

Gravitational Collapse in Anti-de Sitter Space

Brad Cownden
PhD Thesis Defence

June –, 2020
University of Manitoba



University
of Manitoba



THE UNIVERSITY OF
WINNIPEG

- Gravitational Collapse
- Massive Scalars in AdS_5 [arXiv:1711.00454]
 - Scalar Field Collapse in AdS
 - Classifying Phases
 - Phase Diagram & Energy Cascades
- High-Temperature QP Solutions in AdS_4
 - The Two-Time Formalism (TTF)
 - Quasi-Periodic Solutions
 - High-Temperature Families
- Examining Instabilities Due to Driven Scalars in AdS [arXiv:1912.07143]
 - Extending TTF to Driven Scalars
 - Resonant Contributions
 - Special Values of Non-normalizable Frequencies
- Conclusions

Gravitational Collapse

- ▶ AdS/CFT¹ → thermal quench in gauge theory \Leftrightarrow formation of black hole in gravitational theory
- ▶ Massless scalar fields in AdS: unstable against generic initial data, no mass gap² → c.f. Minkowski³
- ▶ Stability for specific initial data below critical energy
- ▶ **Nonlinear theory:** continue⁴ with exploration of phase space
- ▶ **Perturbative theory:** effects of truncation, space of solutions, time-dependent boundary conditions

¹Maldacena [hep-th/9711200]

²Bizoń & Rostworowski [1104.3702]

³Choptuik PRL70 9 (1993)

⁴Deppe & Frey [1508.02709]

Scalar Field Collapse in AdS

- ▶ Minimally-coupled scalar field in AdS₅ (dual to 4D CFT)
- ▶ Spherical symmetry, Schwarzschild-like coordinates → $A(t, x)$, $\delta(t, x)$

$$ds^2 = \frac{\ell^2}{\cos^2(x/\ell)} \left(-Ae^{-2\delta} dt^2 + A^{-1} dx^2 + \sin^2(x/\ell) d\Omega^{d-1} \right)$$

Scalar Field Collapse in AdS

- ▶ Minimally-coupled scalar field in AdS₅ (dual to 4D CFT)
- ▶ Spherical symmetry, Schwarzschild-like coordinates $\rightarrow A(t, x), \delta(t, x)$
- ▶ Einstein + Klein-Gordon \Rightarrow constraint equations
- ▶ Interior gauge $\delta(t, x = 0) = 0$, spherical symmetry $\partial_x \phi(t, x = 0) = 0$
- ▶ Horizon formation when $A(t_H, x_H) \ll 1$

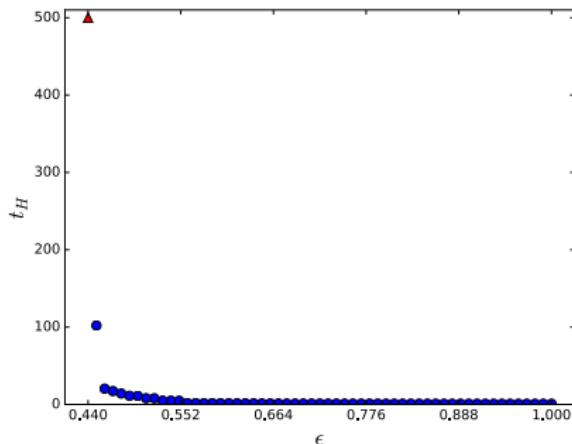
$$\partial_x \delta = -(\Pi^2 + \Phi^2) \sin(x) \cos(x)$$

$$\partial_x M = \frac{\tan^{d-1}(x)}{2} \left(A(\Pi^2 + \Phi^2) + \frac{\mu^2 \phi^2}{\cos^2(x)} \right)$$

$$\Pi(t = 0, x) = \epsilon \exp \left(-\frac{\tan^2(x)}{\sigma^2} \right) \quad \text{Phase space: } \mu, \sigma$$

Stable vs Unstable Profiles

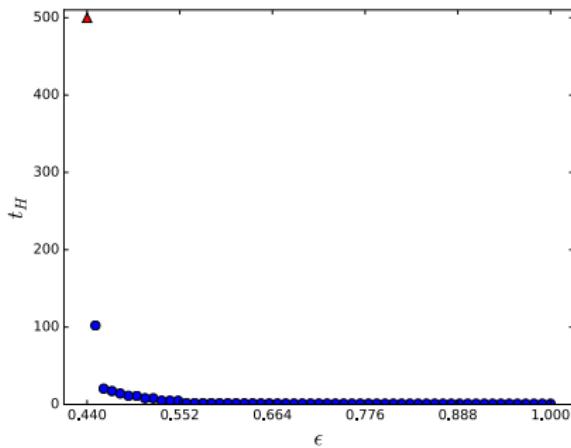
- ▶ Stable: abrupt jump in t_H when $\epsilon < \epsilon_{crit}$



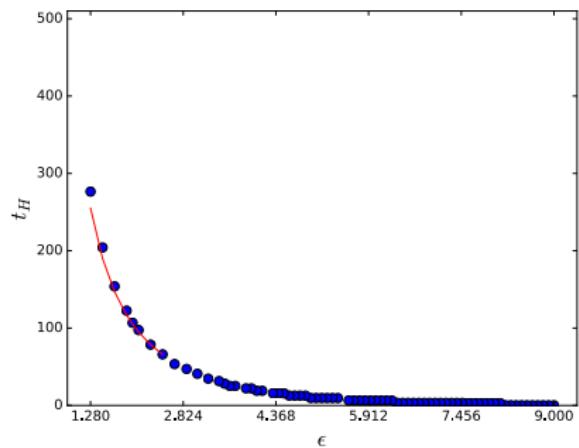
$$\mu = 0, \sigma = 1.5$$

Stable vs Unstable Profiles

- ▶ Stable: abrupt jump in t_H when $\epsilon < \epsilon_{crit}$
- ▶ Unstable: fit $t_H \approx a\epsilon^{-p} + b$ for $t_H \geq 60 \rightarrow$ unstable when $p \approx 2$



