Gravitational Collapse in Anti-de Sitter Space

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PhD Thesis Defence

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Outline

- Gravitational Collapse
- Massive Scalars in AdS_5
  - Scalar Field Collapse in AdS
  - Classifying Phases
  - Phase Diagram
- High-Temperature QP Solutions in AdS_4
  - The Two-Time Formalism (TTF)
  - Quasi-Periodic Solutions
  - High-Temperature Families
- Driven Scalars in AdS
  - Extending TTF to Driven Scalars
  - Resonant Contributions
  - Special Values of Non-normalizable Frequencies
- Conclusions

Gravitational Collapse

- AdS/CFT[1] $\rightarrow$ thermal quench in gauge theory $\Leftrightarrow$ formation of black hole in gravitational theory
- Massless scalar fields in AdS: unstable against generic initial data, no minimum amplitude[2] $\rightarrow$ c.f. Minkowski[3]
- Stability for specific initial data below critical amplitude
- **Nonlinear theory:** continue[4] with exploration of phase space
- **Perturbative theory:** effects of truncation, space of solutions, time-dependent boundary conditions

[1] Maldacena [hep-th/9711200]
[2] Bizoń & Rostworowski [1104.3702]
[3] Choptuik PRL70 9 (1993)
[4] Deppe & Frey [1508.02709]

B Cownden, N Deppe, and AR Frey, *Phase Diagram of Stability for Massive Scalars in Anti-de Sitter Spacetime*, Phys.Rev.D 102 (2020) 026015, [1711.00454].

Scalar Field Collapse in AdS

- Minimally-coupled scalar field in AdS_5 (dual to 4D CFT)
- Spherical symmetry, Schwarzschild-like coordinates $\rightarrow A(t,x)$, $\delta(t,x)$
- Horizon formation when $A(t_H, x_H) \ll 1$

$$ds^2 = \frac{\ell^2}{\cos^2(x/\ell)} \left(-Ae^{-2\delta}dt^2 + A^{-1}dx^2 + \sin^2(x/\ell)d\Omega^{d-1}\right)$$

Scalar Field Collapse in AdS

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- Spherical symmetry, Schwarzschild-like coordinates $\rightarrow A(t,x)$, $\delta(t,x)$
- Horizon formation when $A(t_H, x_H) \ll 1$
- Einstein equations $\Rightarrow$ constraints
- Klein-Gordon equations $\Rightarrow$ dynamics
- Examine behaviour near critical amplitude for different masses, widths

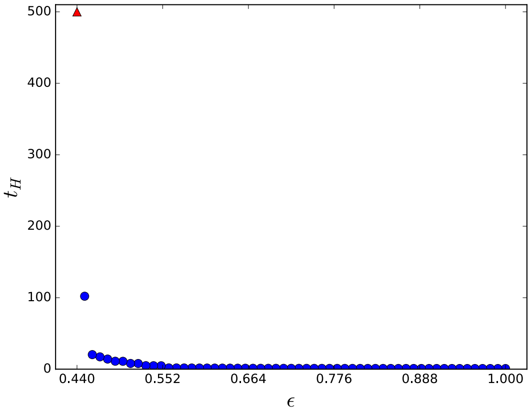
$$\partial_x M_{\mathsf{ADM}} = \frac{\tan^{d-1}(x)}{2}\left(A(\Pi^2+\Phi^2)+\frac{\mu^2\phi^2}{\cos^2(x)}\right)$$

$$\Pi(t=0,x) = \epsilon \exp\left(-\frac{\tan^2(x)}{\sigma^2}\right) \qquad \text{Phase space: } (\mu,\ \sigma)$$

Stable vs Unstable Profiles

Blue dot = collapse detected, red triangle = no collapse detected for $t \leq t_{max}$

- Stable: abrupt jump in t_H when $\epsilon < \epsilon_{crit}$

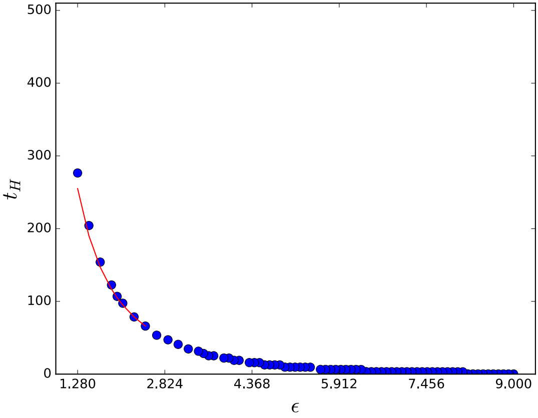


$\mu = 0$, $\sigma = 1.5$

Stable vs Unstable Profiles

Blue dot = collapse detected, red triangle = no collapse detected for $t \leq t_{max}$

