1 Time-Reduced System for the YM Soliton

The non-linear YM soliton outside a unit ball has an equation of motion given by

$$\partial_t^2 f = \partial_r^2 f + \frac{1}{r}\partial_r f + \frac{2}{r^2} f(1-f^2) \tag{1}$$

where $t \in (-\infty, \infty)$ and $r \in [1, \infty)$. We know that there exists a stationary, minimum energy solution known as the *half-kink* given by

$$Q(r) = \frac{r^2-1}{r^1+1}\,. \tag{2}$$

Consider a rescaling of the function $f(t,r)$ by $f \to r^{-1/2} f$. The equation of motion is then

$$\partial_t^2 f = \partial_r^2 f + \frac{2f}{r^2}\left(\frac{9}{8} - \frac{f^2}{r}\right)\,. \tag{3}$$

Now, linearize around the half-kink by expanding f as $f = \bar{Q} + \phi$ where $\bar{Q}$ is the rescaled half-kink. We can then write

$$\partial_t^2 \phi = \partial_r^2 \phi - V\phi + \mathcal{O}(\phi^2) \qquad \text{where} \quad V \equiv \frac{3}{r^2}\left(2Q^2 - \frac{3}{4}\right)\,. \tag{4}$$

This is simply a 2D wave equation and as such we can use the result from Jaramillo *et al.* (PRX 2021) which takes the hyperboloidal compactification

$$t = \lambda(u - h(x)) \quad r = \lambda g(x) \tag{5}$$

and produces the equation

$$\Big[\left(\frac{h'^2}{g'^2} - 1\right)\partial_u^2 + \frac{2}{g'}\left(\frac{h'}{g'}\right)\partial_{ux}^2 + \frac{1}{g'}\partial_x\left(\frac{h'}{g'}\right)\partial_u + \frac{1}{g'}\partial_x\left(\frac{1}{g'}\partial_x\right) - \tilde{V}\Big]\phi = 0 \tag{6}$$

where the potential has been multiplied by the (constant) characteristic scale factor λ such that $\tilde{V} = \lambda^2 V$. After some factoring we can write this is a form that makes the Sturm-Louiville functions immediately obvious:

$$\partial_u^2 \phi = \frac{g'}{g'^2 - h'^2}\left[\frac{2h'}{g'}\partial_{ux}^2\phi + \partial_x\left(\frac{h'}{g'}\right)\partial_u\phi\right] + \frac{g'}{g'^2 - h'^2}\left[\frac{1}{g'}\partial_x^2\phi - \frac{g''}{g'^2}\partial_x\phi - g'\tilde{V}\phi\right]\,. \tag{7}$$

We define the Sturm-Louiville variables by

$$w(x) = \frac{g'^2 - h'^2}{g'}, \quad \gamma(x) = \frac{h'}{g'}, \quad p(x) = \frac{1}{g'}, \quad q(x) = g'\tilde{V} \tag{8}$$

and perform the time-reduction via $\psi = \partial_u \phi$ so that

$$\partial_u \psi = \frac{1}{w}\left[2\gamma\partial_x + \partial_x\gamma\right]\psi + \frac{1}{w}\left[\partial_x(p\partial_x) - q\right]\phi \equiv L_2\psi + L_1\phi\,. \tag{9}$$

There are restrictions on the Sturm-Louiville variables owing to the general definition of the operator. These are:

- $w(x) > 0 \forall x \in (-1, 1)$
- $w(x)$ must be invertible $\forall x \in (-1, 1)$
- To obey the commutivity of the inner product, the combination $p(x)\phi(x)$ must vanish at the endpoints. In our case, the point $x = 1$ corresponds to future null infinity, i.e. the radiative zone, where the function ϕ is non-zero. Thus, p must vanish at this point. At the inner boundary, however, $x = -1$ does not correspond to $\mathscr{I}^-$ and there is indeed a boundary condition on the function that $\phi(x = -1) = 0$; thus, p can take any value here

Additionally, we have restrictions on the compactification choice that ensures future null infinity is mapped to $x = 1$. These are:

- $g(x) : x \in [-1, 1] \to r = g(x) \in [1, \infty)$ is everywhere greater than 1 and $g(x)$ is at least C^0.
- The curvature scalar is $n^\mu n_\mu \propto (dh/dr)^2 - 1$. This means that our approach to infinity must follow a null curve such that $|dh/dr| \to 1$ as $r \to \infty$. Furthermore, we want to consider spacelike curves near $r = 1$ which means $|dh/dr| \leq 1$ as $r \to 1$

