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Examining Instabilities Due to Driven Scalars in AdS

Abstract: We extend the study of the non-linear perturbative theory of energy cascades in AdS_{d+1} to include solutions of driven systems, i.e. those with time-dependent sources on the AdS boundary. This necessitates the activation of non-normalizable modes in the linear solution for the massive bulk scalar field, which couple to the metric and normalizable scalar modes. Analytic expressions for secular terms in the renormalization flow equations are determined for scalars with any mass, and for various driving functions. We then numerically evaluate these sources for $d = 4$ and discuss what role these driven solutions play in the perturbative stability of AdS.

Contents

1 Introduction

Nonlinear instabilities in Anti-de Sitter space have been the subject of examinations on several grounds, including general stability of maximally-symmetric solutions in general relativity **cite**, the study of secular growth of terms in time-dependent perturbation theories **cite**, as well as through the holographic description of quantum quenches via the AdS/CFT correspondence **cite**. Numerical studies in holographic AdS show that the eventual collapse of a scalar field into a black hole in the bulk (which is dual to the thermalization of the boundary theory) is generic to any non-zero perturbation **cite**, but can be avoided or delayed for certain initial conditions **cite** [1]. The method of collapse in such systems is described as a weakly turbulent energy cascade to short length scales. These dynamics can be captured by a non-linear perturbation theory at first non-trivial order through the introduction of a second, "slow time" that describes energy transfer between the fundamental modes **cite**. This is known as the Two-Time Formalism (TTF) **cite** and, through a renormalization group flow procedure, allows for analytic expressions to be determined that absorb secular terms into renormalized amplitudes and phases **cite**. Therefore, stability against a perturbation of order ϵ is maintained over time scales of $t \sim \epsilon^{-2}$.

Conventional examinations of perturbative stability using TTF have focused on the reaction of the bulk space to some initial energy perturbation, and have aimed to study the balance between direct and inverse energy cascades **cite**. However, extensions to include time-dependent perturbations in the quantum theory – corresponding to a driving term on the boundary of the bulk space that injects energy into system **cite** – remain unaddressed. With this in mind, we examine the effects of a time-dependent source on the conformal boundary that couples to the bulk gravity fields. Previous studies of holographic quenches in AdS have used the fact that the linear order solutions to the equations of motion always approach zero near the conformal boundary, thereby limiting the space of solutions to sets of orthogonal polynomials. However, the introduction of a driving term on the boundary means that we must also include a second class of fundamental modes: those that approach constant, non-zero values on the boundary. Since these solutions will have non-finite inner products over the space, they are known as non-normalizable solutions. Non-normalizable modes couple to both the source and the normalizable modes to bring energy into the system, where direct and inverse energy cascades proceed over perturbative time scales.

To capture these dynamics, we expand the fields in powers of a small perturbation and isolate the secular terms that appear at third order in ϵ. Only modes whose frequencies satisfy certain resonance conditions will contribute terms that cannot be absorbed by simple phase shifts. The form of the resonant terms depend on the specific physics of the system, as well as possible symmetries between frequencies. By evaluating the resonant third-order interactions when combinations of normalizable and non-normalizable modes are activated, we can write renormalization flow equations for the slowly varying amplitudes and phases.

This paper is organized as follows: section § 2 involves a brief discussion of how we arrive at the third order source term, as well as additional considerations due to the time-dependent boundary condition. As an exercise – and to provide explicit expressions for the resonant contributions when the scalar field has non-zero mass – § 3 examines the secular terms in the case of a massive scalar field in AdS_{d+1} with any mass-squared, up to and including the Breitenlohner-Freedman mass [2]: $m^2_{BF} \leq m^2$. We demonstrate the natural vanishing of two of the three resonances, and then examine the effects of mass-dependence on the non-vanishing channel. Whenever values are calculated, the choice of $d = 4$ is implied as to draw the most direct comparison to existing literature. In section § 4, we extend the boundary conditions to include a variety of periodic boundary sources that couple to non-normalizable modes in the bulk. For each choice of boundary condition, we derive analytic expressions for applicable resonances and evaluate these expressions for different ranges of scalar field masses. Finally, in § 5 we discuss the implications of non-vanishing resonances on the competing energy cascades, and the implications for the perturbative stability of such systems. For completeness, we include details of our derivation of the general source term in appendix A, as well as a complete list of possible resonance channels and their resulting secular terms in appendix B for the case of two, equal frequency non-normalizable modes.

2 Source Terms and Boundary Conditions

Let us first consider a minimally coupled, massive scalar field coupled to a spherically symmetric, asymptotically AdS_{d+1} spacetime in global coordinates, whose metric is given by

$$ds^2 = \frac{L^2}{\cos(x)}\left(-A(t,x)e^{-2\delta(t,x)}\,dt^2 + A^{-1}(t,x)\,dx^2 + \sin^2(x)\,d\Omega^2_{d-1}\right), \tag{2.1}$$

where L is the AdS curvature (hereafter set to 1), and the radial coordinate $x \in [0, \pi/2)$. The dynamics of the system come from the Einstein and Klein-Gordon equations:

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi\left(\nabla_\mu\phi\nabla_\nu\phi - \frac{1}{2}g_{\mu\nu}\left(\nabla^\rho\phi\nabla_\rho\phi + m^2\phi^2\right)\right), \quad \text{and} \quad \nabla^2\phi - m^2\phi = 0\,, \tag{2.2}$$

with Λ as cosmological constant for AdS, $\Lambda = -d(d-1)/2$.

Perturbing around static AdS, the scalar field is expanded in of odd powers of epsilon

$$\phi(t,x) = \epsilon\phi_1 + \epsilon^3\phi_3 + \ldots \tag{2.3}$$

and the metric functions A and δ in even powers,

$$A(t,x) = 1 + A_2\epsilon^2 + \ldots \quad \text{and} \quad \delta(t,x) = \epsilon^2\delta_2 + \ldots\,. \tag{2.4}$$

We choose to work in the boundary gauge, where $\delta(t, \pi/2) = 0$, for reasons that we discuss below.

At linear order, ϕ_1 satisfies

$$\partial_t^2\phi_1 + \hat{L}\phi_1 = 0 \quad \text{where} \quad \hat{L} \equiv \frac{1}{\mu}(\mu'\partial_x + \mu\partial_x^2) - \frac{m^2}{\cos^2(x)}\,, \tag{2.5}$$

where $\mu \equiv \tan^{d-1}(x)$. Writing the scalar field as the product of time- and position-dependent parts,

$$\phi_1(t,x) = \sum_j c_j(t) e_j(x)\,, \tag{2.6}$$

we find that the basis functions $e_j(x)$ are the solutions to the eigenvalue equation

$$\hat{L} e_j(x) = \omega_j^2 e_j(x). \tag{2.7}$$

The general solution to this eigenvalue equation involves two types of functions: those that vanish as $x \to \pi/2$ and therefore are normalizable, and those that approach a finite value on the boundary and are non-normalizable. In many previous works, the dynamics of scalar fields have been studied using exclusively normalizable functions (as is the focus of § 3); however, we now wish to consider the effects of exciting the second class of basis functions. These driven systems have a time-dependent source term on the boundary that sends energy into bulk space through coupling with non-normalizable modes. Energy is then distributed to both normalizable and non-normalizable scalar modes, as well as the metric functions.

Writing out the full solution to (2.7) involves summing over both normalizable and non-normalizable eigenmodes [3],

$$\phi_1(t,x) = \sum_j c_j(t) e_j(x) + \sum_\alpha \bar{A}_\alpha(t) E_\alpha(x)\,. \tag{2.8}$$

The values of $\bar{A}_\alpha$ are set by the choice of boundary conditions. For example, if the driving term on the boundary is a single, periodic function, then

$$\phi_1(t,\pi/2) = \mathcal{A}\cos\varpi t\,. \tag{2.9}$$

Extending the boundary condition to include the addition of two driving terms – or a more general sum over Fourier modes – would set further $\bar{A}_\alpha$ values.

The normalizable modes have eigenfunctions given by

$$e_j(x) = k_j\,(\cos(x))^{\Delta^+}\,P_j^{(d/2-1,\,\Delta^+-d/2)}\,(\cos(2x)) \tag{2.10}$$

$$k_j = 2\sqrt{\frac{(j+\Delta^+/2)\Gamma(j+1)\Gamma(j+\Delta^+)}{\Gamma(j+d/2)\Gamma(j+\Delta^+-d/2+1)}}\,, \tag{2.11}$$

with fully resonant eigenvalues $\omega_j = 2j+\Delta^+$ with $j \in \mathbb{Z}^*$. We define Δ^+ as the positive root of $\Delta(\Delta-d) = m^2$. This resonant spectrum is responsible for the secular growth of terms at third order, which must be controlled the secular term resummation procedure described below. On the other hand, the non-normalizable eigenfunctions have arbitrary frequencies ω_α, and are given by

$$E_\alpha(x) = (\cos(x))^{\Delta_+}\,{}_2F_1\left(\frac{\Delta_+ + \omega_\alpha}{2}, \frac{\Delta_+ - \omega_\alpha}{2}, d/2; \sin^2(x)\right)\,. \tag{2.12}$$

By allowing frequencies and basis functions to remain unspecified for the time being, we can show that the $\mathcal{O}(\epsilon^3)$ part of the scalar field satisfies the equation

$$\ddot{\phi}_3 + \hat{L}\phi_3 = S = 2(A_2 - \delta_2)\ddot{\phi}_1 + (\dot{A}_2 - \dot{\delta}_2)\dot{\phi}_1 + (A_2' - \delta_2')\phi_1' + m^2 A_2 \phi_1 \sec^2 x\,. \tag{2.13}$$

Following the steps outlined in Appendix A, we project (2.13) onto the basis of normalizable modes and, employing the solution $c_i(t) = a_i \cos(\omega_i t + b_i) = a_i \cos\theta_i$ for the time-dependent portion of the scalar field, find that the general expression for the source term is

$$\begin{aligned}
S_\ell = {} & \frac{1}{4}\sum_{\substack{i,j,k\\k\neq\ell}}^{\infty} \frac{a_i a_j a_k \omega_k}{\omega_\ell^2 - \omega_k^2}\Big[Z^-_{ijk\ell}(\omega_i + \omega_j - 2\omega_k)\cos(\theta_i + \theta_j - \theta_k) - Z^-_{ijk\ell}(\omega_i + \omega_j + 2\omega_k)\cos(\theta_i + \theta_j + \theta_k) - \\
& + Z^+_{ijk\ell}(\omega_i - \omega_j + 2\omega_k)\cos(\theta_i - \theta_j + \theta_k) - Z^+_{ijk\ell}(\omega_i - \omega_j - 2\omega_k)\cos(\theta_i - \theta_j - \theta_k)\Big] \\
& + \frac{1}{2}\sum_{\substack{i,j,k\\i\neq j}}^{\infty} a_i a_j a_k \omega_j \left(H_{ijk\ell} + m^2 V_{jki\ell} - 2\omega_k^2 X_{ijk\ell}\right)\Big[\frac{1}{\omega_i - \omega_j}\left(\cos(\theta_i - \theta_j - \theta_k) + \cos(\theta_i - \theta_j + \theta_k)\right) \\
& - \frac{1}{\omega_i + \omega_j}\left(\cos(\theta_i + \theta_j - \theta_k) + \cos(\theta_i + \theta_j + \theta_k)\right)\Big] \\
& - \frac{1}{4}\sum_{i,j,k}^{\infty} a_i a_j a_k \Big[\left(2\omega_j\omega_k X_{ijk\ell} + m^2 V_{ijk\ell}\right)\cos(\theta_i + \theta_j - \theta_k) - \left(2\omega_j\omega_k X_{ijk\ell} - m^2 V_{ijk\ell}\right)\cos(\theta_i - \theta_j - \theta_k) \\
& + \left(2\omega_j\omega_k X_{ijk\ell} + m^2 V_{ijk\ell}\right)\cos(\theta_i - \theta_j + \theta_k) - \left(2\omega_j\omega_k X_{ijk\ell} - m^2 V_{ijk\ell}\right)\cos(\theta_i + \theta_j + \theta_k)\Big] \\
& + \frac{1}{4}\sum_{i,j}^{\infty} a_i a_j a_\ell \omega_\ell \Big[\tilde{Z}^-_{ij\ell}(\omega_i + \omega_j - 2\omega_\ell)\cos(\theta_i + \theta_j - \theta_\ell) - \tilde{Z}^-_{ij\ell}(\omega_i + \omega_j + 2\omega_\ell)\cos(\theta_i + \theta_j + \theta_\ell) \\
& + \tilde{Z}^+_{ij\ell}(\omega_i - \omega_j + 2\omega_\ell)\cos(\theta_i - \theta_j + \theta_\ell) - \tilde{Z}^+_{ij\ell}(\omega_i - \omega_j - 2\omega_\ell)\cos(\theta_i - \theta_j - \theta_\ell)\Big] \\
& - \frac{1}{4}\sum_{i,j}^{\infty} a_i^2 a_j \left(H_{iij\ell} + m^2 V_{jii\ell} - 2\omega_j^2 X_{iij\ell}\right)\left[\cos(2\theta_i - \theta_j) + \cos(2\theta_i + \theta_j)\right] \\
& - \frac{1}{2}\sum_{i,j}^{\infty} a_i^2 a_j \left(H_{iij\ell} + m^2 V_{jii\ell} - 2\omega_j^2 X_{iij\ell} + 4\omega_i^2\omega_j^2 P_{j\ell i} + 2\omega_i^2 (M_{j\ell i} + m^2 Q_{j\ell i})\right)\cos\theta_j.
\end{aligned} \tag{2.14}$$

Note that sums and restrictions on indices must be interpreted as sums and restrictions on *frequencies* when any of the modes is non-normalizable, since $\omega_\alpha \neq 2\alpha + \Delta^+$ in general.

