

PREPARED FOR SUBMISSION TO JHEP

Arbitrary Dimensions, Massive, Non-normalizable Time-Dependent BCs

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1 Introduction

2 Perturbative Expansion

The backreaction between the metric and the scalar field appears at second order in the perturbation,

$$A'_2 = -\mu\nu \left[(\dot{\phi}_1)^2 + (\phi'_1)^2 + m^2 \phi_1^2 \sec^2 x \right] + \nu' A_2 / \nu \quad (2.1)$$

which can be directly integrated to give

$$A_2 = -\nu \int_0^x dy \mu \left((\dot{\phi}_1)^2 + (\phi'_1)^2 + m^2 \phi_1^2 \sec^2 x \right). \quad (2.2)$$

Furthermore, the first non-trivial contribution to the lapse in the boundary time gauge is

$$\delta_2 = \int_x^{\pi/2} dy \mu \nu \left((\dot{\phi}_1)^2 + (\phi'_1)^2 \right). \quad (2.3)$$

For convenience, we have also defined the functions

$$\mu(x) = (\tan x)^{d-1} \quad \text{and} \quad \nu(x) = (d-1)/\mu'. \quad (2.4)$$

To aide in evaluating integrals, we first derive the following identities: from the equation for the first-order time-dependent coefficients c_i ,

$$\ddot{c}_i + \omega_i^2 c_i = 0 \quad \Rightarrow \quad \partial_t (\dot{c}_i^2 + \omega_i^2 c_i^2) = \partial_t \mathbb{C}_i = 0; \quad (2.5)$$

from the equation definition of \hat{L} ,

$$\hat{L}e_j = -\frac{1}{\mu} (\mu e'_j)' + m^2 \sec^2 x e_j \quad \Rightarrow \quad (\mu e'_j)' = \mu (m^2 \sec^2 x - \omega_j^2) e_j; \quad (2.6)$$

from considering the expression $(\mu e'_i e_j)'$:

$$(\mu e'_i e_j)' = (m^2 \sec^2 x - \omega_i^2) \mu e_i e_j + \mu e'_i e'_j; \quad (2.7)$$

from permuting i, j above and subtracting to give

$$\frac{[\mu(e'_i e_j \omega_j^2 - e_i e'_j \omega_i^2)]'}{(\omega_j^2 - \omega_i^2)} = \mu m^2 \sec^2 x e_i e_j + \mu e'_i e'_j. \quad (2.8)$$

The basis functions $e_j(x)$ are the solutions to the eigenvalue equation

$$\hat{L}e_j(x) = \omega_j^2 e_j(x), \quad (2.9)$$

which, for massive scalars, are (up to some normalization)

$$e_j(x) = (\cos(x))^{\Delta_+} {}_2F_1\left(\frac{\Delta_+ + \omega}{2}, \frac{\Delta_+ - \omega}{2}, d/2; \sin^2(x)\right), \quad (2.10)$$

when ω is arbitrary. However, when the frequency is equal to the resonant frequency $\omega_j = \Delta_+ + 2j$, (2.10) separates into normalizable and non-normalizable solutions

$$\begin{aligned} e_j(x) = & C_1 (\cos(x))^{\Delta_+} {}_2F_1\left(\frac{\Delta_+ + \omega}{2}, \frac{\Delta_+ - \omega}{2}, \Delta_+ - d/2 + 1; \cos^2(x)\right) \\ & + C_2 (\cos(x))^{\Delta_-} {}_2F_1\left(\frac{\Delta_- + \omega}{2}, \frac{\Delta_- - \omega}{2}, \Delta_- - d/2 + 1; \cos^2(x)\right). \end{aligned} \quad (2.11)$$

3 $\mathcal{O}(\epsilon^3)$ Source Terms

At third order in ϵ , the equation for ϕ_3 contains a source S given by

$$\ddot{\phi}_3 + \hat{L}\phi_3 = S = 2(A_2 - \delta_2)\ddot{\phi}_1 + (\dot{A}_2 - \dot{\delta}_2)\dot{\phi}_1 + (A'_2 - \delta'_2)\phi'_1 + m^2 A_2 \phi_1 \sec^2 x \quad (3.1)$$

