Gravitational Collapse in Anti-de Sitter Space

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PhD Thesis Defence

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- Gravitational Collapse
- Massive Scalars in AdS_5 [arXiv:1711.00454]
  - Scalar Field Collapse in AdS
  - Classifying Phases
  - Phase Diagram & Energy Cascades
- High-Temperature QP Solutions in AdS_4
  - The Two-Time Formalism (TTF)
  - Quasi-Periodic Solutions
  - High-Temperature Families
- Examining Instabilities Due to Driven Scalars in AdS [arXiv:1912.07143]
  - Extending TTF to Driven Scalars
  - Resonant Contributions
  - Special Values of Non-normalizable Frequencies
- Conclusions

Gravitational Collapse

- Numerical studies of gravitational collapse in Minkowski spacetime: horizon size = power law[1], mass gap
- AdS/CFT $\rightarrow$ thermal quench in gauge theory $\Leftrightarrow$ formation of black hole in gravitational theory
- Massless scalar fields in AdS: unstable against generic initial data, no mass gap[2] $\rightarrow$ c.f. Minkowski
- Stability for specific initial data below critical energy
- Perturbative theory for stable/nearly-stable solutions
- **Nonlinear theory:** continue with exploration of phase space[3]
- **Perturbative theory:** effects of truncation, space of solutions, evolution of nearly-stable solutions

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[1] Choptuik PRL70 9 (1993)

[2] Bizoń & Rostworowski [1104.3702]

[3] Deppe & Frey [1508.02709]

Scalar Field Collapse in AdS

- Minimally-coupled scalar field in AdS$_5$ (dual to 4D CFT)
- Spherical symmetry, Schwarzschild-like coordinates $\rightarrow A(t,x)$, $\delta(t,x)$

$$ds^2 = \frac{\ell^2}{\cos^2(x/\ell)}\left(-Ae^{-2\delta}dt^2 + A^{-1}dx^2 + \sin^2(x/\ell)d\Omega^{d-1}\right)$$

Scalar Field Collapse in AdS

- Minimally-coupled scalar field in AdS$_5$ (dual to 4D CFT)
- Spherical symmetry, Schwarzschild-like coordinates $\rightarrow A(t,x)$, $\delta(t,x)$
- Einstein + Klein-Gordon $\Rightarrow$ constraint equations
- Interior gauge $\delta(t,x=0)=0$, spherical symmetry $\partial_x\phi(t,x=0)=0$
- Horizon formation when $A(t_H,x_H)\ll 1$

$$\partial_x\delta = -\left(\Pi^2+\Phi^2\right)\sin(x)\cos(x))$$

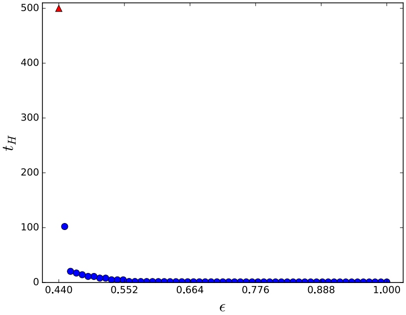
$$\partial_x M = \frac{\tan^{d-1}(x)}{2}\left(A(\Pi^2+\Phi^2)+\frac{\mu^2\phi^2}{\cos^2(x)}\right)$$

$$\Pi(t=0,x)=\epsilon\exp\left(-\frac{\tan^2(x)}{\sigma^2}\right)$$

Phase space: $\mu,\ \sigma$

Stable vs Unstable Profiles

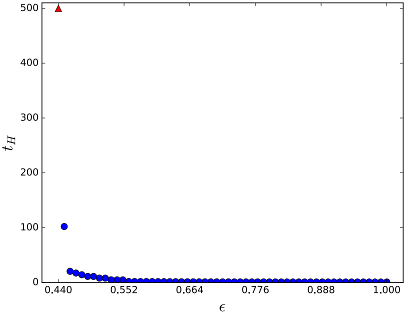
- Stable: abrupt jump in t_H when $\epsilon < \epsilon_{crit}$



$\mu = 0,\ \sigma = 1.5$

Stable vs Unstable Profiles

- Stable: abrupt jump in t_H when $\epsilon < \epsilon_{crit}$
- Unstable: fit $t_H \approx a\epsilon^{-p} + b$ for $t_H \geq 60 \to$ unstable when $p \approx 2$



