Growth Rates

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2/27/2012

1 FTCS 2D Case

$$\mathrm{u}_{l,m}^{n+1} = u_{l,m}^{n} + C[u_{l-1,m}^{n} + u_{l+1,m}^{n} + u_{l,m-1}^{n} + u_{l,m+1}^{n} - 4u_{l,m}^{n}]$$

$$\mathrm{u}^{n} = G(t)e^{ikl\Delta x}e^{ikm\Delta x}$$

$$G(t+1)e^{ikl\Delta x}e^{ikm\Delta x}$$
$$= G(t)e^{ikl\Delta x}e^{ikm\Delta x} + C[G(t)e^{ikl-1\Delta x}e^{ikm\Delta x} + G(t)e^{ikl+1\Delta x}e^{ikm\Delta x} + G(t)e^{ikl\Delta x}e^{ikm-1\Delta x} + G(t)e^{ikl\Delta x}e^{ikm+1}$$

$$\frac{G(t+1)}{G(t)} = 1 + \frac{Ce^{ikl\Delta x}e^{ikm\Delta x}[2e^{1} + 2e^{1} - 4]}{e^{ikl\Delta x}e^{ikm\Delta x}}$$

$$\frac{G(t+1)}{G(t)} = 1 + 2C[e^{-ik\Delta x} + e^{ik\Delta x} - 2]$$

$$\frac{G(t+1)}{G(t)} = 1 + 2C[2cos(k\Delta x) - 2]$$

worst when cos is -1 so

—1-8C—≤ 1

-1 $\leq 1 - 8C \leq 1$

-2 $\leq -8C \leq 0$

$1 \geq 4C \geq 0$

C $\leq \frac{1}{4}$

2 FTCS 3D Case

$$\frac{u_{l,m,n}^{n+1} - u_{l,m,n}^{n}}{\Delta t} = \alpha[u_{l-1,m,n}^{n}+u_{l+1,m,n}^{n}+u_{l,m-1,n}^{n}+u_{l,m+1,n}^{n}+u_{l,m,n-1}^{n}+u_{l,m,n+1}^{n}-6u_{l,m,n}^{n}]$$

$$u_{l,m,n}^{n+1} = u_{l,m,n}^{n}+C[u_{l-1,m,n}^{n}+u_{l+1,m,n}^{n}+u_{l,m-1,n}^{n}+u_{l,m+1,n}^{n}+u_{l,m,n-1}^{n}+u_{l,m,n+1}^{n}-6u_{l,m,n}^{n}]$$

$$u^n = G(t)e^{ikl\Delta x}e^{ikm\Delta x}e^{ikn\Delta x}$$

$$\begin{aligned} &G(t+1)e^{ikl\Delta x}e^{ikm\Delta x}e^{ikn\Delta x} \\ &= G(t)e^{ikl\Delta x}e^{ikm\Delta x}e^{ikn\Delta x}+C[G(t)e^{ikl-1\Delta x}e^{ikm\Delta x}e^{ikn\Delta x}+G(t)e^{ikl+1\Delta x}e^{ikm\Delta x}e^{ikn\Delta x}+G(t)e^{ikl\Delta x}e^{ikm-1\Delta} \end{aligned}$$

$$G(t+1)e^{ikl\Delta x}e^{ikm\Delta x}e^{ikn\Delta x} = G(t)[e^{ikl\Delta x}e^{ikm\Delta x}e^{ikn\Delta x}+C[e^{ikl-1\Delta x}e^{ikm\Delta x}e^{ikn\Delta x}+e^{ikl+1\Delta x}e^{ikm\Delta x}e^{ikn\Delta x}+$$

$$\frac{G(t+1)}{G(t)} = \frac{e^{ikl\Delta x}e^{ikm\Delta x}e^{ikn\Delta x} + C[e^{ikl-1\Delta x}e^{ikm\Delta x}e^{ikn\Delta x} + e^{ikl+1\Delta x}e^{ikm\Delta x}e^{ikn\Delta x} + e^{ikl\Delta x}e^{ikm-1\Delta x}e^{ikn\Delta}}{e^{ikl\Delta x}e}$$

