

Gravitational Collapse in Anti-de Sitter Space

Brad Cownden
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University
of Manitoba



THE UNIVERSITY OF
WINNIPEG

- Gravitational Collapse
- Massive Scalars in AdS_5 [*Phys. Rev. D* **102** (2020) 026015]
 - Scalar Field Collapse in AdS
 - Classifying Phases
 - Phase Diagram & Energy Cascades
- High-Temperature QP Solutions in AdS_4
 - The Two-Time Formalism (TTF)
 - Quasi-Periodic Solutions
 - High-Temperature Families
- Examining Instabilities Due to Driven Scalars in AdS [arXiv:1912.07143]
 - Extending TTF to Driven Scalars
 - Resonant Contributions
 - Special Values of Non-normalizable Frequencies
- Conclusions

Gravitational Collapse

- ▶ AdS/CFT¹ → thermal quench in gauge theory \Leftrightarrow formation of black hole in gravitational theory
- ▶ Massless scalar fields in AdS: unstable against generic initial data, no mass gap² → c.f. Minkowski³
- ▶ Stability for specific initial data below critical energy
- ▶ **Nonlinear theory:** continue⁴ with exploration of phase space
- ▶ **Perturbative theory:** effects of truncation, space of solutions, time-dependent boundary conditions

¹Maldacena [hep-th/9711200]

²Bizoń & Rostworowski [1104.3702]

³Choptuik PRL70 9 (1993)

⁴Deppe & Frey [1508.02709]

Scalar Field Collapse in AdS

- ▶ Minimally-coupled scalar field in AdS₅ (dual to 4D CFT)
- ▶ Spherical symmetry, Schwarzschild-like coordinates → $A(t, x)$, $\delta(t, x)$

$$ds^2 = \frac{\ell^2}{\cos^2(x/\ell)} \left(-Ae^{-2\delta} dt^2 + A^{-1} dx^2 + \sin^2(x/\ell) d\Omega^{d-1} \right)$$

Scalar Field Collapse in AdS

- ▶ Minimally-coupled scalar field in AdS₅ (dual to 4D CFT)
- ▶ Spherical symmetry, Schwarzschild-like coordinates $\rightarrow A(t, x), \delta(t, x)$
- ▶ Einstein + Klein-Gordon \Rightarrow constraint equations
- ▶ Interior gauge $\delta(t, x = 0) = 0$, spherical symmetry $\partial_x \phi(t, x = 0) = 0$
- ▶ Horizon formation when $A(t_H, x_H) \ll 1$

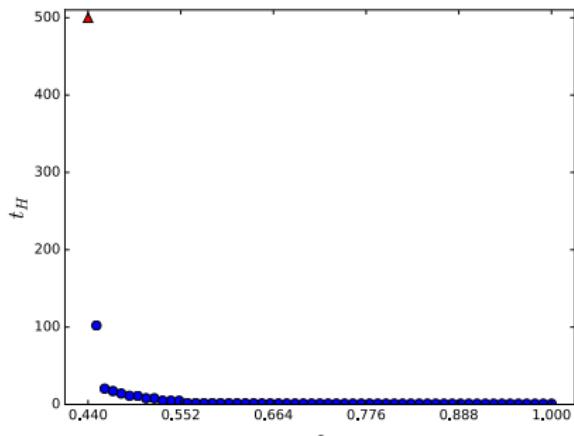
$$\partial_x \delta = -(\Pi^2 + \Phi^2) \sin(x) \cos(x)$$

$$\partial_x M = \frac{\tan^{d-1}(x)}{2} \left(A(\Pi^2 + \Phi^2) + \frac{\mu^2 \phi^2}{\cos^2(x)} \right)$$

$$\Pi(t = 0, x) = \epsilon \exp \left(-\frac{\tan^2(x)}{\sigma^2} \right) \quad \text{Phase space: } \mu, \sigma$$

Stable vs Unstable Profiles

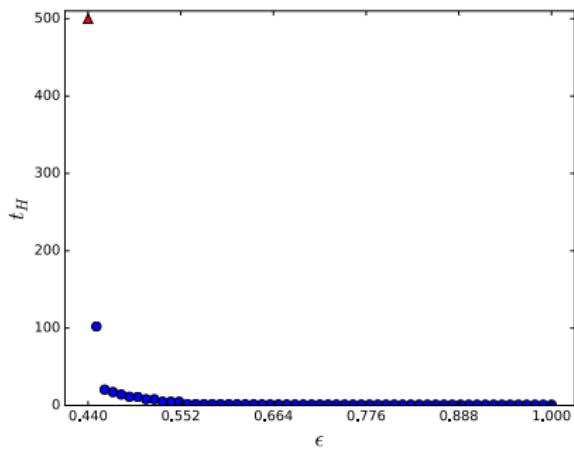
- Stable: abrupt jump in t_H when $\epsilon < \epsilon_{crit}$



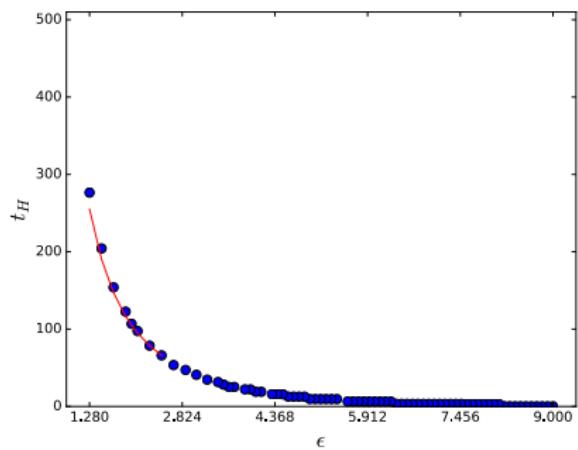
$$\mu = 0, \sigma = 1.5$$

Stable vs Unstable Profiles

- ▶ Stable: abrupt jump in t_H when $\epsilon < \epsilon_{crit}$
- ▶ Unstable: fit $t_H \approx a\epsilon^{-p} + b$ for $t_H \geq 60 \rightarrow$ unstable when $p \approx 2$



