Prepared for submission to JHEP

Arbitrary Dimensions, Massive, Non-normalizable Time-Dependent BCs

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2 Perturbative Expansion

The backreaction between the metric and the scalar field appears at second order in the perturbation,

$$A_2' = -\mu\nu\left[(\dot{\phi}_1)^2 + (\phi_1')^2 + m^2\phi_1^2 \sec^2 x\right] + \nu' A_2/\nu \tag{2.1}$$

which can be directly integrated to give

$$A_2 = -\nu \int_0^x dy\, \mu\left((\dot{\phi}_1)^2 + (\phi_1')^2 + m^2\phi_1^2 \sec^2 x\right)\,. \tag{2.2}$$

Furthermore, the first non-trivial contribution to the lapse in the boundary time gauge is

$$\delta_2 = \int_x^{\pi/2} dy\, \mu\nu\left((\dot{\phi}_1)^2 + (\phi_1')^2\right)\,. \tag{2.3}$$

For convenience, we have also defined the functions

$$\mu(x) = (\tan x)^{d-1} \quad \text{and} \quad \nu(x) = (d-1)/\mu'\,. \tag{2.4}$$

To aide in evaluating integrals, we first derive the following identities: from the equation for the first-order time-dependent coefficients c_i,

$$\ddot{c}_i + \omega_i^2 c_i = 0 \quad \Rightarrow \quad \partial_t\left(\dot{c}_i^2 + \omega_i^2 c_i^2\right) = \partial_t \mathbb{C}_i = 0\,; \tag{2.5}$$

from the equation definition of $\hat{L}$,

$$\hat{L}e_j = -\frac{1}{\mu}\left(\mu e_j'\right)' + m^2 \sec^2 x e_j \quad \Rightarrow \quad \left(\mu e_j'\right)' = \mu\left(m^2\sec^2 x - \omega_j^2\right) e_j\,; \tag{2.6}$$

from considering the expression $(\mu e_i' e_j)'$:

$$\left(\mu e_i' e_j\right)' = \left(m^2 \sec^2 x - \omega_i^2\right)\mu e_i e_j + \mu e_i' e_j'\,; \tag{2.7}$$

from permuting i, j above and subtracting to give

$$\frac{\left[\mu(e_i' e_j \omega_j^2 - e_i e_j' \omega_i^2)\right]'}{(\omega_j^2 - \omega_i^2)} = \mu m^2 \sec^2 x e_i e_j + \mu e_i' e_j'\,. \tag{2.8}$$

The basis functions $e_j(x)$ are the solutions to the eigenvalue equation

$$\hat{L}e_j(x) = \omega_j^2 e_j(x), \tag{2.9}$$

which, for massive scalars, are (up to some normalization)

$$e_j(x) = (\cos(x))^{\Delta_+} \, {}_2F_1\left(\frac{\Delta_+ + \omega}{2}, \frac{\Delta_+ - \omega}{2}, d/2; \sin^2(x)\right), \tag{2.10}$$

when ω is arbitrary. However, when the frequency is equal to the resonant frequency $\omega_j = \Delta_+ + 2j$, (2.10) separates into normalizable and non-normalizable solutions

$$\begin{aligned} e_j(x) = {} & C_1 \, (\cos(x))^{\Delta_+} \, {}_2F_1\left(\frac{\Delta_+ + \omega)}{2}, \frac{\Delta_+ - \omega}{2}, \Delta_+ - d/2 + 1; \cos^2(x)\right) \\ & + C_2 \, (\cos(x))^{\Delta_-} \, {}_2F_1\left(\frac{\Delta_- + \omega)}{2}, \frac{\Delta_- - \omega}{2}, \Delta_- - d/2 + 1; \cos^2(x)\right). \end{aligned} \tag{2.11}$$

3 $\mathcal{O}(\epsilon^3)$ Source Terms

At third order in ϵ, the equation for ϕ_3 contains a source S given by

$$\ddot{\phi}_3 + \hat{L}\phi_3 = S = 2(A_2 - \delta_2)\ddot{\phi}_1 + (\dot{A}_2 - \dot{\delta}_2)\dot{\phi}_1 + (A_2' - \delta_2')\phi_1' + m^2 A_2 \phi_1 \sec^2 x \tag{3.1}$$