$$\frac{G(t+1)}{G(t)} = 1+\frac{C[e^{ikl-1\Delta x}e^{ikm\Delta x}e^{ikn\Delta x} + e^{ikl+1\Delta x}e^{ikm\Delta x}e^{ikn\Delta x} + e^{ikl\Delta x}e^{ikm-1\Delta x}e^{ikn\Delta x} + e^{ikl\Delta x}e^{ikm+1\Delta x}}{e^{ikl\Delta x}e^{ikm\Delta x}e^{ik}}$$

$$\frac{G(t+1)}{G(t)} = 1 + \frac{Ce^{ikl\Delta x}e^{ikm\Delta x}e^{ikn\Delta x}[3e^1 + 3e^1 - 6]}{e^{ikl\Delta x}e^{ikm\Delta x}e^{ikn\Delta x}}$$

$$\frac{G(t+1)}{G(t)} = 1 + 3C[e^{-ik\Delta x} + e^{ik\Delta x} - 2]$$

$$\frac{G(t+1)}{G(t)} = 1 + 3C[2cos(k\Delta x) - 2]$$

worst when cos is -1 so

$$\begin{aligned} |1 - 12C| &\leq 1 \\ -1 \leq 1 - 12C &\leq 1 \\ -2 \leq -12C &\leq 0 \\ 1 \geq 6C &\geq 0 \\ C &\leq \frac{1}{6} \end{aligned}$$

3 BECS 1D Case

$$\frac{u_a^{n+1} - u_a^n}{\Delta t} = \frac{\alpha}{\Delta x^2}[u_{a-1}^{n+1} + u_{a+1}^{n+1} - 2u_a^{n+1}]$$

$$u^n = G(t)e^{ika\Delta x}$$

$$G(t)e^{iklaDeltax} = G(t{+}1)e^{ika\Delta x} - C[G(t{+}1)e^{ika-1\Delta x} + G(t{+}1)e^{ika+1\Delta x} - 2G(t{+}1)e^{ika\Delta x}]$$

$$G(t)e^{iklaDeltax}e^{ikb\Delta x} = G(t{+}1)[e^{ikl\Delta x}e^{ikm\Delta x} - C[e^{ika-1\Delta x} + e^{ika+1\Delta x} - 2e^{ika\Delta x}]]$$

$$G(t) = G(t+1)[1 - C[e^{-ik\Delta x} + e^{ik\Delta x} - 2]]$$

$$G(t) = G(t+1)[1 - C[2cos(ik\Delta x) - 2]]$$

$$G(t) = G(t+1)[1 - C[2cos(ik\Delta x) - 2]]$$

$$\frac{G(t+1)}{G(t)} = \frac{1}{1 - 2C(cos(ik\Delta x) - 1)}$$

$$\begin{aligned} 1 &\geq \frac{1}{1 - 2C(cos(ik\Delta x) - 1)} \\ 1 - 2C(cos(ik\Delta x) - 1) &\geq 1 \\ 1 - 2C(cos(ik\Delta x) - 1) &\geq 1 \\ -2C(cos(ik\Delta x) - 1) &\geq 0 \\ 4C &\geq 0 \\ C &\geq 0 \end{aligned}$$

4 BECS 2D Case

$$\frac{u_{a,b}^{n+1} - u_{a,b}^{n}}{\Delta t} = \frac{\alpha}{\Delta x^2}[u_{a-1,b}^{n+1} + u_{a+1,b}^{n+1} + u_{a,b-1}^{n+1} + u_{a,b+1}^{n+1} - 4u_{a,b}^{n+1}]$$

$$u^n = G(t)e^{ika\Delta x}e^{ikb\Delta x}e$$

$G(t)e^{iklaDeltax}e^{ikb\Delta x} = G(t+1)e^{ika\Delta x}e^{ikb\Delta x} - C[G(t+1)e^{ika-1\Delta x}e^{ikb\Delta x} + G(t+1)e^{ika+1\Delta x}e^{ikb\Delta x} + G(t+1)e^{ik}$

$G(t)e^{iklaDeltax}e^{ikb\Delta x} = G(t+1)[e^{ikl\Delta x}e^{ikm\Delta x} - C[e^{ika-1\Delta x}e^{ikb\Delta x} + e^{ika+1\Delta x}e^{ikb\Delta x} + e^{ika\Delta x}e^{ikb-1\Delta x} + e^{ika\Delta x}e$

