

1 Time-Reduced System for the YM Soliton

The non-linear YM soliton outside a unit ball has an equation of motion given by

$$\partial_t^2 f = \partial_r^2 f + \frac{1}{r} \partial_r f + \frac{2}{r^2} f(1 - f^2) \quad (1)$$

where $t \in (-\infty, \infty)$ and $r \in [1, \infty)$. We know that there exists a stationary, minimum energy solution known as the *half-kink* given by

$$Q(r) = \frac{r^2 - 1}{r^1 + 1}. \quad (2)$$

Consider a rescaling of the function $f(t, r)$ by $f \rightarrow r^{-1/2} f$. The equation of motion is then

$$\partial_t^2 f = \partial_r^2 f + \frac{2f}{r^2} \left(\frac{9}{8} - \frac{f^2}{r} \right). \quad (3)$$

Now, linearize around the half-kink by expanding f as $f = \bar{Q} + \phi$ where \bar{Q} is the rescaled half-kink. We can then write

$$\partial_t^2 \phi = \partial_r^2 \phi - V\phi + \mathcal{O}(\phi^2) \quad \text{where} \quad V \equiv \frac{3}{r^2} \left(2Q^2 - \frac{3}{4} \right). \quad (4)$$

This is simply a 2D wave equation and as such we can use the result from Jaramillo *et al.* (PRX 2021) which takes the hyperboloidal compactification

$$t = \lambda(u - h(x)) \quad r = \lambda g(x) \quad (5)$$

and produces the equation

$$\left[\left(\frac{h'^2}{g'^2} - 1 \right) \partial_u^2 + \frac{2}{g'} \left(\frac{h'}{g'} \right) \partial_{ux}^2 + \frac{1}{g'} \partial_x \left(\frac{h'}{g'} \right) \partial_u + \frac{1}{g'} \partial_x \left(\frac{1}{g'} \partial_x \right) - \tilde{V} \right] \phi = 0 \quad (6)$$

where the potential has been multiplied by the (constant) characteristic scale factor λ such that $\tilde{V} = \lambda^2 V$. After some factoring we can write this in a form that makes the Sturm-Liouville functions immediately obvious:

$$\partial_u^2 \phi = \frac{g'}{g'^2 - h'^2} \left[\frac{2h'}{g'} \partial_{ux}^2 \phi + \partial_x \left(\frac{h'}{g'} \right) \partial_u \phi \right] + \frac{g'}{g'^2 - h'^2} \left[\frac{1}{g'} \partial_x^2 \phi - \frac{g''}{g'^2} \partial_x \phi - g' \tilde{V} \phi \right]. \quad (7)$$

We define the Sturm-Liouville variables by

$$w(x) = \frac{g'^2 - h'^2}{g'}, \quad \gamma(x) = \frac{h'}{g'}, \quad p(x) = \frac{1}{g'}, \quad q(x) = g' \tilde{V} \quad (8)$$

and perform the time-reduction via $\psi = \partial_u \phi$ so that

$$\partial_u \psi = \frac{1}{w} [2\gamma \partial_x + \partial_x \gamma] \psi + \frac{1}{w} [\partial_x (p \partial_x) - q] \phi \equiv L_2 \psi + L_1 \phi. \quad (9)$$

There are restrictions on the Sturm-Liouville variables owing to the general definition of the operator. These are:

Soliton Psuedospectrum Calculations

- $w(x) > 0 \forall x \in (-1, 1)$
- $w(x)$ must be invertible $\forall x \in (-1, 1)$
- To obey the commutivity of the inner product, the combination $p(x)\phi(x)$ must vanish at the endpoints. In our case, the point $x = 1$ corresponds to future null infinity, i.e. the radiative zone, where the function ϕ is non-zero. Thus, p must vanish at this point. At the inner boundary, however, $x = -1$ does not correspond to \mathcal{I}^- and there is indeed a boundary condition on the function that $\phi(x = -1) = 0$; thus, p can take any value here

Additionally, we have restrictions on the compactification choice that ensures future null infinity is mapped to $x = 1$. These are:

- $g(x) : x \in [-1, 1] \rightarrow r = g(x) \in [1, \infty)$ is everywhere greater than 1 and $g(x)$ is at least C^0 .
- The curvature scalar is $n^\mu n_\mu \propto (dh/dr)^2 - 1$. This means that our approach to infinity must follow a null curve such that $|dh/dr| \rightarrow 1$ as $r \rightarrow \infty$. Furthermore, we want to consider spacelike curves near $r = 1$ which means $|dh/dr| \leq 1$ as $r \rightarrow 1$