Projecting each of the terms individually onto the eigenbasis $\{e_\ell\}$:

$$\langle \delta_2 \ddot{\phi}_1, e_\ell \rangle = -\sum_{i=0}^{\infty}\sum_{\substack{j=0\\k\neq\ell}}^{\infty}\sum_{k=0}^{\infty} \frac{\omega_k^2 c_k}{\omega_\ell^2-\omega_k^2}\left[\dot{c}_i\dot{c}_j\left(X_{k\ell ij}-X_{\ell kij}\right)+c_ic_j\left(Y_{ij\ell k}-Y_{ijk\ell}\right)\right]$$
$$-\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\omega_\ell^2 c_\ell\left[\dot{c}_i\dot{c}_j P_{ij\ell}+c_ic_jB_{ij\ell}\right], \tag{3.2}$$

$$\langle A_2 \ddot{\phi}_1, e_\ell \rangle = 2\sum_{i=0}^{\infty}\sum_{\substack{j=0\\i\neq j}}^{\infty}\sum_{k=0}^{\infty} \frac{\omega_k^2 c_k}{\omega_j^2-\omega_i^2} X_{ijk\ell}\left(\dot{c}_i\dot{c}_j+\omega_j^2c_ic_j\right)$$
$$+\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\omega_j^2 c_j\left(\mathbb{C}_iP_{j\ell i}+c_i^2X_{iij\ell}\right), \tag{3.3}$$

$$\langle \dot{\delta}_2 \dot{\phi}_1, e_\ell \rangle = \sum_{i=0}^{\infty}\sum_{\substack{j=0\\k\neq\ell}}^{\infty}\sum_{k=0}^{\infty} \frac{\dot{c}_k}{\omega_\ell^2-\omega_k^2}\left[\partial_t\left(\dot{c}_i\dot{c}_j\right)\left(X_{k\ell ij}-X_{\ell kij}\right)+\partial_t(c_ic_j)\left(Y_{ij\ell k}-Y_{ijk\ell}\right)\right]$$
$$+\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\dot{c}_\ell\left[\partial_t\left(\dot{c}_i\dot{c}_j\right)P_{ij\ell}+\partial_t(c_ic_j)B_{ij\ell}\right], \tag{3.4}$$

$$\langle \dot{A}_2 \dot{\phi}_1, e_\ell \rangle = -2\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\dot{c}_k\dot{c}_jc_iX_{ijk\ell}, \tag{3.5}$$

$$\langle \left(A_2'-\delta_2'\right)\phi_1', e_\ell \rangle = -2\sum_{i=0}^{\infty}\sum_{\substack{j=0\\i\neq j}}^{\infty}\sum_{k=0}^{\infty}\frac{c_k(\dot{c}_i\dot{c}_j+\omega_j^2c_ic_j)}{\omega_j^2-\omega_i^2}H_{ijk\ell}-m^2\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}c_ic_jc_kV_{ijk\ell}$$
$$-\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}c_j\left[c_i^2H_{iij\ell}+\mathbb{C}_iM_{j\ell i}\right], \tag{3.6}$$

$$\langle A_2\phi_1\sec^2 x, e_\ell \rangle = -2\sum_{i=0}^{\infty}\sum_{\substack{j=0\\i\neq j}}^{\infty}\sum_{k=0}^{\infty}\frac{c_k(\dot{c}_i\dot{c}_j+\omega_j^2c_ic_j)}{\omega_j^2-\omega_i^2}V_{jki\ell}$$
$$-\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}c_j\left(c_i^2V_{jii\ell}+\mathbb{C}_iQ_{j\ell i}\right). \tag{3.7}$$

