PREPARED FOR SUBMISSION TO JHEP

Arbitrary Dimensions, Massive, Non-normalizable Time-Dependent BCs

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1 Introduction

2 Perturbative Expansion

The backreaction between the metric and the scalar field appears at second order in the perturbation,

$$A_2' = -\mu\nu \left[(\dot{\phi}_1)^2 + (\phi_1')^2 + m^2\phi_1^2 \sec^2 x \right] + \nu' A_2/\nu$$
 (2.1)

which can be directly integrated to give

$$A_2 = -\nu \int_0^x dy \,\mu \left((\dot{\phi}_1)^2 + (\phi_1')^2 + m^2 \phi_1^2 \sec^2 x \right) \,. \tag{2.2}$$

Furthermore, the first non-trivial contribution to the lapse in the boundary time gauge is

$$\delta_2 = \int_{\tau}^{\pi/2} dy \,\mu\nu \left((\dot{\phi}_1)^2 + (\phi_1')^2 \right) \,. \tag{2.3}$$

For convenience, we have also defined the functions

$$\mu(x) = (\tan x)^{d-1}$$
 and $\nu(x) = (d-1)/\mu'$. (2.4)

To aide in evaluating integrals, we first derive the following identities: from the equation for the first-order time-dependent coefficients c_i ,

$$\ddot{c}_i + \omega_i^2 c_i = 0 \quad \Rightarrow \quad \partial_t \left(\dot{c}_i^2 + \omega_i^2 c_i^2 \right) = \partial_t \mathbb{C}_i = 0 \,; \tag{2.5}$$

from the equation definition of \hat{L} ,

$$\hat{L}e_{j} = -\frac{1}{\mu} (\mu e'_{j})' + m^{2} \sec^{2} x e_{j} \quad \Rightarrow \quad (\mu e'_{j})' = \mu (m^{2} \sec^{2} x - \omega_{j}^{2}) e_{j};$$
 (2.6)

from considering the expression $(\mu e_i' e_j)'$:

$$(\mu e_i' e_j)' = (m^2 \sec^2 x - \omega_i^2) \,\mu e_i e_j + \mu e_i' e_j'; \tag{2.7}$$

from permuting i, j above and subtracting to give

$$\frac{\left[\mu(e_i'e_j\omega_j^2 - e_ie_j'\omega_i^2)\right]'}{(\omega_i^2 - \omega_i^2)} = \mu m^2 \sec^2 x e_i e_j + \mu e_i' e_j'.$$
 (2.8)

3 $\mathcal{O}(\epsilon^3)$ Source Terms

At third order in ϵ , the equation for ϕ_3 contains a source S given by

$$\ddot{\phi}_3 + \hat{L}\phi_3 = S = 2(A_2 - \delta_2)\ddot{\phi}_1 + (\dot{A}_2 - \dot{\delta}_2)\dot{\phi}_1 + (A'_2 - \delta'_2)\phi'_1 + m^2 A_2 \phi_1 \sec^2 x \tag{3.1}$$

Projecting each of the terms individually onto the eigenbasis $\{e_{\ell}\}$:

$$\langle \delta_{2} \ddot{\phi}_{1}, e_{\ell} \rangle = -\sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \frac{\omega_{k}^{2} c_{k}}{\omega_{\ell}^{2} - \omega_{k}^{2}} \left[\dot{c}_{i} \dot{c}_{j} \left(X_{k\ell ij} - X_{\ell kij} \right) + c_{i} c_{j} \left(Y_{ij\ell k} - Y_{ijk\ell} \right) \right]$$

$$-\sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \frac{\omega_{\ell}^{2} c_{\ell}}{\omega_{\ell}^{2} - \omega_{k}^{2}} \left[\dot{c}_{i} \dot{c}_{j} P_{ij\ell} + c_{i} c_{j} B_{ij\ell} \right] , \qquad (3.2)$$

$$\langle A_{2} \ddot{\phi}_{1}, e_{\ell} \rangle = 2 \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \frac{\omega_{k}^{2} c_{k}}{\omega_{j}^{2} - \omega_{i}^{2}} X_{ijk\ell} \left(\dot{c}_{i} \dot{c}_{j} + \omega_{j}^{2} c_{i} c_{j} \right)$$

