Prepared for submission to JHEP

Arbitrary Dimensions, Massive, Non-normalizable Time-Dependent BCs

Contents

1 Introduction

2 Perturbative Expansion

The backreaction between the metric and the scalar field appears at second order in the perturbation,

$$A_2' = -\mu\nu \left[(\dot{\phi}_1)^2 + (\phi_1')^2 + m^2\phi_1^2 \sec^2 x\right] + \nu' A_2/\nu \tag{2.1}$$

which can be directly integrated to give

$$A_2 = -\nu \int_0^x dy\, \mu \left((\dot{\phi}_1)^2 + (\phi_1')^2 + m^2\phi_1^2 \sec^2 x\right) . \tag{2.2}$$

Furthermore, the first non-trivial contribution to the lapse in the boundary time gauge is

$$\delta_2 = \int_x^{\pi/2} dy\, \mu\nu \left((\dot{\phi}_1)^2 + (\phi_1')^2\right) . \tag{2.3}$$

For convenience, we have also defined the functions

$$\mu(x) = (\tan x)^{d-1} \quad \text{and} \quad \nu(x) = (d-1)/\mu' . \tag{2.4}$$

To aide in evaluating integrals, we first derive the following identities: from the equation for the first-order time-dependent coefficients c_i,

$$\ddot{c}_i + \omega_i^2 c_i = 0 \quad \Rightarrow \quad \partial_t \left(\dot{c}_i^2 + \omega_i^2 c_i^2\right) = \partial_t \mathbb{C}_i = 0\,; \tag{2.5}$$

from the equation definition of $\hat{L}$,

$$\hat{L}e_j = -\frac{1}{\mu}\left(\mu e_j'\right)' + m^2 \sec^2 x e_j \quad \Rightarrow \quad \left(\mu e_j'\right)' = \mu\left(m^2 \sec^2 x - \omega_j^2\right) e_j\,; \tag{2.6}$$

from considering the expression $(\mu e_i' e_j)'$:

$$\left(\mu e_i' e_j\right)' = \left(m^2 \sec^2 x - \omega_i^2\right)\mu e_i e_j + \mu e_i' e_j'\,; \tag{2.7}$$

from permuting i, j above and subtracting to give

$$\frac{\left[\mu(e_i' e_j \omega_j^2 - e_i e_j' \omega_i^2)\right]'}{(\omega_j^2 - \omega_i^2)} = \mu m^2 \sec^2 x e_i e_j + \mu e_i' e_j' . \tag{2.8}$$

The basis functions $e_j(x)$ are the solutions to the eigenvalue equation

$$\hat{L}e_j(x) = \omega_j^2 e_j(x), \tag{2.9}$$

which, for massive scalars, are (up to some normalization)

$$e_j(x) = (\cos(x))^{\Delta_+}\, {}_2F_1\left(\frac{\Delta_+ + \omega}{2}, \frac{\Delta_+ - \omega}{2}, d/2; \sin^2(x)\right), \tag{2.10}$$

when ω is arbitrary. However, when the frequency is equal to the resonant frequency $\omega_j = \Delta_+ + 2j$, (2.10) separates into normalizable and non-normalizable solutions

$$\begin{aligned} e_j(x) = {} & C_1 (\cos(x))^{\Delta_+}\, {}_2F_1\left(\frac{\Delta_+ + \omega)}{2}, \frac{\Delta_+ - \omega}{2}, \Delta_+ - d/2 + 1; \cos^2(x)\right) \\ & + C_2 (\cos(x))^{\Delta_-}\, {}_2F_1\left(\frac{\Delta_- + \omega)}{2}, \frac{\Delta_- - \omega}{2}, \Delta_- - d/2 + 1; \cos^2(x)\right). \end{aligned} \tag{2.11}$$

3 $\mathcal{O}(\epsilon^3)$ Source Terms

At third order in ϵ, the equation for ϕ_3 contains a source S given by

$$\ddot{\phi}_3 + \hat{L}\phi_3 = S = 2(A_2 - \delta_2)\ddot{\phi}_1 + (\dot{A}_2 - \dot{\delta}_2)\dot{\phi}_1 + (A_2' - \delta_2')\phi_1' + m^2 A_2 \phi_1 \sec^2 x \tag{3.1}$$

