Gravitational Collapse in Anti-de Sitter Space

Brad Cownden
PhD Thesis Defence

July 28th, 2020

University of Manitoba
The University of Winnipeg

Outline

Gravitational Collapse

- AdS/CFT[1] $\rightarrow$ thermal quench in gauge theory $\Leftrightarrow$ formation of black hole in gravitational theory
- Massless scalar fields in AdS: unstable against generic initial data, no mass gap[2] $\rightarrow$ c.f. Minkowski[3]
- Stability for specific initial data below critical energy
- **Nonlinear theory:** continue[4] with exploration of phase space
- **Perturbative theory:** effects of truncation, space of solutions, time-dependent boundary conditions

[1] Maldacena [hep-th/9711200]
[2] Bizoń & Rostworowski [1104.3702]
[3] Choptuik PRL70 9 (1993)
[4] Deppe & Frey [1508.02709]

B Cownden, N Deppe, and AR Frey, *Phase Diagram of Stability for Massive Scalars in Anti-de Sitter Spacetime*, Phys.Rev.D 102 (2020) 026015, [1711.00454].

Scalar Field Collapse in AdS

- Minimally-coupled scalar field in AdS_5 (dual to 4D CFT)
- Spherical symmetry, Schwarzschild-like coordinates $\rightarrow A(t,x)$, $\delta(t,x)$

$$ds^2 = \frac{\ell^2}{\cos^2(x/\ell)}\left(-Ae^{-2\delta}dt^2 + A^{-1}dx^2 + \sin^2(x/\ell)d\Omega^{d-1}\right)$$

Scalar Field Collapse in AdS

- Minimally-coupled scalar field in AdS$_5$ (dual to 4D CFT)
- Spherical symmetry, Schwarzschild-like coordinates $\rightarrow A(t,x)$, $\delta(t,x)$
- Einstein + Klein-Gordon $\Rightarrow$ constraint equations
- Interior gauge $\delta(t, x=0) = 0$, spherical symmetry $\partial_x \phi(t, x=0) = 0$
- Horizon formation when $A(t_H, x_H) \ll 1$

$$\partial_x \delta = -\left(\Pi^2 + \Phi^2\right) \sin(x) \cos(x)\big)$$

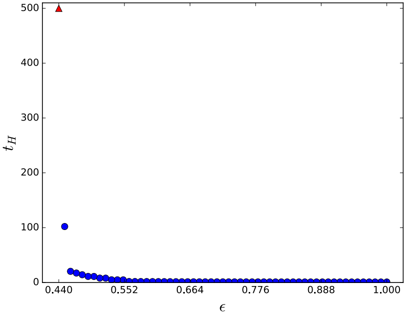
$$\partial_x M = \frac{\tan^{d-1}(x)}{2} \left(A(\Pi^2 + \Phi^2) + \frac{\mu^2 \phi^2}{\cos^2(x)} \right)$$

$$\Pi(t=0, x) = \epsilon \exp\left(-\frac{\tan^2(x)}{\sigma^2} \right)$$

Phase space: μ, σ

Stable vs Unstable Profiles

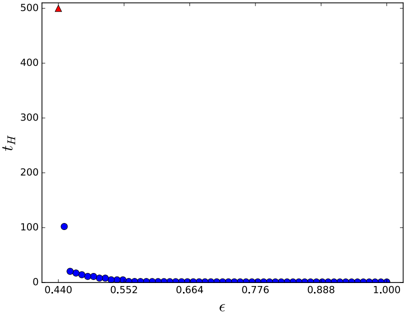
- Stable: abrupt jump in t_H when $\epsilon < \epsilon_{crit}$



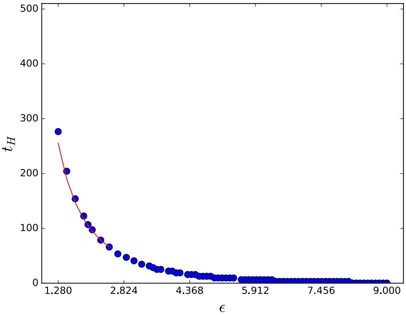
$\mu = 0$, $\sigma = 1.5$

Stable vs Unstable Profiles

- Stable: abrupt jump in t_H when $\epsilon < \epsilon_{crit}$
- Unstable: fit $t_H \approx a\epsilon^{-p} + b$ for $t_H \geq 60 \rightarrow$ unstable when $p \approx 2$