$$+ \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \frac{\omega_{j}^{2} c_{j}}{\omega_{\ell}^{2} - \omega_{k}^{2}} \left[\partial_{t} \left(\dot{c}_{i} \dot{c}_{j} \right) \left(X_{k\ell ij} - X_{\ell kij} \right) + \partial_{t} \left(c_{i} c_{j} \right) \left(Y_{ij\ell k} - Y_{ijk\ell} \right) \right]$$

$$+ \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \frac{\dot{c}_{k}}{\omega_{\ell}^{2} - \omega_{k}^{2}} \left[\partial_{t} \left(\dot{c}_{i} \dot{c}_{j} \right) \left(X_{k\ell ij} - X_{\ell kij} \right) + \partial_{t} \left(c_{i} c_{j} \right) \left(Y_{ij\ell k} - Y_{ijk\ell} \right) \right]$$

$$+ \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \frac{\dot{c}_{k}}{\omega_{\ell}^{2} - \omega_{k}^{2}} \left[\partial_{t} \left(\dot{c}_{i} \dot{c}_{j} \right) \left(X_{k\ell ij} - X_{\ell kij} \right) + \partial_{t} \left(c_{i} c_{j} \right) \left(Y_{ij\ell k} - Y_{ijk\ell} \right) \right]$$

$$+ \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \sum_{k=0}^{\infty} \frac{\dot{c}_{k} \dot{c}_{i} \dot{c}_{i} \dot{c}_{i} \dot{c}_{i} \dot{c}_{i} \right]$$

$$+ \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \sum_{k=0}^{\infty} \frac{\dot{c}_{k} \dot{c}_{i} \dot{c}_{i} \dot{c}_{i} \dot{c}_{j} \right]$$

$$+ \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \sum_{k=0}^{\infty} \frac{\dot{c}_{k} \dot{c}_{i} \dot{c}_{i} \dot{c}_{j} \dot{c}_{i} \dot{c}_{j} \right]$$

$$+ \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \sum_{k=0}^{\infty} \frac{\dot{c}_{k} \dot{c}_{i} \dot{c}_{i} \dot{c}_{j} \dot{c}_{i} \dot{c}_{j} \right]$$

$$+ \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \sum_{k=0}^{\infty} \frac{\dot{c}_{k} \dot{c}_{i} \dot{c}_{i} \dot{c}_{j} \dot{c}_{i} \dot{c}_{j} \right]$$

$$+ \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \sum_{k=0}^{\infty} \sum_{k=0}^{\infty} \frac{\dot{c}_{k} \dot{c}_{i} \dot{c}_{i} \dot{c}_{j} \dot{c}_{i} \dot{c}_{j} \right]$$

$$+ \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \sum_{k=0}^{\infty} \sum_{k=0}^{\infty} \frac{\dot{c}_{k} \dot{c}_{i} \dot{c}_$$

Where the forms of X, Y, V, H, B, M, P, and Q are given by

$$X_{ijk\ell} = \int_0^{\pi/2} dx \,\mu^2 \nu e_i' e_j e_k e_\ell \tag{3.8}$$

$$Y_{ijk\ell} = \int_0^{\pi/2} dx \, \mu^2 \nu e_i' e_j' e_k e_\ell' \tag{3.9}$$

$$V_{ijk\ell} = \int_0^{\pi/2} dx \, \mu^2 \nu e_i e_j e_k' e_\ell \sec^2 x \tag{3.10}$$

$$H_{ijk\ell} = \int_0^{\pi/2} dx \, \mu^2 \nu' e_i' e_j e_k' e_\ell \tag{3.11}$$

$$B_{ij\ell} = \int_0^{\pi/2} dx \,\mu \nu e_i' e_j' \int_0^x dy \,\mu e_\ell^2 \tag{3.12}$$

$$M_{ij\ell} = \int_0^{\pi/2} dx \,\mu \nu' e_i' e_j \int_0^x dy \,\mu e_\ell^2$$
 (3.13)

$$P_{ij\ell} = \int_0^{\pi/2} dx \,\mu \nu e_i e_j \int_0^x dy \,\mu e_\ell^2 \tag{3.14}$$

$$Q_{ij\ell} = \int_0^{\pi/2} dx \,\mu \nu e_i e_j \sec^2 x \int_0^x dy \,\mu e_\ell^2 \tag{3.15}$$

