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Arbitrary Dimensions, Massive, Non-normalizable Time-Dependent BCs

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1 Introduction

2 Perturbative Expansion

The backreaction between the metric and the scalar field appears at second order in the perturbation,

$$A_2' = -\mu\nu\left[(\dot{\phi}_1)^2 + (\phi_1')^2 + m^2\phi_1^2\sec^2 x\right] + \nu' A_2/\nu \tag{2.1}$$

which can be directly integrated to give

$$A_2 = -\nu\int_0^x dy\,\mu\left((\dot{\phi}_1)^2 + (\phi_1')^2 + m^2\phi_1^2\sec^2 x\right)\,. \tag{2.2}$$

Furthermore, the first non-trivial contribution to the lapse in the boundary time gauge is

$$\delta_2 = \int_x^{\pi/2} dy\,\mu\nu\left((\dot{\phi}_1)^2 + (\phi_1')^2\right)\,. \tag{2.3}$$

For convenience, we have also defined the functions

$$\mu(x) = (\tan x)^{d-1} \quad \text{and} \quad \nu(x) = (d-1)/\mu'\,. \tag{2.4}$$

To aide in evaluating integrals, we first derive the following identities: from the equation for the first-order time-dependent coefficients c_i,

$$\ddot{c}_i + \omega_i^2 c_i = 0 \quad \Rightarrow \quad \partial_t\left(\dot{c}_i^2 + \omega_i^2 c_i^2\right) = \partial_t\mathbb{C}_i = 0\,; \tag{2.5}$$

from the equation definition of $\hat{L}$,

$$\hat{L}e_j = -\frac{1}{\mu}\left(\mu e_j'\right)' + m^2\sec^2 x e_j \quad \Rightarrow \quad \left(\mu e_j'\right)' = \mu\left(m^2\sec^2 x - \omega_j^2\right)e_j\,; \tag{2.6}$$

from considering the expression $(\mu e_i' e_j)'$:

$$\left(\mu e_i' e_j\right)' = \left(m^2\sec^2 x - \omega_i^2\right)\mu e_i e_j + \mu e_i' e_j'\,; \tag{2.7}$$

from permuting i, j above and subtracting to give

$$\frac{\left[\mu(e_i' e_j\omega_j^2 - e_i e_j'\omega_i^2)\right]'}{(\omega_j^2 - \omega_i^2)} = \mu m^2\sec^2 x e_i e_j + \mu e_i' e_j'\,. \tag{2.8}$$

The basis functions $e_j(x)$ are the solutions to the eigenvalue equation

$$\hat{L}e_j(x) = \omega_j^2 e_j(x). \tag{2.9}$$

When considering normalizable solutions only, the basis functions become

$$e_j(x) = k_j \left(\cos(x)\right)^{\Delta^+} P_j^{(d/2-1,\Delta^+-d/2)}\left(\cos(2x)\right) \tag{2.10}$$

$$k_j = 2\sqrt{\frac{(j+\Delta^+/2)\Gamma(j+1)\Gamma(j+\Delta^+)}{\Gamma(j+d/2)\Gamma(j+\Delta^+-d/2+1)}}\,, \tag{2.11}$$

with eigenvalues $\omega_j = 2j + \Delta^+$, $j \in \mathbb{Z}^*$, and Δ^+ as the positive root of $\Delta(\Delta - d) = m^2$. On the other hand, for non-normalizable solutions with arbitrary frequency, the basis functions are

$$E_\omega(x) = \left(\cos(x)\right)^{\Delta_+} {}_2F_1\left(\frac{\Delta_+ + \omega}{2}, \frac{\Delta_+ - \omega}{2}, d/2; \sin^2(x)\right). \tag{2.12}$$

3 $\mathcal{O}(\epsilon^3)$ Source Terms

At third order in ϵ, the equation for ϕ_3 contains a source S given by

$$\ddot{\phi}_3 + \hat{L}\phi_3 = S = 2(A_2 - \delta_2)\ddot{\phi}_1 + (\dot{A}_2 - \dot{\delta}_2)\dot{\phi}_1 + (A_2' - \delta_2')\phi_1' + m^2 A_2 \phi_1 \sec^2 x\,. \tag{3.1}$$

Following the steps outlined in Appendix A, and employing the solution $c_i(t) = a_i \cos(\omega_i t + b_i) = a_i \cos\theta_i$, the source term becomes

$$\begin{aligned}
S_\ell = {} & \frac{1}{4}\sum_{\substack{i,j,k\\k\neq\ell}}^{\infty} \frac{a_i a_j a_k \omega_k}{\omega_\ell^2 - \omega_k^2}\Big[Z^-_{ijk\ell}(\omega_i+\omega_j-2\omega_k)\cos(\theta_i+\theta_j-\theta_k) - Z^-_{ijk\ell}(\omega_i+\omega_j+2\omega_k)\cos(\theta_i+\theta_j+\theta_k) - \\
& \quad + Z^+_{ijk\ell}(\omega_i-\omega_j+2\omega_k)\cos(\theta_i-\theta_j+\theta_k) - Z^+_{ijk\ell}(\omega_i-\omega_j-2\omega_k)\cos(\theta_i-\theta_j-\theta_k)\Big] \\
& + \frac{1}{2}\sum_{\substack{i,j,k\\i\neq j}}^{\infty} a_i a_j a_k \omega_j \left(H_{ijk\ell} + m^2 V_{jki\ell} - 2\omega_k^2 X_{ijk\ell}\right)\Big[\frac{1}{\omega_i-\omega_j}\left(\cos(\theta_i-\theta_j-\theta_k) + \cos(\theta_i-\theta_j+\theta_k)\right) \\
& \quad - \frac{1}{\omega_i+\omega_j}\left(\cos(\theta_i+\theta_j-\theta_k) + \cos(\theta_i+\theta_j+\theta_k)\right)\Big] \\
& - \frac{1}{4}\sum_{i,j,k}^{\infty} a_i a_j a_k\Big[\left(2\omega_j\omega_k X_{ijk\ell} + m^2 V_{ijk\ell}\right)\cos(\theta_i+\theta_j-\theta_k) - \left(2\omega_j\omega_k X_{ijk\ell} - m^2 V_{ijk\ell}\right)\cos(\theta_i-\theta_j-\theta_k) \\
& \quad + \left(2\omega_j\omega_k X_{ijk\ell} + m^2 V_{ijk\ell}\right)\cos(\theta_i-\theta_j+\theta_k) - \left(2\omega_j\omega_k X_{ijk\ell} - m^2 V_{ijk\ell}\right)\cos(\theta_i+\theta_j+\theta_k)\Big] \\
& + \frac{1}{4}\sum_{i,j}^{\infty} a_i a_j a_\ell \omega_\ell\Big[\tilde{Z}^-_{ij\ell}(\omega_i+\omega_j-2\omega_\ell)\cos(\theta_i+\theta_j-\theta_\ell) - \tilde{Z}^-_{ij\ell}(\omega_i+\omega_j+2\omega_\ell)\cos(\theta_i+\theta_j+\theta_\ell) \\
& \quad + \tilde{Z}^+_{ij\ell}(\omega_i-\omega_j+2\omega_\ell)\cos(\theta_i-\theta_j+\theta_\ell) - \tilde{Z}^+_{ij\ell}(\omega_i-\omega_j-2\omega_\ell)\cos(\theta_i-\theta_j-\theta_\ell)\Big] \\
& - \frac{1}{4}\sum_{i,j}^{\infty} a_i^2 a_j \left(H_{iij\ell} + m^2 V_{jii\ell} - 2\omega_j^2 X_{iij\ell}\right)\left[\cos(2\theta_i-\theta_j) + \cos(2\theta_i+\theta_j)\right] \\
& - \frac{1}{2}\sum_{i,j}^{\infty} a_i^2 a_j \left(H_{iij\ell} + m^2 V_{jii\ell} - 2\omega_j^2 X_{iij\ell} + 4\omega_i^2\omega_j^2 P_{j\ell i} + 2\omega_i^2(M_{j\ell i} + m^2 Q_{j\ell i})\right)\cos\theta_j.
\end{aligned} \tag{3.2}$$

4 Resonances From Normalizable Solutions

Consider the case where each of the basis functions are given by normalizable solutions. After time-averaging, resonant contributions come from the set of conditions

$$\omega_i \pm \omega_j \pm \omega_k = \pm\omega_\ell \tag{4.1}$$

which separates into three distinct cases

$$\begin{aligned}
&\omega_i + \omega_j + \omega_k = \omega_\ell \qquad (+++) &(4.2)\\
&\omega_i - \omega_j - \omega_k = \omega_\ell \qquad (+--) &(4.3)\\
&\omega_i + \omega_j - \omega_k = \omega_\ell \qquad (++-) &(4.4)
\end{aligned}$$

4.1 $(+++)$

These resonant contributions come from the condition $\omega_i + \omega_j + \omega_k = \omega_\ell$, and are of the form

$$S_\ell = \underbrace{\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}}_{\omega_i+\omega_j+\omega_k=\omega_\ell} \Omega_{ijk\ell}\, a_i a_j a_k \cos\left(\theta_i + \theta_j + \theta_k\right) + \dots\,, \tag{4.5}$$

where

$$\begin{aligned}
\Omega_{ijk\ell} = &-\frac{1}{12}H_{ijk\ell}\frac{\omega_j(\omega_i+\omega_k+2\omega_j)}{(\omega_i+\omega_j)(\omega_j+\omega_k)} - \frac{1}{12}H_{ikj\ell}\frac{\omega_k(\omega_i+\omega_j+2\omega_k)}{(\omega_i+\omega_k)(\omega_j+\omega_k)} - \frac{1}{12}H_{jik\ell}\frac{\omega_i(\omega_j+\omega_k+2\omega_i)}{(\omega_i+\omega_j)(\omega_i+\omega_k)}\\
&-\frac{m^2}{12}V_{ijk\ell}\left(1+\frac{\omega_j}{\omega_j+\omega_k}+\frac{\omega_i}{\omega_i+\omega_k}\right) - \frac{m^2}{12}V_{jki\ell}\left(1+\frac{\omega_j}{\omega_i+\omega_j}+\frac{\omega_k}{\omega_i+\omega_k}\right)\\
&-\frac{m^2}{12}V_{kij\ell}\left(1+\frac{\omega_i}{\omega_i+\omega_j}+\frac{\omega_k}{\omega_j+\omega_k}\right) + \frac{1}{6}\omega_j\omega_k X_{ijk\ell}\left(1+\frac{\omega_j}{\omega_i+\omega_k}+\frac{\omega_k}{\omega_i+\omega_j}\right)\\
&+\frac{1}{6}\omega_i\omega_k X_{jki\ell}\left(1+\frac{\omega_i}{\omega_j+\omega_k}+\frac{\omega_k}{\omega_i+\omega_j}\right) + \frac{1}{6}\omega_i\omega_j X_{kij\ell}\left(1+\frac{\omega_i}{\omega_j+\omega_k}+\frac{\omega_j}{\omega_i+\omega_k}\right)\\
&-\frac{1}{12}Z^-_{ijk\ell}\left(\frac{\omega_k}{\omega_i+\omega_j}\right) - \frac{1}{12}Z^-_{ikj\ell}\left(\frac{\omega_j}{\omega_i+\omega_k}\right) - \frac{1}{12}Z^-_{jki\ell}\left(\frac{\omega_i}{\omega_j+\omega_k}\right)\,.
\end{aligned} \tag{4.6}$$