Given these restrictions on the choices of $h(x)$, $g(x)$ we can identify many compactification schemes that may be of interest. Of course, the choice of compactification should never affect the result of the analysis so exploring multiple compactifications will be a check on our results. Some choices for the height function $h(x)$ are $\sqrt{g^2(x) + 1}$, $\ln(\cosh(g(x) - 1))$, and $g(x) - \ln g(x)$, while some choices for the compactification $g(x)$ are $2/(1 - x)$, and $1 + \exp[\tanh^{-1}(x)]$. Of the possible choices, we choose to focus on the two cases outlined in Table 1.

Choice A	$h(x) = \sqrt{g^2(x) + 1}$	$g(x) = 1 + \exp[\tanh^{-1}(x)]$
Choice B	$h(x) = g(x) - \ln g(x)$	$g(x) = 2/(1 - x)$

Table 1: Two choices of a combination of height function and compactification function will be used to verify that the results are independent of a specific choice of compactification.

By defining $\psi(u, x) \equiv \partial_u \phi(u, x)$ we can write the left-hand side of (??) in terms of a second Sturm-Liouville operator, L_2 ,

$$\partial_u \psi - \frac{1}{\rho} [2ph' \partial_x + \partial_x (ph')] \psi \equiv \partial_u \psi - L_2 \psi \quad (10)$$

where

$$L_2 = \frac{1}{\rho} [2\gamma \partial_x + \partial_x \gamma], \quad \gamma(x) = ph'. \quad (11)$$

Finally, the linearized equation for perturbations around the static kink can be written as

$$\partial_u \psi = L_2 \psi + L_1 \phi. \quad (12)$$

Defining the vector $\Phi = (\phi, \psi)^T$, the system can be written in the time-reduced form

$$\partial_u \Phi = iL\Phi \quad \text{where} \quad L = -i \begin{pmatrix} 0 & 1 \\ L_1 & L_2 \end{pmatrix}. \quad (13)$$

with L_1 given in (??) and L_2 given in (11). Taking the ansatz $\Phi(u, x) = \Phi(x)e^{i\omega u}$ gives the spectral problem

$$\begin{pmatrix} 0 & 1 \\ L_1 & L_2 \end{pmatrix} \begin{pmatrix} \phi \\ \psi \end{pmatrix} = i\omega \begin{pmatrix} \phi \\ \psi \end{pmatrix}. \quad (14)$$

2 Energy Inner Product

To continue further in the calculation of the psuedospectrum, it is important to define a ‘‘proper’’ inner product. Jaramillo proposes the energy inner product which, for a complex scalar on (1+1)-dimensional Minkowski background with a scattering potential V_ℓ , comes from examining the expression for the energy of a constant-time slice:

$$E = \frac{1}{2} \int_a^b [(g'^2 - h'^2) \partial_u \phi^* \partial_u \phi + \partial_x \phi^* \partial_x \phi + g'^2 V_\ell \phi^* \phi] \frac{1}{|g'|} dx. \quad (15)$$

The analogue for the YM soliton comes from multiplying equation (12) by ψ

$$\rho \psi \partial_u \psi = 2\gamma \psi \partial_x \psi + \partial_x \gamma \psi^2 - \partial_x p \psi \partial_x \phi - p \psi \partial_x^2 \phi + Q \psi \phi, \quad (16)$$

and noting that

$$2F(x) \phi \partial_u \phi = \partial_u (F(x) \phi^2) \quad (17)$$

for any function $F(x)$. Then we can write

$$\partial_u \left[\frac{1}{2} \rho \psi^2 - \frac{1}{2} p (\partial_x \phi)^2 - \frac{1}{2} Q \phi^2 \right] = \partial_x [\gamma \psi^2 - p \psi \partial_x \phi]. \quad (18)$$

We identify the terms in square brackets in the left-hand side as the Bondi-type energy

$$E = \frac{1}{2} \int_{-1}^1 dx \left(\rho \psi^2 - p (\partial_x \phi)^2 - Q \phi^2 \right) \quad (19)$$

Following Jaramillo, we define the energy inner product of two solutions to (14) to be

$$\langle \Phi_1, \Phi_2 \rangle_E = \frac{1}{2} \int_{-1}^1 dx \left(p \psi_1^* \psi_2 - p (\partial_x \phi_1)^* (\partial_x \phi_2) - Q \phi_1^* \phi_2 \right) \quad (20)$$

so that $\|\Phi\|_E^2 = \langle \Phi, \Phi \rangle_E$ by construction.