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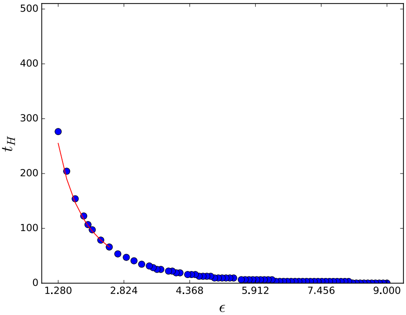
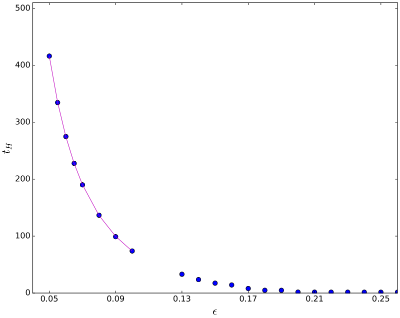


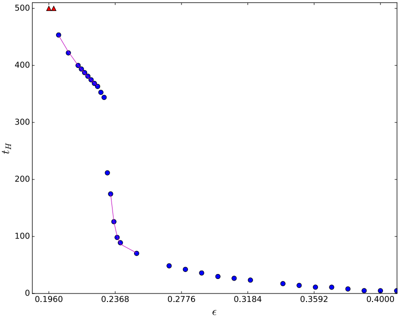
$\mu = 5,\ \sigma = 0.25$

Metastable Profiles

- Scaling of $p > 2$



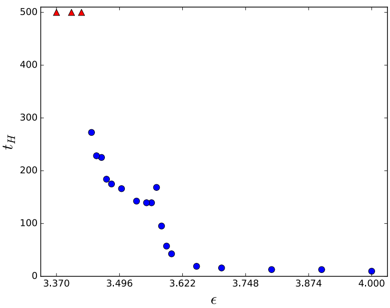
$\mu = 5,\ \sigma = 1.7$



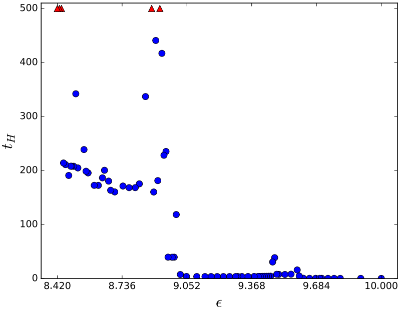
$\mu = 0.5,\ \sigma = 1.7$

Irregular Profiles I

- No scaling



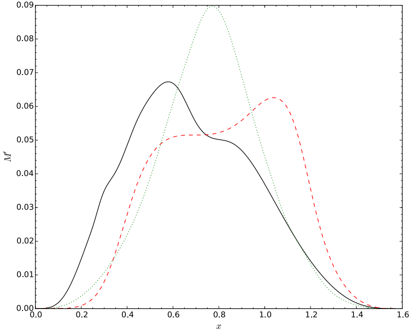
$\mu = 5,\ \sigma = 0.34$



$\mu = 20,\ \sigma = 0.16$

Irregular Profiles II

- Evidence of chaotic behaviour $\rightarrow$ possible self-interaction
- Previous chaotic evolution only seen in thin-shell interactions[4] in AdS, scalar collapse in Gauss-Bonnet gravity[5]



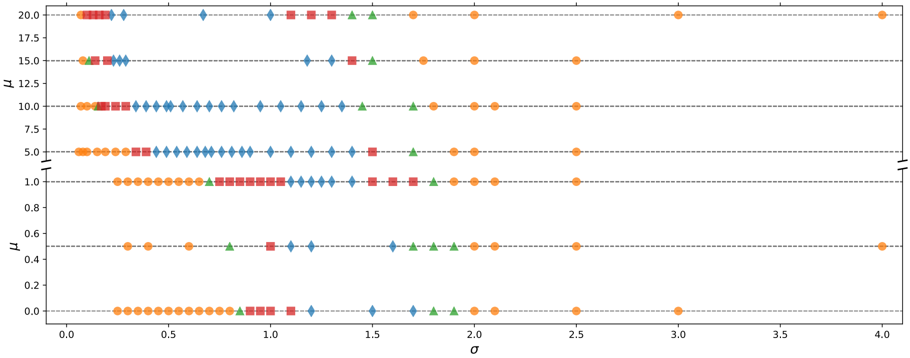
$$\mu = 0,\ \sigma = 1.1,\ \epsilon = 1.01$$

$$t = 60, 62, 64$$

[4] Brito *et al.* [1602.03535]