As mentioned above, the linear growth of resonant terms with time, i.e. secular growth, at $\mathcal{O}(\epsilon^3)$ can be absorbed into the time-dependent part of the scalar field at that order [4]. Thus, (2.13) tells us that

$$\ddot{c}_\ell^{(3)}(t) + \omega_\ell^2 c_\ell^{(3)}(t) = S_\ell^{(3)} \cos\left(\omega_\ell t + \varphi_\ell\right)\,, \tag{2.15}$$

where $S_\ell^{(3)}$ is a polynomial in a_i determined by evaluating the resonant contributions from (2.14), and φ_ℓ is some combination of the b_i. To obtain the renormalization flow equations, we can rewrite the amplitudes and phases in terms of renormalized integration constants that exactly cancel the secular terms at each moment. Doing so yields the renormalization flow equations for the renormalized constants [5]

$$\frac{2\omega_\ell}{\epsilon^2}\frac{da_\ell}{dt} = -S_\ell^{(3)}\sin\left(b_\ell - \varphi_\ell\right)\ , \tag{2.16}$$

$$\frac{2\omega_\ell a_\ell}{\epsilon^2}\frac{db_\ell}{dt} = -S_\ell^{(3)}\cos\left(b_\ell - \varphi_\ell\right)\ . \tag{2.17}$$

Note that the amplitudes and phases evolve with respect to the "slow time" $\tau = \epsilon^2 t$. In practice, once these flow equations can be written down the perturbative evolution of the system is determined up to a timescale of $t \sim \epsilon^{-2}$.

Let us now examine the exact form of $S_\ell^{(3)}$ for the case of a massive scalar field. As an exercise, we first derive the resonant contributions when the boundary source is zero, and therefore only normalizable modes are present. These results agree numerically with previous work on normalizable modes for massless scalars in the interior time gauge ($\delta(t,0) = 0$) [6]. The definitions of the integral functions Z, H, X, etc. differ slightly from other works – in part because of the gauge choice, and in part because of a desire to separate out mass-dependent terms – and so are given explicitly in Appendix A.

3 Resonances From Normalizable Solutions

Consider the case where each of the basis functions are given by normalizable solutions. After time-averaging, the resonant contributions occur for the following combination of normalizable frequencies:

$$\omega_i \pm \omega_j \pm \omega_k = \pm\omega_\ell \tag{3.1}$$

which can be separated into the three distinct cases

$$\omega_i + \omega_j + \omega_k = \omega_\ell \qquad (+++) \tag{3.2}$$

$$\omega_i - \omega_j - \omega_k = \omega_\ell \qquad (+--) \tag{3.3}$$

$$\omega_i + \omega_j - \omega_k = \omega_\ell \qquad (++-)\,. \tag{3.4}$$

We will see that the first two resonances, $(+++)$ and $(+--)$, will non-trivially vanish whenever their respective resonance conditions are satisfied. This is in agreement with the results shown for the massless scalar in the interior time gauge (as they must be, since the choice of time gauge should not change the existence of resonant channels). Here we include the expressions for the naturally vanishing resonances, choosing to explicitly express the mass dependence.

3.1 Naturally Vanishing Resonances: $(+++)$ and $(+--)$

Resonant contributions that come from the condition $\omega_i + \omega_j + \omega_k = \omega_\ell$ contribute to the total source term via

$$S_\ell = \underbrace{\sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty}}_{\omega_i+\omega_j+\omega_k=\omega_\ell} \Omega_{ijk\ell}\, a_i a_j a_k \cos\left(\theta_i + \theta_j + \theta_k\right) + \ldots\,, \tag{3.5}$$

where the dots denote other resonances. $\Omega_{ijk\ell}$ is given by

$$\begin{aligned}
\Omega_{ijk\ell} = &-\frac{1}{12} H_{ijk\ell} \frac{\omega_j(\omega_i + \omega_k + 2\omega_j)}{(\omega_i + \omega_j)(\omega_j + \omega_k)} - \frac{1}{12} H_{ikj\ell} \frac{\omega_k(\omega_i + \omega_j + 2\omega_k)}{(\omega_i + \omega_k)(\omega_j + \omega_k)} - \frac{1}{12} H_{jik\ell} \frac{\omega_i(\omega_j + \omega_k + 2\omega_i)}{(\omega_i + \omega_j)(\omega_i + \omega_k)} \\
&- \frac{m^2}{12} V_{ijk\ell} \left(1 + \frac{\omega_j}{\omega_j + \omega_k} + \frac{\omega_i}{\omega_i + \omega_k}\right) - \frac{m^2}{12} V_{jki\ell} \left(1 + \frac{\omega_j}{\omega_i + \omega_j} + \frac{\omega_k}{\omega_i + \omega_k}\right) \\
&- \frac{m^2}{12} V_{kij\ell} \left(1 + \frac{\omega_i}{\omega_i + \omega_j} + \frac{\omega_k}{\omega_j + \omega_k}\right) + \frac{1}{6} \omega_j \omega_k X_{ijk\ell} \left(1 + \frac{\omega_j}{\omega_i + \omega_k} + \frac{\omega_k}{\omega_i + \omega_j}\right) \\
&+ \frac{1}{6} \omega_i \omega_k X_{jki\ell} \left(1 + \frac{\omega_i}{\omega_j + \omega_k} + \frac{\omega_k}{\omega_i + \omega_j}\right) + \frac{1}{6} \omega_i \omega_j X_{kij\ell} \left(1 + \frac{\omega_i}{\omega_j + \omega_k} + \frac{\omega_j}{\omega_i + \omega_k}\right) \\
&- \frac{1}{12} Z^-_{ijk\ell} \left(\frac{\omega_k}{\omega_i + \omega_j}\right) - \frac{1}{12} Z^-_{ikj\ell} \left(\frac{\omega_j}{\omega_i + \omega_k}\right) - \frac{1}{12} Z^-_{jki\ell} \left(\frac{\omega_i}{\omega_j + \omega_k}\right) .
\end{aligned} \tag{3.6}$$

The second naturally vanishing resonance comes from the condition $\omega_i - \omega_j - \omega_k = \omega_\ell$, and contributes to the total source term via

$$S_\ell = \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \Gamma_{(j+k+\ell+\Delta^+)jk\ell}\, a_j a_k a_{(j+k+\ell+\Delta^+)} \cos\left(\theta_{(j+k+\ell+\Delta^+)} - \theta_j - \theta_k\right) + \ldots\,, \tag{3.7}$$

where

$$\begin{aligned}
\Gamma_{ijk\ell} = &\frac{1}{4} H_{ijk\ell} \frac{\omega_j(\omega_k - \omega_i + 2\omega_j)}{(\omega_i - \omega_j)(\omega_j + \omega_k)} + \frac{1}{4} H_{jki\ell} \frac{\omega_k(\omega_j - \omega_i + 2\omega_k)}{(\omega_i - \omega_k)(\omega_j + \omega_k)} + \frac{1}{4} H_{kij\ell} \frac{\omega_i(\omega_j + \omega_k - 2\omega_i)}{(\omega_i - \omega_j)(\omega_i - \omega_k)} \\
&- \frac{1}{2} \omega_j \omega_k X_{ijk\ell} \left(\frac{\omega_k}{\omega_i - \omega_j} + \frac{\omega_j}{\omega_i - \omega_k} - 1\right) + \frac{1}{2} \omega_i \omega_k X_{jki\ell} \left(\frac{\omega_k}{\omega_i - \omega_j} + \frac{\omega_i}{\omega_j + \omega_k} - 1\right) \\
&+ \frac{1}{2} \omega_i \omega_j X_{kij\ell} \left(\frac{\omega_j}{\omega_i - \omega_k} + \frac{\omega_i}{\omega_j + \omega_k} - 1\right) + \frac{m^2}{4} V_{jki\ell} \left(\frac{\omega_j}{\omega_i - \omega_j} + \frac{\omega_k}{\omega_i - \omega_k} - 1\right) \\
&- \frac{m^2}{4} V_{kij\ell} \left(\frac{\omega_i}{\omega_i - \omega_j} + \frac{\omega_k}{\omega_j + \omega_k} + 1\right) - \frac{m^2}{4} V_{ijk\ell} \left(\frac{\omega_i}{\omega_i - \omega_k} + \frac{\omega_j}{\omega_j + \omega_k} + 1\right) \\
&+ \frac{1}{4} Z^-_{kji\ell} \left(\frac{\omega_i}{\omega_j + \omega_k}\right) - \frac{1}{4} Z^+_{ijk\ell} \left(\frac{\omega_k}{\omega_i - \omega_j}\right) - \frac{1}{4} Z^+_{jki\ell} \left(\frac{\omega_j}{\omega_i - \omega_k}\right) .
\end{aligned} \tag{3.8}$$

Building on the work done with massless scalars, we are able to demonstrate that (3.6) and (3.8) continue to vanish for massive scalars ($m^2 \geq m^2_{BF}$) in the boundary gauge; thus, the dynamics governing the weakly turbulent transfer of energy is determined only from the remaining resonance channel. When non-normalizable modes are introduced, we will see that naturally vanishing resonances are not present and so the total third-order source term is the sum over all resonant channels.

3.2 $(++-)$

The first non-vanishing contributions arise when $\omega_i + \omega_j = \omega_k + \omega_\ell$. This contribution can be split into three coefficients that are evaluated for certain subsets of the allowed values for the indices, namely

$$
\begin{aligned}
S_\ell = T_\ell a_\ell^3 \cos(\theta_\ell + \theta_\ell - \theta_\ell) + \sum_{i\neq\ell}^{\infty} R_{i\ell}\, a_i^2 a_\ell \cos(\theta_i + \theta_\ell - \theta_i) \\
+ \sum_{i\neq\ell}^{\infty} \sum_{j\neq\ell}^{\infty} S_{ij(i+j-\ell)\ell}\, a_i a_j a_{(i+j-\ell)} \cos(\theta_i + \theta_j - \theta_{i+j-\ell}) \,,
\end{aligned} \tag{3.9}
$$

where each of the coefficients is given by

$$
\begin{aligned}
S_{ijk\ell} = & -\frac{1}{4} H_{kij\ell} \frac{\omega_i(\omega_j - \omega_k + 2\omega_i)}{(\omega_i - \omega_k)(\omega_i + \omega_j)} - \frac{1}{4} H_{ijk\ell} \frac{\omega_j(\omega_i - \omega_k + 2\omega_j)}{(\omega_j - \omega_k)(\omega_i + \omega_j)} - \frac{1}{4} H_{jki\ell} \frac{\omega_k(\omega_i + \omega_j - 2\omega_k)}{(\omega_i - \omega_k)(\omega_j - \omega_k)} \\
& - \frac{1}{2}\omega_j\omega_k X_{ijk\ell} \left(\frac{\omega_j}{\omega_i - \omega_k} - \frac{\omega_k}{\omega_i + \omega_j} + 1 \right) - \frac{1}{2}\omega_i\omega_k X_{jki\ell} \left(\frac{\omega_i}{\omega_j - \omega_k} - \frac{\omega_k}{\omega_i + \omega_j} + 1 \right) \\
& + \frac{1}{2}\omega_i\omega_j X_{kij\ell} \left(\frac{\omega_i}{\omega_j - \omega_k} + \frac{\omega_j}{\omega_i - \omega_k} + 1 \right) - \frac{m^2}{4} V_{ijk\ell} \left(\frac{\omega_i}{\omega_i - \omega_k} + \frac{\omega_j}{\omega_j - \omega_k} + 1 \right) \\
& + \frac{m^2}{4} V_{jki\ell} \left(\frac{\omega_k}{\omega_i - \omega_k} - \frac{\omega_j}{\omega_i + \omega_j} - 1 \right) + \frac{m^2}{4} V_{kij\ell} \left(\frac{\omega_k}{\omega_j - \omega_k} - \frac{\omega_i}{\omega_i + \omega_j} - 1 \right) \\
& + \frac{1}{4} Z^-_{ijk\ell} \left(\frac{\omega_k}{\omega_i + \omega_j} \right) + \frac{1}{4} Z^+_{ikj\ell} \left(\frac{\omega_j}{\omega_i - \omega_k} \right) + \frac{1}{4} Z^+_{jki\ell} \left(\frac{\omega_i}{\omega_j - \omega_k} \right) \,,
\end{aligned} \tag{3.10}
$$

$$
\begin{aligned}
R_{i\ell} = & \left(\frac{\omega_i^2}{\omega_\ell^2 - \omega_i^2} \right) \left(Y_{i\ell\ell i} - Y_{i\ell i\ell} + \omega_\ell^2 (X_{i\ell i\ell} - X_{\ell i\ell i}) \right) + \left(\frac{\omega_i^2}{\omega_\ell^2 - \omega_i^2} \right) \left(H_{\ell ii\ell} + m^2 V_{ii\ell\ell} - 2\omega_i^2 X_{\ell ii\ell} \right) \\
& - \left(\frac{\omega_\ell^2}{\omega_\ell^2 - \omega_i^2} \right) \left(H_{i\ell i\ell} + m^2 V_{\ell ii\ell} - 2\omega_i^2 X_{i\ell i\ell} \right) - \frac{m^2}{4} (V_{i\ell i\ell} + V_{ii\ell\ell}) + \omega_i^2 \omega_\ell^2 (P_{ii\ell} - 2P_{\ell\ell i}) \\
& - \omega_i\omega_\ell X_{i\ell i\ell} - \frac{3m^2}{2} V_{\ell ii\ell} - \frac{1}{2} H_{ii\ell\ell} + \omega_\ell^2 B_{ii\ell} - \omega_i^2 M_{\ell\ell i} - m^2 \omega_i^2 Q_{\ell\ell i} \,,
\end{aligned} \tag{3.11}
$$

and

$$
T_\ell = \frac{1}{2}\omega_\ell^2 \left(X_{\ell\ell\ell\ell} + 4B_{\ell\ell\ell} - 2M_{\ell\ell\ell} - 2m^2 Q_{\ell\ell\ell} \right) - \frac{3}{4} \left(H_{\ell\ell\ell\ell} + 3m^2 V_{\ell\ell\ell\ell} \right) . \tag{3.12}
$$