4.2 $(+--)$

These contributions arise from the condition $\omega_i - \omega_j - \omega_k = \omega_\ell$, are of the form

$$S_\ell = \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \Gamma_{(j+k+\ell+\Delta^+)jk\ell}\, a_j a_k a_{(j+k+\ell+\Delta^+)} \cos\left(\theta_{(j+k+\ell+\Delta^+)} - \theta_j - \theta_k\right) + \dots\,, \tag{4.7}$$

where

$$\begin{aligned}\Gamma_{ijk\ell} = &\frac{1}{4}H_{ijk\ell}\frac{\omega_j(\omega_k-\omega_i+2\omega_j)}{(\omega_i-\omega_j)(\omega_j+\omega_k)} + \frac{1}{4}H_{jki\ell}\frac{\omega_k(\omega_j-\omega_i+2\omega_k)}{(\omega_i-\omega_k)(\omega_j+\omega_k)} + \frac{1}{4}H_{kij\ell}\frac{\omega_i(\omega_j+\omega_k-2\omega_i)}{(\omega_i-\omega_j)(\omega_i-\omega_k)} \\ &- \frac{1}{2}\omega_j\omega_k X_{ijk\ell}\left(\frac{\omega_k}{\omega_i-\omega_j}+\frac{\omega_j}{\omega_i-\omega_k}-1\right) + \frac{1}{2}\omega_i\omega_k X_{jki\ell}\left(\frac{\omega_k}{\omega_i-\omega_j}+\frac{\omega_i}{\omega_j+\omega_k}-1\right) \\ &+ \frac{1}{2}\omega_i\omega_j X_{kij\ell}\left(\frac{\omega_j}{\omega_i-\omega_k}+\frac{\omega_i}{\omega_j+\omega_k}-1\right) + \frac{m^2}{4}V_{jki\ell}\left(\frac{\omega_j}{\omega_i-\omega_j}+\frac{\omega_k}{\omega_i-\omega_k}-1\right) \\ &- \frac{m^2}{4}V_{kij\ell}\left(\frac{\omega_i}{\omega_i-\omega_j}+\frac{\omega_k}{\omega_j+\omega_k}+1\right) - \frac{m^2}{4}V_{ijk\ell}\left(\frac{\omega_i}{\omega_i-\omega_k}+\frac{\omega_j}{\omega_j+\omega_k}+1\right) \\ &+ \frac{1}{4}Z^-_{kji\ell}\left(\frac{\omega_i}{\omega_j+\omega_k}\right) - \frac{1}{4}Z^+_{ijk\ell}\left(\frac{\omega_k}{\omega_i-\omega_j}\right) - \frac{1}{4}Z^+_{jki\ell}\left(\frac{\omega_j}{\omega_i-\omega_k}\right) . \end{aligned} \tag{4.8}$$

4.3 Naturally Vanishing Resonances

It has been shown that when $m = 0$, and only normalizable modes are considered, (4.6) and (4.8) vanish by the orthogonality of the basis functions. **Maybe show that mass-dependent terms vanish for normalizable modes?**

4.4 $(++-)$

These contributions arise from the resonant condition $\omega_i + \omega_j = \omega_k + \omega_\ell$, can be written as

$$\begin{aligned} S_\ell = &T_\ell a_\ell^3 \cos(\theta_\ell+\theta_\ell-\theta_\ell) + \sum_{i\neq\ell}^{\infty} R_{i\ell}\, a_i^2 a_\ell \cos(\theta_i+\theta_\ell-\theta_i) \\ &+ \sum_{i\neq\ell}^{\infty}\sum_{j\neq\ell}^{\infty} S_{ij(i+j-\ell)\ell}\, a_i a_j a_{(i+j-\ell)} \cos(\theta_i+\theta_j-\theta_{i+j-\ell}) + \dots \end{aligned} \tag{4.9}$$

where each of the coefficients is given by

$$\begin{aligned} S_{ijk\ell} = &-\frac{1}{4}H_{kij\ell}\frac{\omega_i(\omega_j-\omega_k+2\omega_i)}{(\omega_i-\omega_k)(\omega_i+\omega_j)} - \frac{1}{4}H_{ijk\ell}\frac{\omega_j(\omega_i-\omega_k+2\omega_j)}{(\omega_j-\omega_k)(\omega_i+\omega_j)} - \frac{1}{4}H_{jki\ell}\frac{\omega_k(\omega_i+\omega_j-2\omega_k)}{(\omega_i-\omega_k)(\omega_j-\omega_k)} \\ &- \frac{1}{2}\omega_j\omega_k X_{ijk\ell}\left(\frac{\omega_j}{\omega_i-\omega_k}-\frac{\omega_k}{\omega_i+\omega_j}+1\right) - \frac{1}{2}\omega_i\omega_k X_{jki\ell}\left(\frac{\omega_i}{\omega_j-\omega_k}-\frac{\omega_k}{\omega_i+\omega_j}+1\right) \\ &+ \frac{1}{2}\omega_i\omega_j X_{kij\ell}\left(\frac{\omega_i}{\omega_j-\omega_k}+\frac{\omega_j}{\omega_i-\omega_k}+1\right) - \frac{m^2}{4}V_{ijk\ell}\left(\frac{\omega_i}{\omega_i-\omega_k}+\frac{\omega_j}{\omega_j-\omega_k}+1\right) \\ &+ \frac{m^2}{4}V_{jki\ell}\left(\frac{\omega_k}{\omega_i-\omega_k}-\frac{\omega_j}{\omega_i+\omega_j}-1\right) + \frac{m^2}{4}V_{kij\ell}\left(\frac{\omega_k}{\omega_j-\omega_k}-\frac{\omega_i}{\omega_i+\omega_j}-1\right) \\ &+ \frac{1}{4}Z^-_{ijk\ell}\left(\frac{\omega_k}{\omega_i+\omega_j}\right) + \frac{1}{4}Z^+_{ikj\ell}\left(\frac{\omega_j}{\omega_i-\omega_k}\right) + \frac{1}{4}Z^+_{jki\ell}\left(\frac{\omega_i}{\omega_j-\omega_k}\right) , \end{aligned} \tag{4.10}$$

$$
\begin{aligned}
R_{i\ell} = &\left(\frac{\omega_i^2}{\omega_\ell^2-\omega_i^2}\right)\left(Y_{i\ell\ell i}-Y_{i\ell i\ell}+\omega_\ell^2(X_{i\ell i\ell}-X_{\ell i\ell i})\right)+\left(\frac{\omega_i^2}{\omega_\ell^2-\omega_i^2}\right)\left(H_{\ell ii\ell}+m^2V_{ii\ell\ell}-2\omega_i^2X_{\ell ii\ell}\right)\\
&-\left(\frac{\omega_\ell^2}{\omega_\ell^2-\omega_i^2}\right)\left(H_{i\ell i\ell}+m^2V_{\ell ii\ell}-2\omega_i^2X_{i\ell i\ell}\right)-\frac{m^2}{4}(V_{i\ell i\ell}+V_{ii\ell\ell})+\omega_i^2\omega_\ell^2(P_{ii\ell}-2P_{\ell\ell i})\\
&-\omega_i\omega_\ell X_{i\ell i\ell}-\frac{3m^2}{2}V_{\ell ii\ell}-\frac{1}{2}H_{ii\ell\ell}+\omega_\ell^2B_{ii\ell}-\omega_i^2M_{\ell\ell i}-m^2\omega_i^2Q_{\ell\ell i}\,,
\end{aligned}
\tag{4.11}
$$

$$
T_\ell = \frac{1}{2}\omega_\ell^2\left(X_{\ell\ell\ell\ell}+4B_{\ell\ell\ell}-2M_{\ell\ell\ell}-2m^2Q_{\ell\ell\ell}\right)-\frac{3}{4}\left(H_{\ell\ell\ell\ell}+3m^2V_{\ell\ell\ell\ell}\right)\,. \tag{4.12}
$$

5 Resonances From Non-normalizable Modes

Discuss falloff of A2 and delta2 with three NN modes, but don't calculate anything new. Mention overlap of NN and normalizable cases when omega(i) $\pm$ omega(j) = NN frequency. Then focus on two NN modes.

We now consider the case when at least one of the $e_i(x), e_j(x), e_k(x)$ is a non-normalizable mode. Since the boundary condition has been set to be a single non-normalizable mode, any non-normalizable modes in the source term must exactly cancel; therefore, at least two of the modes must be non-normalizable. This assumption breaks some of the symmetries that contributed to the previous expressions for resonance channels, and so the resonance conditions must be re-examined starting from the source expression (3.2).

An important consideration is also the effect of non-normalizable modes on the perturbative expansion that leads to the source equations. Since non-normalizable solutions do not have well-defined norms, we do not know *a priori* that the inner products described in Appendix A are still finite. To investigate this, consider the second-order metric function A_2

$$
A_2 = -\nu\int_0^x dy\,\mu\left((\dot{\phi}_1)^2+(\phi_1')^2+m^2\phi_1^2\sec^2 x\right)\,, \tag{5.1}
$$

in the limit of $x\to\pi/2$ when the scalar field is given by a generic superposition of normalizable and non-normalizable eigenfunctions:

$$
\phi_1(t,x) = \sum_\alpha e_\alpha\cos(\omega_\alpha t)+\sum_i a_ie_i\cos(\omega_i t+b_i)\,. \tag{5.2}
$$

Focusing on the position-dependence only, we find that

$$
\lim_{\tilde{x}\to 0} A_2(\tilde{x}=\pi/2-x) = \tilde{x}^{-\xi}\left(\frac{2\tilde{x}^{2+d}}{2-\xi}-\frac{\tilde{x}^d(1+(\Delta^-)^2)}{\xi}\right) \tag{5.3}
$$

where we have defined $\xi=\sqrt{d^2+4m^2}$. In the massless case, $\xi=d$ and all powers of $\tilde{x}$ are non-negative and thus the limit is finite; for tachyonic masses, $0\le\xi<d$ and the limit is again finite. However, for massive scalars, at least one of the terms above diverges as $\xi\to 0$. This case would require the addition of counter-terms in the bulk action to cancel such divergences

– we will not consider this case presently. Thus, we will restrict our discussion to $m^2 \leq 0$ to avoid these issues. A similar check on the near-boundary behaviour of δ_2 shows that, in the massless case, the gauge condition $\delta_2(t, x = \pi/2)$ remains unchanged by the addition of non-normalizable modes.

