sigma^2/2)T}{\sigma\sqrt{T}} \\
d_2 &= \frac{\ln \frac{S(0)}{K_1} + (r - \delta - \sigma^2/2)T}{\sigma\sqrt{T}}
\end{aligned}
$$

Question 23 *You wish to purchase a European gap option on an asset, S. The option pays*

$$
\begin{cases} S(1/3) - 40 & \text{if } S(1/3) > 35, \\ 0 & \text{otherwise} \end{cases}
$$

in four months. Determine the price of the option if

- $S(0) = 37$
- $r = 0.08$
- $\delta = 0.02$
- $\sigma = 0.34$

The price is $\boxed{1.32}$.

Solution: The answer must be of the form

$$c_{\text{gap}} = 37e^{-0.02/3}\mathcal{N}(d_1) - 40e^{-0.08/3}\mathcal{N}(d_2),$$

where d_1 and d_2 are given below.

$$\begin{aligned} d_1 &= \frac{\ln\frac{37}{35} + (0.08 - 0.02 + 0.34^2/2)/3}{0.34\sqrt{1/3}} \\ &= 0.483 \\ d_2 &= d_1 - 0.34 \cdot \sqrt{1/3} \\ &= 0.287 \\ c_{\text{gap}} &= 37e^{-0.02/3}\mathcal{N}(0.483) - 40e^{-0.08/3}\mathcal{N}(0.287) \\ &= 36.75 \cdot 0.68545 - 38.95 \cdot 0.61294 \\ &= 1.32 \end{aligned}$$

As you likely noticed, the hardest part when dealing with gap options is the appearance that there are two strikes. I keep this straight in my head by remembering that the payoff is immediately clear from the Black-Scholes formula that you write down. The was that is displayed above is

$$\begin{gathered} S(1/3){-}40 \\ S(1/3)e^{-\delta/3}\mathcal{N}(d_1){-}40e^{-r/3}\mathcal{N}(d_2). \end{gathered}$$

The 40 appears in the payoff and the price. That leaves the value from the “if“ statement for the computations of d_1 and d_2.

We conclude this section with the notion of implied volatility. As you may have noticed, there are aobut six variables that go into every Black-Scholes formula computation: S, K, r, δ, σ and T. In almost all situations you encounter, five of those variables will be public information. That is, all the variables except volatility are known. An argument can be made regarding the risk-free rate and the dividend rate; however, those remain constant in the short term for almost all situations. S, K, and T are all defined in a contract.

What all this means is that if we encounter an option price, then we can deduce the volatility under the Black-Scholes model from that price. This sounds a little daunting since the formula relies on some normal table values. The reality is not so terrible. If you want to use technology, program the Black-Scholes formula into Excel. Then you can use Excel's goal seek feature to determine the volatility given the other five constants from the Black-Schole formula. For example, I used the values from the previous problem. I decided that the price of some call option under similar conditions was 5. This gave me an implied volatility of 0.43.

Obviously, this is not how this kind of question would appear on an exam or homework assignment. You would need to use some sort of algebraic manipulations to back out the appropriate result. Let's see how one might do that in an example.

Example 42. *You observe the price of a six month, strike 59 European call is* 7.64*. The current underlying asset's price is* 57*, the dividend rate is* 0.01*, and the risk-free rate is* 0.079*. Determine the implied volatility.*

Solution: We use the Black-Scholes formula. First, we solve for d_1 and d_2 and hope for the best.

$$
\begin{aligned}
d_1 &= \frac{\ln 5759 + (0.079 - 0.01 + \sigma^2/2)/2}{\sigma\sqrt{1/2}} \\
&= \frac{(\sigma^2/2)/2}{\sigma\sqrt{1/2}} \\
&= \sigma \cdot \frac{\sqrt{2}}{4} \\
d_2 &= d_1 - \sigma\sqrt{1/2} \\
&= -\sigma \cdot \frac{\sqrt{2}}{4} \\
&= -d_1 \\
c &= 7.64 \\
&= 57e^{-0.01/2}\mathcal{N}(d_1) - 59e^{-0.079/2}\mathcal{N}(-d_1) \\
&= 57e^{-0.01/2}[\mathcal{N}(d_1) - \mathcal{N}(-d_1)] \\
&= 57e^{-0.01/2}[2\mathcal{N}(d_1) - 1]
\end{aligned}
$$

The last two steps require some justification. In the first, $57e^{-0.01/2}$ is factored out of the equation. That is because the value 59 is the forward price of the asset. Once we discount that value at the risk-free rate we have $S(0)e^{-0.01/2}$, the prepaid forward price. The last step follows by the symmetry of the standard normal distribution. $\mathcal{N}(-d_1)$ represents the left tail value that is equivalent to the probability $\mathbb{P}(\mathcal{N}(0,1) > d_1) = 1 - \mathbb{P}(\mathcal{N}(0,1) < d_1)$.

Now we just unravel d_1 from the equations we wrote above.

$$
\begin{aligned}
7.64 &= 57e^{-0.01/2}[2\mathcal{N}(d_1) - 1] \\
7.64/56.71 &= [2\mathcal{N}(d_1) - 1] \\
1.13471 &= 2\mathcal{N}(d_1) \\
0.56736 &= \mathcal{N}(d_1) \\
0.16966 &= d_1
\end{aligned}
$$

Now we have a formula for σ!

$$
\begin{aligned}
0.16966 &= \sigma \cdot \frac{\sqrt{2}}{4} \\
0.47987 &= \sigma
\end{aligned}
$$

I actually used $\sigma = 0.48$ in my original formula for the price of this particular option. Any of the steps could have picked up a slight rounding error, but I think 0.47987 is probably good enough!

That concludes our topics that apply the Black-Scholes model in a direct way. Our next section will cover another Black-Scholes formula topic where you buy some asset using an asset (other than cash). This seems more like a barter, but the formula that describes the activity turns out to be identical to the Black-Scholes formula.

1.26 Exchange Options

Usually, we exchange a currency for a financial instrument. In the setting of this section, we will exchange one asset for another. A generalized ideal of volatility will be introduced to handle this situation.

An exchange option uses an interesting concept: you wish to buy some quantity of asset S in the future, but you will use another asset as the currency at that point of time. To measure the price of such an asset today, we need a volatility between the two assets. The reason we need such a volatility is because our normal measures of volatility are measured with respect to some currency.

You may recall in our examples following the Black-Scholes formula that when dealing with currencies we had a volatility of the exchange. Our current framework is very similar to a currency problem. The only difference is that we will usually be given a correlation coefficient that relates the two assets. That coefficient is called rho, ρ.

Suppose that R and S are two assets with volatilities σ_R and σ_S. The volatility between the two assets in our context will be given by the value

$$\sigma^2 = \sigma_R^2 + \sigma_S^2 - 2\rho\sigma_R\sigma_S$$

This may not make sense. It is tempting to think that the signs should all be positive. The way I think about it is that if the prices of the assets are highly correlated (ρ close to 1), then the volatility between the price of these assets should be small. Similarly, when the assets are highly negatively correlated (ρ close to -1), then the volatility between the price of these assets should be large. The minus sign in the formula given accomplishes this.

This explanation is probably not satisfactory for many readers. I know it wouldn't be adequate for me! Let's see why such a formula should even exist. Recall how we estimated volatility when discussing binomial models. We can do something similar here for two assets, say R and S. We will treat R as the currency and S as the asset.

Let r_0, r_1, ..., r_n, s_0, s_1, ..., and s_n be the prices of R and S over some period ov time. In addition, we let $t_0 = s_0/r_0$, $t_1 = s_1/r_1$, ..., and $t_n = s_n/r_n$. This variable represents the number of shares of R necessary to purchase S at a given point in time. It is the volatility of this auxiliary variable that gives us our volatility.