Where the forms of X, Y, V, H, B, M, P, and Q are given by

$$X_{ijk\ell} = \int_0^{\pi/2} dx\, \mu^2 \nu e_i' e_j e_k e_\ell \tag{3.8}$$

$$Y_{ijk\ell} = \int_0^{\pi/2} dx\, \mu^2 \nu e_i' e_j' e_k e_\ell' \tag{3.9}$$

$$V_{ijk\ell} = \int_0^{\pi/2} dx\, \mu^2 \nu e_i e_j e_k' e_\ell \sec^2 x \tag{3.10}$$

$$H_{ijk\ell} = \int_0^{\pi/2} dx\, \mu^2 \nu' e_i' e_j e_k' e_\ell \tag{3.11}$$

$$B_{ij\ell} = \int_0^{\pi/2} dx\, \mu \nu e_i' e_j' \int_0^x dy\, \mu e_\ell^2 \tag{3.12}$$

$$M_{ij\ell} = \int_0^{\pi/2} dx\, \mu \nu' e_i' e_j \int_o^x dy\, \mu e_\ell^2 \tag{3.13}$$

$$P_{ij\ell} = \int_0^{\pi/2} dx\, \mu \nu e_i e_j \int_0^x dy\, \mu e_\ell^2 \tag{3.14}$$

$$Q_{ij\ell} = \int_0^{\pi/2} dx\, \mu \nu e_i e_j \sec^2 x \int_0^x dy\, \mu e_\ell^2 \tag{3.15}$$

Collecting terms together gives the expression for $S_\ell = \langle S, e_\ell \rangle$:

$$\begin{aligned} S_\ell = &\sum_{\substack{i,j,k \\ k\neq\ell}}^{\infty} \frac{1}{\omega_\ell^2 - \omega_k^2} \Big[F_k(\dot{c}_i \dot{c}_j) \left(X_{k\ell ij} - X_{\ell kij} \right) + F_k(c_i c_j) \left(Y_{ij\ell k} - Y_{ijk\ell} \right) \Big] \\ &+ 2 \sum_{\substack{i,j,k \\ i\neq j}}^{\infty} \frac{c_k D_{ij}}{\omega_j^2 - \omega_i^2} \Big[2\omega_k^2 X_{ijk\ell} - H_{ijk\ell} - m^2 V_{jki\ell} \Big] - \sum_{i,j,k}^{\infty} c_i \Big[2\dot{c}_j \dot{c}_k X_{ijk\ell} + m^2 c_j c_k V_{ijk\ell} \Big] \\ &+ \sum_{i,j}^{\infty} \Big[F_\ell(\dot{c}_i \dot{c}_j) P_{ij\ell} + F_\ell(c_i c_j) B_{ij\ell} + 2\omega_j^2 c_j \left(c_i^2 X_{iij\ell} + \mathbb{C}_i P_{j\ell i} \right) \\ &- c_j \left(c_i^2 (H_{iij\ell} + m^2 V_{jii\ell}) + \mathbb{C}_i (M_{j\ell i} + m^2 Q_{j\ell i}) \right) \Big], \end{aligned} \tag{3.16}$$

where $F_k(z) = \dot{c}_k \dot{z} - 2\omega_k^2 c_k z$, $D_{ij} = \dot{c}_i \dot{c}_j + \omega_j^2 c_i c_j$, and $\mathbb{C}_i = \dot{c}_i^2 + \omega_i^2 c_i^2$.

Using the solution $c_i(t) = a_i \cos(\omega_i t + b_i) = a_i \cos\theta_i$, the source term becomes