Projecting each of the terms individually onto the eigenbasis $\{e_\ell\}$:

$$\begin{aligned} \langle \delta_2 \ddot{\phi}_1, e_\ell \rangle = & - \sum_{i=0}^{\infty} \sum_{\substack{j=0 \\ k \neq \ell}}^{\infty} \sum_{k=0}^{\infty} \frac{\omega_k^2 c_k}{\omega_\ell^2 - \omega_k^2} [\dot{c}_i \dot{c}_j (X_{k\ell ij} - X_{\ell k ij}) + c_i c_j (Y_{ij\ell k} - Y_{ijk\ell})] \\ & - \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \omega_\ell^2 c_\ell [\dot{c}_i \dot{c}_j P_{ij\ell} + c_i c_j B_{ij\ell}] , \end{aligned} \quad (3.2)$$

$$\begin{aligned} \langle A_2 \ddot{\phi}_1, e_\ell \rangle = & 2 \sum_{i=0}^{\infty} \sum_{\substack{j=0 \\ i \neq j}}^{\infty} \sum_{k=0}^{\infty} \frac{\omega_k^2 c_k}{\omega_j^2 - \omega_i^2} X_{ijk\ell} (\dot{c}_i \dot{c}_j + \omega_j^2 c_i c_j) \\ & + \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \omega_j^2 c_j (\mathbb{C}_i P_{j\ell i} + c_i^2 X_{iij\ell}) , \end{aligned} \quad (3.3)$$

$$\begin{aligned} \langle \dot{\delta}_2 \dot{\phi}_1, e_\ell \rangle = & \sum_{i=0}^{\infty} \sum_{\substack{j=0 \\ k \neq \ell}}^{\infty} \sum_{k=0}^{\infty} \frac{\dot{c}_k}{\omega_\ell^2 - \omega_k^2} [\partial_t (\dot{c}_i \dot{c}_j) (X_{k\ell ij} - X_{\ell k ij}) + \partial_t (c_i c_j) (Y_{ij\ell k} - Y_{ijk\ell})] \\ & + \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \dot{c}_\ell [\partial_t (\dot{c}_i \dot{c}_j) P_{ij\ell} + \partial_t (c_i c_j) B_{ij\ell}] , \end{aligned} \quad (3.4)$$

$$\langle \dot{A}_2 \dot{\phi}_1, e_\ell \rangle = -2 \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \dot{c}_k \dot{c}_j c_i X_{ijk\ell} , \quad (3.5)$$

$$\begin{aligned} \langle (A'_2 - \delta'_2) \phi'_1, e_\ell \rangle = & -2 \sum_{i=0}^{\infty} \sum_{\substack{j=0 \\ i \neq j}}^{\infty} \sum_{k=0}^{\infty} \frac{c_k (\dot{c}_i \dot{c}_j + \omega_j^2 c_i c_j)}{\omega_j^2 - \omega_i^2} H_{ijk\ell} - m^2 \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} c_i c_j c_k V_{ijk\ell} \\ & - \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} c_j [c_i^2 H_{iij\ell} + \mathbb{C}_i M_{j\ell i}] , \end{aligned} \quad (3.6)$$

$$\begin{aligned} \langle A_2 \phi_1 \sec^2 x, e_\ell \rangle = & -2 \sum_{i=0}^{\infty} \sum_{\substack{j=0 \\ i \neq j}}^{\infty} \sum_{k=0}^{\infty} \frac{c_k (\dot{c}_i \dot{c}_j + \omega_j^2 c_i c_j)}{\omega_j^2 - \omega_i^2} V_{jkil} \\ & - \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} c_j (c_i^2 V_{jii\ell} + \mathbb{C}_i Q_{j\ell i}) . \end{aligned} \quad (3.7)$$