Projecting each of the terms individually onto the eigenbasis $\{e_\ell\}$:

$$\begin{aligned}
\langle \delta_2 \ddot{\phi}_1, e_\ell \rangle = &-\sum_{i=0}^{\infty}\sum_{\substack{j=0\\ k\neq\ell}}^{\infty}\sum_{k=0}^{\infty} \frac{\omega_k^2 c_k}{\omega_\ell^2-\omega_k^2}\left[\dot{c}_i\dot{c}_j\left(X_{k\ell ij}-X_{\ell kij}\right)+c_ic_j\left(Y_{ij\ell k}-Y_{ijk\ell}\right)\right]\\
&-\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\omega_\ell^2 c_\ell\left[\dot{c}_i\dot{c}_jP_{ij\ell}+c_ic_jB_{ij\ell}\right]\,, \qquad (3.2)\\
\langle A_2 \ddot{\phi}_1, e_\ell \rangle = &\;2\sum_{i=0}^{\infty}\sum_{\substack{j=0\\ i\neq j}}^{\infty}\sum_{k=0}^{\infty}\frac{\omega_k^2 c_k}{\omega_j^2-\omega_i^2}X_{ijk\ell}\left(\dot{c}_i\dot{c}_j+\omega_j^2c_ic_j\right)\\
&+\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\omega_j^2c_j\left(\mathbb{C}_iP_{j\ell i}+c_i^2X_{iij\ell}\right)\,, \qquad (3.3)\\
\langle \dot{\delta}_2 \dot{\phi}_1, e_\ell \rangle = &\sum_{i=0}^{\infty}\sum_{\substack{j=0\\ k\neq\ell}}^{\infty}\sum_{k=0}^{\infty}\frac{\dot{c}_k}{\omega_\ell^2-\omega_k^2}\left[\partial_t\left(\dot{c}_i\dot{c}_j\right)\left(X_{k\ell ij}-X_{\ell kij}\right)+\partial_t(c_ic_j)\left(Y_{ij\ell k}-Y_{ijk\ell}\right)\right]\\
&+\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\dot{c}_\ell\left[\partial_t\left(\dot{c}_i\dot{c}_j\right)P_{ij\ell}+\partial_t(c_ic_j)B_{ij\ell}\right]\,, \qquad (3.4)\\
\langle \dot{A}_2 \dot{\phi}_1, e_\ell \rangle = &-2\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\dot{c}_k\dot{c}_jc_iX_{ijk\ell}\,, \qquad (3.5)\\
\langle \left(A_2'-\delta_2'\right)\phi_1', e_\ell \rangle = &-2\sum_{i=0}^{\infty}\sum_{\substack{j=0\\ i\neq j}}^{\infty}\sum_{k=0}^{\infty}\frac{c_k(\dot{c}_i\dot{c}_j+\omega_j^2c_ic_j)}{\omega_j^2-\omega_i^2}H_{ijk\ell}-m^2\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}c_ic_jc_kV_{ijk\ell}\\
&-\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}c_j\left[c_i^2H_{iij\ell}+\mathbb{C}_iM_{j\ell i}\right]\,, \qquad (3.6)\\
\langle A_2\phi_1\sec^2 x, e_\ell \rangle = &-2\sum_{i=0}^{\infty}\sum_{\substack{j=0\\ i\neq j}}^{\infty}\sum_{k=0}^{\infty}\frac{c_k(\dot{c}_i\dot{c}_j+\omega_j^2c_ic_j)}{\omega_j^2-\omega_i^2}V_{jki\ell}\\
&-\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}c_j\left(c_i^2V_{jii\ell}+\mathbb{C}_iQ_{j\ell i}\right). \qquad (3.7)
\end{aligned}$$