Collecting terms together gives the expression for $S_{\ell} = \langle S, e_{\ell} \rangle$:

$$S_{\ell} = \sum_{\substack{i,j,k \\ k \neq \ell}}^{\infty} \frac{1}{\omega_{\ell}^{2} - \omega_{k}^{2}} \Big[F_{k}(\dot{c}_{i}\dot{c}_{j}) \left(X_{k\ell ij} - X_{\ell kij} \right) + F_{k}(c_{i}c_{j}) \left(Y_{ij\ell k} - Y_{ijk\ell} \right) \Big]$$

$$+ 2 \sum_{\substack{i,j,k \\ i \neq j}}^{\infty} \frac{c_{k}D_{ij}}{\omega_{j}^{2} - \omega_{i}^{2}} \Big[2\omega_{k}^{2} X_{ijk\ell} - H_{ijk\ell} - m^{2}V_{jki\ell} \Big] - \sum_{i,j,k}^{\infty} c_{i} \Big[2\dot{c}_{j}\dot{c}_{k} X_{ijk\ell} + m^{2}c_{j}c_{k}V_{ijk\ell} \Big]$$

$$+ \sum_{i,j}^{\infty} \Big[F_{\ell}(\dot{c}_{i}\dot{c}_{j}) P_{ij\ell} + F_{\ell}(c_{i}c_{j}) B_{ij\ell} + 2\omega_{j}^{2}c_{j} \left(c_{i}^{2} X_{iij\ell} + \mathbb{C}_{i}P_{j\ell i} \right)$$

$$- c_{j} \left(c_{i}^{2} (H_{iij\ell} + m^{2}V_{jii\ell}) + \mathbb{C}_{i} (M_{j\ell i} + m^{2}Q_{j\ell i}) \right) \Big], \qquad (3.16)$$

where $F_k(z) = \dot{c}_k \dot{z} - 2\omega_k^2 c_k z$, $D_{ij} = \dot{c}_i \dot{c}_j + \omega_i^2 c_i c_j$, and $\mathbb{C}_i = \dot{c}_i^2 + \omega_i^2 c_i^2$.

Using the solution $c_i(t) = a_i \cos(\omega_i t + b_i) = a_i \cos \theta_i$, the source term becomes

