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Arbitrary Dimensions, Massive, Non-normalizable Time-Dependent BCs

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2 Perturbative Expansion

The backreaction between the metric and the scalar field appears at second order in the perturbation,

$$A_2' = -\mu\nu \left[(\dot{\phi}_1)^2 + (\phi_1')^2 + m^2 \phi_1^2 \sec^2 x \right] + \nu' A_2/\nu \tag{2.1}$$

which can be directly integrated to give

$$A_2 = -\nu \int_0^x dy\, \mu \left((\dot{\phi}_1)^2 + (\phi_1')^2 + m^2 \phi_1^2 \sec^2 x \right) . \tag{2.2}$$

Furthermore, the first non-trivial contribution to the lapse in the boundary time gauge is

$$\delta_2 = \int_x^{\pi/2} dy\, \mu\nu \left((\dot{\phi}_1)^2 + (\phi_1')^2 \right) . \tag{2.3}$$

For convenience, we have also defined the functions

$$\mu(x) = (\tan x)^{d-1} \quad \text{and} \quad \nu(x) = (d-1)/\mu' . \tag{2.4}$$

To aide in evaluating integrals, we first derive the following identities: from the equation for the first-order time-dependent coefficients c_i,

$$\ddot{c}_i + \omega_i^2 c_i = 0 \quad \Rightarrow \quad \partial_t \left(\dot{c}_i^2 + \omega_i^2 c_i^2 \right) = \partial_t \mathbb{C}_i = 0 \, ; \tag{2.5}$$

from the equation definition of $\hat{L}$,

$$\hat{L} e_j = -\frac{1}{\mu} \left(\mu e_j' \right)' + m^2 \sec^2 x e_j \quad \Rightarrow \quad \left(\mu e_j' \right)' = \mu \left(m^2 \sec^2 x - \omega_j^2 \right) e_j \, ; \tag{2.6}$$

from considering the expression $(\mu e_i' e_j)'$:

$$\left(\mu e_i' e_j \right)' = \left(m^2 \sec^2 x - \omega_i^2 \right) \mu e_i e_j + \mu e_i' e_j' \, ; \tag{2.7}$$

from permuting i, j above and subtracting to give

$$\frac{\left[\mu (e_i' e_j \omega_j^2 - e_i e_j' \omega_i^2) \right]'}{(\omega_j^2 - \omega_i^2)} = \mu m^2 \sec^2 x e_i e_j + \mu e_i' e_j' \, . \tag{2.8}$$

The basis functions $e_j(x)$ are the solutions to the eigenvalue equation

$$\hat{L} e_j(x) = \omega_j^2 e_j(x). \tag{2.9}$$

When considering normalizable solutions only, the basis functions become

$$e_j(x) = k_j \left(\cos(x)\right)^{\Delta^+} P_j^{(d/2-1,\Delta^+-d/2)}\left(\cos(2x)\right) \tag{2.10}$$

$$k_j = 2\sqrt{\frac{(j+\Delta^+/2)\Gamma(j+1)\Gamma(j+\Delta^+)}{\Gamma(j+d/2)\Gamma(j+\Delta^+-d/2+1)}}\,, \tag{2.11}$$

with eigenvalues $\omega_j = 2j+\Delta^+$, $j \in \mathbb{Z}^*$, and Δ^+ as the positive root of $\Delta(\Delta-d) = m^2$. On the other hand, for non-normalizable solutions with arbitrary frequency, the basis functions are

$$E_\omega(x) = \left(\cos(x)\right)^{\Delta_+}\,{}_2F_1\left(\frac{\Delta_+ + \omega}{2}, \frac{\Delta_+ - \omega}{2}, d/2; \sin^2(x)\right)\,. \tag{2.12}$$

3 $\mathcal{O}(\epsilon^3)$ Source Terms

At third order in ϵ, the equation for ϕ_3 contains a source S given by

$$\ddot{\phi}_3 + \hat{L}\phi_3 = S = 2(A_2-\delta_2)\ddot{\phi}_1 + (\dot{A}_2-\dot{\delta}_2)\dot{\phi}_1 + (A_2'-\delta_2')\phi_1' + m^2 A_2\phi_1 \sec^2 x \tag{3.1}$$

Projecting each of the terms individually onto the eigenbasis $\{e_\ell\}$:

$$\langle \delta_2 \ddot{\phi}_1, e_\ell \rangle = -\sum_{i=0}^{\infty} \sum_{\substack{j=0 \\ k\neq\ell}}^{\infty} \sum_{k=0}^{\infty} \frac{\omega_k^2 c_k}{\omega_\ell^2 - \omega_k^2} \left[\dot{c}_i \dot{c}_j \left(X_{k\ell ij} - X_{\ell kij} \right) + c_i c_j \left(Y_{ij\ell k} - Y_{ijk\ell} \right) \right]$$

$$- \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \omega_\ell^2 c_\ell \left[\dot{c}_i \dot{c}_j P_{ij\ell} + c_i c_j B_{ij\ell} \right] , \tag{3.2}$$