$$
\begin{aligned}
S_\ell = {} & \frac{1}{4}\sum_{\substack{i,j,k\\k\neq\ell}}^{\infty} \frac{a_i a_j a_k \omega_k}{\omega_\ell^2-\omega_k^2}\Big[Z^-_{ijk\ell}(\omega_i+\omega_j-2\omega_k)\cos(\theta_i+\theta_j-\theta_k) - Z^-_{ijk\ell}(\omega_i+\omega_j+2\omega_k)\cos(\theta_i+\theta_j+\theta_k) - \\
& + Z^+_{ijk\ell}(\omega_i-\omega_j+2\omega_k)\cos(\theta_i-\theta_j+\theta_k) - Z^+_{ijk\ell}(\omega_i-\omega_j-2\omega_k)\cos(\theta_i-\theta_j-\theta_k)\Big] \\
& + \frac{1}{2}\sum_{\substack{i,j,k\\i\neq j}}^{\infty} a_i a_j a_k \omega_j \left(H_{ijk\ell}+m^2V_{jki\ell}-2\omega_k^2 X_{ijk\ell}\right)\Big[\frac{1}{\omega_i-\omega_j}\left(\cos(\theta_i-\theta_j-\theta_k)+\cos(\theta_i-\theta_j+\theta_k)\right) \\
& - \frac{1}{\omega_i+\omega_j}\left(\cos(\theta_i+\theta_j-\theta_k)+\cos(\theta_i+\theta_j+\theta_k)\right)\Big] \\
& - \frac{1}{4}\sum_{i,j,k}^{\infty} a_i a_j a_k\Big[\left(2\omega_j\omega_k X_{ijk\ell}+m^2V_{ijk\ell}\right)\cos(\theta_i+\theta_j-\theta_k) - \left(2\omega_j\omega_k X_{ijk\ell}-m^2V_{ijk\ell}\right)\cos(\theta_i-\theta_j-\theta_k) \\
& + \left(2\omega_j\omega_k X_{ijk\ell}+m^2V_{ijk\ell}\right)\cos(\theta_i-\theta_j+\theta_k) - \left(2\omega_j\omega_k X_{ijk\ell}-m^2V_{ijk\ell}\right)\cos(\theta_i+\theta_j+\theta_k)\Big] \\
& + \frac{1}{4}\sum_{i,j}^{\infty} a_i a_j a_\ell \omega_\ell\Big[\tilde{Z}^-_{ij\ell}(\omega_i+\omega_j-2\omega_\ell)\cos(\theta_i+\theta_j-\theta_\ell) - \tilde{Z}^-_{ij\ell}(\omega_i+\omega_j+2\omega_\ell)\cos(\theta_i+\theta_j+\theta_\ell) \\
& + \tilde{Z}^+_{ij\ell}(\omega_i-\omega_j+2\omega_\ell)\cos(\theta_i-\theta_j+\theta_\ell) - \tilde{Z}^+_{ij\ell}(\omega_i-\omega_j-2\omega_\ell)\cos(\theta_i-\theta_j-\theta_\ell)\Big] \\
& - \frac{1}{4}\sum_{i,j}^{\infty} a_i^2 a_j \left(H_{iij\ell}+m^2V_{jii\ell}-2\omega_j^2X_{iij\ell}\right)\left[\cos(2\theta_i-\theta_j)+\cos(2\theta_i+\theta_j)\right] \\
& - \frac{1}{2}\sum_{i,j}^{\infty} a_i^2 a_j \left(H_{iij\ell}+m^2V_{jii\ell}-2\omega_j^2X_{iij\ell}+4\omega_i^2\omega_j^2P_{j\ell i}+2\omega_i^2(M_{j\ell i}+m^2Q_{j\ell i})\right)\cos\theta_j. \qquad (3.17)
\end{aligned}
$$

To simplify the above expression, we have defined

$$
Z^\pm_{ijk\ell} = \omega_i\omega_j\left(X_{k\ell ij}-X_{\ell kij}\right) \pm \left(Y_{ij\ell k}-Y_{ijk\ell}\right) \quad \text{and} \quad \tilde{Z}^\pm_{ij\ell} = \omega_i\omega_j P_{ij\ell} \pm B_{ij\ell}\,. \qquad (3.18)
$$

Using integration by parts to remove the derivative from ν in the definitions of $H_{ijk\ell}$ and $M_{ij\ell}$, we can show that

$$
H_{ijk\ell} = \omega_i^2 X_{kij\ell} + \omega_k^2 X_{ijk\ell} - Y_{ij\ell k} - Y_{\ell kji} - m^2V_{kji\ell} - m^2V_{ijk\ell} \qquad (3.19)
$$

$$
M_{ij\ell} = \omega_i^2 P_{ij\ell} - B_{ij\ell} - m^2 Q_{ij\ell} \qquad (3.20)
$$