In what follows, s^2 represents sample variance and $\hat{\text{Cov}}$ represents sample covariance. For simplicity, we denote the natural logarithms of the successive quotients by x_j, y_j, and z_j. More concisely:

$$
\begin{aligned}
x_j &= \ln \frac{r_j}{r_{j-1}} \\
y_j &= \ln \frac{s_j}{s_{j-1}} \\
z_j &= \ln \frac{t_j}{t_{j-1}}
\end{aligned}
$$

for $j = 1, 2, \ldots, n$. We need to make an observation regarding the z_j.

$$
\begin{aligned}
z_j &= \ln \frac{t_j}{t_{j-1}} \\
&= \ln \frac{s_j/r_j}{s_{j-1}/r_{j-1}} \\
&= \ln \frac{s_j}{s_{j-1}} - \ln \frac{r_j}{r_{j-1}} \\
&= y_j - x_j
\end{aligned}
$$

Then we know that

$$
\begin{aligned}
s_x^2 &= \frac{1}{n-1} \sum_{j=1}^{n} (x_j - \bar{x})^2 \\
s_y^2 &= \frac{1}{n-1} \sum_{j=1}^{n} (y_j - \bar{y})^2 \\
s_z^2 &= \frac{1}{n-1} \sum_{j=1}^{n} (z_j - \bar{z})^2 \\
&= \frac{1}{n-1} \sum_{j=1}^{n} (y_j - x_j - \bar{y} + \bar{x})^2 \\
&= \frac{1}{n-1} \sum_{j=1}^{n} [(x_j - \bar{x})^2 + (y_j - \bar{y})^2 - 2(x_j - \bar{x})(y_j - \bar{y})] \\
&= \frac{1}{n-1} \sum_{j=1}^{n} (x_j - \bar{x})^2 + \frac{1}{n-1} \sum_{j=1}^{n} (y_j - \bar{y})^2 - 2\frac{1}{n-1} \sum_{j=1}^{n} (x_j - \bar{x})(y_j - \bar{y}) \\
&= s_x^2 + s_y^2 - 2\widehat{\mathrm{Cov}}(x, y) \\
&= s_x^2 + s_y^2 - 2\rho s_x s_y
\end{aligned}
$$

It's at this last stage that we can make the necessary comparison to arrive at our formula. Recall that the sample variance is equal to the volatility squared multiplied by the unit of time for each measurement.

$$\begin{aligned}
s_x &= \sigma_R\sqrt{h}\\
s_y &= \sigma_S\sqrt{h}\\
s_z &= \sigma\sqrt{h}\\
s_z^2 &= \sigma^2 h\\
&= s_x^2 + s_y^2 - 2\rho s_x s_y\\
&= \sigma_R^2 h + \sigma_S^2 h - 2\rho\sigma_R\sqrt{h}\sigma_S\sqrt{h}\\
\sigma^2 &= \sigma_R^2 + \sigma_S^2 - 2\rho\sigma_R\sigma_S
\end{aligned}$$

Now that we have convinced ourselves that this formula is true, we can proceed with an example!

Example 43. *You currently own several shares of XYZ. You wish to purchase a call option to buy 3 shares of ABC using 2 shares of XYZ in four months. Denote XYZ by R and ABC by S. You have the following information:*

- $R(0) = 42$ *and* $S(0) = 29$
- $\delta_R = 0.01$ *and* $\delta_S = 0.04$
- $\sigma_R = 0.3$ *and* $\sigma_S = 0.4$
- $\rho = -0.5$

Determine the price of the call option.

Solution: Everything from our usual Black-Scholes formula is replaced with information regarding R. Also, the quantities must be reflected in our formula. It follows that our modified formulae will be

$$\begin{aligned}
d_1 &= \frac{\ln\frac{3S(0)}{2R(0)} + (\delta_R - \delta_S + \sigma^2/2)/3}{\sigma\sqrt{1/3}}\\
d_2 &= d_1 - \sigma\sqrt{1/3}\\
c &= 3S(0)e^{-\delta_S/3}\mathcal{N}(d_1) - 2R(0)e^{-\delta_R/3}\mathcal{N}(d_1)
\end{aligned}$$

Let's determine σ.

$$\begin{aligned}
\sigma^2 &= \sigma_R^2 + \sigma_S^2 - 2\rho\sigma_R\sigma_S\\
&= 0.09 + 0.16 - 2\cdot(-0.5)\cdot 0.3\cdot 0.4\\
&= 0.37
\end{aligned}$$

Now we can execute the Black-Scholes model.

$$\begin{aligned}
d_1 &= \frac{\ln \frac{87}{84} + (0.01 - 0.04 + 0.37/2)/3}{\sqrt{0.37/3}} \\
&= 0.247 \\
d_2 &= d_1 - \sqrt{0.37/3} \\
&= -0.104 \\
c &= 87e^{-0.04/3}\mathcal{N}(0.247) - 84e^{-0.01/3}\mathcal{N}(-0.104) \\
&= 85.85 \cdot 0.59755 - 83.72 \cdot 0.45858 \\
&= 12.91
\end{aligned}$$

This is really not that much more difficult than a typical Black-Scholes formula problem. There are two extra steps: determine the volatility from the given information, and identify which asset is acting as the currency. Since either asset can be used to purchase or sell the other one, we also have put call duality! The put to sell XYZ using ABC as currency will be intimitely related to the call from the example. See if you can determine the relationship!

We won't go into that here. We will explore a different type of option: one that pays the maximum of the two assets. We will use the same assets as in the previous example.

Example 44. *What is the price of the derivative that pays* $\max\{3S(1/3), 2R(1/3)\}$ *in four months?*

Solution: This really requires some manipulation on maximums.

$$\max\{a, b\} = \max\{a - b, 0\} + b$$

If you don't believe this, consider the two cases: $a \geq b$ and $a < b$. If $a \geq b$, we have

$$\max\{a - b, 0\} + b = a - b + b = a = \max\{a, b\}.$$

If $a < b$, we have

$$\max\{a - b, 0\} + b = 0 + b = b = \max\{a, b\}.$$

We apply this to the payoff of our derivative. The payoff is

$$\max\{3S(1/3), 2R(1/3)\} = \max\{3S(1/3) - 2R(1/3), 0\} + 2R(1/3)$$

Notice that the maximum is of the form of a call option.

$$\max\{3S(1/3) - K, 0\}$$

That means part of our payoff is simply the call from the previous example. The other part of the payoff is an asset. We know how to get that payoff at time 1/3; We buy some quantity of R at time zero that will grow into one share at time 1/3. Thus our price is

$$c + 2R(0)e^{-\delta_R/3} = 12.91 + 84e^{-0.01/3} = 96.63$$

Now try something similar!

Question 24 *Determine the price of a derivative that pays* $\min\{3S(1/3), 2R(1/3)\}$ *in four months. Hint: What happens when you add a minimum and a maximum of two variables?*

Solution: The price of the derivative comes from the relationship

$$\min\{a, b\} + \max\{a, b\} = a + b.$$

We can relate the payoffs of our two derivatives.

$$\begin{aligned}
\min\{3S(1/3), 2R(1/3)\} + \max\{3S(1/3), 2R(1/3)\} &= 3S(1/3) + 2R(1/3) \\
\min\{3S(1/3), 2R(1/3)\} &= 3S(1/3) + 2R(1/3) - \max\{3S(1/3), 2R(1/3)\} \\
&= 3S(1/3) - \max\{3S(1/3) - 2R(1/3), 0\}
\end{aligned}$$

The price of the derivative comes from computing the price of the two positions. We have the second one; it is the call from earlier!

$$3S(0)e^{-\delta_S/3} - c = 85.85 - 12.91 = 72.94$$

The final option we are going to discuss is called a chooser option. The option has a payoff at time t that is the maximum of a related call and put. The call and the put expire at time T, have the same underlying asset, the same strike, and they are both European.

To compute the price of the option, we must make our computations at time t, since that is the time that you choose whether you want the call or the put. Let's go ahead and compute that payoff here. The arguments here will emphasize the varibles that are significant.

$$\begin{aligned}
\max\{c(t,T,K),p(t,T,K)\} &= \max\{c(t,T,K)-p(t,T,K),0\}+p(t,T,K) \\
&= \max\{S(t)e^{-\delta(T-t)}-Ke^{-r(T-t)},0\}+p(t,T,K) \\
&= e^{-\delta(T-t)}\max\{S(t)-Ke^{-(r-\delta)(T-t)},0\}+p(t,T,K)
\end{aligned}$$

The second line follows from put-call parity. The final line is the result of factoring out the constant in front of the asset.

The last line is a payoff that gives us something to work with. The price of the chooser option comes from this. The second term is straight-forward. It is a put option with strike K and expiration at time T. The first is a little trickier. The payoff resembles that of a call option. The option has some different terms. It expires at time t and has strike $Ke^{-(r-\delta)(T-t)}$. It follows that the price of the chooser option is given by

$$e^{-\delta(T-t)}c(0,t,Ke^{-(r-\delta)(T-t)})+p(0,T,K)$$

We will finish with an example of a chooser option.