Where the forms of X, Y, V, H, B, M, P, and Q are given by

$$X_{ijk\ell} = \int_0^{\pi/2} dx\, \mu^2 \nu e_i' e_j e_k e_\ell \tag{3.8}$$

$$Y_{ijk\ell} = \int_0^{\pi/2} dx\, \mu^2 \nu e_i' e_j' e_k e_\ell' \tag{3.9}$$

$$V_{ijk\ell} = \int_0^{\pi/2} dx\, \mu^2 \nu e_i e_j e_k' e_\ell \sec^2 x \tag{3.10}$$

$$H_{ijk\ell} = \int_0^{\pi/2} dx\, \mu^2 \nu' e_i' e_j e_k' e_\ell \tag{3.11}$$

$$B_{ij\ell} = \int_0^{\pi/2} dx\, \mu \nu e_i' e_j' \int_0^x dy\, \mu e_\ell^2 \tag{3.12}$$

$$M_{ij\ell} = \int_0^{\pi/2} dx\, \mu \nu' e_i' e_j \int_o^x dy\, \mu e_\ell^2 \tag{3.13}$$

$$P_{ij\ell} = \int_0^{\pi/2} dx\, \mu \nu e_i e_j \int_0^x dy\, \mu e_\ell^2 \tag{3.14}$$

$$Q_{ij\ell} = \int_0^{\pi/2} dx\, \mu \nu e_i e_j \sec^2 x \int_0^x dy\, \mu e_\ell^2 \tag{3.15}$$

Collecting terms together gives the expression for $S_\ell = \langle S, e_\ell \rangle$:

$$\begin{aligned}
S_\ell = & \sum_{\substack{i,j,k \\ k\neq\ell}}^{\infty} \frac{1}{\omega_\ell^2 - \omega_k^2} \Big[F_k(\dot{c}_i \dot{c}_j) \left(X_{k\ell ij} - X_{\ell kij} \right) + F_k(c_i c_j) \left(Y_{ij\ell k} - Y_{ijk\ell} \right) \Big] \\
& + 2 \sum_{\substack{i,j,k \\ i\neq j}}^{\infty} \frac{c_k D_{ij}}{\omega_j^2 - \omega_i^2} \Big[2\omega_k^2 X_{ijk\ell} - H_{ijk\ell} - m^2 V_{jki\ell} \Big] - \sum_{i,j,k}^{\infty} c_i \Big[2\dot{c}_j \dot{c}_k X_{ijk\ell} + m^2 c_j c_k V_{ijk\ell} \Big] \\
& + \sum_{i,j}^{\infty} \Big[F_\ell(\dot{c}_i \dot{c}_j) P_{ij\ell} + F_\ell(c_i c_j) B_{ij\ell} + 2\omega_j^2 c_j \left(c_i^2 X_{iij\ell} + \mathbb{C}_i P_{j\ell i} \right) \\
& - c_j \left(c_i^2 (H_{iij\ell} + m^2 V_{jii\ell}) + \mathbb{C}_i (M_{j\ell i} + m^2 Q_{j\ell i}) \right) \Big] ,
\end{aligned} \tag{3.16}$$

where $F_k(z) = \dot{c}_k \dot{z} - 2\omega_k^2 c_k z$, $D_{ij} = \dot{c}_i \dot{c}_j + \omega_j^2 c_i c_j$, and $\mathbb{C}_i = \dot{c}_i^2 + \omega_i^2 c_i^2$.

Using the solution $c_i(t) = a_i \cos(\omega_i t + b_i) = a_i \cos\theta_i$, the source term becomes

