Lecture Notes: Growth Theory

1. Brief Reflections on Model Building

1.1. Building Blocks

As is the case with all economic models, the Basic Solow Growth Model consists of:

- ***Variables***: economic quantities of interest that we can calculate and measure, denoted by letters (or maybe two letters) like L for the number of workers in the labor force or Y for the amount of useful goods and services produced in a year—real GDP.

- ***Behavioral relationships***: relationships that (1) describe how humans, making economic decisions given their opportunities and opportunity costs, decide what to do, and (2) that thus determine the values of the economic variables, represented by equations that have an economic variable on the left hand side and, on the right, a set of factors that determine the value of the variable and a rule of thumb for what that value is currently.
    - ***Parameters***: determine which out of a broad family of potential behavioral relations describes the behavior of the particular economic scenario at hand; the ability to work algebraically with parameters allows one to perform an enormous number of potential "what-if?" calculations very quickly and in a very small space
- ***Equilibrium conditions***: conditions that tell us when the economy is in a position of balance, when some subset of the variables are "stable"—that is, are either constant are changing in simple and predictable ways, usually represented by solutions of some system of the equations that are behavioral relationships.

- ***Accounting identities***: statements about the relationships between variables that are automatically and necessarily true because of the way the variables are defined, represented by equations.

1.2. The Key to the Model

In the case of the Solow growth model, the key variable is the capital intensity of the economy, which we will write using a lower-case Greek letter kappa: $ \kappa $. We measure $ \kappa $ as the ratio of the capital stock of the economy $ K $, to the level of output and income $ Y $.

In every economic model economists proceed with their analysis by looking for an equilibrium: a point of balance, a condition of rest, a state of the system toward which the model will converge over time. Economists look for equilibrium for a simple reason: either an economy is at its (or one of its) equilibrium position(s), or it is moving—and probably (or hopefully?) moving rapidly—to an equilibrium position. The Solow growth model is no exception.

Once economists have found the equilibrium position toward which the economy tends to move, they then understand how the model will behave. And, if they have built the right model, it will tell you in broad strokes how the economy will behave.

In the Basic Solow Growth Model, the equilibrium economists look for is an equilibrium in which the economy's capital-output ratio $ \kappa $ is contant, and thus the capital stock per worker, the level of income and output per worker, and the efficiency of labor are all three growing at exactly the same proportional rate—a rate that remains constant over time. We will call this rate g, for growth.

2. The Basics of the Model

2.0. Directions

As you read through this document, you will come across three code cells. Each code cell will have three sections. The last section will begin with a comment line like:

```
# CODE (DON'T TOUCH THIS BELOW!)
```

Do not alter anything in the code cell below that line.

The first of the three sections of each code cell begins with

```
# INTRO
```

and contains a description of the calculations that cell makes. Read it and understand it.

The second of the three sections is contained in a box and begins and ends with a lien of equals signs. We want you to change the right-hand sides of the equation in this box, then run the cell (shift-return or shift-enter), and then see how the change to the parameters of the Solow growth model you made by altering the initial-condition and behavioral-relationship model parameters changed the graph describing the behavior of the model.

The hope is that this will make the algebra of the economic theory real to you—what it does, what it means, what it is.

2.1. Preliminaries

2.1.1. The Production Function

The first behavioral relationship in the Basic Solow Growth Model is the production function: the relationship between the economy's level of income and production $ Y $ and its three determinants: the labor force $ L $, the efficiency of labor $ E $, and the economy's capital-intensity $ \kappa $, which is measured by the quotient of the economy's capital stock $ K $ and its level of income and production $ Y $:

> (2.1.1) $ \kappa = \frac{K}{Y} $

2.1.1.1. Satisfy Three Rules of Thumb: We want this behavioral relationship to satisfy three rules of thumb:

1. A proportional increase in the economy's capital intensity $ \kappa = K/Y $, measured by the capital stock divided by total production, will carry with it the same proportional increase in income and production no matter how rich and productive the economy is. A 1% increase in capital intensity will always increase income and production by the same proportional amount.