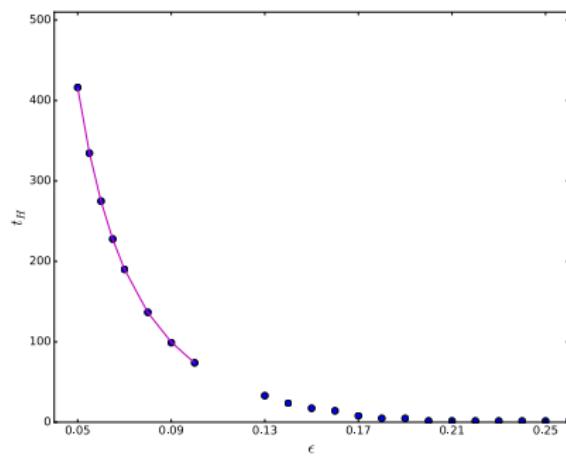
$$\mu = 0, \sigma = 1.5$$



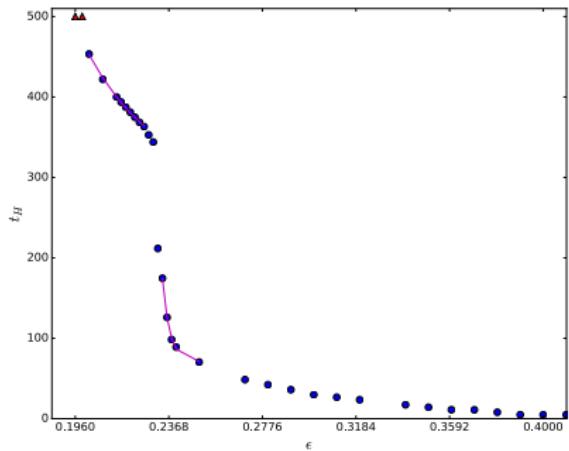
$$\mu = 5, \sigma = 0.25$$

Metastable Profiles

- ▶ Scaling of $p > 2$



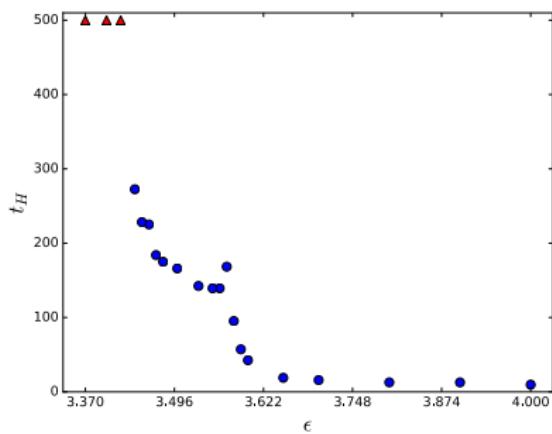
$$\mu = 5, \sigma = 1.7$$



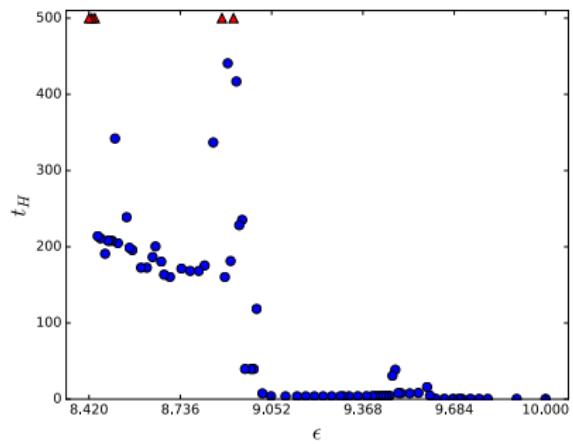
$$\mu = 0.5, \sigma = 1.7$$

Irregular Profiles I

- ▶ No scaling



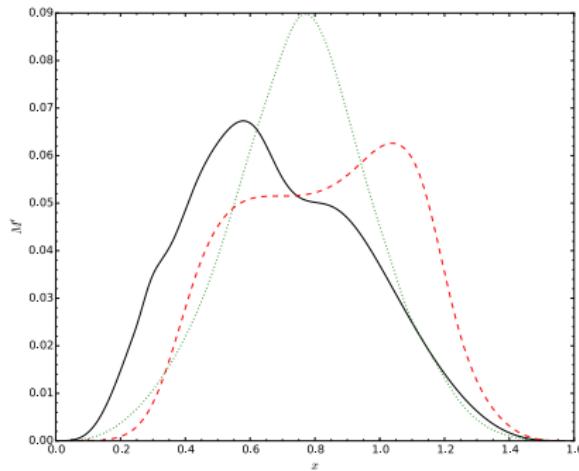
$$\mu = 5, \sigma = 0.34$$



$$\mu = 20, \sigma = 0.16$$

Irregular Profiles II

- ▶ Evidence of chaotic behaviour → possible self-interaction
- ▶ Previous chaotic evolution only seen in thin-shell interactions⁵ in AdS, scalar collapse in Gauss-Bonnet gravity⁶



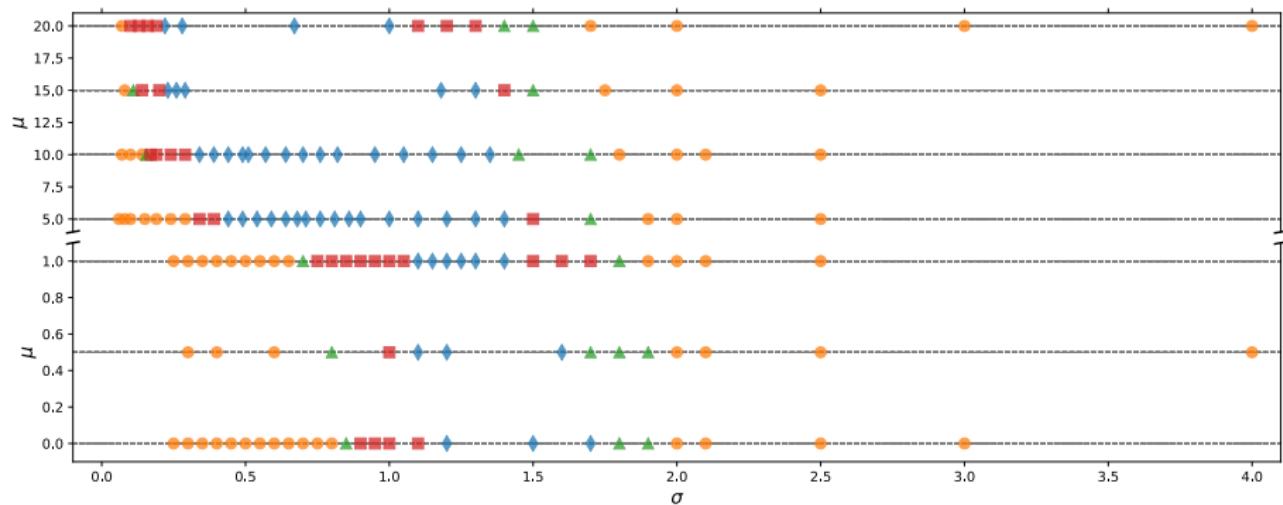
$t = 60, 62, 64$

$$\mu = 0, \sigma = 1.1, \epsilon = 1.01$$

⁵ Brito *et al.* [1602.03535]

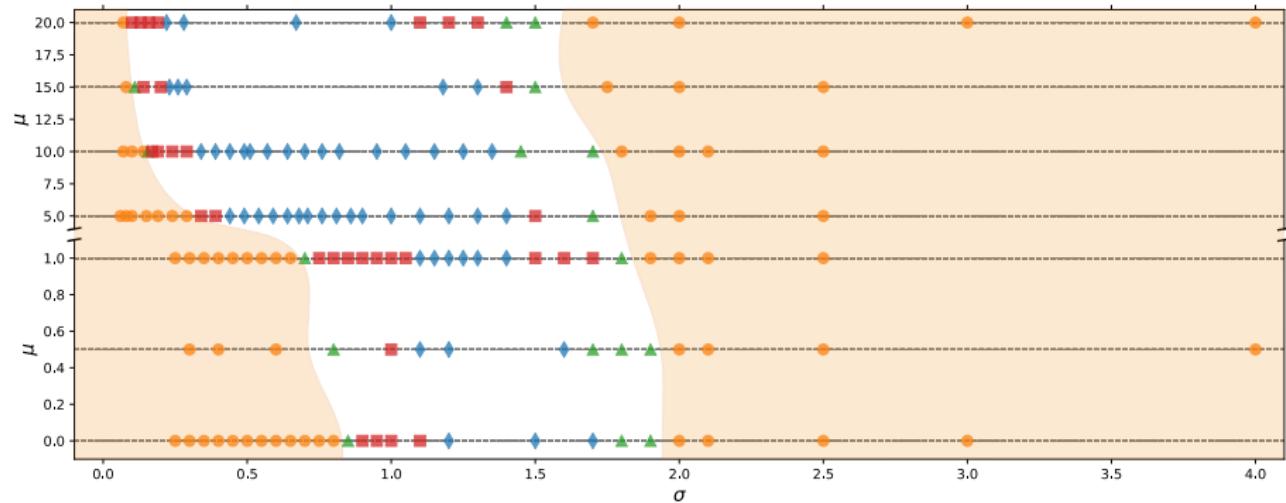
⁶ Deppe, Kolly, *et al.* [1608.05402]