$$\langle A_2 \ddot{\phi}_1, e_\ell \rangle = 2 \sum_{i=0}^{\infty} \sum_{\substack{j=0 \\ i\neq j}}^{\infty} \sum_{k=0}^{\infty} \frac{\omega_k^2 c_k}{\omega_j^2 - \omega_i^2} X_{ijk\ell} \left(\dot{c}_i \dot{c}_j + \omega_j^2 c_i c_j \right)$$

$$+ \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \omega_j^2 c_j \left(\mathbb{C}_i P_{j\ell i} + c_i^2 X_{iij\ell} \right) , \tag{3.3}$$

$$\langle \dot{\delta}_2 \dot{\phi}_1, e_\ell \rangle = \sum_{i=0}^{\infty} \sum_{\substack{j=0 \\ k\neq\ell}}^{\infty} \sum_{k=0}^{\infty} \frac{\dot{c}_k}{\omega_\ell^2 - \omega_k^2} \left[\partial_t \left(\dot{c}_i \dot{c}_j \right) \left(X_{k\ell ij} - X_{\ell kij} \right) + \partial_t (c_i c_j) \left(Y_{ij\ell k} - Y_{ijk\ell} \right) \right]$$

$$+ \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \dot{c}_\ell \left[\partial_t \left(\dot{c}_i \dot{c}_j \right) P_{ij\ell} + \partial_t (c_i c_j) B_{ij\ell} \right] , \tag{3.4}$$

$$\langle \dot{A}_2 \dot{\phi}_1, e_\ell \rangle = -2 \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \dot{c}_k \dot{c}_j c_i X_{ijk\ell} , \tag{3.5}$$

$$\langle \left(A_2' - \delta_2' \right) \phi_1', e_\ell \rangle = -2 \sum_{i=0}^{\infty} \sum_{\substack{j=0 \\ i\neq j}}^{\infty} \sum_{k=0}^{\infty} \frac{c_k (\dot{c}_i \dot{c}_j + \omega_j^2 c_i c_j)}{\omega_j^2 - \omega_i^2} H_{ijk\ell} - m^2 \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} c_i c_j c_k V_{ijk\ell}$$

$$- \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} c_j \left[c_i^2 H_{iij\ell} + \mathbb{C}_i M_{j\ell i} \right] , \tag{3.6}$$

$$\langle A_2 \phi_1 \sec^2 x, e_\ell \rangle = -2 \sum_{i=0}^{\infty} \sum_{\substack{j=0 \\ i\neq j}}^{\infty} \sum_{k=0}^{\infty} \frac{c_k (\dot{c}_i \dot{c}_j + \omega_j^2 c_i c_j)}{\omega_j^2 - \omega_i^2} V_{jki\ell}$$

$$- \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} c_j \left(c_i^2 V_{jii\ell} + \mathbb{C}_i Q_{j\ell i} \right) . \tag{3.7}$$

Where the forms of X, Y, V, H, B, M, P, and Q are given by

$$X_{ijk\ell} = \int_0^{\pi/2} dx\, \mu^2 \nu e_i' e_j e_k e_\ell \tag{3.8}$$

$$Y_{ijk\ell} = \int_0^{\pi/2} dx\, \mu^2 \nu e_i' e_j' e_k e_\ell' \tag{3.9}$$

$$V_{ijk\ell} = \int_0^{\pi/2} dx\, \mu^2 \nu e_i e_j e_k' e_\ell \sec^2 x \tag{3.10}$$

$$H_{ijk\ell} = \int_0^{\pi/2} dx\, \mu^2 \nu' e_i' e_j e_k' e_\ell \tag{3.11}$$

$$B_{ij\ell} = \int_0^{\pi/2} dx\, \mu\nu e_i' e_j' \int_0^x dy\, \mu e_\ell^2 \tag{3.12}$$

$$M_{ij\ell} = \int_0^{\pi/2} dx\, \mu\nu' e_i' e_j \int_o^x dy\, \mu e_\ell^2 \tag{3.13}$$

$$P_{ij\ell} = \int_0^{\pi/2} dx\, \mu\nu e_i e_j \int_0^x dy\, \mu e_\ell^2 \tag{3.14}$$

$$Q_{ij\ell} = \int_0^{\pi/2} dx\, \mu\nu e_i e_j \sec^2 x \int_0^x dy\, \mu e_\ell^2 \tag{3.15}$$

Collecting terms together gives the expression for $S_\ell = \langle S, e_\ell \rangle$:

$$\begin{aligned} S_\ell = &\sum_{\substack{i,j,k \\ k\neq\ell}}^{\infty} \frac{1}{\omega_\ell^2 - \omega_k^2} \Big[F_k(\dot{c}_i\dot{c}_j)\left(X_{k\ell ij} - X_{\ell kij}\right) + F_k(c_i c_j)\left(Y_{ij\ell k} - Y_{ijk\ell}\right) \Big] \\ &+ 2\sum_{\substack{i,j,k \\ i\neq j}}^{\infty} \frac{c_k D_{ij}}{\omega_j^2 - \omega_i^2} \Big[2\omega_k^2 X_{ijk\ell} - H_{ijk\ell} - m^2 V_{jki\ell} \Big] - \sum_{i,j,k}^{\infty} c_i \Big[2\dot{c}_j \dot{c}_k X_{ijk\ell} + m^2 c_j c_k V_{ijk\ell} \Big] \\ &+ \sum_{i,j}^{\infty} \Big[F_\ell(\dot{c}_i\dot{c}_j) P_{ij\ell} + F_\ell(c_i c_j) B_{ij\ell} + 2\omega_j^2 c_j \left(c_i^2 X_{iij\ell} + \mathbb{C}_i P_{j\ell i} \right) \\ &\quad - c_j \left(c_i^2 (H_{iij\ell} + m^2 V_{jii\ell}) + \mathbb{C}_i (M_{j\ell i} + m^2 Q_{j\ell i}) \right) \Big], \end{aligned} \tag{3.16}$$