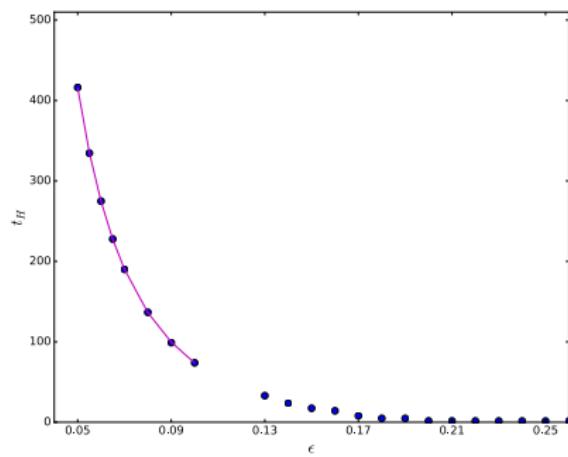
$$\mu = 0, \sigma = 1.5$$



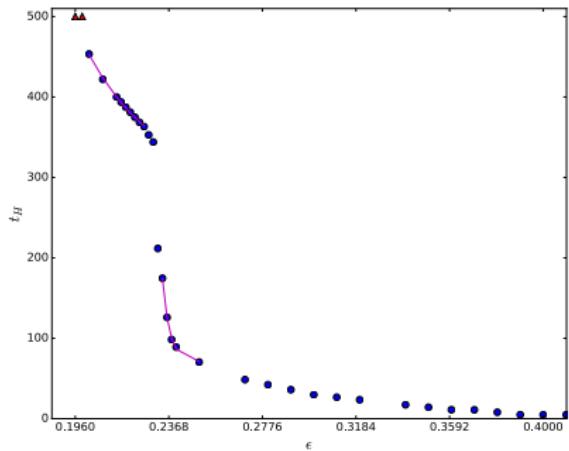
$$\mu = 5, \sigma = 0.25$$

Metastable Profiles

- ▶ Scaling of $p > 2$



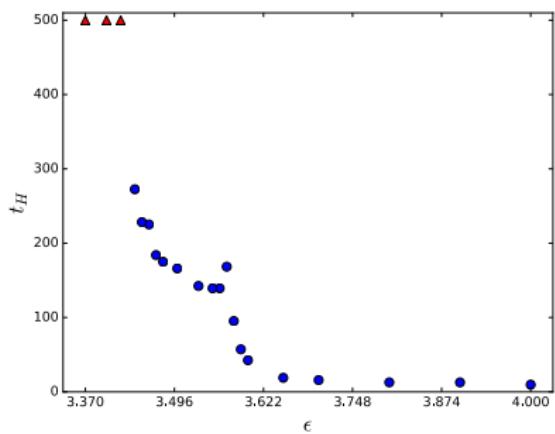
$$\mu = 5, \sigma = 1.7$$



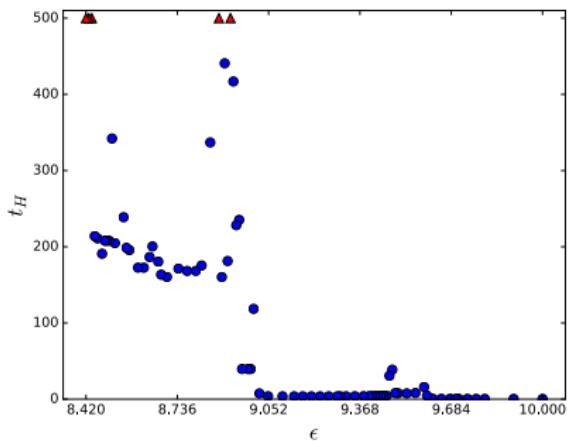
$$\mu = 0.5, \sigma = 1.7$$

Irregular Profiles I

- ▶ No scaling



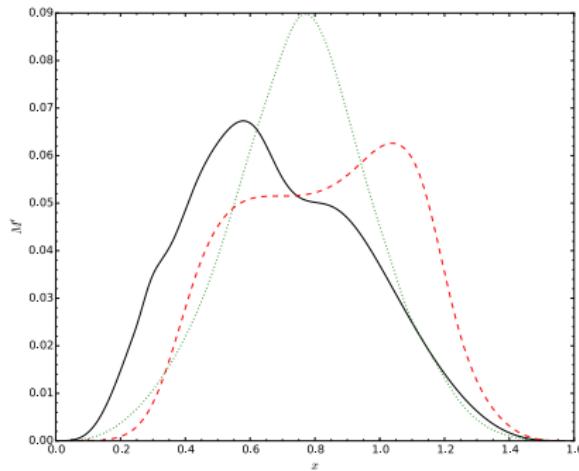
$$\mu = 5, \sigma = 0.34$$



$$\mu = 20, \sigma = 0.16$$

Irregular Profiles II

- ▶ Evidence of chaotic behaviour → possible self-interaction
- ▶ Previous chaotic evolution only seen in thin-shell interactions⁵ in AdS, scalar collapse in Gauss-Bonnet gravity⁶



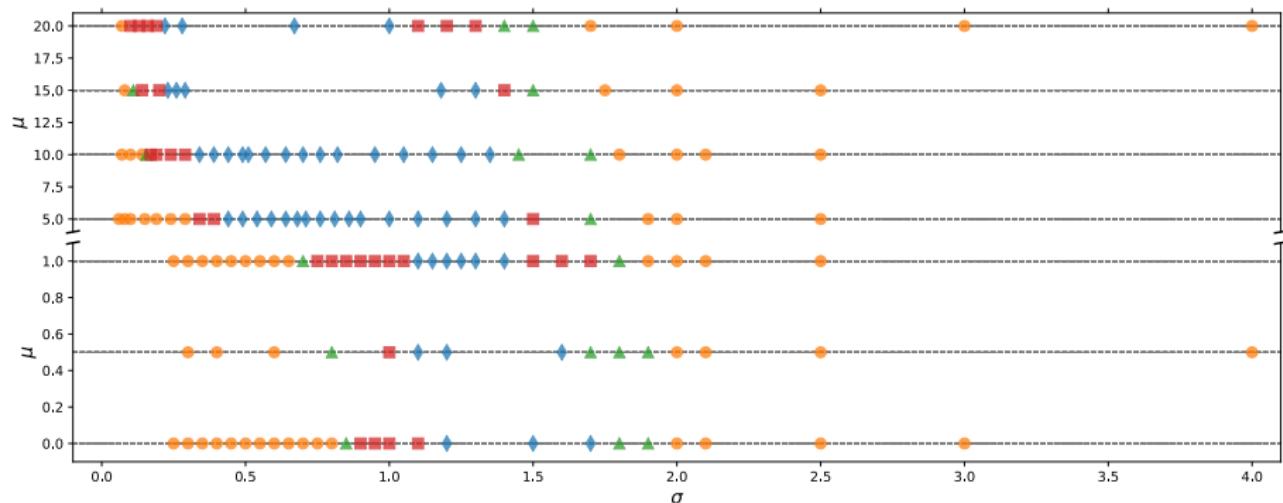
$t = 60, 62, 64$

$$\mu = 0, \sigma = 1.1, \epsilon = 1.01$$

⁵ Brito et al. [1602.03535]