4 Resonances From Normalizable Solutions

Consider the case where each of the basis functions are given by normalizable solutions. After time-averaging, resonant contributions come from the set of conditions

$$
\omega_i \pm \omega_j \pm \omega_k = \pm\omega_\ell \qquad (4.1)
$$

which separates into three distinct cases

$$\omega_i + \omega_j + \omega_k = \omega_\ell \qquad (+++) \tag{4.2}$$

$$\omega_i - \omega_j - \omega_k = \omega_\ell \qquad (+--) \tag{4.3}$$

$$\omega_i + \omega_j - \omega_k = \omega_\ell \qquad (++-) \tag{4.4}$$

4.1 $(+++)$

These resonant contributions come from the condition $\omega_i + \omega_j + \omega_k = \omega_\ell$, and are of the form

$$S_\ell = \underbrace{\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}}_{\omega_i+\omega_j+\omega_k=\omega_\ell} \Omega_{ijk\ell}\, a_i a_j a_k \cos\left(\theta_i + \theta_j + \theta_k\right) + \ldots\,, \tag{4.5}$$

where

$$\begin{aligned}
\Omega_{ijk\ell} = &-\frac{1}{12} H_{ijk\ell} \frac{\omega_j(\omega_i + \omega_k + 2\omega_j)}{(\omega_i + \omega_j)(\omega_j + \omega_k)} - \frac{1}{12} H_{ikj\ell} \frac{\omega_k(\omega_i + \omega_j + 2\omega_k)}{(\omega_i + \omega_k)(\omega_j + \omega_k)} - \frac{1}{12} H_{jik\ell} \frac{\omega_i(\omega_j + \omega_k + 2\omega_i)}{(\omega_i + \omega_j)(\omega_i + \omega_k)} \\
&- \frac{m^2}{12} V_{ijk\ell} \left(1 + \frac{\omega_j}{\omega_j + \omega_k} + \frac{\omega_i}{\omega_i + \omega_k}\right) - \frac{m^2}{12} V_{jki\ell} \left(1 + \frac{\omega_j}{\omega_i + \omega_j} + \frac{\omega_k}{\omega_i + \omega_k}\right) \\
&- \frac{m^2}{12} V_{kij\ell} \left(1 + \frac{\omega_i}{\omega_i + \omega_j} + \frac{\omega_k}{\omega_j + \omega_k}\right) + \frac{1}{6}\omega_j\omega_k X_{ijk\ell} \left(1 + \frac{\omega_j}{\omega_i + \omega_k} + \frac{\omega_k}{\omega_i + \omega_j}\right) \\
&+ \frac{1}{6}\omega_i\omega_k X_{jki\ell} \left(1 + \frac{\omega_i}{\omega_j + \omega_k} + \frac{\omega_k}{\omega_i + \omega_j}\right) + \frac{1}{6}\omega_i\omega_j X_{kij\ell} \left(1 + \frac{\omega_i}{\omega_j + \omega_k} + \frac{\omega_j}{\omega_i + \omega_k}\right) \\
&- \frac{1}{12} Z^-_{ijk\ell} \left(\frac{\omega_k}{\omega_i + \omega_j}\right) - \frac{1}{12} Z^-_{ikj\ell} \left(\frac{\omega_j}{\omega_i + \omega_k}\right) - \frac{1}{12} Z^-_{jki\ell} \left(\frac{\omega_i}{\omega_j + \omega_k}\right)\,.
\end{aligned} \tag{4.6}$$