G(t) = G(t+1)[1 - C[2e$^{-ik\Delta x}$ + 2$e^{ik\Delta x}$ − 4]]

G(t) = G(t+1)[1 - C[4cos(ikΔx) − 4]]

G(t) = G(t+1)[1 - C[4cos(ikΔx) − 4]]

$$\frac{G(t+1)}{G(t)} = \frac{1}{1 - C(4cos(ik\Delta x) - 4)}$$

$$1 \geq \frac{1}{1 - C(4cos(ik\Delta x) - 4)}$$

1 - C(4cos(ikΔx) − 4) ≥ 1

1 - C(4cos(ikΔx) − 4) ≥ 1

-C(4cos(ikΔx) − 4) ≥ 0

8C ≥ 0

C≥ 0

5 BECS 3D Case

$$\frac{u_{a,b,c}^{n+1} - u_{a,b,c}^{n}}{\Delta t} = \frac{\alpha}{\Delta x^2}[u_{a-1,b,c}^{n+1} + u_{a+1,b,c}^{n+1} + u_{a,b-1,c}^{n+1} + u_{a,b+1,c}^{n+1} + u_{a,b,c-1}^{n+1} + u_{a,b,c+1}^{n+1} + -6u_{a,b,c}^{n+1}]$$

$\mathrm{u}^n = G(t)e^{ika\Delta x}e^{ikb\Delta x}e^{ikc\Delta x}$

$G(t)e^{iklaDeltax}e^{ikb\Delta x}e^{ikc\Delta x} = G(t{+}1)e^{ikl\Delta x}e^{ikm\Delta x}e^{ikn\Delta x} - C[G(t{+}1)e^{ika-1\Delta x}e^{ikb\Delta x}e^{ikc\Delta x} + G(t{+}1)e^{ika+1\Delta}$

$G(t)e^{iklaDeltax}e^{ikb\Delta x}e^{ikc\Delta x} = G(t{+}1)[e^{ikl\Delta x}e^{ikm\Delta x}e^{ikn\Delta x} - C[e^{ika-1\Delta x}e^{ikb\Delta x}e^{ikc\Delta x} + e^{ika+1\Delta x}e^{ikb\Delta x}e^{ikc\Delta x}$

$\mathrm{G(t)} = \mathrm{G(t{+}1)[1 - C[e^{-ik\Delta x} + e^{ik\Delta x} - 6]]}$

$\mathrm{G(t)} = \mathrm{G(t{+}1)[1 - C[2cos(ik\Delta x) - 6]]}$

$\mathrm{G(t)} = \mathrm{G(t{+}1)[1 - C[2cos(ik\Delta x) - 6]]}$

$$\frac{G(t+1)}{G(t)} = \frac{1}{1 - C(2cos(ik\Delta x) - 6)}$$

$$1 \geq \frac{1}{1 - C(2cos(ik\Delta x) - 6)}$$

$\mathrm{1 - C(2cos(ik\Delta x) - 6) \geq 1}$

$\mathrm{1 - C(2cos(ik\Delta x) - 6) \geq 1}$

$\mathrm{-C(2cos(ik\Delta x) - 6) \geq 0}$

$\mathrm{8C \geq 0}$

$\mathrm{C \geq 0}$

6 CN 1D Case

$$\frac{u_a^{n+1} - u_a^n}{\Delta t} = \frac{\alpha}{2\Delta x^2}[u_{a-1}^{n+1} + u_{a+1}^{n+1} - 2u_a^{n+1} + u_{a-1}^n + u_{a+1}^n - 2u_a^n]$$

$$u^n = G(t)e^{ika\Delta x}$$

$$G(t+1)e^{ika\Delta x} - G(t)e^{ika\Delta x} = \frac{C}{2}G(t+1)[e^{ika-1\Delta x} + e^{ika+1\Delta x} - 2e^{ika\Delta x} + \frac{C}{2}G(t)[e^{ika-1\Delta x} + e^{ika+1\Delta x} - 2e^{ika\Delta x}]$$

$$G(t+1)[e^{ika\Delta x} - \frac{C}{2}[e^{ika-1\Delta x} + e^{ika+1\Delta x} - 2e^{ika\Delta x}]] = +G(t)[\frac{C}{2}[e^{ika-1\Delta x} + e^{ika+1\Delta x} - 2e^{ika\Delta x}] + e^{ika\Delta x}]$$