$$\begin{split} S_{\ell} &= \frac{1}{4} \sum_{\substack{i,j,k \\ k \neq \ell}}^{\infty} \frac{a_i a_j a_k \omega_k}{\omega_\ell^2 - \omega_k^2} \left[Z_{ijk\ell}^-(\omega_i + \omega_j - 2\omega_k) \cos(\theta_i + \theta_j - \theta_k) - Z_{ijk\ell}^-(\omega_i + \omega_j + 2\omega_k) \cos(\theta_i + \theta_j + \theta_k) - Z_{ijk\ell}^+(\omega_i - \omega_j + 2\omega_k) \cos(\theta_i - \theta_j + \theta_k) - Z_{ijk\ell}^+(\omega_i - \omega_j - 2\omega_k) \cos(\theta_i - \theta_j - \theta_k) \right] \\ &+ \frac{1}{2} \sum_{\substack{i,j,k \\ i \neq j}}^{\infty} a_i a_j a_k \omega_j \left(H_{ijk\ell} + m^2 V_{jki\ell} - 2\omega_k^2 X_{ijk\ell} \right) \left[\frac{1}{\omega_i - \omega_j} \left(\cos(\theta_i - \theta_j - \theta_k) + \cos(\theta_i - \theta_j + \theta_k) \right) - \frac{1}{\omega_i + \omega_j} \left(\cos(\theta_i + \theta_j - \theta_k) + \cos(\theta_i + \theta_j + \theta_k) \right) \right] \\ &- \frac{1}{4} \sum_{\substack{i,j,k \\ i \neq j}}^{\infty} a_i a_j a_k \left[\left(2\omega_j \omega_k X_{ijk\ell} + m^2 V_{ijk\ell} \right) \cos(\theta_i + \theta_j - \theta_k) - \left(2\omega_j \omega_k X_{ijk\ell} - m^2 V_{ijk\ell} \right) \cos(\theta_i - \theta_j - \theta_k) \right. \\ &+ \left(2\omega_j \omega_k X_{ijk\ell} + m^2 V_{ijk\ell} \right) \cos(\theta_i - \theta_j + \theta_k) - \left(2\omega_j \omega_k X_{ijk\ell} - m^2 V_{ijk\ell} \right) \cos(\theta_i + \theta_j + \theta_k) \right] \\ &+ \frac{1}{4} \sum_{\substack{i,j \\ i,j }}^{\infty} a_i a_j a_\ell \omega_\ell \left[\tilde{Z}_{ij\ell}^-(\omega_i + \omega_j - 2\omega_\ell) \cos(\theta_i + \theta_j - \theta_\ell) - \tilde{Z}_{ij\ell}^-(\omega_i + \omega_j + 2\omega_\ell) \cos(\theta_i + \theta_j + \theta_\ell) \right. \\ &+ \tilde{Z}_{ij\ell}^+(\omega_i - \omega_j + 2\omega_\ell) \cos(\theta_i - \theta_j + \theta_\ell) - \tilde{Z}_{ij\ell}^+(\omega_i - \omega_j - 2\omega_\ell) \cos(\theta_i - \theta_j - \theta_\ell) \right] \\ &- \frac{1}{4} \sum_{\substack{i,j \\ i,j }}^{\infty} a_i^2 a_j \left(H_{iij\ell} + m^2 V_{jii\ell} - 2\omega_j^2 X_{iij\ell} \right) \left[\cos(2\theta_i - \theta_j) + \cos(2\theta_i + \theta_j) \right] \\ &- \frac{1}{2} \sum_{\substack{i,j \\ i,j }}^{\infty} a_i^2 a_j \left(H_{iij\ell} + m^2 V_{jii\ell} - 2\omega_j^2 X_{iij\ell} \right) \left[\cos(2\theta_i - \theta_j) + \cos(2\theta_i + \theta_j) \right] \\ &- \frac{1}{2} \sum_{\substack{i,j \\ i,j }}^{\infty} a_i^2 a_j \left(H_{iij\ell} + m^2 V_{jii\ell} - 2\omega_j^2 X_{iij\ell} \right) \left[\cos(2\theta_i - \theta_j) + \cos(2\theta_i + \theta_j) \right] \\ &- \frac{1}{2} \sum_{\substack{i,j \\ i,j }}^{\infty} a_i^2 a_j \left(H_{iij\ell} + m^2 V_{jii\ell} - 2\omega_j^2 X_{iij\ell} \right) \left[\cos(2\theta_i - \theta_j) + \cos(2\theta_i + \theta_j) \right] \\ &- \frac{1}{2} \sum_{\substack{i,j \\ i,j }}^{\infty} a_i^2 a_j \left(H_{iij\ell} + m^2 V_{jii\ell} - 2\omega_j^2 X_{iij\ell} \right) \left[\cos(2\theta_i - \theta_j) + \cos(2\theta_i + \theta_j) \right] \\ &- \frac{1}{2} \sum_{\substack{i,j \\ i,j }}^{\infty} a_i^2 a_j \left(H_{iij\ell} + m^2 V_{jii\ell} - 2\omega_j^2 X_{iij\ell} \right) \left[\cos(2\theta_i - \theta_j) + \cos(2\theta_i + \theta_j) \right] \\ &- \frac{1}{2} \sum_{\substack{i,j \\ i,j }}^{\infty} a_i^2 a_j \left(H_{iij\ell} + m^2 V_{jii\ell} - 2\omega_j^2 X_{iij\ell} \right) \left[a_i + a_i +$$

To simplify the above expression, we have defined

$$Z_{ijk\ell}^{\pm} = \omega_i \omega_j \left(X_{k\ell ij} - X_{\ell kij} \right) \pm \left(Y_{ij\ell k} - Y_{ijk\ell} \right) \quad \text{and} \quad \tilde{Z}_{ij\ell}^{\pm} = \omega_i \omega_j P_{ij\ell} \pm B_{ij\ell} \,. \tag{3.18}$$

Using integration by parts to remove the derivative from ν in the definitions of $H_{ijk\ell}$ and $M_{ii\ell}$, we can show that

$$Z_{ijk\ell} = \omega_i^2 X_{kij\ell} + \omega_k^2 X_{ijk\ell} - Y_{ij\ell k} - Y_{\ell kji} - m^2 V_{kji\ell} - m^2 V_{ijk\ell}$$
 (3.19)