where $F_k(z) = \dot{c}_k \dot{z} - 2\omega_k^2 c_k z$, $D_{ij} = \dot{c}_i\dot{c}_j + \omega_j^2 c_i c_j$, and $\mathbb{C}_i = \dot{c}_i^2 + \omega_i^2 c_i^2$.

Using the solution $c_i(t) = a_i \cos(\omega_i t + b_i) = a_i \cos\theta_i$, the source term becomes

$$
\begin{aligned}
S_\ell = &\frac{1}{4}\sum_{\substack{i,j,k\\k\neq\ell}}^{\infty}\frac{a_ia_ja_k\omega_k}{\omega_\ell^2-\omega_k^2}\Big[Z^-_{ijk\ell}(\omega_i+\omega_j-2\omega_k)\cos(\theta_i+\theta_j-\theta_k)-Z^-_{ijk\ell}(\omega_i+\omega_j+2\omega_k)\cos(\theta_i+\theta_j+\theta_k)-\\
&\quad+Z^+_{ijk\ell}(\omega_i-\omega_j+2\omega_k)\cos(\theta_i-\theta_j+\theta_k)-Z^+_{ijk\ell}(\omega_i-\omega_j-2\omega_k)\cos(\theta_i-\theta_j-\theta_k)\Big]\\
&+\frac{1}{2}\sum_{\substack{i,j,k\\i\neq j}}^{\infty}a_ia_ja_k\omega_j\left(H_{ijk\ell}+m^2V_{jki\ell}-2\omega_k^2X_{ijk\ell}\right)\Big[\frac{1}{\omega_i-\omega_j}\left(\cos(\theta_i-\theta_j-\theta_k)+\cos(\theta_i-\theta_j+\theta_k)\right)\\
&\quad-\frac{1}{\omega_i+\omega_j}\left(\cos(\theta_i+\theta_j-\theta_k)+\cos(\theta_i+\theta_j+\theta_k)\right)\Big]\\
&-\frac{1}{4}\sum_{i,j,k}^{\infty}a_ia_ja_k\Big[\left(2\omega_j\omega_kX_{ijk\ell}+m^2V_{ijk\ell}\right)\cos(\theta_i+\theta_j-\theta_k)-\left(2\omega_j\omega_kX_{ijk\ell}-m^2V_{ijk\ell}\right)\cos(\theta_i-\theta_j-\theta_k)\\
&\quad+\left(2\omega_j\omega_kX_{ijk\ell}+m^2V_{ijk\ell}\right)\cos(\theta_i-\theta_j+\theta_k)-\left(2\omega_j\omega_kX_{ijk\ell}-m^2V_{ijk\ell}\right)\cos(\theta_i+\theta_j+\theta_k)\Big]\\
&+\frac{1}{4}\sum_{i,j}^{\infty}a_ia_ja_\ell\omega_\ell\Big[\tilde{Z}^-_{ij\ell}(\omega_i+\omega_j-2\omega_\ell)\cos(\theta_i+\theta_j-\theta_\ell)-\tilde{Z}^-_{ij\ell}(\omega_i+\omega_j+2\omega_\ell)\cos(\theta_i+\theta_j+\theta_\ell)\\
&\quad+\tilde{Z}^+_{ij\ell}(\omega_i-\omega_j+2\omega_\ell)\cos(\theta_i-\theta_j+\theta_\ell)-\tilde{Z}^+_{ij\ell}(\omega_i-\omega_j-2\omega_\ell)\cos(\theta_i-\theta_j-\theta_\ell)\Big]\\
&-\frac{1}{4}\sum_{i,j}^{\infty}a_i^2a_j\left(H_{iij\ell}+m^2V_{jii\ell}-2\omega_j^2X_{iij\ell}\right)\left[\cos(2\theta_i-\theta_j)+\cos(2\theta_i+\theta_j)\right]\\
&-\frac{1}{2}\sum_{i,j}^{\infty}a_i^2a_j\left(H_{iij\ell}+m^2V_{jii\ell}-2\omega_j^2X_{iij\ell}+4\omega_i^2\omega_j^2P_{j\ell i}+2\omega_i^2(M_{j\ell i}+m^2Q_{j\ell i})\right)\cos\theta_j. \quad (3.17)
\end{aligned}
$$

