
Response to Referee Report JHEP_252P_0420_EDREP032460520

I would like to thank the referee for their detailed and constructive report, and I hope the changes I have outlined here have addressed their concerns. Because the report covered several topics, I will address each of these through the numbering given in the report.

1. (a) The issue of the existence of both types of modes being present at all times has been expanded upon at the end of section 2 following equation (2.19).
- (b) If I have misunderstood the referee's comment here, I apologize and would be happy to reconsider their comment. However, I believe the referee may be mistaken on this point. Let us allow for integer frequency values for the non-normalizable solutions, and examine the general solution for a massless scalar in d dimensions (as per the example put forward in the report). The spatial part of the scalar field is given by

$$E_I(x) = K_I (\cos x)^{\Delta^+} {}_2F_1\left(\frac{\Delta^+ + \omega_I}{2}, \frac{\Delta^+ - \omega_I}{2}; \frac{d}{2}; \sin^2 x\right) \quad (1)$$

where K_I is a constant, and $\Delta^+ = d$ for a massless field. Now, we consider values of $\omega_I = 2i + d$ and examine E_I at the origin:

$$E_I(0) = K_I {}_2F_1\left(i + d, -i; \frac{d}{2}; 0\right) \quad (2)$$

According to the [DLMF](#), the hypergeometric function ${}_2F_1(a, b; c; z)$ exists on the disk $|z| < 1$ provided that $c \neq 0, -1, -2, \dots$ which the number of dimensions, d , always satisfies. Furthermore, we can write

$${}_2F_1(a, b; c; z) = \sum_{s=0}^{\infty} \frac{(a)_s (b)_s}{\Gamma(c+s) s!} z^s \quad (3)$$

for all values of c . This clearly demonstrates that (1) does not diverge at the origin and therefore constitutes a valid field decomposition.

- (c) I have expanded on the issue of including either one or three non-normalizable modes at the end of section 2 and explained that cases where the number of dimensions, the mass, or the driving frequency are tuned to specific values may indeed present resonance channels that are not discussed in the paper. However, the procedure for determining the source terms in the event of such additional resonances is not changed by their presence. The particular case raised by the referee can be addressed using additional information from [31]. As addressed under equation (2.6), we can write the general solution for a scalar field with generic frequency ω as the linear combination of a purely normalizable part Φ^+ with a purely non-normalizable part Φ^- as $\phi_1 = C^+ \Phi^+ + C^- \Phi^-$ where the

coefficients are

$$C^+ = \frac{\Gamma(d/2)\Gamma(-\sqrt{d^2 + 4m^2}/2)}{\Gamma((\Delta^- + \omega)/2)\Gamma((\Delta^- - \omega)/2)} \quad (4)$$

$$C^- = \frac{\Gamma(d/2)\Gamma(\sqrt{d^2 + 4m^2}/2)}{\Gamma((\Delta^+ + \omega)/2)\Gamma((\Delta^+ - \omega)/2)} \quad (5)$$

and the functions Φ^\pm are

$$\Phi^+ = (\cos(x))^{\Delta^+} {}_2F_1\left(\frac{\Delta^+ + \omega}{2}, \frac{\Delta^+ - \omega}{2}, \Delta^+ + 1 - \frac{d}{2}; \cos^2 x\right) \quad (6)$$

$$\Phi^- = (\cos(x))^{\Delta^-} {}_2F_1\left(\frac{\Delta^- + \omega}{2}, \frac{\Delta^- - \omega}{2}, \Delta^- + 1 - \frac{d}{2}; \cos^2 x\right) \quad (7)$$

When the frequency ω is set to an integer that satisfies $\omega = \omega_i = 2i + \Delta^+$, (5) is zero and so $\phi_1 = C^+\Phi^+$, i.e. the field is *entirely normalizable* for $\omega \geq \Delta^+$. Interestingly, for the particular choice of $m^2 = 0$, $d = 4$, and $\omega = 2$ as suggested, it is C^+ that vanishes so that $\phi_1 = C^-\Phi^-$ and there is indeed a non-normalizable mode. However, this requires non-generic values of the parameters and so is not explored in detail.