$$\frac{G(t+1)}{G(t)} = \frac{[\frac{C}{2}[e^{ika-1\Delta x} + e^{ika+1\Delta x} - 2e^{ika\Delta x}] + e^{ika\Delta x}]}{[e^{ika\Delta x} - \frac{C}{2}[e^{ika-1\Delta x} + e^{ika+1\Delta x} - 2e^{ika\Delta x}]]}$$

$$\frac{G(t+1)}{G(t)} = \frac{[\frac{C}{2}[e^{-ik\Delta x} + e^{ik\Delta x} - 2] + 1]}{1 - \frac{C}{2}[e^{-ik\Delta x} + e^{ik\Delta x} - 2]}$$

$$\frac{G(t+1)}{G(t)} = \frac{[C[cos(k\Delta x) - 1] + 1]}{1 - C[cos(k\Delta x) - 1]}$$

$$\begin{aligned} \frac{G(t+1)}{G(t)} &= \frac{1-2C}{1+2C} \\ 1 &\geq \frac{1-2C}{1+2C} \\ 1+2C &\geq 1-2C \\ 4C &\geq 0 \\ C &\geq 0 \end{aligned}$$

7 CN 2D Case

$$\frac{u_{a,b}^{n+1}-u_{a,b}^{n}}{\Delta t}=\frac{\alpha}{2\Delta x^2}[u_{a-1,b}^{n+1}+u_{a+1,b}^{n+1}+u_{a,b-1}^{n+1}+u_{a,b+1}^{n+1}-4u_{a,b}^{n+1}+u_{a-1,b}^{n}+u_{a+1,b}^{n}+u_{a,b-1}^{n}+u_{a,b+1}^{n}-4u_{a,b}^{n}]$$

$$u^n = G(t)e^{ika\Delta x}e^{ikb\Delta x}$$

$$G(t+1)e^{ika\Delta x}e^{ikb\Delta x}-G(t)e^{ika\Delta x}e^{ikb\Delta x}=\frac{C}{2}G(t+1)[e^{ika-1\Delta x}e^{ikb\Delta x}+e^{ika+1\Delta x}e^{ikb\Delta x}+e^{ika\Delta x}e^{ikb-1\Delta x}+e^{ika}$$

$$G(t+1)[e^{ika\Delta x}e^{ikb\Delta x}-\frac{C}{2}[e^{ika-1\Delta x}e^{ikb\Delta x}+e^{ika+1\Delta x}e^{ikb\Delta x}+e^{ika\Delta x}e^{ikb-1\Delta x}+e^{ika\Delta x}e^{ikb+1\Delta x}-4e^{ika\Delta x}e^{ikb\Delta x}]$$

$$\frac{G(t+1)}{G(t)}=\frac{[\frac{C}{2}[e^{ika-1\Delta x}e^{ikb\Delta x}+e^{ika+1\Delta x}e^{ikb\Delta x}+e^{ika\Delta x}e^{ikb-1\Delta x}+e^{ika\Delta x}e^{ikb+1\Delta x}-4e^{ika\Delta x}e^{ikb\Delta x}]+e^{ik}}{[e^{ika\Delta x}e^{ikb\Delta x}-\frac{C}{2}[e^{ika-1\Delta x}e^{ikb\Delta x}+e^{ika+1\Delta x}e^{ikb\Delta x}+e^{ika\Delta x}e^{ikb-1\Delta x}+e^{ika\Delta x}e^{ikb+1\Delta x}-4e^{ika}}$$

$$\frac{G(t+1)}{G(t)}=\frac{[\frac{C}{2}[2e^{-ik\Delta x}+2e^{ik\Delta x}-4]+1]}{1-\frac{C}{2}[2e^{-ik\Delta x}+2e^{ik\Delta x}-4]}$$

$$\frac{G(t+1)}{G(t)}=\frac{[C[e^{-ik\Delta x}+e^{ik\Delta x}-2]+1]}{1-C[e^{-ik\Delta x}+e^{ik\Delta x}-2]}$$