To simplify the above expression, we have defined

$$
Z^{\pm}_{ijk\ell}=\omega_i\omega_j\left(X_{k\ell ij}-X_{\ell kij}\right)\pm\left(Y_{ij\ell k}-Y_{ijk\ell}\right)\quad\text{and}\quad\tilde{Z}^{\pm}_{ij\ell}=\omega_i\omega_jP_{ij\ell}\pm B_{ij\ell}\,. \quad (3.18)
$$

Using integration by parts to remove the derivative from ν in the definitions of $H_{ijk\ell}$ and $M_{ij\ell}$, we can show that

$$
\begin{aligned}
H_{ijk\ell}&=\omega_i^2X_{kij\ell}+\omega_k^2X_{ijk\ell}-Y_{ij\ell k}-Y_{\ell kji}-m^2V_{kji\ell}-m^2V_{ijk\ell} &(3.19)\\
M_{ij\ell}&=\omega_i^2P_{ij\ell}-B_{ij\ell}-m^2Q_{ij\ell} &(3.20)
\end{aligned}
$$

4 Resonances From Normalizable Solutions

Consider the case where each of the basis functions are given by normalizable solutions. After time-averaging, resonant contributions come from the set of conditions

$$
\omega_i\pm\omega_j\pm\omega_k=\pm\omega_\ell \quad (4.1)
$$

which separates into three distinct cases

$$\omega_i + \omega_j + \omega_k = \omega_\ell \qquad (+++) \tag{4.2}$$

$$\omega_i - \omega_j - \omega_k = \omega_\ell \qquad (+--) \tag{4.3}$$

$$\omega_i + \omega_j - \omega_k = \omega_\ell \qquad (++-) \tag{4.4}$$

4.1 $(+++)$

These resonant contributions come from the condition $\omega_i + \omega_j + \omega_k = \omega_\ell$, and are of the form

$$S_\ell = \underbrace{\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}}_{\omega_i+\omega_j+\omega_k=\omega_\ell} \Omega_{ijk\ell}\, a_i a_j a_k \cos\left(\theta_i + \theta_j + \theta_k\right) + \dots\,, \tag{4.5}$$

where

$$\begin{aligned}
\Omega_{ijk\ell} = &-\frac{1}{12}H_{ijk\ell}\frac{\omega_j(\omega_i+\omega_k+2\omega_j)}{(\omega_i+\omega_j)(\omega_j+\omega_k)} - \frac{1}{12}H_{ikj\ell}\frac{\omega_k(\omega_i+\omega_j+2\omega_k)}{(\omega_i+\omega_k)(\omega_j+\omega_k)} - \frac{1}{12}H_{jik\ell}\frac{\omega_i(\omega_j+\omega_k+2\omega_i)}{(\omega_i+\omega_j)(\omega_i+\omega_k)} \\
&- \frac{m^2}{12}V_{ijk\ell}\left(1+\frac{\omega_j}{\omega_j+\omega_k}+\frac{\omega_i}{\omega_i+\omega_k}\right) - \frac{m^2}{12}V_{jki\ell}\left(1+\frac{\omega_j}{\omega_i+\omega_j}+\frac{\omega_k}{\omega_i+\omega_k}\right) \\
&- \frac{m^2}{12}V_{kij\ell}\left(1+\frac{\omega_i}{\omega_i+\omega_j}+\frac{\omega_k}{\omega_j+\omega_k}\right) + \frac{1}{6}\omega_j\omega_k X_{ijk\ell}\left(1+\frac{\omega_j}{\omega_i+\omega_k}+\frac{\omega_k}{\omega_i+\omega_j}\right) \\
&+ \frac{1}{6}\omega_i\omega_k X_{jki\ell}\left(1+\frac{\omega_i}{\omega_j+\omega_k}+\frac{\omega_k}{\omega_i+\omega_j}\right) + \frac{1}{6}\omega_i\omega_j X_{kij\ell}\left(1+\frac{\omega_i}{\omega_j+\omega_k}+\frac{\omega_j}{\omega_i+\omega_k}\right) \\
&- \frac{1}{12}Z^-_{ijk\ell}\left(\frac{\omega_k}{\omega_i+\omega_j}\right) - \frac{1}{12}Z^-_{ikj\ell}\left(\frac{\omega_j}{\omega_i+\omega_k}\right) - \frac{1}{12}Z^-_{jki\ell}\left(\frac{\omega_i}{\omega_j+\omega_k}\right)\,.
\end{aligned} \tag{4.6}$$

4.2 $(+--)$

These contributions arise from the condition $\omega_i - \omega_j - \omega_k = \omega_\ell$, are of the form

$$S_\ell = \sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\Gamma_{(j+k+\ell)jk\ell}\, a_j a_k a_{(j+k+\ell)} \cos\left(\theta_{j+k+\ell} - \theta_j - \theta_k\right) + \dots\,, \tag{4.7}$$