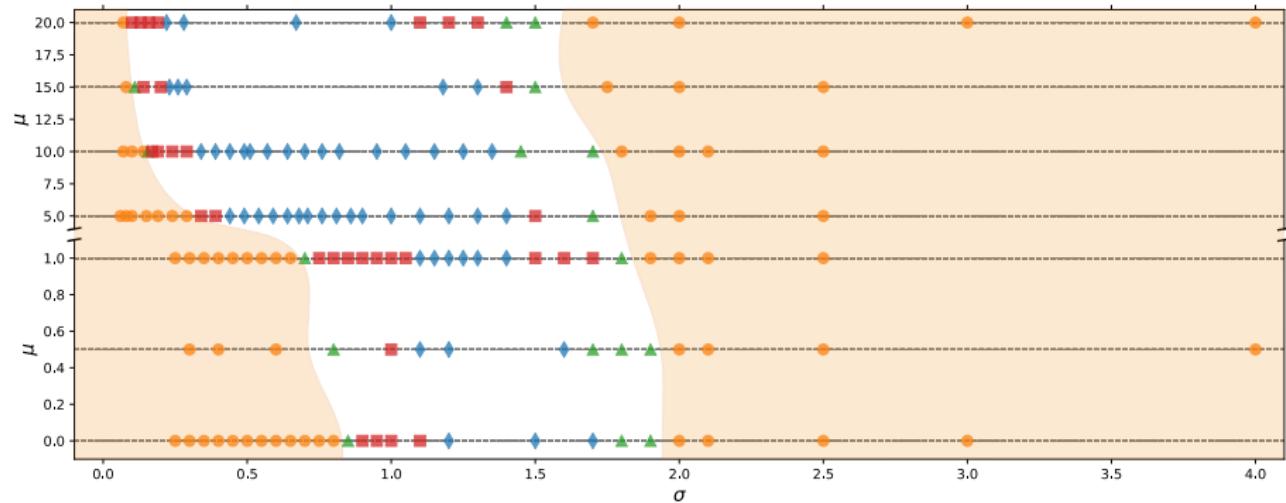
⁶ Deppe, Kolly, et al. [1608.05402]