Example 45. *You expect some price volatility of an asset, S, followed by a period of stability. Therefore, you are uncertain about whether you should purchase call options or put options for the asset. A chooser option may be ideal for you. You are given the following conditions:*

- $S(0) = 53$
- $\delta = 0.02$
- $\sigma = 0.54$
- $r = 0.05$

You would like the options on S to expire in six months and have strike 53. *You would like to choose the maximum of the two options in two months. Determine the price of the chooser option.*

Solution: This requires a bit of work since we are pricing two different options. Let's start with the put option.

$$
\begin{aligned}
d_1 &= \frac{\ln \frac{53}{53} + (0.05 - 0.02 + 0.54^2/2)/2}{0.54\sqrt{1/2}} \\
&= 0.230 \\
d_2 &= d_1 - 0.54\sqrt{1/2} \\
&= -0.152 \\
p &= 53e^{-0.05/2}\mathcal{N}(0.152) - 53e^{-0.02/2}\mathcal{N}(-0.230) \\
&= 7.50
\end{aligned}
$$

Now we can determine the price of the call from the formula.

$$
\begin{aligned}
d_1 &= \frac{\ln \frac{53}{53e^{-(0.05-0.02)/3}} + (0.05 - 0.02 + 0.54^2/2)/6}{0.54\sqrt{1/6}} \\
&= 0.178 \\
d_2 &= d_1 - 0.54\sqrt{1/6} \\
&= -0.042 \\
c &= 53e^{-0.02/6}\mathcal{N}(0.178) - 53e^{-(0.05-0.02)/3}e^{-0.05/6}\mathcal{N}(-0.042) \\
&= 5.01
\end{aligned}
$$

The price of the option is

$$
e^{-0.02/3} \cdot 5.01 + 7.50 = 12.48
$$

That was kind of painful, but it was all using principles we developed in this section in conjunction with parity and the Black-Scholes formula. In some writings, you will see a slightly different formula for the price of the chooser option.

$$
e^{-\delta(T-t)}p(0, t, Ke^{-(r-\delta)(T-t)}) + c(0, T, K)
$$

The development of this version follows the same arguments we used. The only difference is that the put-call parity relationship is used in its negative form. I don't think this is nearly as clear, so I avoided using it. If you want to add some work to your life, see that you get the same result as in the example.

1.27 Portfolio Changes

Much of our work has been dedicated to understanding how to determine the price of a derivative now. Here, we examine how those very same derivatives change in value. There are two variables we focus on: asset value and time value.

Now that we have the tools necessary to determine the price of our derivatives, we will begin to look at what happens to collections of these derivatives. By their very nature, derivative prices are sensitive to the value of the underlying asset. A portfolio is simply a collection of financial instruments. Our portfolios will consist of derivatives, assets, and bonds.

Let's explore what happens to a portfolio's value as the asset changes in value and time varies. Our portfolio will consist of 100 written calls, 60 shares of the underlying asset and a loan of $3,000$. The conditions determining our values will be

- $S(0) = 70$
- $\delta = 0.03$
- $\sigma = 0.35$
- $r = 0.09$

In addition, the calls are European, have strike 68 and expire in one month.

The first thing we should determine is the value of the portfolio. The loan represents a debt at the risk-free rate. My computations below will be more direct than usual. That is because I feel that you are more capable of filling in the pieces!

$$\begin{aligned}
d_1 &= \frac{\ln \frac{70}{68} + (0.09 - 0.03 + 0.35^2/2)/12}{0.35\sqrt{1/12}} \\
&= 0.387 \\
d_2 &= \frac{\ln \frac{70}{68} + (0.09 - 0.03 - 0.35^2/2)/12}{0.35\sqrt{1/12}} \\
&= 0.286 \\
c &= 70e^{-0.03/12}\mathcal{N}(0.387) - 68e^{-0.09/12}\mathcal{N}(0.286) \\
&= 4.0878
\end{aligned}$$

Now we can compute the portfolio value. P will denote the portfolio's value. The first argument is that of time, and the second is that of the asset's value.

$$
\begin{aligned}
P(0,70) &= -100 \cdot 4.0878 + 60 \cdot 70 - 3000 \\
&= 791.22
\end{aligned}
$$

If the underlying asset's value changed immediately, the portfolio's value would also experience a change. Let's examine what happens when the asset both increases or decreases by 2 in value. We'll do the increase first.

$$
\begin{aligned}
d_1 &= \frac{\ln \frac{72}{68} + (0.09 - 0.03 + 0.35^2/2)/12}{0.35\sqrt{1/12}} \\
&= 0.666 \\
d_2 &= \frac{\ln \frac{72}{68} + (0.09 - 0.03 - 0.35^2/2)/12}{0.35\sqrt{1/12}} \\
&= 0.565 \\
c &= 72e^{-0.03/12}\mathcal{N}(0.666) - 68e^{-0.09/12}\mathcal{N}(0.565) \\
&= 5.4850
\end{aligned}
$$

Now the decrease.

$$
\begin{aligned}
d_1 &= \frac{\ln \frac{68}{68} + (0.09 - 0.03 + 0.35^2/2)/12}{0.35\sqrt{1/12}} \\
&= 0.100 \\
d_2 &= \frac{\ln \frac{68}{68} + (0.09 - 0.03 - 0.35^2/2)/12}{0.35\sqrt{1/12}} \\
&= -0.001 \\
c &= 68e^{-0.03/12}\mathcal{N}(0.100) - 68e^{-0.09/12}\mathcal{N}(-0.001) \\
&= 2.8986
\end{aligned}
$$

Now we can compute the portfolio values.

$$
\begin{aligned}
P(0,72) &= -100 \cdot 5.4850 + 60 \cdot 72 - 3000 \\
&= 771.5 \\
P(0,68) &= -100 \cdot 2.8986 + 60 \cdot 68 - 3000 \\
&= 790.14
\end{aligned}
$$

Perhaps instantaneous changes aren't that realistic. Let's assume that the price changes occur over the course of one quarter of a month. We will also want to measure the value of the portfolio if the asset does not change in value. Again, we will determine the price of the call under each of the three scenarios. Since one quarter of a month has moved by, the options are closer to expiration. $1/12 - 1/12 \cdot 1/4 = 3/48 = 1/16$.

The strike 72 call:

$$\begin{aligned}
d_1 &= \frac{\ln \frac{72}{68} + (0.09 - 0.03 + 0.35^2/2)/16}{0.35\sqrt{1/16}} \\
&= 0.740 \\
d_2 &= \frac{\ln \frac{72}{68} + (0.09 - 0.03 - 0.35^2/2)/16}{0.35\sqrt{1/16}} \\
&= 0.652 \\
c &= 72e^{-0.03/16}\mathcal{N}(0.740) - 68e^{-0.09/16}\mathcal{N}(0.652) \\
&= 5.1234
\end{aligned}$$

The strike 70 call:

$$\begin{aligned}
d_1 &= \frac{\ln \frac{70}{68} + (0.09 - 0.03 + 0.35^2/2)/16}{0.35\sqrt{1/16}} \\
&= 0.418 \\
d_2 &= \frac{\ln \frac{70}{68} + (0.09 - 0.03 - 0.35^2/2)/16}{0.35\sqrt{1/16}} \\
&= 0.330 \\
c &= 70e^{-0.03/16}\mathcal{N}(0.418) - 68e^{-0.09/16}\mathcal{N}(0.330) \\
&= 3.6899
\end{aligned}$$

The strike 68 call:

$$\begin{aligned}
d_1 &= \frac{\ln \frac{68}{68} + (0.09 - 0.03 + 0.35^2/2)/16}{0.35\sqrt{1/16}} \\
&= 0.087 \\
d_2 &= \frac{\ln \frac{68}{68} + (0.09 - 0.03 - 0.35^2/2)/16}{0.35\sqrt{1/16}} \\
&= -0.001 \\
c &= 68e^{-0.03/16}\mathcal{N}(0.087) - 68e^{-0.09/16}\mathcal{N}(-0.001) \\
&= 2.4933
\end{aligned}$$

The portfolio values are given below.

$$\begin{aligned}
P(1/48, 72) &= -100 \cdot 5.1234 + 60e^{0.03/48} \cdot 72 - 3000e^{0.09/48} \\
&= 804.73 \\
P(1/48, 70) &= -100 \cdot 3.6899 + 60e^{0.03/48} \cdot 70 - 3000e^{0.09/48} \\
&= 828.01 \\
P(1/48, 68) &= -100 \cdot 2.4933 + 60e^{0.03/48} \cdot 68 - 3000e^{0.09/48} \\
&= 827.59
\end{aligned}$$

I don't know about you, but this does seem a bit strange! The portfolio seemed to gain in value just due to the passage of time. It turns out that this is not a coincidence! There is something called theta-decay, and it is something that we will examine later in this chapter. The idea behind it is that a typical call or put option will decrease in value as it gets closer to expiration (all other terms being held equal). In the following table you will find a variety of portfolio values as both time and asset value change. I chose to use quarter months as in the previous examples, so all of the six values we computed are present. Each quarter month represents 1/48 of a year.