- Stable: abrupt jump in t_H when $\epsilon < \epsilon_{crit}$
- Unstable: fit $t_H \approx a\epsilon^{-p} + b$ for $t_H \geq 60 \rightarrow$ perturbatively unstable when $p \approx 2$ (TTF)

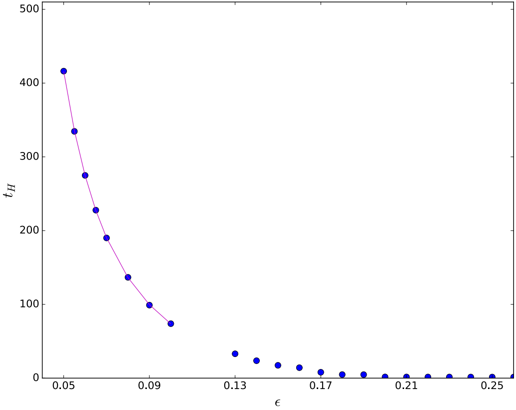


$\mu = 5$, $\sigma = 0.25$

Metastable & Irregular Profiles

Blue dot = collapse detected, red triangle = no collapse detected for $t \leq t_{max}$

- Metastable: fit $t_H \approx a\epsilon^{-p} + b$ for $t_H \geq 60$ $\rightarrow p > 2$

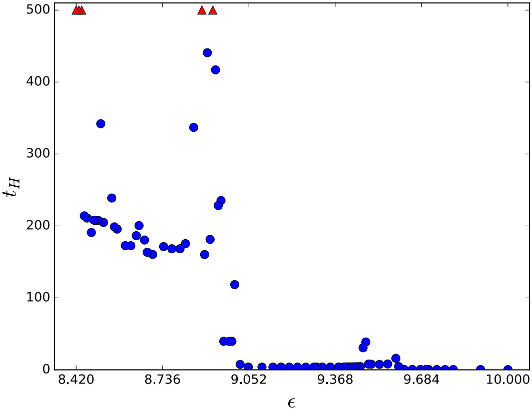


$\mu = 5,\ \sigma = 1.7$

Metastable & Irregular Profiles

Blue dot = collapse detected, red triangle = no collapse detected for $t \leq t_{max}$

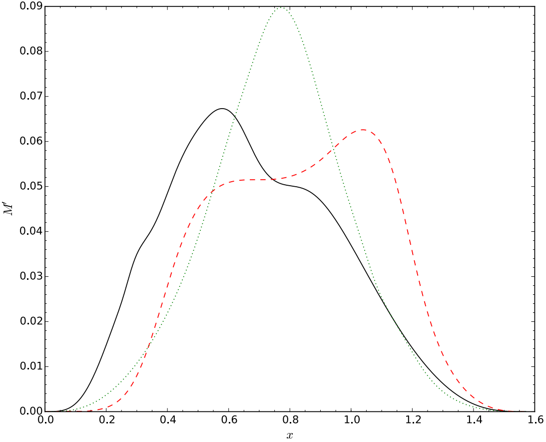
- Metastable: fit $t_H \approx a\epsilon^{-p} + b$ for $t_H \geq 60$ $\rightarrow p > 2$
- Irregular: no scaling



$\mu = 20,\ \sigma = 0.16$

Observations of Chaotic Behaviour

- Possible chaotic evolution $\rightarrow$ scalar self-interaction
- Previous chaotic evolution only seen in thin-shell interactions[5] in AdS, scalar collapse in Gauss-Bonnet gravity[6]



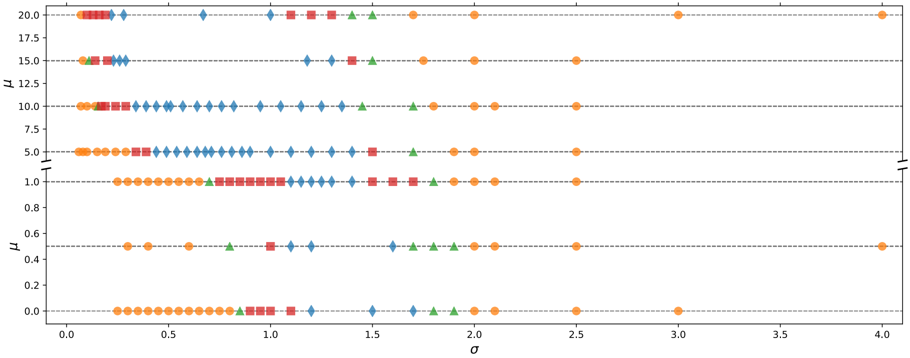
$$\mu = 0,\ \sigma = 1.1,\ \epsilon = 1.01$$

$$t = 60, 62, 64$$

[5] Brito *et al.* [1602.03535]