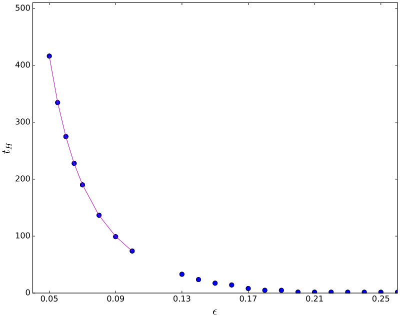
$\mu = 0,\ \sigma = 1.5$



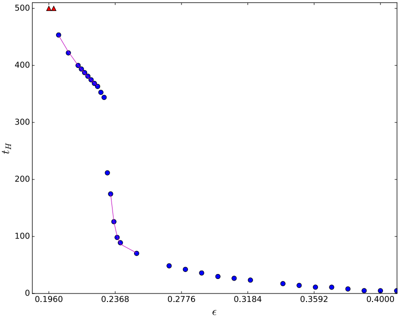
$\mu = 5,\ \sigma = 0.25$

Metastable Profiles

- Scaling of $p > 2$



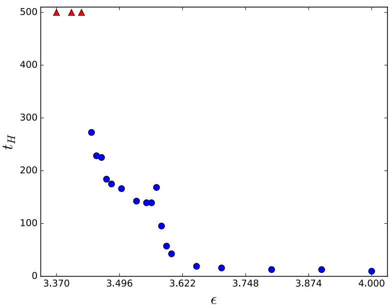
$\mu = 5,\ \sigma = 1.7$



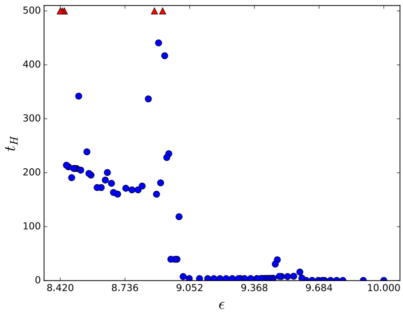
$\mu = 0.5,\ \sigma = 1.7$

Irregular Profiles I

- No scaling



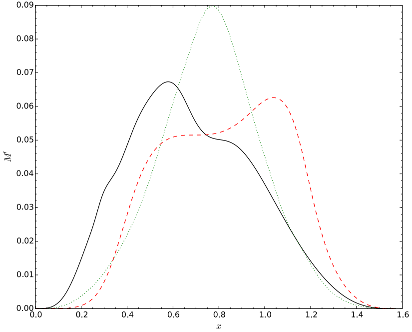
$\mu = 5,\ \sigma = 0.34$



$\mu = 20,\ \sigma = 0.16$

Irregular Profiles II

- Evidence of chaotic behaviour $\rightarrow$ possible self-interaction
- Previous chaotic evolution only seen in thin-shell interactions[5] in AdS, scalar collapse in Gauss-Bonnet gravity[6]



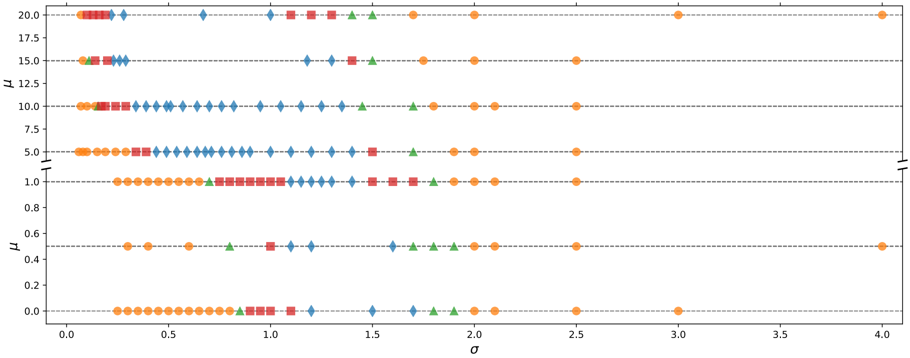
$t = 60, 62, 64$

$\mu = 0,\ \sigma = 1.1,\ \epsilon = 1.01$

[5] Brito *et al.* [1602.03535]

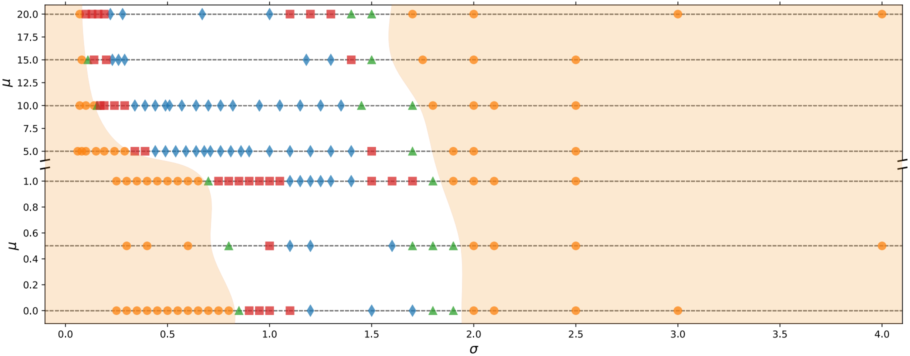
[6] Deppe, Kolly, *et al.* [1608.05402]