Where the forms of X, Y, V, H, B, M, P, and Q are given by

$$X_{ijkl} = \int_0^{\pi/2} dx \mu^2 \nu e'_i e_j e_k e_\ell \quad (3.8)$$

$$Y_{ijkl} = \int_0^{\pi/2} dx \mu^2 \nu e'_i e'_j e_k e'_\ell \quad (3.9)$$

$$V_{ijkl} = \int_0^{\pi/2} dx \mu^2 \nu e_i e_j e'_k e_\ell \sec^2 x \quad (3.10)$$

$$H_{ijkl} = \int_0^{\pi/2} dx \mu^2 \nu' e'_i e'_j e'_k e_\ell \quad (3.11)$$

$$B_{ij\ell} = \int_0^{\pi/2} dx \mu \nu e'_i e'_j \int_0^x dy \mu e_\ell^2 \quad (3.12)$$

$$M_{ij\ell} = \int_0^{\pi/2} dx \mu \nu' e'_i e_j \int_0^x dy \mu e_\ell^2 \quad (3.13)$$

$$P_{ij\ell} = \int_0^{\pi/2} dx \mu \nu e_i e_j \int_0^x dy \mu e_\ell^2 \quad (3.14)$$

$$Q_{ij\ell} = \int_0^{\pi/2} dx \mu \nu e_i e_j \sec^2 x \int_0^x dy \mu e_\ell^2 \quad (3.15)$$

Collecting terms together gives the expression for $S_\ell = \langle S, e_\ell \rangle$:

$$\begin{aligned} S_\ell = & \sum_{\substack{i,j,k \\ k \neq \ell}}^{\infty} \frac{1}{\omega_\ell^2 - \omega_k^2} \left[F_k(\dot{c}_i \dot{c}_j) (X_{k\ell ij} - X_{\ell k ij}) + F_k(c_i c_j) (Y_{ij\ell k} - Y_{ijk\ell}) \right] \\ & + 2 \sum_{\substack{i,j,k \\ i \neq j}}^{\infty} \frac{c_k D_{ij}}{\omega_j^2 - \omega_i^2} \left[2\omega_k^2 X_{ijkl} - H_{ijkl} - m^2 V_{jki\ell} \right] - \sum_{i,j,k}^{\infty} c_i \left[2\dot{c}_j \dot{c}_k X_{ijkl} + m^2 c_j c_k V_{ijkl} \right] \\ & + \sum_{i,j}^{\infty} \left[F_\ell(\dot{c}_i \dot{c}_j) P_{ij\ell} + F_\ell(c_i c_j) B_{ij\ell} + 2\omega_j^2 c_j (c_i^2 X_{iij\ell} + \mathbb{C}_i P_{j\ell i}) \right. \\ & \left. - c_j (c_i^2 (H_{iij\ell} + m^2 V_{jii\ell}) + \mathbb{C}_i (M_{j\ell i} + m^2 Q_{j\ell i})) \right], \end{aligned} \quad (3.16)$$

where $F_k(z) = \dot{c}_k \dot{z} - 2\omega_k^2 c_k z$, $D_{ij} = \dot{c}_i \dot{c}_j + \omega_j^2 c_i c_j$, and $\mathbb{C}_i = \dot{c}_i^2 + \omega_i^2 c_i^2$.

Using the solution $c_i(t) = a_i \cos(\omega_i t + b_i) = a_i \cos \theta_i$, the source term becomes