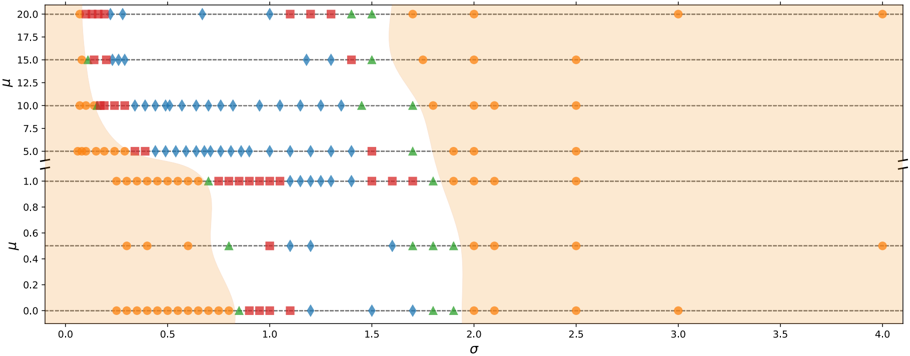
[6] Deppe, Kolly, *et al.* [1608.05402]

Phase Diagram



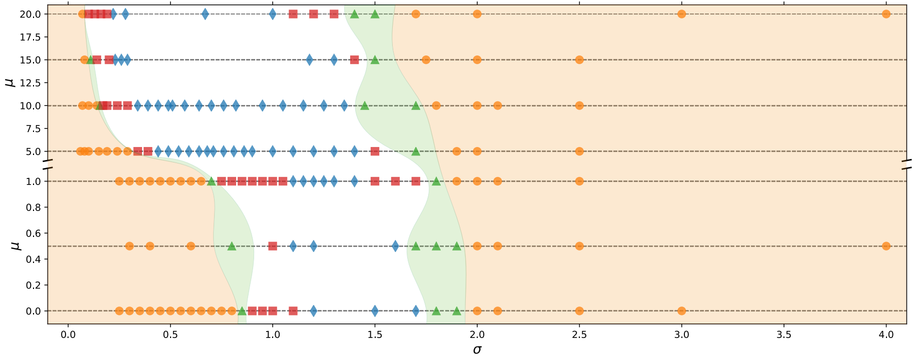
Phase Diagram

- Unstable,



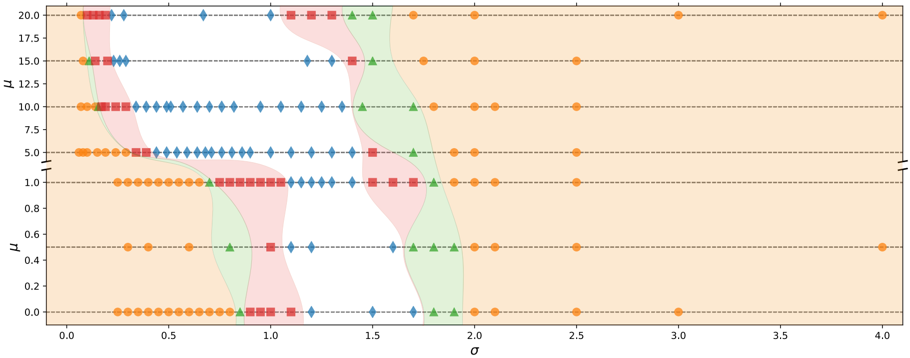
Phase Diagram

- Unstable, metastable,



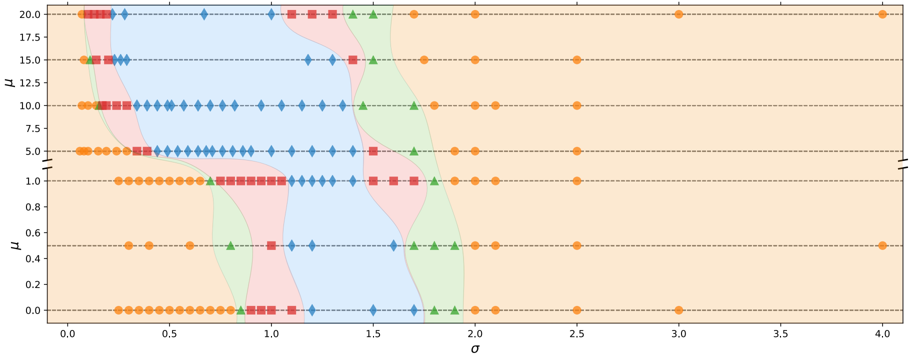
Phase Diagram

- Unstable, metastable, irregular,



Phase Diagram

- Unstable, metastable, irregular, and stable initial data



Results

- First full phase diagram of stability in AdS_5 $\rightarrow$ islands of stability and "shorelines"
- Evidence of metastable and irregular phases at finite ϵ
- Fate of metastable phase as $\epsilon \rightarrow 0$ yet to be determined
- Irregular phase contains quasi-stable initial data[7,8] $\rightarrow$ first evidence for weakly chaotic evolution in massless, spherically-symmetry scalars in AdS
- Metastable and irregular data to be studied in multiscale perturbation theory

[7] Deppe & Frey [1508.02709]

[8] Buchel *et al.* [1304.4166]

B Cownden, N Deppe, and AR Frey, *On the Stability of High-Temperature, Quasi-Periodic Solutions for Massless Scalars in* AdS_4, In progress.

The Two-Time Formalism (TTF)

- Small perturbations in AdS_4: expand scalar field, metric functions in ϵ
- $\mathcal{O}(\epsilon)$: ϕ_1 in terms of eigenfunctions of AdS, $e_j(x)$
- Integer eigenvalues $\omega_j = (2j + d) \rightarrow$ fully resonant spectrum

$$\phi_1(t, x) = \sum_{j=0}^{\infty} \left(A_j(t) e^{i\omega_j t} + \bar{A}_j(t) e^{-i\omega_j t} \right) e_j(x)$$

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- $\mathcal{O}(\epsilon)$: ϕ_1 in terms of eigenfunctions of AdS, $e_j(x)$