Following the form of (2.16) - (2.17), these resonant terms set the evolution of the renormalized integration coefficients to be [7]

$$
\frac{2\omega_\ell}{\epsilon^2} \frac{da_\ell}{dt} = - \sum_{i\neq\ell}^{\infty} \sum_{j\neq\ell}^{\infty} S_{ij(i+j-\ell)\ell}\, a_i a_j a_{(i+j-\ell)} \sin(b_\ell + b_{(i+j-\ell)} - b_i - b_j) \tag{3.13}
$$

$$
\begin{aligned}
\frac{2\omega_\ell a_\ell}{\epsilon^2} \frac{db_\ell}{dt} = & -T_\ell a_\ell^3 - \sum_{i\neq\ell}^{\infty} R_{i\ell}\, a_i^2 a_\ell \\
& - \sum_{i\neq\ell}^{\infty} \sum_{j\neq\ell}^{\infty} S_{ij(i+j-\ell)\ell}\, a_i a_j a_{(i+j-\ell)} \cos(b_\ell + b_{(i+j-\ell)} - b_i - b_j)
\end{aligned} \tag{3.14}
$$

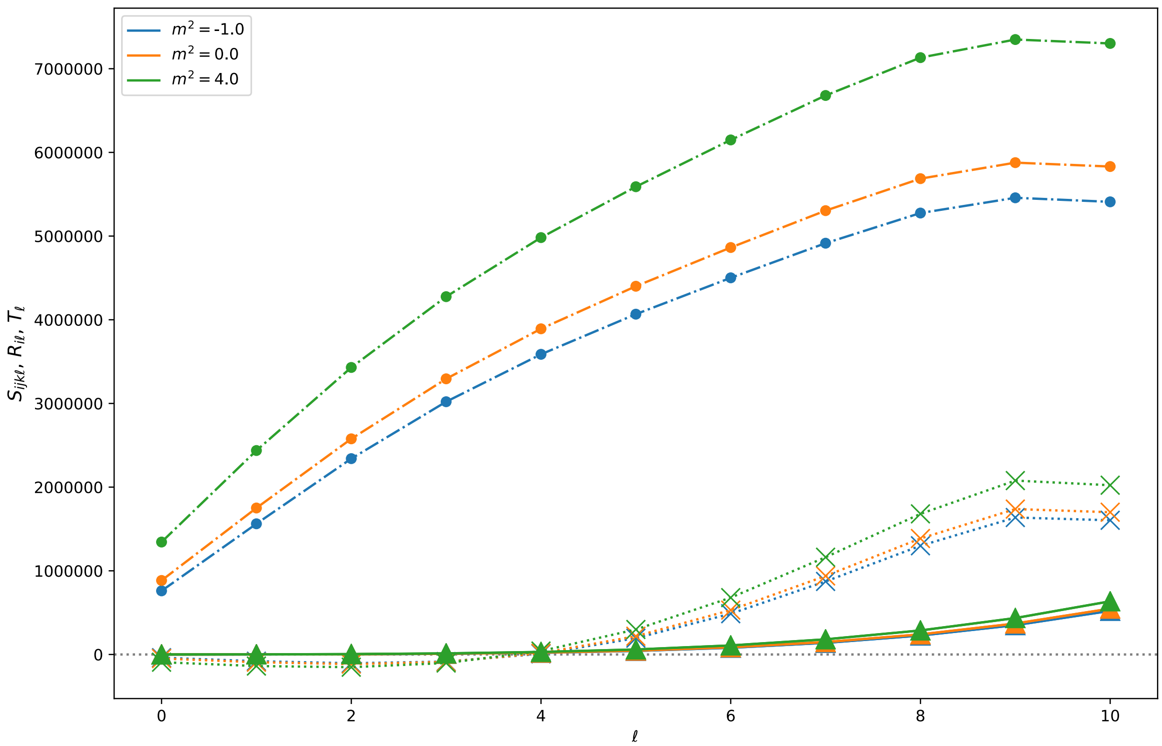


Figure 1: Evaluating (3.10)-(3.12) over different values of m^2 for $\ell \leq 10$. $S_{ij(i+j-\ell)\ell}$ is denoted by filled circles connected by dash-dotted lines; $R_{i\ell}$ is denoted by filled triangles connected by solid lines; T_ℓ is denoted by large Xs connected by dotted lines. Different values of m^2 are denoted by the colour of each series.

To examine the effects of non-zero masses on R, S, and T, we evaluate (3.10)-(3.12) for tachyonic, massless, and massive scalars in figure 1.

4 Resonances From Non-normalizable Modes

Now let us consider the excitation of non-normalizable modes by a driving term on the boundary of AdS. Since this boundary term is set at first order in the perturbative expansion, we must have that $\phi_3(x \to \pi/2) = 0$; therefore, ω_ℓ must correspond to a normalizable mode. What restrictions exist on the other frequencies, $\{\omega_i, \omega_j, \omega_k\}$? Aside from the trivial case where all modes to be normalizable, we could imagine that one of the modes is non-normalizable. However, this would violate the boundary condition on ϕ_3 is violated; thus, at least two modes must be non-normalizable so that the boundary condition can be satisfied. When three non-normalizable modes exist, there are two possibilities: first, that any combination of generically non-integer frequencies gives a non-integer value and does not contribute a secular term when projected onto the ω_ℓ basis; second, some particular combination of the non-normalizable frequencies gives an integer frequency, in which case the procedure for determining the contribution to $S_\ell^{(3)}$ follows the same procedure as the all-normalizable case. Therefore, the pertinent question is what secular contribution to the source term results from two of $\{\omega_i, \omega_j, \omega_j\}$ being non-normalizable. Because this choice breaks some of the symmetries that contributed to the previous expressions for resonance channels, the resonance conditions must be re-examined starting from the source expression (2.14).

Before proceeding further, an important consideration what the effect of non-normalizable modes are on the perturbative expansion that leads to the source equations. Since non-normalizable solutions do not have well-defined norms, we do not know *a priori* that the inner products described in Appendix A are still finite. To investigate this, consider the generic expression for the second-order metric function

$$A_2 = -\nu \int_0^x dy\, \mu \left((\dot{\phi}_1)^2 + (\phi_1')^2 + m^2 \phi_1^2 \sec^2 x \right) , \tag{4.1}$$

in the limit of $x \to \pi/2$, and let the scalar field ϕ_1 be given by a generic superposition of normalizable and non-normalizable eigenfunctions as in (2.8). Ignoring the time-dependent contributions, we find that

$$\lim_{\tilde{x} \to 0} A_2(\tilde{x} \equiv \pi/2 - x) = \tilde{x}^{-\xi} \left(\frac{2\tilde{x}^{2+d}}{2-\xi} - \frac{\tilde{x}^d (1 + (\Delta^-)^2)}{\xi} \right) \tag{4.2}$$

where we have defined $\xi = \sqrt{d^2 + 4m^2}$. In the massless case, $\xi = d$ and all powers of $\tilde{x}$ are non-negative; thus, the limit is finite. For tachyonic masses of $m_{BF}^2 < m^2 < 0$, so $0 < \xi < d$ and the limit is again finite. However, for scalars that either saturate the Breitenlohner-Freedman bound, or have $m^2 > 0$, part of the limit diverges. In order for the boundary to remain asymptotically AdS, counter-terms in the bulk action would be required to cancel such divergences – a case we will not address presently. Thus, we will restrict our discussion to $m_{BF}^2 < m^2 \leq 0$ to avoid these issues. A similar check on the near-boundary behaviour of δ_2 shows that the gauge condition $\delta_2(t, x = \pi/2)$ remains unchanged by the addition of non-normalizable modes given the same restrictions on the mass of the scalar field. With these restrictions in mind, let us now examine the resonances produced by the activation of non-normalizable modes.

4.1 Two Non-normalizable Modes with Equal Frequencies

As a first case, let us assume that the two non-normalizable modes have equal, constant, and arbitrary frequencies, ϖ. Applying the time-averaging procedure to the source S_ℓ once again eliminates all contributions except those that satisfy (3.1). We are now free to choose any one of $\{\omega_i, \omega_j, \omega_k\}$ to be normalizable and consider when the resonance condition is satisfied. In particular, we find that the following combinations are resonant:

$$\omega_i - \omega_j + \omega_k - \omega_\ell = 0 \qquad \Rightarrow \qquad \text{either } \omega_i \text{ or } \omega_k \text{ is normalizable} \tag{4.3}$$

$$\omega_i + \omega_j - \omega_k - \omega_\ell = 0 \qquad \Rightarrow \qquad \text{either } \omega_i \text{ or } \omega_j \text{ is normalizable} \tag{4.4}$$

$$\omega_i - \omega_j - \omega_k + \omega_\ell = 0 \qquad \Rightarrow \qquad \text{either } \omega_j \text{ or } \omega_k \text{ is normalizable.} \tag{4.5}$$

When any of these resonance conditions is met, the remaining normalizable mode will have a frequency equal to ω_ℓ, collapsing all sums over frequencies so that

$$S_\ell = \overline{T}_\ell \, a_\ell \bar{A}_{\varpi}^2 \cos(\theta_\ell) + \dots \, , \tag{4.6}$$

where the non-normalizable modes their amplitudes $\bar{A}_{\varpi}$ set by the choice of boundary condition. Collecting the appropriate terms in (2.14), and evaluating the each possible resonance we find that

$$\begin{aligned}
\overline{T}_\ell = \Bigg[& \frac{1}{2} Z^-_{\ell\varpi\varpi\ell} \left(\frac{\varpi}{\omega_\ell + \varpi} \right) + \frac{1}{2} Z^+_{\ell\varpi\varpi\ell} \left(\frac{\varpi}{\omega_\ell - \varpi} \right) + H_{\ell\varpi\varpi\ell} \left(\frac{\varpi^2}{\omega_\ell^2 - \varpi^2} \right) - H_{\varpi\ell\varpi\ell} \left(\frac{\omega_\ell^2}{\omega_\ell^2 - \varpi^2} \right) \\
& - m^2 V_{\ell\varpi\varpi\ell} \left(\frac{\omega_\ell^2}{\omega_\ell^2 - \varpi^2} \right) + m^2 V_{\varpi\varpi\ell\ell} \left(\frac{\varpi^2}{\omega_\ell^2 - \varpi^2} \right) + 2X_{\varpi\varpi\ell\ell} \left(\frac{\varpi^2 \omega_\ell^2}{\omega_\ell^2 - \varpi^2} \right) - 2X_{\ell\ell\varpi\varpi} \left(\frac{\varpi^4}{\omega_\ell^2 - \varpi^2} \right) \Bigg]_{\varpi \neq \omega_\ell} \\
& + \omega_\ell^2 X_{\varpi\varpi\ell\ell} - \varpi^2 X_{\ell\ell\varpi\varpi} - \frac{3}{2} m^2 V_{\ell\ell\varpi\varpi} - \frac{1}{2} m^2 V_{\varpi\varpi\ell\ell} - \frac{1}{2} H_{\varpi\varpi\ell\ell} + \omega_\ell^2 \tilde{Z}^+_{\varpi\varpi\ell} - 2\varpi^2 \omega_\ell^2 P_{\ell\ell\varpi} \\
& - \varpi^2 \left(\omega_\ell^2 P_{\ell\ell\varpi} - B_{\ell\ell\varpi} \right) \, .
\end{aligned} \tag{4.7}$$

Notice that the terms in the square braces only contribute when $\varpi \neq \omega_\ell$. Beginning from (2.14), only terms in the square braces that are proportional to $Z^\pm$ are limited in this way; the remaining terms have no such restriction. However, it can be shown that integral functions with permuted indices are equal when the non-normalizable frequency equals the normalizable frequency. Upon simplification, factors of $\omega_\ell^2 - \varpi^2$ are canceled and the overall contribution to T_ℓ from the terms in the braces is zero. Thus, these terms are grouped with those that have natural restrictions on the indices.