4.2 $(+--)$

These contributions arise from the condition $\omega_i - \omega_j - \omega_k = \omega_\ell$, are of the form

$$S_\ell = \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \Gamma_{(j+k+\ell)jk\ell}\, a_j a_k a_{(j+k+\ell)} \cos\left(\theta_{j+k+\ell} - \theta_j - \theta_k\right) + \ldots\,, \tag{4.7}$$

where

$$\begin{aligned}
\Gamma_{ijk\ell} = &\frac{1}{4} H_{ijk\ell} \frac{\omega_j(\omega_k - \omega_i + 2\omega_j)}{(\omega_i - \omega_j)(\omega_j + \omega_k)} + \frac{1}{4} H_{jki\ell} \frac{\omega_k(\omega_j - \omega_i + 2\omega_k)}{(\omega_i - \omega_k)(\omega_j + \omega_k)} + \frac{1}{4} H_{kij\ell} \frac{\omega_i(\omega_j + \omega_k - 2\omega_i)}{(\omega_i - \omega_j)(\omega_i - \omega_k)} \\
&- \frac{1}{2}\omega_j\omega_k X_{ijk\ell} \left(\frac{\omega_k}{\omega_i - \omega_j} + \frac{\omega_j}{\omega_i - \omega_k} - 1\right) + \frac{1}{2}\omega_i\omega_k X_{jki\ell} \left(\frac{\omega_k}{\omega_i - \omega_j} + \frac{\omega_i}{\omega_j + \omega_k} - 1\right) \\
&+ \frac{1}{2}\omega_i\omega_j X_{kij\ell} \left(\frac{\omega_j}{\omega_i - \omega_k} + \frac{\omega_i}{\omega_j + \omega_k} - 1\right) + \frac{m^2}{4} V_{jki\ell} \left(\frac{\omega_j}{\omega_i - \omega_j} + \frac{\omega_k}{\omega_i - \omega_k} - 1\right) \\
&- \frac{m^2}{4} V_{kij\ell} \left(\frac{\omega_i}{\omega_i - \omega_j} + \frac{\omega_k}{\omega_j + \omega_k} + 1\right) - \frac{m^2}{4} V_{ijk\ell} \left(\frac{\omega_i}{\omega_i - \omega_k} + \frac{\omega_j}{\omega_j + \omega_k} + 1\right) \\
&+ \frac{1}{4} Z^-_{kji\ell} \left(\frac{\omega_i}{\omega_j + \omega_k}\right) - \frac{1}{4} Z^+_{ijk\ell} \left(\frac{\omega_k}{\omega_i - \omega_j}\right) - \frac{1}{4} Z^+_{jki\ell} \left(\frac{\omega_j}{\omega_i - \omega_k}\right)\,.
\end{aligned} \tag{4.8}$$

4.3 Naturally Vanishing Resonances

It has been shown that when $m = 0$, and only normalizable modes are considered, (4.6) and (4.8) vanish by the orthogonality of the basis functions. **Maybe show that mass-dependent terms vanish for normalizable modes?**

4.4 $(+ + -)$

These contributions arise from the resonant condition $\omega_i + \omega_j = \omega_k + \omega_\ell$, can be written as

$$\begin{aligned} S_\ell &= T_\ell a_\ell^3 \cos(\theta_\ell + \theta_\ell - \theta_\ell) + \sum_{i \neq \ell}^{\infty} R_{i\ell}\, a_i^2 a_\ell \cos(\theta_i + \theta_\ell - \theta_i) \\ &\quad + \sum_{i \neq \ell}^{\infty} \sum_{j \neq \ell}^{\infty} S_{ij(i+j-\ell)\ell}\, a_i a_j a_{(i+j-\ell)} \cos(\theta_i + \theta_j - \theta_{i+j-\ell}) + \ldots \end{aligned} \tag{4.9}$$