- Integer eigenvalues $\omega_j = (2j + d) \to$ fully resonant spectrum
- Secular growth of resonant contributions $\to$ scalar field collapse
- $\mathcal{O}(\epsilon^3)$: source term for resonant contributions
- Complex amplitudes vary with "slow time" $\tau \to$ flow equation to absorb resonances[9]

$$-2i\omega_\ell \frac{dA_\ell(\tau)}{d\tau} = \sum_{i,j,k} f_{ijk}^{(\ell)} \bar{A}_i A_j A_k$$

[9] Balasubramanian *et al.* [1403.6471]

Quasi-Periodic Solutions I

- Renormalization flow techniques to cancel an infinite number of resonances $\rightarrow$ express non-vanishing ones analytically[10]
- Need to truncate number of modes to find solutions: $j_{max} < \infty$ (must be robust as $j_{max} \rightarrow \infty$)

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- Need to truncate number of modes to find solutions: $j_{max} < \infty$ (must be robust as $j_{max} \to \infty$)
- Quasi-periodic[11] solutions $A_j = \alpha_j e^{i\beta_j \tau}$ with $\alpha_j, \beta_j \in \mathbb{R} \rightarrow$ TTF equations become time-independent when $\beta_j = \beta_0 + j(\beta_1 - \beta_0)$
- TTF: conserved quantities[12] $(E, N) \rightarrow$ classify solutions by $T \equiv E/N$
- Solve QP equation using Newton-Raphson method

$$2\omega_\ell \alpha_\ell \beta_\ell = T_\ell \alpha_\ell^3 + \sum_{i \neq \ell} R_{i\ell} \alpha_i^2 \alpha_\ell + \sum_{i \neq \ell}^{\ell \leq i+j} \sum_{j \neq \ell} S_{ij(i+j-\ell)\ell} \alpha_i \alpha_j \alpha_{i+j-\ell}$$

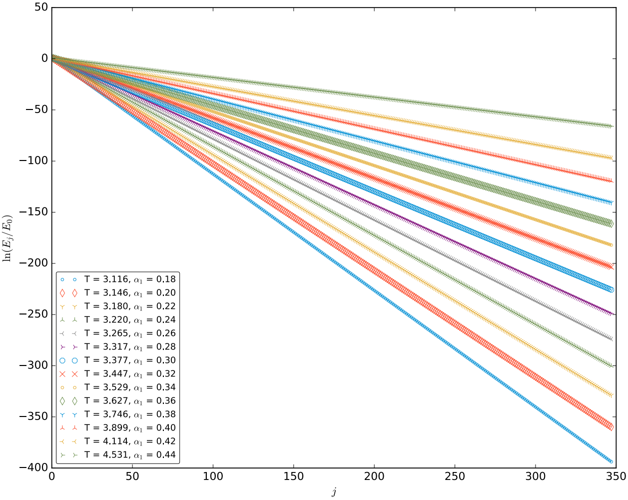
[10] Craps *et al.* [1407.6273]