$$
\begin{aligned}
S_\ell = &\frac{1}{4}\sum_{\substack{i,j,k\\k\neq\ell}}^{\infty}\frac{a_ia_ja_k\omega_k}{\omega_\ell^2-\omega_k^2}\Big[Z^-_{ijk\ell}(\omega_i+\omega_j-2\omega_k)\cos(\theta_i+\theta_j-\theta_k)-Z^-_{ijk\ell}(\omega_i+\omega_j+2\omega_k)\cos(\theta_i+\theta_j+\theta_k)-\\
&\quad+Z^+_{ijk\ell}(\omega_i-\omega_j+2\omega_k)\cos(\theta_i-\theta_j+\theta_k)-Z^+_{ijk\ell}(\omega_i-\omega_j-2\omega_k)\cos(\theta_i-\theta_j-\theta_k)\Big]\\
&+\frac{1}{2}\sum_{\substack{i,j,k\\i\neq j}}^{\infty}a_ia_ja_k\omega_j\left(H_{ijk\ell}+m^2V_{jki\ell}-2\omega_k^2X_{ijk\ell}\right)\Big[\frac{1}{\omega_i-\omega_j}\left(\cos(\theta_i-\theta_j-\theta_k)+\cos(\theta_i-\theta_j+\theta_k)\right)\\
&\quad-\frac{1}{\omega_i+\omega_j}\left(\cos(\theta_i+\theta_j-\theta_k)+\cos(\theta_i+\theta_j+\theta_k)\right)\Big]\\
&-\frac{1}{4}\sum_{i,j,k}^{\infty}a_ia_ja_k\Big[\left(2\omega_j\omega_kX_{ijk\ell}+m^2V_{ijk\ell}\right)\cos(\theta_i+\theta_j-\theta_k)-\left(2\omega_j\omega_kX_{ijk\ell}-m^2V_{ijk\ell}\right)\cos(\theta_i-\theta_j-\theta_k)\\
&\quad+\left(2\omega_j\omega_kX_{ijk\ell}+m^2V_{ijk\ell}\right)\cos(\theta_i-\theta_j+\theta_k)-\left(2\omega_j\omega_kX_{ijk\ell}-m^2V_{ijk\ell}\right)\cos(\theta_i+\theta_j+\theta_k)\Big]\\
&+\frac{1}{4}\sum_{i,j}^{\infty}a_ia_ja_\ell\omega_\ell\Big[\tilde{Z}^-_{ij\ell}(\omega_i+\omega_j-2\omega_\ell)\cos(\theta_i+\theta_j-\theta_\ell)-\tilde{Z}^-_{ij\ell}(\omega_i+\omega_j+2\omega_\ell)\cos(\theta_i+\theta_j+\theta_\ell)\\
&\quad+\tilde{Z}^+_{ij\ell}(\omega_i-\omega_j+2\omega_\ell)\cos(\theta_i-\theta_j+\theta_\ell)-\tilde{Z}^+_{ij\ell}(\omega_i-\omega_j-2\omega_\ell)\cos(\theta_i-\theta_j-\theta_\ell)\Big]\\
&-\frac{1}{4}\sum_{i,j}^{\infty}a_i^2a_j\left(H_{iij\ell}+m^2V_{jii\ell}-2\omega_j^2X_{iij\ell}\right)\left[\cos(2\theta_i-\theta_j)+\cos(2\theta_i+\theta_j)\right]\\
&-\frac{1}{2}\sum_{i,j}^{\infty}a_i^2a_j\left(H_{iij\ell}+m^2V_{jii\ell}-2\omega_j^2X_{iij\ell}+4\omega_i^2\omega_j^2P_{j\ell i}+2\omega_i^2(M_{j\ell i}+m^2Q_{j\ell i})\right)\cos\theta_j. \qquad (3.17)
\end{aligned}
$$

To simplify the above expression, we have defined

$$
Z^\pm_{ijk\ell}=\omega_i\omega_j\left(X_{k\ell ij}-X_{\ell kij}\right)\pm\left(Y_{ij\ell k}-Y_{ijk\ell}\right)\quad\text{and}\quad\tilde{Z}^\pm_{ij\ell}=\omega_i\omega_jP_{ij\ell}\pm B_{ij\ell}\,. \qquad (3.18)
$$

Using integration by parts to remove the derivative from ν in the definitions of $H_{ijk\ell}$ and $M_{ij\ell}$, we can show that

$$
\begin{aligned}
H_{ijk\ell}&=\omega_i^2X_{kij\ell}+\omega_k^2X_{ijk\ell}-Y_{ij\ell k}-Y_{\ell kji}-m^2V_{kji\ell}-m^2V_{ijk\ell} && (3.19)\\
M_{ij\ell}&=\omega_i^2P_{ij\ell}-B_{ij\ell}-m^2Q_{ij\ell} && (3.20)
\end{aligned}
$$