5.1 Two General, Non-normalizable Modes

As a first case, let us assume that the two non-normalizable modes have constant, generic (i.e., non integer) frequency values, $\overline{\omega}$. Applying the time-averaging procedure to the source S_ℓ once again eliminates all contributions except those that satisfy (4.1). Since the basis onto which we are projecting is normalizable, we know that ω_ℓ is given by $\omega_\ell = 2\ell + \Delta^+$. We are now free to choose any one of $\{\omega_i, \omega_j, \omega_k\}$ to be normalizable and consider when the resonance condition is satisfied. In particular, we find that the following combinations are resonant:

$$\omega_i - \omega_j + \omega_k - \omega_\ell = 0 \qquad \Rightarrow \qquad \text{either } \omega_i \text{ or } \omega_k \text{ is normalizable} \tag{5.4}$$

$$\omega_i + \omega_j - \omega_k - \omega_\ell = 0 \qquad \Rightarrow \qquad \text{either } \omega_i \text{ or } \omega_j \text{ is normalizable} \tag{5.5}$$

$$\omega_i - \omega_j - \omega_k + \omega_\ell = 0 \qquad \Rightarrow \qquad \text{either } \omega_j \text{ or } \omega_k \text{ is normalizable.} \tag{5.6}$$

When any of these resonance conditions is met, the remaining normalizable mode will have a frequency equal to ω_ℓ, collapsing all sums over frequencies so that

$$S_\ell = \overline{T}_\ell \, a_\ell A_{\overline{\omega}}^2 \cos(\theta_\ell)\,, \tag{5.7}$$

where the non-normalizable modes have constant amplitudes $A_{\overline{\omega}}$. Collecting the appropriate terms in (3.2), and evaluating the each possible resonance (being careful not to violate restrictions placed on the sums), we find that

$$\begin{aligned}
\overline{T}_\ell = {} & \frac{1}{2} Z^-_{\ell\overline{\omega}\overline{\omega}\ell}\left(\frac{\overline{\omega}}{\omega_\ell + \overline{\omega}}\right) + \frac{1}{2} Z^+_{\ell\overline{\omega}\overline{\omega}\ell}\left(\frac{\overline{\omega}}{\omega_\ell - \overline{\omega}}\right) - H_{\overline{\omega}\ell\overline{\omega}\ell}\left(\frac{\omega_\ell^2}{\omega_\ell^2 - \overline{\omega}^2}\right) + H_{\overline{\omega}\overline{\omega}\ell\ell}\left(\frac{\overline{\omega}^2}{\omega_\ell^2 - \overline{\omega}^2}\right) \\
& + 2\overline{\omega}^2 X_{\overline{\omega}\overline{\omega}\ell\ell} + 2X_{\overline{\omega}\overline{\omega}\ell\ell}\left(\frac{\overline{\omega}^2\omega_\ell^2}{\omega_\ell^2 - \overline{\omega}^2}\right) - \overline{\omega}^2 X_{\ell\ell\overline{\omega}\overline{\omega}} - 2X_{\ell\ell\overline{\omega}\overline{\omega}}\left(\frac{\overline{\omega}^4}{\omega_\ell^2 - \overline{\omega}^2}\right) \\
& + m^2 V_{\overline{\omega}\overline{\omega}\ell\ell}\left(\frac{\overline{\omega}^2}{\omega_\ell^2 - \overline{\omega}^2}\right) - m^2 V_{\ell\ell\overline{\omega}\overline{\omega}}\left(\frac{\omega_\ell^2}{\omega_\ell^2 - \overline{\omega}^2}\right) - 2m^2 V_{\ell\ell\overline{\omega}\overline{\omega}} - \frac{m^2}{2} V_{\overline{\omega}\overline{\omega}\ell\ell} \\
& + 2\omega_\ell^2 \tilde{Z}^+_{\overline{\omega}\overline{\omega}\ell} - H_{\overline{\omega}\overline{\omega}\ell\ell} - 4\overline{\omega}^2\omega_\ell^2 P_{\ell\ell\overline{\omega}} - 2\overline{\omega}^2 M_{\ell\ell\overline{\omega}} - 2m^2\overline{\omega}^2 Q_{\ell\ell\overline{\omega}}\,.
\end{aligned} \tag{5.8}$$

It is easy to see that when $\overline{\omega} = \omega_\ell$, there will be singular terms in (5.8). Using the definitions of $Z^\pm_{ijk\ell}$ and $H_{ijk\ell}$, some simplification is possible; however, $\overline{T}_\ell$ still evaluates to $0/0 +$ (non-singular). Using L'Hôpital's rule to evaluate the undetermined expression, we find that the limit of (5.8) is

$$\lim_{\overline{\omega}\to\omega_\ell} \overline{T}_\ell = 3Y_{\ell\ell\ell\ell} + \omega_\ell^2\left(B_{\ell\ell\ell} - 2m^2 Q_{\ell\ell\ell} - \omega_\ell^2 P_{\ell\ell\ell}\right) - \frac{5m^2}{2} V_{\ell\ell\ell\ell}\,. \tag{5.9}$$

Using these two expressions, we evaluate $\overline{T}_\ell$ for $\ell < 10$ and a variety of $\overline{\omega}$ values for both a massless scalar (figure 1) and a tachyon (figure 2).

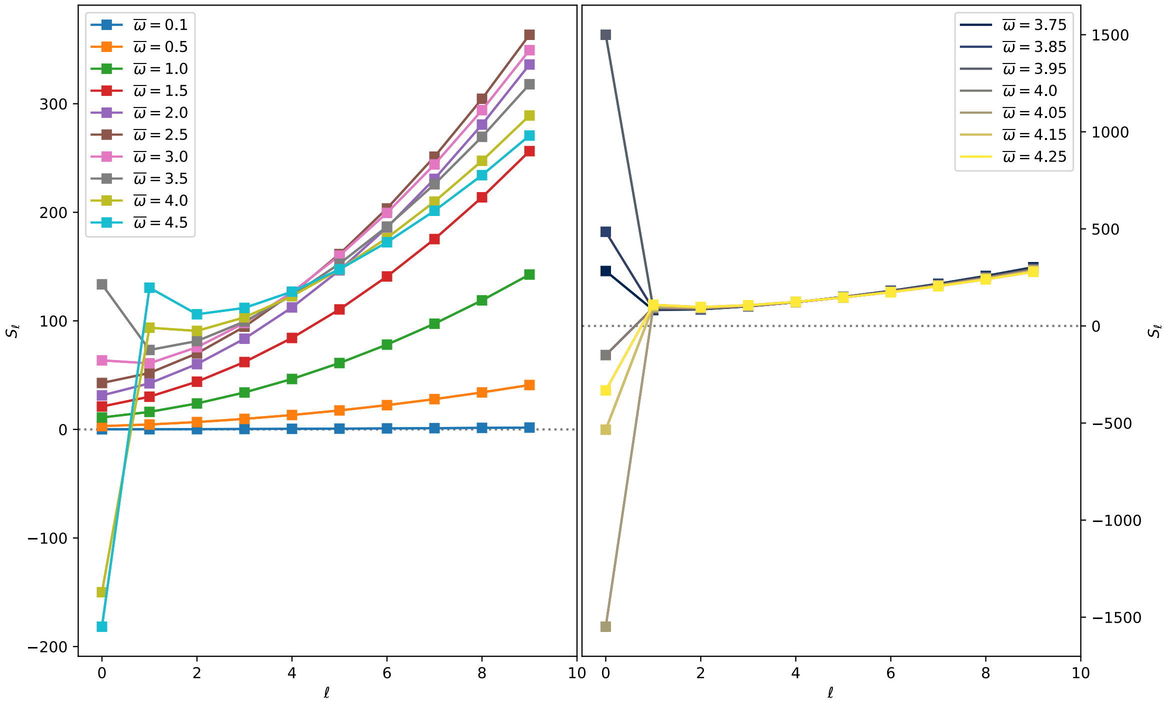

Figure 1: *Left:* Evaluating S_ℓ (rescaled by the amplitudes) when $m^2 = 0$ for various choices of $\overline{\omega}$. *Right*: As $\overline{\omega} \to \omega_\ell^-$, $S_\ell \to \infty$ and as $\overline{\omega} \to \omega_\ell^+$, $S_\ell \to -\infty$. When $\overline{\omega} = \omega_\ell$, S_ℓ is given by (5.8)

5.2 Special Values of Non-normalizable Frequencies

Focus on non-arbitrary values of the non-normalizable frequencies.

5.2.1 Add to an integer

Choose two of the modes to be non-normalizable with frequencies $\overline{\omega}_1$ and $\overline{\omega}_2$ that add to give an integer: $\overline{\omega}_1 + \overline{\omega}_2 = 2n$ where $n = 1, 2, 3, \ldots$ (note that the $n = 0$ case means that both ω_1 and ω_2 would need to be zero by the positive-frequency requirement and so would not contribute). Furthermore, either frequency need not be an integer and therefore the difference $|\overline{\omega}_1 - \overline{\omega}_2|$ will not be an integer.