$$M_{ij\ell} = \omega_i^2 P_{ij\ell} - B_{ij\ell} - m^2 Q_{ij\ell} \tag{3.20}$$

4 Resonances

After time-averaging, resonant contributions come from the set of conditions

$$\omega_i \pm \omega_i \pm \omega_k = \pm \omega_\ell \tag{4.1}$$

which separates into three distinct cases

$$\omega_i + \omega_j + \omega_k = \omega_\ell \qquad (+++) \tag{4.2}$$

$$\omega_i - \omega_j - \omega_k = \omega_\ell \qquad (+ - -) \tag{4.3}$$

$$\omega_i + \omega_j - \omega_k = \omega_\ell \qquad (++-) \tag{4.4}$$

$4.1 \quad (+++)$

These resonant contributions come from the condition $\omega_i + \omega_i + \omega_k = \omega_\ell$, and are of the form

$$S_{\ell} = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \Omega_{ijk\ell} a_i a_j a_k \cos(\theta_i + \theta_j + \theta_k) + \dots,$$

$$(4.5)$$

where

$$\Omega_{ijk\ell} = -\frac{1}{12} H_{ijk\ell} \frac{\omega_j(\omega_i + \omega_k + 2\omega_j)}{(\omega_i + \omega_j)(\omega_j + \omega_k)} - \frac{1}{12} H_{ikj\ell} \frac{\omega_k(\omega_i + \omega_j + 2\omega_k)}{(\omega_i + \omega_k)(\omega_j + \omega_k)} - \frac{1}{12} H_{jik\ell} \frac{\omega_i(\omega_j + \omega_k + 2\omega_i)}{(\omega_i + \omega_j)(\omega_i + \omega_k)} - \frac{m^2}{12} V_{ijk\ell} \left(1 + \frac{\omega_j}{\omega_j + \omega_k} + \frac{\omega_i}{\omega_i + \omega_k} \right) - \frac{m^2}{12} V_{jki\ell} \left(1 + \frac{\omega_j}{\omega_i + \omega_j} + \frac{\omega_k}{\omega_i + \omega_k} \right) - \frac{m^2}{12} V_{kij\ell} \left(1 + \frac{\omega_j}{\omega_i + \omega_k} + \frac{\omega_k}{\omega_i + \omega_j} \right) + \frac{1}{6} \omega_j \omega_k X_{ijk\ell} \left(1 + \frac{\omega_j}{\omega_i + \omega_k} + \frac{\omega_k}{\omega_i + \omega_j} \right) + \frac{1}{6} \omega_i \omega_j X_{kij\ell} \left(1 + \frac{\omega_i}{\omega_j + \omega_k} + \frac{\omega_j}{\omega_i + \omega_k} \right) - \frac{1}{12} Z_{ijk\ell}^{-} \frac{\omega_k}{\omega_i + \omega_j} - \frac{1}{12} Z_{ikj\ell}^{-} \frac{\omega_j}{\omega_i + \omega_k} - \frac{1}{12} Z_{jki\ell}^{-} \frac{\omega_i}{\omega_j + \omega_k} \right. \tag{4.6}$$

4.2 (+--)

These contributions arise from the condition $\omega_i - \omega_k - \omega_k = \omega_\ell$, which are symmetric under exchange of ω_j, ω_k , and are of the form

$$S_{\ell} = \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \Gamma_{(j+k+\ell)jk\ell} a_j a_k a_{(j+k+\ell)} \cos\left(\theta_{j+k+\ell} - \theta_j - \theta_k\right) + \dots, \tag{4.7}$$

where

$$\Gamma_{ijk\ell} = \frac{1}{8} X_{\ell ijk} \left(\omega_i^2 + \omega_j^2 + \omega_k^2 \right) - \frac{1}{8} \omega_i^2 X_{ijk\ell} - \frac{1}{8} \omega_j^2 X_{jki\ell} - \frac{1}{8} \omega_k^2 X_{kij\ell}$$

$$+ \frac{1}{2} \omega_j \omega_k X_{ijk\ell} \left(1 - \frac{\omega_k}{\omega_i - \omega_j} - \frac{\omega_j}{\omega_i - \omega_k} \right) + \frac{1}{2} \omega_i \omega_k X_{jki\ell} \left(1 - \frac{\omega_i}{\omega_j - \omega_k} - \frac{\omega_k}{\omega_j - \omega_i} \right)$$

$$+ \frac{1}{2} \omega_i \omega_j X_{kij\ell} \left(1 - \frac{\omega_j}{\omega_k - \omega_i} - \frac{\omega_i}{\omega_k - \omega_j} \right) - \frac{m^2}{4} V_{jki\ell} \left(1 - \frac{\omega_j}{\omega_i - \omega_j} - \frac{\omega_k}{\omega_i - \omega_k} \right)$$