3 Spectral Methods

For the discretization of operators/derivatives, we use the ‘interior’, ‘roots’, or ‘Gauss-Chebyshev’ abscissa given by

$$\bar{x}_i = \cos\left(\frac{\pi(2i-1)}{2N}\right) \quad i = 1, 2, \dots, N. \quad (21)$$

In this basis, the expression for the first derivative matrix \mathbb{D} and second derivative matrix $\mathbb{D}^{(2)}$ are given by

$$\mathbb{D} = \begin{cases} \frac{\bar{x}_j}{2(1-\bar{x}_j^2)} & \text{if } i = j \\ \frac{(-1)^{(i+j)}}{(\bar{x}_i - \bar{x}_j)} \sqrt{\frac{1-\bar{x}_j^2}{1-\bar{x}_i^2}} & \text{if } i \neq j \end{cases} \quad (22)$$

$$\mathbb{D}^{(2)} = \begin{cases} \frac{\bar{x}_j^2}{(1-\bar{x}_j^2)^2} - \frac{(N^2-1)}{3(1-\bar{x}_j^2)} & \text{if } i = j \\ \mathbb{D}_{ij} \left(\frac{\bar{x}_i}{(1-\bar{x}_i^2)} - \frac{2}{(\bar{x}_i - \bar{x}_j)} \right) & \text{if } i \neq j \end{cases} \quad (23)$$

Integrals can be evaluated using Gaussian quadrature and a decomposition of a function in a basis of Chebyshev polynomials; alternatively, spectral collocation can also be used to evaluate integrals. We introduce Clenshaw-Curtis quadrature.

3.1 Clenshaw-Curtis Quadrature

Following Mason & Handscomb, we want to determine the integral

$$I(f) = \int_{-1}^1 w(x) f(x) dx \quad (24)$$

when the function $f(x)$ is interpolated by

$$f(x) \simeq f_N(x) = \sum_{i=0}^N c_i T_i(\bar{x}_i) \quad \text{where } \bar{x}_i = \cos\left(\frac{(2i-1)\pi}{2(N+1)}\right), \quad i = 1, \dots, N+1. \quad (25)$$

We can then write the approximation to the integral $I(f) \simeq I_N(f)$ as

$$I_N(f) = \sum_{i=1}^{N+1} \omega_i f(\bar{x}_i) \quad (26)$$

$$\text{where } \omega_i = \sum_{j=0}^N \frac{2a_j}{N+1} T_j(\bar{x}_i) \quad (27)$$

$$\text{and } a_j = \int_{-1}^1 w(x) T_j(x) dx. \quad (28)$$

Soliton Psuedospectrum Calculations

Note that the primed sum carries an extra factor of $1/2$ in the first term. To calculate $I_N(f)$, we further note that when $w(x) = 1$, we have

$$\int_{-1}^1 T_j(x) dx = \begin{cases} \frac{(-1)^j + 1}{1 - j^2} & j \neq 1 \\ 0 & j = 1 \end{cases} \quad (29)$$

and so

$$\omega_i = \frac{2}{N+1} \left[1 - 2 \sum_{k=0}^{\lfloor N/2 \rfloor} \frac{T_{2k}(\bar{x}_i)}{4k^2 - 1} \right] \quad (30)$$

$$I_N(f) = \frac{2}{N+1} \sum_{i=1}^{N+1} f(\bar{x}_i) \left[1 - 2 \sum_{k=0}^{\lfloor N/2 \rfloor} \frac{T_{2k}(\bar{x}_i)}{4k^2 - 1} \right]. \quad (31)$$

When the integrand $f(x)$ is a product of functions, each of which is described in terms of an interpolation in the zeros of $T_{N+1}(x)$, the integral can be approximated by

$$I(f(x)g(x)\mu(x)) \simeq I_N(fg\mu) = f^T(\bar{x}) \cdot C_\mu(\bar{x}) \cdot g(\bar{x}) \quad (32)$$

where $f^T(\bar{x})$ is a row vector of the function $f(x)$ evaluated on the \bar{x} abscissa, and the diagonal $N \times N$ matrix $C_\mu(\bar{x})$ has components

$$(C_\mu)_{ii} = \frac{2\mu(\bar{x}_i)}{N+1} \left[1 - 2 \sum_{k=1}^{\lfloor N/2 \rfloor} \frac{T_{2k}(\bar{x}_i)}{4k^2 - 1} \right]. \quad (33)$$