[11] Balasubramanian *et al.* [1403.6471]

[12] Craps *et al.* [1412.3249]

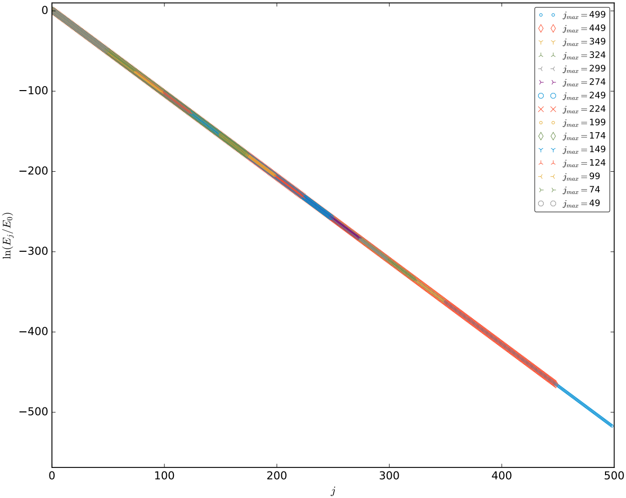
Quasi-Periodic Solutions II

- Solutions found for $3 \leq T \lesssim 5.5$



Quasi-Periodic Solutions II

- Solutions found for $3 \leq T \lesssim 5.5$
- Able to extend existing solutions from $j_{max} \sim 100$ to $j_{max} = 500$
- Robust in $j_{max} \to \infty$ limit



High-Temperature Families I

- Perturb by $\delta E \rightarrow$ new solutions have energy $E + \delta E$, N, and $T + \delta T$
- Solve for updated values of $\alpha_j + \delta\alpha_j, \beta_j + \delta\beta_j$
- Use updated values as seeds to resolve QP equation
- Repeat process up to T_{max}[13]

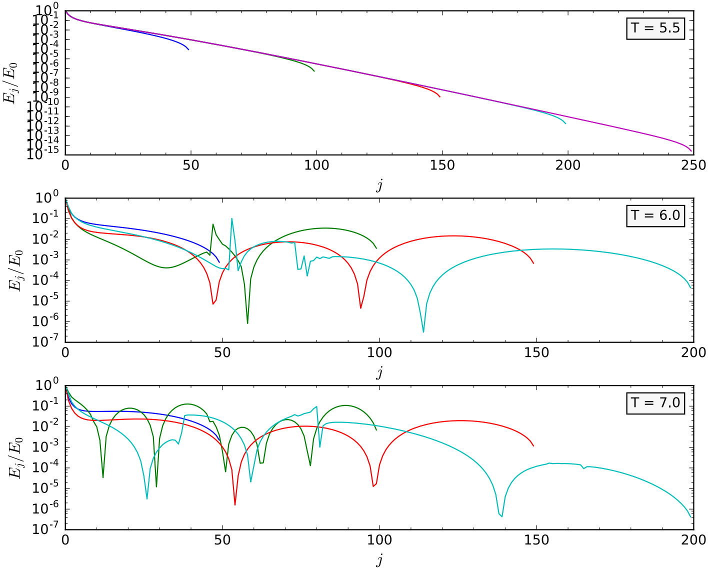
[13] Green *et al.* [1507.08261]

High-Temperature Families I

- Perturb by $\delta E \to$ new solutions have energy $E + \delta E$, N, and $T + \delta T$
- Solve for updated values of $\alpha_j + \delta\alpha_j, \beta_j + \delta\beta_j$
- Use updated values as seeds to resolve QP equation
- Repeat process up to T_{max}[13]
- **Issue**: $\delta\alpha_j, \delta\beta_j$ become larger than α_j, β_j at high temperatures
- No solutions that remain robust as j_{max} increases

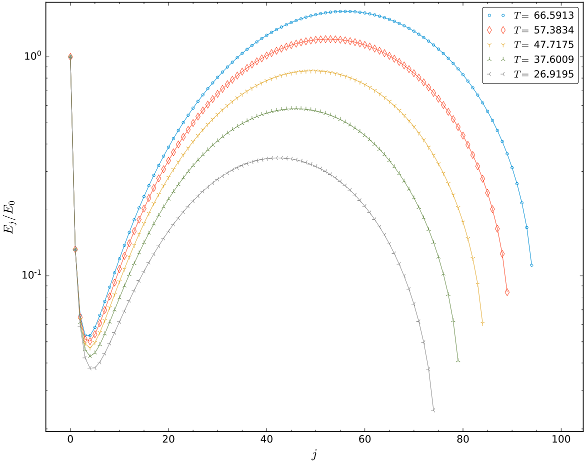
[13] Green *et al.* [1507.08261]

