Arbitrary Dimensions, Massive, Non-normalizable Time-Dependent BCs

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2 Perturbative Expansion

The backreaction between the metric and the scalar field appears at second order in the perturbation,

$$A_2' = -\mu\nu\left[(\dot{\phi}_1)^2 + (\phi_1')^2 + m^2\phi_1^2\sec^2 x\right] + \nu' A_2/\nu \tag{2.1}$$

which can be directly integrated to give

$$A_2 = -\nu\int_0^x dy\,\mu\left((\dot{\phi}_1)^2 + (\phi_1')^2 + m^2\phi_1^2\sec^2 x\right)\,. \tag{2.2}$$

Furthermore, the first non-trivial contribution to the lapse in the boundary time gauge is

$$\delta_2 = \int_x^{\pi/2} dy\,\mu\nu\left((\dot{\phi}_1)^2 + (\phi_1')^2\right)\,. \tag{2.3}$$

For convenience, we have also defined the functions

$$\mu(x) = (\tan x)^{d-1} \quad \text{and} \quad \nu(x) = (d-1)/\mu'\,. \tag{2.4}$$

To aide in evaluating integrals, we first derive the following identities: from the equation for the first-order time-dependent coefficients c_i,

$$\ddot{c}_i + \omega_i^2 c_i = 0 \quad \Rightarrow \quad \partial_t\left(\dot{c}_i^2 + \omega_i^2 c_i^2\right) = \partial_t\mathbb{C}_i = 0\,; \tag{2.5}$$

from the equation definition of $\hat{L}$,

$$\hat{L}e_j = -\frac{1}{\mu}\left(\mu e_j'\right)' + m^2\sec^2 x\,e_j \quad \Rightarrow \quad \left(\mu e_j'\right)' = \mu\left(m^2\sec^2 x - \omega_j^2\right)e_j\,; \tag{2.6}$$

from considering the expression $(\mu e_i' e_j)'$:

$$\left(\mu e_i' e_j\right)' = \left(m^2\sec^2 x - \omega_i^2\right)\mu e_i e_j + \mu e_i' e_j'\,; \tag{2.7}$$

from permuting i, j above and subtracting to give

$$\frac{\left[\mu(e_i' e_j\omega_j^2 - e_i e_j'\omega_i^2)\right]'}{(\omega_j^2 - \omega_i^2)} = \mu m^2\sec^2 x\,e_i e_j + \mu e_i' e_j'\,. \tag{2.8}$$

3 $\mathcal{O}(\epsilon^3)$ Source Terms

At third order in ϵ, the equation for ϕ_3 contains a source S given by

$$\ddot{\phi}_3 + \hat{L}\phi_3 = S = 2(A_2 - \delta_2)\ddot{\phi}_1 + (\dot{A}_2 - \dot{\delta}_2)\dot{\phi}_1 + (A_2' - \delta_2')\phi_1' + m^2 A_2\phi_1\sec^2 x \tag{3.1}$$

Projecting each of the terms individually onto the eigenbasis $\{e_\ell\}$:

$$\langle\delta_2\ddot{\phi}_1, e_\ell\rangle = -\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{\substack{k=0 \\ k\neq\ell}}^{\infty}\frac{\omega_k^2 c_k}{\omega_\ell^2 - \omega_k^2}\left[\dot{c}_i\dot{c}_j\left(X_{k\ell ij} - X_{\ell kij}\right) + c_ic_j\left(Y_{ij\ell k} - Y_{ijk\ell}\right)\right]$$

$$-\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\omega_\ell^2 c_\ell\left[\dot{c}_i\dot{c}_j P_{ij\ell} + c_ic_j B_{ij\ell}\right], \tag{3.2}$$

$$\langle A_2\ddot{\phi}_1, e_\ell\rangle = 2\sum_{i=0}^{\infty}\sum_{\substack{j=0 \\ i\neq j}}^{\infty}\sum_{k=0}^{\infty}\frac{\omega_k^2 c_k}{\omega_j^2 - \omega_i^2}X_{ijk\ell}\left(\dot{c}_i\dot{c}_j + \omega_j^2 c_ic_j\right)$$

$$+\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\omega_j^2 c_j\left(\mathbb{C}_i P_{j\ell i} + c_i^2 X_{iij\ell}\right), \tag{3.3}$$

$$\langle\dot{\delta}_2\dot{\phi}_1, e_\ell\rangle = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{\substack{k=0 \\ k\neq\ell}}^{\infty}\frac{\dot{c}_k}{\omega_\ell^2 - \omega_k^2}\left[\partial_t\left(\dot{c}_i\dot{c}_j\right)\left(X_{k\ell ij} - X_{\ell kij}\right) + \partial_t(c_ic_j)\left(Y_{ij\ell k} - Y_{ijk\ell}\right)\right]$$