Phase Diagram



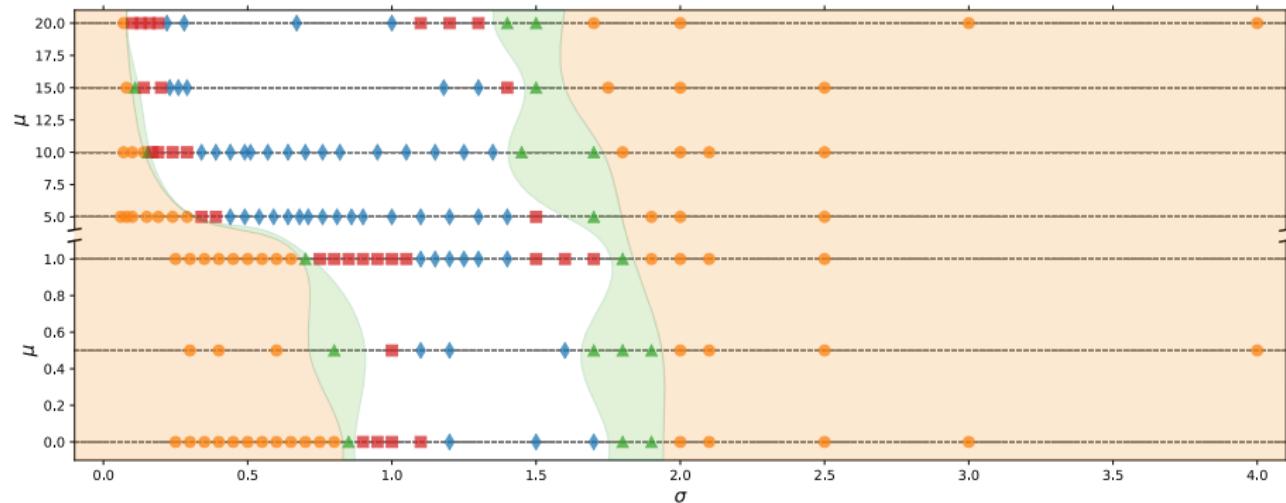
Phase Diagram

► Unstable



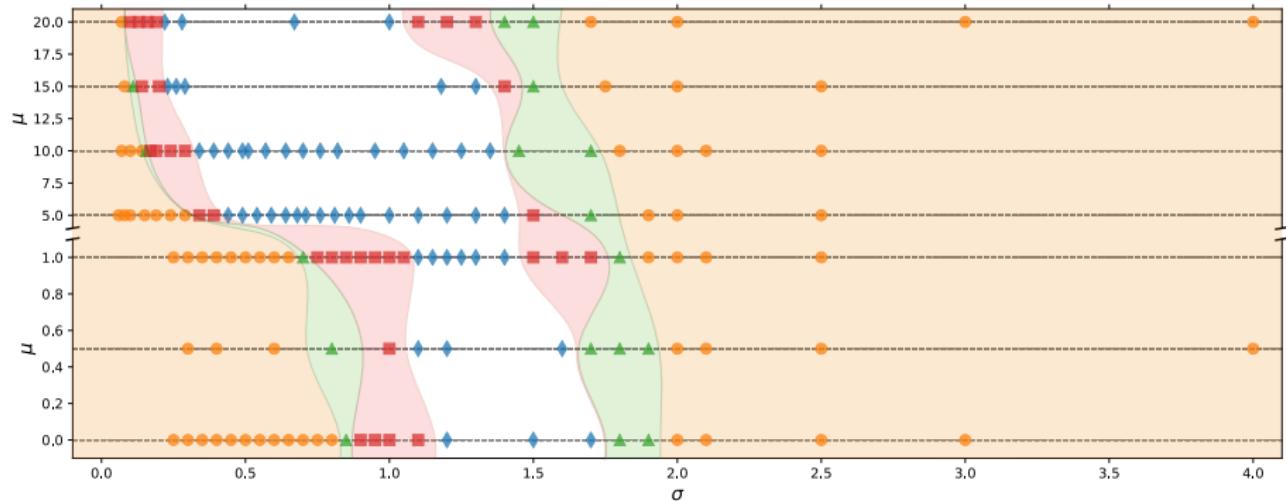
Phase Diagram

- Unstable, metastable,



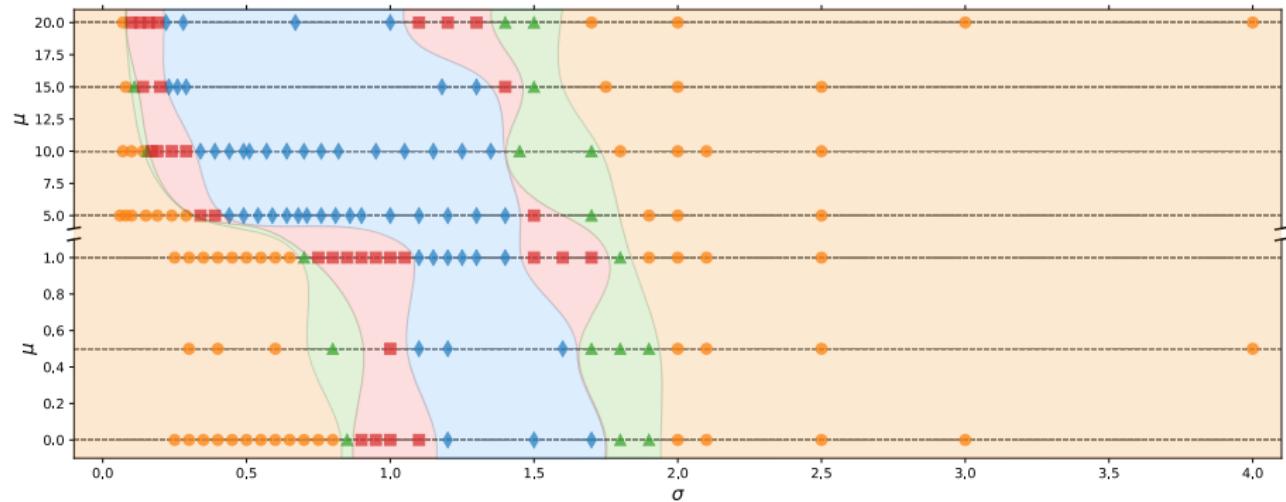
Phase Diagram

- Unstable, metastable, irregular,



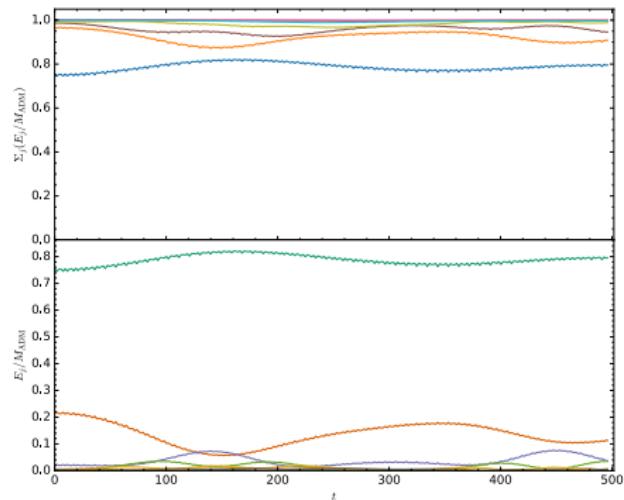
Phase Diagram

- Unstable, metastable, irregular, and stable initial data

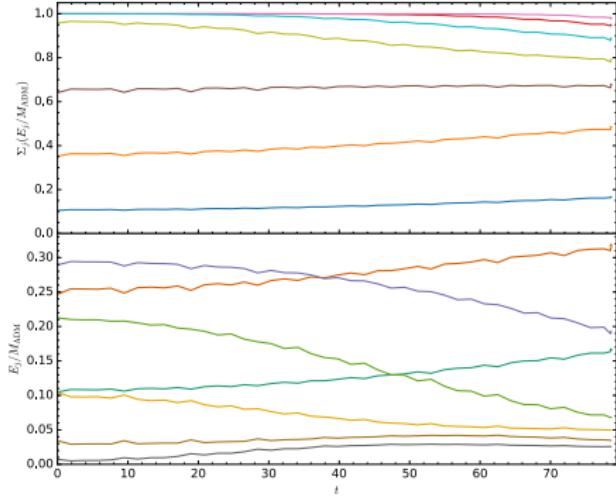


Energy Cascades

- ▶ Stable solutions: **direct** and **inverse** energy cascades



$\mu = 0, \sigma = 1.8, \epsilon = 0.13$
Stable



$\mu = 0, \sigma = 0.25, \epsilon = 2.28$
Unstable

Results

- ▶ First full phase diagram of stability in AdS₅ → islands of stability and “shorelines”
- ▶ Evidence of metastable and irregular phases at finite ϵ
- ▶ Fate of metastable phase as $\epsilon \rightarrow 0$ yet to be determined
- ▶ Irregular phase contains quasi-stable initial data^{7,8} → first evidence for weakly chaotic evolution in massless, spherically-symmetry scalars in AdS
- ▶ Metastable and irregular data to be studied in multiscale perturbation theory

⁷Deppe & Frey [1508.02709]

⁸Buchel *et al.* [1304.4166]

The Two-Time Formalism (TTF)

- ▶ Small perturbations in AdS₄: expand scalar field, metric functions in ϵ
- ▶ $\mathcal{O}(\epsilon)$: ϕ_1 in terms of eigenfunctions of AdS, $e_j(x)$
- ▶ Integer eigenvalues $\omega_j = (2j + d) \rightarrow$ fully resonant spectrum

$$\phi_1(t, x) = \sum_{j=0}^{\infty} a_j(t) \cos(\omega_j t + b_j(t)) e_j(x)$$

The Two-Time Formalism (TTF)

- ▶ Small perturbations in AdS₄: expand scalar field, metric functions in ϵ
- ▶ $\mathcal{O}(\epsilon)$: ϕ_1 in terms of eigenfunctions of AdS, $e_j(x)$
- ▶ Integer eigenvalues $\omega_j = (2j + d) \rightarrow$ fully resonant spectrum