Phase Diagram



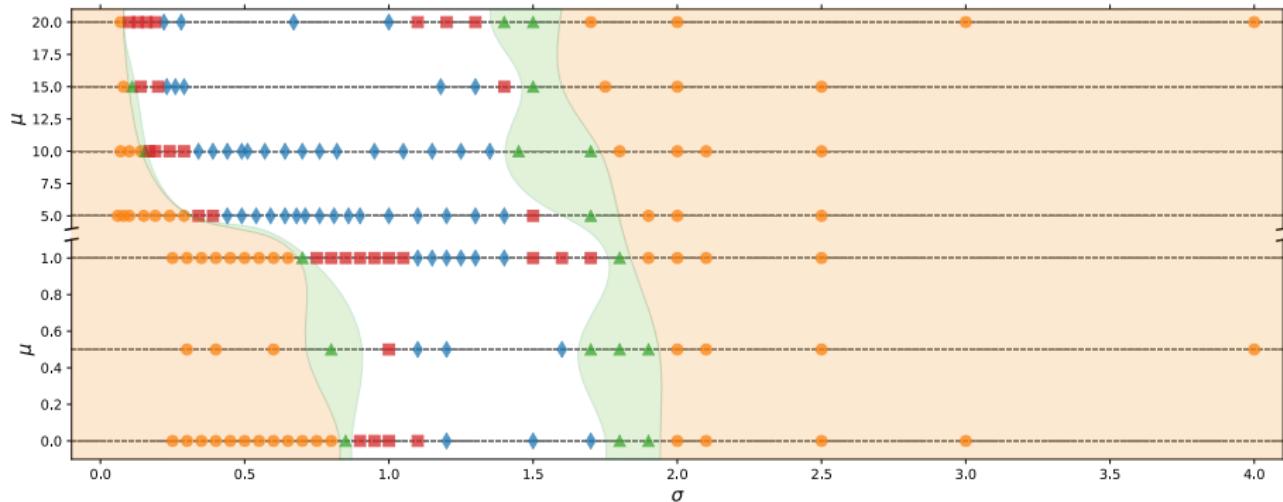
Phase Diagram

► Unstable



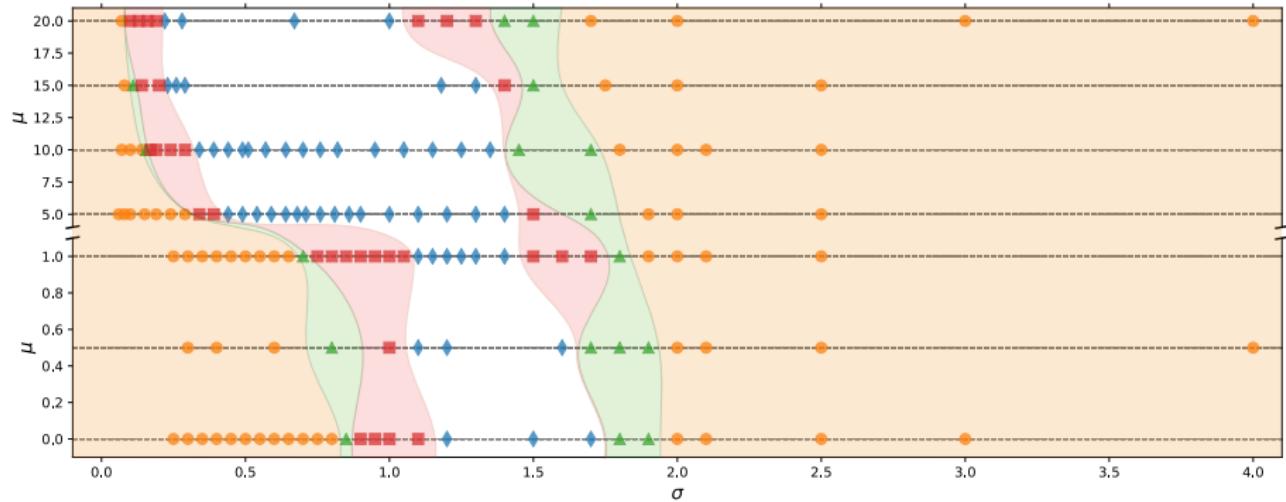
Phase Diagram

- Unstable, metastable,



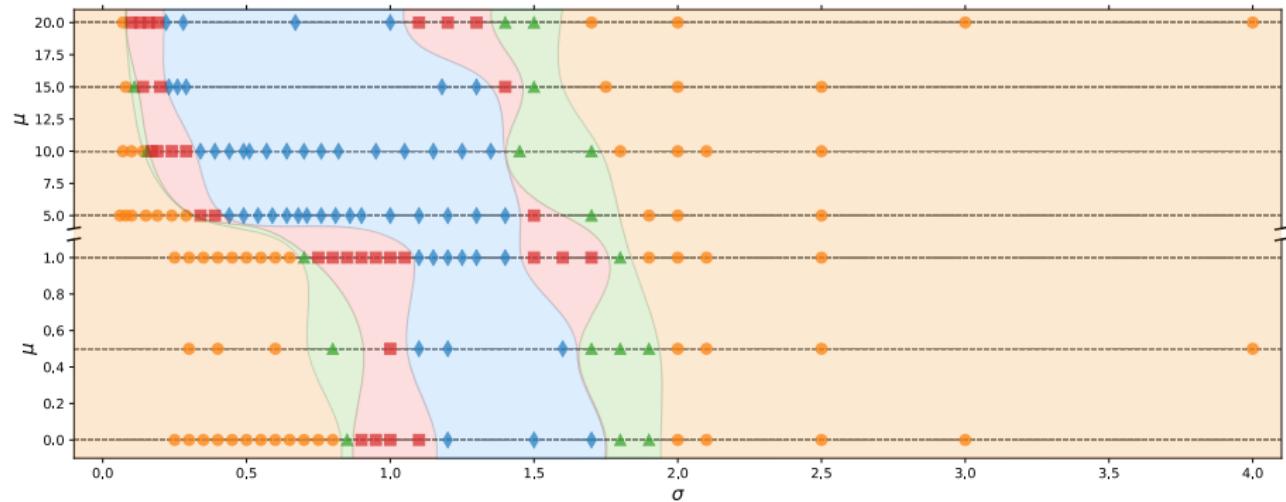
Phase Diagram

- Unstable, metastable, irregular,



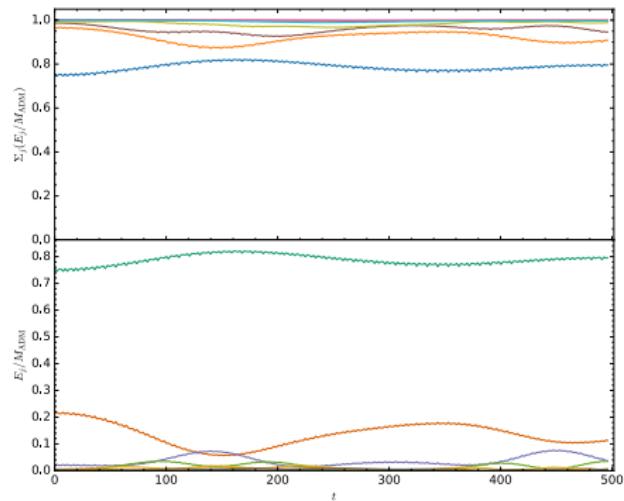
Phase Diagram

- Unstable, metastable, irregular, and stable initial data



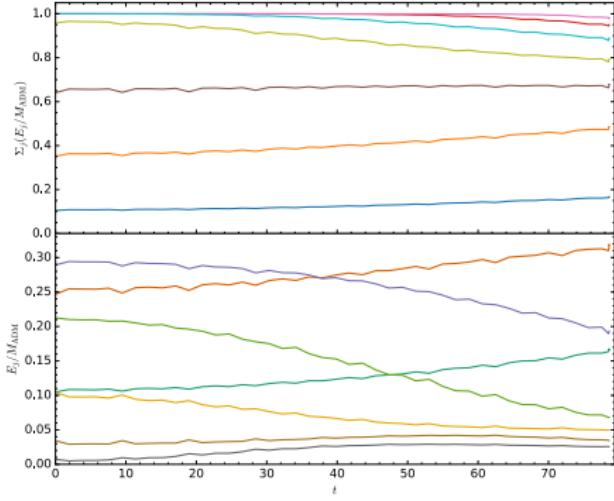
Energy Cascades

- ▶ Stable solutions: direct and inverse energy cascades



$$\mu = 0, \sigma = 1.8, \epsilon = 0.13$$

Stable



$$\mu = 0, \sigma = 0.25, \epsilon = 2.28$$

Unstable

Results

- ▶ First full phase diagram of stability in $\text{AdS}_5 \rightarrow$ islands of stability and “shorelines”
- ▶ Evidence of metastable and irregular phases at finite ϵ
- ▶ Fate of metastable phase as $\epsilon \rightarrow 0$ yet to be determined
- ▶ Irregular phase contains quasi-stable initial data^{7,8} \rightarrow first evidence for weakly chaotic evolution in massless, spherically-symmetry scalars in AdS
- ▶ Metastable and irregular data to be studied in multiscale perturbation theory

⁷Deppe & Frey [1508.02709]

⁸Buchel *et al.* [1304.4166]

The Two-Time Formalism (TTF)