where

$$\begin{aligned}
\Gamma_{ijk\ell} = &\frac{1}{4}H_{ijk\ell}\frac{\omega_j(\omega_k-\omega_i+2\omega_j)}{(\omega_i-\omega_j)(\omega_j+\omega_k)} + \frac{1}{4}H_{jki\ell}\frac{\omega_k(\omega_j-\omega_i+2\omega_k)}{(\omega_i-\omega_k)(\omega_j+\omega_k)} + \frac{1}{4}H_{kij\ell}\frac{\omega_i(\omega_j+\omega_k-2\omega_i)}{(\omega_i-\omega_j)(\omega_i-\omega_k)} \\
&- \frac{1}{2}\omega_j\omega_k X_{ijk\ell}\left(\frac{\omega_k}{\omega_i-\omega_j}+\frac{\omega_j}{\omega_i-\omega_k}-1\right) + \frac{1}{2}\omega_i\omega_k X_{jki\ell}\left(\frac{\omega_k}{\omega_i-\omega_j}+\frac{\omega_i}{\omega_j+\omega_k}-1\right) \\
&+ \frac{1}{2}\omega_i\omega_j X_{kij\ell}\left(\frac{\omega_j}{\omega_i-\omega_k}+\frac{\omega_i}{\omega_j+\omega_k}-1\right) + \frac{m^2}{4}V_{jki\ell}\left(\frac{\omega_j}{\omega_i-\omega_j}+\frac{\omega_k}{\omega_i-\omega_k}-1\right) \\
&- \frac{m^2}{4}V_{kij\ell}\left(\frac{\omega_i}{\omega_i-\omega_j}+\frac{\omega_k}{\omega_j+\omega_k}+1\right) - \frac{m^2}{4}V_{ijk\ell}\left(\frac{\omega_i}{\omega_i-\omega_k}+\frac{\omega_j}{\omega_j+\omega_k}+1\right) \\
&+ \frac{1}{4}Z^-_{kji\ell}\left(\frac{\omega_i}{\omega_j+\omega_k}\right) - \frac{1}{4}Z^+_{ijk\ell}\left(\frac{\omega_k}{\omega_i-\omega_j}\right) - \frac{1}{4}Z^+_{jki\ell}\left(\frac{\omega_j}{\omega_i-\omega_k}\right)\,.
\end{aligned} \tag{4.8}$$

4.3 Naturally Vanishing Resonances

It has been shown that when $m = 0$, and only normalizable modes are considered, (4.6) and (4.8) vanish by the orthogonality of the basis functions. **Maybe show that mass-dependent terms vanish for normalizable modes?**

4.4 $(++-)$

These contributions arise from the resonant condition $\omega_i + \omega_j = \omega_k + \omega_\ell$, can be written as

$$S_\ell = T_\ell a_\ell^3 \cos(\theta_\ell + \theta_\ell - \theta_\ell) + \sum_{i\neq\ell}^{\infty} R_{i\ell}\, a_i^2 a_\ell \cos(\theta_i + \theta_\ell - \theta_i)$$
$$+ \sum_{i\neq\ell}^{\infty}\sum_{j\neq\ell}^{\infty} S_{ij(i+j-\ell)\ell}\, a_i a_j a_{(i+j-\ell)} \cos(\theta_i + \theta_j - \theta_{i+j-\ell}) + \dots \tag{4.9}$$

where each of the coefficients is given by

$$\begin{aligned} S_{ijk\ell} = &-\frac{1}{4}H_{kij\ell}\frac{\omega_i(\omega_j - \omega_k + 2\omega_i)}{(\omega_i - \omega_k)(\omega_i + \omega_j)} - \frac{1}{4}H_{ijk\ell}\frac{\omega_j(\omega_i - \omega_k + 2\omega_j)}{(\omega_j - \omega_k)(\omega_i + \omega_j)} - \frac{1}{4}H_{jki\ell}\frac{\omega_k(\omega_i + \omega_j - 2\omega_k)}{(\omega_i - \omega_k)(\omega_j - \omega_k)} \\ &- \frac{1}{2}\omega_j\omega_k X_{ijk\ell}\left(\frac{\omega_j}{\omega_i - \omega_k} - \frac{\omega_k}{\omega_i + \omega_j} + 1\right) - \frac{1}{2}\omega_i\omega_k X_{jki\ell}\left(\frac{\omega_i}{\omega_j - \omega_k} - \frac{\omega_k}{\omega_i + \omega_j} + 1\right) \\ &+ \frac{1}{2}\omega_i\omega_j X_{kij\ell}\left(\frac{\omega_i}{\omega_j - \omega_k} + \frac{\omega_j}{\omega_i - \omega_k} + 1\right) - \frac{m^2}{4}V_{ijk\ell}\left(\frac{\omega_i}{\omega_i - \omega_k} + \frac{\omega_j}{\omega_j - \omega_k} + 1\right) \\ &+ \frac{m^2}{4}V_{jki\ell}\left(\frac{\omega_k}{\omega_i - \omega_k} - \frac{\omega_j}{\omega_i + \omega_j} - 1\right) + \frac{m^2}{4}V_{kij\ell}\left(\frac{\omega_k}{\omega_j - \omega_k} - \frac{\omega_i}{\omega_i + \omega_j} - 1\right) \\ &+ \frac{1}{4}Z^-_{ijk\ell}\left(\frac{\omega_k}{\omega_i + \omega_j}\right) + \frac{1}{4}Z^+_{ikj\ell}\left(\frac{\omega_j}{\omega_i - \omega_k}\right) + \frac{1}{4}Z^+_{jki\ell}\left(\frac{\omega_i}{\omega_j - \omega_k}\right), \end{aligned} \tag{4.10}$$