When we consider possible resonance channels, we see that resonances can be grouped into

$$(++):\ \omega_I + 2n = \omega_\ell \quad I \in \{i, j, k\}\ \forall\ \ell \geq n \tag{5.10}$$

$$(+-):\ \omega_I - 2n = \omega_\ell \quad I \in \{i, j, k\}\ \forall\ n \tag{5.11}$$

for any $m_{BF}^2 \leq m^2 < 0$. However, for a massless scalar, we have an additional channel

$$(-+):\ -\omega_I + 2n = \omega_\ell \quad I \in \{i, j, k\}\ \forall\ n \geq \ell + d \tag{5.12}$$

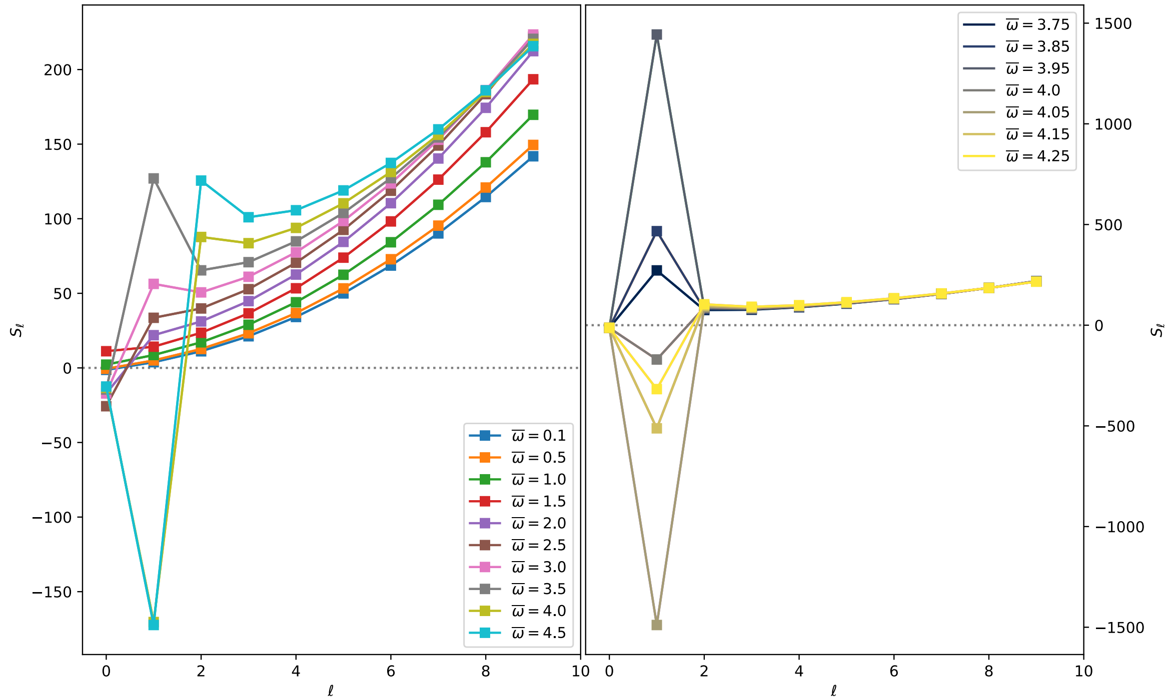

Figure 2: *Left:* Evaluating (5.8) for the same choices of $\overline{\omega}$ as in figure 1 but for a tachyon with $m^2 = -4.0$.

Adding the channels together, the total source term is

$$\begin{aligned} S_\ell = {} & \Theta\left(\ell - n\right) \overline{R}^{(++)}_{(\ell-n)12\ell} \cos\left(\theta_{(\ell-n)} + 2nt\right) + \overline{R}^{(+-)}_{(\ell+n)12\ell} \cos\left(\theta_{(\ell+n)} - 2nt\right) \\ & + \delta(m^2)\Theta\left(n - \ell - d\right) \overline{R}^{(-+)}_{(n-\ell-d)12\ell} \cos\left(\theta_{(n-\ell-d)} + 2nt\right) + \overline{T}_{12\ell} \cos\left(\theta_\ell\right) , \end{aligned} \tag{5.13}$$

where the Heaviside step function $\Theta(x)$ enforces the restrictions on the indices in (5.10) and (5.12).

Examining each channel individually, we find that

$$\begin{aligned}
\overline{R}^{(++)}_{i12\ell} = &-\frac{1}{4}\sum_{\overline{\omega}_1+\overline{\omega}_2=2n}^{\overline{\omega}_2\neq\omega_\ell}\frac{\overline{\omega}_2}{\omega_\ell-\overline{\omega}_2}Z^{-}_{i12\ell}-\frac{1}{4}\sum_{\overline{\omega}_1+\overline{\omega}_2=2n}^{\overline{\omega}_1\neq\omega_\ell}\frac{\overline{\omega}_1}{\omega_\ell-\overline{\omega}_1}Z^{-}_{i21\ell}-\frac{1}{8n}\sum_{\overline{\omega}_1+\overline{\omega}_2=2n}(\omega_\ell-2n)\,Z^{-}_{12i\ell}\\
&+\frac{1}{2}\sum_{\overline{\omega}_1+\overline{\omega}_2=2n}\Big\{\overline{\omega}_1\overline{\omega}_2X_{i12\ell}+(\omega_\ell-2n)\left(\overline{\omega}_1X_{21i\ell}+\overline{\omega}_2X_{12i\ell}\right)-\frac{m^2}{2}\left(V_{i12\ell}+V_{i21\ell}+V_{12i\ell}\right)\Big\}\\
&-\frac{1}{4}\sum_{\overline{\omega}_1+\overline{\omega}_2=2n}^{\omega_i\neq\overline{\omega}_1}\frac{1}{\omega_\ell-\overline{\omega}_2}\Big\{\overline{\omega}_1\left(H_{i12\ell}+m^2V_{12i\ell}-2\overline{\omega}_2^2X_{i12\ell}\right)+(\omega_\ell-2n)\left(H_{1i2\ell}+m^2V_{i21\ell}-2\overline{\omega}_2^2X_{1i2\ell}\right)\Big\}\\
&-\frac{1}{4}\sum_{\overline{\omega}_1+\overline{\omega}_2=2n}^{\omega_i\neq\overline{\omega}_2}\frac{1}{\omega_\ell-\overline{\omega}_1}\Big\{\overline{\omega}_2\left(H_{i21\ell}+m^2V_{21i\ell}-2\overline{\omega}_1^2X_{i21\ell}\right)+(\omega_\ell-2n)\left(H_{2i1\ell}+m^2V_{i12\ell}-2\overline{\omega}_1^2X_{2i1\ell}\right)\Big\}\\
&-\frac{1}{8n}\sum_{\overline{\omega}_1+\overline{\omega}_2=2n}^{\overline{\omega}_1\neq\overline{\omega}_2}\Big\{\overline{\omega}_1H_{21i\ell}+\overline{\omega}_2H_{12i\ell}+m^2\left(\overline{\omega}_1V_{1i2\ell}+\overline{\omega}_2V_{2i1\ell}\right)-(\omega_\ell-2n)^2\left(\overline{\omega}_1X_{21i\ell}+\overline{\omega}_2X_{12i\ell}\right)\Big\}
\end{aligned}\tag{5.14}$$

The notation $X_{i12\ell}$ corresponds to evaluating $X_{ijk\ell}$ with $\omega_j=\overline{\omega}_1$ and $\omega_k=\overline{\omega}_2$.

Next, we find that

$$\begin{aligned}
\overline{R}^{(+-)}_{i12\ell} = &-\frac{1}{4}\sum_{\overline{\omega}_1+\overline{\omega}_2=2n}\Big\{\frac{(\omega_\ell+2n)}{2n}Z^{-}_{12i\ell}+2(\omega_\ell+2n)\left(\overline{\omega}_1X_{21i\ell}+\overline{\omega}_2X_{12i\ell}\right)\\
&-\frac{\overline{\omega}_1}{(\omega_\ell+\overline{\omega}_2)}\left(H_{i12\ell}+m^2V_{12i\ell}-2\overline{\omega}_2^2X_{i12\ell}\right)+\frac{(\omega_\ell+2n)}{(\omega_\ell+\overline{\omega}_2)}\left(H_{1i2\ell}+m^2V_{i21\ell}-2\overline{\omega}_2^2X_{1i2\ell}\right)\\
&-\frac{\overline{\omega}_2}{(\omega_\ell+\overline{\omega}_1)}\left(H_{i21\ell}+m^2V_{21i\ell}-2\overline{\omega}_1^2X_{i21\ell}\right)+\frac{(\omega_\ell+2n)}{(\omega_\ell+\overline{\omega}_1)}\left(H_{2i1\ell}+m^2V_{i12\ell}-2\overline{\omega}_1^2X_{2i1\ell}\right)\\
&-2\overline{\omega}_1\overline{\omega}_2X_{i12\ell}+m^2\left(V_{12i\ell}+V_{i12\ell}+V_{i21\ell}\right)\Big\}+\frac{1}{4}\sum_{\overline{\omega}_1+\overline{\omega}_2=2n}^{\overline{\omega}_2\neq\omega_\ell}\frac{\overline{\omega}_1\overline{\omega}_2(\omega_\ell+2n)}{\omega_\ell+\overline{\omega}_2}\left(X_{21i\ell}-X_{\ell i12}\right)\\
&+\frac{1}{4}\sum_{\overline{\omega}_1+\overline{\omega}_2=2n}^{\overline{\omega}_1\neq\omega_\ell}\frac{\overline{\omega}_1\overline{\omega}_2(\omega_\ell+2n)}{\omega_\ell+\overline{\omega}_1}\left(X_{12i\ell}-X_{\ell i12}\right).
\end{aligned}\tag{5.15}$$

In figure 5, values of $\overline{R}^{(+-)}_{(\ell+n)12\ell}$ are calculated up to $\ell=10$ for $n=2$ and steps of $\Delta\overline{\omega}=0.25$.

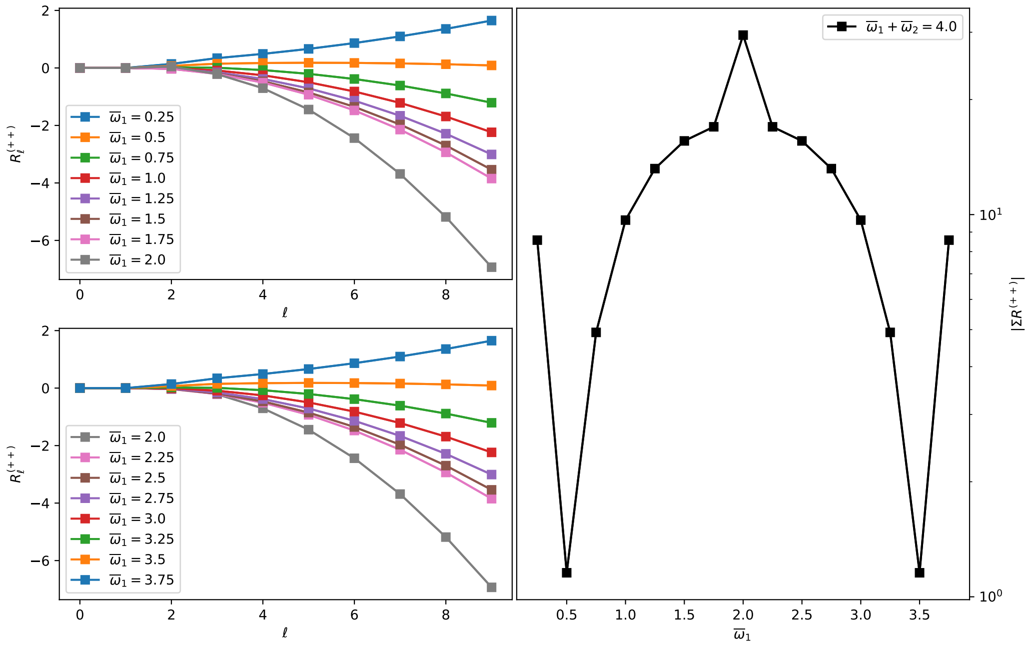

Figure 3: *Left, above*: Individual values of $\overline{R}^{(++)}_{(\ell-n)12\ell}$ for $m^2 = -4.0$ as a function of ℓ for $\overline{\omega}_1 + \overline{\omega}_2 = 2n$ with $n = 4$ when $\overline{\omega}_1 \leq n$. *Left, below*: The same function, but for $\overline{\omega}_1 \geq n$; note the symmetry in values as $\overline{\omega}_1 \leftrightarrow \overline{\omega}_2$. *Right*: The absolute value of the sum of the $(++)$ source terms for choices of $\overline{\omega}_1$.