- ▶ Small perturbations in AdS₄: expand scalar field, metric functions in ϵ
- ▶ $\mathcal{O}(\epsilon)$: ϕ_1 in terms of eigenfunctions of AdS, $e_j(x)$
- ▶ Integer eigenvalues $\omega_j = (2j + d) \rightarrow$ fully resonant spectrum

$$\phi_1(t, x) = \sum_{j=0}^{\infty} a_j(t) \cos(\omega_j t + b_j(t)) e_j(x)$$

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- ▶ Integer eigenvalues $\omega_j = (2j + d) \rightarrow$ fully resonant spectrum
- ▶ $\mathcal{O}(\epsilon^3)$: **source terms** for resonant contributions
- ▶ Absorb resonances into amplitudes & phases⁹

$$-2\omega_\ell \frac{da_\ell(\tau)}{d\tau} = \sum_{i \neq \ell} \sum_{j \neq \ell}^{\ell \leq i+j} f_1(a_i, a_j, a_{i+j-\ell}, b_i, b_j, b_{i+j-\ell})$$

$$-2\omega_\ell a_\ell \frac{db_\ell(\tau)}{d\tau} = \sum_{i \neq \ell} \sum_{j \neq \ell}^{\ell \leq i+j} f_2(a_i, a_j, a_{i+j-\ell}, b_i, b_j, b_{i+j-\ell})$$

⁹Balasubramanian *et al.* [1403.6471]

Quasi-Periodic Solutions I

- ▶ Renormalization flow techniques to cancel an infinite number of resonances
→ express non-vanishing ones analytically¹⁰
- ▶ Need to truncate number of modes to find solutions: $j_{max} < \infty$ (must be robust as $j_{max} \rightarrow \infty$)

¹⁰Craps *et al.* [1407.6273]

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- ▶ Quasi-periodic¹¹ solutions $a_j = \alpha_j e^{i\beta_j \tau} \rightarrow$ TTF equations become time-independent when $\beta_j = \beta_0 + j(\beta_1 - \beta_0)$
- ▶ TTF: conserved quantities¹² (E, N) → classify solutions by $T \equiv E/N$
- ▶ Solve QP equation using Newton-Raphson method

$$2\omega_\ell \alpha_\ell \beta_\ell = T_\ell \alpha_\ell^3 + \sum_{i \neq \ell} R_{i\ell} \alpha_i^2 \alpha_\ell + \sum_{i \neq \ell} \sum_{j \neq \ell}^{\ell \leq i+j} S_{ij(i+j-\ell)\ell} \alpha_i \alpha_j \alpha_{i+j-\ell}$$

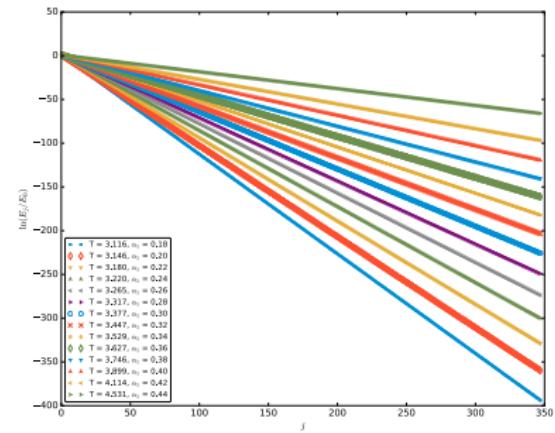
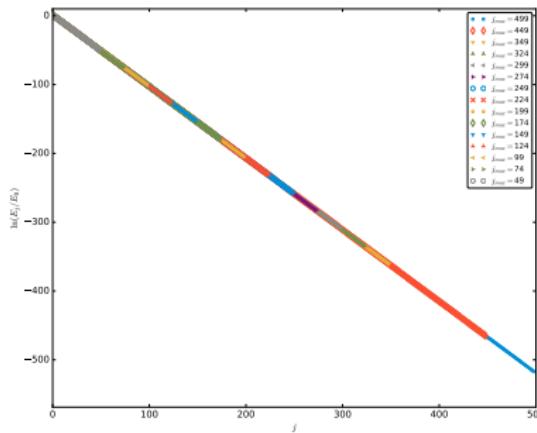
¹⁰Craps et al. [1407.6273]

¹¹Balasubramanian et al. [1403.6471]

¹²Craps et al. [1412.3249]

Quasi-Periodic Solutions II

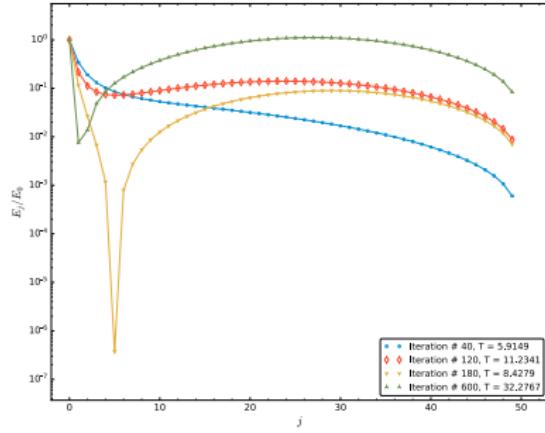
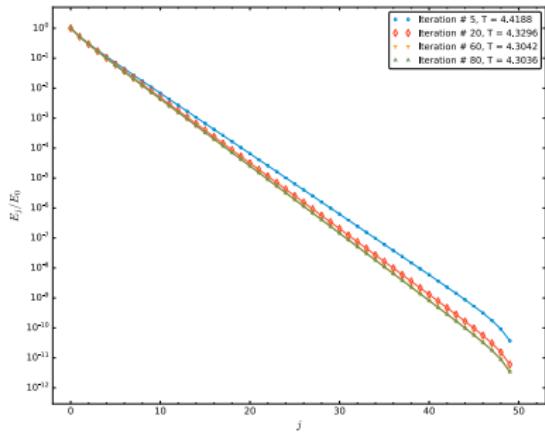
- ▶ Solutions found for $3 \geq T \gtrsim 5.5$
- ▶ Able to extend existing solutions from $j_{max} \sim 100$ to $j_{max} = 500$
- ▶ Robust in $j_{max} \rightarrow \infty$ limit