Asset Value	Time 0	Time 1/48	Time 2/48	Time 3/48
64	718.95	747.47	778.38	810.65
66	766.17	800.75	840.85	890.31
68	790.14	827.59	871.95	929.59
70	791.22	828.01	870.76	923.75
72	771.50	804.73	841.37	881.30
74	734.30	762.27	790.79	816.73

After examining this table, I am really tempted to start writing some call options! Before you or I become an options trader, it would probably be beneficial to observe options and portfolio prices in the market before engaging in such risky behavior. It is worthwhile to examine which positions are profitable. Remember, to measure profit you take the current position and subtract the value of the initial investment grown at the risk-free rate. The table below give the profit information for all times beyond time 0.

Asset Value	Time 1/48	Time 2/48	Time 3/48
64	−45.23	−15.81	14.97
66	8.05	46.66	94.63
68	34.89	77.76	133.91
70	35.31	76.57	128.07
72	12.03	47.18	85.62
74	−30.43	−3.40	21.05

It would seem that this portfolio is almost always profitable near expiration! Why don't you determine the risk-free probability that the asset will fall in this range ($64 \leq S(1/16) \leq 74$) under the Black-Scholes model.

To proceed from here we will need to develop some tools from differential calculus.

1.28 Delta, Gamma, and Theta

Our last section demonstrated how a portfolio changes in value with respect to changes to time and asset value. Here we learn about the tools that will be used to estimate those changes. At face value, they are as simple as taking partial derivatives. The only difficulty lies in recollection of the chain rule and the fundamental theorem of calculus.

In our previous section, we discussed how a portfolio changes with respect to changes in value of the underlying asset and changes in time. We treated these as independent variables, and that will continue in this section. Here, we are going to examine the instantaneous rate of change with respect to these variables.

Instantaneous change with respect to the asset is called the delta. If we differentiate the portfolio's value with respect to S, then we have the change desired. The instantaneous change is denoted

$$\Delta_P = \frac{\mathrm{d}P}{\mathrm{d}S}.$$

We could go one step further and differentiate again to arrive at the gamma value. That is denoted

$$\Gamma_P = \frac{\mathrm{d}^2P}{\mathrm{d}S^2}.$$

We also examined what happened with respect to changes in time. While changes in asset value aren't guaranteed, changes in time are inevitable. The instantaneous change is called theta, and it is denoted

$$\Theta_P = \frac{\mathrm{d}P}{\mathrm{d}t}.$$

I want to emphasize that this derivative is with respect to the time that varies. That is t, not T.

These values are some of what are called the Greeks. There are a few more that we will see later, but they aren't as useful in what we do in this class. Let's see how one might compute some of these values. We'll begin by computing the delta value of a call option. This may sound easy, but remember that the price of a call option is a mess.

$$c = Se^{-\delta(T-t)}\mathcal{N}(d_1) - Ke^{-r(T-t)}\mathcal{N}(d_2)$$

I wrote the more general formula since it has t in it, and we will need that later. I also didn't write the argument in S since it will go away when we differentiate with respect to S. One might mistakenly differentiate with respect to S and arrive at

$$\begin{aligned}
\Delta_c &= \frac{\mathrm{d}}{\mathrm{d}S}\left[Se^{-\delta(T-t)}\mathcal{N}(d_1) - Ke^{-r(T-t)}\mathcal{N}(d_2)\right] \\
&= e^{-\delta(T-t)}\mathcal{N}(d_1) - 0 \\
&= e^{-\delta(T-t)}\mathcal{N}(d_1)
\end{aligned}$$

The nice part about this is that the result is correct. The unfortunate part is that we cheated! We didn't use the fundamental theorem of calculus. Remember that $\mathcal{N}(d_1)$ is actually defined by an integral. Also, d_1 is a function of S! We really cheated. Let's try to compute the value of Δ_c correctly. Before we make that computation, let's try to compute the derivative of $\mathcal{N}(d_1)$ and $\mathcal{N}(d_2)$.

$$\begin{aligned}
\mathcal{N}(d_1) &= \int_{-\infty}^{d_1} \frac{1}{\sqrt{2\pi}} e^{-z^2/2} \mathrm{d}z \\
\frac{\mathrm{d}}{\mathrm{d}S}\mathcal{N}(d_1) &= \frac{1}{\sqrt{2\pi}} e^{-d_1^2/2} \frac{\mathrm{d}}{\mathrm{d}S} d_1 \\
&= \frac{1}{\sqrt{2\pi}} e^{-d_1^2/2} \frac{1}{S\sigma\sqrt{T-t}}
\end{aligned}$$

Similarly, it can be shown that

$$\frac{\mathrm{d}}{\mathrm{d}S}\mathcal{N}(d_2) = \frac{1}{\sqrt{2\pi}} e^{-d_2^2/2} \frac{1}{S\sigma\sqrt{T-t}}$$

$$\begin{aligned}
\Delta_c &= \frac{\mathrm{d}}{\mathrm{d}S}\left[Se^{-\delta(T-t)}\mathcal{N}(d_1) - Ke^{-r(T-t)}\mathcal{N}(d_2)\right] \\
&= e^{-\delta(T-t)}\mathcal{N}(d_1) + Se^{-\delta(T-t)}\frac{1}{\sqrt{2\pi}}e^{-d_1^2/2}\frac{1}{S\sigma\sqrt{T-t}} - Ke^{-r(T-t)}\frac{1}{\sqrt{2\pi}}e^{-d_2^2/2}\frac{1}{S\sigma\sqrt{T-t}} \\
&= e^{-\delta(T-t)}\mathcal{N}(d_1) + \frac{1}{S\sigma\sqrt{2\pi(T-t)}}\left(Se^{-\delta(T-t)}e^{-d_1^2/2} - Ke^{-r(T-t)}e^{-d_2^2/2}\right)
\end{aligned}$$

For the earlier formula to be correct, something must happen in the parenthetical expression in the last line! Before we can show that, it may be useful to further simplify that expression. For convenience (and to save my poor fingers), we will make the following identifications:

$$\begin{aligned}
a &= \left[\ln\frac{S}{K} + (r-\delta)(T-t)\right] \\
b &= \left(\frac{\sigma^2}{2}\right)(T-t) \\
c &= \sigma\sqrt{T-t}
\end{aligned}$$

Now we can simplify our expressions.

$$
\begin{aligned}
d_1 &= \frac{\ln \frac{S}{K} + (r - \delta + \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}} \\
&= \frac{a+b}{c} \\
d_1^2 &= \frac{a^2 + 2ab + b^2}{c^2} \\
d_2 &= \frac{\ln \frac{S}{K} + (r - \delta - \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}} \\
&= \frac{a-b}{c} \\
d_2^2 &= \frac{a^2 - 2ab + b^2}{c^2} \\
e^{-d_1^2/2} &= e^{-(a^2+b^2)/2c^2} e^{-ab/c^2} \\
e^{-d_2^2/2} &= e^{-(a^2+b^2/2c^2} e^{ab/c^2}
\end{aligned}
$$

It is the differences that we care about here. We can factor out any like terms as we did earlier. Notice that there is a relationship between b and c^2. In particular,

$$\frac{b}{c^2} = \frac{1}{2}.$$

Believe it or not, that is exactly what we wanted!

$$
\begin{aligned}
e^{-ab/c^2} &= e^{-a/2} \\
&= \sqrt{\frac{K}{S}} e^{-(r-\delta)(T-t)/2} \\
e^{ab/c^2} &= e^{a/2} \\
&= \sqrt{\frac{S}{K}} e^{(r-\delta)(T-t)/2}
\end{aligned}
$$

Now, we can show that the parenthetical statement from earlier is actually 0!

$$
\begin{aligned}
\left(Se^{-\delta(T-t)}e^{-d_1^2/2} - Ke^{-r(T-t)}e^{-d_2^2/2}\right) =& e^{-(a^2+b^2)/2c^2} \left(Se^{-\delta(T-t)} \sqrt{\frac{K}{S}} e^{-(r-\delta)(T-t)/2} \right. \\
& \left. - Ke^{-r(T-t)} \sqrt{\frac{S}{K}} e^{(r-\delta)(T-t)/2} \right) \\
=& e^{-(a^2+b^2)/2c^2} (\sqrt{SK}e^{-(r+\delta)/2} - \sqrt{SK}e^{-(r+\delta)/2}) \\
=& 0
\end{aligned}
$$

That really is remarkable and unexpected. We have shown how to compute the value of Δ_c. We could go through all of this again to compute the delta value of the corresponding put, but that is really too much. Let's use put-call parity and the additivity of the derivative to determine Δ_p.