When $m^2 = 0$, we have contributions from

$$\begin{aligned}
\overline{R}^{(-+)}_{i12\ell} = &\frac{1}{4} \sum_{\overline{\omega}_1+\overline{\omega}_2=2n}^{\overline{\omega}_2\neq\omega_\ell} \frac{\overline{\omega}_2}{\omega_\ell - \overline{\omega}_2} Z^+_{i12\ell} + \frac{1}{4} \sum_{\overline{\omega}_1+\overline{\omega}_2=2n}^{\overline{\omega}_1\neq\omega_\ell} \frac{\overline{\omega}_1}{\omega_\ell - \overline{\omega}_1} Z^+_{i21\ell} + \frac{1}{4} \sum_{\overline{\omega}_1+\overline{\omega}_2=2n}^{i\neq\ell} \left(\frac{2n-\omega_\ell}{2n}\right) Z^-_{12i\ell} \\
&+ \frac{1}{4} \sum_{\overline{\omega}_1+\overline{\omega}_2=2n}^{\overline{\omega}_1\neq\omega_i} \frac{1}{\omega_i - \overline{\omega}_1} \Big\{ \overline{\omega}_1 \left(H_{i12\ell} - 2\overline{\omega}_2^2 X_{i12\ell}\right) - (2n-\omega_\ell)\left(H_{1i2\ell} - 2\overline{\omega}_2^2 X_{1i2\ell}\right) \Big\} \\
&+ \frac{1}{4} \sum_{\overline{\omega}_1+\overline{\omega}_2=2n}^{\overline{\omega}_2\neq\omega_i} \frac{1}{\omega_i - \overline{\omega}_2} \Big\{ \overline{\omega}_2 \left(H_{i21\ell} - 2\overline{\omega}_1^2 X_{i21\ell}\right) - (2n-\omega_\ell)\left(H_{2i1\ell} - 2\overline{\omega}_1^2 X_{2i1\ell}\right) \Big\} \\
&- \frac{1}{8n} \sum_{\overline{\omega}_1+\overline{\omega}_2=2n}^{\overline{\omega}_1\neq\overline{\omega}_2} \Big\{ \overline{\omega}_1 H_{21i\ell} + \overline{\omega}_2 H_{12i\ell} - 2(2n-\omega_\ell)^2 \left(\overline{\omega}_1 X_{21i\ell} + \overline{\omega}_2 X_{12i\ell}\right) \Big\} \\
&- \frac{1}{2} \sum_{\overline{\omega}_1+\overline{\omega}_2=2n} \Big\{ (2n-\omega_\ell)\left(\overline{\omega}_1 X_{21i\ell} + \overline{\omega}_2 X_{12i\ell}\right) - \overline{\omega}_1\overline{\omega}_2 X_{i12\ell} \Big\},
\end{aligned} \tag{5.16}$$

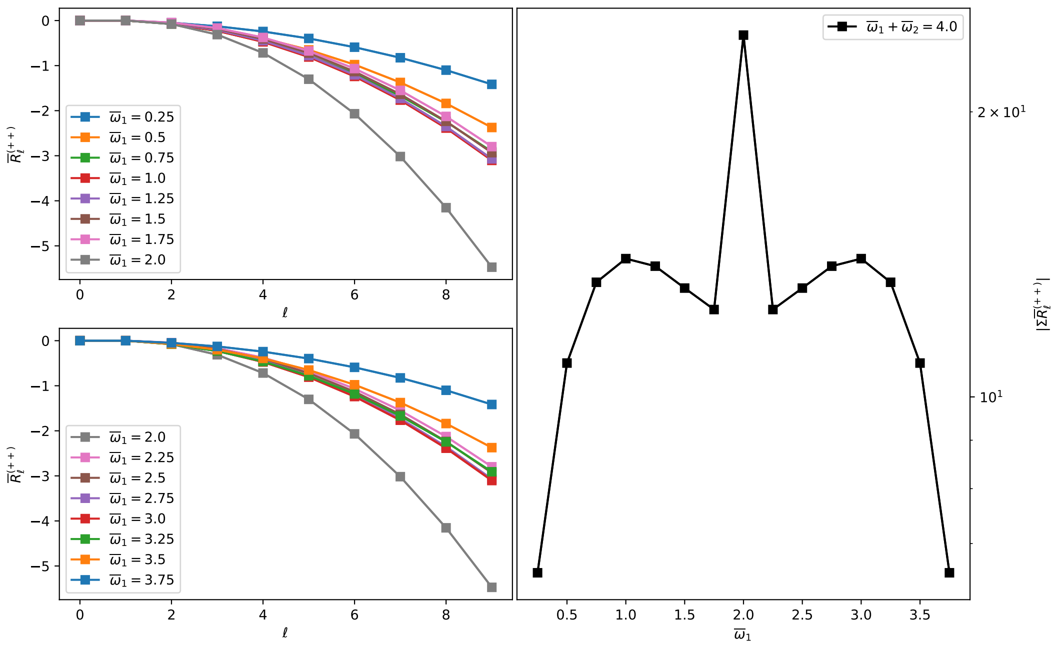

Figure 4: $\overline{R}^{(++)}_{(\ell-n)12\ell}$ for $m^2 = 0$.

NB. In (5.16) *only*, $\omega_i = 2i + d$ since this term requires that $m^2 = 0$ to contribute. We maintain the same notation despite the special case out of convenience.

Finally,

$$\overline{T}_{12\ell} = \frac{1}{2}\omega_\ell^2 \left(\tilde{Z}^+_{11\ell} + \tilde{Z}^+_{22\ell}\right) - \frac{1}{2}\Big\{ H_{11\ell\ell} + H_{22\ell\ell} + m^2 \left(V_{\ell 11\ell} + V_{\ell 22\ell}\right) - 2\omega_\ell^2 \left(X_{11\ell\ell} + X_{22\ell\ell}\right)$$
$$+ 4\omega_\ell^2 \left(\overline{\omega}_1^2 P_{\ell\ell 1} + \overline{\omega}_2^2 P_{\ell\ell 2}\right) + 2\overline{\omega}_1^2 M_{\ell\ell 1} + 2\overline{\omega}_2^2 M_{\ell\ell 2} + 2m^2 \left(\overline{\omega}_1^2 Q_{\ell\ell 1} + \overline{\omega}_2^2 Q_{\ell\ell 2}\right) \Big\}. \quad (5.17)$$

5.3 Integer Plus χ

This is a case where the non-normalizable frequencies are non-integer, but differ from integer values by a specific amount. In analogue to the case where all modes are normalizable, we consider setting any two of the non-normalizable frequencies to

$$\omega_i = 2i + \chi\,, \quad (5.18)$$

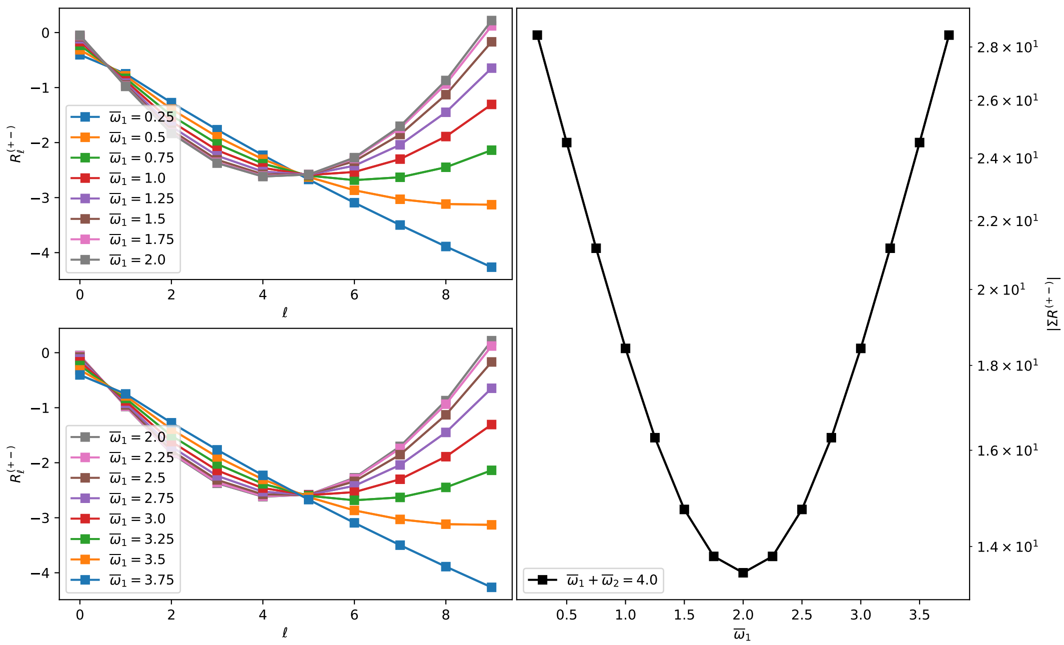

Figure 5: *Left, above*: Individual values of $\overline{R}^{(+-)}_{(\ell-n)12\ell}$ for $m^2 = -4.0$ as a function of ℓ for $\overline{\omega}_1 + \overline{\omega}_2 = 2n$ with $n = 4$ when $\overline{\omega}_1 \leq n$. *Left, below*: The same function, but for $\overline{\omega}_1 \geq n$; note the symmetry in values as $\overline{\omega}_1 \leftrightarrow \overline{\omega}_2$. *Right*: The absolute value of the sum of the $(+-)$ source terms for choices of $\overline{\omega}_1$.

where m^2 is *not* chosen to be a special value[1], i.e. $\chi \notin \mathbb{Z}^*$. For this choice of non-normalizable frequencies, there are no resonant contributions from the all-plus channel; only the $(++-)$ and $(+--)$ channels can contribute resonant terms.