High-Temperature Families II



High-Temperature Families III

- Alternative methods for finding high-T solutions explored
- E.g. fit low j_{max}, high-T data to generate seeds for Newton-Raphson solver



Results

- Low-T QP solutions robust as j_{max} increases
- Not able to find evidence that high-T solutions continued to exist at large j_{max} $\rightarrow$ possible reduction of space of QP
- **Caveat**: focused on configurations where $\alpha_0 = 1$ $\rightarrow$ free to set dominant energy in any α_j $\rightarrow$ other configurations required for high temperatures?
- **To do**: Motivation for temperature limit of $T \sim 5.5$?
- Perturbative system: massless scalar, static boundary conditions at $x = \pi/2$
- Extend to massive scalars, time-dependent boundary conditions $\rightarrow$ activation of non-normalizable modes

B Cownden, *Examining Instabilities Due to Driven Scalars in AdS*, `JHEP_252P_0420` , [1912.07143].

Extending TTF to Driven Scalars

- Driven scalars $\rightarrow$ equation for ϕ_1 has inhomogeneous term at $x = \pi/2$

$$\partial_t^2 \phi_1 + \hat{L}\phi_1 = \delta(x - \pi/2)\mathcal{F}(t)$$

Extending TTF to Driven Scalars

- Driven scalars $\rightarrow$ equation for ϕ_1 has inhomogeneous term at $x = \pi/2$
- Examine scaling behaviour as $x \rightarrow \pi/2$: $\Phi^+(x) \sim (\cos x)^{\Delta^+}$ and $\Phi^-(x) \sim (\cos x)^{\Delta^-}$

$\Phi^+(x) \equiv$ "normalizable" $\qquad$ $\Phi^-(x) \equiv$ "non-normalizable"

Extending TTF to Driven Scalars

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- Examine scaling behaviour as $x \rightarrow \pi/2$: $\Phi^{+}(x) \sim (\cos x)^{\Delta^{+}}$ and $\Phi^{-}(x) \sim (\cos x)^{\Delta^{-}}$
- Scalar field is linear combination of both kinds of modes
- $e_j(x)$ are same eigenfunctions of AdS & have eigenvalues $\omega_j = (2j + \Delta^{+})$

$$\phi_1(t,x) = \sum_{j=0}^{\infty} a_j(t) \cos\left(\omega_j t + b_j(t)\right) e_j(x) + \sum_{\alpha=0}^{\infty} \mathcal{A}_\alpha(t) \cos\left(\omega_\alpha t + \mathcal{B}_\alpha\right) E_\alpha(x)$$

$$\Delta^{\pm} = \frac{d}{2} \pm \frac{1}{2}\sqrt{d^2 + 4m^2}$$

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- Scalar field is linear combination of both kinds of modes
- $e_j(x)$ are same eigenfunctions of AdS & have eigenvalues $\omega_j = (2j + \Delta^+)$
- $E_\alpha(x)$ are hypergeometric functions with frequencies ω_α from $\mathcal{F}(t)$

$$\phi_1(t,x) = \sum_{j=0}^{\infty} a_j(t) \cos\left(\omega_j t + b_j(t)\right) e_j(x) + \sum_{\alpha=0}^{\infty} \mathcal{A}_\alpha(t) \cos\left(\omega_\alpha t + \mathcal{B}_\alpha\right) E_\alpha(x)$$

$$\Delta^\pm = \frac{d}{2} \pm \frac{1}{2}\sqrt{d^2 + 4m^2}$$

Resonant Contributions I

- $\mathcal{O}(\epsilon^3)$: source terms for resonant contributions $\rightarrow$ examine resonance conditions

$$\omega_i + \omega_j + \omega_k = \omega_\ell$$
$$\omega_i - \omega_j - \omega_k = \omega_\ell$$
$$\omega_i + \omega_j - \omega_k = \omega_\ell$$

Resonant Contributions I

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- **Unforced**: restrictions on indices and mass value

$$
\begin{aligned}
\omega_i + \omega_j + \omega_k = \omega_\ell \quad &\Rightarrow \quad i + j + k = \ell - \Delta^+ \in \mathbb{Z}^+ \\
\omega_i - \omega_j - \omega_k = \omega_\ell \quad &\Rightarrow \quad i - j - k = \ell + \Delta^+ \in \mathbb{Z}^+ \\
\omega_i + \omega_j - \omega_k = \omega_\ell \quad &\Rightarrow \quad i + j = k + \ell \in \mathbb{Z}^+
\end{aligned}
$$