Phase Diagram



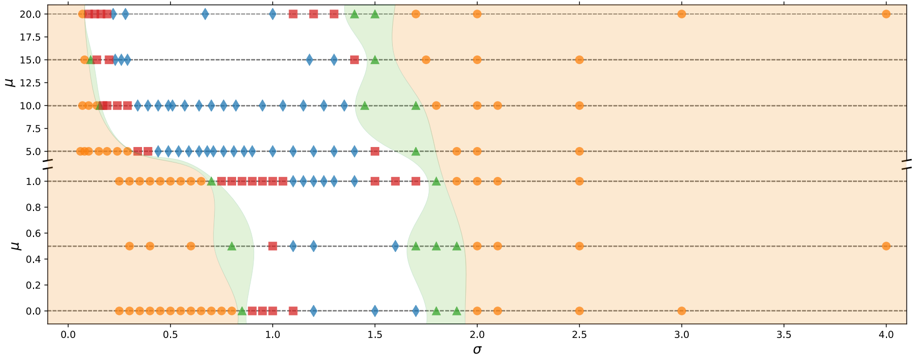
Phase Diagram

- Unstable



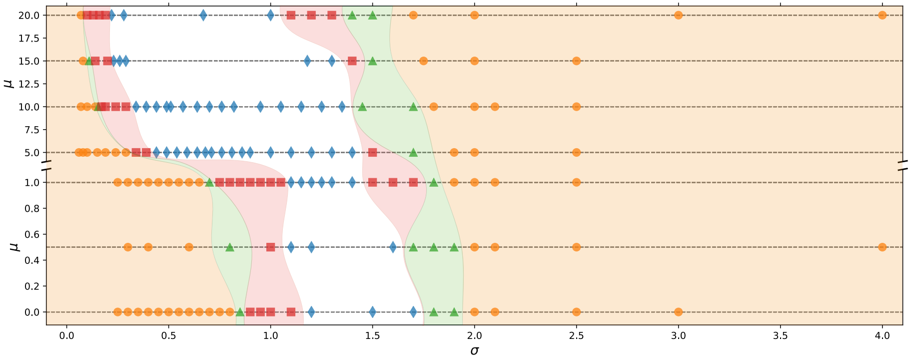
Phase Diagram

- Unstable, metastable,



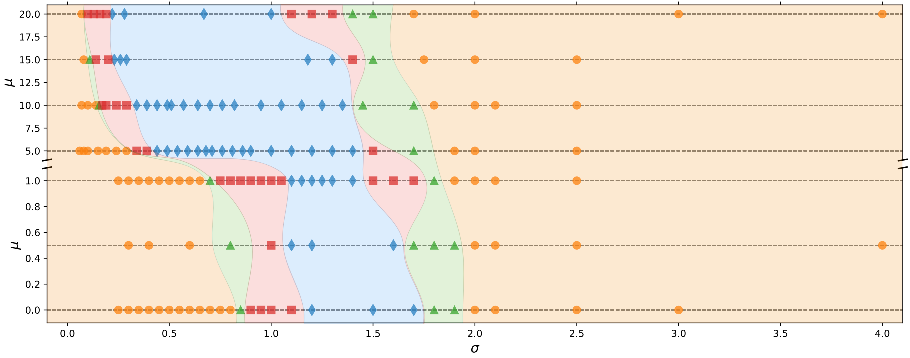
Phase Diagram

- Unstable, metastable, irregular,



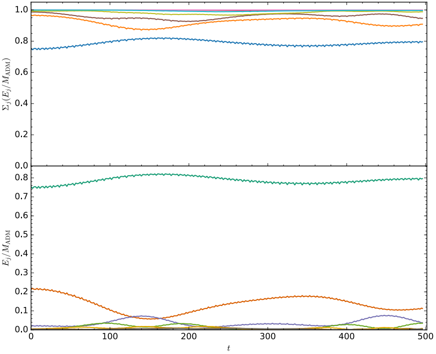
Phase Diagram

- Unstable, metastable, irregular, and stable initial data

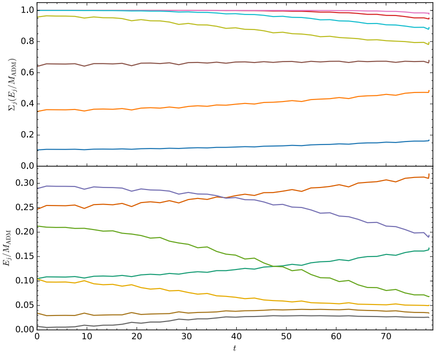


Energy Cascades

- Stable solutions: direct and inverse energy cascades



$\mu = 0,\ \sigma = 1.8,\ \epsilon = 0.13$
Stable



$\mu = 0,\ \sigma = 0.25,\ \epsilon = 2.28$
Unstable

Results

- First full phase diagram of stability in AdS_5 $\rightarrow$ islands of stability and "shorelines"
- Evidence of metastable and irregular phases at finite ϵ
- Fate of metastable phase as $\epsilon \rightarrow 0$ yet to be determined
- Irregular phase contains quasi-stable initial data[7,8] $\rightarrow$ first evidence for weakly chaotic evolution in massless, spherically-symmetry scalars in AdS
- Metastable and irregular data to be studied in multiscale perturbation theory

[7] Deppe & Frey [1508.02709]
[8] Buchel *et al.* [1304.4166]

B Cownden, N Deppe, and AR Frey, *On the Stability of High-Temperature, Quasi-Periodic Solutions for Massless Scalars in AdS$_4$*, In progress.