$$- \frac{m^2}{4} V_{kij\ell} \left(1 - \frac{\omega_k}{\omega_j - \omega_k} - \frac{\omega_i}{\omega_j - \omega_i} \right) - \frac{m^2}{4} V_{ijk\ell} \left(1 - \frac{\omega_i}{\omega_k - \omega_i} - \frac{\omega_j}{\omega_k - \omega_j} \right)$$

$$+ \frac{1}{4} H_{ijk\ell} \frac{\omega_j (\omega_i + \omega_k - 2\omega_j)}{(\omega_i - \omega_j) (\omega_k - \omega_j)} + \frac{1}{4} H_{ikj\ell} \frac{\omega_k (\omega_i + \omega_j - 2\omega_k)}{(\omega_i - \omega_k) (\omega_j - \omega_k)}$$

$$+ \frac{1}{4} H_{jik\ell} \frac{\omega_i (\omega_j + \omega_k - 2\omega_i)}{(\omega_j - \omega_i) (\omega_k - \omega_i)}. \tag{4.8}$$

4.3 (++-)

These contributions arise from the resonant condition $\omega_i + \omega_j = \omega_k + \omega_\ell$, which is symmetric under $\omega_i \leftrightarrow \omega_j$. These terms can be written as

$$S_{\ell} = T_{\ell} a_{\ell}^{3} \cos(\theta_{\ell} + \theta_{\ell} - \theta_{\ell}) + \sum_{i \neq \ell}^{\infty} R_{i\ell} a_{i}^{2} a_{\ell} \cos(\theta_{i} + \theta_{\ell} - \theta_{i})$$

$$+ \sum_{i \neq \ell}^{\infty} \sum_{j \neq \ell}^{\infty} S_{ij(i+j-\ell)\ell} a_{i} a_{j} a_{(i+j-\ell)} \cos(\theta_{i} + \theta_{j} - \theta_{i+j-\ell}) + \dots$$

$$(4.9)$$

where each of the coefficients is given by

$$S_{ijk\ell} = \frac{1}{4} \left\{ \omega_i \omega_j X_{kij\ell} \left(1 - \frac{\omega_k}{\omega_i + \omega_j} \right) + X_{\ell ijk} \left(\frac{\omega_i \omega_j \omega_k}{\omega_i + \omega_j} \right) \right.$$

$$\left. + \omega_j \omega_k X_{ijk\ell} \left(1 - \frac{3\omega_k}{\omega_i + \omega_j} \right) + \omega_i \omega_k X_{jki\ell} \left(1 - \frac{3\omega_k}{\omega_i + \omega_j} \right) \right.$$

$$\left. + Y_{ij\ell k} \left(\frac{\omega_i + \omega_j + \omega_k}{\omega_i + \omega_j} \right) + \frac{\omega_i Y_{\ell kij} + \omega_j Y_{\ell kji} - \omega_k Y_{ijk\ell}}{\omega_i + \omega_j} \right.$$

$$\left. + m^2 \left[B_{kij\ell} - W_{ijk\ell} + 2B_{ijk\ell} + 2B_{jki\ell} \right] \right\}, \tag{4.10}$$

$$R_{i\ell} = \frac{1}{4} \left\{ \omega_i \omega_\ell X_{i\ell i\ell} \left(1 - \frac{5\omega_i \omega_\ell}{\omega_\ell^2 - \omega_i^2} \right) + \omega_i^2 X_{\ell\ell ii} \left(2 + \frac{\omega_i \omega_\ell}{\omega_\ell^2 - \omega_i^2} \right) \right.$$

$$\left. + \omega_\ell Y_{i\ell\ell i} \left(\frac{4\omega_\ell - \omega_i}{\omega_\ell^2 - \omega_i^2} \right) + Y_{\ell ii\ell} \left(\frac{\omega_i \omega_\ell}{\omega_\ell^2 - \omega_i^2} \right) \right.$$

$$\left. + m^2 \left[6B_{i\ell i\ell} \frac{\omega_\ell^2}{\omega_\ell^2 - \omega_i^2} - W_{i\ell i\ell} - 2\omega_\ell^2 (\omega_i^2 P_{ii\ell} + Q_{ii\ell}) \right] \right\}, \tag{4.11}$$

$$T_{\ell} = \tag{4.12}$$

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