Given these restrictions on the choices of $h(x)$, $g(x)$ we can identify many compactification schemes that may be of interest. Of course, the choice of compactification should never affect the result of the analysis so exploring multiple compactifications will be a check on our results. Some choices for the height function $h(x)$ are $\sqrt{g^2(x) + 1}$, $\ln(\cosh(g(x) - 1))$, and $g(x) - \ln g(x)$, while some choices for the compactification $g(x)$ are $2/(1 - x)$, and $1 + \exp\left[\tanh^{-1}(x)\right]$. Of the possible choices, we choose to focus on the two cases outlined in Table 1.

Choice A	$h(x) = \sqrt{g^2(x) + 1}$	$g(x) = 1 + \exp\left[\tanh^{-1}(x)\right]$
Choice B	$h(x) = g(x) - \ln g(x)$	$g(x) = 2/(1 - x)$

Table 1: Two choices of a combination of height function and compactification function will be used to verify that the results are independent of a specific choice of compactification.

By defining $\psi(u, x) \equiv \partial_u \phi(u, x)$ we can write the left-hand side of (**??**) in terms of a second Sturm-Liouville operator, L_2,

$$\partial_u \psi - \frac{1}{\rho}\left[2ph'\partial_x + \partial_x\left(ph'\right)\right]\psi \equiv \partial_u \psi - L_2 \psi \tag{10}$$

where

$$L_2 = \frac{1}{\rho}\left[2\gamma\partial_x + \partial_x \gamma\right], \qquad \gamma(x) = ph'. \tag{11}$$

Finally, the linearized equation for perturbations around the static kink can be written as

$$\partial_u \psi = L_2 \psi + L_1 \phi. \tag{12}$$

Defining the vector $\Phi = (\phi, \psi)^T$, the system can be written in the time-reduced form

$$\partial_u \Phi = iL\Phi \qquad \text{where} \quad L = -i \begin{pmatrix} 0 & 1 \\ L_1 & L_2 \end{pmatrix} . \tag{13}$$

with L_1 given in (**??**) and L_2 given in (11). Taking the ansatz $\Phi(u,x) = \Phi(x)e^{i\omega u}$ gives the spectral problem

$$\begin{pmatrix} 0 & 1 \\ L_1 & L_2 \end{pmatrix} \begin{pmatrix} \phi \\ \psi \end{pmatrix} = i\omega \begin{pmatrix} \phi \\ \psi \end{pmatrix} . \tag{14}$$

2 Energy Inner Product

To continue further in the calculation of the psuedospectrum, it is important to define a "proper" inner product. Jaramillo proposes the energy inner product which, for a complex scalar on (1+1)-dimensional Minkowski background with a scattering potential V_ℓ, comes from examining the expression for the energy of a constant-time slice:

$$E = \frac{1}{2}\int_a^b \left[(g'^2 - h'^2)\partial_u\phi^*\partial_u\phi + \partial_x\phi^*\partial_x\phi + g'^2 V_\ell \phi^*\phi\right] \frac{1}{|g'|} dx \, . \tag{15}$$

The analogue for the YM soliton comes from multiplying equation (12) by ψ

$$\rho\psi\partial_u\psi = 2\gamma\psi\partial_x\psi + \partial_x\gamma\psi^2 - \partial_x p\psi\partial_x\phi - p\psi\partial_x^2\phi + Q\psi\phi \, , \tag{16}$$

and noting that

$$2F(x)\phi\partial_u\phi = \partial_u\left(F(x)\phi^2\right) \tag{17}$$

for any function $F(x)$. Then we can write

$$\partial_u\left[\frac{1}{2}\rho\psi^2 - \frac{1}{2}p\left(\partial_x\phi\right)^2 - \frac{1}{2}Q\phi^2\right] = \partial_x\left[\gamma\psi^2 - p\psi\partial_x\phi\right] . \tag{18}$$

We identify the terms in square brackets in the left-hand side as the Bondi-type energy

$$E = \frac{1}{2}\int_{-1}^{1} dx \, \left(\rho\psi^2 - p\left(\partial_x\phi\right)^2 - Q\phi^2\right) \tag{19}$$

Following Jaramillo, we define the energy inner product of two solutions to (14) to be

$$\langle\Phi_1, \Phi_2\rangle_E = \frac{1}{2}\int_{-1}^{1} dx \left(p\psi_1^*\psi_2 - p(\partial_x\phi_1)^*(\partial_x\phi_2) - Q\phi_1^*\phi_2^*\right) \tag{20}$$

so that $||\Phi||_E^2 = \langle\Phi, \Phi\rangle_E$ by construction.