$$\begin{aligned}
S_\ell = & \frac{1}{4} \sum_{\substack{i,j,k \\ k \neq \ell}}^{\infty} \frac{a_i a_j a_k \omega_k}{\omega_\ell^2 - \omega_k^2} \left[Z_{ijk\ell}^-(\omega_i + \omega_j - 2\omega_k) \cos(\theta_i + \theta_j - \theta_k) - Z_{ijk\ell}^-(\omega_i + \omega_j + 2\omega_k) \cos(\theta_i + \theta_j + \theta_k) - \right. \\
& \left. + Z_{ijk\ell}^+(\omega_i - \omega_j + 2\omega_k) \cos(\theta_i - \theta_j + \theta_k) - Z_{ijk\ell}^+(\omega_i - \omega_j - 2\omega_k) \cos(\theta_i - \theta_j - \theta_k) \right] \\
& + \frac{1}{2} \sum_{\substack{i,j,k \\ i \neq j}}^{\infty} a_i a_j a_k \omega_j \left(H_{ijk\ell} + m^2 V_{jki\ell} - 2\omega_k^2 X_{ijk\ell} \right) \left[\frac{1}{\omega_i - \omega_j} (\cos(\theta_i - \theta_j - \theta_k) + \cos(\theta_i - \theta_j + \theta_k)) \right. \\
& \left. - \frac{1}{\omega_i + \omega_j} (\cos(\theta_i + \theta_j - \theta_k) + \cos(\theta_i + \theta_j + \theta_k)) \right] \\
& - \frac{1}{4} \sum_{i,j,k}^{\infty} a_i a_j a_k \left[(2\omega_j \omega_k X_{ijk\ell} + m^2 V_{ijk\ell}) \cos(\theta_i + \theta_j - \theta_k) - (2\omega_j \omega_k X_{ijk\ell} - m^2 V_{ijk\ell}) \cos(\theta_i - \theta_j - \theta_k) \right. \\
& \left. + (2\omega_j \omega_k X_{ijk\ell} + m^2 V_{ijk\ell}) \cos(\theta_i - \theta_j + \theta_k) - (2\omega_j \omega_k X_{ijk\ell} - m^2 V_{ijk\ell}) \cos(\theta_i + \theta_j + \theta_k) \right] \\
& + \frac{1}{4} \sum_{i,j}^{\infty} a_i a_j a_\ell \omega_\ell \left[\tilde{Z}_{ij\ell}^-(\omega_i + \omega_j - 2\omega_\ell) \cos(\theta_i + \theta_j - \theta_\ell) - \tilde{Z}_{ij\ell}^-(\omega_i + \omega_j + 2\omega_\ell) \cos(\theta_i + \theta_j + \theta_\ell) \right. \\
& \left. + \tilde{Z}_{ij\ell}^+(\omega_i - \omega_j + 2\omega_\ell) \cos(\theta_i - \theta_j + \theta_\ell) - \tilde{Z}_{ij\ell}^+(\omega_i - \omega_j - 2\omega_\ell) \cos(\theta_i - \theta_j - \theta_\ell) \right] \\
& - \frac{1}{4} \sum_{i,j}^{\infty} a_i^2 a_j \left(H_{iij\ell} + m^2 V_{jii\ell} - 2\omega_j^2 X_{iij\ell} \right) [\cos(2\theta_i - \theta_j) + \cos(2\theta_i + \theta_j)] \\
& - \frac{1}{2} \sum_{i,j}^{\infty} a_i^2 a_j \left(H_{iij\ell} + m^2 V_{jii\ell} - 2\omega_j^2 X_{iij\ell} + 4\omega_i^2 \omega_j^2 P_{j\ell i} + 2\omega_i^2 (M_{j\ell i} + m^2 Q_{j\ell i}) \right) \cos \theta_j. \quad (3.17)
\end{aligned}$$

To simplify the above expression, we have defined

$$Z_{ijk\ell}^\pm = \omega_i \omega_j (X_{k\ell ij} - X_{\ell k ij}) \pm (Y_{ij\ell k} - Y_{ij k\ell}) \quad \text{and} \quad \tilde{Z}_{ij\ell}^\pm = \omega_i \omega_j P_{ij\ell} \pm B_{ij\ell}. \quad (3.18)$$