4 Resonances From Normalizable Solutions

Consider the case where each of the basis functions are given by normalizable solutions. After time-averaging, resonant contributions come from the set of conditions

$$
\omega_i\pm\omega_j\pm\omega_k=\pm\omega_\ell \qquad (4.1)
$$

which separates into three distinct cases

$$\omega_i + \omega_j + \omega_k = \omega_\ell \qquad (+++) \tag{4.2}$$

$$\omega_i - \omega_j - \omega_k = \omega_\ell \qquad (+--) \tag{4.3}$$

$$\omega_i + \omega_j - \omega_k = \omega_\ell \qquad (++-) \tag{4.4}$$

4.1 $(+++)$

These resonant contributions come from the condition $\omega_i + \omega_i + \omega_k = \omega_\ell$, and are of the form

$$S_\ell = \underbrace{\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}}_{\omega_i+\omega_j+\omega_k=\omega_\ell} \Omega_{ijk\ell} a_i a_j a_k \cos(\theta_i + \theta_j + \theta_k) + \ldots , \tag{4.5}$$

where

$$\begin{aligned}
\Omega_{ijk\ell} = &-\frac{1}{12} H_{ijk\ell} \frac{\omega_j(\omega_i + \omega_k + 2\omega_j)}{(\omega_i + \omega_j)(\omega_j + \omega_k)} - \frac{1}{12} H_{ikj\ell} \frac{\omega_k(\omega_i + \omega_j + 2\omega_k)}{(\omega_i + \omega_k)(\omega_j + \omega_k)} - \frac{1}{12} H_{jik\ell} \frac{\omega_i(\omega_j + \omega_k + 2\omega_i)}{(\omega_i + \omega_j)(\omega_i + \omega_k)} \\
&- \frac{m^2}{12} V_{ijk\ell} \left(1 + \frac{\omega_j}{\omega_j + \omega_k} + \frac{\omega_i}{\omega_i + \omega_k}\right) - \frac{m^2}{12} V_{jki\ell} \left(1 + \frac{\omega_j}{\omega_i + \omega_j} + \frac{\omega_k}{\omega_i + \omega_k}\right) \\
&- \frac{m^2}{12} V_{kij\ell} \left(1 + \frac{\omega_i}{\omega_i + \omega_j} + \frac{\omega_k}{\omega_j + \omega_k}\right) + \frac{1}{6} \omega_j \omega_k X_{ijk\ell} \left(1 + \frac{\omega_j}{\omega_i + \omega_k} + \frac{\omega_k}{\omega_i + \omega_j}\right) \\
&+ \frac{1}{6} \omega_i \omega_k X_{jki\ell} \left(1 + \frac{\omega_i}{\omega_j + \omega_k} + \frac{\omega_k}{\omega_i + \omega_j}\right) + \frac{1}{6} \omega_i \omega_j X_{kij\ell} \left(1 + \frac{\omega_i}{\omega_j + \omega_k} + \frac{\omega_j}{\omega_i + \omega_k}\right) \\
&- \frac{1}{12} Z^-_{ijk\ell} \left(\frac{\omega_k}{\omega_i + \omega_j}\right) - \frac{1}{12} Z^-_{ikj\ell} \left(\frac{\omega_j}{\omega_i + \omega_k}\right) - \frac{1}{12} Z^-_{jki\ell} \left(\frac{\omega_i}{\omega_j + \omega_k}\right) .
\end{aligned} \tag{4.6}$$

4.2 $(+--)$

These contributions arise from the condition $\omega_i - \omega_k - \omega_k = \omega_\ell$, are of the form

$$S_\ell = \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \Gamma_{(j+k+\ell)jk\ell} a_j a_k a_{(j+k+\ell)} \cos(\theta_{j+k+\ell} - \theta_j - \theta_k) + \ldots , \tag{4.7}$$