BC, Deppe, & Frey: In progress

High-Temperature Families I

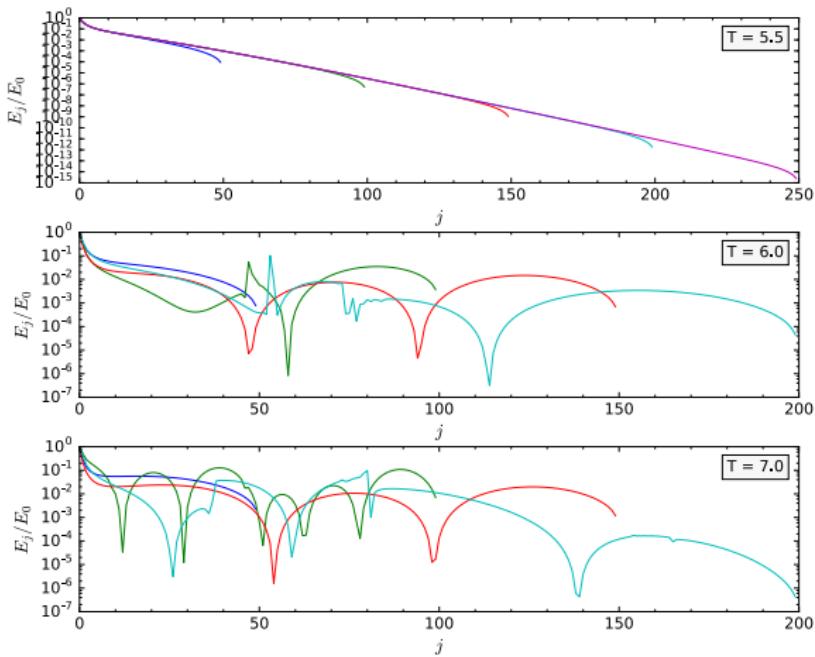
- ▶ Perturb by $\delta E \rightarrow$ new solutions have energy $E + \delta E$, N , and $T + \delta T$
- ▶ Repeat process to determine T_{max}
- ▶ Project back to QP solution surface at constant α_1 or T
- ▶ Loss of smooth profile above a certain temperature at **constant α_1**



Cownden, Deppe, & Frey: In progress

High-Temperature Families II

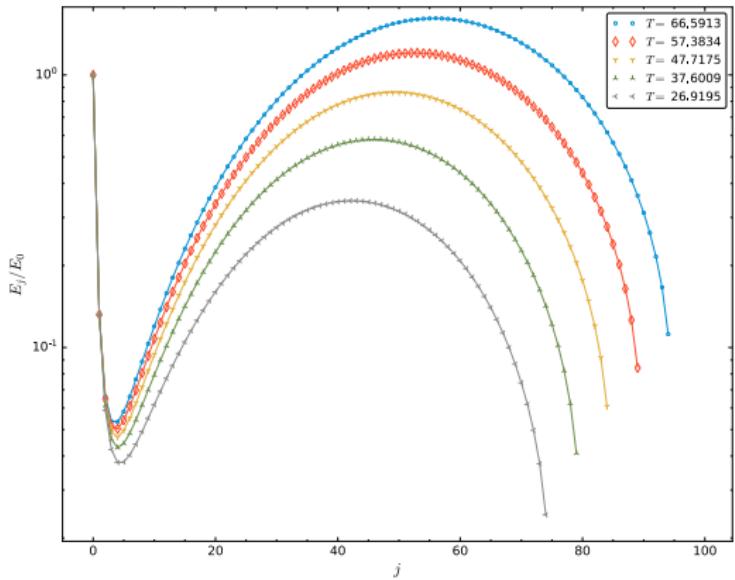
- And at constant T



Cownden, Deppe, & Frey: In progress

High-Temperature Families III

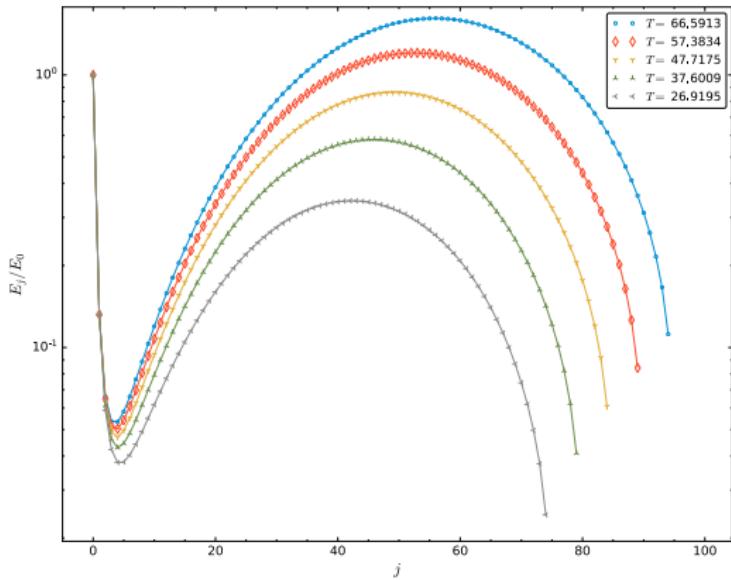
- ▶ Alternative methods for finding high-T solutions
- ▶ Fit low j_{max} , high-T data to generate seeds for Newton-Raphson solver



Cownden, Deppe, & Frey: In progress

High-Temperature Families III

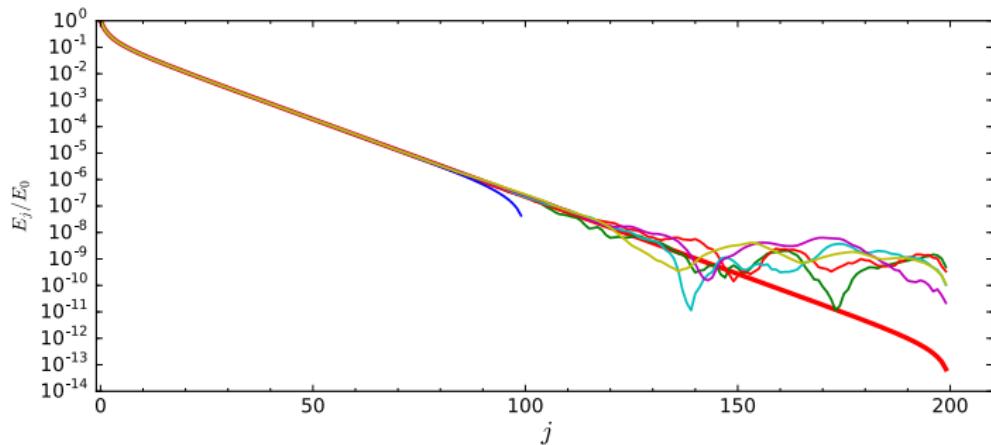
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Cownden, Deppe, & Frey: In progress