Resonant Contributions I

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Resonant Contributions I

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- **Unforced**: restrictions on indices and mass value $\rightarrow$ two channels vanish numerically
- One non-vanishing channel

$$
\begin{aligned}
\omega_i + \omega_j + \omega_k = \omega_\ell \quad &\Rightarrow \quad i + j + k = \ell - \Delta^+ \in \mathbb{Z}^+ \\
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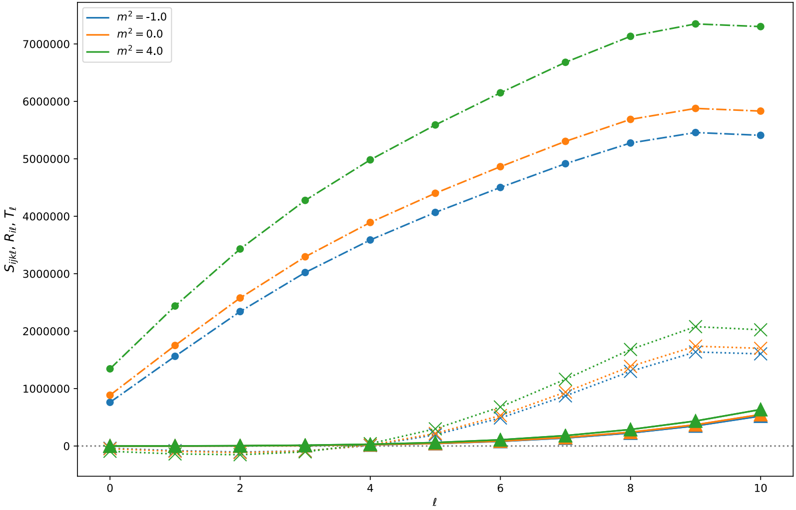
\omega_i + \omega_j - \omega_k = \omega_\ell \quad &\Rightarrow \quad i + j = k + \ell \in \mathbb{Z}^+
\end{aligned}
$$

Resonant Contributions II

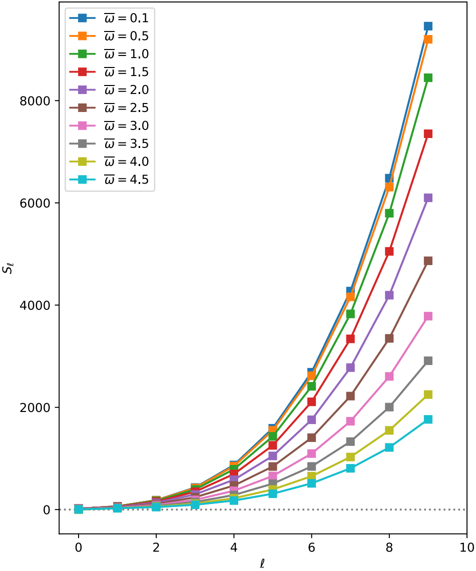
- Sum over i, j with $i + j \leq \ell$ (dots: a_ℓ^3, triangles: $a_i^2 a_\ell$, X: $a_i a_j a_{i+j-\ell}$)



Special Values of Non-normalizable Frequencies

- **Forced**: ω_α set by driving term
- Single frequency: $\omega_\alpha = \bar{\omega}$ $\rightarrow$ one channel

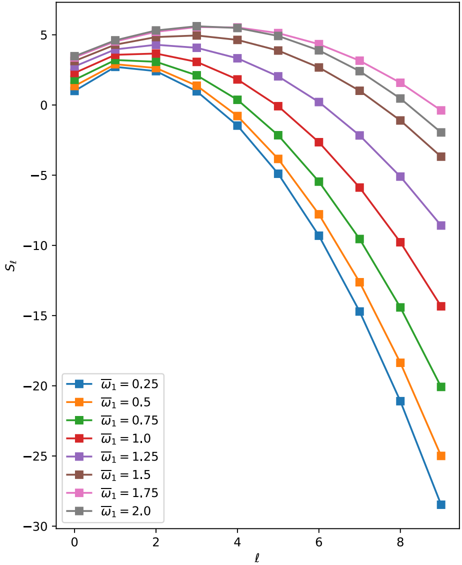
$$\omega_i + \bar{\omega} - \bar{\omega} = \omega_\ell$$