$$\begin{aligned}
c - p &= Se^{-\delta(T-t)} - Ke^{-r(T-t)} \\
\Delta_c - \Delta_p &= \frac{\mathrm{d}}{\mathrm{d}S}\left[Se^{-\delta(T-t)} - Ke^{-r(T-t)}\right] \\
&= e^{-\delta(T-t)} \\
\Delta_c - e^{-\delta(T-t)} &= \Delta_p \\
e^{-\delta(T-t)}\mathcal{N}(d_1) - e^{-\delta(T-t)} &= \Delta_p \\
-e^{-\delta(T-t)}\mathcal{N}(-d_1) &= \Delta_p
\end{aligned}$$

That was a lot easier! Let's go a bit further and differentiate the equation again.

$$\begin{aligned}
c - p &= Se^{-\delta(T-t)} - Ke^{-r(T-t)} \\
\Delta_c - \Delta_p &= \frac{\mathrm{d}}{\mathrm{d}S}\left[Se^{-\delta(T-t)} - Ke^{-r(T-t)}\right] \\
&= e^{-\delta(T-t)} \\
\Gamma_c - \Gamma_p &= \frac{\mathrm{d}}{\mathrm{d}S}e^{-\delta(T-t)} \\
&= 0 \\
\Gamma_c &= \Gamma_p
\end{aligned}$$

Let's go ahead and determine what this value is. We can do this rather easily after all the work we have put in.

$$\begin{aligned}
\Gamma_c &= \frac{\mathrm{d}}{\mathrm{d}S}e^{-\delta(T-t)}\mathcal{N}(d_1) \\
&= e^{-\delta(T-t)}\frac{1}{\sqrt{2\pi}}e^{-d_1^2/2}\frac{1}{S\sigma\sqrt{T-t}} \\
&= e^{-\delta(T-t)}\frac{e^{-d_1^2/2}}{S\sigma\sqrt{2\pi(T-t)}}
\end{aligned}$$

Let's package up all the work we have done into a theorem.

Theorem 9 (Delta, Gamma, and Theta)**.** *For a European call and put options, we have the following formula for their delta and gamma values.*

(a) $\Delta_c = e^{-\delta(T-t)}\mathcal{N}(d_1)$

(b) $\Delta_p = -e^{-\delta(T-t)}\mathcal{N}(-d_1)$

(c) $\Gamma_c = e^{-\delta(T-t)}\dfrac{e^{-d_1^2/2}}{S\sigma\sqrt{2\pi(T-t)}} = \Gamma_p$

(d) $\Theta_c = rc + (\delta - r)S\Delta_c - \dfrac{\sigma^2 S^2}{2}\Gamma_c$

(e) $\Theta_p = \Theta_c + rKe^{-r(T-t)} - \delta S e^{-\delta(T-t)}$

Notice that I didn't actually prove the last two formulae. That will be done at in a moment. The formula for Θ_p follows from the formula for Θ_c and the use of put-call parity. I will leave that formulation to you. It is much easier than the development of Θ_c.

You will probably find writing t in each of the formula just as annoying as I do. Just remember, we will almost always be computing delta and gamma when $t = 0$. There are some important observations that need to be made.

- $0 < \Delta_c < 1$,
- $-1 < \Delta_p < 0$, and
- Γ_c and Γ_p are positive.

These almost all follow from properties of exponential functions. The caveat here is that $\delta \geq 0$. I've never heard of negative dividends, and I think it would be in poor taste for a company to say that owners must pay the company dividends! In strange times, some governments have resorted to negative interest rates. Switzerland and Japan in 2020 are two such countries. In a currency problem, it is possible to have some of the above inequalities violated. In normal conditions, this shouldn't happen.

Now let's prove the formula for Θ_c. Be warned that I swore never to do this when I was a student of financial economics. It was only when I was writing this section that I decided to really sit down and do it myself.

In the proof, we will use the values of a, b, and c from earlier in this section. Let's begin.

Proof Remember our objective

$$\Theta_c = rc + (\delta - r)S\Delta_c - \frac{\sigma^2 S^2}{2}\Gamma_c.$$

We start by differentiation of the Black-Scholes formula for the price of a call option.

$$
\begin{aligned}
c &= Se^{-\delta(T-t)}\mathcal{N}(d_1) - Ke^{-r(T-t)}\mathcal{N}(d_2) \\
\Theta_c &= \frac{\mathrm{d}}{\mathrm{d}t}\left[Se^{-\delta(T-t)}\mathcal{N}(d_1) - Ke^{-r(T-t)}\mathcal{N}(d_2)\right] \\
&= \delta Se^{-\delta(T-t)}\mathcal{N}(d_1) + Se^{-\delta(T-t)}\mathcal{N}'(d_1)\frac{\mathrm{d}d_1}{\mathrm{d}t} - rKe^{-r(T-t)}\mathcal{N}(d_2) - Ke^{-r(T-t)}\mathcal{N}'(d_2)\frac{\mathrm{d}d_2}{\mathrm{d}t} \\
&= \delta S\Delta_c + Se^{-\delta(T-t)}\mathcal{N}'(d_1)\frac{\mathrm{d}d_1}{\mathrm{d}t} - rKe^{-r(T-t)}\mathcal{N}(d_2) - Ke^{-r(T-t)}\mathcal{N}'(d_2)\frac{\mathrm{d}d_2}{\mathrm{d}t} \\
&= (\delta - r)S\Delta_c + rS\Delta_c + Se^{-\delta(T-t)}\mathcal{N}'(d_1)\frac{\mathrm{d}d_1}{\mathrm{d}t} - rKe^{-r(T-t)}\mathcal{N}(d_2) - Ke^{-r(T-t)}\mathcal{N}'(d_2)\frac{\mathrm{d}d_2}{\mathrm{d}t} \\
&= (\delta - r)S\Delta_c + rS\Delta_c - rKe^{-r(T-t)}\mathcal{N}(d_2) + Se^{-\delta(T-t)}\mathcal{N}'(d_1)\frac{\mathrm{d}d_1}{\mathrm{d}t} - Ke^{-r(T-t)}\mathcal{N}'(d_2)\frac{\mathrm{d}d_2}{\mathrm{d}t} \\
&= (\delta - r)S\Delta_c + rc + Se^{-\delta(T-t)}\mathcal{N}'(d_1)\frac{\mathrm{d}d_1}{\mathrm{d}t} - Ke^{-r(T-t)}\mathcal{N}'(d_2)\frac{\mathrm{d}d_2}{\mathrm{d}t}
\end{aligned}
$$

We are almost done! We have two out of the three terms from the claimed formula (just in a different order). It's hard to believe that the remaining terms all combine into one nice compact form, but they do. It is here that we use a, b and c. Let's focus on the remaining piece.

$$
\begin{aligned}
Se^{-\delta(T-t)}\mathcal{N}'(d_1)\frac{\mathrm{d}d_1}{\mathrm{d}t} - Ke^{-r(T-t)}\mathcal{N}'(d_2)\frac{\mathrm{d}d_2}{\mathrm{d}t} &= Se^{-\delta(T-t)}\frac{e^{-d_1^2/2}}{\sqrt{2\pi}}\frac{\mathrm{d}d_1}{\mathrm{d}t} - Ke^{-r(T-t)}e^{-d_2^2/2}\sqrt{2\pi}\frac{\mathrm{d}d_2}{\mathrm{d}t} && (4) \\
&= \frac{e^{-d_1^2/2}}{\sqrt{2\pi}}\left[Se^{-\delta(T-t)}\frac{\mathrm{d}d_1}{\mathrm{d}t} - Ke^{-r(T-t)}e^{4ab/2c^2}\frac{\mathrm{d}d_2}{\mathrm{d}t}\right] && (5)
\end{aligned}
$$

This last factorization works since we have

$$
\begin{aligned}
d_1^2 &= \frac{a^2 + 2ab + b^2}{c^2} \\
d_2^2 &= \frac{a^2 - 2ab + b^2}{c^2} \\
e^{-d_2^2/2} &= e^{-d_1^2/2}e^{4ab/2c^2}
\end{aligned}
$$

Recall that $b/c^2 = 1/2$. We continue from line (2) above.