- ▶ $\mathcal{O}(\epsilon^3)$: **source terms** for resonant contributions
- ▶ Absorb resonances into amplitudes & phases⁹

$$-2\omega_\ell \frac{da_\ell(\tau)}{d\tau} = \sum_{i \neq \ell} \sum_{j \neq \ell}^{\ell \leq i+j} f_1(a_i, a_j, a_{i+j-\ell}, b_i, b_j, b_{i+j-\ell})$$

$$-2\omega_\ell a_\ell \frac{db_\ell(\tau)}{d\tau} = \sum_{i \neq \ell} \sum_{j \neq \ell}^{\ell \leq i+j} f_2(a_i, a_j, a_{i+j-\ell}, b_i, b_j, b_{i+j-\ell})$$

⁹Balasubramanian *et al.* [1403.6471]

Quasi-Periodic Solutions I

- ▶ Renormalization flow techniques to cancel an infinite number of resonances → express non-vanishing ones analytically¹⁰
- ▶ Need to truncate number of modes to find solutions: $j_{max} < \infty$ (must be robust as $j_{max} \rightarrow \infty$)

¹⁰Craps *et al.* [1407.6273]

Quasi-Periodic Solutions I

- ▶ Renormalization flow techniques to cancel an infinite number of resonances → express non-vanishing ones analytically¹⁰
- ▶ Need to truncate number of modes to find solutions: $j_{max} < \infty$ (must be robust as $j_{max} \rightarrow \infty$)
- ▶ Quasi-periodic¹¹ solutions $a_j = \alpha_j e^{i\beta_j \tau} \rightarrow$ TTF equations become time-independent when $\beta_j = \beta_0 + j(\beta_1 - \beta_0)$
- ▶ TTF: conserved quantities¹² (E, N) → classify solutions by $T \equiv E/N$
- ▶ Solve QP equation using Newton-Raphson method

$$2\omega_\ell \alpha_\ell \beta_\ell = T_\ell \alpha_\ell^3 + \sum_{i \neq \ell} R_{i\ell} \alpha_i^2 \alpha_\ell + \sum_{i \neq \ell} \sum_{j \neq \ell}^{\ell \leq i+j} S_{ij(i+j-\ell)\ell} \alpha_i \alpha_j \alpha_{i+j-\ell}$$

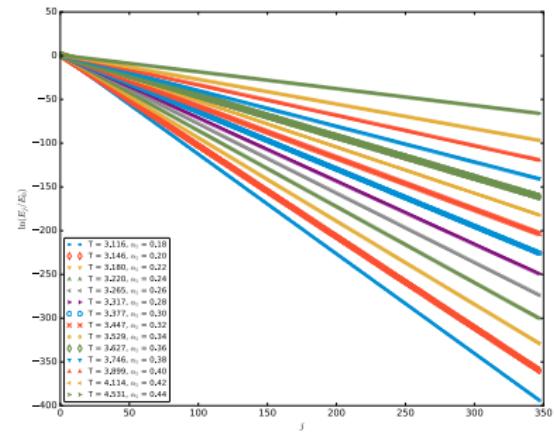
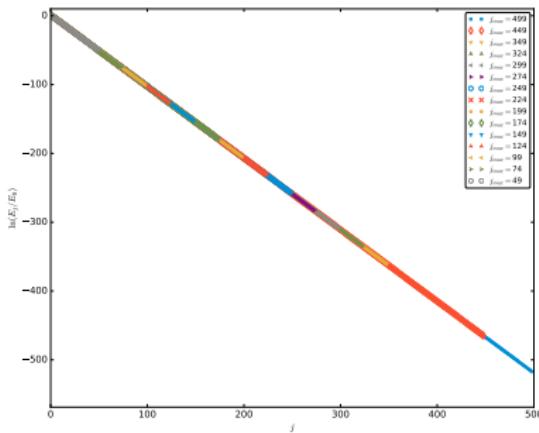
¹⁰Craps et al. [1407.6273]