2. If two economies have the same capital intensity, defined as the same capital-output ratio $ \kappa $, and have the same level of technology- and organization-driven efficiency-of-labor $ E $, then the ratio of their levels of income and output will be equal to the ratio of their labor forces $ L $.

3. If two economies have the same capital intensity, defined as the same capital-output ratio $ \kappa $, and have the same labor forces, then the ratio of their levels of income and output will be equal to the ratio of their technology- and organization-driven efficiencies-of-labor $ E $.

There is one and only one way to write an algebraic expression that satisfies these two rule-of-thumb conditions. It is:

> (2.1.2) $ Y = \kappa^\theta E L $

or with a lower-case $ y $ denoting income and production per worker:

> (2.1.3) $ y = \kappa^\theta E $

We call this Greek letter $ \theta $ the *salience* of capital in the production function.

> **Note**: You might object that this is circular: output and income $ Y $ depend on capital intensity $ \kappa $ defined as the capital-output ratio $ K/Y $, but how can you calculate $ \kappa $ and thus calculate $ Y $ when you need to know $ Y $ to calculate $ \kappa $? If this worries you, you could start elsewhere, with a different-looking production function:
>
> > (2.1.4) $ Y = K^\alpha (E L)^{1-\alpha} $
>
> Then divide both sides by $ Y^\alpha $:

> > (2.1.5) $ \frac{Y}{Y^\alpha} = \frac{K^\alpha}{Y^\alpha} (E L)^{1-\alpha} $
> >
> > (2.1.6) $ Y^{1-\alpha} = \left(\frac{K}{Y}\right)^\alpha (E L)^{1-\alpha} $
> >
> > (2.1.7) $ Y^{1-\alpha} = \kappa^\alpha (E L)^{1-\alpha} $
> >
> > (2.1.8) $ Y^{\left(\frac{1-\alpha}{1-\alpha}\right)} = \kappa^{\left(\frac{\alpha}{1-\alpha}\right)} (E L)^{\left(\frac{1-\alpha}{1-\alpha}\right)} $
> >
> > (2.1.9) $ Y = \kappa^{\left(\frac{\alpha}{1-\alpha}\right)} E L $
>
> which is exactly the same as (2.1.2), once we define $ \alpha = \theta/(1+\theta) $ and thus $ \theta = \alpha/(1-\alpha) $.
>
> For historical reasons, every single other book uses $ \alpha $. Since $ \theta = \alpha/(1-\alpha) $, $\alpha = \theta/(1+\theta) $, and $ 1 - \alpha = \theta/(1+\theta) $, moving back and forth between the forms is not a problem. We will work with $ \theta $ as it produces simpler formulas.
>
> We call θ the *salience* of capital in the production function? What is $ \alpha $ called? In a simple model in which labor, capital, and output markets are all perfectly competitive, the production-function parameter $ \alpha $ is also equal to the share of total gross income received by the owners of capital, so it is often called the "capital share" parameter. I think that this is a mistake, because we have no reason to believe that any real economy is close enough to perfectly competitive for us to think that this production-function parameter $ \alpha $ has any relationship at all with the share of gross income received by capital owners.

2.1.1.2. Why These Rules of Thumb?: Why did Robert Solow back in 1956 look for an algebraic formula for his production function that would satisfy these three rules-of-thumb? Economists like to simplify, ruthlessly, when they can get away with it. Rule-of-thumb (1) is a simplifying assumption: an intellectual bet that the process of aggregate economic growth is likely to look very similar as an economy goes from an income-per-capita level of 10,000 to 20,000 dollars per worker per year as when it goes from 40,000 to 80,000 dollars per worker per year. It is worth making only as long as that is in fact true—that the similarities in the aggregate overall economic growth process in different decades and at different income-per-worker levels outweigh the differences. If that were not or were to cease being the case, we should drop that rule-of-thumb assumption. So far, so good.