The Two-Time Formalism (TTF)

- Small perturbations in AdS_4: expand scalar field, metric functions in ϵ
- $\mathcal{O}(\epsilon)$: ϕ_1 in terms of eigenfunctions of AdS, $e_j(x)$
- Integer eigenvalues $\omega_j = (2j + d) \rightarrow$ fully resonant spectrum

$$\phi_1(t,x) = \sum_{j=0}^{\infty} a_j(t) \cos(\omega_j t + b_j(t))\, e_j(x)$$

The Two-Time Formalism (TTF)

- Small perturbations in AdS_4: expand scalar field, metric functions in ϵ
- $\mathcal{O}(\epsilon)$: ϕ_1 in terms of eigenfunctions of AdS, $e_j(x)$
- Integer eigenvalues $\omega_j = (2j + d) \to$ fully resonant spectrum
- $\mathcal{O}(\epsilon^3)$: source terms for resonant contributions
- Absorb resonances into amplitudes & phases[9]

$$-2\omega_\ell \frac{da_\ell(\tau)}{d\tau} = \sum_{i\neq\ell}^{\ell\leq i+j} \sum_{j\neq\ell} f_1\left(a_i, a_j, a_{i+j-\ell}, b_i, b_j, b_{i+j-\ell}\right)$$

$$-2\omega_\ell a_\ell \frac{db_\ell(\tau)}{d\tau} = \sum_{i\neq\ell}^{\ell\leq i+j} \sum_{j\neq\ell} f_2\left(a_i, a_j, a_{i+j-\ell}, b_i, b_j, b_{i+j-\ell}\right)$$

[9] Balasubramanian *et al.* [1403.6471]

Quasi-Periodic Solutions I

- Renormalization flow techniques to cancel an infinite number of resonances $\rightarrow$ express non-vanishing ones analytically[10]
- Need to truncate number of modes to find solutions: $j_{max} < \infty$ (must be robust as $j_{max} \rightarrow \infty$)

[10] Craps *et al.* [1407.6273]

Quasi-Periodic Solutions I

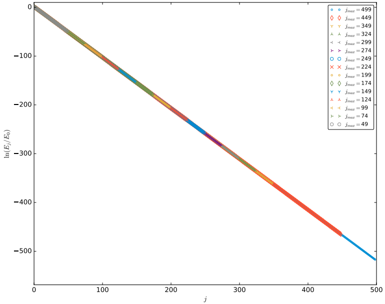
- Renormalization flow techniques to cancel an infinite number of resonances $\to$ express non-vanishing ones analytically[10]
- Need to truncate number of modes to find solutions: $j_{max} < \infty$ (must be robust as $j_{max} \to \infty$)
- Quasi-periodic[11] solutions $a_j = \alpha_j e^{i\beta_j \tau}$ $\to$ TTF equations become time-independent when $\beta_j = \beta_0 + j(\beta_1 - \beta_0)$
- TTF: conserved quantities[12] (E, N) $\to$ classify solutions by $T \equiv E/N$
- Solve QP equation using Newton-Raphson method

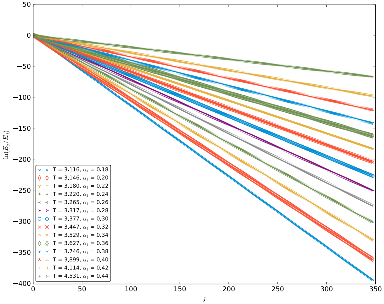
$$2\omega_\ell \alpha_\ell \beta_\ell = T_\ell \alpha_\ell^3 + \sum_{i \neq \ell} R_{i\ell} \alpha_i^2 \alpha_\ell + \sum_{i \neq \ell}^{\ell \leq i+j} \sum_{j \neq \ell} S_{ij(i+j-\ell)\ell} \alpha_i \alpha_j \alpha_{i+j-\ell}$$

[10] Craps *et al.* [1407.6273]
[11] Balasubramanian *et al.* [1403.6471]
[12] Craps *et al.* [1412.3249]

Quasi-Periodic Solutions II

- Solutions found for $3 \geq T \gtrsim 5.5$
- Able to extend existing solutions from $j_{max} \sim 100$ to $j_{max} = 500$
- Robust in $j_{max} \to \infty$ limit

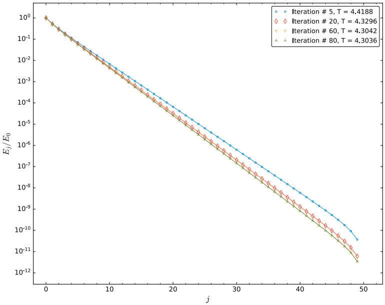


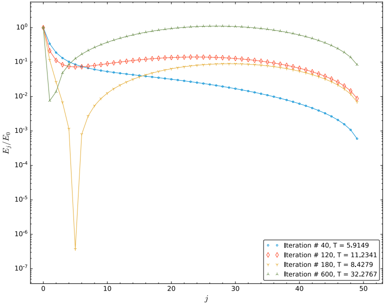


BC, Deppe, & Frey: In progress

High-Temperature Families I

- Perturb by $\delta E \rightarrow$ new solutions have energy $E + \delta E$, N, and $T + \delta T$
- Repeat process to determine T_{max}
- Project back to QP solution surface at constant α_1 or T
- Loss of smooth profile above a certain temperature at constant α_1