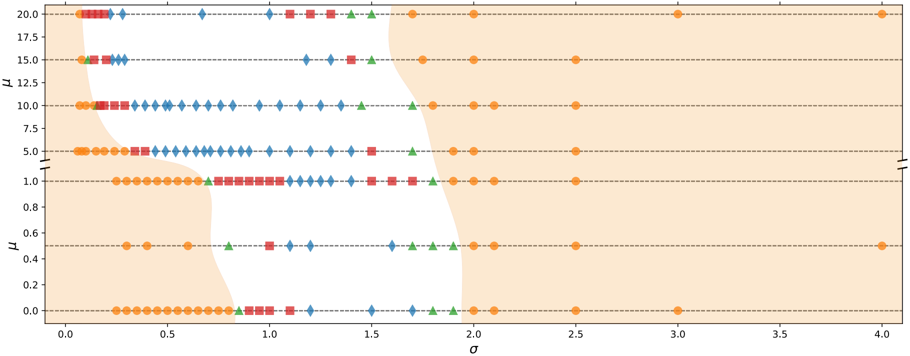
[5] Deppe, Kolly, *et al.* [1608.05402]

Phase Diagram



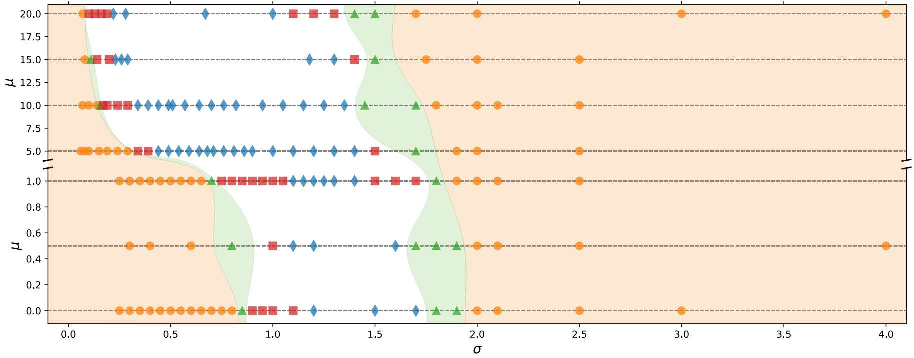
Phase Diagram

- Unstable



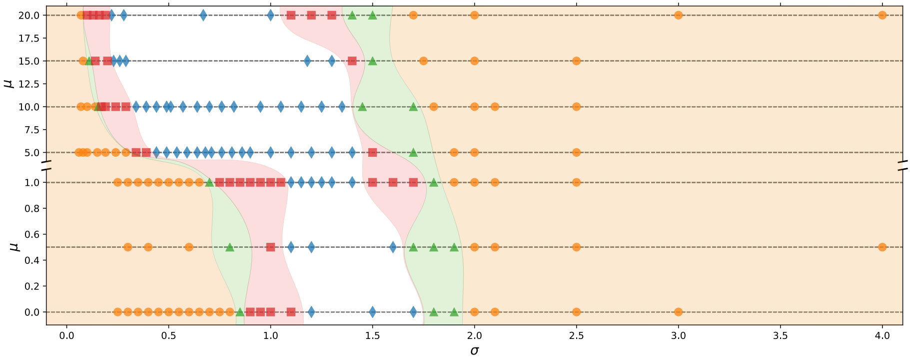
Phase Diagram

- Unstable, metastable,



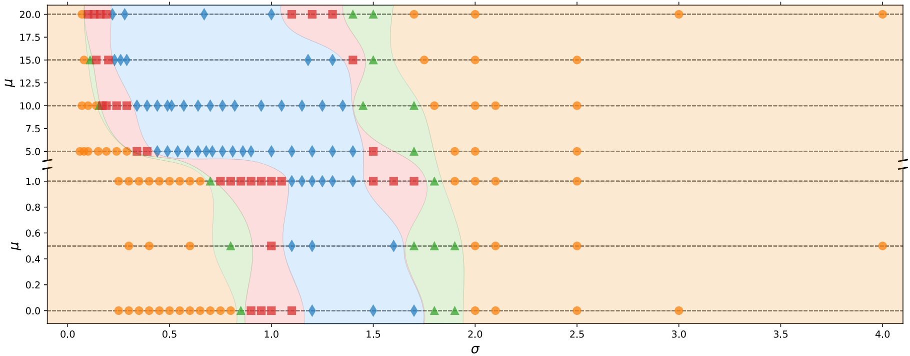
Phase Diagram

- Unstable, metastable, irregular,



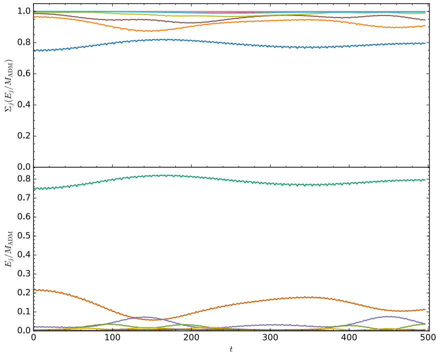
Phase Diagram

- Unstable, metastable, irregular, and stable initial data

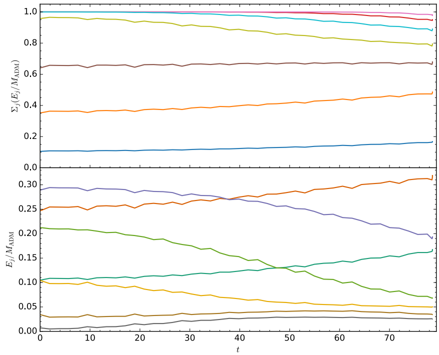


Energy Cascades

- Stable solutions: direct and inverse energy cascades



$\mu = 0,\ \sigma = 1.8,\ \epsilon = 0.13$
Stable



$\mu = 0,\ \sigma = 0.25,\ \epsilon = 2.28$
Unstable

Results

- First full phase diagram of stability in AdS$_5$ $\rightarrow$ islands of stability and "shorelines"
- Evidence of metastable and irregular phases at finite ϵ
- Fate of metastable phase as $\epsilon \rightarrow 0$ yet to be determined
- Irregular phase contains quasi-stable initial data[6,7] $\rightarrow$ first evidence for weakly chaotic evolution in massless, spherically-symmetry scalars in AdS
- Metastable and irregular data to be studied in multiscale perturbation theory