Using integration by parts to remove the derivative from ν in the definitions of $H_{ijk\ell}$ and $M_{ij\ell}$, we can show that

$$H_{ijk\ell} = \omega_i^2 X_{kij\ell} + \omega_k^2 X_{ijk\ell} - Y_{ij\ell k} - Y_{\ell k ji} - m^2 V_{kji\ell} - m^2 V_{ijk\ell} \quad (3.19)$$

$$M_{ij\ell} = \omega_i^2 P_{ij\ell} - B_{ij\ell} - m^2 Q_{ij\ell} \quad (3.20)$$

4 Resonances From Normalizable Solutions

Consider the case where each of the basis functions are given by normalizable solutions. After time-averaging, resonant contributions come from the set of conditions

$$\omega_i \pm \omega_j \pm \omega_k = \pm \omega_\ell \quad (4.1)$$

which separates into three distinct cases

$$\omega_i + \omega_j + \omega_k = \omega_\ell \quad (+ + +) \quad (4.2)$$

$$\omega_i - \omega_j - \omega_k = \omega_\ell \quad (+ - -) \quad (4.3)$$

$$\omega_i + \omega_j - \omega_k = \omega_\ell \quad (+ + -) \quad (4.4)$$

4.1 (+ + +)

These resonant contributions come from the condition $\omega_i + \omega_i + \omega_k = \omega_\ell$, and are of the form

$$S_\ell = \underbrace{\sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty}}_{\omega_i + \omega_j + \omega_k = \omega_\ell} \Omega_{ijk\ell} a_i a_j a_k \cos(\theta_i + \theta_j + \theta_k) + \dots, \quad (4.5)$$

where

$$\begin{aligned} \Omega_{ijk\ell} = & -\frac{1}{12} H_{ijk\ell} \frac{\omega_j(\omega_i + \omega_k + 2\omega_j)}{(\omega_i + \omega_j)(\omega_j + \omega_k)} - \frac{1}{12} H_{ikj\ell} \frac{\omega_k(\omega_i + \omega_j + 2\omega_k)}{(\omega_i + \omega_k)(\omega_j + \omega_k)} - \frac{1}{12} H_{jik\ell} \frac{\omega_i(\omega_j + \omega_k + 2\omega_i)}{(\omega_i + \omega_j)(\omega_i + \omega_k)} \\ & - \frac{m^2}{12} V_{ijk\ell} \left(1 + \frac{\omega_j}{\omega_j + \omega_k} + \frac{\omega_i}{\omega_i + \omega_k}\right) - \frac{m^2}{12} V_{jki\ell} \left(1 + \frac{\omega_j}{\omega_i + \omega_j} + \frac{\omega_k}{\omega_i + \omega_k}\right) \\ & - \frac{m^2}{12} V_{kij\ell} \left(1 + \frac{\omega_i}{\omega_i + \omega_j} + \frac{\omega_k}{\omega_j + \omega_k}\right) + \frac{1}{6} \omega_j \omega_k X_{ijk\ell} \left(1 + \frac{\omega_j}{\omega_i + \omega_k} + \frac{\omega_k}{\omega_i + \omega_j}\right) \\ & + \frac{1}{6} \omega_i \omega_k X_{jki\ell} \left(1 + \frac{\omega_i}{\omega_j + \omega_k} + \frac{\omega_k}{\omega_i + \omega_j}\right) + \frac{1}{6} \omega_i \omega_j X_{kij\ell} \left(1 + \frac{\omega_i}{\omega_j + \omega_k} + \frac{\omega_j}{\omega_i + \omega_k}\right) \\ & - \frac{1}{12} Z_{ijk\ell}^- \left(\frac{\omega_k}{\omega_i + \omega_j}\right) - \frac{1}{12} Z_{ikj\ell}^- \left(\frac{\omega_j}{\omega_i + \omega_k}\right) - \frac{1}{12} Z_{jik\ell}^- \left(\frac{\omega_i}{\omega_j + \omega_k}\right). \end{aligned} \quad (4.6)$$

4.2 (+ - -)