$$\frac{G(t+1)}{G(t)}=\frac{[2C[cos(k\Delta x)-1]+1]}{1-2C[cos(k\Delta x)-1]}$$

$$\frac{G(t+1)}{G(t)}=\frac{1-4C}{1+4C}$$

$$1\geq\frac{1-4C}{1+4C}$$

$$1+4C\geq 1-4C$$

$$8C\geq 0$$

$$C\geq 0$$

8 CN 3D Case

$$\frac{u_{a,b,c}^{n+1}-u_{a,b,c}^{n}}{\Delta t}=\frac{\alpha}{2\Delta x^2}[u_{a-1,b,c}^{n+1}+u_{a+1,b,c}^{n+1}+u_{a,b-1,c}^{n+1}+u_{a,b+1,c}^{n+1}+u_{a,b,c-1}^{n+1}+u_{a,b,c+1}^{n+1}+-6u_{a,b,c}^{n+1}+u_{a-1,b,c}^{n}+u_{a+1,b,c}^{n}+$$

$$u^n=G(t)e^{ika\Delta x}e^{ikb\Delta x}e^{ikc\Delta x}$$

$$G(t+1)e^{ika\Delta x}e^{ikb\Delta x}e^{ikc\Delta x}-G(t)e^{ika\Delta x}e^{ikb\Delta x}e^{ikc\Delta x}=\frac{C}{2}G(t+1)[e^{ika-1\Delta x}e^{ikb\Delta x}e^{ikc\Delta x}+e^{ika+1\Delta x}e^{ikb\Delta x}e^{ikc\Delta x}$$

$$G(t+1)e^{ika\Delta x}e^{ikb\Delta x}e^{ikc\Delta x}-\frac{C}{2}G(t+1)[e^{ika-1\Delta x}e^{ikb\Delta x}e^{ikc\Delta x}+e^{ika+1\Delta x}e^{ikb\Delta x}e^{ikc\Delta x}+e^{ika\Delta x}e^{ikb-1\Delta x}e^{ikc\Delta x}+$$

$$G(t+1)[e^{ika\Delta x}e^{ikb\Delta x}e^{ikc\Delta x}-\frac{C}{2}[e^{ika-1\Delta x}e^{ikb\Delta x}e^{ikc\Delta x}+e^{ika+1\Delta x}e^{ikb\Delta x}e^{ikc\Delta x}+e^{ika\Delta x}e^{ikb-1\Delta x}e^{ikc\Delta x}+e^{ika\Delta x}$$

$$\frac{G(t+1)}{G(t)}=\frac{[\frac{C}{2}[e^{ika-1\Delta x}e^{ikb\Delta x}e^{ikc\Delta x}+e^{ika+1\Delta x}e^{ikb\Delta x}e^{ikc\Delta x}+e^{ika\Delta x}e^{ikb-1\Delta x}e^{ikc\Delta x}+e^{ika\Delta x}e^{ikb+1\Delta x}e^{ikc\Delta}}{[e^{ika\Delta x}e^{ikb\Delta x}e^{ikc\Delta x}-\frac{C}{2}[e^{ika-1\Delta x}e^{ikb\Delta x}e^{ikc\Delta x}+e^{ika+1\Delta x}e^{ikb\Delta x}e^{ikc\Delta x}+e^{ika\Delta x}e^{ikb-1\Delta x}e^{ikc\Delta x}$$

$$\frac{G(t+1)}{G(t)}=\frac{[\frac{C}{2}[3e^{-ik\Delta x}+3e^{ik\Delta x}-6]+1]}{1-\frac{C}{2}[3e^{-ik\Delta x}+3e^{ik\Delta x}-6]}$$

$$\frac{G(t+1)}{G(t)}=\frac{[\frac{3C}{2}[e^{-ik\Delta x}+e^{ik\Delta x}-2]+1]}{1-\frac{3C}{2}[e^{-ik\Delta x}+e^{ik\Delta x}-2]}$$

$$\frac{G(t+1)}{G(t)}=\frac{[3C[cos(k\Delta x)-1]+1]}{1-3C[cos(k\Delta x)-1]}$$

$$\frac{G(t+1)}{G(t)}=\frac{1-6C}{1+6C}$$

$$\begin{aligned}1&\geq\frac{1-6C}{1+6C}\\1+6C&\geq 1-6C\\12C&\geq 0\\C&\geq 0\end{aligned}$$