1 Perturbative Descriptions of Driven Instabilities in AdS

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1.1 Contributions

All contributions are mine.

Examining Instabilities Due to Driven Scalars in AdS

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We extend the study of the non-linear perturbative theory of weakly turbulent energy cascades in AdS_{d+1} to include solutions of driven systems, i.e. those with time-dependent sources on the AdS boundary. This necessitates the activation of non-normalizable modes in the linear solution for the massive bulk scalar field, which couple to the metric and normalizable scalar modes. We determine analytic expressions for secular terms in the renormalization flow equations for any mass, and for various driving functions. Finally, we numerically evaluate these sources for $d = 4$ and discuss what role these driven solutions play in the perturbative stability of AdS.

1.2 Introduction

Nonlinear instabilities in Anti-de Sitter space have been the subject of examinations on several grounds in addition to the holographic description of quantum quenches via the AdS/CFT correspondence [?, ?], including general stability of maximally-symmetric solutions in general relativity [?, ?, ?], and the study of the growth of secular terms in time-dependent perturbation theories [?,?]. Numerical studies in holographic AdS show that the eventual collapse of a scalar field into a black hole in the bulk (which is dual to the thermalization of the boundary theory) is generic to any finite sized perturbation [?,?,?], but can be avoided or delayed for certain initial conditions [?,?,?,?]. The mechanism of collapse in such systems is described as a weakly turbulent energy cascade to short length scales. These dynamics can be captured by a non-linear perturbation theory at first non-trivial order through the introduction of a second, "slow time" that describes energy transfer between the fundamental modes. This is known as the Two-Time Formalism (TTF) [?] and yields a renormalization flow equation that allows for the absorption of secular terms into renormalized amplitudes and phases [?,?,?,?]. Therefore, stability against a perturbation of order ϵ is maintained over time scales of $t \sim \epsilon^{-2}$.

Conventional examinations of perturbative stability using TTF have focused on the reaction of the bulk space to some initial energy perturbation, and have aimed to study the balance between direct and inverse energy cascades [?,?,?,?,?]. Furthermore, numerical examinations of "pumped" scalars and their implications for thermalization of the dual theory have also been examined [?, ?, ?, ?, ?]. However, extensions of the perturbative description to include time-dependent sources – corresponding to a driving term on the boundary of the bulk space – remain unaddressed.

With this in mind, we examine the effects that a time-dependent source on the conformal boundary has on the non-linear perturbative theory. The introduction of a driving term on the boundary means that we must include a second class of fundamental modes with arbitrary frequencies. Since these solutions will have non-finite inner products over the bulk space, they are known as non-normalizable. Non-normalizable modes couple to both the source on the boundary and the regular normalizable modes to bring energy into the system, where direct and inverse energy cascades proceed over perturbative time scales.

To capture these dynamics, we expand the fields in powers of a small perturbation and isolate the secular terms that appear at third order in ϵ. Only modes whose frequencies satisfy certain resonance conditions will contribute terms that cannot be absorbed by simple frequency shifts. The form of the resonant terms depends on the specific physics of the system, as well as possible symmetries between frequencies. Finally, by evaluating the resonant third-order interactions when combinations of normalizable and non-normalizable modes are activated, we can write renormalization flow equations for the slowly varying amplitudes and phases.

This paper is organized as follows: section §1.3 involves a brief discussion of how we arrive at the third order source term, as well as additional considerations due to the time-dependent boundary condition. As an exercise – and to provide explicit expressions for the resonant contributions when the scalar field has non-zero mass – § 1.4 examines the secular terms in the case of a massive scalar field in AdS$_{d+1}$ with any mass-squared, up to and including the Breitenlohner-Freedman mass [?]: $m^2_{BF} \leq m^2$. We demonstrate the natural vanishing of two of the three resonances, and then examine the effects of mass-dependence on the non-vanishing channel. Whenever values are calculated, the choice of $d = 4$ is implied as to draw the most direct comparison to existing literature. In section § 1.5, we extend the boundary conditions to include a variety of periodic boundary sources that couple to non-normalizable modes in the bulk. For each choice of boundary condition, we derive analytic expressions for applicable resonances and evaluate these expressions for different ranges of scalar field masses. Finally, in § 1.6 we discuss the implications of non-vanishing resonances on the competing energy cascades, and the implications for the perturbative stability of such systems. For completeness, we include details of our derivation of the general source term in appendix 1.A, as well as a complete list of possible resonance channels and their resulting secular terms in appendix 1.B for the case of two, equal frequency non-normalizable modes.

1.3 Source Terms and Boundary Conditions

Let us first consider a minimally coupled, massive scalar field coupled to a spherically symmetric, asymptotically AdS$_{d+1}$ spacetime in global coordinates, whose metric is given by

$$ds^2 = \frac{L^2}{\cos(x)}\left(-A(t,x)e^{-2\delta(t,x)}\,dt^2 + A^{-1}(t,x)\,dx^2 + \sin^2(x)\,d\Omega^2_{d-1}\right)\,, \tag{1.1}$$

where L is the AdS curvature (hereafter set to 1), and the radial coordinate $x \in [0, \pi/2)$. The dynamics of the system come from the Einstein and Klein-Gordon equations:

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi\left(\nabla_\mu\phi\nabla_\nu\phi - \frac{1}{2}g_{\mu\nu}\left(\nabla^\rho\phi\nabla_\rho\phi + m^2\phi^2\right)\right) \quad \text{and} \quad \nabla^2\phi - m^2\phi = 0\,, \tag{1.2}$$

with the cosmological constant for AdS given by $\Lambda = -d(d-1)/2$.

Perturbing around static AdS, the scalar field is expanded in odd powers of epsilon

$$\phi(t,x) = \epsilon\phi_1(t,x) + \epsilon^3\phi_3(t,x) + \dots \tag{1.3}$$

and the metric functions A and δ in even powers,

$$A(t,x) = 1 + \epsilon^2 A_2(t,x) + \dots \tag{1.4}$$
$$\delta(t,x) = \epsilon^2 \delta_2(t,x) + \dots . \tag{1.5}$$

We choose to work in the boundary gauge, where $\delta(t, \pi/2) = 0$, for reasons that we discuss below.

At linear order, ϕ_1 satisfies

$$\partial_t^2 \phi_1 + \hat{L}\phi_1 = 0 \quad \text{where} \quad \hat{L} \equiv \frac{1}{\mu}(\mu' \partial_x + \mu \partial_x^2) - \frac{m^2}{\cos^2(x)}\,, \tag{1.6}$$

and $\mu \equiv \tan^{d-1}(x)$. The general solution for (1.6) in the bulk is a linear combination of the eigenfunctions $\Phi_I^{\pm}(x)$, whose frequencies ω_I are arbitrary. Examining each function's scaling when $x \to \pi/2$, we see that Φ_I^+ is normalizable and goes as $(\cos x)^{\Delta^+}$ while Φ_I^- is non-normalizable and goes as $(\cos x)^{\Delta^-}$. We denote the positive (negative) root of $\Delta(\Delta - d) = m^2$ as $\Delta^+(\Delta^-)$.

For an arbitrary frequency, requiring regularity at the origin means that we must choose the linear combination [?]

$$E_I(x) = K_I \left(\cos(x)\right)^{\Delta^+} {}_2F_1\left(\frac{\Delta^+ + \omega_I}{2}, \frac{\Delta^+ - \omega_I}{2}, d/2; \sin^2(x)\right), \tag{1.7}$$

that solves the eigenvalue equation

$$\hat{L}E_I(x) = \omega_I^2 E_I(x)\,. \tag{1.8}$$

For special integer values of the frequencies $\omega_I = \omega_i = 2i + \Delta^+$ with $i \in \mathbb{Z}^+$, the functions $\Phi_i^{\pm}(x)$ are individually regular at the origin. In this case, the normalizable part of the solution in (1.7) can be written as

$$E_I(x) = e_i(x) = k_i \left(\cos(x)\right)^{\Delta^+} P_i^{(d/2-1,\, \Delta^+ - d/2)}\left(\cos(2x)\right)\,, \tag{1.9}$$

with the Jacobi polynomials $P_n^{(a,b)}(x)$ providing an orthogonal basis so that $\langle e_i(x), e_j(x)\rangle = \delta_{ij}$ when

$$k_i = 2\sqrt{\frac{(i + \Delta^+/2)\Gamma(i+1)\Gamma(i+\Delta^+)}{\Gamma(i+d/2)\Gamma(i+\Delta^+ - d/2 + 1)}}\,. \tag{1.10}$$

For consistency with other frequency values, we choose to write the non-normalizable contributions in the general form of (1.7).

The interpretation of the driving term through the AdS/CFT dictionary is the addition of a time-dependent part of the boundary Hamiltonian. Therefore, the presence of non-normalizable modes corresponds to pumping energy in and out of the system. We will find it useful when calculating the third-order source term – which requires a triple sum over first-order modes – to be able to separate the contributions from either kind of mode. To that end, we write the first-order part of the scalar field as a sum over both normalizable and non-normalizable modes:

$$\begin{aligned}\phi_1(t,x) &= \sum_I c_I(t) E_I(x) \\ &= \sum_j a_j(t) \cos\left(\omega_i t + b_i(t)\right) e_j(x) + \sum_\alpha \bar{A}_\alpha \cos\left(\omega_\alpha t + \bar{B}_\alpha\right) E_\alpha(x)\,. \end{aligned} \tag{1.11}$$

The values of $\bar{A}_\alpha$ and $\bar{B}_\alpha$ will be set by the driving term. This informs our choice of working in the boundary gauge; the time t is the proper time measured on the boundary, as well as the time scale of oscillations from the driving term. In the simplest example, the driving term on the boundary is a single, periodic function

$$\phi_1(t, \pi/2) = \mathcal{A} \cos \varpi t \,. \tag{1.12}$$

In this case, (1.11) collapses into a single term so that

$$\sum_\alpha \bar{A}_\alpha \cos(\omega_\alpha t + \bar{B}_\alpha) E_\alpha(\pi/2) = \mathcal{A} \cos \varpi t \ \Rightarrow \ \bar{A}_\varpi \, E_\varpi(\pi/2) = \mathcal{A} \quad \text{and} \quad \bar{B}_\varpi = 0 \,. \tag{1.13}$$

Generalizing the boundary condition to a sum over Fourier modes would set further $\bar{A}_\alpha$ and $\bar{B}_\alpha$ to non-zero values.

Without specifying whether frequencies or basis functions have been chosen to be either normalizable or non-normalizable for the time being, we can show that the $\mathcal{O}(\epsilon^3)$ part of the scalar field satisfies the equation

$$\ddot{\phi}_3 + \hat{L}\phi_3 = S = 2(A_2 - \delta_2)\ddot{\phi}_1 + (\dot{A}_2 - \dot{\delta}_2)\dot{\phi}_1 + (A_2' - \delta_2')\phi_1' + m^2 A_2 \phi_1 \sec^2 x \,. \tag{1.14}$$

Following the steps outlined in appendix 1.A, we project (1.14) onto the basis of normalizable modes since all non-normalizable contributions have been fixed by the $\mathcal{O}(\epsilon)$ boundary condition. Employing a ubiquitous time-dependent solution $c_I(t) = a_I \cos(\omega_I t + b_I) = a_I \cos \theta_I$ with $I \in \{i, \alpha\}$, we find that the source term for the ℓ^{th} mode is

$$
\begin{aligned}
S_\ell = &\frac{1}{4}\sum_{\substack{I,J,K\\K\neq\ell}}^{\infty}\frac{a_I a_J a_K \omega_K}{\omega_l^2-\omega_K^2}\Big[Z^-_{IJK\ell}(\omega_I+\omega_J-2\omega_K)\cos(\theta_I+\theta_J-\theta_K)\\
&\quad - Z^-_{IJK\ell}(\omega_I+\omega_J+2\omega_K)\cos(\theta_I+\theta_J+\theta_K) + Z^+_{IJK\ell}(\omega_I-\omega_J+2\omega_K)\cos(\theta_I-\theta_J+\theta_K)\\
&\quad - Z^+_{IJK\ell}(\omega_I-\omega_J-2\omega_K)\cos(\theta_I-\theta_J-\theta_K)\Big]\\
&+\frac{1}{2}\sum_{\substack{I,J,K\\I\neq J}}^{\infty} a_I a_J a_K \omega_J\left(H_{IJK\ell}+m^2 V_{JKI\ell}-2\omega_K^2 X_{IJK\ell}\right)\left[\frac{1}{\omega_I-\omega_J}\big(\cos(\theta_I-\theta_J-\theta_K)\right.\\
&\quad\left.+\cos(\theta_I-\theta_J+\theta_K)\big)-\frac{1}{\omega_I+\omega_J}\big(\cos(\theta_I+\theta_J-\theta_K)+\cos(\theta_I+\theta_J+\theta_K)\big)\right]\\
&-\frac{1}{4}\sum_{I,J,K}^{\infty} a_I a_J a_K\Big[\left(2\omega_J\omega_K X_{IJK\ell}+m^2 V_{IJK\ell}\right)\cos(\theta_I+\theta_J-\theta_K)\\
&\quad-\left(2\omega_J\omega_K X_{IJK\ell}-m^2 V_{IJK\ell}\right)\cos(\theta_I-\theta_J-\theta_K)+\left(2\omega_J\omega_K X_{IJK\ell}+m^2 V_{IJK\ell}\right)\cos(\theta_I-\theta_J+\theta_K)\\
&\quad-\left(2\omega_J\omega_K X_{IJK\ell}-m^2 V_{IJK\ell}\right)\cos(\theta_I+\theta_J+\theta_K)\Big]\\
&+\frac{1}{4}\sum_{I,J}^{\infty} a_I a_J a_\ell \omega_l\Big[\tilde{Z}^-_{IJ\ell}(\omega_I+\omega_J-2\omega_l)\cos(\theta_I+\theta_J-\theta_\ell)-\tilde{Z}^-_{IJ\ell}(\omega_I+\omega_J+2\omega_l)\cos(\theta_I+\theta_J+\theta_\ell)\\
&\quad+\tilde{Z}^+_{IJ\ell}(\omega_I-\omega_J+2\omega_l)\cos(\theta_I-\theta_J+\theta_\ell)-\tilde{Z}^+_{IJ\ell}(\omega_I-\omega_J-2\omega_l)\cos(\theta_I-\theta_J-\theta_\ell)\Big]\\
&-\frac{1}{4}\sum_{I,J}^{\infty} a_I^2 a_J\Big[H_{IIJ\ell}+m^2 V_{JII\ell}-2\omega_J^2 X_{IIJ\ell}\Big]\big(\cos(2\theta_I-\theta_J)+\cos(2\theta_I+\theta_J)\big)\\
&-\frac{1}{2}\sum_{I,J}^{\infty} a_I^2 a_J\Big[H_{IIJ\ell}+m^2 V_{JII\ell}-2\omega_J^2 X_{IIJ\ell}+4\omega_I^2\omega_J^2 P_{J\ell I}+2\omega_I^2(M_{J\ell I}+m^2 Q_{J\ell I})\Big]\cos\theta_J. \qquad (1.15)
\end{aligned}
$$

Note that sums and restrictions on indices must be interpreted as sums and restrictions on *frequencies* when any of the modes is non-normalizable, since $\omega_\alpha \neq 2\alpha + \Delta^+$ in general.