Rule-of-thumb (2) is simply that holding other things—capital intensity and efficiency-of-labor—constant, you can always duplicate what you are doing and produce and earn twice as much. Again, it is worth making only as long as that is in fact true, or approximately true. Again, so far, so good.

Rule-of-thumb (3) is a recognition that better-organized economies making better use of technology will be more productive: it is best thought of as a definition of how we are going to construct our quantitative index of the level of applied technological knowledge combined with efficient economic organization—the variable $ E $ that we call the efficiency-of-labor. It has no deeper implications.

2.1.2. Investment and Capital Accumulation

Following economists' custom of ruthless simplification, assume that individuals, households, and businesses desire to save a fraction $ s $ of their income $ Y $, so that total savings are:

> (2.1.10) $ S = sY $

We call s the economy's saving rate or, more completely, its saving-investment rate (to remind us that s is measuring both the flow of saving into the economy's financial markets and also the share of total production that is invested and used to build up and increase the economy's capital stock).

Assume that there are no problems in translating individuals', households', and businesses' desires to save some of their income $ Y $ into investment I:

> (2.1.11) $ I = S = sY $

While the saving-investment rate s is constant, the economy's capital stock K is not. It changes from year to year from investment and also from depreciation $ \Delta $:

> (2.1.12) $ \frac{dK}{dt} = I - \Delta $

Assume that:

> (2.1.13) $ \Delta = \delta K $

Each year a fraction $ \delta $ of the existing capital stock depreciates and wears out, so that the rate of change of the capital stock is:

The growth of the economy's capital stock $ K $ is thus determined by investment, a share $ s $ of income $ Y $, minus depreciation, a share $ \delta $ of the current capital stock $ K $:

> (2.1.14) $ \frac{dK}{dt} = sY - \delta K = \left(\frac{s}{\kappa} - \delta \right)K $

We typically assume that s is constant. We do, however, think about the consequences of its taking a permanent upward or downward jump at some particular moment of time. The background assumption, however—made because it makes formulas much simpler—will always be that s will then remain at its jumped-to value as far as we look into the future.

2.1.3. The Labor Force and the Efficiency-of-Labor

If the labor force L were constant and technological and organizational progress plus education that add to the efficiency-of-labor E were non-existent, we could immediately move on. But the economy's labor force grows as more people turn 18 or so and join the labor force than retire, and as immigrants continue to arrive. And the efficiency of labor rises as science and technology progress, people keep thinking of new and more efficient forms of business organization, and people go to school and learn on the job

We assume—once again making a simplifying leap—that the economy's labor force L's proportional rate g_L is a constant n.

Note that n is not the same across countries. Note that it can and does shift over time in any one country. Since we want to tackle simple cases first, our background assumption will be that n is constant now as far as we can see into the future. But we will drop and vary this when we want to.

Thus between this year and the next the labor force grows according to the formula:

> (2.1.15) $\frac{dL}{dt} = g_LL = nL$

Next year's labor force will thus be n percent higher than this year's labor force.

We also assume—once again making a simplifying leap—that the economy's efficiency of labor E's proportional growth rate g is a constant every year. Note that g is not the same across countries. Note that it can and does shift over time in any one country. But we want to tackle simple cases first. A constant efficiency-of-labor growth rate g is simple. Thus our background assumption will be that g is constant as far as we can see into the future. Then between this year and the next the efficiency of labor grows according to the formula:

> (2.1.16) $\frac{dE}{dt} = gE$

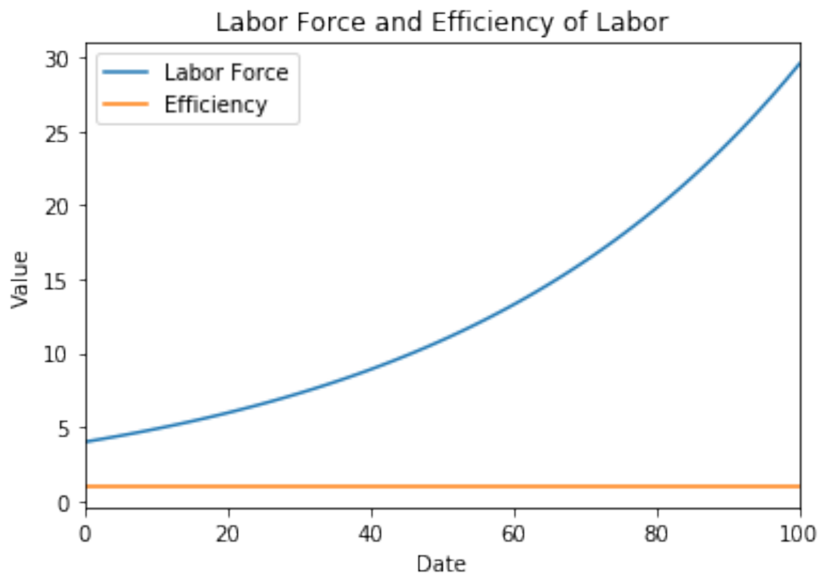
2.1.4. Gaining Intuition for n, g, L, E, Y, $ \kappa $, and $ \theta $

Immediately below this paragraph are two Python code cells. The purpose of these two code cells is to make the above algebra real. What does it mean to say that the proportional growth rate of the labor force $ L$ is $ n = 0.02 $—2% per year? It means that the labor force would double every thirty-five years. What does it mean to say that labor efficiency E's proportional growth rate is $ g = 0.05 $—5% per year? It means that labor efficiency doubles every twelve years. What are the implications of $ \theta = 1 $? It means that if you were to compare two economies with no differences save that the capital-intensity $ \kappa $ of one were twice that of the other, output and output per worker in the first would also be twice as great as in the other. What are the implications of $ \theta = 3 $? It means that if you were to compare two economies with no differences save that the capital-intensity $ \kappa $ of one were twice that of the other, output and output per worker in the first would also be nine times as great as in the other. Play with the code cells—it is the only way to make the algebra real:

In [1]:

```
# CODE CELL 2.1.4.a GAINING INTUITION FOR n & g & L AND E
#
# INTRO
#
# getting a sense of how the Solow growth model
# works...
#
# what are the implications of saying that the
# labor force L increases at a gross rate n?
# Or that the efficiency-of-labor E increases
# at a growth rate g? or that the elasticity
# of output Y with respect to the economy's
# capital-intensity κ is a parameter θ?
# these code cells are for you to experiment,
# and find out. they set out simple finger
# exercises for you to investigate.
#
# this code cell allows you to explore what
# various values for the parameters n and g
# (and also initial values L_0 and E_0) really
# mean.
#
#
# ===========================================================================
# BEGIN MODIFICATION BLOCK                                                  =
#                                                                           =
# change the right-hand-sides of the equations                              =
# in this block to what you wish. simply substitute                         =
# your own choices of values for L_0, n, E_0,                               =
# g, and the length T of the period you want                                =
# to examine. then execute this code cell, and                              =
```