These contributions arise from the condition $\omega_i - \omega_k - \omega_k = \omega_\ell$, are of the form

$$S_\ell = \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \Gamma_{(j+k+\ell)jk\ell} a_j a_k a_{(j+k+\ell)} \cos(\theta_{j+k+\ell} - \theta_j - \theta_k) + \dots, \quad (4.7)$$

where

$$\begin{aligned} \Gamma_{ijk\ell} = & \frac{1}{4} H_{ijk\ell} \frac{\omega_j(\omega_k - \omega_i + 2\omega_j)}{(\omega_i - \omega_j)(\omega_j + \omega_k)} + \frac{1}{4} H_{jki\ell} \frac{\omega_k(\omega_j - \omega_i + 2\omega_k)}{(\omega_i - \omega_k)(\omega_j + \omega_k)} + \frac{1}{4} H_{kij\ell} \frac{\omega_i(\omega_j + \omega_k - 2\omega_i)}{(\omega_i - \omega_j)(\omega_i - \omega_k)} \\ & - \frac{1}{2} \omega_j \omega_k X_{ijk\ell} \left(\frac{\omega_k}{\omega_i - \omega_j} + \frac{\omega_j}{\omega_i - \omega_k} - 1\right) + \frac{1}{2} \omega_i \omega_k X_{jki\ell} \left(\frac{\omega_k}{\omega_i - \omega_j} + \frac{\omega_i}{\omega_j + \omega_k} - 1\right) \\ & + \frac{1}{2} \omega_i \omega_j X_{kij\ell} \left(\frac{\omega_j}{\omega_i - \omega_k} + \frac{\omega_i}{\omega_j + \omega_k} - 1\right) + \frac{m^2}{4} V_{jki\ell} \left(\frac{\omega_j}{\omega_i - \omega_j} + \frac{\omega_k}{\omega_i - \omega_k} - 1\right) \\ & - \frac{m^2}{4} V_{kij\ell} \left(\frac{\omega_i}{\omega_i - \omega_j} + \frac{\omega_k}{\omega_j + \omega_k} + 1\right) - \frac{m^2}{4} V_{ijk\ell} \left(\frac{\omega_i}{\omega_i - \omega_k} + \frac{\omega_j}{\omega_j + \omega_k} + 1\right) \\ & + \frac{1}{4} Z_{kji\ell}^- \left(\frac{\omega_i}{\omega_j + \omega_k}\right) - \frac{1}{4} Z_{ijk\ell}^+ \left(\frac{\omega_k}{\omega_i - \omega_j}\right) - \frac{1}{4} Z_{jik\ell}^+ \left(\frac{\omega_j}{\omega_i - \omega_k}\right). \end{aligned} \quad (4.8)$$

4.3 Naturally Vanishing Resonances

It has been shown that when $m = 0$, and only normalizable modes are considered, (4.6) and (4.8) vanish by the orthogonality of the basis functions. **Maybe show that mass-dependent terms vanish for normalizable modes?**

4.4 (+ + -)

These contributions arise from the resonant condition $\omega_i + \omega_j = \omega_k + \omega_\ell$, can be written as

$$S_\ell = T_\ell a_\ell^3 \cos(\theta_\ell + \theta_\ell - \theta_\ell) + \sum_{i \neq \ell}^\infty R_{i\ell} a_i^2 a_\ell \cos(\theta_i + \theta_\ell - \theta_i) \\ + \sum_{i \neq \ell}^\infty \sum_{j \neq \ell}^\infty S_{ij(i+j-\ell)\ell} a_i a_j a_{(i+j-\ell)} \cos(\theta_i + \theta_j - \theta_{i+j-\ell}) + \dots \quad (4.9)$$