$$
\begin{aligned}
&\frac{e^{-d_1^2/2}}{\sqrt{2\pi}}\left[Se^{-\delta(T-t)}\frac{\mathrm{d}d_1}{\mathrm{d}t}\right.\\
&\left.-Ke^{-r(T-t)}e^{4ab/2c^2}\frac{\mathrm{d}d_2}{\mathrm{d}t}\right] = \frac{e^{-d_1^2/2}}{\sqrt{2\pi}}\left[Se^{-\delta(T-t)}\frac{\mathrm{d}d_1}{\mathrm{d}t} - Ke^{-r(T-t)}e^{a}\frac{\mathrm{d}d_2}{\mathrm{d}t}\right]\\
&= \frac{e^{-d_1^2/2}}{\sqrt{2\pi}}\left[Se^{-\delta(T-t)}\frac{\mathrm{d}d_1}{\mathrm{d}t} - Ke^{-r(T-t)}e^{\ln\frac{S}{K}+(r-\delta)(T-t)}\frac{\mathrm{d}d_2}{\mathrm{d}t}\right]\\
&= \frac{e^{-d_1^2/2}}{\sqrt{2\pi}}\left[Se^{-\delta(T-t)}\frac{\mathrm{d}d_1}{\mathrm{d}t} - Se^{-\delta(T-t)}\frac{\mathrm{d}d_2}{\mathrm{d}t}\right]\\
&= \frac{e^{-d_1^2/2}}{\sqrt{2\pi}}Se^{-\delta(T-t)}\frac{\mathrm{d}}{\mathrm{d}t}\left[d_1-d_2\right]\\
&= \frac{e^{-d_1^2/2}}{\sqrt{2\pi}}Se^{-\delta(T-t)}\frac{\mathrm{d}}{\mathrm{d}t}[\sigma\sqrt{T-t}]\\
&= \frac{e^{-d_1^2/2}}{\sqrt{2\pi}}Se^{-\delta(T-t)}\left(-\frac{\sigma}{2\sqrt{T-t}}\right)\\
&= -e^{-\delta(T-t)}\frac{e^{-d_1^2/2}}{S\sigma\sqrt{2\pi(T-t)}}\frac{\sigma^2S^2}{2}\\
&= -\Gamma_c\frac{\sigma^2S^2}{2}
\end{aligned}
$$

We have shown the last of the three parts, and the proof is complete. ■

I know what you're thinking. Do I have to know this? You won't see anything like this on an exam. I just wanted to show you how to properly compute Θ. It is so cumbersome because t appears in every expression in our formula, so there is a chain rule or product rule at every step along the way. Fortunately, our auxillary variables saved the day.

The reason I wrote the formula the way I did was that it is much more reminiscent of the way you might do it if you knew the Black-Scholes Equation. That equation states the following:

$$
rD = \Theta_D + (r-\delta)S\Delta_D + \frac{\sigma^2S^2}{2}\Gamma_D
$$

for any derivative, D. The proof of this equation is not within the scope of our studies. Some details may be found here. Unfortunately, some background reading may be necessary to fully understand what the author is doing.

In the last section, we examined a portfolio that had 100 written calls. Let's examine how delta, gamma, and theta evolve for that call here. I will give a table for each Greek value. Since we aren't as worried with initial conditions, I will allow S to vary at time 0 (this was not allowed for the profit table).

Call Price	Time 0	Time 1/48	Time 2/48	Time 3/48
64	1.21	0.89	0.55	0.20
66	1.94	1.56	1.13	0.60
68	2.90	2.49	2.02	1.41
70	4.09	3.69	3.23	2.67
72	5.48	5.12	4.73	4.30
74	7.06	6.75	6.43	6.15

Delta	Time 0	Time 1/48	Time 2/48	Time 3/48
64	0.3078	0.2717	0.2181	0.1250
66	0.4215	0.3988	0.3638	0.2941
68	0.5385	0.5335	0.5275	0.5196
70	0.6490	0.6607	0.6823	0.7332
72	0.7453	0.7689	0.8070	0.8807
74	0.8235	0.8522	0.8940	0.9570

Gamma	Time 0	Time 1/48	Time 2/48	Time 3/48
64	0.0543	0.0592	0.0644	0.0637
66	0.0585	0.0668	0.0796	0.1033
68	0.0576	0.0667	0.0818	0.1159
70	0.0522	0.0596	0.0711	0.0928
72	0.0438	0.0481	0.0530	0.0545
74	0.0343	0.0353	0.0343	0.0241

Theta	Time 0	Time 1/48	Time 2/48	Time 3/48
64	−14.70	−15.82	−16.94	−16.44
66	−17.10	−19.26	−22.58	−28.67
68	−18.25	−20.84	−25.14	−34.82
70	−18.02	−20.33	−23.91	−30.69
72	−16.63	−18.13	−19.89	−20.72
74	−14.53	−15.02	−14.89	−11.78

Notice that the call values decrease as you move from left to right. This is a result of the negative values of theta. This is what is called theta decay. Also, notice that the call values increase as you move down the table. This is because the delta value is positive. One final thing, as you move down the table, the increase in call value is also increasing! This is the result of a positive gamma value.

The theta table may suggest something that is not true: theta values are always negative. The reality is that they are often negative. To see that they can be positive, we can check by example.

Example 46. *Let* $S = 50$*,* $r = 0.07$*,* $\delta = 0.03$*,* $\sigma = 0.35$*. Suppose that you have a three month, strike 10 European call option. Then* $\Theta_c = 0.80$*.*

Let $S = 50$*,* $r = 0.07$*,* $\delta = 0.03$*,* $\sigma = 0.35$*. Suppose that you have a three month, strike 70 European put option. Then* $\Theta_p = 1.84$*.*

There was nothing special about the values I chose in the example. The only part that is significant is that the options are deeply in-the-money. That is, if you could exercise the option today, the payoff would be large.

We will revisit many of these concepts in our section on approximation. Before that we will take a slightly different perspective: percentage change.

1.29 Elasticity

We explore the economic concept of elasticity and how it applies to our call and put options. This can be viewed as a first order estimate in changes to portfolio values.

In our last section, we derived the values of delta, gamma, and theta for our call and put options. Delta gives a way of determining the approximate change in the derivative's price relative to the change in the underlying asset. We will see more of that when we cover approximations. Before that, though, we can take another perspective: what is the percentage change in our derivatives relative to percentage changes in the price of the underlying asset? In words, this is volatility. Mathematically, we have the definition below.

Definition 21. *The* **elasticity** *of a portfolio is denoted Ω_P. It is given by the following:*

$$\Omega_P = \frac{\Delta_P}{P} \cdot S$$

Perhaps you are not convinced that the english above translates to the term from the definition. Let's explore why this holds.

$$\begin{aligned}
\Omega_P &= \frac{\%\text{change in portfolio}}{\%\text{change in } S} \\
&= \frac{\text{change in portfolio/original portfolio value}}{\text{change in } S\text{/original value of } S} \\
&= \frac{\text{change in portfolio}}{\text{change in } S} \cdot \frac{S}{P} \\
&= \Delta_P \cdot \frac{S}{P}
\end{aligned}$$

The last step is the conversion from the change over a period of time to that of an instantaneous change as we would see in calculus 1.

Before we see some examples of elasticity, let's work out a couple of properties. Let D_1, D_2, ... and D_n be n derivatives in a portfolio with quantities a_1, a_2, ... and a_n, respectively. Then the value of the portfolio will be

$$P = \sum_{j=1}^{n} a_j D_j.$$

From this and properties of the derivative, we can easily deduce the delta, gamma, and theta values of the portfolio:

$$
\begin{aligned}
\Delta_P &= \sum_{j=1}^{n} a_j \Delta_j, \\
\Gamma_P &= \sum_{j=1}^{n} a_j \Gamma_j, \text{ and} \\
\Theta_P &= \sum_{j=1}^{n} a_j \Theta_j.
\end{aligned}
$$

Unfortunately, the same does not hold for elasticity. Recall from the definition, we have that the elasticity of the portfolio has the delta value in the numerator and the whole portfolio in the denominator. We will assume that the portfolio's value is not 0 for our argument.

$$
\begin{aligned}
\Omega_P &= \frac{\Delta_P}{P} \cdot S \\
&= \frac{\sum_{j=1}^{n} a_j \Delta_j}{P} \cdot S \\
&= \left[\sum_{j=1}^{n} \frac{a_j \Delta_j}{P} \right] \cdot S \\
&= \left[\sum_{j=1}^{n} \frac{a_j \Delta_j}{a_j D_j} \cdot \frac{a_j D_j}{P} \right] \cdot S \\
&= \left[\sum_{j=1}^{n} \frac{\Delta_j}{D_j} \cdot S \cdot \frac{a_j D_j}{P} \right] \\
&= \sum_{j=1}^{n} a_j \Omega_j \cdot \frac{D_j}{P} \\
&\neq \sum_{j=1}^{n} a_j \Omega_j
\end{aligned}
$$

While the argument above demonstrates the lack of additivity of elasticity, I feel that the result may stick a little better if we had a concrete example.