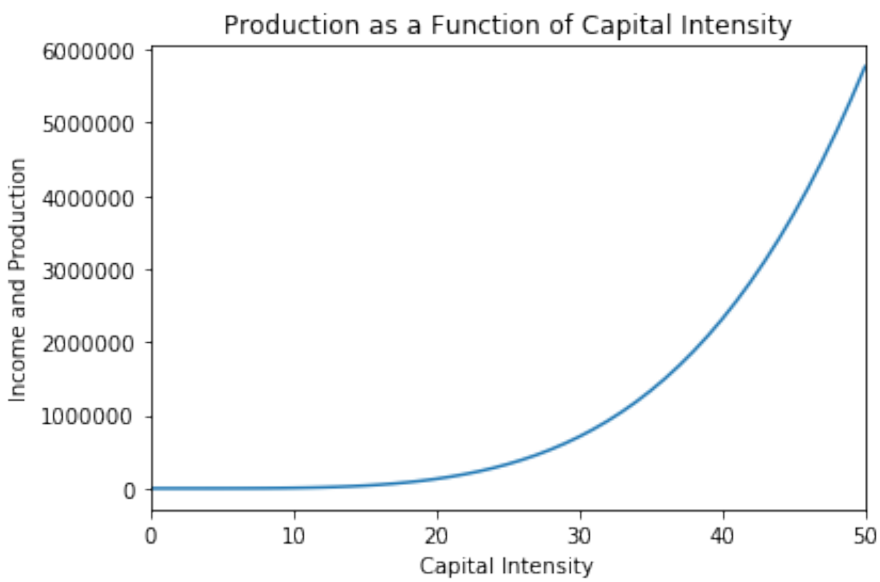
```
# see what results:                                                          =
#                                                                            =
T = 100      # the time-span lenth of the simulation                         =
L_0 = 4      # this is the initial time-0 level of the labor force           =
n = 0.02     # this is the labor force growth rate (0.02=2%/year)            =
E_0 = 1      # this is the initial time-0 level of labor efficiency          =
g = 0.00     # this is the labor efficiency growth rate (0=0%/year)          =
#                                                                            =
# END MODIFICATION BLOCK                                                     =
# ============================================================================
#
# CODE (DON'T TOUCH THIS BELOW!)

import numpy as np
import pandas as pd
import matplotlib.pyplot as plt

L_series = [L_0]
E_series = [E_0]
L = L_0
E = E_0

for i in range(T):
    L = L*np.exp(n)
    E = E *np.exp(g)
    L_series = L_series + [L]
    E_series = E_series + [E]

n_and_g_df = pd.DataFrame()
n_and_g_df['Labor Force'] = L_series
n_and_g_df['Efficiency'] = E_series

ax = n_and_g_df.plot()
ax.set_title("Labor Force and Efficiency of Labor")
ax.set_xlabel("Date")
ax.set_ylabel("Value")

plt.show()
```



In [2]:

```
# CODE CELL 2.1.4.b. GAINING INTUITION FOR Y & θ & κ
#
# INTRO
#
# this code cell allows you to explore what
# various values for the parameter θ mean for
# the relationship between Y and κ.
#
# choose your values for the efficiency-of-
# labor E, for the lafor force L; choose a
# maximum value for capital-intensity κ and
# a value for θ; or accept the values given
# in the block. then execute this code cell,
# and see what results:
#
# ===================================================================
# BEGIN MODIFICATION BLOCK                                          =
#                                                                   =
# change the right-hand-sides of the equations                      =
# in this block to what you wish. simply substitute                 =
# your own choices of values for L, E, θ, and κ_{max}—              =
# the maximum capital-intensity you want to examine.                =
# then execute this code cell, and see what results:                =
#                                                                   =
L = 1        # this is the assumed & constant level of the labor force    =
E = 1        # this is the assumed & constant level of labor efficiency   =
K_max = 50   # this is the maximum x-axis value for your graph            =
θ = 4        # this is the salience of capital in the production function =
#                                                                   =
# END MODIFICATION BLOCK                                            =
# ===================================================================
#
#
#
```

```
# CODE (DON'T TOUCH THIS BELOW!)

import numpy as np
import pandas as pd
import matplotlib.pyplot as plt

K_series = [0]
Y_series = [0]

for K in range(K_max):
    Y = K**θ * E * L
    K_series = K_series + [K]
    Y_series = Y_series + [Y]

Y_and_K_df = pd.DataFrame()
Y_and_K_df['capital_intensity'] = K_series
Y_and_K_df['production'] = Y_series

Y_and_K_df.set_index('capital_intensity')

ax = Y_and_K_df.production.plot()
ax.set_title("Production as a Function of Capital Intensity")
ax.set_xlabel("Capital Intensity")
ax.set_ylabel("Income and Production")

plt.show()
```



2.2. The Equilibrium Condition

2.2.1. Balanced Growth

Multiply the economy's capital-intensity $ \kappa $ by the economy's level of total income and production $ Y$ and you get the economy's capital stock $ K$:

> (2.2.1) $ K = \kappa Y $

the amount of produced means of production that the economy has inherited from its past.