[6] Deppe & Frey [1508.02709]

[7] Buchel *et al.* [1304.4166]

The Two-Time Formalism (TTF) I

- Small perturbations in AdS_4: expand scalar field, metric functions in ϵ
- $\mathcal{O}(\epsilon)$: ϕ_1 in terms of eigenfunctions of AdS, $e_j(x)$
- Integer eigenvalues $\omega_j = (2j + d) \to$ fully resonant spectrum
- $\mathcal{O}(\epsilon^2)$: backreaction on metric in terms of ϕ_1

$$\phi_1(t,x) = \sum_{j=0}^{\infty} A_j(t)\cos(\omega_j t + B_j(t))\, e_j(x)$$

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- $\mathcal{O}(\epsilon)$: ϕ_1 in terms of eigenfunctions of AdS, $e_j(x)$
- Integer eigenvalues $\omega_j = (2j + d) \to$ fully resonant spectrum
- $\mathcal{O}(\epsilon^2)$: backreaction on metric in terms of ϕ_1
- $\mathcal{O}(\epsilon^3)$: source terms for resonant contributions
- Define slow time $\tau \equiv \epsilon^2 t$

$$-2\omega_\ell \frac{dA_\ell(\tau)}{d\tau} = \sum_{i\neq\ell} \sum_{j\neq\ell}^{\ell\leq i+j} f_1 \left(A_i, A_j, A_{i+j-\ell}, B_i, B_j, B_{i+j-\ell}\right)$$

$$-2\omega_\ell A_\ell \frac{dB_\ell(\tau)}{d\tau} = \sum_{i\neq\ell} \sum_{j\neq\ell}^{\ell\leq i+j} f_2 \left(A_i, A_j, A_{i+j-\ell}, B_i, B_j, B_{i+j-\ell}\right)$$