where

$$\begin{aligned}
\Gamma_{ijk\ell} = &\frac{1}{4} H_{ijk\ell} \frac{\omega_j(\omega_k - \omega_i + 2\omega_j)}{(\omega_i - \omega_j)(\omega_j + \omega_k)} + \frac{1}{4} H_{jki\ell} \frac{\omega_k(\omega_j - \omega_i + 2\omega_k)}{(\omega_i - \omega_k)(\omega_j + \omega_k)} + \frac{1}{4} H_{kij\ell} \frac{\omega_i(\omega_j + \omega_k - 2\omega_i)}{(\omega_i - \omega_j)(\omega_i - \omega_k)} \\
&- \frac{1}{2} \omega_j \omega_k X_{ijk\ell} \left(\frac{\omega_k}{\omega_i - \omega_j} + \frac{\omega_j}{\omega_i - \omega_k} - 1\right) + \frac{1}{2} \omega_i \omega_k X_{jki\ell} \left(\frac{\omega_k}{\omega_i - \omega_j} + \frac{\omega_i}{\omega_j + \omega_k} - 1\right) \\
&+ \frac{1}{2} \omega_i \omega_j X_{kij\ell} \left(\frac{\omega_j}{\omega_i - \omega_k} + \frac{\omega_i}{\omega_j + \omega_k} - 1\right) + \frac{m^2}{4} V_{jki\ell} \left(\frac{\omega_j}{\omega_i - \omega_j} + \frac{\omega_k}{\omega_i - \omega_k} - 1\right) \\
&- \frac{m^2}{4} V_{kij\ell} \left(\frac{\omega_i}{\omega_i - \omega_j} + \frac{\omega_k}{\omega_j + \omega_k} + 1\right) - \frac{m^2}{4} V_{ijk\ell} \left(\frac{\omega_i}{\omega_i - \omega_k} + \frac{\omega_j}{\omega_j + \omega_k} + 1\right) \\
&+ \frac{1}{4} Z^-_{kji\ell} \left(\frac{\omega_i}{\omega_j + \omega_k}\right) - \frac{1}{4} Z^+_{ijk\ell} \left(\frac{\omega_k}{\omega_i - \omega_j}\right) - \frac{1}{4} Z^+_{jki\ell} \left(\frac{\omega_j}{\omega_i - \omega_k}\right) .
\end{aligned} \tag{4.8}$$

4.3 Naturally Vanishing Resonances

It has been shown that when $m = 0$, and only normalizable modes are considered, (4.6) and (4.8) vanish by the orthogonality of the basis functions. **Maybe show that mass-dependent terms vanish for normalizable modes?**

4.4 $(++-)$

These contributions arise from the resonant condition $\omega_i + \omega_j = \omega_k + \omega_\ell$, can be written as

$$S_\ell = T_\ell a_\ell^3 \cos(\theta_\ell + \theta_\ell - \theta_\ell) + \sum_{i\neq\ell}^{\infty} R_{i\ell} a_i^2 a_\ell \cos(\theta_i + \theta_\ell - \theta_i)$$
$$+ \sum_{i\neq\ell}^{\infty}\sum_{j\neq\ell}^{\infty} S_{ij(i+j-\ell)\ell} a_i a_j a_{(i+j-\ell)} \cos(\theta_i + \theta_j - \theta_{i+j-\ell}) + \dots \tag{4.9}$$

where each of the coefficients is given by

$$\begin{aligned}
S_{ijk\ell} = &-\frac{1}{4}H_{kij\ell}\frac{\omega_i(\omega_j - \omega_k + 2\omega_i)}{(\omega_i - \omega_k)(\omega_i + \omega_j)} - \frac{1}{4}H_{ijk\ell}\frac{\omega_j(\omega_i - \omega_k + 2\omega_j)}{(\omega_j - \omega_k)(\omega_i + \omega_j)} - \frac{1}{4}H_{jki\ell}\frac{\omega_k(\omega_i + \omega_j - 2\omega_k)}{(\omega_i - \omega_k)(\omega_j - \omega_k)} \\
&- \frac{1}{2}\omega_j\omega_k X_{ijk\ell}\left(\frac{\omega_j}{\omega_i - \omega_k} - \frac{\omega_k}{\omega_i + \omega_j} + 1\right) - \frac{1}{2}\omega_i\omega_k X_{jki\ell}\left(\frac{\omega_i}{\omega_j - \omega_k} - \frac{\omega_k}{\omega_i + \omega_j} + 1\right) \\
&+ \frac{1}{2}\omega_i\omega_j X_{kij\ell}\left(\frac{\omega_i}{\omega_j - \omega_k} + \frac{\omega_j}{\omega_i - \omega_k} + 1\right) - \frac{m^2}{4}V_{ijk\ell}\left(\frac{\omega_i}{\omega_i - \omega_k} + \frac{\omega_j}{\omega_j - \omega_k} + 1\right) \\
&+ \frac{m^2}{4}V_{jki\ell}\left(\frac{\omega_k}{\omega_i - \omega_k} - \frac{\omega_j}{\omega_i + \omega_j} - 1\right) + \frac{m^2}{4}V_{kij\ell}\left(\frac{\omega_k}{\omega_j - \omega_k} - \frac{\omega_i}{\omega_i + \omega_j} - 1\right) \\
&+ \frac{1}{4}Z^-_{ijk\ell}\left(\frac{\omega_k}{\omega_i + \omega_j}\right) + \frac{1}{4}Z^+_{ikj\ell}\left(\frac{\omega_j}{\omega_i - \omega_k}\right) + \frac{1}{4}Z^+_{jki\ell}\left(\frac{\omega_i}{\omega_j - \omega_k}\right),
\end{aligned} \tag{4.10}$$