¹¹Craps et al. [1412.3249]

¹²Green et al. [1507.08261]

Quasi-Periodic Solutions II

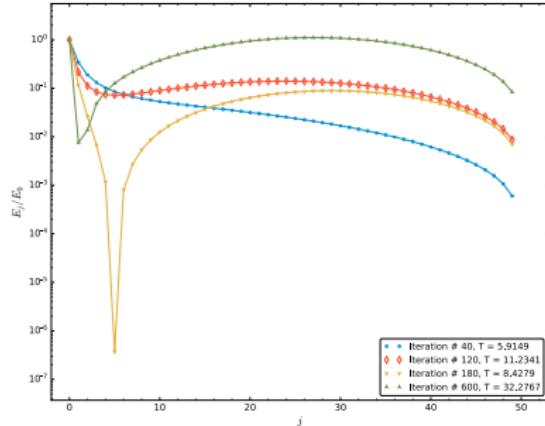
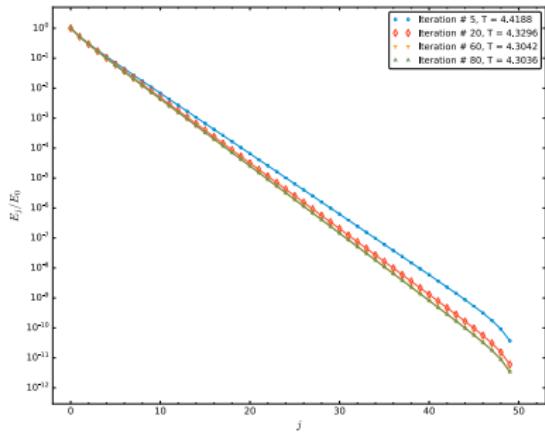
- ▶ Solutions found for $3 \geq T \gtrsim 5.5$
- ▶ Able to extend existing solutions from $j_{max} \sim 100$ to $j_{max} = 500$
- ▶ Robust in $j_{max} \rightarrow \infty$ limit



BC, Deppe, & Frey: In progress

High-Temperature Families I

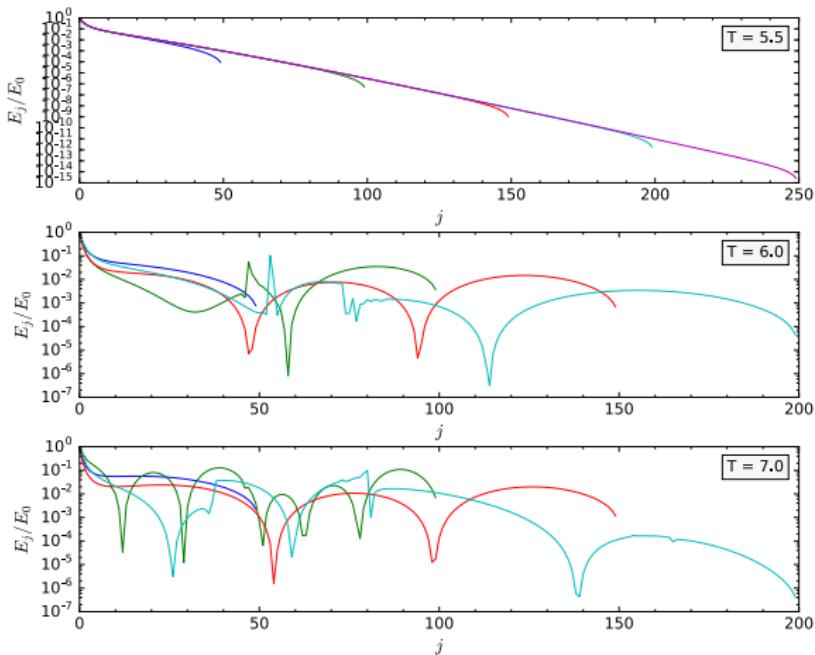
- ▶ Perturb by $\delta E \rightarrow$ new solutions have energy $E + \delta E$, N , and $T + \delta T$
- ▶ Repeat process to determine T_{max}
- ▶ Project back to QP solution surface at constant α_1 or T
- ▶ Loss of smooth profile above a certain temperature at **constant α_1**