The Two-Time Formalism (TTF) II

- Energy exchange between modes through slowly varying amplitude $A_j(\tau)$ and phase $B_j(\tau)$ to resist collapse[8]
- Resummation techniques absorb resonances into amplitude/phase variables[9]
- Solve by truncating series to $j_{max} < \infty$
- Solutions must be robust as $j_{max} \to \infty$
- Examine quasi-periodic families of solutions[10]
- Develop numerical techniques for extending $j_{max} \gtrsim 100$
- Verify families of solutions remain valid as j_{max} increases

[8] Balasubramanian *et al.* [1403.6471]
[9] Craps *et al.* [1407.6273]
[10] Green *et al.* [1507.08261]

Quasi-Periodic Solutions I

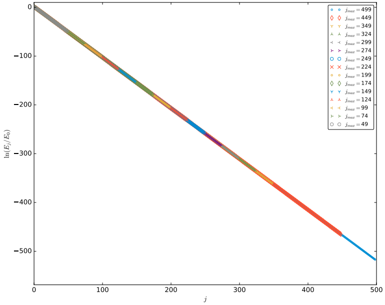
- Quasi-periodic ansatz $A_j = \alpha_j e^{i\beta_j\tau} \to$ TTF equations become time-independent when $\beta_j = \beta_0 + j(\beta_1 - \beta_0)$
- Solve QP equation with Newton-Raphson $\to$ seed equation $\alpha_j \propto e^{-j}$, $j \neq 0$ for low j_{max}
- TTF: conserved quantities[11] E, $N \to$ classify solutions by $T \equiv E/N$
- T_i, R_{ij}, $S_{ijk\ell}$ calculated numerically

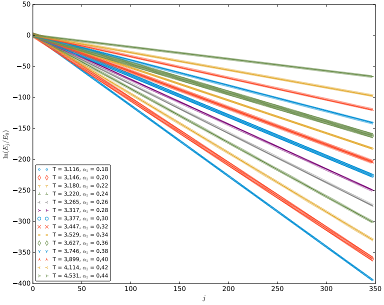
$$2\omega_\ell\alpha_\ell\beta_\ell = T_\ell\alpha_\ell^3 + \sum_{i\neq\ell} R_{i\ell}\alpha_i^2\alpha_\ell + \sum_{i\neq\ell}^{\ell\leq i+j}\sum_{j\neq\ell} S_{ij(i+j-\ell)\ell}\alpha_i\alpha_j\alpha_{i+j-\ell}$$

[11] Craps *et al.* [1412.3249]

Quasi-Periodic Solutions II

- Solutions found for $3 \geq T \gtrsim 4.6$
- Able to extend existing solutions from $j_{max} \sim 100$ to $j_{max} = 500$
- Robust in $j_{max} \to \infty$ limit

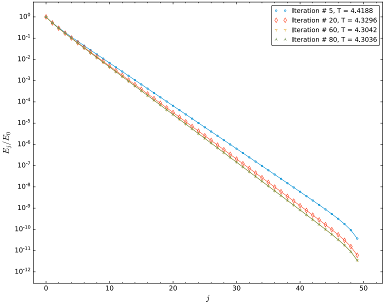


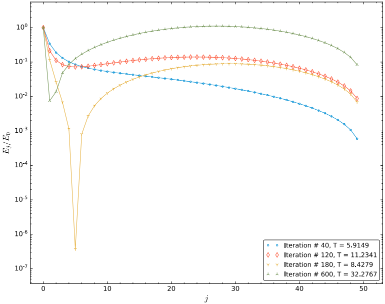


BC, Deppe, & Frey: In progress

High-Temperature Solutions I

- Perturb by $\delta E \rightarrow$ new solutions have energy $E + \delta E$, N, and $T + \delta T$
- Repeat process to $T_{max} = (2j_{max} + d)$
- Project back to QP solution surface at constant α_1 or T
- Loss of smooth profile above a certain temperature at constant α_1