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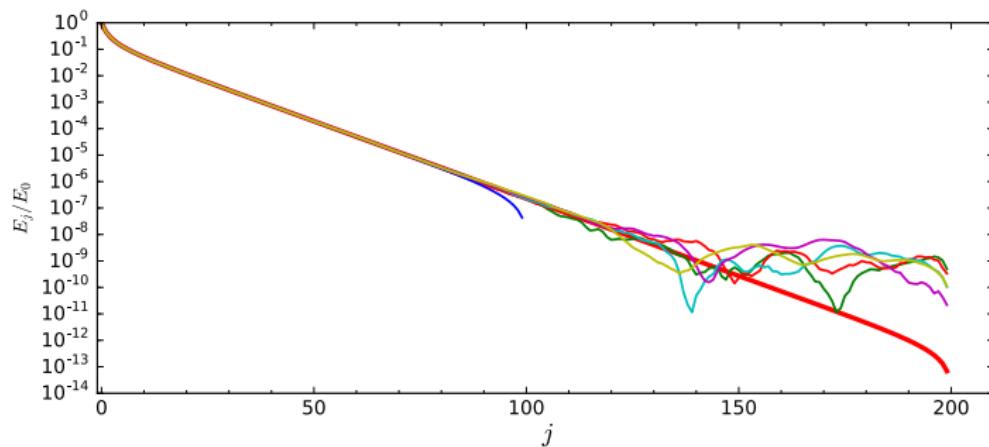
- ▶ Alternative methods for finding high-T solutions
- ▶ Fit low j_{max} , high-T data to generate seeds for Newton-Raphson solver → not robust as j_{max} increases
- ▶ Pad low j_{max} data with zeros, evolve within the TTF



Cownden, Deppe, & Frey: In progress

High-Temperature Families III

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- ▶ Fit low j_{max} , high-T data to generate seeds for Newton-Raphson solver → not robust as j_{max} increases
- ▶ Pad low j_{max} data with zeros, evolve within the TTF → **isothermal evolution does not match known solution**



Cownden, Deppe, & Frey: In progress

Results

- ▶ Low-T QP solutions robust as j_{max} increases
- ▶ Not able to find evidence that high-T solutions continued to exist at large $j_{max} \rightarrow$ possible reduction of space of QP
- ▶ **Caveat:** focused on configurations where $\alpha_0 = 1 \rightarrow$ free to set dominant energy in any $\alpha_j \rightarrow$ other configurations required for high temperatures?
- ▶ Motivation for temperature limit of $T \sim 5.5$?
- ▶ Perturbative system: massless scalar, static boundary conditions at $x = \pi/2$
- ▶ Extend to massive scalars, time-dependent boundary conditions \rightarrow activation of non-normalizable modes

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$\Phi^+(x) \equiv \text{"normalizable"} \quad \Phi^-(x) \equiv \text{"non-normalizable"}$

Extending TTF to Driven Scalars

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- ▶ Scalar field is linear combination of both kinds of modes
- ▶ $e_j(x)$ are same eigenfunctions of AdS & have eigenvalues $\omega_j = (2j + \Delta^+)$

$$\phi_1(t, x) = \sum_{j=0}^{\infty} a_j(t) \cos(\omega_j t + b_j(t)) e_j(x) + \sum_{\alpha=0}^{\infty} \bar{A}_{\alpha}(t) \cos(\omega_{\alpha} t + \bar{B}_{\alpha}) E_{\alpha}(x)$$
$$\Delta^{\pm} = \frac{d}{2} \pm \frac{1}{2} \sqrt{d^2 + 4m^2}$$

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- ▶ Scalar field is linear combination of both kinds of modes
- ▶ $e_j(x)$ are same eigenfunctions of AdS & have eigenvalues $\omega_j = (2j + \Delta^+)$
- ▶ $E_\alpha(x)$ are hypergeometric functions & have continuous eigenvalues ω_α

$$\phi_1(t, x) = \sum_{j=0}^{\infty} a_j(t) \cos(\omega_j t + b_j(t)) e_j(x) + \sum_{\alpha=0}^{\infty} \bar{A}_\alpha(t) \cos(\omega_\alpha t + \bar{B}_\alpha) E_\alpha(x)$$
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Resonant Contributions

- ▶ $\mathcal{O}(\epsilon^3)$: source terms for resonant contributions → examine resonance conditions

$$\omega_i + \omega_j + \omega_k = \omega_\ell$$

$$\omega_i - \omega_j - \omega_k = \omega_\ell$$

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Resonant Contributions

- ▶ $\mathcal{O}(\epsilon^3)$: source terms for resonant contributions → examine resonance conditions
- ▶ **Unforced**: restrictions on indices and mass value

$$\omega_i + \omega_j + \omega_k = \omega_\ell \quad \Rightarrow \quad i + j + k = \ell - \Delta^+ \in \mathbb{Z}^+$$

$$\omega_i - \omega_j - \omega_k = \omega_\ell \quad \Rightarrow \quad i - j - k = \ell + \Delta^+ \in \mathbb{Z}^+$$

$$\omega_i + \omega_j - \omega_k = \omega_\ell \quad \Rightarrow \quad i + j = k + \ell \in \mathbb{Z}^+$$

Resonant Contributions

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Resonant Contributions

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- ▶ One **non-vanishing** channel

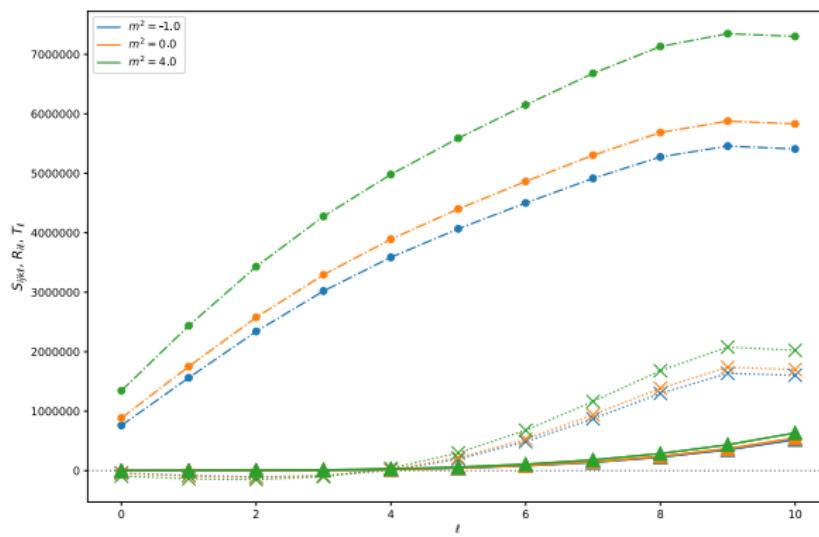
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Resonant Contributions