Cownden, Deppe, & Frey: In progress

High-Temperature Families II

- And at constant T



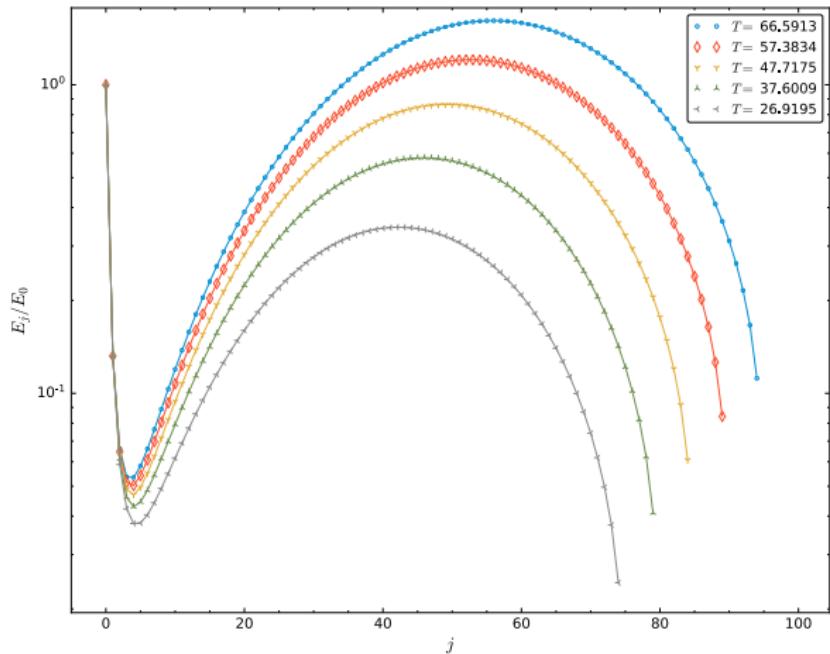
Cownden, Deppe, & Frey: In progress

High-Temperature Families III

- ▶ Alternative methods for finding high-T solutions

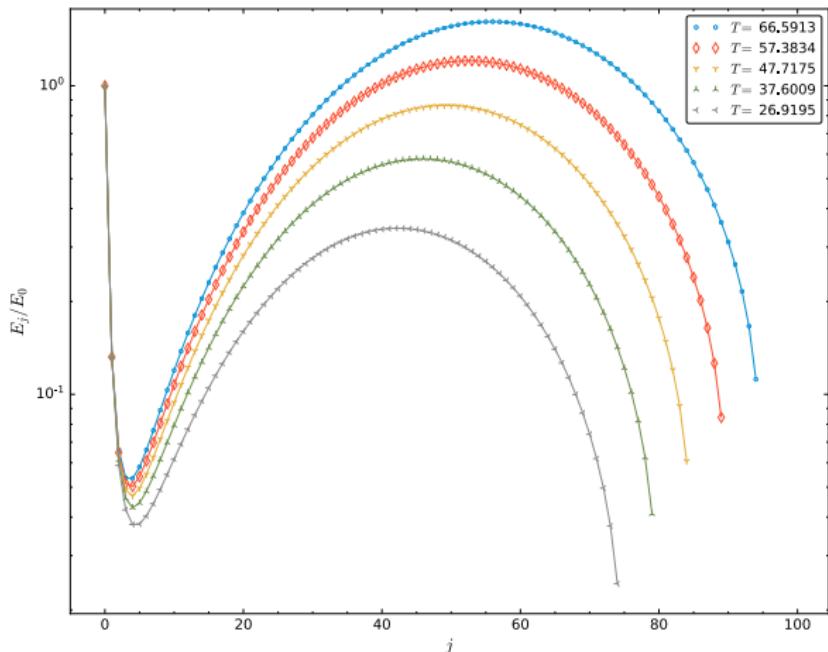
High-Temperature Families III

- ▶ Alternative methods for finding high-T solutions
- ▶ Fit low j_{max} , high-T data to generate seeds for Newton-Raphson solver



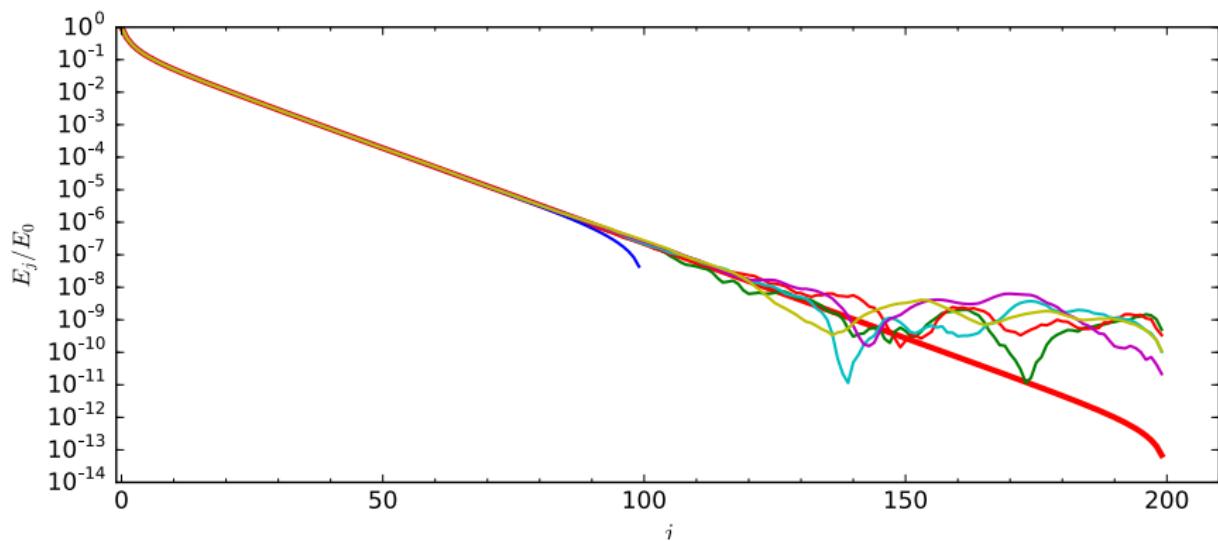
High-Temperature Families III

- ▶ Alternative methods for finding high-T solutions
- ▶ Fit low j_{max} , high-T data to generate seeds for Newton-Raphson solver → not robust as j_{max} increases



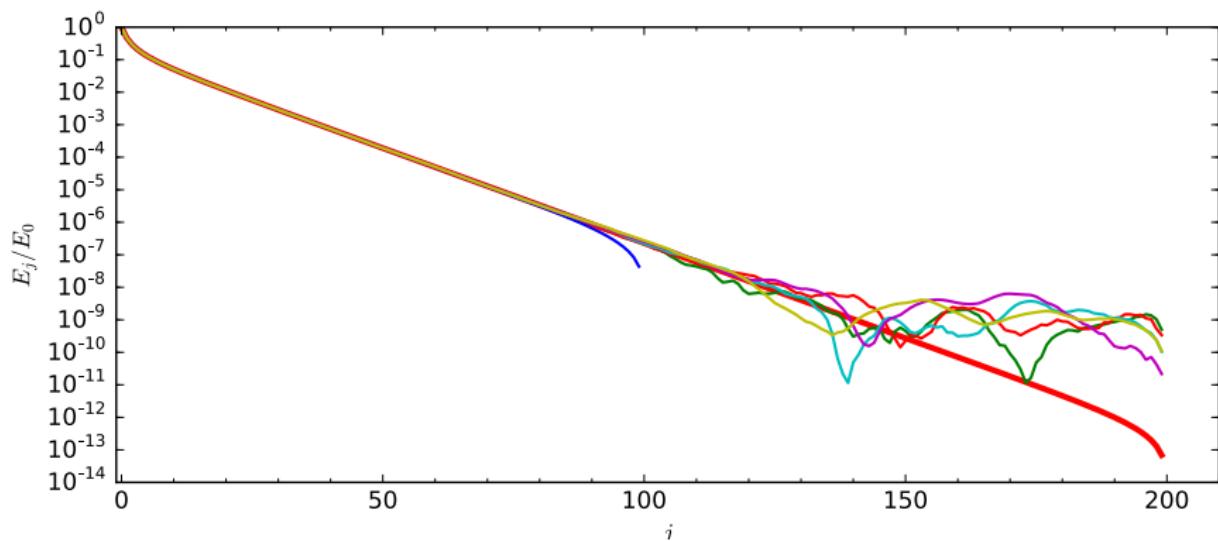
High-Temperature Families III

- ▶ Alternative methods for finding high-T solutions
- ▶ Fit low j_{max} , high-T data to generate seeds for Newton-Raphson solver → not robust as j_{max} increases
- ▶ Pad low j_{max} data with zeros, evolve within the TTF



High-Temperature Families III

- ▶ Alternative methods for finding high-T solutions
- ▶ Fit low j_{max} , high-T data to generate seeds for Newton-Raphson solver → not robust as j_{max} increases
- ▶ Pad low j_{max} data with zeros, evolve within the TTF → **isothermal evolution does not match known solution**



Results

- ▶ Low-T QP solutions robust as j_{max} increases
- ▶ Not able to find evidence that high-T solutions continued to exist at large $j_{max} \rightarrow$ possible reduction of space of QP
- ▶ **Caveat:** focused on configurations where $\alpha_0 = 1 \rightarrow$ free to set dominant energy in any $\alpha_j \rightarrow$ other configurations required for high temperatures?
- ▶ Motivation for temperature limit of $T \sim 5.5$?
- ▶ Perturbative system: massless scalar, static boundary conditions at $x = \pi/2$
- ▶ Extend to massive scalars, time-dependent boundary conditions \rightarrow activation of non-normalizable modes

Extending TTF to Driven Scalars

- ▶ Driven scalars → scalars with time-dependent boundary conditions at $x = \pi/2$