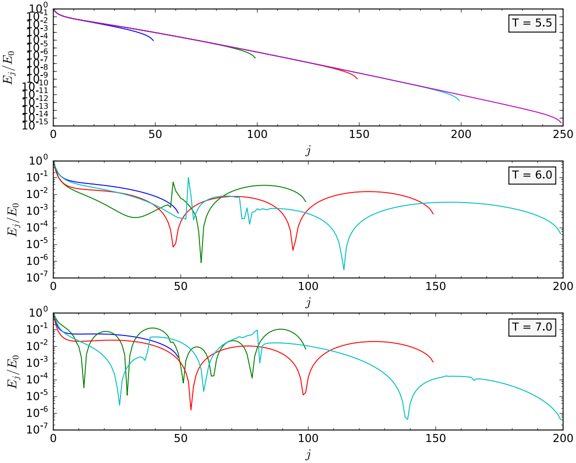




Cownden, Deppe, & Frey: In progress

High-Temperature Solutions II

- And at constant T



Cownden, Deppe, & Frey: In progress

High-Temperature Solutions III

- Alternative methods for finding high-T solutions

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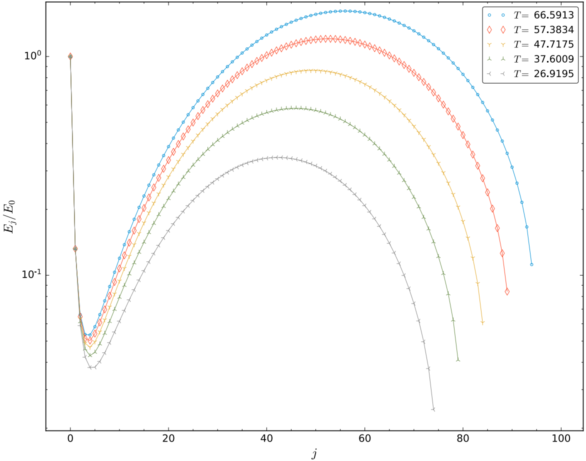
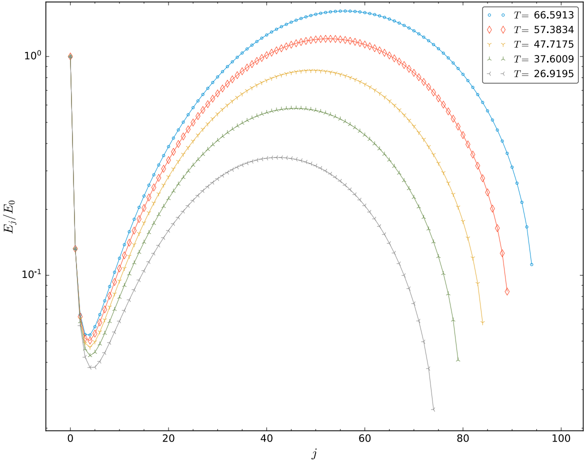
- Fit low j_{max}, high-T data to generate seeds for Newton-Raphson solver



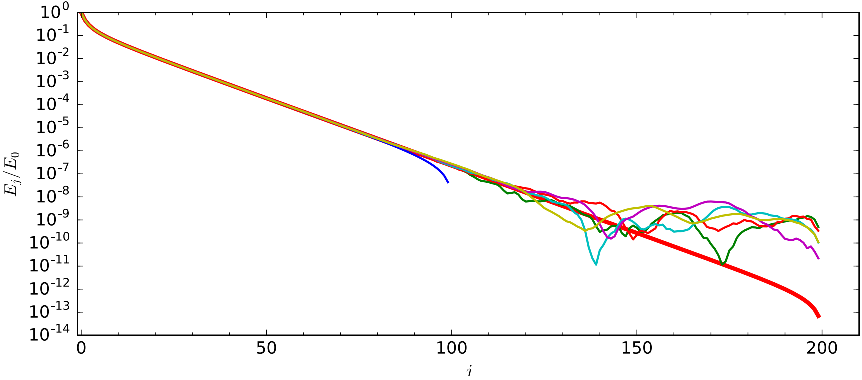
High-Temperature Solutions III

- Alternative methods for finding high-T solutions
- Fit low j_{max}, high-T data to generate seeds for Newton-Raphson solver $\rightarrow$ not robust as j_{max} increases