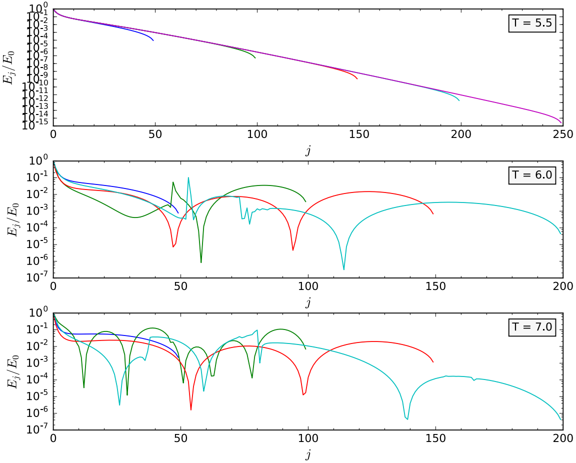




Cownden, Deppe, & Frey: In progress

High-Temperature Families II

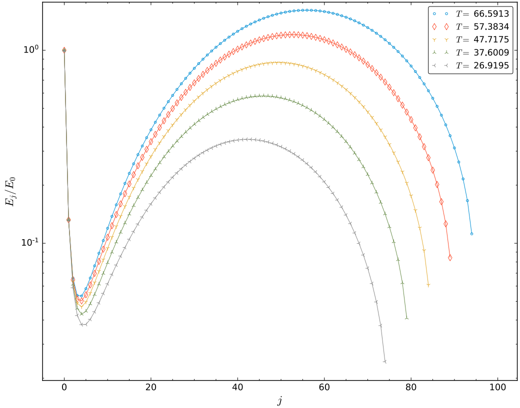
- And at constant T



Cownden, Deppe, & Frey: In progress

High-Temperature Families III

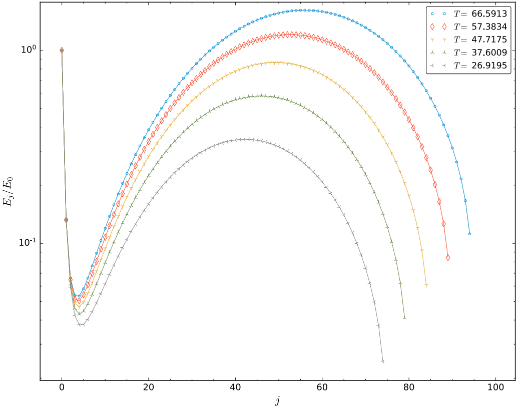
- Alternative methods for finding high-T solutions
- Fit low j_{max}, high-T data to generate seeds for Newton-Raphson solver



Cownden, Deppe, & Frey: In progress

High-Temperature Families III

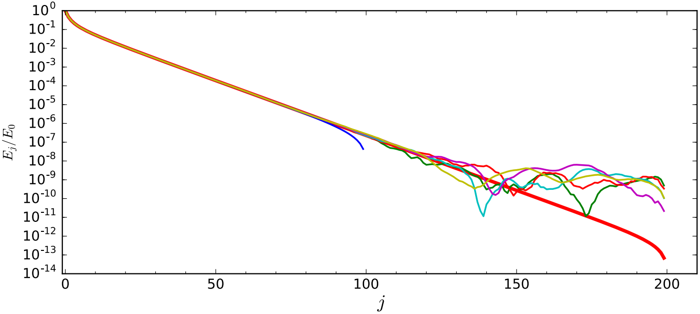
- Alternative methods for finding high-T solutions
- Fit low j_{max}, high-T data to generate seeds for Newton-Raphson solver $\rightarrow$ not robust as j_{max} increases



Cownden, Deppe, & Frey: In progress

High-Temperature Families III

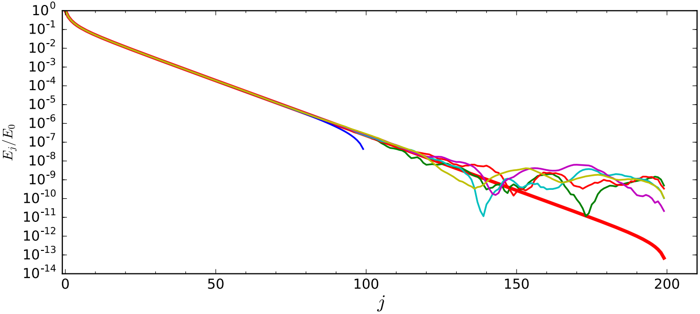
- Alternative methods for finding high-T solutions
- Fit low j_{max}, high-T data to generate seeds for Newton-Raphson solver $\rightarrow$ not robust as j_{max} increases
- Pad low j_{max} data with zeros, evolve within the TTF



Cownden, Deppe, & Frey: In progress

High-Temperature Families III

- Alternative methods for finding high-T solutions
- Fit low j_{max}, high-T data to generate seeds for Newton-Raphson solver $\rightarrow$ not robust as j_{max} increases
- Pad low j_{max} data with zeros, evolve within the TTF $\rightarrow$ isothermal evolution does not match known solution