As mentioned above, the growth of resonant terms with time, i.e. secular growth, at $\mathcal{O}(\epsilon^3)$ can be absorbed into the time-dependent part of the scalar field at that order [?]. Thus, (1.14) tells us that

$$
\ddot{c}_\ell^{(3)}(t)+\omega_l^2 c_\ell^{(3)}(t) = S_\ell^{(3)}\cos\left(\omega_l t+\varphi_\ell\right)\,, \qquad (1.16)
$$

where $S_\ell^{(3)}$ is a polynomial in a_I determined by evaluating the resonant contributions from (1.15), and φ_ℓ is some combination of the b_I. To obtain the renormalization flow equations, we can rewrite the amplitudes and phases in terms of renormalized integration constants that exactly cancel the secular terms at each instant. Doing so yields the renormalization flow equations for the renormalized constants [?]

$$\frac{2\omega_l}{\epsilon^2}\frac{da_\ell}{dt} = -S_\ell^{(3)}\sin\left(b_\ell - \varphi_\ell\right) \tag{1.17}$$

$$\frac{2\omega_l a_\ell}{\epsilon^2}\frac{db_\ell}{dt} = -S_\ell^{(3)}\cos\left(b_\ell - \varphi_\ell\right)\ . \tag{1.18}$$

Note that the amplitudes and phases evolve with respect to the "slow time" $\tau = \epsilon^2 t$. In practice, once these flow equations can be written down, the perturbative evolution of the system is determined up to a timescale of $t \sim \epsilon^{-2}$.

To determine the exact form of $S_\ell^{(3)}$, we must consider all combinations of the frequencies $\{\omega_I, \omega_J, \omega_K\}$ that satisfy the resonance condition

$$\omega_I \pm \omega_J \pm \omega_K = \pm\omega_l\,. \tag{1.19}$$

As an exercise, we first derive the resonant contributions when the boundary source is zero, and therefore only normalizable modes are present. These results agree numerically with previous work on normalizable modes for massless scalars in the interior time gauge ($\delta(t, 0) = 0$) [?]. The definitions of the functions Z, H, X, etc. in (1.15) differ slightly from other works – in part because of the gauge choice, and in part because of a desire to separate out mass-dependent terms – and so are given explicitly in appendix 1.A.

1.4 Resonances From Normalizable Solutions

Consider the case where each of the basis functions are given by normalizable solutions. The possible combinations of frequencies that satisfy (1.19) can be separated into the three distinct cases:

$$\omega_i + \omega_j + \omega_k = \omega_l \qquad (+++) \tag{1.20}$$

$$\omega_i - \omega_j - \omega_k = \omega_l \qquad (+--) \tag{1.21}$$

$$\omega_i + \omega_j - \omega_k = \omega_l \qquad (++-)\,. \tag{1.22}$$

Note that the $(+++)$ and $(+--)$ resonances produce restrictions on the allowed values of the indices $\{i, j, k\}$, as well as on values of the mass, since $\omega_i = 2i + \Delta^+$. In the first case, the indices are restricted by $i + j + k = \ell - \Delta^+$, and so Δ^+ must be an integer and greater than ℓ for resonance to occur. Similarly, the $(+--)$ resonance condition becomes $i - j - k = \ell + \Delta^+$, which is resonant for any integer value of Δ^+. We will see that these two resonance channels will non-trivially vanish whenever their respective resonance conditions are satisfied. This is in agreement with the results shown for the massless scalar in the interior time gauge (as they must be, since the choice of time gauge should not change the existence of resonant channels). Here we include the expressions for the naturally vanishing resonances, choosing to explicitly express the mass dependence.

1.4.1 Naturally Vanishing Resonances: $(+++)$ and $(+--)$

Resonant contributions that come from the condition $\omega_i + \omega_j + \omega_k = \omega_l$ contribute to the total source term via

$$S_\ell = \underbrace{\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}}_{\omega_i+\omega_j+\omega_k=\omega_l} \Omega_{ijk\ell}\, a_i a_j a_k \cos\left(\theta_i + \theta_j + \theta_k\right) + \dots\,, \tag{1.23}$$

where the ellipsis denotes other resonances. $\Omega_{ijk\ell}$ is given by

$$\begin{aligned}\Omega_{ijk\ell} = &-\frac{1}{12}H_{ijk\ell}\frac{\omega_j(\omega_i+\omega_k+2\omega_j)}{(\omega_i+\omega_j)(\omega_j+\omega_k)} - \frac{1}{12}H_{ikj\ell}\frac{\omega_k(\omega_i+\omega_j+2\omega_k)}{(\omega_i+\omega_k)(\omega_j+\omega_k)} - \frac{1}{12}H_{jik\ell}\frac{\omega_i(\omega_j+\omega_k+2\omega_i)}{(\omega_i+\omega_j)(\omega_i+\omega_k)} \\ &- \frac{m^2}{12}V_{ijk\ell}\left(1+\frac{\omega_j}{\omega_j+\omega_k}+\frac{\omega_i}{\omega_i+\omega_k}\right) - \frac{m^2}{12}V_{jki\ell}\left(1+\frac{\omega_j}{\omega_i+\omega_j}+\frac{\omega_k}{\omega_i+\omega_k}\right) \\ &- \frac{m^2}{12}V_{kij\ell}\left(1+\frac{\omega_i}{\omega_i+\omega_j}+\frac{\omega_k}{\omega_j+\omega_k}\right) + \frac{1}{6}\omega_j\omega_k X_{ijk\ell}\left(1+\frac{\omega_j}{\omega_i+\omega_k}+\frac{\omega_k}{\omega_i+\omega_j}\right) \\ &+ \frac{1}{6}\omega_i\omega_k X_{jki\ell}\left(1+\frac{\omega_i}{\omega_j+\omega_k}+\frac{\omega_k}{\omega_i+\omega_j}\right) + \frac{1}{6}\omega_i\omega_j X_{kij\ell}\left(1+\frac{\omega_i}{\omega_j+\omega_k}+\frac{\omega_j}{\omega_i+\omega_k}\right) \\ &- \frac{1}{12}Z^-_{ijk\ell}\left(\frac{\omega_k}{\omega_i+\omega_j}\right) - \frac{1}{12}Z^-_{ikj\ell}\left(\frac{\omega_j}{\omega_i+\omega_k}\right) - \frac{1}{12}Z^-_{jki\ell}\left(\frac{\omega_i}{\omega_j+\omega_k}\right). \end{aligned} \tag{1.24}$$

The second naturally vanishing resonance comes from the condition $\omega_i - \omega_j - \omega_k = \omega_l$, and contributes to the total source term via

$$S_\ell = \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \Gamma_{(j+k+\ell+\Delta^+)jk\ell}\, a_j a_k a_{(j+k+\ell+\Delta^+)} \cos\left(\theta_{(j+k+\ell+\Delta^+)} - \theta_j - \theta_k\right) + \dots, \tag{1.25}$$

where

$$\begin{aligned}\Gamma_{ijk\ell} = &\frac{1}{4}H_{ijk\ell}\frac{\omega_j(\omega_k-\omega_i+2\omega_j)}{(\omega_i-\omega_j)(\omega_j+\omega_k)} + \frac{1}{4}H_{jki\ell}\frac{\omega_k(\omega_j-\omega_i+2\omega_k)}{(\omega_i-\omega_k)(\omega_j+\omega_k)} + \frac{1}{4}H_{kij\ell}\frac{\omega_i(\omega_j+\omega_k-2\omega_i)}{(\omega_i-\omega_j)(\omega_i-\omega_k)} \\ &- \frac{1}{2}\omega_j\omega_k X_{ijk\ell}\left(\frac{\omega_k}{\omega_i-\omega_j}+\frac{\omega_j}{\omega_i-\omega_k}-1\right) + \frac{1}{2}\omega_i\omega_k X_{jki\ell}\left(\frac{\omega_k}{\omega_i-\omega_j}+\frac{\omega_i}{\omega_j+\omega_k}-1\right) \\ &+ \frac{1}{2}\omega_i\omega_j X_{kij\ell}\left(\frac{\omega_j}{\omega_i-\omega_k}+\frac{\omega_i}{\omega_j+\omega_k}-1\right) + \frac{m^2}{4}V_{jki\ell}\left(\frac{\omega_j}{\omega_i-\omega_j}+\frac{\omega_k}{\omega_i-\omega_k}-1\right) \\ &- \frac{m^2}{4}V_{kij\ell}\left(\frac{\omega_i}{\omega_i-\omega_j}+\frac{\omega_k}{\omega_j+\omega_k}+1\right) - \frac{m^2}{4}V_{ijk\ell}\left(\frac{\omega_i}{\omega_i-\omega_k}+\frac{\omega_j}{\omega_j+\omega_k}+1\right) \\ &+ \frac{1}{4}Z^-_{kji\ell}\left(\frac{\omega_i}{\omega_j+\omega_k}\right) - \frac{1}{4}Z^+_{ijk\ell}\left(\frac{\omega_k}{\omega_i-\omega_j}\right) - \frac{1}{4}Z^+_{jki\ell}\left(\frac{\omega_j}{\omega_i-\omega_k}\right). \end{aligned} \tag{1.26}$$

Building on the work done with massless scalars, we are able to show numerically that (1.24) and (1.26) continue to vanish for massive scalars ($m^2 \geq m^2_{BF}$) in the boundary gauge; thus, the dynamics governing the weakly turbulent transfer of energy are determined only from the remaining resonance channel. When non-normalizable modes are introduced, we will see that naturally vanishing resonances are not present and so the total third-order source term is the sum over all resonant channels.

1.4.2 Non-vanishing Resonance: $(++-)$

The first non-vanishing contributions arise when $\omega_i + \omega_j = \omega_k + \omega_l$. This contribution can be split into three coefficients that are evaluated for certain subsets of the allowed values for the indices, namely

$$S_\ell = T_\ell a_\ell^3 \cos(\theta_\ell + \theta_\ell - \theta_\ell) + \sum_{i\neq\ell}^{\infty} R_{i\ell}\, a_i^2 a_\ell \cos(\theta_i + \theta_\ell - \theta_i)$$
$$+ \sum_{i\neq\ell}^{\infty}\sum_{j\neq\ell}^{\infty} S_{ij(i+j-\ell)\ell}\, a_i a_j a_{(i+j-\ell)} \cos(\theta_i + \theta_j - \theta_{i+j-\ell}) \,, \tag{1.27}$$

where the coefficients are given by

$$S_{ijk\ell} = -\frac{1}{4}H_{kij\ell}\frac{\omega_i(\omega_j-\omega_k+2\omega_i)}{(\omega_i-\omega_k)(\omega_i+\omega_j)} - \frac{1}{4}H_{ijk\ell}\frac{\omega_j(\omega_i-\omega_k+2\omega_j)}{(\omega_j-\omega_k)(\omega_i+\omega_j)} - \frac{1}{4}H_{jki\ell}\frac{\omega_k(\omega_i+\omega_j-2\omega_k)}{(\omega_i-\omega_k)(\omega_j-\omega_k)}$$
$$- \frac{1}{2}\omega_j\omega_k X_{ijk\ell}\left(\frac{\omega_j}{\omega_i-\omega_k} - \frac{\omega_k}{\omega_i+\omega_j} + 1\right) - \frac{1}{2}\omega_i\omega_k X_{jki\ell}\left(\frac{\omega_i}{\omega_j-\omega_k} - \frac{\omega_k}{\omega_i+\omega_j} + 1\right)$$
$$+ \frac{1}{2}\omega_i\omega_j X_{kij\ell}\left(\frac{\omega_i}{\omega_j-\omega_k} + \frac{\omega_j}{\omega_i-\omega_k} + 1\right) - \frac{m^2}{4}V_{ijk\ell}\left(\frac{\omega_i}{\omega_i-\omega_k} + \frac{\omega_j}{\omega_j-\omega_k} + 1\right)$$
$$+ \frac{m^2}{4}V_{jki\ell}\left(\frac{\omega_k}{\omega_i-\omega_k} - \frac{\omega_j}{\omega_i+\omega_j} - 1\right) + \frac{m^2}{4}V_{kij\ell}\left(\frac{\omega_k}{\omega_j-\omega_k} - \frac{\omega_i}{\omega_i+\omega_j} - 1\right)$$
$$+ \frac{1}{4}Z^-_{ijk\ell}\left(\frac{\omega_k}{\omega_i+\omega_j}\right) + \frac{1}{4}Z^+_{ikj\ell}\left(\frac{\omega_j}{\omega_i-\omega_k}\right) + \frac{1}{4}Z^+_{jki\ell}\left(\frac{\omega_i}{\omega_j-\omega_k}\right) \,, \tag{1.28}$$

$$R_{i\ell} = \left(\frac{\omega_i^2}{\omega_l^2-\omega_i^2}\right)\left(Y_{i\ell\ell i} - Y_{i\ell i\ell} + \omega_l^2(X_{i\ell i\ell} - X_{\ell i\ell i})\right) + \left(\frac{\omega_i^2}{\omega_l^2-\omega_i^2}\right)\left(H_{\ell ii\ell} + m^2 V_{ii\ell\ell} - 2\omega_i^2 X_{\ell ii\ell}\right)$$
$$- \left(\frac{\omega_l^2}{\omega_l^2-\omega_i^2}\right)\left(H_{i\ell i\ell} + m^2 V_{\ell ii\ell} - 2\omega_i^2 X_{i\ell i\ell}\right) - \frac{m^2}{4}(V_{i\ell i\ell} + V_{ii\ell\ell}) + \omega_i^2\omega_l^2(P_{ii\ell} - 2P_{\ell\ell i})$$
$$- \omega_i\omega_l X_{i\ell i\ell} - \frac{3m^2}{2}V_{\ell ii\ell} - \frac{1}{2}H_{ii\ell\ell} + \omega_l^2 B_{ii\ell} - \omega_i^2 M_{\ell\ell i} - m^2\omega_i^2 Q_{\ell\ell i} \,, \tag{1.29}$$

and

$$T_\ell = \frac{1}{2}\omega_l^2\left(X_{\ell\ell\ell\ell} + 4B_{\ell\ell\ell} - 2M_{\ell\ell\ell} - 2m^2 Q_{\ell\ell\ell}\right) - \frac{3}{4}\left(H_{\ell\ell\ell\ell} + 3m^2 V_{\ell\ell\ell\ell}\right) \,. \tag{1.30}$$