$$\begin{aligned}
R_{i\ell} = &\left(\frac{\omega_i^2}{\omega_\ell^2 - \omega_i^2}\right)\left(Y_{i\ell\ell i} - Y_{i\ell i\ell} + \omega_\ell^2(X_{i\ell i\ell} - X_{\ell i\ell i})\right) + \left(\frac{\omega_i^2}{\omega_\ell^2 - \omega_i^2}\right)\left(H_{\ell ii\ell} + m^2 V_{ii\ell\ell} - 2\omega_i^2 X_{\ell ii\ell}\right) \\
&- \left(\frac{\omega_\ell^2}{\omega_\ell^2 - \omega_i^2}\right)\left(H_{i\ell i\ell} + m^2 V_{\ell ii\ell} - 2\omega_i^2 X_{i\ell i\ell}\right) - \frac{m^2}{4}(V_{i\ell i\ell} + V_{ii\ell\ell}) + \omega_i^2\omega_\ell^2(P_{ii\ell} - 2P_{\ell\ell i}) \\
&- \omega_i\omega_\ell X_{i\ell i\ell} - \frac{3m^2}{2}V_{\ell ii\ell} - \frac{1}{2}H_{ii\ell\ell} + \omega_\ell^2 B_{ii\ell} - \omega_i^2 M_{\ell\ell i} - m^2\omega_i^2 Q_{\ell\ell i},
\end{aligned} \tag{4.11}$$

$$T_\ell = \frac{1}{2}\omega_\ell^2\left(X_{\ell\ell\ell\ell} + 4B_{\ell\ell\ell} - 2M_{\ell\ell\ell} - 2m^2 Q_{\ell\ell\ell}\right) - \frac{3}{4}\left(H_{\ell\ell\ell\ell} + 3m^2 V_{\ell\ell\ell\ell}\right). \tag{4.12}$$

5 Resonances From Non-normalizable Modes

We now consider the case when at least one of the $e_i(x), e_j(x), e_k(x)$ is a non-normalizable mode. Since the boundary condition has been set to be a single non-normalizable mode, any non-normalizable modes in the source term must exactly cancel; therefore, at least two of the modes must be non-normalizable. This assumption breaks some of the symmetries that contributed to the previous expressions for resonance channels, and so each resonance must be re-examined starting from the source expression (3.17).

5.1 Two General, Equal-frequency Modes

As a first case, let us assume that the two non-normalizable modes have equal, constant frequencies $\overline{\omega}$ that is equal to the driving frequency of the boundary value. We are projecting onto a basis of normalizable modes, and so $\omega_\ell = 2\ell + \Delta^+$. We can now choose two of the remaining modes to be non-normalizable and the final mode to be normalizable.