Extending TTF to Driven Scalars

- ▶ Driven scalars → scalars with time-dependent boundary conditions at $x = \pi/2$
- ▶ Examine scaling behaviour as $x \rightarrow \pi/2$: $\Phi^+(x) \sim (\cos x)^{\Delta^+}$ and $\Phi^-(x) \sim (\cos x)^{\Delta^-}$

$\Phi^+(x) \equiv \text{"normalizable"} \quad \Phi^-(x) \equiv \text{"non-normalizable"}$

Extending TTF to Driven Scalars

- ▶ Driven scalars → scalars with time-dependent boundary conditions at $x = \pi/2$
- ▶ Examine scaling behaviour as $x \rightarrow \pi/2$: $\Phi^+(x) \sim (\cos x)^{\Delta^+}$ and $\Phi^-(x) \sim (\cos x)^{\Delta^-}$
- ▶ Scalar field is linear combination of both kinds of modes
- ▶ $e_j(x)$ are same eigenfunctions of AdS & have eigenvalues $\omega_j = (2j + \Delta^+)$

$$\phi_1(t, x) = \sum_{j=0}^{\infty} a_j(t) \cos(\omega_j t + b_j(t)) e_j(x) + \sum_{\alpha=0}^{\infty} \bar{A}_{\alpha}(t) \cos(\omega_{\alpha} t + \bar{B}_{\alpha}) E_{\alpha}(x)$$
$$\Delta^{\pm} = \frac{d}{2} \pm \frac{1}{2} \sqrt{d^2 + 4m^2}$$

Extending TTF to Driven Scalars

- ▶ Driven scalars → scalars with time-dependent boundary conditions at $x = \pi/2$
- ▶ Examine scaling behaviour as $x \rightarrow \pi/2$: $\Phi^+(x) \sim (\cos x)^{\Delta^+}$ and $\Phi^-(x) \sim (\cos x)^{\Delta^-}$
- ▶ Scalar field is linear combination of both kinds of modes
- ▶ $e_j(x)$ are same eigenfunctions of AdS & have eigenvalues $\omega_j = (2j + \Delta^+)$
- ▶ $E_\alpha(x)$ are hypergeometric functions & have continuous eigenvalues ω_α

$$\phi_1(t, x) = \sum_{j=0}^{\infty} a_j(t) \cos(\omega_j t + b_j(t)) e_j(x) + \sum_{\alpha=0}^{\infty} \bar{A}_\alpha(t) \cos(\omega_\alpha t + \bar{B}_\alpha) E_\alpha(x)$$
$$\Delta^\pm = \frac{d}{2} \pm \frac{1}{2} \sqrt{d^2 + 4m^2}$$

Resonant Contributions

- ▶ $\mathcal{O}(\epsilon^3)$: source terms for resonant contributions → examine resonance conditions

$$\omega_i + \omega_j + \omega_k = \omega_\ell$$

$$\omega_i - \omega_j - \omega_k = \omega_\ell$$

$$\omega_i + \omega_j - \omega_k = \omega_\ell$$

Resonant Contributions

- ▶ $\mathcal{O}(\epsilon^3)$: source terms for resonant contributions → examine resonance conditions
- ▶ **Unforced**: restrictions on indices and mass value

$$\omega_i + \omega_j + \omega_k = \omega_\ell \quad \Rightarrow \quad i + j + k = \ell - \Delta^+ \in \mathbb{Z}^+$$

$$\omega_i - \omega_j - \omega_k = \omega_\ell \quad \Rightarrow \quad i - j - k = \ell + \Delta^+ \in \mathbb{Z}^+$$

$$\omega_i + \omega_j - \omega_k = \omega_\ell \quad \Rightarrow \quad i + j = k + \ell \in \mathbb{Z}^+$$

Resonant Contributions

- ▶ $\mathcal{O}(\epsilon^3)$: source terms for resonant contributions → examine resonance conditions
- ▶ **Unforced**: restrictions on indices and mass value → two channels vanish numerically

$$\omega_i + \omega_j + \omega_k = \omega_\ell$$

$$\omega_i - \omega_j - \omega_k = \omega_\ell$$

$$\omega_i + \omega_j - \omega_k = \omega_\ell$$

Resonant Contributions

- ▶ $\mathcal{O}(\epsilon^3)$: source terms for resonant contributions → examine resonance conditions
- ▶ **Unforced**: restrictions on indices and mass value → two channels vanish numerically
- ▶ One **non-vanishing** channel

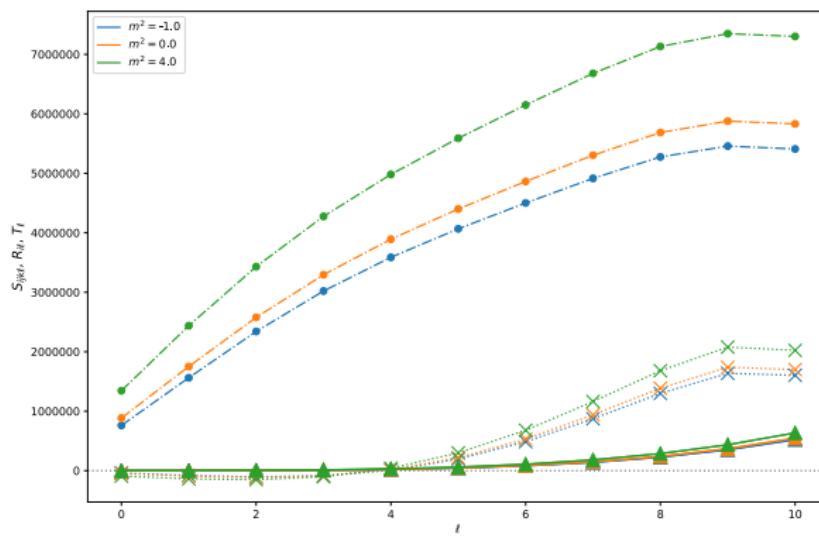
$$\omega_i + \omega_j + \omega_k = \omega_\ell$$

$$\omega_i - \omega_j - \omega_k = \omega_\ell$$

$$\omega_i + \omega_j - \omega_k = \omega_\ell$$

Resonant Contributions

- ▶ $\mathcal{O}(\epsilon^3)$: source terms for resonant contributions → examine resonance conditions
- ▶ **Unforced**: restrictions on indices and mass value → two channels vanish numerically
- ▶ One non-vanishing channel → sum over i, j with $i + j \leq \ell$



Special Values of Non-normalizable Frequencies

- ▶ **Forced:** choose values of ω_α that satisfy resonance condition \rightarrow two normalizable, two non-normalizable