Example 47. *Let $S = 60$, $\delta = 0.03$, $\sigma = 0.22$, $r = 0.07$. A portfolio consists of a call and a put with identical terms. They are both European, expire in two months, and they have strike 58. Verify that the portfolio's elasticity is not the sum of the two elasticities.*

Solution: We need the values c, p, Δ_c, and Δ_p. From there, we can compute everything necessary. I will spare the details of all the computations and just give the necessary values.

$$\begin{aligned}
c &= 3.49 \\
p &= 1.12 \\
\Delta_c &= 0.6868 \\
\Delta_p &= -0.3082 \\
\Omega_c &= \frac{0.6868}{3.49} \cdot 60 \\
&= 11.81 \\
\Omega_p &= \frac{-0.3082}{1.12} \cdot 60 \\
&= -16.51
\end{aligned}$$

We have the preliminary values. Now, we will determine the elasticity of the portfolio.

$$\begin{aligned}
P &= c + p \\
&= 4.61 \\
\Delta_P &= \Delta_c + \Delta_P \\
&= 0.3786 \\
\Omega_P &= \frac{0.3786}{4.61} \cdot 60 \\
&= 4.93 \\
&\neq 11.81 - 16.51
\end{aligned}$$

We can use this example a little more. That will help in our interpretation of elasticity. Suppose that the asset rose in value by 3. That should have an effect on the call and the put from our example. Since elasticity is a quotient of percentages, we can estimate the change of the call and the put.

The change in the asset's value (as a percent) is $3/60 = 5$. We would estimate that the change in the call's value should be

$$5 \cdot \Omega_c = 59.05$$

percent. In cash value, this would be a change from 3.49 to $1.5905 \cdot 3.49 = 5.55$. Using the Black-Scholes value, we should have 5.81. The estimate from volatility isn't too bad! It is a little lacking since we aren't using higher order derivatives. For completeness, let's do the same computation for the put.

$$5 \cdot \Omega_p = -82.55$$

The put's estimated value is $0.1745 \cdot 1.12 = 0.20$. Again, the actual Black-Scholes value is 0.45.

Let's conclude with some properties of of Ω_c and Ω_p.

Theorem 10. *For European calls and puts, we always have the following:*

(a) $\Omega_c > 1$

(b) $\Omega_p < 0$

Proof We begin with the elasticity of a call.

$$\begin{aligned}
\Omega_c &= \frac{\Delta_c}{c} S \\
&= \frac{S\Delta_c}{c} \\
&> \frac{S\Delta_c - Ke^{-rT}\mathcal{N}(d_2)}{c} \\
&= \frac{c}{c} \\
&= 1
\end{aligned}$$

The same argument does not readily apply to a put option. The result follows since $\Delta_p < 0$. ∎

It may seem like there should be a better bound for the elasticity of a put option; however, the elasticity of a put can be very close to 0. By taking our example earlier and changing the value of the asset to a number very close to zero gives an elasticity that is also close to zero. The theorem is as good as we can get. Also, taking S to be very large gives a call elasticity that is close to one.

Since we did some approximation in this section, it is only fitting that we explore approximation to a greater extent. Our approximation theorems are the subject of our next section.

1.30 Approximation

In the first section of this chapter, it was clear that it can be difficult to compute the value of a portfolio over time. This section covers approximations that will simplify our work. These theorems are considered second order estimates.

In our last section, we used elasticity to estimate changes. In reality, this was just a first order approximation that you might have done in calculus. In the estimate for the change in the call's value, we had the following:

$$(1 + \text{percent change in } S/100 \cdot \Omega_c)c = \text{approximate value of } c$$

We can manipulate the left hand side to see that this is just a linear approximation. Just to be clear, a linear approximation usually looks like

$$y = f(a) + f'(a)(x - a).$$

Let's make the necessary manipulations.

$$\begin{aligned}
(1 + \text{percent change in } S/100 \cdot \Omega_c)c &= c + \text{percent change in } S/100 \cdot \Omega_c \cdot c \\
&= c + \text{percent change in } S/100 \cdot \Delta_c \cdot S \\
&= c + \frac{\text{change in value of } S}{S} \cdot \Delta_c \cdot S \\
&= c + \text{change in value of } S \cdot \Delta_c \\
&= c + \Delta_c \epsilon \\
&= c(S) + \frac{\mathrm{d}c}{\mathrm{d}S}\epsilon
\end{aligned}$$

In our current notation, c represents the value $f(a)$, Δ_c represents the first derivative of our option with respect to S, and ϵ is the difference of the new value of S from the old value of S. I would use ΔS, but I think that notation is really confusing in our context!

Now that we have established that the content of the previous section was just a first order approximation, we may as well go to second order approximations.

Theorem 11 (Delta-Gamma Approximation)**.** *Suppose that we are given a derivative D with underlying asset S. Then we can approximate the derivative's value given an immediate change in the asset's value of ϵ using the formula*

$$D(S + \epsilon) \approx D(S) + \Delta_D \epsilon + \Gamma_D \frac{\epsilon^2}{2}.$$

Normally, a statement like this would be supported with a proof. Fortunately, that isn't necessary here. This is just a second order Taylor approximation from Calculus using the notation of financial mathematics. Let's see how this formula applies to things we have already computed! We had the following table in an earlier section. Recall that this was a specific portfolio consisting of 100 written calls, 60 shares of the asset, and a debt of 3000. The conditions were that $S(0) = 70$, $\delta = 0.03$, $\sigma = 0.35$, and $r = 0.09$. The call was European, had strike 68, and it expired in one month.

Asset Value	Time 0	Time 1/48	Time 2/48	Time 3/48
64	718.95	747.47	778.38	810.65
66	766.17	800.75	840.85	890.31
68	790.14	827.59	871.95	929.59
70	791.22	828.01	870.76	923.75
72	771.50	804.73	841.37	881.30
74	734.30	762.27	790.79	816.73

To use this particular form of approximation, we can only use the entry corresponding to time 0 and an asset value of 70 as our base point. We also need Δ and Γ of our portfolio. Those aren't too hard to compute! I will just give the values for the call, and I will use them in my computation of the portfolio's Greek values.

$$\begin{aligned}
\Delta_c &= 0.6490 \\
\Gamma_c &= 0.0522 \\
\Delta_P &= -100\Delta_c + 60 \\
&= -4.9 \\
\Gamma_P &= -100\Gamma_c + 0 \\
&= -5.22
\end{aligned}$$

Some of those computations may not be clear. The justification is this: $\Delta_S = 1$ and $\Gamma_S = 0$. That is because we are differentiating S with respect to itself! Let's approximate the first column from our table above using the base point $P(70) = 791.22$.

$$
\begin{aligned}
P(64) &\approx P(70) + \Delta_P \cdot (-6) + \Gamma_P \cdot \frac{(-6)^2}{2} \\
&= 726.66 \\
P(66) &\approx P(70) + \Delta_P \cdot (-4) + \Gamma_P \cdot \frac{(-4)^2}{2} \\
&= 769.06 \\
P(68) &\approx P(70) + \Delta_P \cdot (-2) + \Gamma_P \cdot \frac{(-2)^2}{2} \\
&= 790.58 \\
P(72) &\approx P(70) + \Delta_P \cdot 2 + \Gamma_P \cdot \frac{2^2}{2} \\
&= 770.98 \\
P(74) &\approx P(70) + \Delta_P \cdot 4 + \Gamma_P \cdot \frac{4^2}{2} \\
&= 729.86
\end{aligned}
$$

Let's put all this into a table so we can better see the two together.

Asset Value	Portfolio Value	Portfolio Approximation
64	718.95	726.66
66	766.17	769.06
68	790.14	790.58
70	791.22	791.22
72	771.50	770.98
74	734.30	729.86

The important thing to notice is that the approximation is better when small changes are made and worse when large changes are made. There should be something nagging you now. This theorem seems rather limited for two reasons: the first is that the values could be better in our approximation, and the second should have to do with time. Usually time passes between changes in stock prices. We should have a way to account for this. That is the topic of our next theorem.