Following the form of (1.17) - (1.18), these resonant terms set the evolution of the renormalized integration coefficients to be [?]

$$\frac{2\omega_l}{\epsilon^2}\frac{da_\ell}{dt} = -\sum_{i\neq\ell}^{\infty}\sum_{j\neq\ell}^{\infty} S_{ij(i+j-\ell)\ell}\, a_i a_j a_{(i+j-\ell)} \sin(b_\ell + b_{(i+j-\ell)} - b_i - b_j) \,, \tag{1.31}$$

$$\frac{2\omega_l a_\ell}{\epsilon^2}\frac{db_\ell}{dt} = -T_\ell a_\ell^3 - \sum_{i\neq\ell}^{\infty} R_{i\ell}\, a_i^2 a_\ell$$
$$- \sum_{i\neq\ell}^{\infty}\sum_{j\neq\ell}^{\infty} S_{ij(i+j-\ell)\ell}\, a_i a_j a_{(i+j-\ell)} \cos(b_\ell + b_{(i+j-\ell)} - b_i - b_j) \,. \tag{1.32}$$

To examine the effects of non-zero masses on R, S, and T, we evaluate (1.28)-(1.30) for tachyonic, massless, and massive scalars in figure 1.1. The result is a vertical shift in the coefficient value that is proportional to the choice of mass-squared. By inspection, there is an indication that this shift increases with increasing ℓ values; however, a numerical fit of the data would be needed to claim this definitively.

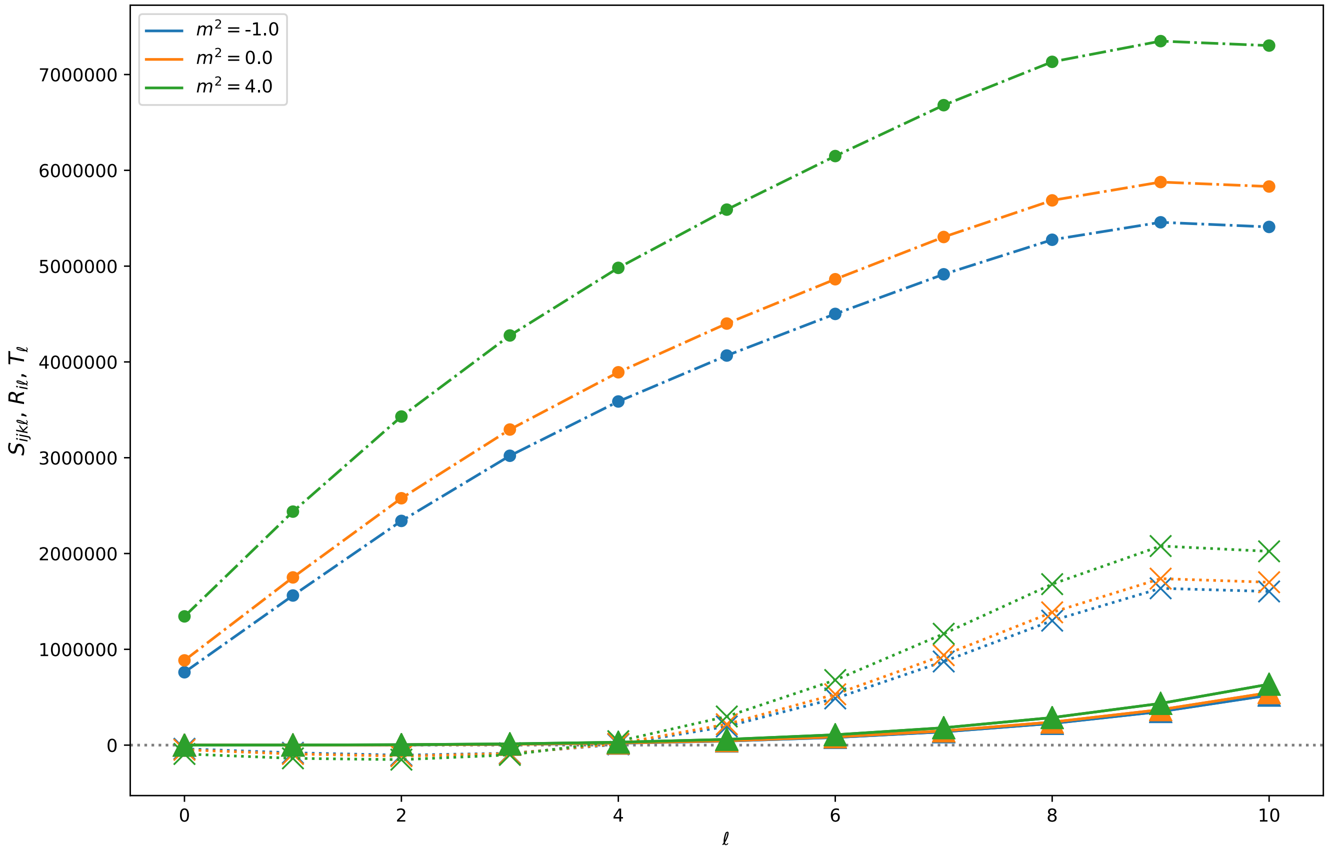

Figure 1.1: *Evaluating* (1.28)-(1.30) *over different values of* m^2 *for* $\ell \leq 10$. $S_{ij(i+j-\ell)\ell}$ *is denoted by filled circles connected by dash-dotted lines;* $R_{i\ell}$ *is denoted by filled triangles connected by solid lines;* T_ℓ *is denoted by large Xs connected by dotted lines. Different values of* m^2 *are denoted by the colour of each series.*

1.5 Resonances From Non-normalizable Modes

Now let us consider the excitation of non-normalizable modes by a driving term on the boundary of AdS. Having set ω_l to be a normalizable mode, we may ask what restrictions exist on our choices for the other frequencies, $\{\omega_i, \omega_j, \omega_k\}$. Aside from the trivial case where all modes are normalizable, we could imagine that one of the modes is non-normalizable. However, this would violate the resonance condition (1.19); thus, at least two modes must be non-normalizable. When three non-normalizable modes exist, there are two possibilities: first, that any combination of generically non-integer frequencies gives a non-integer value and so does not contribute a secular term when projected onto the ω_ℓ basis; second, some particular combination of the non-normalizable frequencies gives an integer frequency, in which case there are resonant contributions to $S_\ell^{(3)}$. Therefore, the pertinent question is what resonances are possible when two of $\{\omega_i, \omega_j, \omega_j\}$ are non-normalizable? Because this choice breaks some of the symmetries that contributed to the previous expressions for resonance channels, the resonance conditions must be re-examined starting from the source expression (1.15).

Before proceeding further, an important consideration is what the effect of non-normalizable modes are on the perturbative expansion that leads to the source equations. Since non-normalizable solutions do not have well-defined norms, we do not know *a priori* that the inner products described in appendix 1.A are still finite. To investigate this, consider the generic expression for the second-order metric function

$$A_2 = -\nu \int_0^x dy\, \mu \left((\dot{\phi}_1)^2 + (\phi_1')^2 + m^2 \phi_1^2 \sec^2 x \right) , \tag{1.33}$$

in the limit of $x \to \pi/2$, and let the scalar field ϕ_1 be given by a generic superposition of normalizable and non-normalizable eigenfunctions as in (1.11). Ignoring the time-dependent contributions, we find that

$$\lim_{\tilde{x}\to 0} A_2(\tilde{x} \equiv \pi/2 - x) = \tilde{x}^{-\xi} \left(\frac{2\tilde{x}^{2+d}}{2-\xi} - \frac{\tilde{x}^d (1 + (\Delta^-)^2)}{\xi} \right) , \tag{1.34}$$

where we have defined $\xi = \sqrt{d^2 + 4m^2}$. In the massless case, $\xi = d$ and all powers of $\tilde{x}$ are non-negative; thus, the limit is finite. For tachyonic masses, $m_{BF}^2 < m^2 < 0$ so that $0 < \xi < d$ and the limit is again finite. However, when $m^2 > 0$, part of the limit diverges. In order for the boundary to remain asymptotically AdS, counter-terms in the bulk action would be required to cancel such divergences – a case we will not address presently. Furthermore, for masses that saturate the Breitenlohner-Freedman bound, the limit would have to be re-evaluated. We will therefore restrict our discussion to $m_{BF}^2 < m^2 \leq 0$ to avoid these issues. A similar check on the near-boundary behaviour of δ_2 shows that the gauge condition $\delta_2(t, \pi/2) = 0$ remains unchanged by the addition of non-normalizable modes given the same restrictions on the mass of the scalar field. With these restrictions in mind, let us now examine the resonances produced by the activation of non-normalizable modes.

1.5.1 Two Non-normalizable Modes with Equal Frequencies

As a first case, let us assume that the two non-normalizable modes have equal, constant, and arbitrary frequencies, $\overline{\omega}$ (and therefore amplitudes $\bar{A}_{\overline{\omega}}$). The resonance condition (1.19) will only be satisfied when one of $\{\omega_I, \omega_I, \omega_K\}$ are normalizable. In particular, we find that the following combinations are resonant:

$$\omega_i - \omega_j + \omega_k - \omega_l = 0 \quad \Rightarrow \quad \text{either } \omega_i \text{ or } \omega_k \text{ is normalizable} \tag{1.35}$$
$$\omega_i + \omega_j - \omega_k - \omega_l = 0 \quad \Rightarrow \quad \text{either } \omega_i \text{ or } \omega_j \text{ is normalizable} \tag{1.36}$$
$$\omega_i - \omega_j - \omega_k + \omega_l = 0 \quad \Rightarrow \quad \text{either } \omega_j \text{ or } \omega_k \text{ is normalizable.} \tag{1.37}$$

When any of these resonance conditions is met, the remaining normalizable mode will have a frequency equal to ω_l, collapsing all sums over frequencies so that

$$S_\ell = \overline{T}_\ell \, a_\ell \bar{A}_{\varpi}^2 \cos(\theta_\ell) + \dots \,, \tag{1.38}$$

where the amplitudes of the non-normalizable modes $\bar{A}_{\varpi}$ are set by the choice of boundary condition. Collecting the appropriate terms in (1.15) and evaluating each possible resonance, we find that

$$\begin{aligned}\overline{T}_\ell = \Bigg[& \frac{1}{2} Z^-_{\ell\varpi\varpi\ell}\left(\frac{\varpi}{\omega_l + \varpi}\right) + \frac{1}{2} Z^+_{\ell\varpi\varpi\ell}\left(\frac{\varpi}{\omega_l - \varpi}\right) + H_{\ell\varpi\varpi\ell}\left(\frac{\varpi^2}{\omega_l^2 - \varpi^2}\right) - H_{\varpi\ell\varpi\ell}\left(\frac{\omega_l^2}{\omega_l^2 - \varpi^2}\right) \\ & - m^2 V_{\ell\varpi\varpi\ell}\left(\frac{\omega_l^2}{\omega_l^2 - \varpi^2}\right) + m^2 V_{\varpi\varpi\ell\ell}\left(\frac{\varpi^2}{\omega_l^2 - \varpi^2}\right) + 2X_{\varpi\varpi\ell\ell}\left(\frac{\varpi^2\omega_l^2}{\omega_l^2 - \varpi^2}\right) - 2X_{\ell\ell\varpi\varpi}\left(\frac{\varpi^4}{\omega_l^2 - \varpi^2}\right)\Bigg]_{\varpi \neq \omega_l} \\ & + \omega_l^2 X_{\varpi\varpi\ell\ell} - \varpi^2 X_{\ell\ell\varpi\varpi} - \frac{3}{2} m^2 V_{\ell\ell\varpi\varpi} - \frac{1}{2} m^2 V_{\varpi\varpi\ell\ell} - \frac{1}{2} H_{\varpi\varpi\ell\ell} + \omega_l^2 \tilde{Z}^+_{\varpi\varpi\ell} - 2\varpi^2\omega_l^2 P_{\ell\ell\varpi} \\ & - \varpi^2\left(\omega_l^2 P_{\ell\ell\varpi} - B_{\ell\ell\varpi}\right) \,. \end{aligned} \tag{1.39}$$

Notice that the terms in the square braces only contribute when $\varpi \neq \omega_l$. Beginning from (1.15), only terms in the square braces that are proportional to $Z^{\pm}$ are limited in this way; the remaining terms have no such restriction. However, it can be shown that integral functions with permuted indices are equal when the non-normalizable frequency equals the normalizable frequency. Upon simplification, factors of $\omega_l^2 - \varpi^2$ are cancelled, and the overall contribution to T_ℓ from the terms in the braces is zero. Thus, these terms are grouped with those that have natural restrictions on the indices.