The renormalization flow equations for two equal, constant, non-normalizable frequencies are then

$$\frac{2\omega_\ell}{\epsilon^2} \frac{da_\ell}{dt} = 0, \quad \text{and} \quad \frac{2\omega_\ell a_\ell}{\epsilon^2} \frac{db_\ell}{dt} = -\overline{T}_\ell a_\ell \bar{A}_{\varpi}^2 \, . \tag{4.8}$$

Qualitatively, we see that instead of both the amplitude and the phase running with respect to τ, only the phase changes in time. Indeed, (4.8) tells us that b_ℓ is a linear function of τ with a slope that is determined by the $\mathcal{O}(\epsilon^3)$ physics encapsulated by $\overline{T}_\ell$.

Other resonant contributions become possible for more restrictive values of the non-normalizable frequency, such as if ϖ is allowed to be an integer. These contributions are denoted by the dots in (4.6) and are discussed briefly in Appendix B. In figures 2 and 3, we evaluate (4.7) for $\ell < 10$ over a variety of ϖ values first for a massless scalar, then for a tachyonic scalar.

4.2 Special Values of Non-normalizable Frequencies

Let us now consider special values of non-normalizable frequencies that will lead to a greater number of resonance channels. While general non-normalizable frequencies do not require any such restrictions, we will find it informative to examine these special cases as they possess more symmetry in index/frequency values than the case of equal non-normalizable frequencies, but less than all-normalizable modes.

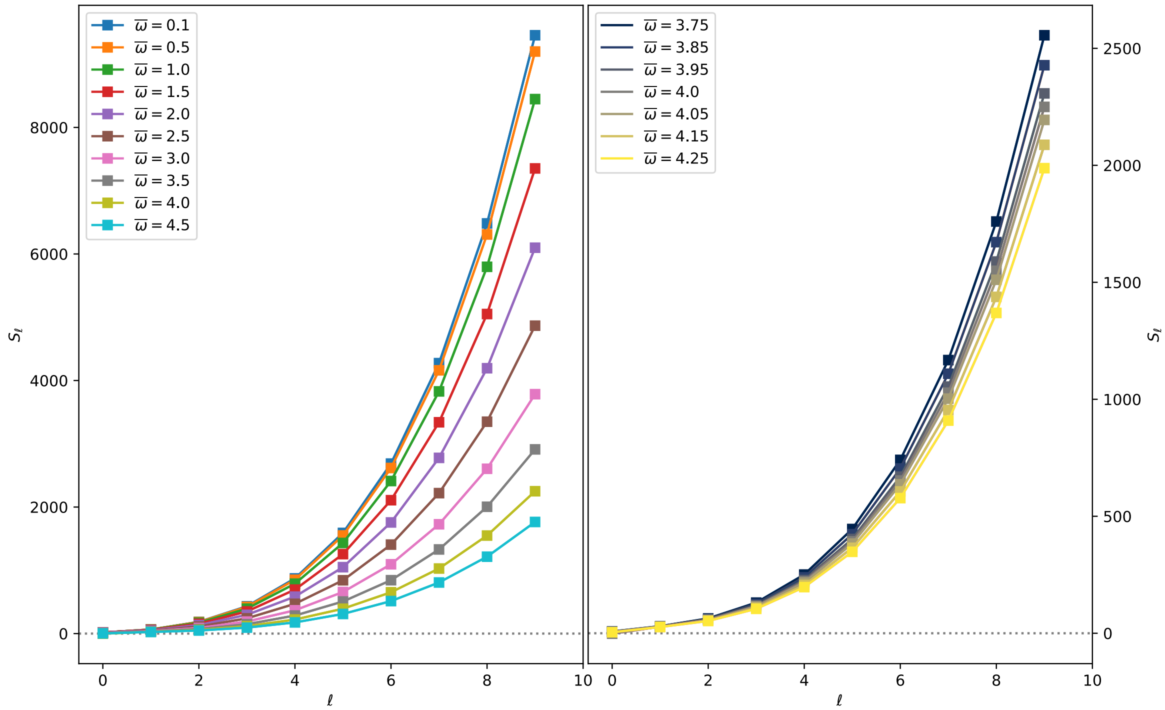


Figure 2: *Left:* Evaluating (4.7) when $m^2 = 0$ for various choices of $\overline{\omega}$. *Right*: The behaviour of S_ℓ for $\overline{\omega}$ values near ω_0.

4.2.1 Add to an integer

First, we choose two of the modes to be non-normalizable with frequencies $\overline{\omega}_1$ and $\overline{\omega}_2$ that add to give an integer: $\overline{\omega}_1 + \overline{\omega}_2 = 2n$ where $n = 1, 2, 3, \ldots$ (note that the $n = 0$ case means that both $\overline{\omega}_1$ and $\overline{\omega}_2$ would need to be zero by the positive-frequency requirement and so would not contribute). Furthermore, either frequency need not be an integer and therefore the difference $|\overline{\omega}_1 - \overline{\omega}_2|$ will, in general, not be an integer. In § 4.3, we examine the case when the difference of non-normalizable frequencies is an integer.

When we consider possible resonance channels, we see that resonances can be grouped into

$$(++) : \ \omega_i + 2n = \omega_\ell \quad \forall\, \ell \geq n \tag{4.9}$$

$$(+-) : \ \omega_i - 2n = \omega_\ell \quad \forall\, n \tag{4.10}$$

for any $m^2_{BF} < m^2 < 0$. However, for a massless scalar, we have an additional channel

$$(-+) : \ -\omega_i + 2n = \omega_\ell \quad \forall\, n \geq \ell + d \tag{4.11}$$

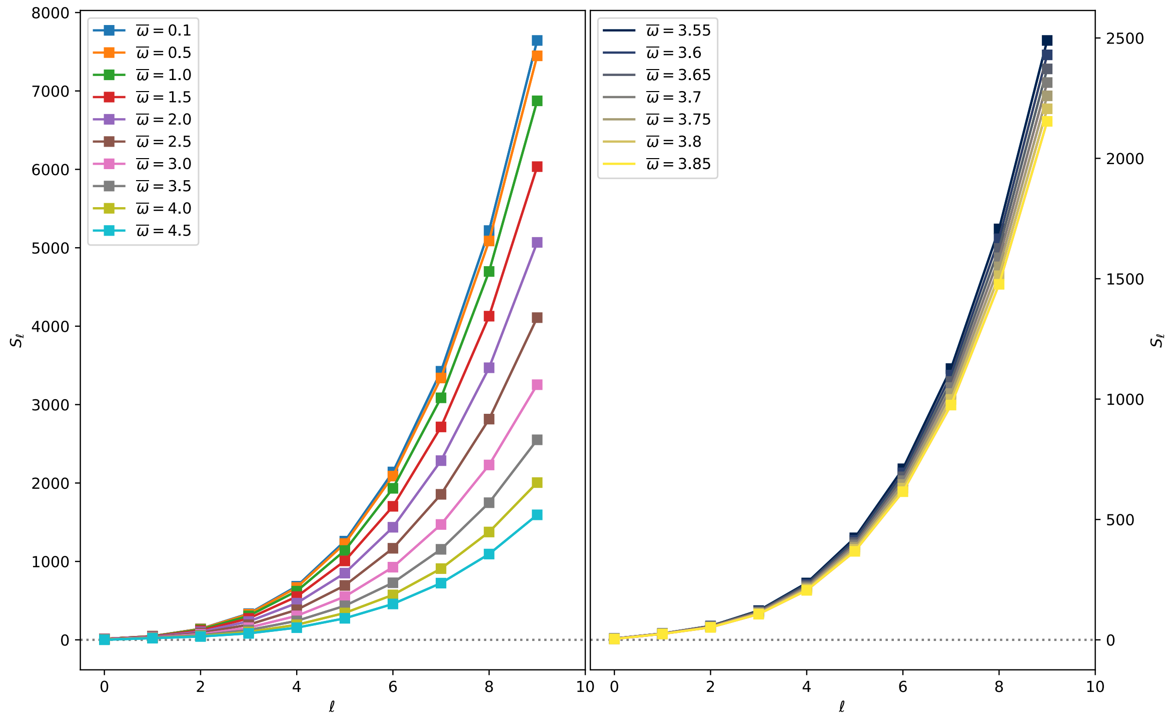


Figure 3: *Left:* Evaluating $\overline{T}_\ell$ for a tachyon with $m^2 = -1.0$. *Right:* The behaviour of S_ℓ near $\omega_0 = \Delta^+ \approx 3.7$.

Adding the channels together, the total source term is

$$\begin{aligned} S_\ell = &\sum_{\overline{\omega}_1+\overline{\omega}_2=2n} \left[\Theta\left(n-\ell-d\right)\overline{R}^{(-+)}_{(n-\ell-d)\ell}\,\bar{A}_1\bar{A}_2\,a_{(n-\ell-d)}\cos\left(\theta_{(n-\ell-d)}-\theta_1-\theta_2\right)\right]_{m^2=0} \\ &+\sum_{\overline{\omega}_1+\overline{\omega}_2=2n}\Theta\left(\ell-n\right)\overline{R}^{(++)}_{(\ell-n)\ell}\,\bar{A}_1\bar{A}_2\,a_{(\ell-n)}\cos\left(\theta_{(\ell-n)}+\theta_1+\theta_2\right) \\ &+\sum_{\overline{\omega}_1+\overline{\omega}_2=2n}\overline{R}^{(+-)}_{(\ell+n)\ell}\,\bar{A}_1\bar{A}_2\,a_{(\ell+n)}\cos\left(\theta_{(\ell+n)}-\theta_1-\theta_2\right) \\ &+\overline{T}_\ell\,\bar{A}_1\bar{A}_2\,a_\ell\cos\left(\theta_\ell\right) \end{aligned} \tag{4.12}$$

where the Heaviside step function $\Theta(x)$ enforces the restrictions on the indices in (4.9) and (4.11) and $\theta_1 = \overline{\omega}_1 t + b_1$, etc.

In the following expressions, the sum over all $\overline{\omega}_1$, $\overline{\omega}_2$ such that $\overline{\omega}_1 + \overline{\omega}_2 = 2n$ is implied, and only the restrictions on individual frequencies are included. Examining each channel in

(4.12) individually, we find

$$\begin{aligned}
\overline{R}_{i\ell}^{(++)} &= -\frac{1}{4}\sum_{\overline{\omega}_2\neq\omega_\ell}\frac{\overline{\omega}_2}{\omega_\ell-\overline{\omega}_2}Z^-_{i12\ell}-\frac{1}{4}\sum_{\overline{\omega}_1\neq\omega_\ell}\frac{\overline{\omega}_1}{\omega_\ell-\overline{\omega}_1}Z^-_{i21\ell}-\frac{1}{8n}\left(\omega_\ell-2n\right)Z^-_{12i\ell}\\
&\quad-\frac{1}{4}\sum_{\omega_i\neq\overline{\omega}_1}\frac{1}{\omega_\ell-\overline{\omega}_2}\Big[\overline{\omega}_1\left(H_{i12\ell}+m^2V_{12i\ell}-2\overline{\omega}_2^2X_{i12\ell}\right)+\left(\omega_\ell-2n\right)\left(H_{1i2\ell}+m^2V_{i21\ell}-2\overline{\omega}_2^2X_{1i2\ell}\right)\Big]\\
&\quad-\frac{1}{4}\sum_{\omega_i\neq\overline{\omega}_2}\frac{1}{\omega_\ell-\overline{\omega}_1}\Big[\overline{\omega}_2\left(H_{i21\ell}+m^2V_{21i\ell}-2\overline{\omega}_1^2X_{i21\ell}\right)+\left(\omega_\ell-2n\right)\left(H_{2i1\ell}+m^2V_{i12\ell}-2\overline{\omega}_1^2X_{2i1\ell}\right)\Big]\\
&\quad-\frac{1}{8n}\sum_{\overline{\omega}_1\neq\overline{\omega}_2}\Big[\overline{\omega}_1H_{21i\ell}+\overline{\omega}_2H_{12i\ell}+m^2\left(\overline{\omega}_1V_{1i2\ell}+\overline{\omega}_2V_{2i1\ell}\right)-\left(\omega_\ell-2n\right)^2\left(\overline{\omega}_1X_{21i\ell}+\overline{\omega}_2X_{12i\ell}\right)\Big]\\
&\quad+\frac{1}{2}\Big[\overline{\omega}_1\overline{\omega}_2X_{i12\ell}+\left(\omega_\ell-2n\right)\left(\overline{\omega}_1X_{21i\ell}+\overline{\omega}_2X_{12i\ell}\right)-\frac{m^2}{2}\left(V_{i12\ell}+V_{i21\ell}+V_{12i\ell}\right)\Big]
\end{aligned}\tag{4.13}$$