Special Values of Non-normalizable Frequencies

- **Forced**: ω_α set by driving term
- Single frequency: $\omega_\alpha = \bar{\omega}$ $\rightarrow$ one channel
- Add to integer: $\bar{\omega}_1 + \bar{\omega}_2 = 2n \rightarrow$ three channels

$$
\begin{aligned}
(++) &: \omega_i + 2n = \omega_\ell \quad \forall\, \ell \geq n \\
(+-) &: \omega_i - 2n = \omega_\ell \quad \forall\, n \\
(-+) &: -\omega_i + 2n = \omega_\ell \quad \forall\, n \geq \ell + d
\end{aligned}
$$



Flow Equations

- Source terms give flow equations for amplitude/phase of normalizable modes
- No naturally vanishing resonances, *c.f.* static boundary conditions
- E.g. single frequency $\rightarrow$ single channel $\rightarrow$ equations decouple

$$\frac{2\omega_\ell}{\epsilon^2}\frac{da_\ell}{dt} = 0 \qquad \text{and} \qquad \frac{2\omega_\ell}{\epsilon^2}\frac{db_\ell}{dt} = f^{(\ell)}\mathcal{A}_{\bar{\omega}}^2$$

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- No naturally vanishing resonances, *c.f.* static boundary conditions
- E.g. single frequency $\rightarrow$ single channel $\rightarrow$ equations decouple
- E.g. add to integer $\rightarrow$ sum all three channels $\rightarrow$ equations are coupled with single power of normalizable amplitude

$$\frac{2\omega_\ell}{\mathcal{A}_1\mathcal{A}_2\epsilon^2}\frac{da_\ell}{dt} = \sum_{(++)} f^{(\ell)}_{(++)} a_{n-\ell-d} \sin\left(b_{n-\ell-d} - \mathcal{B}_1 - \mathcal{B}_2\right)$$

$$+ \sum_{(+-)} f^{(\ell)}_{(+-)} a_{\ell-n} \sin\left(b_{\ell-n} - \mathcal{B}_1 - \mathcal{B}_2\right) + \sum_{(-+)} f^{(\ell)}_{(-+)} a_{\ell+n} \sin\left(b_{\ell+n} - \mathcal{B}_1 - \mathcal{B}_2\right)$$

$$\frac{2\omega_\ell a_\ell}{\mathcal{A}_1\mathcal{A}_2\epsilon^2}\frac{db_\ell}{dt} = f^{(\ell)} a_\ell + \sum_{(++)} f^{(\ell)}_{(++)} a_{n-\ell-d} \cos\left(b_{n-\ell-d} - \mathcal{B}_1 - \mathcal{B}_2\right)$$

$$+ \sum_{(+-)} f^{(\ell)}_{(+-)} a_{\ell-n} \cos\left(b_{\ell-n} - \mathcal{B}_1 - \mathcal{B}_2\right) + \sum_{(-+)} f^{(\ell)}_{(-+)} a_{\ell+n} \cos\left(b_{\ell+n} - \mathcal{B}_1 - \mathcal{B}_2\right)$$

Results

- Confirm two of three resonant channels vanish for massive scalar (all normalizable)[14]
- First TTF formulation with time-dependent boundary conditions
- No naturally-vanishing source terms $\rightarrow$ sum resonant channels
- Some flow equations decouple amplitude/phase variables $a_\ell(t)$, $b_\ell(t)$
- *N.B.* normalizable modes are still present $\rightarrow$ sum resonances from both types of modes
- **Further work**: quasi-periodic solutions[15]? Conserved quantities? Energy cascades?

[14] Biasi *et al.* [1810.04753]
[15] Carracedo *et al.* [1612.07701]

Conclusions

- Examine the stability of AdS to scalar field collapse in various dimensions in perturbative & non-perturbative regimes
- Addition of time-dependent boundary conditions to TTF description

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- First evidence of weakly chaotic evolution of scalars in AdS_5

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- First evidence of weakly chaotic evolution of scalars in AdS$_5$
- Verified low-T QP solutions are robust in the $j_{max} \rightarrow \infty$ limit
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- First evidence of weakly chaotic evolution of scalars in AdS_5
- Verified low-T QP solutions are robust in the $j_{max} \rightarrow \infty$ limit
- No evidence of high-T QP solutions for $\alpha_0 = 1$ family
- Developed perturbative theory for massive scalars with time-dependent boundary conditions
- Derived flow equations for amplitude/phase variables for some choices of driving term $\rightarrow$ evaluated source terms numerically

Thanks

- Supervisor: Andrew Frey
- PhD Committee: Derek Krepski, Gabor Kunstatter, Robert Mann, Khodr Shamseddine
- Co-authors: Nils Deppe
- University of Winnipeg and University of Manitoba
- Westgrid & Compute Canada

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