With the resonant contributions determined, the renormalization flow equations for two equal, constant, non-normalizable frequencies follow from (1.17) - (1.18) and are

$$\frac{2\omega_l}{\epsilon^2}\frac{da_\ell}{dt} = 0 \quad \text{and} \quad \frac{2\omega_l a_\ell}{\epsilon^2}\frac{db_\ell}{dt} = -\overline{T}_\ell a_\ell \bar{A}_{\varpi}^2 \,. \tag{1.40}$$

Qualitatively, we see that instead of both the amplitude and the phase running with respect to τ, only the phase changes in time. Indeed, (1.40) tells us that b_ℓ is a linear function of τ with a slope that is determined by the $\mathcal{O}(\epsilon^3)$ physics encapsulated by $\overline{T}_\ell$.

Other resonant contributions become possible for more restrictive values of the non-normalizable frequency, such as if ϖ is allowed to be an integer. These contributions are denoted by the ellipsis in (1.38) and are listed in appendix 1.B. In figures 1.2 and 1.3, we evaluate (1.39) for $\ell < 10$ over a variety of ϖ values first for a massless scalar, then for a tachyonic scalar. For both values of mass-squared, T_ℓ demonstrates power law-type behaviour as a function of ℓ with a leading coefficient that is proportional to the non-normalizable frequency ϖ. We also see that the limit of (1.39) as $\varpi \to \omega_0$ is well-defined in both cases.

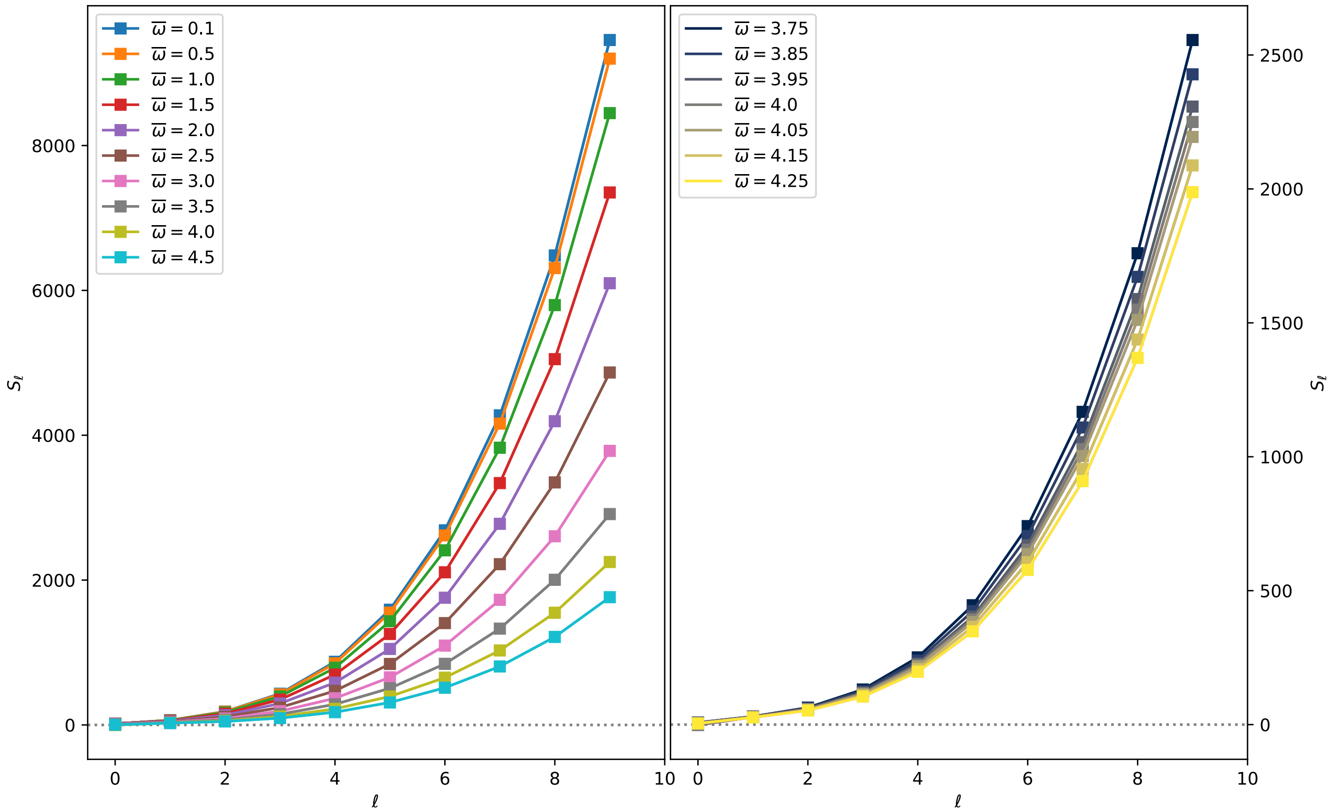

Figure 1.2: *Left: Evaluating* (1.39) *when* $m^2 = 0$ *for various choices of* $\overline{\omega}$*. Right: The behaviour of* S_ℓ *for* $\overline{\omega}$ *values near* ω_0*.*

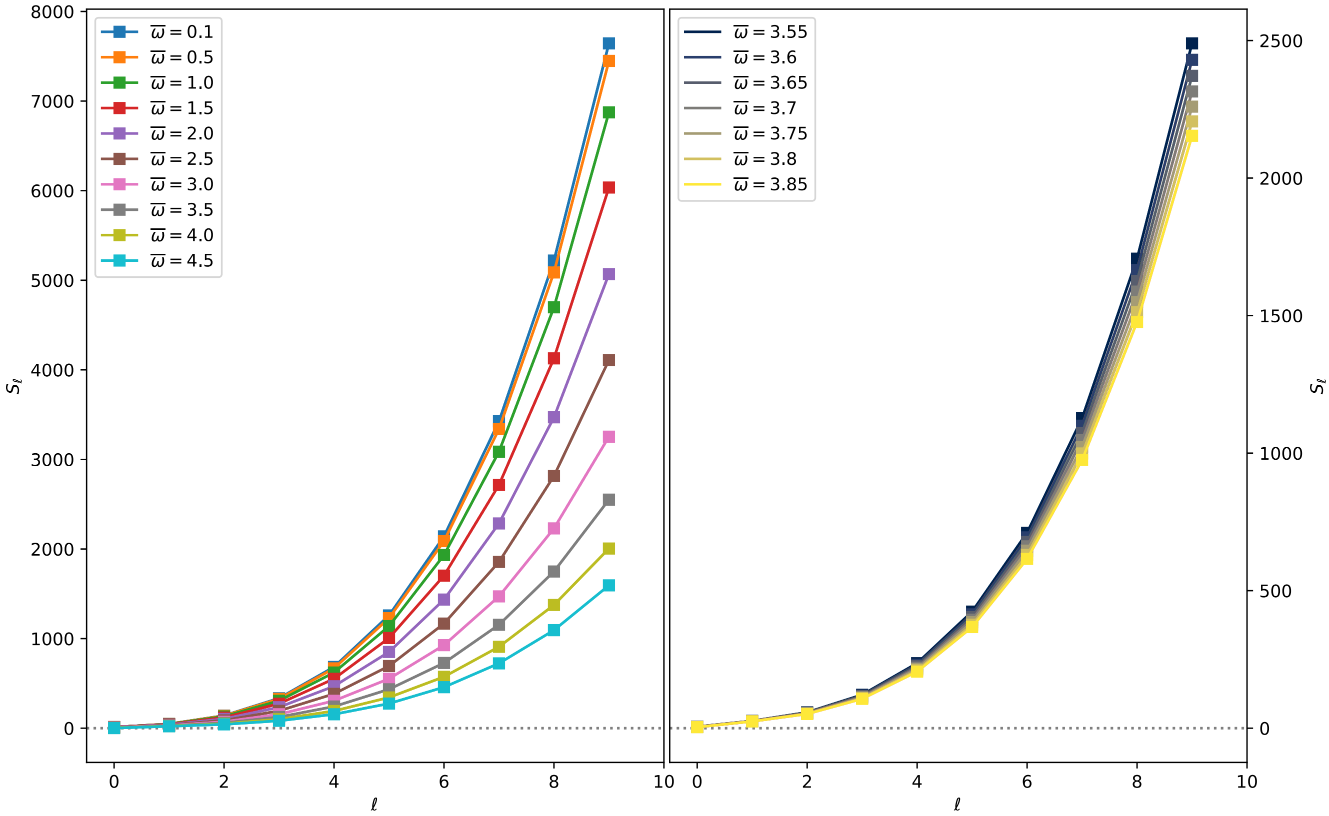

Figure 1.3: *Left: Evaluating $\overline{T}_\ell$ for a tachyon with $m^2 = -1.0$. Right: The behaviour of S_ℓ near $\omega_0 = \Delta^+ \approx 3.7$.*

1.5.2 Special Values of Non-normalizable Frequencies

Let us now consider special values of non-normalizable frequencies that will lead to a greater number of resonance channels. While general non-normalizable frequencies do not require any such restrictions, we will find it informative to examine these special cases as they possess more symmetry in index/frequency values than the case of equal non-normalizable frequencies, but less than all-normalizable modes.

Add to an integer

First, we choose two of the modes to be non-normalizable with frequencies $\overline{\omega}_1$ and $\overline{\omega}_2$ that add to give an integer: $\overline{\omega}_1 + \overline{\omega}_2 = 2n$ where $n = 1, 2, 3, \ldots$ (note that the $n = 0$ case means that both $\overline{\omega}_1$ and $\overline{\omega}_2$ would need to be zero by the positive-frequency requirement and so would not contribute). Furthermore, either frequency need not be an integer and therefore the difference $|\overline{\omega}_1 - \overline{\omega}_2|$ will, in general, not be an integer. In § 1.5.3, we examine the case when the difference of non-normalizable frequencies is an integer.

When we consider possible resonance channels, we see that resonances can be grouped into

$$(++) : \ \omega_i + 2n = \omega_\ell \quad \forall\, \ell \geq n \tag{1.41}$$
$$(+-) : \ \omega_i - 2n = \omega_\ell \quad \forall\, n \tag{1.42}$$

for any $m_{BF}^2 < m^2 < 0$. However, for a massless scalar, we have an additional channel

$$(-+): \ -\omega_i + 2n = \omega_\ell \quad \forall\, n \geq \ell + d\,. \tag{1.43}$$

Adding the channels together, the total source term is

$$\begin{aligned} S_\ell = & \sum_{\overline{\omega}_1+\overline{\omega}_2=2n} \left[\Theta\left(n-\ell-d\right) \overline{R}^{(-+)}_{(n-\ell-d)\ell}\, \bar{A}_1 \bar{A}_2\, a_{(n-\ell-d)} \cos\left(\theta_{(n-\ell-d)} - \theta_1 - \theta_2\right)\right]_{m^2=0} \\ & + \sum_{\overline{\omega}_1+\overline{\omega}_2=2n} \Theta\left(\ell-n\right) \overline{R}^{(++)}_{(\ell-n)\ell}\, \bar{A}_1 \bar{A}_2\, a_{(\ell-n)} \cos\left(\theta_{(\ell-n)} + \theta_1 + \theta_2\right) \\ & + \sum_{\overline{\omega}_1+\overline{\omega}_2=2n} \overline{R}^{(+-)}_{(\ell+n)\ell}\, \bar{A}_1 \bar{A}_2\, a_{(\ell+n)} \cos\left(\theta_{(\ell+n)} - \theta_1 - \theta_2\right) \\ & + \overline{T}_\ell\, \bar{A}_1 \bar{A}_2\, a_\ell \cos\left(\theta_\ell\right) \end{aligned} \tag{1.44}$$

where the Heaviside step function $\Theta(x)$ enforces the restrictions on the indices in (1.41) and (1.43) and $\theta_1 = \overline{\omega}_1 t + \bar{B}_1$, etc.

In the following expressions, the sum over all $\overline{\omega}_1$, $\overline{\omega}_2$ such that $\overline{\omega}_1 + \overline{\omega}_2 = 2n$ is implied, and only the restrictions on individual frequencies are included. Examining each channel in (1.44) individually, we find

$$\begin{aligned} \overline{R}^{(++)}_{i\ell} = & -\frac{1}{4} \sum_{\overline{\omega}_2 \neq \omega_l} \frac{\overline{\omega}_2}{\omega_l - \overline{\omega}_2} Z^-_{i12\ell} - \frac{1}{4} \sum_{\overline{\omega}_1 \neq \omega_l} \frac{\overline{\omega}_1}{\omega_l - \overline{\omega}_1} Z^-_{i21\ell} - \frac{1}{8n} \left(\omega_l - 2n\right) Z^-_{12i\ell} \\ & - \frac{1}{4} \sum_{\omega_i \neq \overline{\omega}_1} \frac{1}{\omega_l - \overline{\omega}_2} \Big[\overline{\omega}_1 \left(H_{i12\ell} + m^2 V_{12i\ell} - 2\overline{\omega}_2^2 X_{i12\ell}\right) + \left(\omega_l - 2n\right) \left(H_{1i2\ell} + m^2 V_{i21\ell} - 2\overline{\omega}_2^2 X_{1i2\ell}\right) \Big] \\ & - \frac{1}{4} \sum_{\omega_i \neq \overline{\omega}_2} \frac{1}{\omega_l - \overline{\omega}_1} \Big[\overline{\omega}_2 \left(H_{i21\ell} + m^2 V_{21i\ell} - 2\overline{\omega}_1^2 X_{i21\ell}\right) + \left(\omega_l - 2n\right) \left(H_{2i1\ell} + m^2 V_{i12\ell} - 2\overline{\omega}_1^2 X_{2i1\ell}\right) \Big] \\ & - \frac{1}{8n} \sum_{\overline{\omega}_1 \neq \overline{\omega}_2} \Big[\overline{\omega}_1 H_{21i\ell} + \overline{\omega}_2 H_{12i\ell} + m^2 \left(\overline{\omega}_1 V_{1i2\ell} + \overline{\omega}_2 V_{2i1\ell}\right) - \left(\omega_l - 2n\right)^2 \left(\overline{\omega}_1 X_{21i\ell} + \overline{\omega}_2 X_{12i\ell}\right) \Big] \\ & + \frac{1}{2} \Big[\overline{\omega}_1 \overline{\omega}_2 X_{i12\ell} + \left(\omega_l - 2n\right) \left(\overline{\omega}_1 X_{21i\ell} + \overline{\omega}_2 X_{12i\ell}\right) - \frac{m^2}{2} \left(V_{i12\ell} + V_{i21\ell} + V_{12i\ell}\right) \Big]\,. \end{aligned} \tag{1.45}$$