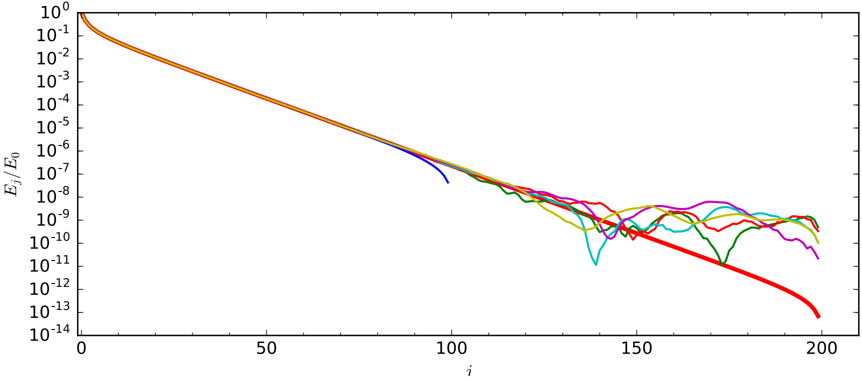
High-Temperature Solutions III

- Alternative methods for finding high-T solutions
- Fit low j_{max}, high-T data to generate seeds for Newton-Raphson solver $\rightarrow$ not robust as j_{max} increases
- Pad low j_{max} data with zeros, evolve within the TTF



High-Temperature Solutions III

- Alternative methods for finding high-T solutions
- Fit low j_{max}, high-T data to generate seeds for Newton-Raphson solver $\rightarrow$ not robust as j_{max} increases
- Pad low j_{max} data with zeros, evolve within the TTF $\rightarrow$ isothermal evolution does not match known solution



Results

- Low-T QP solutions robust as j_{max} increases
- Not able to find evidence that high-T solutions continued to exist at large j_{max} $\rightarrow$ possible reduction of space of QP
- **Caveat**: focused on configurations where $\alpha_0 = 1$ $\rightarrow$ free to set dominant energy in any α_j $\rightarrow$ other configurations required for high temperatures?
- Motivation for temperature limit of $T \sim 5.5$?
- Perturbative system: massless scalar, static boundary conditions at $x = \pi/2$
- Extend to massive scalars, time-dependent boundary conditions $\rightarrow$ activation of non-normalizable modes

Extending TTF to Driven Scalars

- Driven scalars $\rightarrow$ scalars with time-dependent boundary conditions at $x = \pi/2$
- $\mathcal{O}(\epsilon)$: $\phi_1(t, x = \pi/2) = f(t)$

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- $\mathcal{O}(\epsilon)$: $\phi_1(t, x = \pi/2) = f(t)$
- Examine scaling behaviour as $x \rightarrow \pi/2$: $\Phi^+(x) \sim (\cos x)^{\Delta^+}$ and $\Phi^-(x) \sim (\cos x)^{\Delta^-}$

$\Phi^+(x) \equiv$ "normalizable" $\Phi^-(x) \equiv$ "non-normalizable"

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- Examine scaling behaviour as $x \rightarrow \pi/2$: $\Phi^+(x) \sim (\cos x)^{\Delta^+}$ and $\Phi^-(x) \sim (\cos x)^{\Delta^-}$
- Scalar field is linear combination of both kinds of modes
- $e_j(x)$ are same eigenfunctions of AdS & have eigenvalues $\omega_j = (2j + \Delta^+)$

$$\phi_1(t, x) = \sum_{j=0}^{\infty} A_j(t) \cos\left(\omega_j t + B_j(t)\right) e_j(x) + \sum_{\alpha=0}^{\infty} \bar{A}_\alpha(t) \cos\left(\omega_\alpha t + \bar{B}_\alpha\right) E_\alpha(x)$$

$$\Delta^{\pm} = \frac{d}{2} \pm \frac{1}{2}\sqrt{d^2 - 4m^2}$$

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- Scalar field is linear combination of both kinds of modes
- $e_j(x)$ are same eigenfunctions of AdS & have eigenvalues $\omega_j = (2j + \Delta^+)$
- $E_\alpha(x)$ are hypergeometric functions & have continuous eigenvalues ω_α

$$\phi_1(t,x) = \sum_{j=0}^{\infty} A_j(t) \cos\left(\omega_j t + B_j(t)\right) e_j(x) + \sum_{\alpha=0}^{\infty} \bar{A}_\alpha(t) \cos\left(\omega_\alpha t + \bar{B}_\alpha\right) E_\alpha(x)$$

$$\Delta^\pm = \frac{d}{2} \pm \frac{1}{2}\sqrt{d^2 - 4m^2}$$

Resonant Contributions

- $\mathcal{O}(\epsilon^2)$: backreaction on metric in terms of ϕ_1
- $\mathcal{O}(\epsilon^3)$: source terms for resonant contributions $\rightarrow$ examine resonance conditions

$$\omega_i + \omega_j + \omega_k = \omega_\ell$$
$$\omega_i - \omega_j - \omega_k = \omega_\ell$$
$$\omega_i + \omega_j - \omega_k = \omega_\ell$$