Theorem 12 (Delta-Gamma-Theta Approximation)**.** *Suppose that we are given a derivative D with underlying asset S. Then we can approximate the derivative's value given a change in the asset's value of ϵ and change in time h using the formula*

$$D(S+\epsilon, t+h) \approx D(S,t) + \Delta_D \epsilon + \Gamma_D \frac{\epsilon^2}{2} + \Theta_D h.$$

Many sources will attribute this approximation to some sort of multidimensional Taylor's theorem; however, that is glossing over some important details. For example, what happens to the higher order terms for time and what about the mixed terms? We will not do anything like that. This approximation theorem is a result of the Itô-Doeblin formula found here and written below.

$$\mathrm{d}f(t, \mathcal{Z}(t)) = f_t(S(t), t)\mathrm{d}t + f_S(S(t), t)\mathrm{d}S + \frac{1}{2} f_{SS}(S(t), t)[\mathrm{d}S]^2$$

The equality above is giving us the change of the function. In our context, the function is the price of a derivative. We are saying that the new value is approximately equal to the old value plus the differential. The reason the higher order time terms and the cross terms cancel out is because of some results regarding quadratic variation. We won't dive into that here. Let's use the approximation theorem to derive approximations of our table from earlier. The first row of computations will be displayed below, and the rest will be filled into the table.

$$\begin{aligned}
P(64, 1/48) &\approx P(70) + \Delta_P \cdot (-6) + \Gamma_P \cdot \frac{(-6)^2}{2} + \Theta_P \cdot 1/48 \\
&= 764.20 \\
P(64, 2/48) &\approx P(70) + \Delta_P \cdot (-6) + \Gamma_P \cdot \frac{(-6)^2}{2} + \Theta_P \cdot 2/48 \\
&= 801.74 \\
P(64, 3/48) &\approx P(70) + \Delta_P \cdot (-6) + \Gamma_P \cdot \frac{(-6)^2}{2} + \Theta_P \cdot 3/48 \\
&= 839.29
\end{aligned}$$

Let's add all the values to a table with the true portfolio values juxtaposed with their approximations.

Asset Value	Time 0	Approx.	Time 1/48	Approx.
64	718.95	726.66	747.47	764.20
66	766.17	769.06	800.75	806.60
68	790.14	790.58	827.59	828.12
70	791.22	791.22	828.01	828.76
72	771.50	770.98	804.73	808.52
74	734.30	729.86	762.27	767.40

Asset Value	Time 2/48	Approx.	Time 3/48	Approx.
64	778.38	801.74	810.65	839.29
66	840.85	844.14	890.31	881.69
68	871.95	865.66	929.59	903.21
70	870.76	866.30	923.75	903.85
72	841.37	846.06	881.30	883.61
74	790.79	804.94	816.73	842.49

As we can see, this process is not too complicated. Our initial portfolio value and three different Greeks provided us with everything in the table. The time zero approximations come from Delta-Gamma approximation since $h = 0$. As you can tell, some of the best approximations come from

This concludes our section covering approximation. In the next section, we will discuss hedging strategies.

1.31 Elasticity

We explore the economic concept of elasticity and how it applies to our call and put options. This can be viewed as a first order estimate in changes to portfolio values.

In our last section, we derived the values of delta, gamma, and theta for our call and put options. Delta gives a way of determining the approximate change in the derivative's price relative to the change in the underlying asset. We will see more of that when we cover approximations. Before that, though, we can take another perspective: what is the percentage change in our derivatives relative to percentage changes in the price of the underlying asset? In words, this is volatility. Mathematically, we have the definition below.

Definition 22. *The* **elasticity** *of a portfolio is denoted Ω_P. It is given by the following:*

$$\Omega_P = \frac{\Delta_P}{P} \cdot S$$

Perhaps you are not convinced that the english above translates to the term from the definition. Let's explore why this holds.

$$\begin{aligned}
\Omega_P &= \frac{\%\text{change in portfolio}}{\%\text{change in } S} \\
&= \frac{\text{change in portfolio/original portfolio value}}{\text{change in } S/\text{original value of } S} \\
&= \frac{\text{change in portfolio}}{\text{change in } S} \cdot \frac{S}{P} \\
&= \Delta_P \cdot \frac{S}{P}
\end{aligned}$$

The last step is the conversion from the change over a period of time to that of an instantaneous change as we would see in calculus 1.

Before we see some examples of elasticity, let's work out a couple of properties. Let D_1, D_2, ... and D_n be n derivatives in a portfolio with quantities a_1, a_2, ... and a_n, respectively. Then the value of the portfolio will be

$$P = \sum_{j=1}^{n} a_j D_j.$$

From this and properties of the derivative, we can easily deduce the delta, gamma, and theta values of the portfolio:

$$\Delta_P = \sum_{j=1}^{n} a_j \Delta_j,$$
$$\Gamma_P = \sum_{j=1}^{n} a_j \Gamma_j, \text{ and}$$
$$\Theta_P = \sum_{j=1}^{n} a_j \Theta_j.$$

Unfortunately, the same does not hold for elasticity. Recall from the definition, we have that the elasticity of the portfolio has the delta value in the numerator and the whole portfolio in the denominator. We will assume that the portfolio's value is not 0 for our argument.

$$\begin{aligned}
\Omega_P &= \frac{\Delta_P}{P} \cdot S \\
&= \frac{\sum_{j=1}^{n} a_j \Delta_j}{P} \cdot S \\
&= \left[\sum_{j=1}^{n} \frac{a_j \Delta_j}{P} \right] \cdot S \\
&= \left[\sum_{j=1}^{n} \frac{a_j \Delta_j}{a_j D_j} \cdot \frac{a_j D_j}{P} \right] \cdot S \\
&= \left[\sum_{j=1}^{n} \frac{\Delta_j}{D_j} \cdot S \cdot \frac{a_j D_j}{P} \right] \\
&= \sum_{j=1}^{n} a_j \Omega_j \cdot \frac{D_j}{P} \\
&\neq \sum_{j=1}^{n} a_j \Omega_j
\end{aligned}$$

While the argument above demonstrates the lack of additivity of elasticity, I feel that the result may stick a little better if we had a concrete example.

Example 48. *Let $S = 60$, $\delta = 0.03$, $\sigma = 0.22$, $r = 0.07$. A portfolio consists of a call and a put with identical terms. They are both European, expire in two months, and they have strike 58. Verify that the portfolio's elasticity is not the sum of the two elasticities.*

Solution: We need the values c, p, Δ_c, and Δ_p. From there, we can compute everything necessary. I will spare the details of all the computations and just give the necessary values.

$$\begin{aligned} c &= 3.49 \\ p &= 1.12 \\ \Delta_c &= 0.6868 \\ \Delta_p &= -0.3082 \\ \Omega_c &= \frac{0.6868}{3.49} \cdot 60 \\ &= 11.81 \\ \Omega_p &= \frac{-0.3082}{1.12} \cdot 60 \\ &= -16.51 \end{aligned}$$

We have the preliminary values. Now, we will determine the elasticity of the portfolio.

$$\begin{aligned} P &= c + p \\ &= 4.61 \\ \Delta_P &= \Delta_c + \Delta_P \\ &= 0.3786 \\ \Omega_P &= \frac{0.3786}{4.61} \cdot 60 \\ &= 4.93 \\ &\neq 11.81 - 16.51 \end{aligned}$$

We can use this example a little more. That will help in our interpretation of elasticity. Suppose that the asset rose in value by 3. That should have an effect on the call and the put from our example. Since elasticity is a quotient of percentages, we can estimate the change of the call and the put.

The change in the asset's value (as a percent) is $3/60 = 5$. We would estimate that the change in the call's value should be

$$5 \cdot \Omega_c = 59.05$$

percent. In cash value, this would be a change from 3.49 to $1.5905 \cdot 3.49 = 5.55$. Using the Black-Scholes value, we should have 5.81. The estimate from volatility isn't too bad! It is a little lacking since we aren't using higher order derivatives. For completeness, let's do the same computation for the put.

$$5 \cdot \Omega_p = -82.55$$

The put's estimated value is $0.1745 \cdot 1.12 = 0.20$. Again, the actual Black-Scholes value is 0.45.

Let's conclude with some properties of of Ω_c and Ω_p.

Theorem 13. *For European calls and puts, we always have the following:*

(a) $\Omega_c > 1$

(b) $\Omega_p < 0$

Proof We begin with the elasticity of a call.

$$\begin{aligned}
\Omega_c &= \frac{\Delta_c}{c} S \\
&= \frac{S\Delta_c}{c} \\
&> \frac{S\Delta_c - Ke^{-rT}\mathcal{N}(d_2)}{c} \\
&= \frac{c}{c} \\
&= 1
\end{aligned}$$

The same argument does not readily apply to a put option. The result follows since $\Delta_p < 0$. ■

It may seem like there should be a better bound for the elasticity of a put option; however, the elasticity of a put can be very close to 0. By taking our example earlier and changing the value of the asset to a number very close to zero gives an elasticity that is also close to zero. The theorem is as good as we can get. Also, taking S to be very large gives a call elasticity that is close to one.

Since we did some approximation in this section, it is only fitting that we explore approximation to a greater extent. Our approximation theorems are the subject of our next section.