Cownden, Deppe, & Frey: In progress

Results

- Low-T QP solutions robust as j_{max} increases
- Not able to find evidence that high-T solutions continued to exist at large j_{max} $\rightarrow$ possible reduction of space of QP
- **Caveat**: focused on configurations where $\alpha_0 = 1$ $\rightarrow$ free to set dominant energy in any α_j $\rightarrow$ other configurations required for high temperatures?
- Motivation for temperature limit of $T \sim 5.5$?
- Perturbative system: massless scalar, static boundary conditions at $x = \pi/2$
- Extend to massive scalars, time-dependent boundary conditions $\rightarrow$ activation of non-normalizable modes

B Cownden, *Examining Instabilities Due to Driven Scalars in AdS*, [1912.07143].

Extending TTF to Driven Scalars

- Driven scalars $\rightarrow$ scalars with time-dependent boundary conditions at $x = \pi/2$

Extending TTF to Driven Scalars

- Driven scalars $\rightarrow$ scalars with time-dependent boundary conditions at $x = \pi/2$
- Examine scaling behaviour as $x \rightarrow \pi/2$: $\Phi^+(x) \sim (\cos x)^{\Delta^+}$ and $\Phi^-(x) \sim (\cos x)^{\Delta^-}$

$\Phi^+(x) \equiv$ "normalizable" $\qquad$ $\Phi^-(x) \equiv$ "non-normalizable"

Extending TTF to Driven Scalars

- Driven scalars $\rightarrow$ scalars with time-dependent boundary conditions at $x = \pi/2$
- Examine scaling behaviour as $x \rightarrow \pi/2$: $\Phi^{+}(x) \sim (\cos x)^{\Delta^{+}}$ and $\Phi^{-}(x) \sim (\cos x)^{\Delta^{-}}$
- Scalar field is linear combination of both kinds of modes
- $e_j(x)$ are same eigenfunctions of AdS & have eigenvalues $\omega_j = (2j + \Delta^{+})$

$$\phi_1(t,x) = \sum_{j=0}^{\infty} a_j(t) \cos\left(\omega_j t + b_j(t)\right) e_j(x) + \sum_{\alpha=0}^{\infty} \bar{A}_\alpha(t) \cos\left(\omega_\alpha t + \bar{B}_\alpha\right) E_\alpha(x)$$

$$\Delta^{\pm} = \frac{d}{2} \pm \frac{1}{2}\sqrt{d^2 + 4m^2}$$

Extending TTF to Driven Scalars

- Driven scalars $\rightarrow$ scalars with time-dependent boundary conditions at $x = \pi/2$
- Examine scaling behaviour as $x \rightarrow \pi/2$: $\Phi^{+}(x) \sim (\cos x)^{\Delta^{+}}$ and $\Phi^{-}(x) \sim (\cos x)^{\Delta^{-}}$
- Scalar field is linear combination of both kinds of modes
- $e_j(x)$ are same eigenfunctions of AdS & have eigenvalues $\omega_j = (2j + \Delta^{+})$
- $E_\alpha(x)$ are hypergeometric functions & have continuous eigenvalues ω_α

$$\phi_1(t,x) = \sum_{j=0}^{\infty} a_j(t) \cos\left(\omega_j t + b_j(t)\right) e_j(x) + \sum_{\alpha=0}^{\infty} \bar{A}_\alpha(t) \cos\left(\omega_\alpha t + \bar{B}_\alpha\right) E_\alpha(x)$$

$$\Delta^{\pm} = \frac{d}{2} \pm \frac{1}{2}\sqrt{d^2 + 4m^2}$$

Resonant Contributions

- $\mathcal{O}(\epsilon^3)$: source terms for resonant contributions $\rightarrow$ examine resonance conditions

$$\omega_i + \omega_j + \omega_k = \omega_\ell$$
$$\omega_i - \omega_j - \omega_k = \omega_\ell$$
$$\omega_i + \omega_j - \omega_k = \omega_\ell$$

Resonant Contributions

- $\mathcal{O}(\epsilon^3)$: source terms for resonant contributions $\rightarrow$ examine resonance conditions
- **Unforced**: restrictions on indices and mass value

$$
\begin{aligned}
\omega_i + \omega_j + \omega_k = \omega_\ell \quad &\Rightarrow \quad i + j + k = \ell - \Delta^+ \in \mathbb{Z}^+ \\
\omega_i - \omega_j - \omega_k = \omega_\ell \quad &\Rightarrow \quad i - j - k = \ell + \Delta^+ \in \mathbb{Z}^+ \\
\omega_i + \omega_j - \omega_k = \omega_\ell \quad &\Rightarrow \quad i + j = k + \ell \in \mathbb{Z}^+
\end{aligned}
$$

