Response to Referee Report
JHEP_252P_0420_EDREP032460520

I would like to thank the referee for their detailed and constructive report, and I hope the changes I have outlined here have addressed their concerns. Because the report covered several topics, I will address each of these through the numbering given in the report.

1. (a) The issue of the existence of both types of modes being present at all times has been expanded upon at the end of section 2 following equation (2.19).

   (b) If I have misunderstood the referee's comment here, I apologize and would be happy to reconsider their comment. However, I believe the referee may be mistaken on this point. Let us allow for integer frequency values for the non-normalizable solutions, and examine the general solution for a massless scalar in d dimensions (as per the example put forward in the report). The spatial part of the scalar field is given by

$$E_I(x) = K_I \left(\cos x\right)^{\Delta^+} {}_2F_1\left(\frac{\Delta^+ + \omega_I}{2}, \frac{\Delta^+ - \omega_I}{2}; \frac{d}{2}; \sin^2 x\right) \tag{1}$$

   where K_I is a constant, and $\Delta^+ = d$ for a massless field. Now, we consider values of $\omega_I = 2i + d$ and examine E_I the origin:

$$E_I(0) = K_I \, {}_2F_1\left(i + d, -i; \frac{d}{2}; 0\right) \tag{2}$$

   According to the DLMF, the hypergeometric function ${}_2F_1(a, b; c; z)$ exists on the disk $|z| < 1$ provided that $c \neq 0, -1, -2, \ldots$ which the number of dimensions, d, always satisfies. Furthermore, we can write

$$ {}_2F_1(a, b; c; z) = \sum_{s=0}^{\infty} \frac{(a)_s (b)_s}{\Gamma(c + s) s!} z^s \tag{3}$$

   for all values of c. This clearly demonstrates that (1) does not diverge at the origin and therefore constitutes a valid field decomposition.

   (c) I have expanded on the issue of including either one or three non-normalizable modes at the end of section 2. The particular case raised by the referee can be addressed using additional information from [31]. As addressed under equation (2.6), we can write the general solution for a scalar field with generic frequency ω as the linear combination of a purely normalizable part Φ^+ with a purely non-normalizable part Φ^- as $\phi_\omega = C^+\Phi^+ + C^-\Phi^-$ where the coefficients are

$$C^+ = \frac{\Gamma(d/2)\Gamma(-\sqrt{d^2 + 4m^2}/2)}{\Gamma\Big((\Delta^- + \omega)/2\Big)\Gamma\Big((\Delta^- - \omega)/2\Big)} \tag{4}$$

$$C^- = \frac{\Gamma(d/2)\Gamma(\sqrt{d^2 + 4m^2}/2)}{\Gamma\Big((\Delta^+ + \omega)/2\Big)\Gamma\Big((\Delta^+ - \omega)/2\Big)} \tag{5}$$

and the functions $\Phi^{\pm}$ are

$$\Phi^{+} = (\cos(x))^{\Delta^{+}} {}_2F_1\left(\frac{\Delta^{+}+\omega}{2}, \frac{\Delta^{+}-\omega}{2}, \Delta^{+}+1-\frac{d}{2}; \cos^2 x\right) \tag{6}$$

$$\Phi^{-} = (\cos(x))^{\Delta^{-}} {}_2F_1\left(\frac{\Delta^{-}+\omega}{2}, \frac{\Delta^{-}-\omega}{2}, \Delta^{-}+1-\frac{d}{2}; \cos^2 x\right) \tag{7}$$

When the frequency ω is set to an integer that satisfies $\omega = \omega_i = 2i + \Delta^{+}$, (5) is zero and so $\phi_{\omega_i} = C^{+}\Phi^{+}$, i.e. the field is *entirely normalizable*. Interestingly, for the particular choices of $m^2 = 0$, $d = 4$, and $\omega = 2$ as suggested, it is C^{+} than vanishes so that $\phi_{\omega} = C^{-}\Phi^{-}$. However, in this case $\Delta^{-} = 0$ so that

$$\phi\Big|_{\omega=2} \propto {}_2F_1\left(1, -1, -1; \cos^2 x\right) . \tag{8}$$

Since ${}_2F_1(a, b, b; z) = (1-z)^{-a}\ \forall\ b$,

$$\phi\Big|_{\omega=2} \propto \sin^2 x\,, \tag{9}$$

which clearly goes to one as $x \to \pi/2$, making this a normalizable mode.

(d)

2. (a) The final paragraph of section 4 was added to an expanded discussion of the holographic mappings between different components of the bulk field to the energy of the boundary CFT in section 2. Additional references to preliminary material on the topic of bulk/boundary mappings that underpin the theoretical motivation for studying non-normalizable modes were also added.

   (b)

   (c)

   (d)

   (e)

   (f)

   (g)

   (h)

3. (a)

   (b)

   (c)

   (d)

4. (a)

   (b)

   (c)

   (d)

(e)

5. (a) Please see the response to 1(b) where I have addressed the issue of the regularity of the basis functions $\Phi_i^{\pm}(x)$.

   (b) Since the basis functions for the normalizable modes $e_j(x)$ vanish at the boundary $x \to \pi/2 \; \forall \; j$, the first term in equation (2.11) is exactly zero while the summation in the second term collapses to a single expression. I believe the wording here accurately describes the simplification of (2.11).

   (c) This sentence describes the existence of *naturally vanishing* resonances, such as those given in equations (3.5) and (3.7). These terms evaluate to zero for several choices of the scalar field mass $m \neq 0$. When $m = 0$, [17] has shown that Ω and Γ vanish for all allowed frequencies by orthogonality of the basis functions. Since chapter 4 demonstrates that there are no resonances that naturally vanish in this fashion, I believe this sentence is correct as it stands.

   (d) I have rewritten equation (4.2) as an expansion in $\tilde{x}$ to make the expression consistent.

   (e) I have reduced the three expressions to a single expression that allows for relabeling of the normalizable index.

   (f) I have corrected T_ℓ to read $\overline{T}_\ell$ in the paragraph below (4.2).

   (g)

   (h) Yes, (4.9) and (4.10) are indeed applicable for $m^2 = 0$. I have changed the language around (4.9) and (4.10) to make this clearer.

   (i)