where each of the coefficients is given by

$$\begin{aligned} S_{ijk\ell} = &-\frac{1}{4} H_{kij\ell} \frac{\omega_i(\omega_j - \omega_k + 2\omega_i)}{(\omega_i - \omega_k)(\omega_i + \omega_j)} - \frac{1}{4} H_{ijk\ell} \frac{\omega_j(\omega_i - \omega_k + 2\omega_j)}{(\omega_j - \omega_k)(\omega_i + \omega_j)} - \frac{1}{4} H_{jki\ell} \frac{\omega_k(\omega_i + \omega_j - 2\omega_k)}{(\omega_i - \omega_k)(\omega_j - \omega_k)} \\ &- \frac{1}{2}\omega_j\omega_k X_{ijk\ell} \left(\frac{\omega_j}{\omega_i - \omega_k} - \frac{\omega_k}{\omega_i + \omega_j} + 1 \right) - \frac{1}{2}\omega_i\omega_k X_{jki\ell} \left(\frac{\omega_i}{\omega_j - \omega_k} - \frac{\omega_k}{\omega_i + \omega_j} + 1 \right) \\ &+ \frac{1}{2}\omega_i\omega_j X_{kij\ell} \left(\frac{\omega_i}{\omega_j - \omega_k} + \frac{\omega_j}{\omega_i - \omega_k} + 1 \right) - \frac{m^2}{4} V_{ijk\ell} \left(\frac{\omega_i}{\omega_i - \omega_k} + \frac{\omega_j}{\omega_j - \omega_k} + 1 \right) \\ &+ \frac{m^2}{4} V_{jki\ell} \left(\frac{\omega_k}{\omega_i - \omega_k} - \frac{\omega_j}{\omega_i + \omega_j} - 1 \right) + \frac{m^2}{4} V_{kij\ell} \left(\frac{\omega_k}{\omega_j - \omega_k} - \frac{\omega_i}{\omega_i + \omega_j} - 1 \right) \\ &+ \frac{1}{4} Z^-_{ijk\ell} \left(\frac{\omega_k}{\omega_i + \omega_j} \right) + \frac{1}{4} Z^+_{ikj\ell} \left(\frac{\omega_j}{\omega_i - \omega_k} \right) + \frac{1}{4} Z^+_{jki\ell} \left(\frac{\omega_i}{\omega_j - \omega_k} \right), \end{aligned} \tag{4.10}$$

$$\begin{aligned} R_{i\ell} = &\left(\frac{\omega_i^2}{\omega_\ell^2 - \omega_i^2} \right) \left(Y_{i\ell\ell i} - Y_{i\ell i\ell} + \omega_\ell^2 (X_{i\ell i\ell} - X_{\ell i\ell i}) \right) + \left(\frac{\omega_i^2}{\omega_\ell^2 - \omega_i^2} \right) \left(H_{\ell ii\ell} + m^2 V_{ii\ell\ell} - 2\omega_i^2 X_{\ell ii\ell} \right) \\ &- \left(\frac{\omega_\ell^2}{\omega_\ell^2 - \omega_i^2} \right) \left(H_{i\ell i\ell} + m^2 V_{\ell ii\ell} - 2\omega_i^2 X_{i\ell i\ell} \right) - \frac{m^2}{4} (V_{i\ell i\ell} + V_{ii\ell\ell}) + \omega_i^2 \omega_\ell^2 (P_{ii\ell} - 2P_{\ell\ell i}) \\ &- \omega_i \omega_\ell X_{i\ell i\ell} - \frac{3m^2}{2} V_{\ell ii\ell} - \frac{1}{2} H_{ii\ell\ell} + \omega_\ell^2 B_{ii\ell} - \omega_i^2 M_{\ell\ell i} - m^2 \omega_i^2 Q_{\ell\ell i}, \end{aligned} \tag{4.11}$$

$$T_\ell = \frac{1}{2}\omega_\ell^2 \left(X_{\ell\ell\ell\ell} + 4B_{\ell\ell\ell} - 2M_{\ell\ell\ell} - 2m^2 Q_{\ell\ell\ell} \right) - \frac{3}{4} \left(H_{\ell\ell\ell\ell} + 3m^2 V_{\ell\ell\ell\ell} \right). \tag{4.12}$$

5 Resonances From Non-normalizable Modes

We now consider the case when at least one of the $e_i(x), e_j(x), e_k(x)$ is a non-normalizable mode. Since the boundary condition has been set to be a single non-normalizable mode, any non-normalizable modes in the source term must exactly cancel; therefore, at least two of the modes must be non-normalizable. This assumption breaks some of the symmetries that contributed to the previous expressions for resonance channels, and so the resonance conditions must be re-examined starting from the source expression (3.17).