The notation $X_{i12\ell}$ corresponds to evaluating $X_{ijk\ell}$ with $\omega_j=\overline{\omega}_1$ and $\omega_k=\overline{\omega}_2$. Next, we find that

$$\begin{aligned}
\overline{R}_{i\ell}^{(+-)} &= -\frac{1}{4}\Big[\frac{(\omega_\ell+2n)}{2n}Z^-_{12i\ell}+2(\omega_\ell+2n)\left(\overline{\omega}_1X_{21i\ell}+\overline{\omega}_2X_{12i\ell}\right)\\
&\quad-\frac{\overline{\omega}_1}{(\omega_\ell+\overline{\omega}_2)}\left(H_{i12\ell}+m^2V_{12i\ell}-2\overline{\omega}_2^2X_{i12\ell}\right)+\frac{(\omega_\ell+2n)}{(\omega_\ell+\overline{\omega}_2)}\left(H_{1i2\ell}+m^2V_{i21\ell}-2\overline{\omega}_2^2X_{1i2\ell}\right)\\
&\quad-\frac{\overline{\omega}_2}{(\omega_\ell+\overline{\omega}_1)}\left(H_{i21\ell}+m^2V_{21i\ell}-2\overline{\omega}_1^2X_{i21\ell}\right)+\frac{(\omega_\ell+2n)}{(\omega_\ell+\overline{\omega}_1)}\left(H_{2i1\ell}+m^2V_{i12\ell}-2\overline{\omega}_1^2X_{2i1\ell}\right)\\
&\quad-2\overline{\omega}_1\overline{\omega}_2X_{i12\ell}+m^2\left(V_{12i\ell}+V_{i12\ell}+V_{i21\ell}\right)\Big]+\frac{1}{4}\sum_{\overline{\omega}_2\neq\omega_\ell}\frac{\overline{\omega}_1\overline{\omega}_2(\omega_\ell+2n)}{\omega_\ell+\overline{\omega}_2}\left(X_{21i\ell}-X_{\ell i12}\right)\\
&\quad+\frac{1}{4}\sum_{\overline{\omega}_1\neq\omega_\ell}\frac{\overline{\omega}_1\overline{\omega}_2(\omega_\ell+2n)}{\omega_\ell+\overline{\omega}_1}\left(X_{12i\ell}-X_{\ell i12}\right).
\end{aligned}\tag{4.14}$$

When $m^2=0$, we have contributions from

$$\begin{aligned}
\overline{R}_{i\ell}^{(-+)} &= \frac{1}{4}\sum_{\overline{\omega}_2\neq\omega_\ell}\frac{\overline{\omega}_2}{\omega_\ell-\overline{\omega}_2}Z^+_{i12\ell}+\frac{1}{4}\sum_{\overline{\omega}_1\neq\omega_\ell}\frac{\overline{\omega}_1}{\omega_\ell-\overline{\omega}_1}Z^+_{i21\ell}+\frac{1}{4}\sum_{i\neq\ell}\left(\frac{2n-\omega_\ell}{2n}\right)Z^-_{12i\ell}\\
&\quad+\frac{1}{4}\sum_{\overline{\omega}_1\neq\omega_i}\frac{1}{\omega_i-\overline{\omega}_1}\Big[\overline{\omega}_1\left(H_{i12\ell}-2\overline{\omega}_2^2X_{i12\ell}\right)-\left(2n-\omega_\ell\right)\left(H_{1i2\ell}-2\overline{\omega}_2^2X_{1i2\ell}\right)\Big]\\
&\quad+\frac{1}{4}\sum_{\overline{\omega}_2\neq\omega_i}\frac{1}{\omega_i-\overline{\omega}_2}\Big[\overline{\omega}_2\left(H_{i21\ell}-2\overline{\omega}_1^2X_{i21\ell}\right)-\left(2n-\omega_\ell\right)\left(H_{2i1\ell}-2\overline{\omega}_1^2X_{2i1\ell}\right)\Big]\\
&\quad-\frac{1}{8n}\sum_{\overline{\omega}_1\neq\overline{\omega}_2}\Big[\overline{\omega}_1H_{21i\ell}+\overline{\omega}_2H_{12i\ell}-2\left(2n-\omega_\ell\right)^2\left(\overline{\omega}_1X_{21i\ell}+\overline{\omega}_2X_{12i\ell}\right)\Big]\\
&\quad-\frac{1}{2}\Big[\left(2n-\omega_\ell\right)\left(\overline{\omega}_1X_{21i\ell}+\overline{\omega}_2X_{12i\ell}\right)-\overline{\omega}_1\overline{\omega}_2X_{i12\ell}\Big].
\end{aligned}\tag{4.15}$$

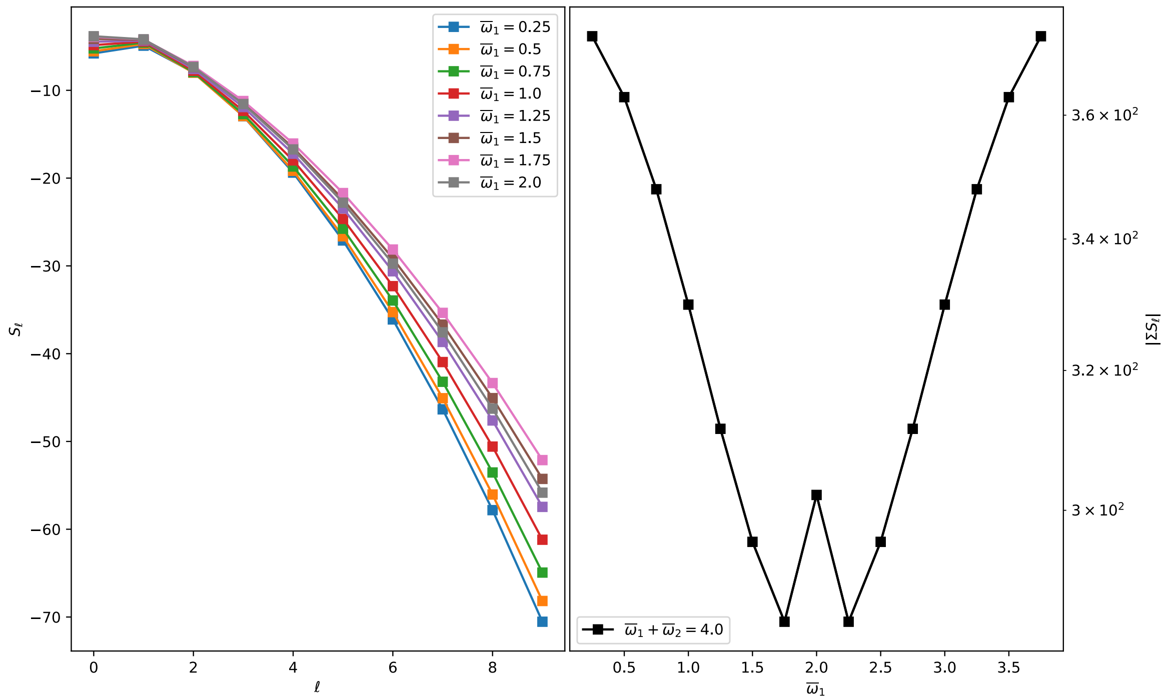


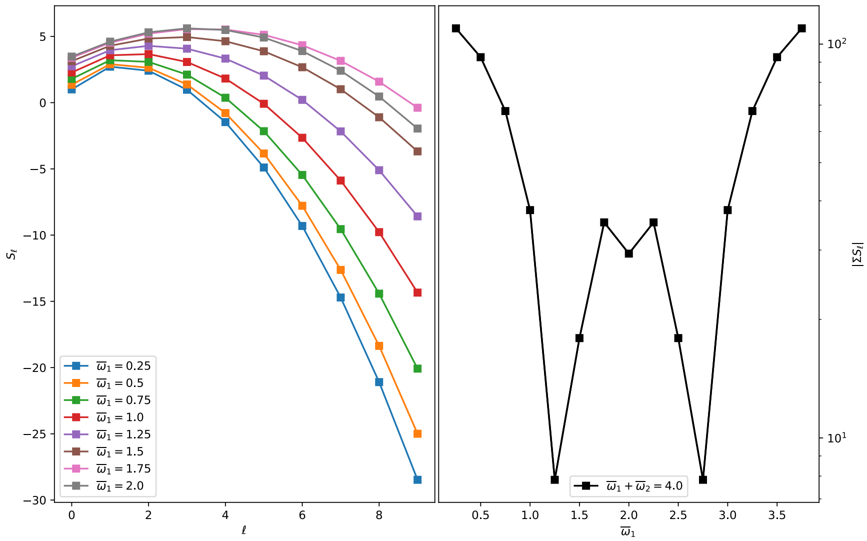
Figure 4: *Left*: Source term values for a tachyonic scalar with $m^2 = -1.0$ when the frequencies of non-normalizable modes sum to 4.0. *Right*: The absolute value of the sum of the source terms for each choice of $\overline{\omega}_1$, $\overline{\omega}_2$.

NB. In (4.15) only, $\omega_i = 2i + \Delta^+ = 2i + d$ since this term requires that $m^2 = 0$ to contribute. We maintain the same notation out of convenience, despite the special case. Finally,

$$\begin{aligned}\overline{T}_\ell = \frac{1}{2}\omega_\ell^2\left(\tilde{Z}^+_{11\ell} + \tilde{Z}^+_{22\ell}\right) - \frac{1}{2}\Big[&H_{11\ell\ell} + H_{22\ell\ell} + m^2\left(V_{\ell 11\ell} + V_{\ell 22\ell}\right) - 2\omega_\ell^2\left(X_{11\ell\ell} + X_{22\ell\ell}\right) \\ &+ 4\omega_\ell^2\left(\overline{\omega}_1^2 P_{\ell\ell 1} + \overline{\omega}_2^2 P_{\ell\ell 2}\right) + 2\overline{\omega}_1^2 M_{\ell\ell 1} + 2\overline{\omega}_2^2 M_{\ell\ell 2} + 2m^2\left(\overline{\omega}_1^2 Q_{\ell\ell 1} + \overline{\omega}_2^2 Q_{\ell\ell 2}\right)\Big].\end{aligned} \tag{4.16}$$

In figure 4, we compute the total source term – modulo the amplitudes a_i and $\bar{A}_\alpha$ – for a tachyonic scalar with $n = 2$. Figure 5 provides a comparison between the value of the source term for a massless scalar between two choices of n: one that includes contributions from $\overline{R}^{(-+)}_{i\ell}$ and one that does not.

The renormalization flow equations include the sum of all the channels (none of which



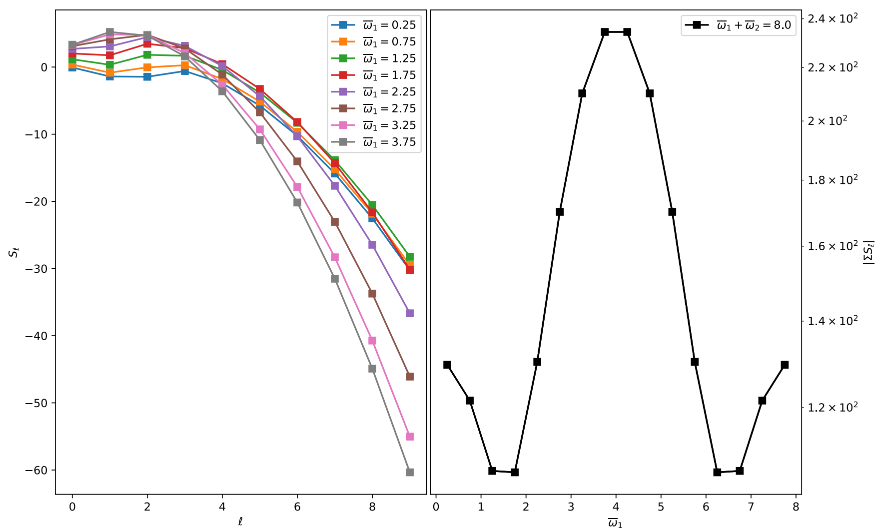


Figure 5: *Above:* The value of (4.12) as a function of ℓ for a massless scalar with values of $\overline{\omega}_1$ and $\overline{\omega}_2$ chosen so that $\overline{\omega}_1 + \overline{\omega}_2 = 4$. *Below:* The same plot but with values chosen to satisfy $\overline{\omega}_1 + \overline{\omega}_2 = 8$.