The notation $X_{i12\ell}$ corresponds to evaluating $X_{ijk\ell}$ with $\omega_j = \overline{\omega}_1$ and $\omega_k = \overline{\omega}_2$. Next, we find that

$$\begin{aligned} \overline{R}^{(+-)}_{i\ell} = & -\frac{1}{4} \Big[\frac{(\omega_l + 2n)}{2n} Z^-_{12i\ell} + 2(\omega_l + 2n) \left(\overline{\omega}_1 X_{21i\ell} + \overline{\omega}_2 X_{12i\ell}\right) \\ & - \frac{\overline{\omega}_1}{(\omega_l + \overline{\omega}_2)} \left(H_{i12\ell} + m^2 V_{12i\ell} - 2\overline{\omega}_2^2 X_{i12\ell}\right) + \frac{(\omega_l + 2n)}{(\omega_l + \overline{\omega}_2)} \left(H_{1i2\ell} + m^2 V_{i21\ell} - 2\overline{\omega}_2^2 X_{1i2\ell}\right) \\ & - \frac{\overline{\omega}_2}{(\omega_l + \overline{\omega}_1)} \left(H_{i21\ell} + m^2 V_{21i\ell} - 2\overline{\omega}_1^2 X_{i21\ell}\right) + \frac{(\omega_l + 2n)}{(\omega_l + \overline{\omega}_1)} \left(H_{2i1\ell} + m^2 V_{i12\ell} - 2\overline{\omega}_1^2 X_{2i1\ell}\right) \\ & - 2\overline{\omega}_1 \overline{\omega}_2 X_{i12\ell} + m^2 \left(V_{12i\ell} + V_{i12\ell} + V_{i21\ell}\right) \Big] + \frac{1}{4} \sum_{\overline{\omega}_2 \neq \omega_l} \frac{\overline{\omega}_1 \overline{\omega}_2 (\omega_l + 2n)}{\omega_l + \overline{\omega}_2} \left(X_{21i\ell} - X_{\ell i12}\right) \\ & + \frac{1}{4} \sum_{\overline{\omega}_1 \neq \omega_l} \frac{\overline{\omega}_1 \overline{\omega}_2 (\omega_l + 2n)}{\omega_l + \overline{\omega}_1} \left(X_{12i\ell} - X_{\ell i12}\right)\,. \end{aligned} \tag{1.46}$$

When $m^2 = 0$, we have contributions from

$$\begin{aligned}
\overline{R}_{i\ell}^{(-+)} &= \frac{1}{4}\sum_{\overline{\omega}_2 \neq \omega_l} \frac{\overline{\omega}_2}{\omega_l - \overline{\omega}_2} Z^+_{i12\ell} + \frac{1}{4}\sum_{\overline{\omega}_1 \neq \omega_l} \frac{\overline{\omega}_1}{\omega_l - \overline{\omega}_1} Z^+_{i21\ell} + \frac{1}{4}\sum_{i\neq \ell}\left(\frac{2n-\omega_l}{2n}\right) Z^-_{12i\ell} \\
&\quad + \frac{1}{4}\sum_{\overline{\omega}_1 \neq \omega_i} \frac{1}{\omega_i - \overline{\omega}_1}\Big[\overline{\omega}_1\left(H_{i12\ell} - 2\overline{\omega}_2^2 X_{i12\ell}\right) - (2n-\omega_l)\left(H_{1i2\ell} - 2\overline{\omega}_2^2 X_{1i2\ell}\right)\Big] \\
&\quad + \frac{1}{4}\sum_{\overline{\omega}_2 \neq \omega_i} \frac{1}{\omega_i - \overline{\omega}_2}\Big[\overline{\omega}_2\left(H_{i21\ell} - 2\overline{\omega}_1^2 X_{i21\ell}\right) - (2n-\omega_l)\left(H_{2i1\ell} - 2\overline{\omega}_1^2 X_{2i1\ell}\right)\Big] \\
&\quad - \frac{1}{8n}\sum_{\overline{\omega}_1 \neq \overline{\omega}_2}\Big[\overline{\omega}_1 H_{21i\ell} + \overline{\omega}_2 H_{12i\ell} - 2\left(2n-\omega_l\right)^2\left(\overline{\omega}_1 X_{21i\ell} + \overline{\omega}_2 X_{12i\ell}\right)\Big] \\
&\quad - \frac{1}{2}\Big[(2n-\omega_l)\left(\overline{\omega}_1 X_{21i\ell} + \overline{\omega}_2 X_{12i\ell}\right) - \overline{\omega}_1\overline{\omega}_2 X_{i12\ell}\Big]. \qquad (1.47)
\end{aligned}$$

NB. In (1.47) only, $\omega_i = 2i + \Delta^+ = 2i + d$ since this term requires that $m^2 = 0$ to contribute. We maintain the same notation out of convenience, despite the special case. Finally,

$$\begin{aligned}
\overline{T}_\ell &= \frac{1}{2}\omega_l^2\left(\tilde{Z}^+_{11\ell} + \tilde{Z}^+_{22\ell}\right) - \frac{1}{2}\Big[H_{11\ell\ell} + H_{22\ell\ell} + m^2\left(V_{\ell 11\ell} + V_{\ell 22\ell}\right) - 2\omega_l^2\left(X_{11\ell\ell} + X_{22\ell\ell}\right) \\
&\quad + 4\omega_l^2\left(\overline{\omega}_1^2 P_{\ell\ell 1} + \overline{\omega}_2^2 P_{\ell\ell 2}\right) + 2\overline{\omega}_1^2 M_{\ell\ell 1} + 2\overline{\omega}_2^2 M_{\ell\ell 2} + 2m^2\left(\overline{\omega}_1^2 Q_{\ell\ell 1} + \overline{\omega}_2^2 Q_{\ell\ell 2}\right)\Big]. \qquad (1.48)
\end{aligned}$$

In figure 1.4, we compute the total source term (modulo the amplitudes a_i and $\bar{A}_\alpha$) for a tachyonic scalar with $n = 2$. Figure 1.5 provides a comparison between the value of the source term for a massless scalar between two choices of n: one that includes contributions from $\overline{R}_{i\ell}^{(-+)}$ and one that does not. As expected, the source terms are symmetric in $\overline{\omega}_1 \leftrightarrow \overline{\omega}_2$, hence only $\overline{\omega}_1 \leq n$ data are shown. As a function of ℓ, (1.44) starts near zero before becoming increasingly negative as ℓ becomes large. As a check for naturally vanishing channels, the absolute value of the sum of S_ℓ is also plotted; however, there is no indication that any channel vanishes for any of the $\overline{\omega}_1$, $\overline{\omega}_2$ values considered.

The renormalization flow equations include the sum of all the channels (none of which vanish naturally), and are

$$\begin{aligned}
\frac{2\omega_l}{\epsilon^2}\frac{da_\ell}{dt} &= -\sum_{\overline{\omega}_1+\overline{\omega}_2=2n}\Big[\Theta\left(n-\ell-d\right)\overline{R}^{(-+)}_{(n-\ell-d)\ell}\,\bar{A}_1\bar{A}_2\,a_{(n-\ell-d)}\sin\left(b_{(n-\ell-d)} - \bar{B}_1 - \bar{B}_2\right)\Big]_{m^2=0} \\
&\quad - \sum_{\overline{\omega}_1+\overline{\omega}_2=2n}\Theta\left(\ell-n\right)\overline{R}^{(++)}_{(\ell-n)\ell}\,\bar{A}_1\bar{A}_2\,a_{(\ell-n)}\sin\left(b_{(\ell-n)} + \bar{B}_1 + \bar{B}_2\right) \\
&\quad - \sum_{\overline{\omega}_1+\overline{\omega}_2=2n}\overline{R}^{(+-)}_{(\ell+n)\ell}\,\bar{A}_1\bar{A}_2\,a_{(\ell+n)}\sin\left(b_{(\ell+n)} - \bar{B}_1 - \bar{B}_2\right), \qquad (1.49)
\end{aligned}$$

and

$$\begin{aligned}
\frac{2\omega_l a_\ell}{\epsilon^2}\frac{db_\ell}{dt} &= -\sum_{\overline{\omega}_1+\overline{\omega}_2=2n}\Big[\Theta\left(n-\ell-d\right)\overline{R}^{(-+)}_{(n-\ell-d)\ell}\,\bar{A}_1\bar{A}_2\,a_{(n-\ell-d)}\cos\left(b_{(n-\ell-d)} - \bar{B}_1 - \bar{B}_2\right)\Big]_{m^2=0} \\
&\quad - \sum_{\overline{\omega}_1+\overline{\omega}_2=2n}\Theta\left(\ell-n\right)\overline{R}^{(++)}_{(\ell-n)\ell}\,\bar{A}_1\bar{A}_2\,a_{(\ell-n)}\cos\left(b_{(\ell-n)} + \bar{B}_1 + \bar{B}_2\right) \\
&\quad - \sum_{\overline{\omega}_1+\overline{\omega}_2=2n}\overline{R}^{(+-)}_{(\ell+n)\ell}\,\bar{A}_1\bar{A}_2\,a_{(\ell+n)}\cos\left(b_{(\ell+n)} - \bar{B}_1 - \bar{B}_2\right) - \overline{T}_\ell\,\bar{A}_1\bar{A}_2\,a_\ell. \qquad (1.50)
\end{aligned}$$

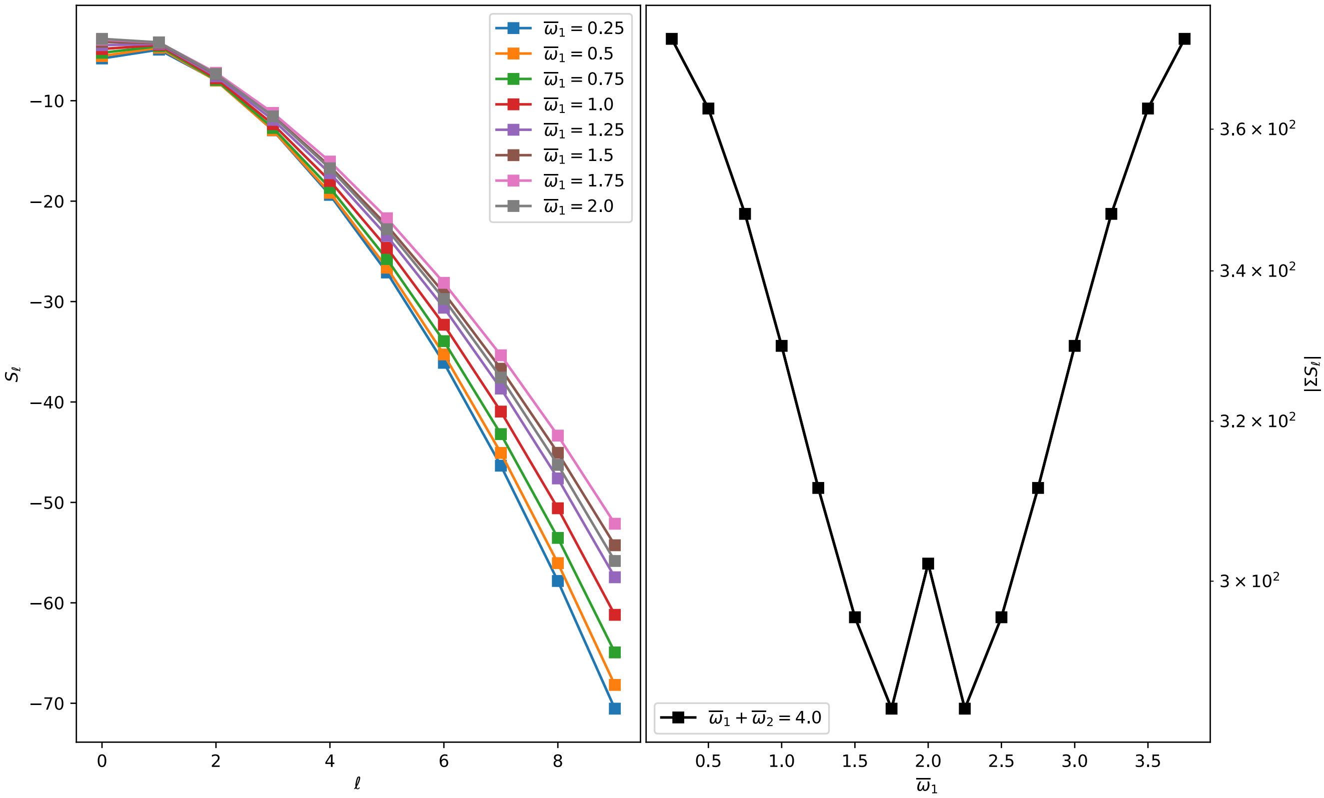

Figure 1.4: *Left: Source term values for a tachyonic scalar with* $m^2 = -1.0$ *when the frequencies of non-normalizable modes sum to* 4.0*. Right: The absolute value of the sum of the source terms for each choice of* $\overline{\omega}_1$, $\overline{\omega}_2$.