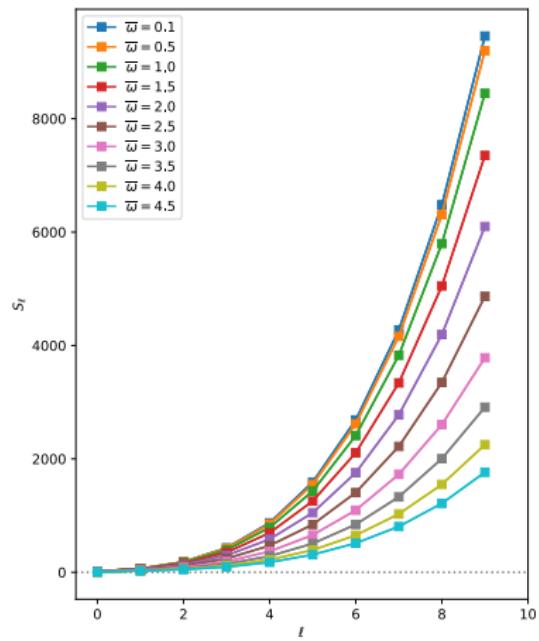
$$\pm\omega_I \pm \omega_J \pm \omega_K \pm \omega_\ell = 0 \quad I \in \{i, \alpha\}$$

Special Values of Non-normalizable Frequencies

- ▶ **Forced:** choose values of ω_α that satisfy resonance condition \rightarrow two normalizable, two non-normalizable
- ▶ **Equal frequencies:** $\omega_\alpha = \bar{\omega}$

$$\pm\omega_I \pm \omega_J \pm \omega_K \pm \omega_\ell = 0 \quad I \in \{i, \alpha\}$$

$$\omega_i + \bar{\omega} - \bar{\omega} = \omega_\ell$$

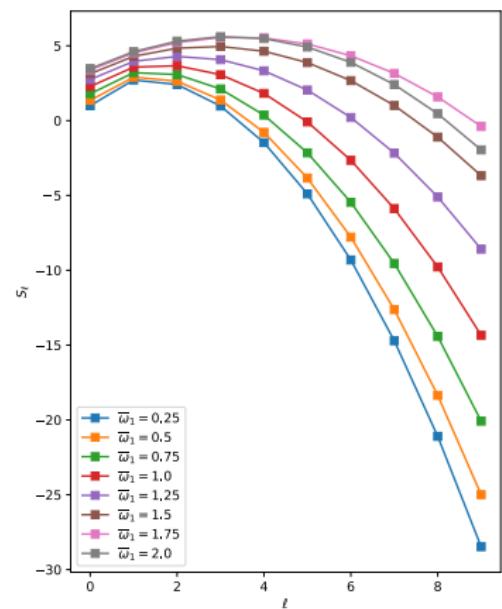


Special Values of Non-normalizable Frequencies

- ▶ **Forced:** choose values of ω_α that satisfy resonance condition \rightarrow two normalizable, two non-normalizable
- ▶ Equal frequencies: $\omega_\alpha = \bar{\omega}$
- ▶ Add to integer: $\bar{\omega}_1 + \bar{\omega}_2 = 2n$

$$\pm\omega_I \pm \omega_J \pm \omega_K \pm \omega_\ell = 0 \quad I \in \{i, \alpha\}$$

$$\left\{ \begin{array}{l} (++): \omega_i + 2n = \omega_\ell \quad \forall \ell \geq n \\ (+-): \omega_i - 2n = \omega_\ell \quad \forall n \\ (-+): -\omega_i + 2n = \omega_\ell \quad \forall n \geq \ell + d \end{array} \right\}$$



Special Values of Non-normalizable Frequencies

- ▶ **Forced:** choose values of ω_α that satisfy resonance condition \rightarrow two normalizable, two non-normalizable
- ▶ Equal frequencies: $\omega_\alpha = \bar{\omega}$
- ▶ Add to integer: $\bar{\omega}_1 + \bar{\omega}_2 = 2n$
- ▶ Source terms give flow equations for amplitude/phase of ϕ_1 , e.g. two equal non-normalizable frequencies \rightarrow c.f. normalizable modes only

$$\frac{2\omega_\ell}{\epsilon^2} \frac{da_\ell}{dt} = 0$$

$$\frac{2\omega_\ell a_\ell}{\epsilon^2} \frac{db_\ell}{dt} = -S_\ell a_\ell \bar{A}_{\bar{\omega}}^2$$

Results

- ▶ Confirm two of three resonant channels vanish for massive scalar (all normalizable)¹³
- ▶ First TTF formulation with time-dependent boundary conditions
- ▶ No naturally-vanishing source terms for ω_α considered → sum resonant channels
- ▶ Flow equations are linear in amplitude/phase variables $a_\ell(t)$, $b_\ell(t)$
- ▶ **Further work:** quasi-periodic solutions? Conserved quantities? Energy cascades? Relation to driven holographic CFTs¹⁴?

¹²Biasi *et al.* [1810.04753]

¹³Rangamani *et al.* [1502.05726]

Conclusions



Thanks

- ▶ Supervisor: Andrew Frey (University of Winnipeg)
- ▶ PhD Committee: (), (), (), ()
- ▶ Co-authors: Nils Deppe (Cornell)
- ▶ University of Winnipeg and University of Manitoba
- ▶ Westgrid & Compute Canada

References

- ▶ J. Maldacena, *The Large N limit of superconformal field theories and supergravity*, Int. J. Theor. Phys. 38 (1999) 1113-1133, [hep-th/9711200].
- ▶ P. Bizoń and A. Rostworowski, *On weakly turbulent instability of anti-de Sitter space*, Phys. Rev. Lett. 107 (2011) 031102, [1104.3702].
- ▶ M. Choptuik, *Universality and scaling in gravitational collapse of a massless scalar field*, Phys. Rev. Lett. 70 (1993) 9-12.
- ▶ N. Deppe and A. R. Frey, *Classes of Stable Initial Data for Massless and Massive Scalars in Anti-de Sitter Spacetime*, JHEP 12 (2015) 004, [1508.02709].
- ▶ R. Brito, V. Cardoso, and J. V. Rocha, *Interacting shells in AdS spacetime and chaos*, Phys. Rev. D94 (2016), no. 2 024003, [1602.03535].
- ▶ N. Deppe, A. Kolly, A. R. Frey, and G. Kunstatter, *Black Hole Formation in AdS Einstein-Gauss-Bonnet Gravity*, J. High Energ. Phys. (2016) 2016: 87, [1608.05402].
- ▶ A. Buchel, S. L. Liebling, and L. Lehner, *Boson Stars in AdS Spacetime*, Phys. Rev. D87 (2013) 123006, [1304.4166].

References

- ▶ V. Balasubramanian, A. Buchel, S. R. Green, L. Lehner, and S. L. Liebling, *Holographic Thermalization, Stability of Anti-de Sitter Space, and the Fermi-Pasta-Ulam Paradox*, Phys. Rev. Lett. 113 (2014) 071601, [1403.6471].
- ▶ B. Craps, O. Evnin, and J. Vanhoof, *Renormalization group, secular term resummation and AdS (in)stability*, JHEP 10 (2014) 048, [1407.6273].
- ▶ S. R. Green, A. Maillard, L. Lehner, and S. L. Liebling, *Islands of stability and recurrence times in AdS*, Phys. Rev. D92 (2015) 084001, [1507.08261].
- ▶ B. Craps, O. Evnin, and J. Vanhhof, *Renormalization, averaging, conservation laws and AdS (in)stability*, J. High Energ. Phys. 1501 (2015) 108, [1412.3249].
- ▶ A. Biasi, B. Craps, and O. Evnin, *Energy Returns in Global AdS_4* , [1810.04753].
- ▶ M. Rangamani, M. Rozali and A. Wong, *Driven Holographic CFTs*, JHEP 04 (2015) 093, [1502.05726].