$$\begin{aligned} R_{i\ell} = &\left(\frac{\omega_i^2}{\omega_\ell^2 - \omega_i^2}\right)\left(Y_{i\ell\ell i} - Y_{i\ell i\ell} + \omega_\ell^2(X_{i\ell i\ell} - X_{\ell i\ell i})\right) + \left(\frac{\omega_i^2}{\omega_\ell^2 - \omega_i^2}\right)\left(H_{\ell ii\ell} + m^2 V_{ii\ell\ell} - 2\omega_i^2 X_{\ell ii\ell}\right) \\ &- \left(\frac{\omega_\ell^2}{\omega_\ell^2 - \omega_i^2}\right)\left(H_{i\ell i\ell} + m^2 V_{\ell ii\ell} - 2\omega_i^2 X_{i\ell i\ell}\right) - \frac{m^2}{4}(V_{i\ell i\ell} + V_{ii\ell\ell}) + \omega_i^2\omega_\ell^2(P_{ii\ell} - 2P_{\ell\ell i}) \\ &- \omega_i\omega_\ell X_{i\ell i\ell} - \frac{3m^2}{2}V_{\ell ii\ell} - \frac{1}{2}H_{ii\ell\ell} + \omega_\ell^2 B_{ii\ell} - \omega_i^2 M_{\ell\ell i} - m^2\omega_i^2 Q_{\ell\ell i}, \end{aligned} \tag{4.11}$$

$$T_\ell = \frac{1}{2}\omega_\ell^2\left(X_{\ell\ell\ell\ell} + 4B_{\ell\ell\ell} - 2M_{\ell\ell\ell} - 2m^2 Q_{\ell\ell\ell}\right) - \frac{3}{4}\left(H_{\ell\ell\ell\ell} + 3m^2 V_{\ell\ell\ell\ell}\right). \tag{4.12}$$

5 Resonances From Non-normalizable Modes

We now consider the case when at least one of the $e_i(x), e_j(x), e_k(x)$ is a non-normalizable mode. Since the boundary condition has been set to be a single non-normalizable mode, any non-normalizable modes in the source term must exactly cancel; therefore, at least two of the modes must be non-normalizable. This assumption breaks some of the symmetries that contributed to the previous expressions for resonance channels, and so the resonance conditions must be re-examined starting from the source expression (3.17).

5.1 Two General, Non-normalizable Modes

As a first case, let us assume that the two non-normalizable modes have constant, generic (i.e., non integer) frequency values, $\overline{\omega}$. Applying the time-averaging procedure to the source S_ℓ once again eliminates all contributions except those that satisfy (4.1). Since the basis onto which we are projecting is normalizable, we know that ω_ℓ is given by $\omega_\ell = 2\ell + \Delta^+$. We are now free to choose any one of $\{\omega_i, \omega_j, \omega_k\}$ to be normalizable and consider when the resonance condition is satisfied. In particular, we find that the following combinations are resonant:

$$\omega_i - \omega_j + \omega_k - \omega_\ell = 0 \qquad \Rightarrow \qquad \text{either } \omega_i \text{ or } \omega_k \text{ is normalizable} \tag{5.1}$$
$$\omega_i + \omega_j - \omega_k - \omega_\ell = 0 \qquad \Rightarrow \qquad \text{either } \omega_i \text{ or } \omega_j \text{ is normalizable} \tag{5.2}$$
$$\omega_i - \omega_j - \omega_k + \omega_\ell = 0 \qquad \Rightarrow \qquad \text{either } \omega_j \text{ or } \omega_k \text{ is normalizable.} \tag{5.3}$$

When any of these resonance conditions is met, the remaining normalizable mode will have a frequency equal to ω_ℓ, collapsing all sums over frequencies so that

$$S_\ell = \overline{T}_\ell \, a_\ell^3 \cos(\theta_\ell)\,. \tag{5.4}$$

Collecting the appropriate terms in (3.17), and evaluating the each possible resonance (being careful not to violate restrictions placed on the sums), we find that