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- ▶ One non-vanishing channel → sum over i, j with $i + j \leq \ell$



Special Values of Non-normalizable Frequencies

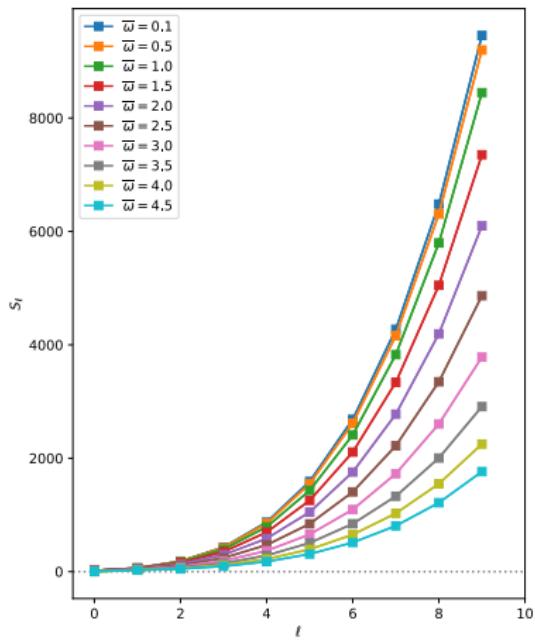
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- ▶ **Single frequency:** $\omega_\alpha = \bar{\omega}$

$$\pm\omega_I \pm \omega_J \pm \omega_K \pm \omega_\ell = 0 \quad I \in \{i, \alpha\}$$

$$\omega_i + \bar{\omega} - \bar{\omega} = \omega_\ell$$

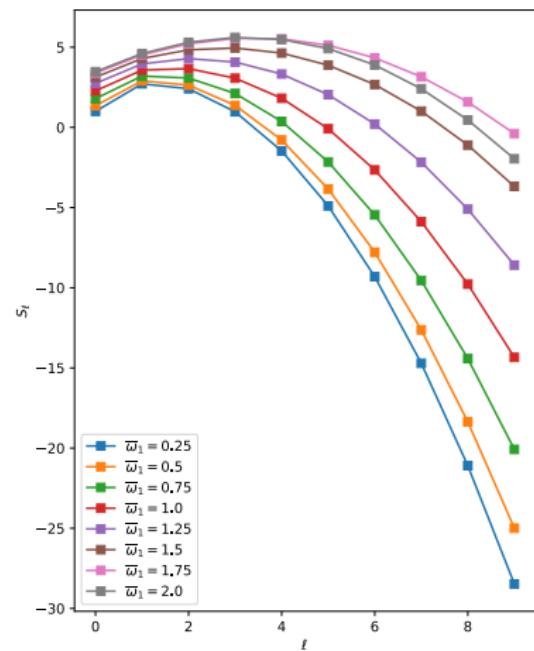


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- ▶ Single frequency: $\omega_\alpha = \bar{\omega}$
- ▶ Add to integer: $\bar{\omega}_1 + \bar{\omega}_2 = 2n$

$$\pm\omega_I \pm \omega_J \pm \omega_K \pm \omega_\ell = 0 \quad I \in \{i, \alpha\}$$

$$\left\{ \begin{array}{l} (++): \omega_i + 2n = \omega_\ell \quad \forall \ell \geq n \\ (+-): \omega_i - 2n = \omega_\ell \quad \forall n \\ (-+): -\omega_i + 2n = \omega_\ell \quad \forall n \geq \ell + d \end{array} \right\}$$



Special Values of Non-normalizable Frequencies

- ▶ **Forced:** ω_α set by driving term → explore several scenarios
- ▶ Single frequency: $\omega_\alpha = \bar{\omega}$
- ▶ Add to integer: $\bar{\omega}_1 + \bar{\omega}_2 = 2n$
- ▶ $\omega_\alpha = 2i + \chi$, $\chi \notin \mathbb{Z}^+$
- ▶ Source terms give flow equations for amplitude/phase of ϕ_1 , e.g. single frequency → equations decouple

$$\frac{2\omega_\ell}{\epsilon^2} \frac{da_\ell}{dt} = 0$$

$$\frac{2\omega_\ell}{\epsilon^2} \frac{db_\ell}{dt} = -S_\ell \bar{A}_{\bar{\omega}}^2$$

Results

- ▶ Confirm two of three resonant channels vanish for massive scalar (all normalizable)¹³
- ▶ First TTF formulation with time-dependent boundary conditions
- ▶ No naturally-vanishing source terms → sum resonant channels
- ▶ Some flow equations decouple amplitude/phase variables $a_\ell(t)$, $b_\ell(t)$
- ▶ **Note:** normalizable modes are still present → sum all contributions
- ▶ **Further work:** quasi-periodic solutions¹⁴? Conserved quantities? Energy cascades?

¹³Biasi *et al.* [1810.04753]

¹⁴Carracedo *et al.* [1612.07701]

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- ▶ Verified low-T QP solutions are robust in the $j_{max} \rightarrow \infty$ limit
- ▶ No evidence of high-T QP solutions for $\alpha_0 = 1$ family
- ▶ Developed perturbative theory for massive scalars with time-dependent boundary conditions
- ▶ Derived flow equations for amplitude/phase variables for some choices of driving term → evaluated source terms numerically

Thanks

- ▶ Supervisor: Andrew Frey
- ▶ PhD Committee: Derek Krepski, Gabor Kunstatter, Robert Mann, Khodr Shamseddine
- ▶ Co-authors: Nils Deppe
- ▶ University of Winnipeg and University of Manitoba
- ▶ Westgrid & Compute Canada

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