Resonant Contributions

- $\mathcal{O}(\epsilon^3)$: source terms for resonant contributions $\rightarrow$ examine resonance conditions
- **Unforced**: restrictions on indices and mass value $\rightarrow$ two channels vanish numerically

$$\omega_i + \omega_j + \omega_k = \omega_\ell$$

$$\omega_i - \omega_j - \omega_k = \omega_\ell$$

$$\omega_i + \omega_j - \omega_k = \omega_\ell$$

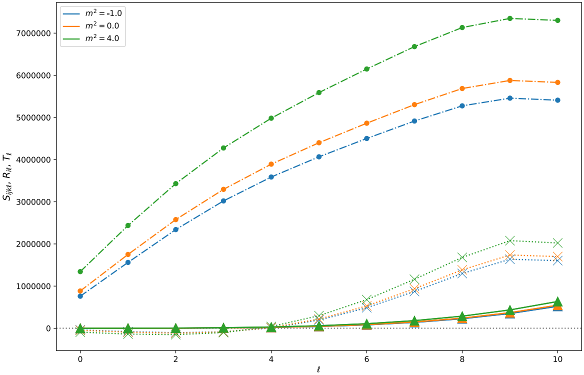
Resonant Contributions

- $\mathcal{O}(\epsilon^3)$: source terms for resonant contributions $\rightarrow$ examine resonance conditions
- **Unforced**: restrictions on indices and mass value $\rightarrow$ two channels vanish numerically
- One non-vanishing channel

$$\omega_i + \omega_j + \omega_k = \omega_\ell$$
$$\omega_i - \omega_j - \omega_k = \omega_\ell$$
$$\omega_i + \omega_j - \omega_k = \omega_\ell$$

Resonant Contributions

- $\mathcal{O}(\epsilon^3)$: source terms for resonant contributions $\rightarrow$ examine resonance conditions
- **Unforced**: restrictions on indices and mass value $\rightarrow$ two channels vanish numerically
- One non-vanishing channel $\rightarrow$ sum over i, j with $i + j \leq \ell$



Special Values of Non-normalizable Frequencies

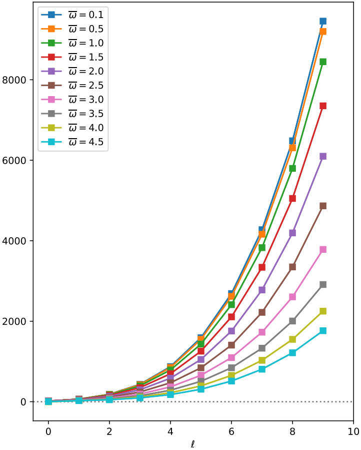
- **Forced**: ω_α set by driving term $\rightarrow$ explore several scenarios

Special Values of Non-normalizable Frequencies

- **Forced**: ω_α set by driving term $\rightarrow$ explore several scenarios
- Single frequency: $\omega_\alpha = \bar{\omega}$

$$\pm\omega_I \pm \omega_J \pm \omega_K \pm \omega_\ell = 0 \quad I \in \{i, \alpha\}$$

$$\omega_i + \bar{\omega} - \bar{\omega} = \omega_\ell$$

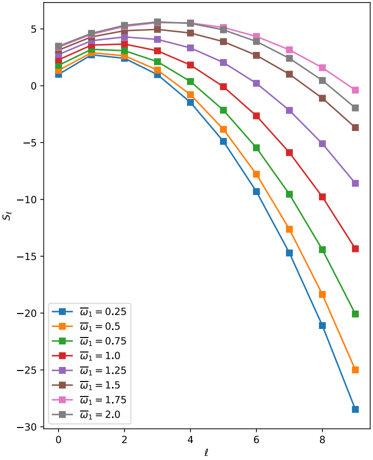


Special Values of Non-normalizable Frequencies

- **Forced**: ω_α set by driving term $\rightarrow$ explore several scenarios
- Single frequency: $\omega_\alpha = \bar{\omega}$
- Add to integer: $\bar{\omega}_1 + \bar{\omega}_2 = 2n$

$$\pm\omega_I \pm \omega_J \pm \omega_K \pm \omega_\ell = 0 \quad I \in \{i, \alpha\}$$

$$\left\{ \begin{array}{l} (++) : \omega_i + 2n = \omega_\ell \quad \forall\, \ell \geq n \\ (+-) : \omega_i - 2n = \omega_\ell \quad \forall\, n \\ (-+) : -\omega_i + 2n = \omega_\ell \quad \forall\, n \geq \ell + d \end{array} \right\}$$



Special Values of Non-normalizable Frequencies

- **Forced**: ω_α set by driving term $\rightarrow$ explore several scenarios
- Single frequency: $\omega_\alpha = \bar{\omega}$
- Add to integer: $\bar{\omega}_1 + \bar{\omega}_2 = 2n$
- $\omega_\alpha = 2i + \chi,\ \chi \notin \mathbb{Z}^+$
- Source terms give flow equations for amplitude/phase of ϕ_1, e.g. single frequency $\rightarrow$ equations decouple

$$\frac{2\omega_\ell}{\epsilon^2}\frac{da_\ell}{dt} = 0$$
$$\frac{2\omega_\ell}{\epsilon^2}\frac{db_\ell}{dt} = -S_\ell \bar{A}_{\bar{\omega}}^2$$

Results

- Confirm two of three resonant channels vanish for massive scalar (all normalizable)[13]
- First TTF formulation with time-dependent boundary conditions
- No naturally-vanishing source terms $\rightarrow$ sum resonant channels
- Some flow equations decouple amplitude/phase variables $a_\ell(t)$, $b_\ell(t)$
- **Note**: normalizable modes are still present $\rightarrow$ sum all contributions
- **Further work**: quasi-periodic solutions[14]? Conserved quantities? Energy cascades?