$$+\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\dot{c}_\ell\left[\partial_t\left(\dot{c}_i\dot{c}_j\right)P_{ij\ell} + \partial_t(c_ic_j)B_{ij\ell}\right], \tag{3.4}$$

$$\langle\dot{A}_2\dot{\phi}_1, e_\ell\rangle = -2\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\dot{c}_k\dot{c}_j c_i X_{ijk\ell}, \tag{3.5}$$

$$\langle\left(A_2' - \delta_2'\right)\phi_1', e_\ell\rangle = -2\sum_{i=0}^{\infty}\sum_{\substack{j=0 \\ i\neq j}}^{\infty}\sum_{k=0}^{\infty}\frac{c_k(\dot{c}_i\dot{c}_j + \omega_j^2 c_ic_j)}{\omega_j^2 - \omega_i^2}H_{ijk\ell} - m^2\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}c_ic_jc_k V_{ijk\ell}$$

$$-\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}c_j\left[c_i^2 H_{iij\ell} + \mathbb{C}_i M_{j\ell i}\right], \tag{3.6}$$

$$\langle A_2\phi_1\sec^2 x, e_\ell\rangle = -2\sum_{i=0}^{\infty}\sum_{\substack{j=0 \\ i\neq j}}^{\infty}\sum_{k=0}^{\infty}\frac{c_k(\dot{c}_i\dot{c}_j + \omega_j^2 c_ic_j)}{\omega_j^2 - \omega_i^2}V_{jki\ell}$$

$$-\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}c_j\left(c_i^2 V_{jii\ell} + \mathbb{C}_i Q_{j\ell i}\right). \tag{3.7}$$

Where the forms of X, Y, V, H, B, M, P, and Q are given by

$$X_{ijk\ell} = \int_0^{\pi/2} dx \, \mu^2 \nu e_i' e_j e_k e_\ell \tag{3.8}$$

$$Y_{ijk\ell} = \int_0^{\pi/2} dx \, \mu^2 \nu e_i' e_j' e_k e_\ell' \tag{3.9}$$

$$V_{ijk\ell} = \int_0^{\pi/2} dx \, \mu^2 \nu e_i e_j e_k' e_\ell \sec^2 x \tag{3.10}$$

$$H_{ijk\ell} = \int_0^{\pi/2} dx \, \mu^2 \nu' e_i' e_j e_k' e_\ell \tag{3.11}$$

$$B_{ij\ell} = \int_0^{\pi/2} dx \, \mu \nu e_i' e_j' \int_0^x dy \, \mu e_\ell^2 \tag{3.12}$$

$$M_{ij\ell} = \int_0^{\pi/2} dx \, \mu \nu' e_i' e_j \int_o^x dy \, \mu e_\ell^2 \tag{3.13}$$

$$P_{ij\ell} = \int_0^{\pi/2} dx \, \mu \nu e_i e_j \int_0^x dy \, \mu e_\ell^2 \tag{3.14}$$

$$Q_{ij\ell} = \int_0^{\pi/2} dx \, \mu \nu e_i e_j \sec^2 x \int_0^x dy \, \mu e_\ell^2 \tag{3.15}$$

Collecting terms together gives the expression for $S_\ell = \langle S, e_\ell \rangle$:

$$
\begin{aligned}
S_\ell = \sum_{\substack{i,j,k \\ k \neq \ell}}^{\infty} \frac{1}{\omega_\ell^2 - \omega_k^2} & \left[F_k(\dot{c}_i \dot{c}_j)(X_{k\ell ij} - X_{\ell kij}) + F_k(c_i c_j)(Y_{ij\ell k} - Y_{ijk\ell}) \right] \\
& + 2 \sum_{\substack{i,j,k \\ i \neq j}}^{\infty} \frac{c_k D_{ij}}{\omega_j^2 - \omega_i^2} \left[2\omega_k^2 X_{ijk\ell} - H_{ijk\ell} - m^2 V_{jki\ell} \right] - \sum_{i,j,k}^{\infty} c_i \left[2\dot{c}_j \dot{c}_k X_{ijk\ell} + m^2 c_j c_k V_{ijk\ell} \right] \\
& + \sum_{i,j}^{\infty} \left[F_\ell(\dot{c}_i \dot{c}_j) P_{ij\ell} + F_\ell(c_i c_j) B_{ij\ell} + 2\omega_j^2 c_j \left(c_i^2 X_{iij\ell} + \mathbb{C}_i P_{j\ell i} \right) \right. \\
& \left. - c_j \left(c_i^2 (H_{iij\ell} + m^2 V_{jii\ell}) + \mathbb{C}_i (M_{j\ell i} + m^2 Q_{j\ell i}) \right) \right],
\end{aligned}
\tag{3.16}
$$

where $F_k(z) = \dot{c}_k \dot{z} - 2\omega_k^2 c_k z$, $D_{ij} = \dot{c}_i \dot{c}_j + \omega_j^2 c_i c_j$, and $\mathbb{C}_i = \dot{c}_i^2 + \omega_i^2 c_i^2$.