5.1.1 $(+++)$

Consider the resonance condition $\omega_i + \omega_i + \omega_k = \omega_\ell$. Below we choose ω_i to be the frequency of the normalizable mode, with the other choices following immediately under $\omega_i \to \omega_j$ or $\omega_i \to \omega_k$. These resonant contributions are of the form

$$S_\ell = \sum_{\substack{i=0 \\ \omega_i \neq \overline{\omega} \\ \omega_\ell \geq 2\overline{\omega}}}^{\infty} \overline{\Omega}_{i\overline{\omega}\overline{\omega}\ell}\, a_i a_{\overline{\omega}} a_{\overline{\omega}} \cos\left(\theta_i + \theta_{\overline{\omega}} + \theta_{\overline{\omega}}\right) + \ldots \tag{5.1}$$

where

$$\begin{aligned}\overline{\Omega}_{ijk\ell} = &-\frac{1}{8} Z^-_{ijk\ell}\left(\frac{\omega_k}{\omega_i+\omega_j}\right) - \frac{1}{8} Z^-_{ikj\ell}\left(\frac{\omega_j}{\omega_i+\omega_k}\right) + \frac{1}{2}\omega_j\omega_k X_{ijk\ell} - \frac{m^2}{8}\left(V_{ijk\ell}+V_{ikj\ell}\right) \\ &- \frac{1}{4} H_{ijk\ell}\left(\frac{\omega_j}{\omega_i+\omega_j}\right) - \frac{1}{4} H_{ikj\ell}\left(\frac{\omega_k}{\omega_i+\omega_k}\right) - \frac{1}{2}\omega_j\omega_k X_{ijk\ell}\left(\frac{\omega_k}{\omega_i+\omega_j} + \frac{\omega_j}{\omega_i+\omega_k}\right) \\ &- \frac{m^2}{4} V_{jki\ell}\left(\frac{\omega_j}{\omega_i+\omega_j} + \frac{\omega_k}{\omega_i+\omega_k}\right).\end{aligned} \tag{5.2}$$

5.1.2 $(+--)$

Next, consider the resonance condition $\omega_i - \omega_j - \omega_k = \omega_\ell$. When we choose either ω_j or ω_k to be normalizable, we find, for example, that $\omega_i - \omega_j - \omega_k = \overline{\omega} - \overline{\omega} - \omega_k = \omega_\ell$. However, we have chosen all frequencies to be strictly positive, and therefore there are no possible solutions. Next, we choose ω_i to be the frequency of a normalizable mode, with $\omega_j = \omega_k = \overline{\omega}$ being non-normalizable. The resonant contribution takes the form

$$S_\ell = \sum_{\substack{i=0 \\ \omega_i, \omega_\ell \neq \overline{\omega}}}^{\infty} \overline{\Gamma}^{(i)}_{i\overline{\omega}\overline{\omega}\ell}\, a_i a_{\overline{\omega}} a_{\overline{\omega}} \cos\left(\theta_i - \theta_{\overline{\omega}} - \theta_{\overline{\omega}}\right) + \ldots \tag{5.3}$$

where

$$\begin{aligned}\overline{\Gamma}^{(i)}_{ijk\ell} = &-\frac{1}{8} Z^+_{ijk\ell}\left(\frac{\omega_k}{\omega_i-\omega_j}\right) - \frac{1}{8} Z^+_{ikj\ell}\left(\frac{\omega_j}{\omega_i-\omega_k}\right) + \frac{1}{4} H_{ijk\ell}\left(\frac{\omega_j}{\omega_i-\omega_j}\right) + \frac{1}{4} H_{ikj\ell}\left(\frac{\omega_k}{\omega_i-\omega_k}\right) \\ &+ \frac{m^2}{4} V_{jki\ell}\left(\frac{\omega_j}{\omega_i-\omega_j} + \frac{\omega_k}{\omega_i-\omega_k}\right) - \frac{m^2}{2}\left(V_{ijk\ell}+V_{ikj\ell}\right) \\ &- \frac{1}{2}\omega_j\omega_k X_{ijk\ell}\left(\frac{\omega_k}{\omega_i-\omega_j} + \frac{\omega_j}{\omega_i-\omega_k} - 1\right).\end{aligned} \tag{5.4}$$

5.1.3 $(++-)$

5.2 Special Values of Non-normalizable Frequencies

5.2.1 Differ by an integer

5.2.2 Resonant Values

5.3 Boundary Condition is a Superposition/Fourier Integral of Non-normalizable Modes

Acknowledgments

This research was enabled in part by support provided by WestGrid (www.westgrid.ca) and Compute Canada (www.computecanada.ca).