5.3.1 $(++-)$

As in the case of all normalizable modes, the contribution from the $(++-)$ is equal to that of the $(+-+)$ and thus only one will be considered. This channel contributes secular terms of the form

$$S_\ell = \overline{S}_{i\beta\gamma\ell} \cos\left(\theta_i + \theta_\beta - \theta_\gamma\right)\Big|_{\omega_i+\omega_\beta-\omega_\gamma=\omega_\ell} + \overline{R}_{i\beta} \cos\left(\theta_i + \theta_\beta - \theta_\beta\right)\Big|_{i=\ell} + \overline{T}_\ell \cos\left(\theta_\ell\right) \quad (5.19)$$

[1] By tuning the value of the mass so that χ is an integer, additional resonant terms are possible; however, this scenario is addressed in §5.2.1. Furthermore, we do not consider the case when the Breitenlohmer-Freeman bound is saturated. This would place further restrictions on the allowed values of the indices in certain terms since the difference between the frequencies of normalizable and non-normalizable modes could then be zero.

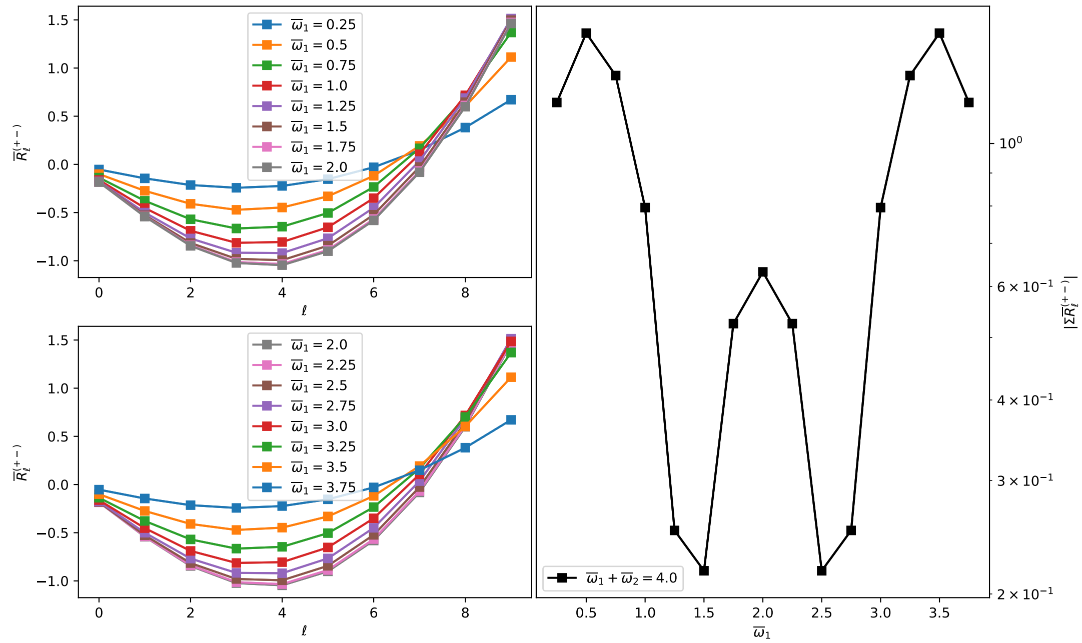

Figure 6: $\overline{R}^{(+-)}_{(\ell-n)12\ell}$ for $m^2 = 0$.

where (sums over the indices that are not ℓ are implied, and only restrictions on the indices are explicitly included)

$$
\begin{aligned}
\overline{S}_{i\beta\gamma\ell} = & \frac{1}{4}\sum_{\gamma\neq\ell}\frac{\omega_\gamma}{\omega_i+\omega_\beta}Z^-_{i\beta\gamma\ell} + \frac{1}{4}\sum_{\beta\neq\ell}\frac{\omega_\beta}{\omega_i-\omega_\gamma}Z^+_{i\gamma\beta\ell} + \sum_{i\neq\ell}\frac{\omega_i}{\omega_\beta-\omega_\gamma}Z^+_{\beta\gamma i\ell} \\
& + \frac{1}{4}\sum_{i\neq\gamma}\left[\frac{\omega_\gamma}{\omega_i-\omega_\gamma}(H_{i\gamma\beta\ell}-2\omega_\beta^2X_{i\gamma\beta\ell}) - \frac{\omega_i}{\omega_i+\omega_\gamma}(H_{\gamma i\beta\ell}-2\omega_\beta^2X_{\gamma i\beta\ell})\right] \\
& + \frac{1}{4}\sum_{\beta\neq\gamma}\left[\frac{\omega_\gamma}{\omega_\beta-\omega_\gamma}(H_{\beta\gamma i\ell}-2\omega_i^2X_{\beta\gamma i\ell}) - \frac{\omega_\beta}{\omega_\beta+\omega_\gamma}(H_{\gamma\beta i\ell}-2\omega_i^2X_{\gamma\beta i\ell})\right] \\
& - \frac{1}{4}\sum_{i\neq\beta}\left[\frac{\omega_\beta}{\omega_i+\omega_\beta}(H_{i\beta\gamma\ell}-2\omega_\gamma^2X_{i\beta\gamma\ell}) + \frac{\omega_i}{\omega_i+\omega_\beta}(H_{\beta i\gamma\ell}-2\omega_\gamma^2X_{\beta i\gamma\ell})\right] \\
& - \frac{1}{2}\sum\left(\omega_\beta\omega_\gamma X_{i\beta\gamma\ell} + \omega_i\omega_\gamma X_{\beta\gamma i\ell} - \omega_i\omega_\beta X_{\gamma i\beta\ell}\right)\,,
\end{aligned}
\tag{5.20}
$$

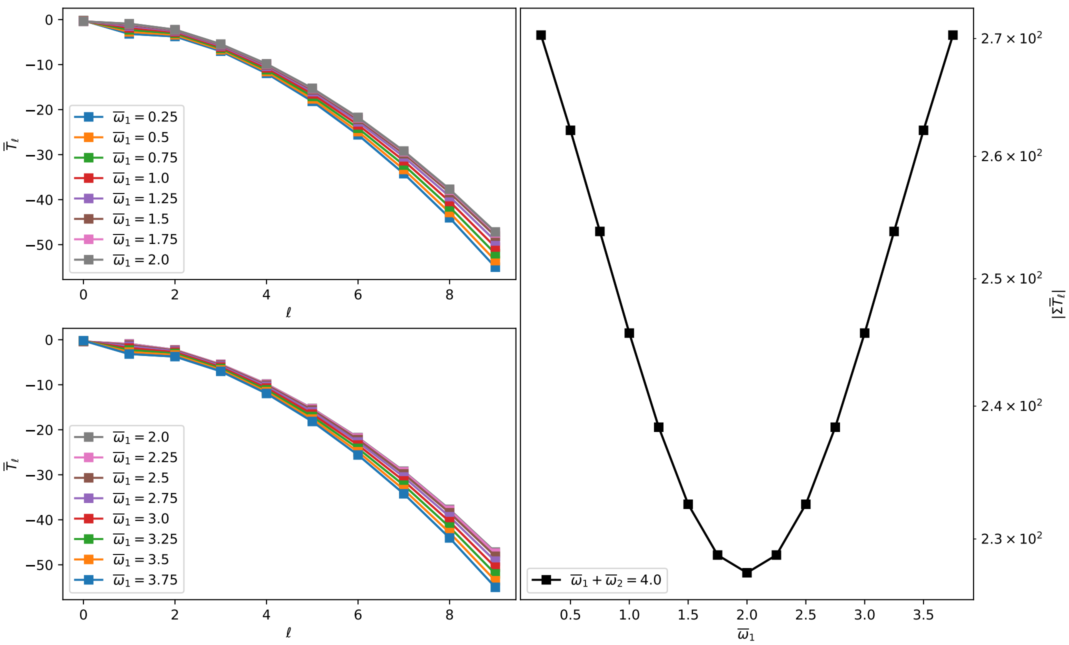

Figure 7: *Left, above*: . *Left, below*: .*Right*: $m^2 = -4.0$.

$$
\begin{aligned}
\overline{R}_{i\beta} = & \frac{1}{2} \sum_{\beta \neq \ell} \left[\frac{\omega_\beta}{\omega_\ell^2 - \omega_\beta^2} \left((\omega_i - \omega_\beta) Z^-_{i\beta\beta\ell} + 2(\omega_i + \omega_\beta) Z^+_{i\beta\beta\ell} \right) \right] + \sum_{i \neq \ell} \frac{\omega_i^2}{\omega_\ell^2 - \omega_i^2} Z^+_{\beta\beta i\ell} \\
& + \sum_{i \neq \beta} \left[H_{i\beta\beta\ell} \left(\frac{\omega_\beta^2}{\omega_i^2 - \omega_\beta^2} \right) - H_{\beta i\beta\ell} \left(\frac{\omega_i}{\omega_i + \omega_\beta} \right) - 2\omega_\beta^2 X_{i\beta\beta\ell} \left(\frac{\omega_\beta^2}{\omega_i^2 - \omega_\beta^2} \right) \right. \\
& \left. + \omega_\beta^2 X_{\beta\beta i\ell} \left(\frac{\omega_i}{\omega_i + \omega_\beta} \right) \right] - \sum \omega_\beta^2 X_{i\beta\beta\ell} \,, \qquad (5.21)
\end{aligned}
$$

$$
\overline{T}_\ell = -\frac{1}{2} \sum \left(4\omega_\beta^2 \omega_\ell^2 P_{\ell\ell\beta} + H_{\beta\beta\ell\ell} + 2\omega_\beta^2 M_{\ell\ell\beta} - 2\omega_\ell^2 X_{\beta\beta\ell\ell} \right) . \qquad (5.22)
$$

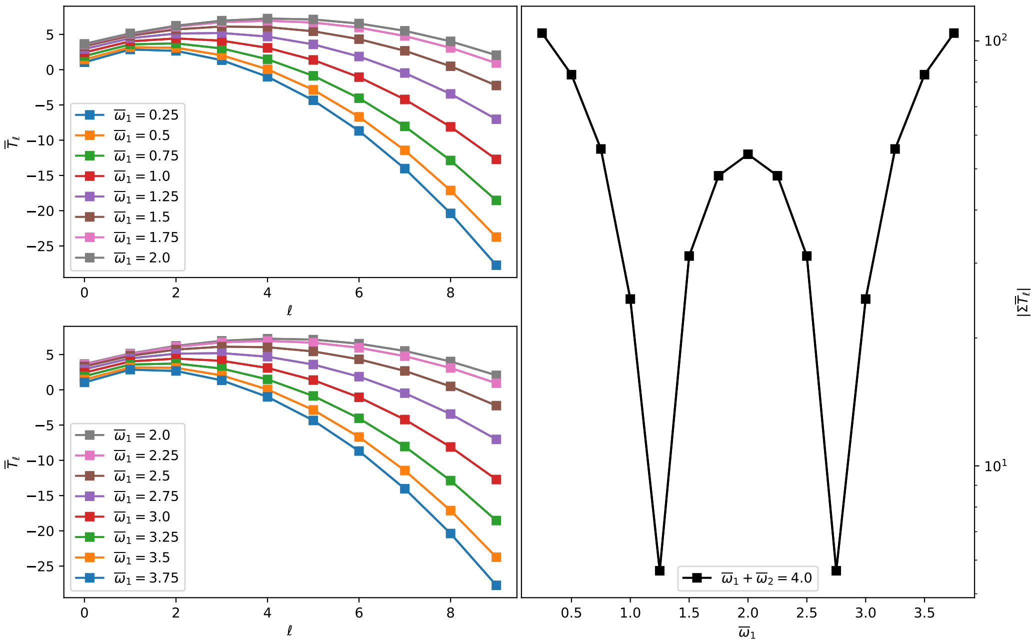

Figure 8: $\overline{T}$ for $m^2 = 0$.