Using the solution $c_i(t) = a_i \cos(\omega_i t + b_i) = a_i \cos\theta_i$, the source term becomes

$$S_\ell = \frac{1}{4} \sum_{\substack{i,j,k \\ k \neq \ell}}^{\infty} \frac{a_i a_j a_k \omega_k}{\omega_\ell^2 - \omega_k^2} \Big[Z_{jk}^-(\omega_i + \omega_j + 2\omega_k) \cos(\theta_i + \theta_j + \theta_k) - Z_{jk}^-(\omega_i + \omega_j - 2\omega_k) \cos(\theta_i + \theta_j - \theta_k)$$

$$+ Z_{jk}^+(\omega_i - \omega_j - 2\omega_k) \cos(\theta_i - \theta_j - \theta_k) - Z_{jk}^+(\omega_i - \omega_j + 2\omega_k) \cos(\theta_i - \theta_j + \theta_k) \Big]$$

$$+ \frac{1}{2} \sum_{\substack{i,j,k \\ i \neq j}}^{\infty} a_i a_j a_k \omega_j \left(H_{ijk\ell} + m^2 V_{jki\ell} - 2\omega_k^2 X_{ijk\ell} \right) \Big[\frac{1}{\omega_i - \omega_j} \left(\cos(\theta_i - \theta_j - \theta_k) + \cos(\theta_i - \theta_j + \theta_k) \right)$$

$$- \frac{1}{\omega_i + \omega_j} \left(\cos(\theta_i + \theta_j - \theta_k) + \cos(\theta_i + \theta_j + \theta_k) \right) \Big]$$

$$- \frac{1}{4} \sum_{i,j,k}^{\infty} a_i a_j a_k \Big[\left(2\omega_j \omega_k X_{ijk\ell} - m^2 V_{ijk\ell} \right) \cos(\theta_i + \theta_j - \theta_k) - \left(2\omega_j \omega_k X_{ijk\ell} - m^2 V_{ijk\ell} \right) \cos(\theta_i - \theta_j - \theta_k)$$

$$+ \left(2\omega_j \omega_k X_{ijk\ell} + m^2 V_{ijk\ell} \right) \cos(\theta_i - \theta_j + \theta_k) - \left(2\omega_j \omega_k X_{ijk\ell} - m^2 V_{ijk\ell} \right) \cos(\theta_i + \theta_j + \theta_k) \Big]$$

$$+ \frac{1}{4} \sum_{i,j}^{\infty} a_i a_j a_\ell \omega_\ell \Big[\tilde{Z}_{ij}^-(\omega_i + \omega_j + 2\omega_\ell) \cos(\theta_i + \theta_j + \theta_\ell) - \tilde{Z}_{ij}^-(\omega_i + \omega_j - 2\omega_\ell) \cos(\theta_i + \theta_j - \theta_\ell)$$

$$+ \tilde{Z}_{ij}^+(\omega_i - \omega_j - 2\omega_\ell) \cos(\theta_i - \theta_j - \theta_\ell) - \tilde{Z}_{ij}^+(\omega_i - \omega_j + 2\omega_\ell) \cos(\theta_i - \theta_j + \theta_\ell) \Big]$$

$$- \frac{1}{4} \sum_{i,j}^{\infty} a_i^2 a_j \left(H_{iij\ell} + m^2 V_{jii\ell} - 2\omega_j^2 X_{iij\ell} \right) \big[\cos(2\theta_i - \theta_j) + \cos(2\theta_i + \theta_j) \big]$$

$$- \frac{1}{2} \sum_{i,j}^{\infty} a_i^2 a_j \left(H_{iij\ell} + m^2 V_{jii\ell} - 2\omega_j^2 X_{iij\ell} + 4\omega_i^2 \omega_j^2 P_{j\ell i} + 2\omega_i^2 (M_{j\ell i} + m^2 Q_{j\ell i}) \right) \cos\theta_j. \quad (3.17)$$