Resonant Contributions

- $\mathcal{O}(\epsilon^2)$: backreaction on metric in terms of ϕ_1
- $\mathcal{O}(\epsilon^3)$: source terms for resonant contributions $\rightarrow$ examine resonance conditions
- All normalizable: restrictions on indices **and** mass value

$$
\begin{aligned}
\omega_i + \omega_j + \omega_k = \omega_\ell \quad &\Rightarrow \quad i + j + k = \ell - \Delta^+ \in \mathbb{Z}^+ \\
\omega_i - \omega_j - \omega_k = \omega_\ell \quad &\Rightarrow \quad i - j - k = \ell + \Delta^+ \in \mathbb{Z}^+ \\
\omega_i + \omega_j - \omega_k = \omega_\ell \quad &\Rightarrow \quad i + j = k + \ell \in \mathbb{Z}^+
\end{aligned}
$$

Resonant Contributions

- $\mathcal{O}(\epsilon^2)$: backreaction on metric in terms of ϕ_1
- $\mathcal{O}(\epsilon^3)$: source terms for resonant contributions $\rightarrow$ examine resonance conditions
- All normalizable: restrictions on indices **and** mass value $\rightarrow$ two channels vanish numerically

$$\omega_i + \omega_j + \omega_k = \omega_\ell$$

$$\omega_i - \omega_j - \omega_k = \omega_\ell$$

$$\omega_i + \omega_j - \omega_k = \omega_\ell$$

Resonant Contributions

- $\mathcal{O}(\epsilon^2)$: backreaction on metric in terms of ϕ_1
- $\mathcal{O}(\epsilon^3)$: source terms for resonant contributions $\rightarrow$ examine resonance conditions
- All normalizable: restrictions on indices **and** mass value $\rightarrow$ two channels vanish numerically
- One non-vanishing channel

$$\omega_i + \omega_j + \omega_k = \omega_\ell$$
$$\omega_i - \omega_j - \omega_k = \omega_\ell$$
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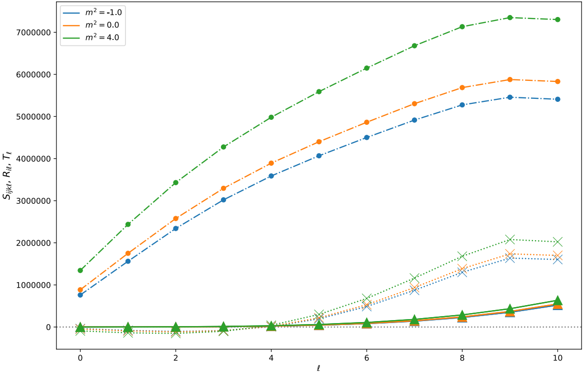
Resonant Contributions

- $\mathcal{O}(\epsilon^2)$: backreaction on metric in terms of ϕ_1
- $\mathcal{O}(\epsilon^3)$: source terms for resonant contributions $\rightarrow$ examine resonance conditions
- All normalizable: restrictions on indices **and** mass value $\rightarrow$ two channels vanish numerically
- One non-vanishing channel $\rightarrow$ evaluate numerically



Special Values of Non-normalizable Frequencies

Results

Conclusions

- Collapse of scalar field in AdS $\Leftrightarrow$ thermalization of dual CFT
- **Nonlinear theory:** "islands of stability", metastable & irregular phases, chaotic behaviour from self-interaction
- Weakly turbulent energy cascade to short length scales $\rightarrow$ TTF for inverse cascades
- **Perturbative theory:** QP solutions robust in $j_{max} \rightarrow \infty$, high-T solutions are not; space of stable solutions is restricted by T_{th}
- **Next steps**
  - Physical interpretation of T_{th}
  - Ways to construct robust QP solutions with $T > T_{th}$
- **Future:** develop theory for *massive* TTF $\rightarrow$ less symmetry in equations $\therefore$ fewer cancelations of resonant terms; TTF in AdS_5; time-dependent boundary conditions

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- University of Winnipeg and University of Manitoba
- Westgrid & Compute Canada

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