5.3.2 $(+--)$

Resonant terms proportional to $\cos(\theta_\alpha - \theta_\beta - \theta_k)$ must be evaluated subject to $\omega_\alpha - \omega_\beta - \omega_k + \omega_\ell = 0$ and are of the form

$$
\begin{aligned}
\overline{S}_{\alpha\beta k\ell} = &\frac{1}{4}\sum_{\alpha\neq\ell}\frac{\omega_\alpha}{\omega_\beta+\omega_k}Z^-_{\beta k\alpha\ell} + \frac{1}{4}\sum_{\beta\neq\ell}\frac{\omega_\beta}{\omega_k-\omega_\alpha}Z^+_{\alpha k\beta\ell} + \frac{1}{4}\sum_{k\neq\ell}\frac{\omega_k}{\omega_\beta-\omega_\alpha}Z^+_{\alpha\beta k\ell} \\
&+ \frac{1}{4}\sum_{k\neq\alpha}\left[\frac{\omega_\alpha}{\omega_k-\omega_\alpha}(H_{k\alpha\beta\ell} - 2\omega_\beta^2 X_{k\alpha\beta\ell}) - \frac{\omega_k}{\omega_k+\omega_\alpha}(H_{\alpha k\beta\ell} - 2\omega_\beta^2 X_{\alpha k\beta\ell})\right] \\
&+ \frac{1}{4}\sum_{\alpha\neq\beta}\left[\frac{\omega_\alpha}{\omega_\beta-\omega_\alpha}(H_{\beta\alpha k\ell} - 2\omega_k^2 X_{\beta\alpha k\ell}) - \frac{\omega_\beta}{\omega_\alpha+\omega_\beta}(H_{\alpha\beta k\ell} - 2\omega_k^2 X_{\alpha\beta k\ell})\right] \\
&- \frac{1}{4}\sum_{k\neq\beta}\left[\frac{\omega_\beta}{\omega_k+\omega_\beta}(H_{k\beta\alpha\ell} - 2\omega_\alpha^2 X_{k\beta\alpha\ell}) + \frac{\omega_k}{\omega_k+\omega_\beta}(H_{\beta k\alpha\ell} - 2\omega_\alpha^2 X_{\beta k\alpha\ell})\right] \\
&- \frac{1}{2}\sum\left(\omega_\alpha\omega_\beta X_{k\alpha\beta\ell} + \omega_\alpha\omega_k X_{\beta k\alpha\ell} - \omega_\beta\omega_k X_{\alpha\beta k\ell}\right) . \qquad (5.23)
\end{aligned}
$$

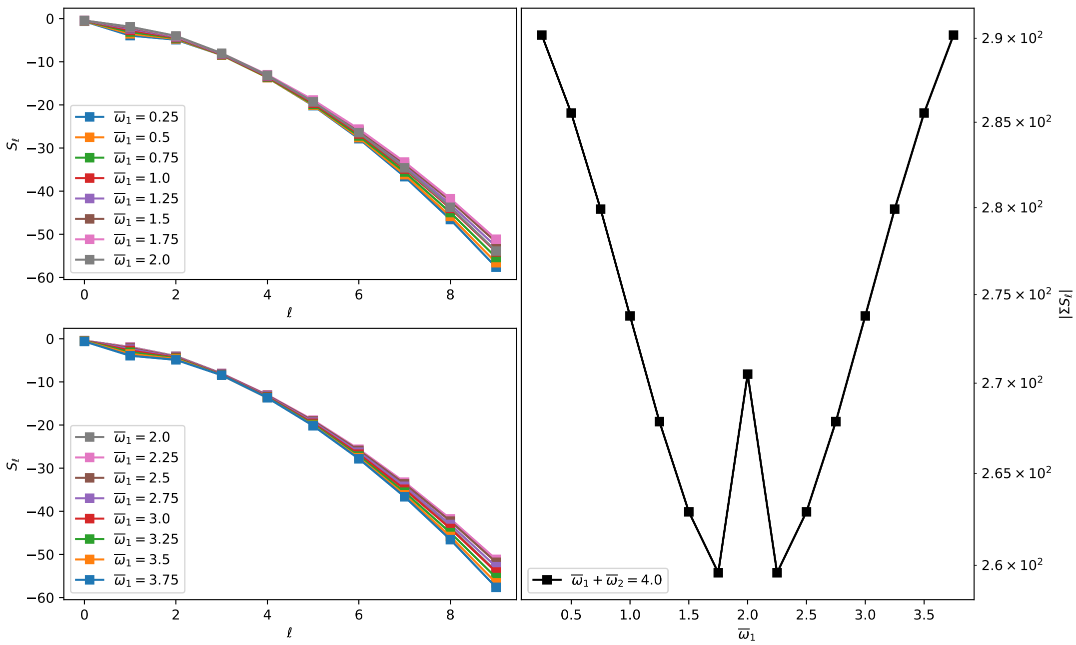

Figure 9: *Left, above*: . *Left, below*: .*Right*: sum of all source terms for $m^2 = -4.0$.

Other resonances are proportional to $\cos(\theta_\alpha - \theta_\alpha - \theta_k)$, and can be written as $\overline{R}_{\ell\alpha}$ where

$$\begin{aligned}\overline{R}_{k\alpha} = \frac{1}{2}\sum_{\alpha\neq\ell}\left[\frac{\omega_\alpha}{\omega_\ell^2-\omega_\alpha^2}\left(\omega_k(Z^-_{k\alpha\alpha\ell}+Z^+_{k\alpha\alpha\ell})+\omega_\alpha(Z^+_{k\alpha\alpha\ell}-Z^-_{k\alpha\alpha\ell})\right)\right]+\sum_{k\neq\ell}\frac{\omega_k^2}{\omega_\ell^2-\omega_k^2}Z^+_{\alpha\alpha k\ell}\\ +\sum_{k\neq\alpha}\left[H_{k\alpha\alpha\ell}\left(\frac{\omega_\alpha^2}{\omega_\ell^2-\omega_\alpha^2}\right)-2\omega_\alpha^2X_{k\alpha\alpha\ell}\left(\frac{\omega_\alpha\omega_k}{\omega_k^2-\omega_\alpha^2}\right)-H_{\alpha k\alpha\ell}\left(\frac{\omega_k}{\omega_k+\omega_\alpha}\right)\right.\\ \left.-2\omega_\alpha^2X_{\alpha k\alpha\ell}\left(\frac{\omega_k}{\omega_k+\omega_\alpha}\right)\right]-\sum_{k,\alpha}\omega_\alpha^2X_{k\alpha\alpha\ell} \qquad (5.24)\end{aligned}$$

Finally, the last set of contributions comes from terms that go as $\overline{T}_\ell\cos\theta_\ell$ where

$$\overline{T}_\ell = -\frac{1}{2}\sum_\alpha\cos\theta_\ell\left(4\omega_\alpha^2\omega_\ell^2P_{\ell\ell\alpha}+H_{\alpha\alpha\ell\ell}+2\omega_\alpha^2M_{\ell\ell\alpha}-2\omega_\ell^2X_{\alpha\alpha\ell\ell}\right). \qquad (5.25)$$

6 QP Equations

7 Numerics

Acknowledgments

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A Derivation of Source Terms For Massive Scalars

The derivation of the source terms for massive scalars closely follows the massless case, particularly if one chooses not to write out the explicit mass dependence as was done in [1]. However, since we have chosen to write our equations in a slightly different way – and in a different gauge – than previous authors, one may find it instructive to see the differences in the derivations. Below we have included the intermediate steps involved in deriving the third-order source term S_ℓ.

Projecting each of the terms individually onto the eigenbasis $\{e_\ell\}$:

$$\langle \delta_2 \ddot{\phi}_1, e_\ell \rangle = -\sum_{i=0}^{\infty} \sum_{\substack{j=0 \\ k\neq\ell}}^{\infty} \sum_{k=0}^{\infty} \frac{\omega_k^2 c_k}{\omega_\ell^2 - \omega_k^2} \left[\dot{c}_i \dot{c}_j \left(X_{k\ell ij} - X_{\ell kij} \right) + c_i c_j \left(Y_{ij\ell k} - Y_{ijk\ell} \right) \right]$$

$$- \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \omega_\ell^2 c_\ell \left[\dot{c}_i \dot{c}_j P_{ij\ell} + c_i c_j B_{ij\ell} \right] , \tag{A.1}$$

$$\langle A_2 \ddot{\phi}_1, e_\ell \rangle = 2 \sum_{i=0}^{\infty} \sum_{\substack{j=0 \\ i\neq j}}^{\infty} \sum_{k=0}^{\infty} \frac{\omega_k^2 c_k}{\omega_j^2 - \omega_i^2} X_{ijk\ell} \left(\dot{c}_i \dot{c}_j + \omega_j^2 c_i c_j \right)$$

$$+ \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \omega_j^2 c_j \left(\mathbb{C}_i P_{j\ell i} + c_i^2 X_{iij\ell} \right) , \tag{A.2}$$