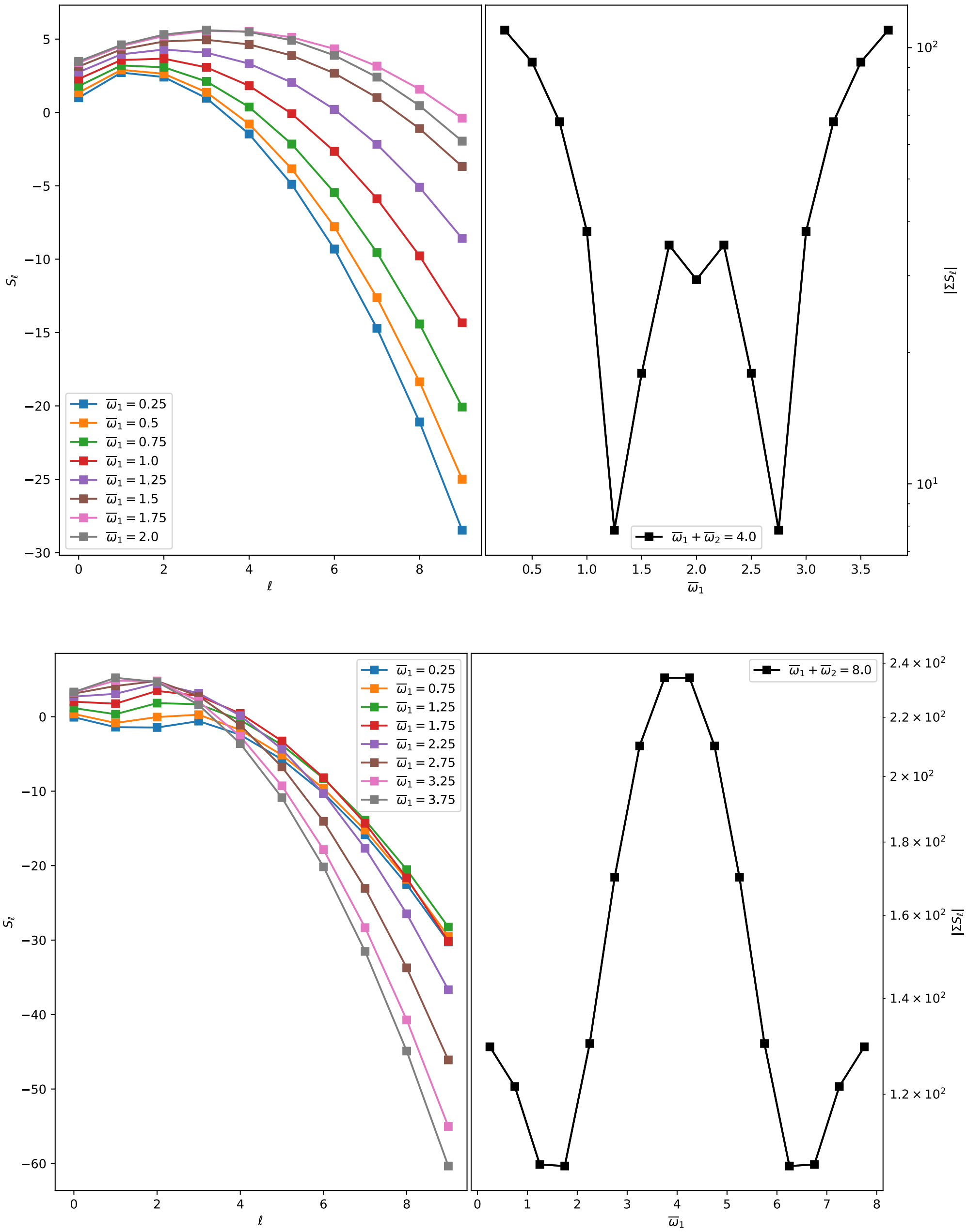

Figure 1.5: *Above: The value of* (1.44) *as a function of ℓ for a massless scalar with values of $\overline{\omega}_1$ and $\overline{\omega}_2$ chosen so that $\overline{\omega}_1 + \overline{\omega}_2 = 4$. Below: The same plot but with values chosen to satisfy $\overline{\omega}_1 + \overline{\omega}_2 = 8$.*

1.5.3 Integer Plus χ

Finally, let us consider the case where the non-normalizable frequencies are non-integer, but differ from integer values by a set amount. In analogue to the case where all modes are normalizable, we consider the non-normalizable frequencies to be shifted away from integer values by

$$\omega_\gamma = 2\gamma + \chi \,, \tag{1.51}$$

where $\gamma \in \mathbb{Z}^+$ (greek letters are chosen to differentiate these non-normalizable modes from normalizable modes with integer frequencies, which use roman letters). We furthermore limit χ to be non-integer[1] and set $m^2 = 0$ throughout. For this choice of non-normalizable frequencies there are no resonant contributions from the all-plus channel, unlike the naturally vanishing resonance found in § 1.4.1. Only when either $\omega_i + \omega_\gamma = \omega_\beta - \omega_l$, or $\omega_i + \omega_\gamma = \omega_\beta + \omega_l$ with $i + \gamma \geq \ell$, are resonant terms present. Let us examine each case separately.

$\omega_i + \omega_\gamma = \omega_\beta - \omega_l$

When the resonance condition $\omega_i + \omega_\gamma = \omega_\beta - \omega_l$ is met, the contribution to the source term is of the form

$$S_\ell = \sum_{i \neq \ell} \sum_{\gamma \neq \beta} \overline{S}^{(1)}_{i(i+\gamma+\ell)\gamma\ell}\, a_i \bar{A}_{(i+\gamma+\ell)} \bar{A}_\gamma \cos\left(\theta_i - \theta_{(i+\gamma+\ell)} + \theta_\gamma\right) + \sum_\beta \overline{R}^{(1)}_{\beta\ell}\, a_\ell \bar{A}^2_\beta \cos\left(\theta_\ell + \theta_\beta - \theta_\beta\right) + \dots\,, \tag{1.52}$$

where

$$\begin{aligned}
\overline{S}^{(1)}_{i\beta\gamma\ell} = {} & \frac{1}{4} H_{\beta\gamma i\ell} \frac{\omega_\gamma(\omega_i - \omega_\beta + 2\omega_\gamma)}{(\omega_\beta - \omega_\gamma)(\omega_i + \omega_\gamma)} - \frac{1}{4} H_{\gamma\beta i\ell} \frac{\omega_\beta(\omega_i + \omega_\gamma - 2\omega_\beta)}{(\omega_i - \omega_\beta)(\omega_\beta - \omega_\gamma)} - \frac{1}{4} H_{\gamma i\beta\ell} \frac{\omega_i(\omega_\gamma - \omega_\beta + 2\omega_i)}{(\omega_i - \omega_\beta)(\omega_i + \omega_\gamma)} \\
& + \frac{1}{2}\omega_i\omega_\gamma X_{\beta\gamma i\ell}\left(\frac{\omega_\gamma}{\omega_i - \omega_\beta} - \frac{\omega_i}{\omega_\beta + \omega_\gamma} + 1\right) + \frac{1}{2}\omega_i\omega_\beta X_{\gamma\beta i\ell}\left(\frac{\omega_i}{\omega_\beta - \omega_\gamma} + \frac{\omega_\beta}{\omega_i + \omega_\gamma} - 1\right) \\
& + \frac{1}{2}\omega_\beta\omega_\gamma X_{i\beta\gamma\ell}\left(\frac{\omega_\beta}{\omega_i + \omega_\gamma} - \frac{\omega_\gamma}{\omega_i - \omega_\beta} - 1\right) - \frac{1}{4} Z^+_{\beta\gamma i\ell}\left(\frac{\omega_i}{\omega_i + \omega_l}\right) \\
& + \frac{1}{4} Z^-_{i\gamma\beta\ell}\left(\frac{\omega_\beta}{\omega_l - \omega_\beta}\right) + \frac{1}{4} Z^+_{i\beta\gamma\ell}\left(\frac{\omega_\gamma}{\omega_l + \omega_\gamma}\right)\,,
\end{aligned} \tag{1.53}$$

and

$$\begin{aligned}
\overline{R}^{(1)}_{\beta\ell} = {} & \frac{1}{4} Z^-_{\ell\beta\beta\ell}\left(\frac{\omega_\beta}{\omega_l + \omega_\beta}\right) + \frac{1}{4} Z^+_{\ell\beta\beta\ell}\left(\frac{\omega_\beta}{\omega_l - \omega_\beta}\right) + \frac{1}{2} H_{\ell\beta\beta\ell}\left(\frac{\omega_\beta^2}{\omega_l^2 - \omega_\beta^2}\right) - \frac{1}{2} H_{\beta\ell\beta\ell}\left(\frac{\omega_l^2}{\omega_l^2 - \omega_\beta^2}\right) \\
& + X_{\beta\ell\beta\ell}\left(\frac{\omega_l^4}{\omega_l^2 - \omega_\beta^2}\right) - \frac{1}{2}\omega_\beta^2 X_{\ell\beta\beta\ell}\left(\frac{\omega_l^2 + \omega_\beta^2}{\omega_l^2 - \omega_\beta^2}\right) - \frac{1}{2} H_{\ell\beta\beta\ell} + \omega_l^2 \tilde{Z}^+_{\beta\beta\ell} - 2\omega_\beta^2\omega_l^2 P_{\ell\ell\beta} - \omega_\beta^2 M_{\ell\ell\beta}\,.
\end{aligned} \tag{1.54}$$

$\omega_i + \omega_\gamma = \omega_\beta + \omega_l$

Similarly, when the resonance condition $\omega_i + \omega_\gamma = \omega_\beta + \omega_l$ is met, the contribution to the source term is

$$S_\ell = \underbrace{\sum_{i \neq \ell} \sum_{\gamma \neq \beta}}_{i+\gamma \geq \ell} \overline{S}^{(2)}_{i(i+\gamma-\ell)\gamma\ell}\, a_i \bar{A}_{(i+\gamma-\ell)} \bar{A}_\gamma \cos\left(\theta_i - \theta_{(i+\gamma-\ell)} + \theta_\gamma\right) + \sum_\beta \overline{R}^{(2)}_{\beta\ell}\, a_\ell \bar{A}^2_\beta \cos\left(\theta_\ell + \theta_\beta - \theta_\beta\right) + \dots\,, \tag{1.55}$$

[1] Indeed, for integer values of χ, the sum or difference of two non-normalizable modes could be an integer. This would either be covered by the work in § 1.5.2, or be a slight variation of it.

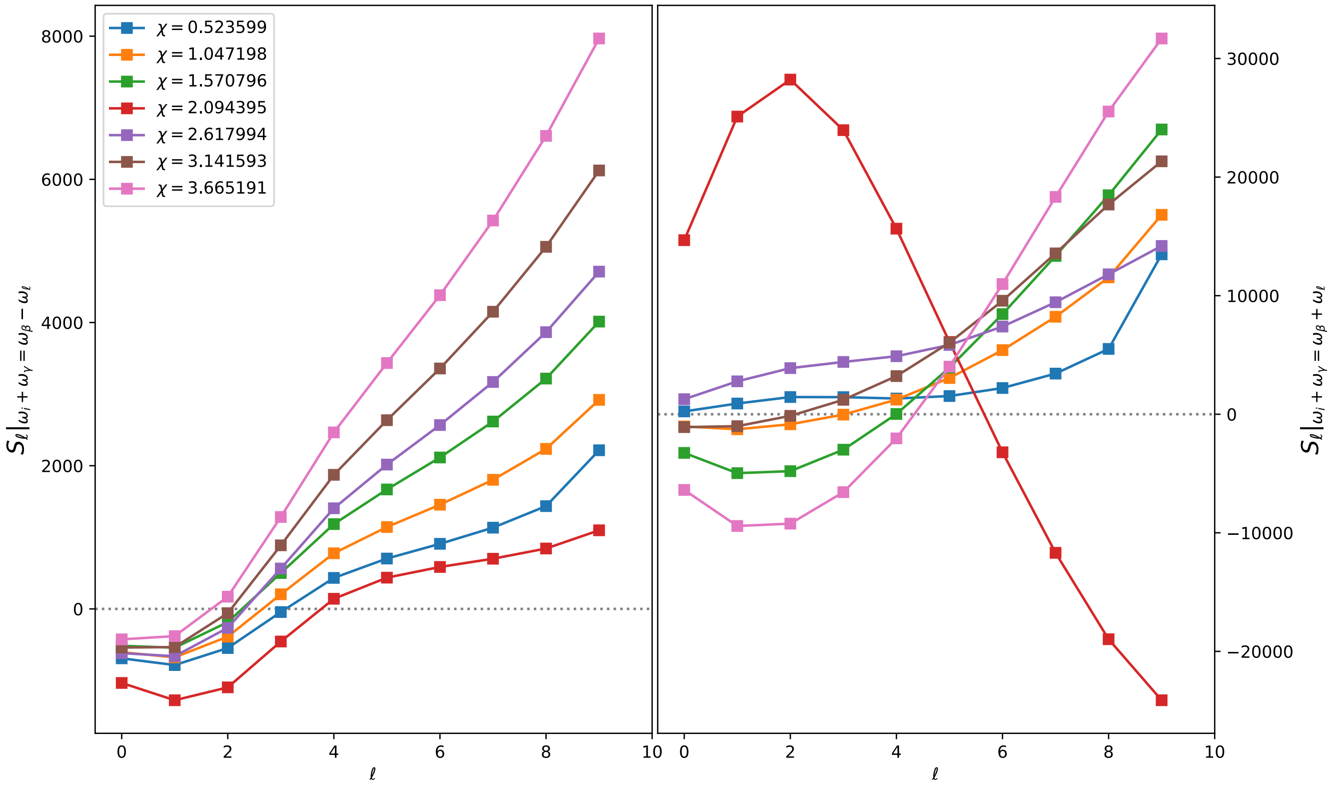

Figure 1.6: *Left: Evaluating the source term (1.52) for various values of χ for $\ell < 10$. Right: Evaluating the source term (1.55) subject to $i + \gamma \geq \ell$ for the same values of χ and the same range of ℓ.*