$$\begin{aligned}
\overline{T}_\ell = {}& \frac{1}{2} Z^-_{\ell\overline{\omega}\overline{\omega}\ell}\left(\frac{\overline{\omega}}{\omega_\ell + \overline{\omega}}\right) + \frac{1}{2} Z^+_{\ell\overline{\omega}\overline{\omega}\ell}\left(\frac{\overline{\omega}}{\omega_\ell - \overline{\omega}}\right) - H_{\overline{\omega}\ell\overline{\omega}\ell}\frac{\omega_\ell^2}{\omega_\ell^2 - \overline{\omega}^2} + \frac{1}{2} H_{\ell\overline{\omega}\overline{\omega}\ell}\left(\frac{2\overline{\omega}^2}{\omega_\ell^2 - \overline{\omega}^2} - 1\right) \\
& + \overline{\omega}\omega_\ell X_{\overline{\omega}\overline{\omega}\ell\ell}\left(\frac{\overline{\omega}}{\omega_\ell - \overline{\omega}} + \frac{\overline{\omega}}{\omega_\ell + \overline{\omega}} + \frac{\omega_\ell}{\overline{\omega}}\right) - \overline{\omega}^2 X_{\ell\ell\overline{\omega}\overline{\omega}}\left(\frac{\overline{\omega}}{\omega_\ell - \overline{\omega}} - \frac{\overline{\omega}}{\omega_\ell + \overline{\omega}} + 1\right) \\
& + \frac{m^2}{2} V_{\ell\ell\overline{\omega}\overline{\omega}}\left(\frac{\omega_\ell}{\omega_\ell - \overline{\omega}} + \frac{\omega_\ell}{\omega_\ell + \overline{\omega}} + \frac{5}{2}\right) + \frac{m^2}{2} V_{\overline{\omega}\overline{\omega}\ell\ell}\left(\frac{\overline{\omega}}{\omega_\ell - \overline{\omega}} - \frac{\overline{\omega}}{\omega_\ell + \overline{\omega}} - 1\right) \\
& + \omega_\ell^2 \tilde{Z}^+_{\overline{\omega}\overline{\omega}\ell} - 2\overline{\omega}^2\omega_\ell^2 P_{\ell\ell\overline{\omega}} - \overline{\omega}^2 M_{\ell\ell\overline{\omega}} - m^2\overline{\omega}^2 Q_{\ell\ell\overline{\omega}}\,.
\end{aligned} \tag{5.5}$$

5.2 Special Values of Non-normalizable Frequencies

Focus on non-arbitrary values of the non-normalizable frequencies.

5.2.1 Add to an integer

Choose two of the modes to be non-normalizable with frequencies ω_1 and ω_2 that add to give an integer: $\omega_1 + \omega_2 = 2n$ where $n = 1, 2, 3, \ldots$ (note that the $n = 0$ case means that both ω_1 and ω_2 would need to be zero by the positive-frequency requirement and so would not contribute). Keeping the values as general as possible, the difference of the frequencies will not necessarily be an integer. Once again, consider each channel and determine the conditions for resonance.

$$\omega_i + \omega_j + \omega_k - \omega_\ell = 0 \quad \Rightarrow \quad \text{any of } \{\omega_i, \omega_j, \omega_k\} \text{ is normalizable with } \ell \geq n \tag{5.6}$$
$$\omega_i - \omega_j + \omega_k + \omega_\ell = 0 \quad \Rightarrow \quad \omega_j \text{ is normalizable and } \forall\, n \in \mathbb{Z}^+ \tag{5.7}$$
$$\omega_i + \omega_j - \omega_k + \omega_\ell = 0 \quad \Rightarrow \quad \omega_k \text{ is normalizable and } \forall\, n \in \mathbb{Z}^+ \tag{5.8}$$
$$\omega_i - \omega_j - \omega_k - \omega_\ell = 0 \quad \Rightarrow \quad \omega_i \text{ is normalizable and } \forall\, n \in \mathbb{Z}^+. \tag{5.9}$$

These resonances will contribute to the source term in the form

$$S_\ell = \left[\overline{R}^{(1)}_{(\ell-n)\ell}\Theta\left(\ell-n\right) + \overline{R}^{(2)}_{(\ell+n)\ell} + \overline{T}_\ell\right]\cos(\theta_\ell)\,, \tag{5.10}$$

where the Heaviside step function $\Theta(x)$ enforces the restriction on n that comes from (5.6), and

$$\begin{aligned}
\overline{R}^{(1)}_{i\ell} = {} & \frac{1}{4}Z^{-}_{12i\ell}\left(\frac{2n-\omega_\ell}{2n}\right) - \frac{1}{4}Z^{-}_{i21\ell}\frac{\omega_1(\omega_\ell+\omega_1)}{\omega_\ell^2-\omega_1^2} - \frac{1}{4}Z^{-}_{i12\ell}\frac{\omega_2(\omega_\ell+\omega_2)}{\omega_\ell^2-\omega_2^2} - \frac{1}{4}\omega_1 H_{i12\ell}\left(\frac{1}{\omega_\ell-\omega_2}+\frac{1}{2n}\right) \\
& - \frac{1}{4}\omega_2 H_{i21\ell}\left(\frac{1}{\omega_\ell-\omega_1}+\frac{1}{2n}\right) - \frac{1}{4}(\omega_\ell-2n)H_{1i2\ell}\left(\frac{1}{\omega_\ell-\omega_1}+\frac{1}{\omega_\ell-\omega_2}\right) \\
& + \frac{1}{2}\omega_1\omega_2 X_{i12\ell}\left(\frac{\omega_2}{\omega_\ell-\omega_2}+\frac{\omega_1}{\omega_\ell-\omega_1}+1\right) + \frac{1}{2}(\omega_\ell-2n)\omega_2 X_{12i\ell}\left(\frac{\omega_2}{\omega_\ell-\omega_2}+\frac{\omega_\ell}{2n}\right) \\
& + \frac{1}{2}(\omega_\ell-2n)\omega_1 X_{21i\ell}\left(\frac{\omega_1}{\omega_\ell-\omega_1}+\frac{\omega_\ell}{2n}\right) - \frac{m^2}{4}V_{12i\ell}\left(\frac{\omega_1}{\omega_\ell-\omega_2}+\frac{\omega_2}{\omega_\ell-\omega_1}+1\right) \\
& - \frac{m^2}{4}V_{i21\ell}\left(\frac{\omega_\ell-2n}{\omega_\ell-\omega_2}+\frac{\omega_\ell-2n}{\omega_\ell-\omega_1}+2\right) - \frac{m^2}{4}V_{i12\ell}\,,
\end{aligned} \tag{5.11}$$