3 Spectral Methods

For the discretization of operators/derivatives, we use the 'interior', 'roots', or 'Gauss-Chebyshev' abscissa given by

$$\bar{x}_i = \cos\left(\frac{\pi(2i-1)}{2N}\right) \quad i = 1, 2, \ldots, N\,. \tag{21}$$

In this basis, the expression for the first derivative matrix $\mathbb{D}$ and second derivative matrix $\mathbb{D}^{(2)}$ are given by

$$\mathbb{D} = \begin{cases} \dfrac{\bar{x}_j}{2(1-\bar{x}_j^2)} & \text{if } i = j \\ \dfrac{(-1)^{(i+j)}}{(\bar{x}_i - \bar{x}_j)}\sqrt{\dfrac{1-\bar{x}_j^2}{1-\bar{x}_i^2}} & \text{if } i \neq j \end{cases} \tag{22}$$

$$\mathbb{D}^{(2)} = \begin{cases} \dfrac{\bar{x}_j^2}{(1-\bar{x}_j^2)^2} - \dfrac{(N^2-1)}{3(1-\bar{x}_j^2)} & \text{if } i = j \\ \mathbb{D}_{ij}\left(\dfrac{\bar{x}_i}{(1-\bar{x}_i^2)} - \dfrac{2}{(\bar{x}_i - \bar{x}_j)}\right) & \text{if } i \neq j \end{cases} \tag{23}$$

Integrals can be evaluated using Gaussian quadrature and a decomposition of a function in a basis of Chebyshev polynomials; alternatively, spectral collocation can also be used to evaluate integrals. We introduce Clenshaw-Curtis quadrature.

3.1 Clenshaw-Curtis Quadrature

Following Mason & Handscomb, we want to determine the integral

$$I(f) = \int_{-1}^{1} w(x) f(x) dx \tag{24}$$

when the function $f(x)$ is interpolated by

$$f(x) \simeq f_N(x) = \sum_{i=0}^{N} c_i T_i(\bar{x}_i) \quad \text{where } \bar{x}_i = \cos\left(\frac{(2i-1)\pi}{2(N+1)}\right)\,,\ i = 1, \ldots, N+1\,. \tag{25}$$

We can then write the approximation to the integral $I(f) \simeq I_N(f)$ as

$$I_N(f) = \sum_{i=1}^{N+1} \omega_i f(\bar{x}_i) \tag{26}$$

$$\text{where } \omega_i = \sum_{j=0}^{N}{}' \frac{2a_j}{N+1} T_j(\bar{x}_i) \tag{27}$$

$$\text{and } a_j = \int_{-1}^{1} w(x) T_j(x) dx\,. \tag{28}$$

Note that the primed sum carries an extra factor of 1/2 in the first term. To calculate $I_N(f)$, we further note that when $w(x) = 1$, we have

$$\int_{-1}^{1} T_j(x)dx = \begin{cases} \dfrac{(-1)^j + 1}{1 - j^2} & j \neq 1 \\ 0 & j = 1 \end{cases} \tag{29}$$

and so

$$\omega_i = \frac{2}{N+1}\left[1 - 2\sum_{k=0}^{\lfloor N/2 \rfloor} \frac{T_{2k}(\bar{x}_i)}{4k^2 - 1}\right] \tag{30}$$

$$I_N(f) = \frac{2}{N+1}\sum_{i=1}^{N+1} f(\bar{x}_i)\left[1 - 2\sum_{k=0}^{\lfloor N/2 \rfloor} \frac{T_{2k}(\bar{x}_i)}{4k^2 - 1}\right]. \tag{31}$$

When the integrand $f(x)$ is a product of functions, each of which is described in terms of an interpolation in the zeros of $T_{N+1}(x)$, the integral can be approximated by

$$I(f(x)g(x)\mu(x)) \simeq I_N(fg\mu) = f^T(\bar{x}) \cdot C_\mu(\bar{x}) \cdot g(\bar{x}) \tag{32}$$

where $f^T(\bar{x})$ is a row vector of the function $f(x)$ evaluated on the $\bar{x}$ abscissa, and the diagonal $N \times N$ matrix $C_\mu(\bar{x})$ has components

$$(C_\mu)_{ii} = \frac{2\mu(\bar{x}_i)}{N+1}\left[1 - 2\sum_{k=1}^{\lfloor N/2 \rfloor} \frac{T_{2k}(\bar{x}_i)}{4k^2 - 1}\right]. \tag{33}$$