vanish naturally), and are

$$\begin{aligned} \frac{2\omega_\ell}{\epsilon^2}\frac{da_\ell}{dt} = &- \sum_{\overline{\omega}_1+\overline{\omega}_2=2n} \left[\Theta\left(n-\ell-d\right) \overline{R}^{(-+)}_{(n-\ell-d)\ell}\, \bar{A}_1 \bar{A}_2\, a_{(n-\ell-d)} \sin\left(b_{(n-\ell-d)} - b_1 - b_2\right) \right]_{m^2=0} \\ &- \sum_{\overline{\omega}_1+\overline{\omega}_2=2n} \Theta\left(\ell-n\right) \overline{R}^{(++)}_{(\ell-n)\ell}\, \bar{A}_1 \bar{A}_2\, a_{(\ell-n)} \sin\left(b_{(\ell-n)} + b_1 + b_2\right) \\ &- \sum_{\overline{\omega}_1+\overline{\omega}_2=2n} \overline{R}^{(+-)}_{(\ell+n)\ell}\, \bar{A}_1 \bar{A}_2\, a_{(\ell+n)} \sin\left(b_{(\ell+n)} - b_1 - b_2\right) \,, \end{aligned} \tag{4.17}$$

$$
\begin{aligned}
\frac{2\omega_\ell a_\ell}{\epsilon^2}\frac{db_\ell}{dt} = &-\sum_{\overline{\omega}_1+\overline{\omega}_2=2n}\left[\Theta\left(n-\ell-d\right)\overline{R}^{(-+)}_{(n-\ell-d)\ell}\;\bar{A}_1\bar{A}_2\,a_{(n-\ell-d)}\cos\left(b_{(n-\ell-d)}-b_1-b_2\right)\right]_{m^2=0} \\
&-\sum_{\overline{\omega}_1+\overline{\omega}_2=2n}\Theta\left(\ell-n\right)\overline{R}^{(++)}_{(\ell-n)\ell}\,\bar{A}_1\bar{A}_2\,a_{(\ell-n)}\cos\left(b_{(\ell-n)}+b_1+b_2\right) \\
&-\sum_{\overline{\omega}_1+\overline{\omega}_2=2n}\overline{R}^{(+-)}_{(\ell+n)\ell}\,\bar{A}_1\bar{A}_2\,a_{(\ell+n)}\cos\left(b_{(\ell+n)}-b_1-b_2\right)-\overline{T}_\ell\,\bar{A}_1\bar{A}_2\,a_\ell\,.
\end{aligned}
\tag{4.18}
$$

4.3 Integer Plus χ

Finally, let us consider the case where the non-normalizable frequencies are non-integer, but differ from integer values by a set amount. In analogue to the case where all modes are normalizable, we consider the non-normalizable frequencies to be shifted away from integer values by

$$
\omega_\gamma = 2\gamma+\chi\,,
\tag{4.19}
$$

where $\gamma\in\mathbb{Z}^*$ (greek letters are chosen to differentiate these non-normalizable modes from normalizable modes with integer frequencies, which use roman letters). We furthermore limit χ to be non-integer[1] and set $m^2=0$ throughout. For this choice of non-normalizable frequencies there are no resonant contributions from the all-plus channel, unlike the naturally vanishing resonance found in § 3.1. Only when either $\omega_i+\omega_\gamma=\omega_\beta-\omega_\ell$, or $\omega_i+\omega_\gamma=\omega_\beta+\omega_\ell$ with $i+\gamma\geq\ell$, are resonant terms present. Let us examine each case separately.

4.3.1 $\omega_i+\omega_\gamma=\omega_\beta-\omega_\ell$

When the resonance condition $\omega_i+\omega_\gamma=\omega_\beta-\omega_\ell$ is met, the contribution to the source term is of the form

$$
\begin{aligned}
S_\ell = &\sum_{i\neq\ell}\sum_{\gamma\neq\beta}\overline{S}^{(1)}_{i(i+\gamma+\ell)\gamma\ell}\,a_i\bar{A}_{(i+\gamma+\ell)}\bar{A}_\gamma\cos\left(\theta_i-\theta_{(i+\gamma+\ell)}+\theta_\gamma\right) \\
&+\sum_\beta\overline{R}^{(1)}_{\beta\ell}\,a_\ell\bar{A}^2_\beta\cos\left(\theta_\ell+\theta_\beta-\theta_\beta\right)+\dots\,,
\end{aligned}
\tag{4.20}
$$

where

$$
\begin{aligned}
\overline{S}^{(1)}_{i\beta\gamma\ell} = &\frac{1}{4}H_{\beta\gamma i\ell}\frac{\omega_\gamma(\omega_i-\omega_\beta+2\omega_\gamma)}{(\omega_\beta-\omega_\gamma)(\omega_i+\omega_\gamma)}-\frac{1}{4}H_{\gamma\beta i\ell}\frac{\omega_\beta(\omega_i+\omega_\gamma-2\omega_\beta)}{(\omega_i-\omega_\beta)(\omega_\beta-\omega_\gamma)}-\frac{1}{4}H_{\gamma i\beta\ell}\frac{\omega_i(\omega_\gamma-\omega_\beta+2\omega_i)}{(\omega_i-\omega_\beta)(\omega_i+\omega_\gamma)} \\
&+\frac{1}{2}\omega_i\omega_\gamma X_{\beta\gamma i\ell}\left(\frac{\omega_\gamma}{\omega_i-\omega_\beta}-\frac{\omega_i}{\omega_\beta+\omega_\gamma}+1\right)+\frac{1}{2}\omega_i\omega_\beta X_{\gamma\beta i\ell}\left(\frac{\omega_i}{\omega_\beta-\omega_\gamma}+\frac{\omega_\beta}{\omega_i+\omega_\gamma}-1\right) \\
&+\frac{1}{2}\omega_\beta\omega_\gamma X_{i\beta\gamma\ell}\left(\frac{\omega_\beta}{\omega_i+\omega_\gamma}-\frac{\omega_\gamma}{\omega_i-\omega_\beta}-1\right)-\frac{1}{4}Z^+_{\beta\gamma i\ell}\left(\frac{\omega_i}{\omega_i+\omega_\ell}\right) \\
&+\frac{1}{4}Z^-_{i\gamma\beta\ell}\left(\frac{\omega_\beta}{\omega_\ell-\omega_\beta}\right)+\frac{1}{4}Z^+_{i\beta\gamma\ell}\left(\frac{\omega_\gamma}{\omega_\ell+\omega_\gamma}\right)\,,
\end{aligned}
\tag{4.21}
$$

[1] Indeed, for integer values of χ, the sum or difference of two non-normalizable modes could be an integer. This would either be covered by the work in § 4.2.1, or be a slight variation of it.

and

$$\overline{R}^{(1)}_{\beta\ell} = \frac{1}{4}Z^{-}_{\ell\beta\beta\ell}\left(\frac{\omega_\beta}{\omega_\ell+\omega_\beta}\right) + \frac{1}{4}Z^{+}_{\ell\beta\beta\ell}\left(\frac{\omega_\beta}{\omega_\ell-\omega_\beta}\right) + \frac{1}{2}H_{\ell\beta\beta\ell}\left(\frac{\omega_\beta^2}{\omega_\ell^2-\omega_\beta^2}\right) - \frac{1}{2}H_{\beta\ell\beta\ell}\left(\frac{\omega_\ell^2}{\omega_\ell^2-\omega_\beta^2}\right)$$
$$+ X_{\beta\ell\beta\ell}\left(\frac{\omega_\ell^4}{\omega_\ell^2-\omega_\beta^2}\right) - \frac{1}{2}\omega_\beta^2 X_{\ell\beta\beta\ell}\left(\frac{\omega_\ell^2+\omega_\beta^2}{\omega_\ell^2-\omega_\beta^2}\right) - \frac{1}{2}H_{\ell\beta\beta\ell} + \omega_\ell^2\tilde{Z}^{+}_{\beta\beta\ell} - 2\omega_\beta^2\omega_\ell^2 P_{\ell\ell\beta} - \omega_\beta^2 M_{\ell\ell\beta}\,. \tag{4.22}$$

4.3.2 $\omega_i + \omega_\gamma = \omega_\beta + \omega_\ell$

Similarly, when the resonance condition $\omega_i + \omega_\gamma = \omega_\beta + \omega_\ell$ is met, the contribution to the source term is

$$S_\ell = \underbrace{\sum_{i\neq\ell}\sum_{\gamma\neq\beta}}_{i+\gamma\geq\ell} \overline{S}^{(2)}_{i(i+\gamma-\ell)\gamma\ell}\, a_i \bar{A}_{(i+\gamma-\ell)}\bar{A}_\gamma \cos\left(\theta_i - \theta_{(i+\gamma-\ell)} + \theta_\gamma\right)$$
$$+ \sum_\beta \overline{R}^{(2)}_{\beta\ell}\, a_\ell \bar{A}_\beta^2 \cos\left(\theta_\ell + \theta_\beta - \theta_\beta\right) + \dots\,, \tag{4.23}$$

where

$$\overline{S}^{(2)}_{i\beta\gamma\ell} = \frac{1}{4}H_{\beta\gamma i\ell}\frac{\omega_\gamma(\omega_i-\omega_\beta)}{(\omega_\beta-\omega_\gamma)(\omega_i-\omega_\gamma)} - \frac{1}{4}H_{\gamma\beta i\ell}\frac{\omega_\beta(\omega_\ell-\omega_\beta)}{(\omega_\beta-\omega_\gamma)(\omega_i-\omega_\beta)} + \frac{1}{4}H_{\beta i\gamma\ell}\frac{\omega_i(\omega_\gamma-\omega_\beta)}{(\omega_i-\omega_\beta)(\omega_i-\omega_\gamma)}$$
$$+ \frac{1}{2}\omega_i\omega_\gamma X_{\beta\gamma i\ell}\left(\frac{\omega_\gamma}{\omega_i-\omega_\beta} - \frac{\omega_i}{\omega_\beta-\omega_\gamma} + 1\right) + \frac{1}{2}\omega_i\omega_\beta X_{\gamma\beta i\ell}\left(\frac{\omega_i}{\omega_\beta-\omega_\gamma} - \frac{\omega_\beta}{\omega_i-\omega_\gamma} - 1\right)$$
$$+ \frac{1}{2}\omega_\beta\omega_\gamma X_{i\beta\gamma\ell}\left(\frac{\omega_\beta}{\omega_i-\omega_\gamma} - \frac{\omega_\gamma}{\omega_i-\omega_\beta} - 1\right) + \frac{1}{4}Z^{-}_{i\gamma\beta\ell}\left(\frac{\omega_\beta}{\omega_\ell+\omega_\beta}\right)$$
$$+ \frac{1}{4}Z^{+}_{i\beta\gamma\ell}\left(\frac{\omega_\gamma}{\omega_\ell-\omega_\gamma}\right) - \frac{1}{4}Z^{+}_{\beta\gamma i\ell}\left(\frac{\omega_i}{\omega_i-\omega_\ell}\right)\,, \tag{4.24}$$

and

$$\overline{R}^{(2)}_{\beta\ell} = \frac{1}{4}Z^{-}_{\ell\beta\beta\ell}\left(\frac{\omega_\beta}{\omega_\ell+\omega_\beta}\right) + \frac{1}{4}Z^{+}_{\ell\beta\beta\ell}\left(\frac{\omega_\beta}{\omega_\ell-\omega_\beta}\right) + \frac{1}{2}H_{\ell\beta\beta\ell}\left(\frac{\omega_\beta^2}{\omega_\ell^2-\omega_\beta^2}\right) - \frac{1}{2}H_{\beta\ell\beta\ell}\left(\frac{\omega_\ell^2}{\omega_\ell^2-\omega_\beta^2}\right)$$
$$+ X_{\beta\beta\ell\ell}\left(\frac{\omega_\ell^2}{\omega_\ell^2-\omega_\beta^2}\right) + \frac{1}{2}\omega_\beta^2 X_{\ell\beta\beta\ell}\left(\frac{\omega_\ell^2+\omega_\beta^2}{\omega_\ell^2-\omega_\beta^2}\right) - \frac{1}{2}H_{\beta\beta\ell\ell} + \omega_\ell^2\tilde{Z}^{+}_{\beta\beta\ell} - 2\omega_\beta^2\omega_\ell^2 P_{\ell\ell\beta} - \omega_\beta^2 M_{\ell\ell\beta} \tag{4.25}$$

Unlike the case with all normalizable modes where two of the three resonance channels naturally vanished, both of the resonant channels contribute when the non-normalizable modes have frequencies given by (4.19). Therefore, the renormalization flow equations will