where

$$\begin{aligned}
\overline{S}^{(2)}_{i\beta\gamma\ell} &= \frac{1}{4} H_{\beta\gamma i\ell} \frac{\omega_\gamma(\omega_i - \omega_\beta)}{(\omega_\beta - \omega_\gamma)(\omega_i - \omega_\gamma)} - \frac{1}{4} H_{\gamma\beta i\ell} \frac{\omega_\beta(\omega_l - \omega_\beta)}{(\omega_\beta - \omega_\gamma)(\omega_i - \omega_\beta)} + \frac{1}{4} H_{\beta i\gamma\ell} \frac{\omega_i(\omega_\gamma - \omega_\beta)}{(\omega_i - \omega_\beta)(\omega_i - \omega_\gamma)} \\
&+ \frac{1}{2}\omega_i\omega_\gamma X_{\beta\gamma i\ell}\left(\frac{\omega_\gamma}{\omega_i - \omega_\beta} - \frac{\omega_i}{\omega_\beta - \omega_\gamma} + 1\right) + \frac{1}{2}\omega_i\omega_\beta X_{\gamma\beta i\ell}\left(\frac{\omega_i}{\omega_\beta - \omega_\gamma} - \frac{\omega_\beta}{\omega_i - \omega_\gamma} - 1\right) \\
&+ \frac{1}{2}\omega_\beta\omega_\gamma X_{i\beta\gamma\ell}\left(\frac{\omega_\beta}{\omega_i - \omega_\gamma} - \frac{\omega_\gamma}{\omega_i - \omega_\beta} - 1\right) + \frac{1}{4} Z^-_{i\gamma\beta\ell}\left(\frac{\omega_\beta}{\omega_l + \omega_\beta}\right) \\
&+ \frac{1}{4} Z^+_{i\beta\gamma\ell}\left(\frac{\omega_\gamma}{\omega_l - \omega_\gamma}\right) - \frac{1}{4} Z^+_{\beta\gamma i\ell}\left(\frac{\omega_i}{\omega_i - \omega_l}\right), \qquad (1.56)
\end{aligned}$$

and

$$\begin{aligned}
\overline{R}^{(2)}_{\beta\ell} &= \frac{1}{4} Z^-_{\ell\beta\beta\ell}\left(\frac{\omega_\beta}{\omega_l + \omega_\beta}\right) + \frac{1}{4} Z^+_{\ell\beta\beta\ell}\left(\frac{\omega_\beta}{\omega_l - \omega_\beta}\right) + \frac{1}{2} H_{\ell\beta\beta\ell}\left(\frac{\omega_\beta^2}{\omega_l^2 - \omega_\beta^2}\right) - \frac{1}{2} H_{\beta\ell\beta\ell}\left(\frac{\omega_l^2}{\omega_l^2 - \omega_\beta^2}\right) \\
&+ X_{\beta\beta\ell\ell}\left(\frac{\omega_l^2}{\omega_l^2 - \omega_\beta^2}\right) + \frac{1}{2}\omega_\beta^2 X_{\ell\beta\beta\ell}\left(\frac{\omega_l^2 + \omega_\beta^2}{\omega_l^2 - \omega_\beta^2}\right) - \frac{1}{2} H_{\beta\beta\ell\ell} + \omega_l^2 \tilde{Z}^+_{\beta\beta\ell} - 2\omega_\beta^2\omega_l^2 P_{\ell\ell\beta} - \omega_\beta^2 M_{\ell\ell\beta}. \qquad (1.57)
\end{aligned}$$

Unlike the case with all normalizable modes where two of the three resonance channels naturally vanished, both of the resonant channels contribute when the non-normalizable modes have frequencies given by (1.51). Therefore, the renormalization flow equations will contain contributions from both channels:

$$\begin{aligned}\frac{2\omega_l}{\epsilon^2}\frac{da_\ell}{dt} = &-\sum_{i\neq\ell}\sum_{\gamma\neq\beta}\overline{S}^{(1)}_{i(i+\gamma+\ell)\gamma\ell}\,a_i\bar{A}_{(i+\gamma+\ell)}\bar{A}_\gamma\sin\left(b_\ell+\bar{B}_{(i+\gamma+\ell)}-b_i-\bar{B}_\gamma\right)\\ &-\underbrace{\sum_{i\neq\ell}\sum_{\gamma\neq\beta}}_{i+\gamma\geq\ell}\overline{S}^{(2)}_{i(i+\gamma-\ell)\gamma\ell}\,a_i\bar{A}_{(i+\gamma-\ell)}\bar{A}_\gamma\sin\left(b_\ell+\bar{B}_{(i+\gamma-\ell)}-b_i-\bar{B}_\gamma\right)\,,\end{aligned} \tag{1.58}$$

$$\begin{aligned}\frac{2\omega_l a_\ell}{\epsilon^2}\frac{db_\ell}{dt} = &-\sum_{\beta}\overline{R}^{(1)}_{\beta\ell}\,a_\ell\bar{A}^2_\beta-\sum_{\beta}\overline{R}^{(2)}_{\beta\ell}\,a_\ell\bar{A}^2_\beta\\ &-\sum_{i\neq\ell}\sum_{\gamma\neq\beta}\overline{S}^{(1)}_{i(i+\gamma+\ell)\gamma\ell}\,a_i\bar{A}_{(i+\gamma+\ell)}\bar{A}_\gamma\cos\left(b_\ell+\bar{B}_{(i+\gamma+\ell)}-b_i-\bar{B}_\gamma\right)\\ &-\underbrace{\sum_{i\neq\ell}\sum_{\gamma\neq\beta}}_{i+\gamma\geq\ell}\overline{S}^{(2)}_{i(i+\gamma-\ell)\gamma\ell}\,a_i\bar{A}_{(i+\gamma-\ell)}\bar{A}_\gamma\cos\left(b_\ell+\bar{B}_{(i+\gamma-\ell)}-b_i-\bar{B}_\gamma\right)\,.\end{aligned} \tag{1.59}$$

In figure 1.6, we evaluate both resonant contributions channels' and plot their contributions for various values of χ. In particular, we examine the values $\chi\in\{\pi/6,\ldots,7\pi/6\}$. Again, there is no indication of any channel vanishing naturally. Interestingly, both sources demonstrate anomalous behaviour when $\chi\sim 2$ for reasons that are not immediately clear. The source term (1.52) is generally more positive for larger χ except for $\chi=2\pi/3$, which is translated negatively with respect to the source terms produced by other χ values. Again, when (1.55) is evaluated for $\chi=2\pi/3$, the result differs significantly from other choices of χ: seemingly reflected through the x axis with respect to other results. The significance of the choice $\chi=2\pi/3\sim d/2$ is possibly explained by the non-normalizable modes being *nearly* equal to the normalizable ones. In this event, S_ℓ would contain additional terms, such as those present in § 1.4. The departure of the $\chi=2\pi/3$ data from other data sets is perhaps a signal of these missing resonances.

1.6 Discussion

We have seen that the inclusion of a time-dependent boundary term in the holographic dual of a quantum quench allows energy to enter the bulk spacetime through coupling with non-normalizable modes. The dynamics of the weakly turbulent energy cascades that trigger instability were captured by secular terms at third-order that could not be removed by frequency shifts alone. Using the Two-Time Formalism, we have determined the renormalization group flow equations for the slowly varying amplitudes and phases that are tuned to cancel the secular terms that give rise to instability.

Unlike when only normalizable modes are considered, the introduction of non-normalizable modes results in no naturally vanishing resonance channels for the frequencies considered. The flow equations for a_ℓ and b_ℓ are now linear, since the non-normalizable amplitudes and phases are set by the first-order boundary condition and thus remain constant. In practice, this means the evolution of the system will be different than in the case where only normalizable modes are activated. Furthermore, periodic pumping of energy into and out of the bulk theory will undoubtably add interesting dynamics to the evolution already observed for quasi-periodic solutions with static boundary conditions [**?**].

With the renormalization flow equations established, future work will examine whether equilibrium solutions can be derived. Then, general non-collapsing solutions will be constructed out of perturbations of the equilibrium solutions and their numerical evolution will be examined. Comparisons to established numerical pumped solutions in the full theory may be instructive in understanding the space of stable and nearly-stable data.

Properties of the boundary CFT can also be determined from the perturbative theory in the bulk. For instance, the AdS/CFT dictionary relates the leading coefficient of the normalizable modes of the scalar field at the boundary to the expectation value of an operator $\langle \mathcal{O}_\phi \rangle$; the leading part of the non-normalizable modes are related to a time-dependent driving term in the boundary Hamiltonian $s(t)$. The Ward identity for time translations gives the time dependence of the energy density in the CFT in terms of these quantities

$$\partial_t \langle T_{tt} \rangle = -\partial_t s(t) \langle \mathcal{O}_\phi \rangle \,. \tag{1.60}$$

The evolution of the energy density can then be examined via the slowly varying amplitude and phase variables and compared with fully numeric results.

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Appendix

1.A Derivation of Source Terms For Massive Scalars

The derivation of the general expression for the $\mathcal{O}(\epsilon^3)$ source term for massive scalars closely follows the massless case, particularly if one chooses not to write out the explicit mass dependence as was done in [?]. However, since we have chosen to write our equations in a slightly different way – and in a different gauge – than previous authors, one may find it instructive to see the differences in the derivations. Below we have included the intermediate steps involved in deriving the third-order source term S_ℓ.

Continuing the expansion of the equations of motion in powers of ϵ, we see that the backreaction between the metric and the scalar field appears at second order in the perturbation,

$$A_2' = -\mu\nu\left[(\dot{\phi}_1)^2 + (\phi_1')^2 + m^2\phi_1^2\sec^2 x\right] + \nu' A_2/\nu\,, \tag{1.61}$$

which can be directly integrated to give

$$A_2 = -\nu\int_0^x dy\,\mu\left((\dot{\phi}_1)^2 + (\phi_1')^2 + m^2\phi_1^2\sec^2 x\right)\,. \tag{1.62}$$

For convenience, we have also defined the functions

$$\mu(x) = (\tan x)^{d-1} \quad \text{and} \quad \nu(x) = (d-1)/\mu'\,. \tag{1.63}$$

Similarly, the first non-trivial contribution to the lapse (in the boundary time gauge) is

$$\delta_2 = \int_x^{\pi/2} dy\,\mu\nu\left((\dot{\phi}_1)^2 + (\phi_1')^2\right)\,. \tag{1.64}$$

Projecting each of the terms in (1.14) individually onto the eigenbasis $\{e_\ell\}$ will involve evaluating inner products involving multiple integrals. To aide in evaluating these expressions, it is useful to derive several identities. First, from the equation for the scalar field's time-dependent coefficients c_i,

$$\ddot{c}_i + \omega_i^2 c_i = 0 \quad \Rightarrow \quad \partial_t\left(\dot{c}_i^2 + \omega_i^2 c_i^2\right) = \partial_t \mathbb{C}_i = 0\,. \tag{1.65}$$

Next, from the definition of $\hat{L}$,

$$\hat{L}e_j = -\frac{1}{\mu}\left(\mu e_j'\right)' + m^2\sec^2 x e_j \quad \Rightarrow \quad \left(\mu e_j'\right)' = \mu\left(m^2\sec^2 x - \omega_j^2\right)e_j\,. \tag{1.66}$$

By considering the expression $(\mu e_i' e_j)'$, we see that

$$(\mu e_i' e_j)' = \left(m^2 \sec^2 x - \omega_i^2\right) \mu e_i e_j + \mu e_i' e_j' \,, \tag{1.67}$$

which, after permuting i, j and subtracting from above, gives

$$\frac{\left[\mu(e_i' e_j \omega_j^2 - e_i e_j' \omega_i^2)\right]'}{(\omega_j^2 - \omega_i^2)} = \mu m^2 \sec^2 x e_i e_j + \mu e_i' e_j' \,. \tag{1.68}$$

Using these identities, we evaluate each of the inner products and find that

$$\begin{aligned}
\langle \delta_2 \ddot{\phi}_1, e_\ell \rangle &= -\sum_{i=0}^{\infty} \sum_{\substack{j=0 \\ k\neq\ell}}^{\infty} \sum_{k=0}^{\infty} \frac{\omega_k^2 c_k}{\omega_l^2 - \omega_k^2} \left[\dot{c}_i \dot{c}_j \left(X_{k\ell ij} - X_{\ell kij}\right) + c_i c_j \left(Y_{ij\ell k} - Y_{ijk\ell}\right)\right] \\
&\quad - \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \omega_l^2 c_\ell \left[\dot{c}_i \dot{c}_j P_{ij\ell} + c_i c_j B_{ij\ell}\right] \,, && (1.69) \\
\langle A_2 \ddot{\phi}_1, e_\ell \rangle &= 2 \sum_{i=0}^{\infty} \sum_{\substack{j=0 \\ i\neq j}}^{\infty} \sum_{k=0}^{\infty} \frac{\omega_k^2 c_k}{\omega_j^2 - \omega_i^2} X_{ijk\ell} \left(\dot{c}_i \dot{c}_j + \omega_j^2 c_i c_j\right) \\
&\quad + \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \omega_j^2 c_j \left(\mathbb{C}_i P_{j\ell i} + c_i^2 X_{iij\ell}\right) \,, && (1.70) \\
\langle \dot{\delta}_2 \dot{\phi}_1, e_\ell \rangle &= \sum_{i=0}^{\infty} \sum_{\substack{j=0 \\ k\neq\ell}}^{\infty} \sum_{k=0}^{\infty} \frac{\dot{c}_k}{\omega_l^2 - \omega_k^2} \left[\partial_t \left(\dot{c}_i \dot{c}_j\right) \left(X_{k\ell ij} - X_{\ell kij}\right) + \partial_t (c_i c_j) \left(Y_{ij\ell k} - Y_{ijk\ell}\right)\right] \\
&\quad + \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \dot{c}_\ell \left[\partial_t \left(\dot{c}_i \dot{c}_j\right) P_{ij\ell} + \partial_t (c_i c_j) B_{ij\ell}\right] \,, && (1.71) \\
\langle \dot{A}_2 \dot{\phi}_1, e_\ell \rangle &= -2 \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \dot{c}_k \dot{c}_j c_i X_{ijk\ell} \,, && (1.72) \\
\langle \left(A_2' - \delta_2'\right) \phi_1', e_\ell \rangle &= -2 \sum_{i=0}^{\infty} \sum_{\substack{j=0 \\ i\neq j}}^{\infty} \sum_{k=0}^{\infty} \frac{c_k (\dot{c}_i \dot{c}_j + \omega_j^2 c_i c_j)}{\omega_j^2 - \omega_i^2} H_{ijk\ell} - m^2 \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} c_i c_j c_k V_{ijk\ell} \\
&\quad - \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} c_j \left[c_i^2 H_{iij\ell} + \mathbb{C}_i M_{j\ell i}\right] \,, && (1.73) \\
\langle A_2 \phi_1 \sec^2 x, e_\ell \rangle &= -2 \sum_{i=0}^{\infty} \sum_{\substack{j=0 \\ i\neq j}}^{\infty} \sum_{k=0}^{\infty} \frac{c_k (\dot{c}_i \dot{c}_j + \omega_j^2 c_i c_j)}{\omega_j^2 - \omega_i^2} V_{jki\ell} \\
&\quad - \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} c_j \left(c_i^2 V_{jii\ell} + \mathbb{C}_i Q_{j\ell i}\right) \,, && (1.74)
\end{aligned}$$