3.2 Gram Matrices

Now we wish to apply this method to the evaluation of the energy inner product given in (20). Using Clenshaw-Curtis quadrature with the Boyd transformation, we can see that

$$\langle \Phi_1, \Phi_2 \rangle_E = \frac{1}{2} \int_{-1}^1 \rho \psi_1^\dagger \psi_2 - \frac{1}{2} \int_{-1}^1 p (\mathbb{D}\phi_1)^\dagger \mathbb{D}\phi_2 - \frac{1}{2} \int_{-1}^1 Q \phi_1^\dagger \phi_2 \quad (34)$$

$$\begin{aligned} &= \frac{1}{2} \sum_{j=1}^N w_j \rho(\cos t_j) \psi_1^\dagger(\cos t_j) \psi_2(\cos t_j) - \frac{1}{2} \sum_{j=1}^N w_j p(\cos t_j) (\mathbb{D}\phi_1)^\dagger(\cos t_j) \mathbb{D}\phi_2(\cos t_j) \\ &\quad - \frac{1}{2} \sum_{j=1}^N w_j Q(\cos t_j) \phi_1^\dagger(\cos t_j) \phi_2(\cos t_j) \end{aligned} \quad (35)$$

$$\begin{aligned} &= \psi_1^T(\bar{x}) \cdot \text{diag} \left[\frac{1}{2} \sum_{j=1}^N w_j \rho(\cos t_j) \right] \cdot \psi_2(\bar{x}) + \phi_1^T(\bar{x}) \cdot \mathbb{D}^T \cdot \text{diag} \left[-\frac{1}{2} \sum_{j=1}^N w_j p(\cos t_j) \right] \cdot \mathbb{D} \cdot \phi_2(\bar{x}) \\ &\quad + \phi_1^T(\bar{x}) \cdot \text{diag} \left[-\frac{1}{2} \sum_{j=1}^N w_j Q(\cos t_j) \right] \cdot \phi_2(\bar{x}). \end{aligned} \quad (36)$$

Soliton Psuedospectrum Calculations

After defining the Gram matrix, G , this is

$$\langle \Phi_1, \Phi_2 \rangle_E = \Phi_1 \cdot G \cdot \Phi_2 = (\phi_1^*, \psi_1^*) \begin{pmatrix} G_1^E & 0 \\ 0 & G_2^E \end{pmatrix} \begin{pmatrix} \phi_2 \\ \psi_2 \end{pmatrix}, \quad (37)$$

with the diagonals given by

$$G_1^E = (\mathbb{D})^T \cdot \text{diag} \left[-\frac{1}{2} \sum_{j=1}^N w_j p(\cos t_j) \right] \cdot \mathbb{D} + \text{diag} \left[-\frac{1}{2} \sum_{j=1}^N w_j Q(\cos t_j) \right] \quad (38)$$

$$G_2^E = \text{diag} \left[\frac{1}{2} \sum_{j=1}^N w_j \rho(\cos t_j) \right]. \quad (39)$$

3.3 Pseudospectrum

In terms of the energy inner product given by the Gram matrix G , the pseudospectrum is

$$\sigma_G^\epsilon(M) = \{ \lambda \in \mathbb{C} : s_{\min}(\sqrt{\lambda} : \lambda \in \sigma(M^\dagger M)) < \epsilon \}, \quad (40)$$

where s_{\min} is the smallest singular value from Singular Value Decomposition of the expression $M^\dagger M$. Finally, the adjoint of the matrix in the basis of the energy inner product is given by

$$M^\dagger = G^{-1} M^* G. \quad (41)$$

Note that M^* is the conjugate transpose, i.e. $M_{ij}^* = \bar{M}_{ji}$. Thus, the prescription for calculating the pseudospectrum with respect to the energy inner product (20) is

- For a given degree of discretization, N , calculate the Gram matrices G_1^E and G_2^E using Clenshaw-Curtis quadrature.
- Calculate the inverses of each Gram matrix to construct the inverse of G .
- For each shifted matrix \tilde{L} repeat the following:
 - Calculate the adjoint of \tilde{L} using (41).
 - Determine the smallest singular value of $\tilde{L}^\dagger \tilde{L}$.