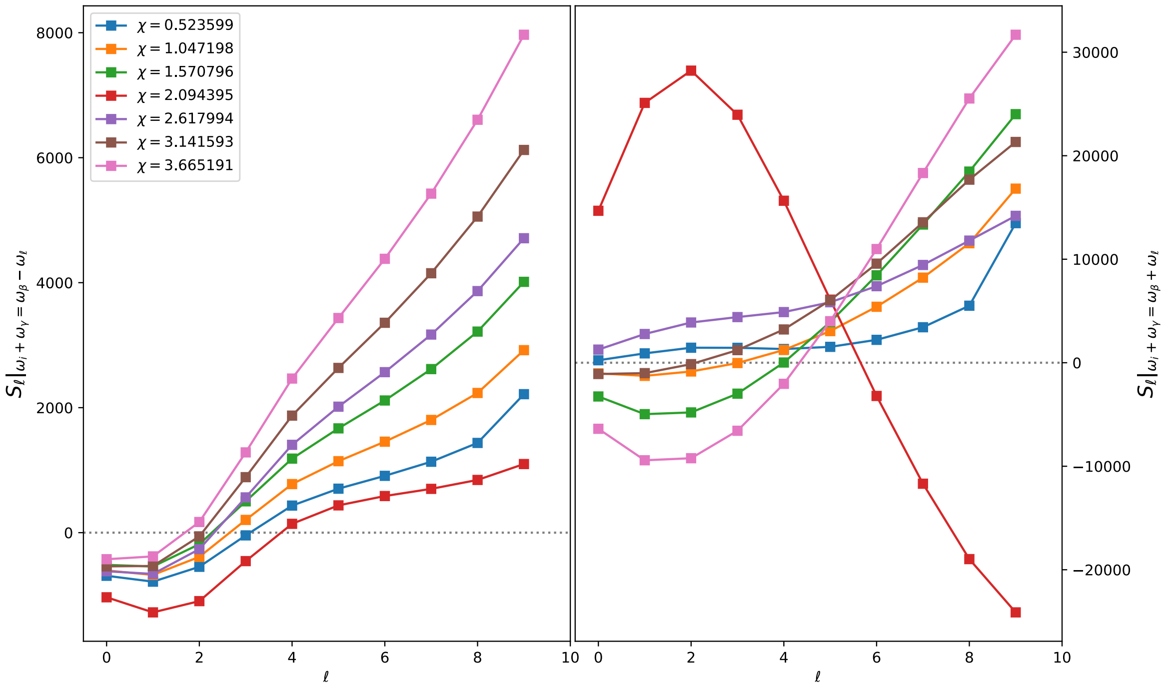


Figure 6: *Left:* Evaluating the source term (4.20) for various values of χ for $\ell < 10$. *Right:* Evaluating the source term (4.23) subject to $i + \gamma \geq \ell$ for the same values of χ and the same range of ℓ.

contain contributions from both channels:

$$\begin{aligned}\frac{2\omega_\ell}{\epsilon^2}\frac{da_\ell}{dt} = &-\sum_{i\neq\ell}\sum_{\gamma\neq\beta}\overline{S}^{(1)}_{i(i+\gamma+\ell)\gamma\ell}\,a_i\bar{A}_{(i+\gamma+\ell)}\bar{A}_\gamma\sin\left(b_\ell + b_{(i+\gamma+\ell)} - b_i - b_\gamma\right)\\ &-\underbrace{\sum_{i\neq\ell}\sum_{\gamma\neq\beta}}_{i+\gamma\geq\ell}\overline{S}^{(2)}_{i(i+\gamma-\ell)\gamma\ell}\,a_i\bar{A}_{(i+\gamma-\ell)}\bar{A}_\gamma\sin\left(b_\ell + b_{(i+\gamma-\ell)} - b_i - b_\gamma\right)\end{aligned} \tag{4.26}$$

$$\begin{aligned}\frac{2\omega_\ell a_\ell}{\epsilon^2}\frac{db_\ell}{dt} = &-\sum_{\beta}\overline{R}^{(1)}_{\beta\ell}\,a_\ell\bar{A}^2_\beta - \sum_{\beta}\overline{R}^{(2)}_{\beta\ell}\,a_\ell\bar{A}^2_\beta\\ &-\sum_{i\neq\ell}\sum_{\gamma\neq\beta}\overline{S}^{(1)}_{i(i+\gamma+\ell)\gamma\ell}\,a_i\bar{A}_{(i+\gamma+\ell)}\bar{A}_\gamma\cos\left(b_\ell + b_{(i+\gamma+\ell)} - b_i - b_\gamma\right)\\ &-\underbrace{\sum_{i\neq\ell}\sum_{\gamma\neq\beta}}_{i+\gamma\geq\ell}\overline{S}^{(2)}_{i(i+\gamma-\ell)\gamma\ell}\,a_i\bar{A}_{(i+\gamma-\ell)}\bar{A}_\gamma\cos\left(b_\ell + b_{(i+\gamma-\ell)} - b_i - b_\gamma\right)\end{aligned} \tag{4.27}$$

In figure 6, we evaluate both resonant contributions channels and plot their contributions for various values of χ. In particular, we examine the values $\chi \in \{\pi/6, \ldots, 7\pi/6\}$.

5 Discussion

We have seen that the inclusion of a time-dependent boundary term in the holographic dual of a quantum quench allows energy to enter the bulk spacetime through coupling between normalizable and non-normalizable modes. The dynamics of the weakly turbulent energy cascades that trigger instability were captured by secular terms at third-order that could not be removed by phase shifts alone. By using the Two-Time Formalism, we have determined the renormalization group flow equations for the slowly varying amplitudes and phases that are tuned as to cancel the secular terms.

Unlike when only normalizable modes are considered, the introduction of non-normalizable modes results in no naturally vanishing resonance channels for the frequencies considered. Most importantly, the flow equations for a_ℓ and b_ℓ are now *linear*, since the non-normalizable amplitudes and phases are set by the first-order boundary condition and thus remain constant. In practice, this means the evolution of the system will be different than in the case where only normalizable modes are activated [8], however how exactly the behaviour differs will require further study except in the simplest cases. We have evaluated the expressions for the resonant contributions over a variety of scalar field masses, the range of which is restricted by ensuring the correct boundary behaviour for the second-order metric functions.

Acknowledgments

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A Derivation of Source Terms For Massive Scalars

The derivation of the general expression for the $\mathcal{O}(\epsilon^3)$ source term for massive scalars closely follows the massless case, particularly if one chooses not to write out the explicit mass dependence as was done in [6]. However, since we have chosen to write our equations in a slightly different way – and in a different gauge – than previous authors, one may find it instructive to see the differences in the derivations. Below we have included the intermediate steps involved in deriving the third-order source term S_ℓ.

Continuing the expansion of the equations of motion in powers of ϵ, we see that the back-reaction between the metric and the scalar field appears at second order in the perturbation,

$$A_2' = -\mu\nu \left[(\dot{\phi}_1)^2 + (\phi_1')^2 + m^2\phi_1^2 \sec^2 x\right] + \nu' A_2/\nu \,, \tag{A.1}$$

which can be directly integrated to give

$$A_2 = -\nu \int_0^x dy\, \mu \left((\dot{\phi}_1)^2 + (\phi_1')^2 + m^2\phi_1^2 \sec^2 x\right) . \tag{A.2}$$

For convenience, we have also defined the functions

$$\mu(x) = (\tan x)^{d-1} \quad \text{and} \quad \nu(x) = (d-1)/\mu' \,. \tag{A.3}$$

Similarly, the first non-trivial contribution to the lapse (in the boundary time gauge) is

$$\delta_2 = \int_x^{\pi/2} dy\, \mu\nu \left((\dot{\phi}_1)^2 + (\phi_1')^2 \right) . \tag{A.4}$$

To aide in evaluating integrals and inner products, it is useful to derive several identities. First, from the equation for the scalar field's time-dependent coefficients c_i,

$$\ddot{c}_i + \omega_i^2 c_i = 0 \quad \Rightarrow \quad \partial_t \left(\dot{c}_i^2 + \omega_i^2 c_i^2 \right) = \partial_t \mathbb{C}_i = 0 \, . \tag{A.5}$$

Next, from the definition of $\hat{L}$,

$$\hat{L} e_j = -\frac{1}{\mu} \left(\mu e_j' \right)' + m^2 \sec^2 x e_j \quad \Rightarrow \quad \left(\mu e_j' \right)' = \mu \left(m^2 \sec^2 x - \omega_j^2 \right) e_j \, . \tag{A.6}$$

By considering the expression $(\mu e_i' e_j)'$, we see that

$$\left(\mu e_i' e_j \right)' = \left(m^2 \sec^2 x - \omega_i^2 \right) \mu e_i e_j + \mu e_i' e_j' \, , \tag{A.7}$$

which, after permuting i, j and subtracting from above, gives

$$\frac{\left[\mu (e_i' e_j \omega_j^2 - e_i e_j' \omega_i^2) \right]'}{(\omega_j^2 - \omega_i^2)} = \mu m^2 \sec^2 x e_i e_j + \mu e_i' e_j' \, . \tag{A.8}$$

Projecting each of the terms in (2.13) individually onto the eigenbasis $\{e_\ell\}$ will involve evaluating inner products involving multiple integrals. To aide in evaluating these expressions, it is useful to derive several identities. First, from the equation for the scalar field's time-dependent coefficients c_i,

$$\ddot{c}_i + \omega_i^2 c_i = 0 \quad \Rightarrow \quad \partial_t \left(\dot{c}_i^2 + \omega_i^2 c_i^2 \right) = \partial_t \mathbb{C}_i = 0 \, . \tag{A.9}$$

Next, from the definition of $\hat{L}$,

$$\hat{L} e_j = -\frac{1}{\mu} \left(\mu e_j' \right)' + m^2 \sec^2 x e_j \quad \Rightarrow \quad \left(\mu e_j' \right)' = \mu \left(m^2 \sec^2 x - \omega_j^2 \right) e_j \, . \tag{A.10}$$

By considering the expression $(\mu e_i' e_j)'$, we see that

$$\left(\mu e_i' e_j \right)' = \left(m^2 \sec^2 x - \omega_i^2 \right) \mu e_i e_j + \mu e_i' e_j' \, , \tag{A.11}$$

which, after permuting i, j and subtracting from above, gives

$$\frac{\left[\mu (e_i' e_j \omega_j^2 - e_i e_j' \omega_i^2) \right]'}{(\omega_j^2 - \omega_i^2)} = \mu m^2 \sec^2 x e_i e_j + \mu e_i' e_j' \, . \tag{A.12}$$

Using these identities, we evaluate each of the inner products and find that

$$\langle \delta_2 \ddot{\phi}_1, e_\ell \rangle = -\sum_{i=0}^{\infty} \sum_{\substack{j=0 \\ k\neq \ell}}^{\infty} \sum_{k=0}^{\infty} \frac{\omega_k^2 c_k}{\omega_\ell^2 - \omega_k^2} \left[\dot{c}_i \dot{c}_j \left(X_{k\ell ij} - X_{\ell kij} \right) + c_i c_j \left(Y_{ij\ell k} - Y_{ijk\ell} \right) \right]$$

$$- \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \omega_\ell^2 c_\ell \left[\dot{c}_i \dot{c}_j P_{ij\ell} + c_i c_j B_{ij\ell} \right] , \tag{A.13}$$

$$\langle A_2 \ddot{\phi}_1, e_\ell \rangle = 2 \sum_{i=0}^{\infty} \sum_{\substack{j=0 \\ i\neq j}}^{\infty} \sum_{k=0}^{\infty} \frac{\omega_k^2 c_k}{\omega_j^2 - \omega_i^2} X_{ijk\ell} \left(\dot{c}_i \dot{c}_j + \omega_j^2 c_i c_j \right)$$

$$+ \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \omega_j^2 c_j \left(\mathbb{C}_i P_{j\ell i} + c_i^2 X_{iij\ell} \right) , \tag{A.14}$$

$$\langle \dot{\delta}_2 \dot{\phi}_1, e_\ell \rangle = \sum_{i=0}^{\infty} \sum_{\substack{j=0 \\ k\neq \ell}}^{\infty} \sum_{k=0}^{\infty} \frac{\dot{c}_k}{\omega_\ell^2 - \omega_k^2} \left[\partial_t \left(\dot{c}_i \dot{c}_j \right) \left(X_{k\ell ij} - X_{\ell kij} \right) + \partial_t (c_i c_j) \left(Y_{ij\ell k} - Y_{ijk\ell} \right) \right]$$

$$+ \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \dot{c}_\ell \left[\partial_t \left(\dot{c}_i \dot{c}_j \right) P_{ij\ell} + \partial_t (c_i c_j) B_{ij\ell} \right] , \tag{A.15}$$

$$\langle \dot{A}_2 \dot{\phi}_1, e_\ell \rangle = -2 \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \dot{c}_k \dot{c}_j c_i X_{ijk\ell} , \tag{A.16}$$

$$\langle \left(A_2' - \delta_2' \right) \phi_1', e_\ell \rangle = -2 \sum_{i=0}^{\infty} \sum_{\substack{j=0 \\ i\neq j}}^{\infty} \sum_{k=0}^{\infty} \frac{c_k (\dot{c}_i \dot{c}_j + \omega_j^2 c_i c_j)}{\omega_j^2 - \omega_i^2} H_{ijk\ell} - m^2 \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} c_i c_j c_k V_{ijk\ell}$$

$$- \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} c_j \left[c_i^2 H_{iij\ell} + \mathbb{C}_i M_{j\ell i} \right] , \tag{A.17}$$

$$\langle A_2 \phi_1 \sec^2 x, e_\ell \rangle = -2 \sum_{i=0}^{\infty} \sum_{\substack{j=0 \\ i\neq j}}^{\infty} \sum_{k=0}^{\infty} \frac{c_k (\dot{c}_i \dot{c}_j + \omega_j^2 c_i c_j)}{\omega_j^2 - \omega_i^2} V_{jki\ell}$$

$$- \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} c_j \left(c_i^2 V_{jii\ell} + \mathbb{C}_i Q_{j\ell i} \right) , \tag{A.18}$$

where the forms of X, Y, V, H, B, M, P, and Q are given by

$$X_{ijk\ell} = \int_0^{\pi/2} dx \, \mu^2 \nu e_i' e_j e_k e_\ell \tag{A.19}$$

$$Y_{ijk\ell} = \int_0^{\pi/2} dx \, \mu^2 \nu e_i' e_j' e_k e_\ell' \tag{A.20}$$

$$V_{ijk\ell} = \int_0^{\pi/2} dx \, \mu^2 \nu e_i e_j e_k' e_\ell \sec^2 x \tag{A.21}$$

$$H_{ijk\ell} = \int_0^{\pi/2} dx\, \mu^2 \nu' e_i' e_j e_k' e_\ell \tag{A.22}$$

$$B_{ij\ell} = \int_0^{\pi/2} dx\, \mu\nu e_i' e_j' \int_0^x dy\, \mu e_\ell^2 \tag{A.23}$$

$$M_{ij\ell} = \int_0^{\pi/2} dx\, \mu\nu' e_i' e_j \int_o^x dy\, \mu e_\ell^2 \tag{A.24}$$

$$P_{ij\ell} = \int_0^{\pi/2} dx\, \mu\nu e_i e_j \int_0^x dy\, \mu e_\ell^2 \tag{A.25}$$

$$Q_{ij\ell} = \int_0^{\pi/2} dx\, \mu\nu e_i e_j \sec^2 x \int_0^x dy\, \mu e_\ell^2 \,. \tag{A.26}$$