[13] Biasi *et al.* [1810.04753]
[14] Carracedo *et al.* [1612.07701]

Conclusions

- Examine the stability of AdS to scalar field collapse in various dimensions in perturbative & non-perturbative regimes $\rightarrow$ addition of time-dependent boundary conditions

Conclusions

- Examine the stability of AdS to scalar field collapse in various dimensions in perturbative & non-perturbative regimes $\rightarrow$ addition of time-dependent boundary conditions
- Constructed phase diagram of scalar field collapse in AdS_5 $\rightarrow$ two new phases on "shorelines" $\rightarrow$ non-perturbative regime only
- First evidence of weakly chaotic evolution of scalars in AdS_5

Conclusions

- Examine the stability of AdS to scalar field collapse in various dimensions in perturbative & non-perturbative regimes $\rightarrow$ addition of time-dependent boundary conditions
- Constructed phase diagram of scalar field collapse in AdS$_5$ $\rightarrow$ two new phases on "shorelines" $\rightarrow$ non-perturbative regime only
- First evidence of weakly chaotic evolution of scalars in AdS$_5$
- Verified low-T QP solutions are robust in the $j_{max} \rightarrow \infty$ limit
- No evidence of high-T QP solutions for $\alpha_0 = 1$ family

Conclusions

- Examine the stability of AdS to scalar field collapse in various dimensions in perturbative & non-perturbative regimes $\rightarrow$ addition of time-dependent boundary conditions
- Constructed phase diagram of scalar field collapse in AdS$_5$ $\rightarrow$ two new phases on "shorelines" $\rightarrow$ non-perturbative regime only
- First evidence of weakly chaotic evolution of scalars in AdS$_5$
- Verified low-T QP solutions are robust in the $j_{max} \rightarrow \infty$ limit
- No evidence of high-T QP solutions for $\alpha_0 = 1$ family
- Developed perturbative theory for massive scalars with time-dependent boundary conditions
- Derived flow equations for amplitude/phase variables for some choices of driving term $\rightarrow$ evaluated source terms numerically

Thanks

- Supervisor: Andrew Frey
- PhD Committee: Derek Krepski, Gabor Kunstatter, Robert Mann, Khodr Shamseddine
- Co-authors: Nils Deppe
- University of Winnipeg and University of Manitoba
- Westgrid & Compute Canada

References

- J. Maldacena, *The Large N limit of superconformal field theories and supergravity*, Int. J. Theor. Phys. 38 (1999) 1113-1133, [hep-th/9711200].
- P. Bizoń and A. Rostworowski, *On weakly turbulent instability of anti-de Sitter space*, Phys. Rev. Lett. 107 (2011) 031102, [1104.3702].
- M. Choptuik, *Universality and scaling in gravitational collapse of a massless scalar field*, Phys. Rev. Lett. 70 (1993) 9-12.
- N. Deppe and A. R. Frey, *Classes of Stable Initial Data for Massless and Massive Scalars in Anti-de Sitter Spacetime*, JHEP 12 (2015) 004, [1508.02709].
- R. Brito, V. Cardoso, and J. V. Rocha, *Interacting shells in AdS spacetime and chaos*, Phys. Rev. D94 (2016), no. 2 024003, [1602.03535].
- N. Deppe, A. Kolly, A. R. Frey, and G. Kunstatter, *Black Hole Formation in AdS Einstein-Gauss-Bonnet Gravity*, J. High Energ. Phys. (2016) 2016: 87, [1608.05402].
- A. Buchel, S. L. Liebling, and L. Lehner, *Boson Stars in AdS Spacetime*, Phys. Rev. D87 (2013) 123006, [1304.4166].

References

- V. Balasubramanian, A. Buchel, S. R. Green, L. Lehner, and S. L. Liebling, *Holographic Thermalization, Stability of Anti-de Sitter Space, and the Fermi-Pasta-Ulam Paradox*, Phys. Rev. Lett. 113 (2014) 071601, [1403.6471].
- B. Craps, O. Evnin, and J. Vanhoof, *Renormalization group, secular term resummation and AdS (in)stability*, JHEP 10 (2014) 048, [1407.6273].
- S. R. Green, A. Maillard, L. Lehner, and S. L. Liebling, *Islands of stability and recurrence times in AdS*, Phys. Rev. D92 (2015) 084001, [1507.08261].
- B. Craps, O. Evnin, and J. Vanhhof, *Renormalization, averaging, conservation laws and AdS (in)stability*, J. High Energ. Phys. 1501 (2015) 108, [1412.3249].
- A. Biasi, B. Craps, and O. Evnin, *Energy Returns in Global* AdS_4, [1810.04753].
- P. Carracedo, J. Mas, D. Musso and A. Serantes, *Adiabatic pumping solutions in global AdS*, JHEP 05 (2017) 141, [1612.07701].