$$\langle \dot{\delta}_2 \dot{\phi}_1, e_\ell \rangle = \sum_{i=0}^{\infty} \sum_{\substack{j=0 \\ k\neq\ell}}^{\infty} \sum_{k=0}^{\infty} \frac{\dot{c}_k}{\omega_\ell^2 - \omega_k^2} \left[\partial_t \left(\dot{c}_i \dot{c}_j \right) \left(X_{k\ell ij} - X_{\ell kij} \right) + \partial_t (c_i c_j) \left(Y_{ij\ell k} - Y_{ijk\ell} \right) \right]$$

$$+ \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \dot{c}_\ell \left[\partial_t \left(\dot{c}_i \dot{c}_j \right) P_{ij\ell} + \partial_t (c_i c_j) B_{ij\ell} \right] , \tag{A.3}$$

$$\langle \dot{A}_2 \dot{\phi}_1, e_\ell \rangle = -2 \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \dot{c}_k \dot{c}_j c_i X_{ijk\ell} , \tag{A.4}$$

$$\langle \left(A_2' - \delta_2' \right) \phi_1', e_\ell \rangle = -2 \sum_{i=0}^{\infty} \sum_{\substack{j=0 \\ i\neq j}}^{\infty} \sum_{k=0}^{\infty} \frac{c_k (\dot{c}_i \dot{c}_j + \omega_j^2 c_i c_j)}{\omega_j^2 - \omega_i^2} H_{ijk\ell} - m^2 \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} c_i c_j c_k V_{ijk\ell}$$

$$- \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} c_j \left[c_i^2 H_{iij\ell} + \mathbb{C}_i M_{j\ell i} \right] , \tag{A.5}$$

$$\langle A_2 \phi_1 \sec^2 x, e_\ell \rangle = -2 \sum_{i=0}^{\infty} \sum_{\substack{j=0 \\ i\neq j}}^{\infty} \sum_{k=0}^{\infty} \frac{c_k (\dot{c}_i \dot{c}_j + \omega_j^2 c_i c_j)}{\omega_j^2 - \omega_i^2} V_{jki\ell}$$

$$- \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} c_j \left(c_i^2 V_{jii\ell} + \mathbb{C}_i Q_{j\ell i} \right) . \tag{A.6}$$

Where the forms of X, Y, V, H, B, M, P, and Q are given by

$$X_{ijk\ell} = \int_0^{\pi/2} dx\, \mu^2 \nu e_i' e_j e_k e_\ell \tag{A.7}$$

$$Y_{ijk\ell} = \int_0^{\pi/2} dx\, \mu^2 \nu e_i' e_j' e_k e_\ell' \tag{A.8}$$

$$V_{ijk\ell} = \int_0^{\pi/2} dx\, \mu^2 \nu e_i e_j e_k' e_\ell \sec^2 x \tag{A.9}$$

$$H_{ijk\ell} = \int_0^{\pi/2} dx\, \mu^2 \nu' e_i' e_j e_k' e_\ell \tag{A.10}$$

$$B_{ij\ell} = \int_0^{\pi/2} dx\, \mu\nu e_i' e_j' \int_0^x dy\, \mu e_\ell^2 \tag{A.11}$$

$$M_{ij\ell} = \int_0^{\pi/2} dx\, \mu\nu' e_i' e_j \int_o^x dy\, \mu e_\ell^2 \tag{A.12}$$

$$P_{ij\ell} = \int_0^{\pi/2} dx\, \mu\nu e_i e_j \int_0^x dy\, \mu e_\ell^2 \tag{A.13}$$

$$Q_{ij\ell} = \int_0^{\pi/2} dx\, \mu\nu e_i e_j \sec^2 x \int_0^x dy\, \mu e_\ell^2 \tag{A.14}$$

Collecting terms together gives the expression for $S_\ell = \langle S, e_\ell \rangle$:

$$\begin{aligned} S_\ell = &\sum_{\substack{i,j,k \\ k\neq\ell}}^{\infty} \frac{1}{\omega_\ell^2 - \omega_k^2} \Big[F_k(\dot{c}_i\dot{c}_j)\left(X_{k\ell ij} - X_{\ell kij}\right) + F_k(c_ic_j)\left(Y_{ij\ell k} - Y_{ijk\ell}\right) \Big] \\ &+ 2\sum_{\substack{i,j,k \\ i\neq j}}^{\infty} \frac{c_k D_{ij}}{\omega_j^2 - \omega_i^2} \Big[2\omega_k^2 X_{ijk\ell} - H_{ijk\ell} - m^2 V_{jki\ell} \Big] - \sum_{i,j,k}^{\infty} c_i \Big[2\dot{c}_j\dot{c}_k X_{ijk\ell} + m^2 c_j c_k V_{ijk\ell} \Big] \\ &+ \sum_{i,j}^{\infty} \Big[F_\ell(\dot{c}_i\dot{c}_j) P_{ij\ell} + F_\ell(c_ic_j) B_{ij\ell} + 2\omega_j^2 c_j \left(c_i^2 X_{iij\ell} + \mathbb{C}_i P_{j\ell i} \right) \\ &\quad - c_j \left(c_i^2 (H_{iij\ell} + m^2 V_{jii\ell}) + \mathbb{C}_i (M_{j\ell i} + m^2 Q_{j\ell i}) \right) \Big], \end{aligned} \tag{A.15}$$

where $F_k(z) = \dot{c}_k\dot{z} - 2\omega_k^2 c_k z$, $D_{ij} = \dot{c}_i\dot{c}_j + \omega_j^2 c_i c_j$, and $\mathbb{C}_i = \dot{c}_i^2 + \omega_i^2 c_i^2$.

To simplify the above expression, we have defined

$$Z^{\pm}_{ijk\ell} = \omega_i\omega_j \left(X_{k\ell ij} - X_{\ell kij}\right) \pm \left(Y_{ij\ell k} - Y_{ijk\ell}\right) \quad \text{and} \quad \tilde{Z}^{\pm}_{ij\ell} = \omega_i\omega_j P_{ij\ell} \pm B_{ij\ell}\,. \tag{A.16}$$

Using integration by parts to remove the derivative from ν in the definitions of $H_{ijk\ell}$ and $M_{ij\ell}$, we can show that

$$H_{ijk\ell} = \omega_i^2 X_{kij\ell} + \omega_k^2 X_{ijk\ell} - Y_{ij\ell k} - Y_{\ell kji} - m^2 V_{kji\ell} - m^2 V_{ijk\ell} \tag{A.17}$$

$$M_{ij\ell} = \omega_i^2 P_{ij\ell} - B_{ij\ell} - m^2 Q_{ij\ell} \tag{A.18}$$

References

[1] A. Biasi, B. Craps and O. Evnin, *Energy returns in global AdS_4*, 1810.04753.

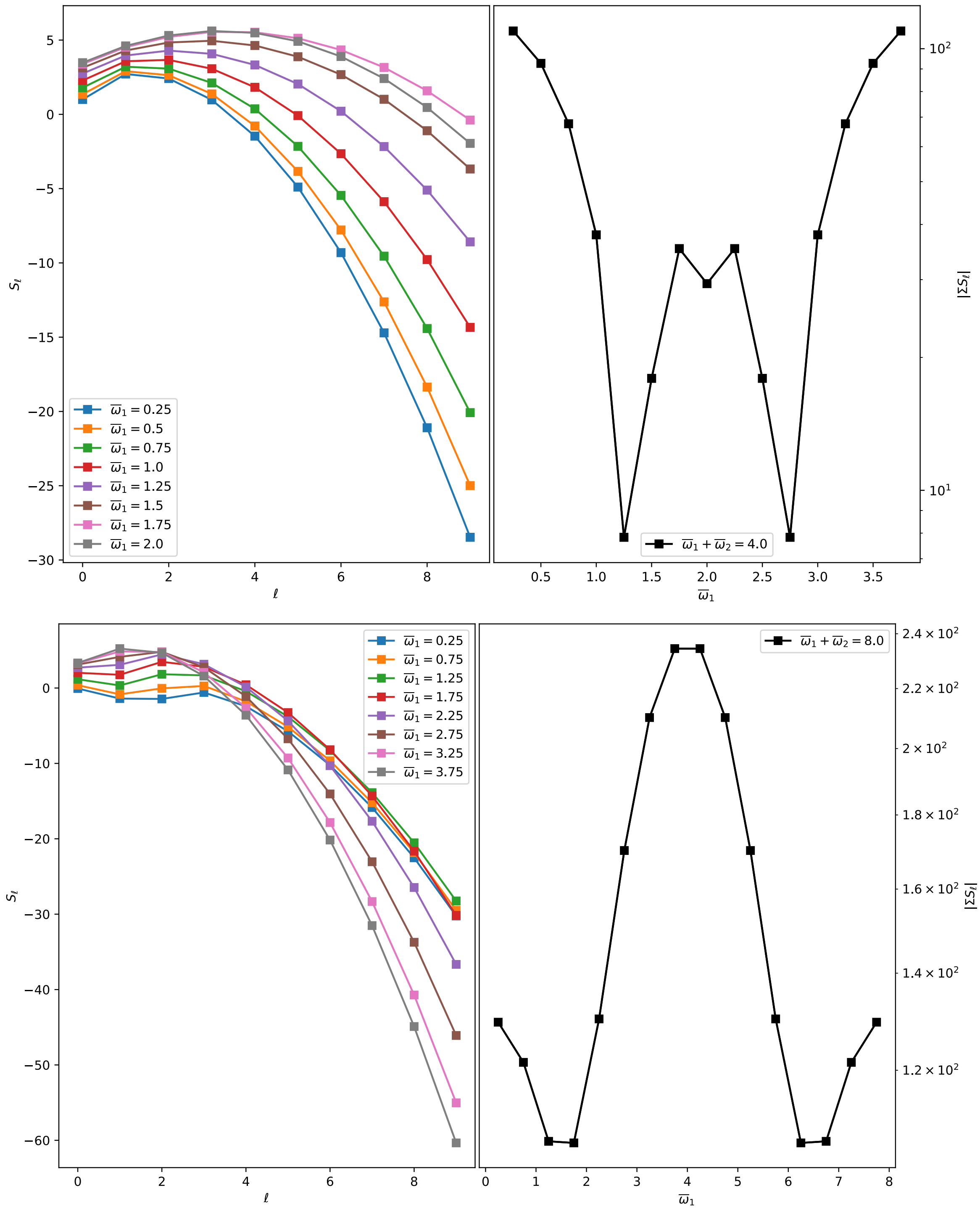

Figure 10: *Above:* The value of (5.13) as a function of ℓ for a massless scalar with values of $\overline{\omega}_1$ and $\overline{\omega}_2$ chosen so that $\overline{\omega}_1 + \overline{\omega}_2 = 4$. *Below:* The same plot but with values chosen to satisfy $\overline{\omega}_1 + \overline{\omega}_2 = 8$.