Note that, using integration by parts to remove the derivative from ν in the definitions of $H_{ijk\ell}$ and $M_{ij\ell}$, we can show that

$$H_{ijk\ell} = \omega_i^2 X_{kij\ell} + \omega_k^2 X_{ijk\ell} - Y_{ij\ell k} - Y_{\ell kji} - m^2 V_{kji\ell} - m^2 V_{ijk\ell} \,, \tag{A.27}$$

$$M_{ij\ell} = \omega_i^2 P_{ij\ell} - B_{ij\ell} - m^2 Q_{ij\ell} \,. \tag{A.28}$$

Collecting (A.13) - (A.17) gives the expression for $S_\ell = \langle S, e_\ell \rangle$:

$$\begin{aligned} S_\ell = &\sum_{\substack{i,j,k \\ k\neq\ell}}^{\infty} \frac{1}{\omega_\ell^2 - \omega_k^2} \Big[F_k(\dot{c}_i \dot{c}_j)\,(X_{k\ell ij} - X_{\ell kij}) + F_k(c_i c_j)\,(Y_{ij\ell k} - Y_{ijk\ell}) \Big] \\ &+ 2 \sum_{\substack{i,j,k \\ i\neq j}}^{\infty} \frac{c_k D_{ij}}{\omega_j^2 - \omega_i^2} \Big[2\omega_k^2 X_{ijk\ell} - H_{ijk\ell} - m^2 V_{jki\ell} \Big] - \sum_{i,j,k}^{\infty} c_i \Big[2\dot{c}_j \dot{c}_k X_{ijk\ell} + m^2 c_j c_k V_{ijk\ell} \Big] \\ &+ \sum_{i,j}^{\infty} \Big[F_\ell(\dot{c}_i \dot{c}_j) P_{ij\ell} + F_\ell(c_i c_j) B_{ij\ell} + 2\omega_j^2 c_j \left(c_i^2 X_{iij\ell} + \mathbb{C}_i P_{j\ell i} \right) \\ &\quad - c_j \left(c_i^2 (H_{iij\ell} + m^2 V_{jii\ell}) + \mathbb{C}_i (M_{j\ell i} + m^2 Q_{j\ell i}) \right) \Big] \,, \end{aligned} \tag{A.29}$$

where $F_k(z) = \dot{c}_k \dot{z} - 2\omega_k^2 c_k z$, $D_{ij} = \dot{c}_i \dot{c}_j + \omega_j^2 c_i c_j$, and $\mathbb{C}_i = \dot{c}_i^2 + \omega_i^2 c_i^2$. Additionally, we have combined some integrals into their own expressions, namely

$$Z^{\pm}_{ijk\ell} = \omega_i \omega_j \,(X_{k\ell ij} - X_{\ell kij}) \pm (Y_{ij\ell k} - Y_{ijk\ell}) \quad \text{and} \quad \tilde{Z}^{\pm}_{ij\ell} = \omega_i \omega_j P_{ij\ell} \pm B_{ij\ell} \,. \tag{A.30}$$

Finally, using the solution for the time-dependent coefficients, $c_i(t) = a_i(t) \cos(\omega_i t + b_i(t)) \equiv a_i \cos\theta_i$, we arrive at (2.14).

B Two Non-normalizable Modes with Equal Frequencies

Let us return to the case of two, equal, non-normalizable modes with frequency ϖ. Within the space of resonant frequency values, there are frequencies that happen to satisfy $\varpi = \omega_\ell$ numerically and may produce extra resonances subject to restrictions on the normalizable frequency. These instances were excluded from the discussion in § 4.1, and we address them here. When considering special integer values of $\overline{\omega}$ each choice of $\overline{\omega}$ below will contribute a $\overline{T}$-type term to the total source:

$$\overline{T}_i^{(1)}: \quad \omega_i = \omega_\ell + 2\overline{\omega} \quad \forall\, \overline{\omega} \in \mathbb{Z}^* \tag{B.1}$$

$$\overline{T}_i^{(2)}: \quad \omega_i = \omega_\ell - 2\overline{\omega} \quad \forall\, \overline{\omega} \in \mathbb{Z}^* \text{ such that } \ell \geq \overline{\omega} \tag{B.2}$$

$$\overline{T}_i^{(3)}: \quad \omega_i = 2\overline{\omega} - \omega_\ell \quad \forall\, \overline{\omega} \in \mathbb{Z}^* \text{ such that } \overline{\omega} \leq \ell + \Delta^+\,, \tag{B.3}$$

with $\omega_i \neq \omega_\ell$ in each case. These special values contribute to the case of two, equal non-normalizable modes via

$$\begin{aligned} S_\ell = {} & \bar{A}_{\overline{\omega}}^2\, \overline{T}_{(\ell+\overline{\omega})}^{(1)}\, a_{(\ell+\overline{\omega})} \cos\left(\theta_{(\ell+\overline{\omega})} - 2\overline{\omega}t\right) + \bar{A}_{\overline{\omega}}^2\, \overline{T}_{(\ell-\overline{\omega})}^{(2)}\, a_{(\ell-\overline{\omega})} \cos\left(\theta_{(\ell-\overline{\omega})} + 2\overline{\omega}t\right) \\ & + \bar{A}_{\overline{\omega}}^2\, \overline{T}_{(\overline{\omega}-\ell-\Delta^+)}^{(3)}\, a_{(\overline{\omega}-\ell-\Delta^+)} \cos\left(2\overline{\omega}t - \theta_{(\overline{\omega}-\ell-\Delta^+)}\right) \end{aligned} \tag{B.4}$$

under their respective conditions on the value of $\overline{\omega}$. The total resonant contribution for all possible $\overline{\omega}$ values is the addition of (B.4) and (4.6). Evaluating (2.14) in each case of the cases described by (B.1) - (B.3), we find that

$$\begin{aligned} \overline{T}_i^{(1)} = {} & \frac{1}{2}\Bigg[H_{i\overline{\omega}\overline{\omega}\ell}\left(\frac{\overline{\omega}}{\omega_i - \overline{\omega}}\right) - H_{\overline{\omega}i\overline{\omega}\ell}\left(\frac{\omega_i}{\omega_i - \overline{\omega}}\right) + m^2 V_{\overline{\omega}\overline{\omega}i\ell}\left(\frac{\overline{\omega}}{\omega_i - \overline{\omega}}\right) \\ & - m^2 V_{i\overline{\omega}\overline{\omega}\ell}\left(\frac{\omega_i}{\omega_i - \overline{\omega}}\right) - 2\overline{\omega}^2 X_{i\overline{\omega}\overline{\omega}\ell}\left(\frac{\overline{\omega}}{\omega_i - \overline{\omega}}\right) + 2\overline{\omega}^2 X_{\overline{\omega}i\overline{\omega}\ell}\left(\frac{\omega_i}{\omega_i - \overline{\omega}}\right)\Bigg]_{\omega_i \neq \overline{\omega}} \\ & - \frac{1}{2}\left[Z^+_{i\overline{\omega}\overline{\omega}\ell}\left(\frac{\overline{\omega}}{\omega_\ell + \overline{\omega}}\right)\right]_{\omega_\ell \neq \overline{\omega}} + \frac{1}{4} Z^-_{\overline{\omega}\overline{\omega}i\ell}\left(\frac{\omega_\ell + 2\overline{\omega}}{2\overline{\omega}}\right) + \frac{1}{2}\overline{\omega}^2 X_{i\overline{\omega}\overline{\omega}\ell} - \frac{m^2}{4} V_{\overline{\omega}\overline{\omega}i\ell} \\ & - \overline{\omega}\omega_i X_{\overline{\omega}\overline{\omega}i\ell} - \frac{m^2}{2} V_{i\overline{\omega}\overline{\omega}\ell}\,, \end{aligned} \tag{B.5}$$

$$\begin{aligned} \overline{T}_i^{(2)} = {} & -\frac{1}{2}\Bigg[H_{i\overline{\omega}\overline{\omega}\ell}\left(\frac{\overline{\omega}}{\omega_i + \overline{\omega}}\right) + H_{\overline{\omega}i\overline{\omega}\ell}\left(\frac{\omega_i}{\omega_i + \overline{\omega}}\right) + m^2 V_{\overline{\omega}\overline{\omega}i\ell}\left(\frac{\overline{\omega}}{\omega_i + \overline{\omega}}\right) \\ & + m^2 V_{i\overline{\omega}\overline{\omega}\ell}\left(\frac{\omega_i}{\omega_i + \overline{\omega}}\right) - 2\overline{\omega}^2 X_{i\overline{\omega}\overline{\omega}\ell}\left(\frac{\overline{\omega}}{\omega_i + \overline{\omega}}\right) - 2\overline{\omega}^2 X_{\overline{\omega}\overline{\omega}i\ell}\left(\frac{\omega_i}{\omega_i + \overline{\omega}}\right)\Bigg]_{\omega_i \neq \overline{\omega}} \\ & - \frac{1}{2}\left[Z^-_{i\overline{\omega}\overline{\omega}\ell}\left(\frac{\overline{\omega}}{\omega_\ell - \overline{\omega}}\right)\right]_{\omega_\ell \neq \overline{\omega}} - \frac{1}{4} Z^-_{\overline{\omega}\overline{\omega}i\ell}\left(\frac{\omega_\ell - 2\overline{\omega}}{\overline{\omega}}\right) + \frac{1}{2}\overline{\omega}^2 X_{i\overline{\omega}\overline{\omega}\ell} + \frac{m^2}{4} V_{\overline{\omega}\overline{\omega}i\ell} \\ & + \overline{\omega}\omega_i X_{\overline{\omega}\overline{\omega}i\ell} + \frac{m^2}{2} V_{i\overline{\omega}\overline{\omega}\ell}\,, \end{aligned} \tag{B.6}$$

and

$$\begin{aligned}\overline{T}_i^{(3)} &= \frac{1}{2}\Bigg[H_{i\overline{\omega}\overline{\omega}\ell}\left(\frac{\overline{\omega}}{\omega_i - \overline{\omega}}\right) - H_{\overline{\omega}i\overline{\omega}\ell}\left(\frac{\omega_i}{\omega_i - \overline{\omega}}\right) + m^2 V_{\overline{\omega}\overline{\omega}i\ell}\left(\frac{\overline{\omega}}{\omega_i - \overline{\omega}}\right) \\ &\quad - m^2 V_{i\overline{\omega}\overline{\omega}\ell}\left(\frac{\omega_i}{\omega_i - \overline{\omega}}\right) - 2\overline{\omega}^2 X_{i\overline{\omega}\overline{\omega}\ell}\left(\frac{\overline{\omega}}{\omega_i - \overline{\omega}}\right) + 2\omega_i^2 X_{\overline{\omega}\overline{\omega}i\ell}\left(\frac{\overline{\omega}}{\omega_i - \overline{\omega}}\right) \\ &\quad - Z^{+}_{i\overline{\omega}\overline{\omega}\ell}\left(\frac{\overline{\omega}}{\omega_i - \overline{\omega}}\right)\Bigg]_{\omega_i \neq \overline{\omega}} + \frac{1}{4} Z^{-}_{\overline{\omega}\overline{\omega}i\ell}\left(\frac{2\overline{\omega} - \omega_\ell}{2\overline{\omega}}\right) + \frac{1}{2}\overline{\omega}^2 X_{i\overline{\omega}\overline{\omega}\ell} - \frac{m^2}{4} V_{\overline{\omega}\overline{\omega}i\ell} \\ &\quad - \overline{\omega}\omega_i X_{\overline{\omega}\overline{\omega}i\ell} - \frac{m^2}{2} V_{i\overline{\omega}\overline{\omega}\ell}\,. \end{aligned} \tag{B.7}$$

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