To simplify the above expression, we have defined

$$Z_{jk}^\pm = \omega_j \omega_k \left(X_{k\ell ij} - X_{\ell kij} \right) \pm \left(Y_{ij\ell k} - Y_{ijk\ell} \right) \quad \text{and} \quad \tilde{Z}_{ij}^\pm = \omega_i \omega_j P_{ij\ell} \pm B_{ij\ell}. \quad (3.18)$$

Using integration by parts to remove the derivative from ν in the definitions of $H_{ijk\ell}$ and $M_{ij\ell}$, we can show that

$$Z_{ijk\ell} = \omega_i^2 X_{kij\ell} + \omega_k^2 X_{ijk\ell} - Y_{ij\ell k} - Y_{\ell kji} - m^2 V_{kji\ell} - m^2 V_{ijk\ell} \quad (3.19)$$

$$M_{ij\ell} = \omega_i^2 P_{ij\ell} - B_{ij\ell} - m^2 Q_{ij\ell} \quad (3.20)$$

4 Resonances

After time-averaging, resonant contributions come from the set of conditions

$$\omega_i \pm \omega_j \pm \omega_k = \pm \omega_\ell \quad (4.1)$$

which separates into three distinct cases

$$\omega_i + \omega_j + \omega_k = \omega_\ell \qquad (+++) \tag{4.2}$$

$$\omega_i - \omega_j - \omega_k = \omega_\ell \qquad (+--) \tag{4.3}$$

$$\omega_i + \omega_j - \omega_k = \omega_\ell \qquad (++-) \tag{4.4}$$

4.1 $(+++)$

These resonant contributions come from the condition $\omega_i + \omega_i + \omega_k = \omega_\ell$, and are of the form

$$S_\ell = \underbrace{\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}}_{\omega_i+\omega_j+\omega_k=\omega_\ell} \Omega_{ijk\ell}\, a_i a_j a_k \cos\left(\theta_i + \theta_j + \theta_k\right) + \dots, \tag{4.5}$$

where

$$
\begin{aligned}
\Omega_{ijk\ell} = \frac{1}{12}\Bigg\{ &\omega_j\omega_k X_{ijk\ell}\left(\frac{\omega_i}{\omega_j+\omega_k} - \frac{3\omega_k}{\omega_i+\omega_j} - \frac{3\omega_j}{\omega_i+\omega_k} - 1\right)\\
&+ \omega_i\omega_k X_{jki\ell}\left(\frac{\omega_j}{\omega_i+\omega_k} - \frac{3\omega_k}{\omega_i+\omega_j} - \frac{3\omega_i}{\omega_j+\omega_k} - 1\right)\\
&+ \omega_i\omega_j X_{kij\ell}\left(\frac{\omega_k}{\omega_i+\omega_j} - \frac{3\omega_i}{\omega_j+\omega_k} - \frac{3\omega_j}{\omega_i+\omega_k} - 1\right)\\
&- \omega_i\omega_j\omega_k X_{\ell ijk}\left(\frac{1}{\omega_i+\omega_j} + \frac{1}{\omega_j+\omega_k} + \frac{1}{\omega_i+\omega_k}\right)\\
&- Y_{ji\ell k}\left(\frac{\omega_k}{\omega_i+\omega_j} + \frac{\omega_j}{\omega_i+\omega_k} + \frac{\omega_i}{\omega_j+\omega_k} - 3\right)\\
&+ \omega_k Y_{ijk\ell}\left(\frac{1}{\omega_i+\omega_j} + \frac{1}{\omega_j+\omega_k} + \frac{1}{\omega_i+\omega_k}\right)\\
&+ \omega_i Y_{jki\ell}\left(\frac{1}{\omega_i+\omega_j} + \frac{1}{\omega_j+\omega_k} + \frac{1}{\omega_i+\omega_k}\right)\\
&+ \omega_j Y_{kij\ell}\left(\frac{1}{\omega_i+\omega_j} + \frac{1}{\omega_j+\omega_k} + \frac{1}{\omega_i+\omega_k}\right)\\
&+ m^2\Bigg[2B_{ijk\ell}\left(\frac{\omega_j}{\omega_i+\omega_j} + \frac{\omega_k}{\omega_i+\omega_k}\right)\\
&+ 2B_{jki\ell}\left(\frac{\omega_i}{\omega_i+\omega_j} + \frac{\omega_k}{\omega_j+\omega_k}\right) + 2B_{kij\ell}\left(\frac{\omega_j}{\omega_j+\omega_k} + \frac{\omega_i}{\omega_i+\omega_k}\right)\\
&+ (B_{ijk\ell} - W_{ijk\ell}) + (B_{jki\ell} - W_{jki\ell}) + (B_{kij\ell} - W_{kij\ell})\Bigg]\Bigg\}.
\end{aligned}
\tag{4.6}
$$