Finally, I have addressed the case of three non-normalizable modes as an additional example of resonances that require special choices of either mass, number of dimensions, or driving frequency. Such cases are not addressed in this work, as they are not the most broadly applicable. I have also included a caveat below equation (2.19) to caution that not all possible resonances are addressed in this work. However, the procedure for deriving the contributions from special resonances is the same as the procedure for the cases that are included, and so I believe that including such resonances would not produce distinctly new results.

- (d) I agree that (4.11) is satisfied for the choice of $d = 3$ and $m^2 = -2$. Based on my response above, I have not included this special case; however, I have added a footnote above equation (4.11) that addresses this case.
- 2. (a) The final paragraph of section 4 was added to an expanded discussion of the holographic mappings between different components of the bulk field to the energy of the boundary CFT in section 2. Additional references to preliminary material on the topic of bulk/boundary mappings that underpin the theoretical motivation for studying non-normalizable modes were also added.
- (b)
- (c)
- (d) The details of deriving the flow equations for the amplitudes and phases have been presented extensively in the literature from the point of view of both the Two-Time Formalism [14] and renormalization group method [17]. Since the new physics is encapsulated in the specific nonlinearities of S_ℓ , I believe that including additional details in how one arrives at (2.17)-(2.18) is best left to the works that are cited.

The definition of $c_\ell^{(3)}(t)$ follows from the general expression for $c_I^{(3)}(t)$ given below (2.14). I have added a clarifying sentence below (2.16) that links the amplitudes and phases at this order to their first-order counterparts. Finally, I have amended the wording about (2.16) to reflect the fact that secular terms are absorbed into the definition of the first-order amplitudes and phases.

- (e) The material concerning resonances from only normalizable modes has been moved to an appendix.
- (f) To make the choice of boundary condition more clear, I have included an explicit form of $\mathcal{F}(t)$ for each of the cases explored in this section. I have also reiterated the choice of considering a single non-normalizable mode in the general expression for resonances (2.19).

The discussion of static boundary conditions ($\mathcal{F}(t) = 0$) for a massive scalar has been moved to an appendix. I believe this will clarify the restriction of equations (3.5), (3.7), and (3.9-3.11) to the case of all normalizable modes. I have added an additional comment at the beginning of the appendix to remind the reader of the case being considered.

- (g)
- (h)
- 3. (a) References added to other literature that examine TTF in $d = 4$ dimensions.
- (b) References have been changed to reflect this correction.
- (c) References have been added to the discussion resonances (both vanishing and non-vanishing) for massive scalars with static boundary conditions.
- (d)
- 4. (a)
- (b)
- (c)
- (d)
- (e)
- 5. (a) Please see the response to 1(b) where I have addressed the issue of the regularity of the basis functions $\Phi_i^\pm(x)$.
- (b) Since the basis functions for the normalizable modes $e_j(x)$ vanish at the boundary $x \rightarrow \pi/2 \forall j$, the first term in equation (2.11) is exactly zero while the summation in the second term collapses to a single expression. I believe the wording here accurately describes the simplification of (2.11).
- (c) This sentence describes the existence of *naturally vanishing* resonances, such as those given in equations (3.5) and (3.7). These terms evaluate to zero for several choices of the scalar field mass $m \neq 0$. When $m = 0$, [17] has shown that Ω and Γ vanish for all allowed frequencies by orthogonality of the basis functions. Since chapter 4 demonstrates that there are no resonances that naturally vanish in this fashion, I believe this sentence is correct as it stands.

-
- (d) I have rewritten equation (4.2) as an expansion in \tilde{x} to make the expression consistent.
 - (e) I have reduced the three expressions to a single expression that allows for relabeling of the normalizable index and removed the repeated index on the frequencies.
 - (f) I have corrected T_ℓ to read \bar{T}_ℓ in the paragraph below (4.2).
 - (g) I have corrected “source equations” to “source term” and provided an equation reference.
 - (h) Yes, (4.9) and (4.10) are indeed applicable for $m^2 = 0$. I have changed the language around (4.9) and (4.10) to make this clearer.
 - (i)

Once again, I would like to thank the referee for their diligent reading of this draft. I hope I have satisfactorily addressed their concerns.