where the forms of X, Y, V, H, B, M, P, and Q are given by

$$X_{ijk\ell} = \int_0^{\pi/2} dx\, \mu^2 \nu e_i' e_j e_k e_\ell \tag{1.75}$$

$$Y_{ijk\ell} = \int_0^{\pi/2} dx\, \mu^2 \nu e_i' e_j' e_k e_\ell' \tag{1.76}$$

$$V_{ijk\ell} = \int_0^{\pi/2} dx\, \mu^2 \nu e_i e_j e_k' e_\ell \sec^2 x \tag{1.77}$$

$$H_{ijk\ell} = \int_0^{\pi/2} dx\, \mu^2 \nu' e_i' e_j e_k' e_\ell \tag{1.78}$$

$$B_{ij\ell} = \int_0^{\pi/2} dx\, \mu\nu e_i' e_j' \int_0^x dy\, \mu e_\ell^2 \tag{1.79}$$

$$M_{ij\ell} = \int_0^{\pi/2} dx\, \mu\nu' e_i' e_j \int_o^x dy\, \mu e_\ell^2 \tag{1.80}$$

$$P_{ij\ell} = \int_0^{\pi/2} dx\, \mu\nu e_i e_j \int_0^x dy\, \mu e_\ell^2 \tag{1.81}$$

$$Q_{ij\ell} = \int_0^{\pi/2} dx\, \mu\nu e_i e_j \sec^2 x \int_0^x dy\, \mu e_\ell^2\,. \tag{1.82}$$

Note that, using integration by parts to remove the derivative from ν in the definitions of $H_{ijk\ell}$ and $M_{ij\ell}$, we can show that

$$H_{ijk\ell} = \omega_i^2 X_{kij\ell} + \omega_k^2 X_{ijk\ell} - Y_{ij\ell k} - Y_{\ell kji} - m^2 V_{kji\ell} - m^2 V_{ijk\ell}\,, \tag{1.83}$$

$$M_{ij\ell} = \omega_i^2 P_{ij\ell} - B_{ij\ell} - m^2 Q_{ij\ell}\,. \tag{1.84}$$

Collecting (1.69) - (1.74) gives the expression for $S_\ell = \langle S, e_\ell \rangle$:

$$\begin{aligned} S_\ell = &\sum_{\substack{i,j,k \\ k\neq\ell}}^{\infty} \frac{1}{\omega_l^2 - \omega_k^2} \Big[F_k(\dot{c}_i \dot{c}_j)\left(X_{k\ell ij} - X_{\ell kij}\right) + F_k(c_i c_j)\left(Y_{ij\ell k} - Y_{ijk\ell}\right) \Big] \\ &+ 2 \sum_{\substack{i,j,k \\ i\neq j}}^{\infty} \frac{c_k D_{ij}}{\omega_j^2 - \omega_i^2} \Big[2\omega_k^2 X_{ijk\ell} - H_{ijk\ell} - m^2 V_{jki\ell} \Big] - \sum_{i,j,k}^{\infty} c_i \Big[2\dot{c}_j \dot{c}_k X_{ijk\ell} + m^2 c_j c_k V_{ijk\ell} \Big] \\ &+ \sum_{i,j}^{\infty} \Big[F_\ell(\dot{c}_i \dot{c}_j) P_{ij\ell} + F_\ell(c_i c_j) B_{ij\ell} + 2\omega_j^2 c_j \left(c_i^2 X_{iij\ell} + \mathbb{C}_i P_{j\ell i} \right) \\ &\quad - c_j \left(c_i^2 (H_{iij\ell} + m^2 V_{jii\ell}) + \mathbb{C}_i (M_{j\ell i} + m^2 Q_{j\ell i}) \right) \Big]\,, \end{aligned} \tag{1.85}$$

where $F_k(z) = \dot{c}_k \dot{z} - 2\omega_k^2 c_k z$, $D_{ij} = \dot{c}_i \dot{c}_j + \omega_j^2 c_i c_j$, and $\mathbb{C}_i = \dot{c}_i^2 + \omega_i^2 c_i^2$. Additionally, we have combined some integrals into their own expressions, namely

$$Z^{\pm}_{ijk\ell} = \omega_i \omega_j \left(X_{k\ell ij} - X_{\ell kij}\right) \pm \left(Y_{ij\ell k} - Y_{ijk\ell}\right) \quad \text{and} \quad \tilde{Z}^{\pm}_{ij\ell} = \omega_i \omega_j P_{ij\ell} \pm B_{ij\ell}\,. \tag{1.86}$$

Finally, using the solution for the time-dependent coefficients, $c_i(t) = a_i(t) \cos\left(\omega_i t + b_i(t)\right) \equiv a_i \cos\theta_i$, we arrive at (1.15).

1.B Two Non-normalizable Modes with Equal Frequencies

Let us return to the case of two, equal, non-normalizable modes with frequency $\overline{\omega}$. Within the space of resonant frequency values, there are frequencies that happen to satisfy $\overline{\omega} = \omega_l$ numerically and may produce extra resonances subject to restrictions on the normalizable frequency. These instances were excluded from the discussion in § 1.5.1, and we address them here. When considering special integer values of $\overline{\omega}$ each choice of $\overline{\omega}$ below will contribute a $\overline{T}$-type term to the total source:

$$\overline{T}_i^{(1)}: \quad \omega_i = \omega_l + 2\overline{\omega} \quad \forall\, \overline{\omega} \in \mathbb{Z}^+ \tag{1.87}$$

$$\overline{T}_i^{(2)}: \quad \omega_i = \omega_l - 2\overline{\omega} \quad \forall\, \overline{\omega} \in \mathbb{Z}^+ \text{ such that } \ell \geq \overline{\omega} \tag{1.88}$$

$$\overline{T}_i^{(3)}: \quad \omega_i = 2\overline{\omega} - \omega_l \quad \forall\, \overline{\omega} \in \mathbb{Z}^+ \text{ such that } \overline{\omega} \leq \ell + \Delta^+\,, \tag{1.89}$$

with $\omega_i \neq \omega_\ell$ in each case. These special values contribute to the case of two, equal non-normalizable modes via

$$S_\ell = \bar{A}_{\overline{\omega}}^2\, \overline{T}^{(1)}_{(\ell+\overline{\omega})}\, a_{(\ell+\overline{\omega})} \cos\left(\theta_{(\ell+\overline{\omega})} - 2\overline{\omega}t\right) + \bar{A}_{\overline{\omega}}^2\, \overline{T}^{(2)}_{(\ell-\overline{\omega})}\, a_{(\ell-\overline{\omega})} \cos\left(\theta_{(\ell-\overline{\omega})} + 2\overline{\omega}t\right)$$
$$+ \bar{A}_{\overline{\omega}}^2\, \overline{T}^{(3)}_{(\overline{\omega}-\ell-\Delta^+)}\, a_{(\overline{\omega}-\ell-\Delta^+)} \cos\left(2\overline{\omega}t - \theta_{(\overline{\omega}-\ell-\Delta^+)}\right) \tag{1.90}$$

under their respective conditions on the value of $\overline{\omega}$. The total resonant contribution for all possible $\overline{\omega}$ values is the addition of (1.90) and (1.38). Evaluating (1.15) in each case of the cases described by (1.87) - (1.89), we find that

$$\begin{aligned}\overline{T}_i^{(1)} = {} & \frac{1}{2}\Bigg[H_{i\overline{\omega}\overline{\omega}\ell}\left(\frac{\overline{\omega}}{\omega_i - \overline{\omega}}\right) - H_{\overline{\omega}i\overline{\omega}\ell}\left(\frac{\omega_i}{\omega_i - \overline{\omega}}\right) + m^2 V_{\overline{\omega}\overline{\omega}i\ell}\left(\frac{\overline{\omega}}{\omega_i - \overline{\omega}}\right) \\ & - m^2 V_{i\overline{\omega}\overline{\omega}\ell}\left(\frac{\omega_i}{\omega_i - \overline{\omega}}\right) - 2\overline{\omega}^2 X_{i\overline{\omega}\overline{\omega}\ell}\left(\frac{\overline{\omega}}{\omega_i - \overline{\omega}}\right) + 2\overline{\omega}^2 X_{\overline{\omega}i\overline{\omega}\ell}\left(\frac{\omega_i}{\omega_i - \overline{\omega}}\right)\Bigg]_{\omega_i \neq \overline{\omega}} \\ & - \frac{1}{2}\left[Z^+_{i\overline{\omega}\overline{\omega}\ell}\left(\frac{\overline{\omega}}{\omega_l + \overline{\omega}}\right)\right]_{\omega_l \neq \overline{\omega}} + \frac{1}{4} Z^-_{\overline{\omega}\overline{\omega}i\ell}\left(\frac{\omega_l + 2\overline{\omega}}{2\overline{\omega}}\right) + \frac{1}{2}\overline{\omega}^2 X_{i\overline{\omega}\overline{\omega}\ell} - \frac{m^2}{4} V_{\overline{\omega}\overline{\omega}i\ell} \\ & - \overline{\omega}\omega_i X_{\overline{\omega}\overline{\omega}i\ell} - \frac{m^2}{2} V_{i\overline{\omega}\overline{\omega}\ell}\,, \end{aligned} \tag{1.91}$$

$$\begin{aligned}\overline{T}_i^{(2)} = {} & -\frac{1}{2}\Bigg[H_{i\overline{\omega}\overline{\omega}\ell}\left(\frac{\overline{\omega}}{\omega_i + \overline{\omega}}\right) + H_{\overline{\omega}i\overline{\omega}\ell}\left(\frac{\omega_i}{\omega_i + \overline{\omega}}\right) + m^2 V_{\overline{\omega}\overline{\omega}i\ell}\left(\frac{\overline{\omega}}{\omega_i + \overline{\omega}}\right) \\ & + m^2 V_{i\overline{\omega}\overline{\omega}\ell}\left(\frac{\omega_i}{\omega_i + \overline{\omega}}\right) - 2\overline{\omega}^2 X_{i\overline{\omega}\overline{\omega}\ell}\left(\frac{\overline{\omega}}{\omega_i + \overline{\omega}}\right) - 2\overline{\omega}^2 X_{\overline{\omega}\overline{\omega}i\ell}\left(\frac{\omega_i}{\omega_i + \overline{\omega}}\right)\Bigg]_{\omega_i \neq \overline{\omega}} \\ & - \frac{1}{2}\left[Z^-_{i\overline{\omega}\overline{\omega}\ell}\left(\frac{\overline{\omega}}{\omega_l - \overline{\omega}}\right)\right]_{\omega_l \neq \overline{\omega}} - \frac{1}{4} Z^-_{\overline{\omega}\overline{\omega}i\ell}\left(\frac{\omega_l - 2\overline{\omega}}{\overline{\omega}}\right) + \frac{1}{2}\overline{\omega}^2 X_{i\overline{\omega}\overline{\omega}\ell} + \frac{m^2}{4} V_{\overline{\omega}\overline{\omega}i\ell} \\ & + \overline{\omega}\omega_i X_{\overline{\omega}\overline{\omega}i\ell} + \frac{m^2}{2} V_{i\overline{\omega}\overline{\omega}\ell}\,, \end{aligned} \tag{1.92}$$

and

$$\begin{aligned}
\overline{T}_i^{(3)} &= \frac{1}{2}\Bigg[H_{i\overline{\omega}\overline{\omega}\ell}\left(\frac{\overline{\omega}}{\omega_i-\overline{\omega}}\right) - H_{\overline{\omega}i\overline{\omega}\ell}\left(\frac{\omega_i}{\omega_i-\overline{\omega}}\right) + m^2 V_{\overline{\omega}\overline{\omega}i\ell}\left(\frac{\overline{\omega}}{\omega_i-\overline{\omega}}\right) \\
&\quad - m^2 V_{i\overline{\omega}\overline{\omega}\ell}\left(\frac{\omega_i}{\omega_i-\overline{\omega}}\right) - 2\overline{\omega}^2 X_{i\overline{\omega}\overline{\omega}\ell}\left(\frac{\overline{\omega}}{\omega_i-\overline{\omega}}\right) + 2\omega_i^2 X_{\overline{\omega}\overline{\omega}i\ell}\left(\frac{\overline{\omega}}{\omega_i-\overline{\omega}}\right) \\
&\quad - Z^+_{i\overline{\omega}\overline{\omega}\ell}\left(\frac{\overline{\omega}}{\omega_i-\overline{\omega}}\right)\Bigg]_{\omega_i\neq\overline{\omega}} + \frac{1}{4} Z^-_{\overline{\omega}\overline{\omega}i\ell}\left(\frac{2\overline{\omega}-\omega_l}{2\overline{\omega}}\right) + \frac{1}{2}\overline{\omega}^2 X_{i\overline{\omega}\overline{\omega}\ell} - \frac{m^2}{4} V_{\overline{\omega}\overline{\omega}i\ell} \\
&\quad - \overline{\omega}\omega_i X_{\overline{\omega}\overline{\omega}i\ell} - \frac{m^2}{2} V_{i\overline{\omega}\overline{\omega}\ell}\,. \qquad (1.93)
\end{aligned}$$

These resonance channels can then be added into the right hand side of the equation for da_ℓ/dt in (1.40).