4.2 $(+--)$

These contributions arise from the condition $\omega_i - \omega_k - \omega_k = \omega_\ell$, which are symmetric under exchange of ω_j, ω_k, and are of the form

$$S_\ell = \sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\Gamma_{(j+k+\ell)jk\ell}\, a_j a_k a_{(j+k+\ell)} \cos\left(\theta_{j+k+\ell} - \theta_j - \theta_k\right) + \dots, \tag{4.7}$$

where

$$\Gamma_{ijk\ell} = \frac{1}{4}\left\{ m^2\left(B_{kij\ell} - W_{ijk\ell}\right) + Y_{ij\ell k} - (1+m^2)B_{ijk\ell}\left(\frac{\omega_j}{\omega_i - \omega_j}\right)\right.$$
$$- 3\omega_k^2 X_{jik\ell}\left(\frac{\omega_i}{\omega_i - \omega_j}\right) + 3\omega_k^2 X_{ijk\ell}\left(\frac{\omega_j}{\omega_i - \omega_j}\right)$$
$$\left. + Y_{\ell kij}\left(\frac{\omega_i}{\omega_i - \omega_j}\right) - Y_{\ell kji}\left(\frac{\omega_j}{\omega_i - \omega_j}\right)\right\}. \tag{4.8}$$

4.3 $(++-)$

These contributions arise from the resonant condition $\omega_i + \omega_j = \omega_k + \omega_\ell$, which is symmetric under $\omega_i \leftrightarrow \omega_j$. These terms can be written as

$$S_\ell = T_\ell a_\ell^3 \cos(\theta_\ell + \theta_\ell - \theta_\ell) + \sum_{i\neq\ell}^{\infty} R_{i\ell} a_i^2 a_\ell \cos(\theta_i + \theta_\ell - \theta_i)$$
$$+ \sum_{i\neq\ell}^{\infty}\sum_{j\neq\ell}^{\infty} S_{ij(i+j-\ell)\ell} a_i a_j a_{(i+j-\ell)} \cos(\theta_i + \theta_j - \theta_{i+j-\ell}) + \ldots \tag{4.9}$$

where each of the coefficients is given by

$$S_{ijk\ell} = \frac{1}{4}\left\{ \omega_i\omega_j X_{kij\ell}\left(1 - \frac{\omega_k}{\omega_i + \omega_j}\right) + X_{\ell ijk}\left(\frac{\omega_i\omega_j\omega_k}{\omega_i + \omega_j}\right)\right.$$
$$+ \omega_j\omega_k X_{ijk\ell}\left(1 - \frac{3\omega_k}{\omega_i + \omega_j}\right) + \omega_i\omega_k X_{jki\ell}\left(1 - \frac{3\omega_k}{\omega_i + \omega_j}\right)$$
$$+ Y_{ij\ell k}\left(\frac{\omega_i + \omega_j + \omega_k}{\omega_i + \omega_j}\right) + \frac{\omega_i Y_{\ell kij} + \omega_j Y_{\ell kji} - \omega_k Y_{ijk\ell}}{\omega_i + \omega_j}$$
$$\left. + m^2\left[B_{kij\ell} - W_{ijk\ell} + 2B_{ijk\ell} + 2B_{jki\ell}\right]\right\}, \tag{4.10}$$

$$R_{i\ell} = \frac{1}{4}\left\{ \omega_i\omega_\ell X_{i\ell i\ell}\left(1 - \frac{5\omega_i\omega_\ell}{\omega_\ell^2 - \omega_i^2}\right) + \omega_i^2 X_{\ell\ell ii}\left(2 + \frac{\omega_i\omega_\ell}{\omega_\ell^2 - \omega_i^2}\right)\right.$$
$$+ \omega_\ell Y_{i\ell\ell i}\left(\frac{4\omega_\ell - \omega_i}{\omega_\ell^2 - \omega_i^2}\right) + Y_{\ell ii\ell}\left(\frac{\omega_i\omega_\ell}{\omega_\ell^2 - \omega_i^2}\right)$$
$$\left. + m^2\left[6B_{i\ell i\ell}\frac{\omega_\ell^2}{\omega_\ell^2 - \omega_i^2} - W_{i\ell i\ell} - 2\omega_\ell^2(\omega_i^2 P_{ii\ell} + Q_{ii\ell})\right]\right\}, \tag{4.11}$$

$$T_\ell = \tag{4.12}$$

Acknowledgments