3.2 Gram Matrices

Now we wish to apply this method to the evaluation of the energy inner product given in (20). Using Clenshaw-Curtis quadrature with the Boyd transformation, we can see that

$$\langle \Phi_1, \Phi_2 \rangle_E = \frac{1}{2}\int_{-1}^{1} \rho \psi_1^\dagger \psi_2 - \frac{1}{2}\int_{-1}^{1} p\,(\mathbb{D}\phi_1)^\dagger\,\mathbb{D}\phi_2 - \frac{1}{2}\int_{-1}^{1} Q\phi_1^\dagger \phi_2 \tag{34}$$

$$= \frac{1}{2}\sum_{j=1}^{N} w_j \rho(\cos t_j)\psi_1^\dagger(\cos t_j)\psi_2(\cos t_j) - \frac{1}{2}\sum_{j=1}^{N} w_j p(\cos t_j)\,(\mathbb{D}\phi_1)^\dagger(\cos t_j)\mathbb{D}\phi_2(\cos t_j)$$
$$- \frac{1}{2}\sum_{j=1}^{N} w_j Q(\cos t_j)\phi_1^\dagger(\cos t_j)\phi_2(\cos t_j) \tag{35}$$

$$= \psi_1^T(\bar{x}) \cdot \mathrm{diag}\left[\frac{1}{2}\sum_{j=1}^{N} w_j \rho(\cos t_j)\right] \cdot \psi_2(\bar{x}) + \phi_1^T(\bar{x}) \cdot \mathbb{D}^T \cdot \mathrm{diag}\left[-\frac{1}{2}\sum_{j=1}^{N} w_j p(\cos t_j)\right] \cdot \mathbb{D} \cdot \phi_2(\bar{x})$$
$$+ \phi_1^T(\bar{x}) \cdot \mathrm{diag}\left[-\frac{1}{2}\sum_{j=1}^{N} w_j Q(\cos t_j)\right] \cdot \phi_2(\bar{x}). \tag{36}$$

After defining the Gram matrix, G, this is

$$\langle \Phi_1, \Phi_2 \rangle_E = \Phi_1 \cdot G \cdot \Phi_2 = (\phi_1^*, \psi_1^*) \begin{pmatrix} G_1^E & 0 \\ 0 & G_2^E \end{pmatrix} \begin{pmatrix} \phi_2 \\ \psi_2 \end{pmatrix} , \tag{37}$$

with the diagonals given by

$$G_1^E = (\mathbb{D})^T \cdot \mathrm{diag}\left[-\frac{1}{2} \sum_{j=1}^{N} w_j p(\cos t_j) \right] \cdot \mathbb{D} + \mathrm{diag}\left[-\frac{1}{2} \sum_{j=1}^{N} w_j Q(\cos t_j) \right] \tag{38}$$

$$G_2^E = \mathrm{diag}\left[\frac{1}{2} \sum_{j=1}^{N} w_j \rho(\cos t_j) \right] . \tag{39}$$

3.3 Pseudospectrum

In terms of the energy inner product given by the Gram matrix G, the pseudospectrum is

$$\sigma_G^\epsilon(M) = \{\lambda \in \mathbb{C} : s_{\min} \left(\sqrt{\lambda} : \lambda \in \sigma(M^\dagger M) \right) < \epsilon \} , \tag{40}$$

where $s_{\min}$ is the smallest singular value from Singular Value Decomposition of the expression $M^\dagger M$. Finally, the adjoint of the matrix in the basis of the energy inner product is given by

$$M^\dagger = G^{-1} M^* G . \tag{41}$$

Note that M^* is the conjugate transpose, i.e. $M_{ij}^* = \bar{M}_{ji}$. Thus, the prescription for calculating the pseudospectrum with respect to the energy inner product (20) is

- For a given degree of discretization, N, calculate the Gram matrices G_1^E and G_2^E using Clenshaw-Curtis quadrature.
- Calculate the inverses of each Gram matrix to construct the inverse of G.
- For each shifted matrix $\tilde{L}$ repeat the following:
  - Calculate the adjoint of $\tilde{L}$ using (41).
  - Determine the smallest singular value of $\tilde{L}^\dagger \tilde{L}$.