5.1 Two General, Non-normalizable Modes

As a first case, let us assume that the two non-normalizable modes have constant, generic (i.e., non integer) frequency values, $\overline{\omega}$. Applying the time-averaging procedure to the source S_ℓ once again eliminates all contributions except those that satisfy (4.1). Since the basis onto which we are projecting is normalizable, we know that ω_ℓ is given by $\omega_\ell = 2\ell + \Delta^+$. We are now free to choose any one of $\{\omega_i, \omega_j, \omega_k\}$ to be normalizable and consider when the resonance condition is satisfied. In particular, we find that the following combinations are resonant:

$$\omega_i - \omega_j + \omega_k - \omega_\ell = 0 \qquad \Rightarrow \qquad \text{either } \omega_i \text{ or } \omega_k \text{ is normalizable} \tag{5.1}$$

$$\omega_i + \omega_j - \omega_k - \omega_\ell = 0 \qquad \Rightarrow \qquad \text{either } \omega_i \text{ or } \omega_j \text{ is normalizable} \tag{5.2}$$

$$\omega_i - \omega_j - \omega_k + \omega_\ell = 0 \qquad \Rightarrow \qquad \text{either } \omega_j \text{ or } \omega_k \text{ is normalizable.} \tag{5.3}$$

When any of these resonance conditions is met, the remaining normalizable mode will have a frequency equal to ω_ℓ, collapsing all sums over frequencies so that

$$S_\ell = \overline{T}_\ell \, a_\ell^3 \cos(\theta_\ell)\,. \tag{5.4}$$

Collecting the appropriate terms in (3.17), and evaluating the each possible resonance (being careful not to violate restrictions placed on the sums), we find that

$$\begin{aligned}\overline{T}_\ell = &\frac{3\omega_\ell^2\overline{\omega}^2}{\omega_\ell^2 - \overline{\omega}^2} X_{\overline{\omega}\overline{\omega}\ell\ell} - \frac{\overline{\omega}^2(\omega_\ell^2 + \overline{\omega}^2)}{\omega_\ell^2 - \overline{\omega}^2} X_{\ell\ell\overline{\omega}\overline{\omega}} + \omega_\ell^2 X_{\overline{\omega}\overline{\omega}\ell\ell} - \overline{\omega}^2 X_{\ell\ell\overline{\omega}\overline{\omega}} + \frac{2\omega_\ell^2}{\omega_\ell^2 - \overline{\omega}^2} Y_{\overline{\omega}\overline{\omega}\ell\ell} \\ &- \frac{2\overline{\omega}^2}{\omega_\ell^2 - \overline{\omega}^2} Y_{\ell\ell\overline{\omega}\overline{\omega}} - \frac{1}{2} H_{\ell\overline{\omega}\overline{\omega}\ell} + 2m^2\left(\frac{4\omega_\ell^2 - 3\overline{\omega}^2}{\omega_\ell^2 - \overline{\omega}^2}\right) V_{\ell\ell\overline{\omega}\overline{\omega}} - \frac{m^2}{2} V_{\overline{\omega}\overline{\omega}\ell\ell} \\ &+ \omega_\ell^2\overline{\omega}^2 P_{\overline{\omega}\overline{\omega}\ell} - 3\overline{\omega}^2\omega_\ell^2 P_{\ell\ell\overline{\omega}} + \omega_\ell^2 B_{\overline{\omega}\overline{\omega}\ell} + \overline{\omega}^2 B_{\ell\ell\overline{\omega}}\,,\end{aligned} \tag{5.5}$$

after expanding out $Z^\pm_{ijk\ell}$, $H_{ijk\ell}$, $\tilde{Z}^\pm_{ij\ell}$, and $M_{ij\ell}$.

5.2 Special Values of Non-normalizable Frequencies

5.2.1 Differ by an integer

5.2.2 Resonant Values

5.3 Boundary Condition is a Superposition/Fourier Integral of Non-normalizable Modes

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