where each of the coefficients is given by

$$S_{ijk\ell} = -\frac{1}{4} H_{kij\ell} \frac{\omega_i(\omega_j - \omega_k + 2\omega_i)}{(\omega_i - \omega_k)(\omega_i + \omega_j)} - \frac{1}{4} H_{ijk\ell} \frac{\omega_j(\omega_i - \omega_k + 2\omega_j)}{(\omega_j - \omega_k)(\omega_i + \omega_j)} - \frac{1}{4} H_{jkil} \frac{\omega_k(\omega_i + \omega_j - 2\omega_k)}{(\omega_i - \omega_k)(\omega_j - \omega_k)} \\ - \frac{1}{2} \omega_j \omega_k X_{ijk\ell} \left(\frac{\omega_j}{\omega_i - \omega_k} - \frac{\omega_k}{\omega_i + \omega_j} + 1 \right) - \frac{1}{2} \omega_i \omega_k X_{jkil} \left(\frac{\omega_i}{\omega_j - \omega_k} - \frac{\omega_k}{\omega_i + \omega_j} + 1 \right) \\ + \frac{1}{2} \omega_i \omega_j X_{kij\ell} \left(\frac{\omega_i}{\omega_j - \omega_k} + \frac{\omega_j}{\omega_i - \omega_k} + 1 \right) - \frac{m^2}{4} V_{ijk\ell} \left(\frac{\omega_i}{\omega_i - \omega_k} + \frac{\omega_j}{\omega_j - \omega_k} + 1 \right) \\ + \frac{m^2}{4} V_{jkil} \left(\frac{\omega_k}{\omega_i - \omega_k} - \frac{\omega_j}{\omega_i + \omega_j} - 1 \right) + \frac{m^2}{4} V_{kij\ell} \left(\frac{\omega_k}{\omega_j - \omega_k} - \frac{\omega_i}{\omega_i + \omega_j} - 1 \right) \\ + \frac{1}{4} Z_{ijk\ell}^- \left(\frac{\omega_k}{\omega_i + \omega_j} \right) + \frac{1}{4} Z_{ikj\ell}^+ \left(\frac{\omega_j}{\omega_i - \omega_k} \right) + \frac{1}{4} Z_{jki\ell}^+ \left(\frac{\omega_i}{\omega_j - \omega_k} \right), \quad (4.10)$$

$$R_{i\ell} = \left(\frac{\omega_i^2}{\omega_\ell^2 - \omega_i^2} \right) (Y_{i\ell i\ell} - Y_{i\ell i\ell} + \omega_\ell^2 (X_{i\ell i\ell} - X_{i\ell i\ell})) + \left(\frac{\omega_i^2}{\omega_\ell^2 - \omega_i^2} \right) (H_{i\ell i\ell} + m^2 V_{i\ell i\ell} - 2\omega_i^2 X_{i\ell i\ell}) \\ - \left(\frac{\omega_\ell^2}{\omega_\ell^2 - \omega_i^2} \right) (H_{i\ell i\ell} + m^2 V_{i\ell i\ell} - 2\omega_i^2 X_{i\ell i\ell}) - \frac{m^2}{4} (V_{i\ell i\ell} + V_{i\ell i\ell}) + \omega_i^2 \omega_\ell^2 (P_{i\ell} - 2P_{\ell i}) \\ - \omega_i \omega_\ell X_{i\ell i\ell} - \frac{3m^2}{2} V_{i\ell i\ell} - \frac{1}{2} H_{i\ell i\ell} + \omega_\ell^2 B_{i\ell} - \omega_i^2 M_{\ell i} - m^2 \omega_i^2 Q_{\ell i}, \quad (4.11)$$

$$T_\ell = \frac{1}{2} \omega_\ell^2 (X_{\ell\ell\ell\ell} + 4B_{\ell\ell} - 2M_{\ell\ell} - 2m^2 Q_{\ell\ell}) - \frac{3}{4} (H_{\ell\ell\ell\ell} + 3m^2 V_{\ell\ell\ell\ell}). \quad (4.12)$$

5 Resonances From Non-normalizable Modes

5.1 Two General, Equal-frequency Modes

5.1.1 $m^2 = 0$

5.1.2 $m^2 = 1$

5.2 Special Values of Non-normalizable Frequencies

5.2.1 Differ by an integer

5.2.2 Resonant Values

5.3 Boundary Condition is a Superposition/Fourier Integral of Non-normalizable Modes

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