$$\begin{aligned}
\overline{R}^{(2)}_{i\ell} = {} & \frac{1}{4}Z^{-}_{12i\ell}\left(\frac{\omega_\ell+2n}{2n}\right) - \frac{1}{4}Z^{+}_{i21\ell}\frac{\omega_1(\omega_\ell-\omega_1)}{\omega_\ell^2-\omega_1^2} - \frac{1}{4}Z^{+}_{i12\ell}\frac{\omega_2(\omega_\ell-\omega_2)}{\omega_\ell^2-\omega_2^2} + \frac{1}{4}\omega_1 H_{i12\ell}\left(\frac{1}{\omega_\ell+\omega_2}-\frac{1}{2n}\right) \\
& + \frac{1}{4}\omega_2 H_{i21\ell}\left(\frac{1}{\omega_\ell+\omega_1}-\frac{1}{2n}\right) - \frac{1}{4}(\omega_\ell+2n)H_{1i2\ell}\left(\frac{1}{\omega_\ell+\omega_1}+\frac{1}{\omega_\ell+\omega_2}\right) \\
& - \frac{1}{2}\omega_1\omega_2 X_{i12\ell}\left(\frac{\omega_2}{\omega_\ell+\omega_2}+\frac{\omega_1}{\omega_\ell+\omega_1}-1\right) + \frac{1}{2}(\omega_\ell+2n)\omega_2 X_{12i\ell}\left(\frac{\omega_2}{\omega_\ell+\omega_2}+\frac{\omega_\ell}{2n}\right) \\
& + \frac{1}{2}(\omega_\ell+2n)\omega_1 X_{21i\ell}\left(\frac{\omega_1}{\omega_\ell+\omega_1}+\frac{\omega_\ell}{2n}\right) + \frac{m^2}{4}V_{12i\ell}\left(\frac{\omega_1}{\omega_\ell+\omega_2}+\frac{\omega_2}{\omega_\ell+\omega_1}-1\right) \\
& - \frac{m^2}{4}V_{i21\ell}\left(\frac{\omega_\ell+2n}{\omega_\ell+\omega_2}+\frac{\omega_2}{2n}+1\right) - \frac{m^2}{4}V_{i12\ell}\left(\frac{\omega_\ell+2n}{\omega_\ell+\omega_1}+\frac{\omega_1}{2n}+1\right)\,,
\end{aligned} \tag{5.12}$$

and

$$\overline{T}_\ell = -\frac{3}{4}H_{\ell\ell\ell\ell} + \frac{3}{4}\omega_\ell^2 X_{\ell\ell\ell\ell} - \frac{3}{4}m^2 V_{\ell\ell\ell\ell} - 2\omega_\ell^4 P_{\ell\ell\ell} - \omega_\ell^2 M_{\ell\ell\ell} - m^2\omega_\ell^2 Q_{\ell\ell\ell}\,. \tag{5.13}$$

The notation $X_{i12\ell}$ corresponds to evaluating $X_{ijk\ell}$ with $\omega_j = \omega_1$ and $\omega_k = \omega_2$.

5.2.2 Integer Plus/Minus Δ^-

This is a case where the non-normalizable frequencies are non-integer, but differ from integer values by a specific amount. In analogue to the case where all modes are normalizable, we consider setting any two of the non-normalizable frequencies to

$$\omega_i = 2i + \Delta^-\,. \tag{5.14}$$

An examination of the resonance channels shows that the same terms as those found in § 4 appear. The derivation of the resonant terms is therefore trivial, and the solution would only differ by the form of the basis functions **See if source terms still vanish for massless scalars in this case.**

However, when the form of the frequencies of the non-normalizable modes is

$$\omega_i = 2i - \Delta^- \tag{5.15}$$

there are different contributions to consider. In addition to the resonance conditions described in (5.1)-(5.3), there are

$$\omega_i + \omega_j - \omega_k - \omega_\ell = 0 \quad \Rightarrow \quad \omega_k \text{ is normalizable} \tag{5.16}$$

$$\omega_i - \omega_j + \omega_k - \omega_\ell = 0 \quad \Rightarrow \quad \omega_j \text{ is normalizable} \tag{5.17}$$

$$\omega_i - \omega_j - \omega_k + \omega_\ell = 0 \quad \Rightarrow \quad \omega_i \text{ is normalizable}\,. \tag{5.18}$$

These resonance conditions are satisfied not when, using (5.16) as an example, $i + j - k = \ell$, but rather when $i + j - k = \ell + d$. Thus, these resonances will be satisfied under the shifted condition $\ell \to \ell + d$.

6 QP Equations

7 Numerics

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