Now take the natural log of (2.1.1) and then take the time derivative of the result:

> (2.2.2) $ \ln(Y) = \theta\ln(\kappa) + \ln(L) + \ln(E) $
>
> (2.2.3) $ \frac{1}{Y}\frac{dY}{dt} = g_Y = \theta \left(\frac{1}{\kappa}\frac{d\kappa}{dt} \right) + \frac{1}{L}\frac{dL}{dt} + \frac{1}{E}\frac{dE}{dt} $

We assumed that the second term on the right-hand-side of (2.2.3) is n and that the third term is g. If the capital-intensity $ \kappa $ is constant, then the left-hand-side will be equal to n+g: that will then be the proportional growth rate $ g_Y $ of income and production $ Y $. If Y is growing at rate $n+g$ and $ \kappa $ is constant, then the economy's capital-stock K will also be growing at $n+g$. Everything will then be in *balanced growth*. And if the economy is in *balanced growth* it will stay there. And if the economy is not in *balanced growth*, it will head for a configuration that is.

2.2.2. When is Capital-Intensity $ \kappa $ Constant?

We had assumed that a constant fraction $ s $ of income $ Y $ was saved and invested to add to the capital stock. We had assumed that a share $ \delta $ of the capital stock rusts and erodes and disappears each year. Thus we had assumed that the capital-stock $ K$ was changing at:

> (2.2.4) $ \frac{dK}{dt} = sY - \delta K $

And the capital stock was growing at a proportional growth rate:

> (2.2.5) $ \frac{1}{K}\frac{dK}{dt} = g_{K} = \frac{s}{\kappa} - \delta $

If we take the time derivative of (2.2.1) and substitute it into (2.2.3) we find:

> (2.2.6) $ \frac{1}{Y}\frac{dY}{dt} = g_Y = \theta \left(\frac{1}{K}\frac{dK}{dt} - \frac{1}{Y}\frac{dY}{dt} \right) + \frac{1}{L}\frac{dL}{dt} + \frac{1}{E}\frac{dE}{dt} $
>
> (2.2.7) $ (1+\theta)\frac{1}{Y}\frac{dY}{dt} = \theta \left(\frac{s}{\kappa} - \delta \right) + n + g $

So the proportional rate of growth of capital-intensity $ \kappa $ is:

> (2.2.8) $ \frac{1}{\kappa}\frac{d\kappa}{dt} = g_\kappa = \frac{s}{\kappa} - \delta - \left(\frac{\theta}{1+\theta} \right)\left(\frac{s}{\kappa} - \delta \right) - \frac{n + g}{1+\theta} $
>
> (2.2.9) $ \frac{1}{\kappa}\frac{d\kappa}{dt} = \frac{s/\kappa - (n+g+\delta)}{1+\theta} $

Thus the capital-stock will be growing at the rate $n+g$ required for balanced growth if and only if:

> (2.2.10) $ \frac{s}{\kappa} - \delta = n + g $

2.2.3. Gaining Intuition with Respect to the Rate of Change of Capital-Intensity $\kappa $

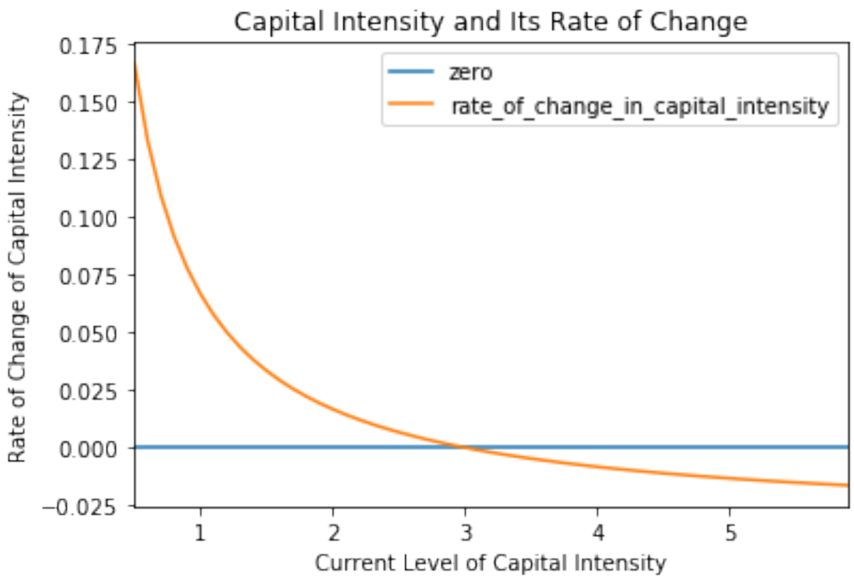
Immediately below this paragraph is a Python code cell to help you gain some intuition with respect to what equation (2.2.9) is telling us about the rate of change of the capital-intensity $ \kappa $ of an economy following the Solow growth model. Once again, play with the code cell—it is the only way to make the algebra real:

In [3]:

```
# CODE CELL 2.2.3.a. κ, κ*, AND THE DYNAMIC BEHAVIOR OF CAPITAL INTENSITY
#
# INTRO
#
# this code cell plots how the proportional growth rate
# g_κ of capital-intensity κ varies with the level
# of capital-intensity κ.
#
# it will show you that the growth rate of capital-
# intensity κ is equal to zero if and only if:
#
```

```
#    κ = κ^* = s/(n+g+δ)
#
# either accept the values given below for s, n, g, and
# δ, or substitute your own in the relevant code lines
# in the block. then execute this code cell, and see
# what results:
#
# ===========================================================================
# BEGIN MODIFICATION BLOCK                                                  =
#                                                                           =
# change the right-hand-sides of the equations                              =
# in this block to what you wish. simply substitute                         =
# your own choices of values for s, n, g, δ, and θ.                         =
# then execute this code cell, and see what results:                        =
#                                                                           =
s = 0.30    # this is the share of output that is saved & invested          =
n = 0.01    # this is the population & labor-force growth rate              =
            # (0.01=1%/year)                                                =
g = 0.015   # this is the labor-efficiency growth rate (0.015=1.5%/year)    =
δ = 0.075   # this is the depreciation rate (0.075=7.5%/year)               =
θ = 2       # this is the salience of capital in production                 =
#                                                                           =
# END MODIFICATION BLOCK                                                    =
# ===========================================================================
#
# CODE (DON'T TOUCH THIS BELOW!)
#

import numpy as np
import pandas as pd
import matplotlib.pyplot as plt

κ_star = s/(n+g+δ)
κ_max = 2*κ_star
κ_min = 0.5

dκ_series = []
κ_series = []
zero_series = []

for j in range(round(10*κ_min), 10*round(κ_max)):
    κ = j/10
    dκ_series = dκ_series + [(s/κ - (n+g+δ))/(1+θ)]
    κ_series = κ_series + [κ]
    zero_series = zero_series + [0]

κ_and_dκ_df = pd.DataFrame()
κ_and_dκ_df['rate_of_change_in_capital_intensity'] = dκ_series
κ_and_dκ_df['capital_intensity'] = κ_series
κ_and_dκ_df['zero'] = zero_series

ax = plt.gca()
```

```
κ_and_dκ_df.plot(kind='line', x='capital_intensity', y='zero', ax=ax)
κ_and_dκ_df.plot(kind='line', x='capital_intensity',
             y='rate_of_change_in_capital_intensity',
             title = 'Capital Intensity and Its Rate of Change',
             ax=ax)

ax.set_xlabel("Current Level of Capital Intensity")
ax.set_ylabel("Rate of Change of Capital Intensity")

plt.show()
```



In [ ]:

3. Growing Along and Converging to the Balanced-Growth Equilibrium Path

3.1. The Balanced-Growth Equilibrium Path

3.1.1. The Balanced-Growth Equilbrium Capital Intensity κ^*

We define κ^* as that value of capital-intensity κ for which, at the current levels of the parameters n, g, δ, s, and θ:

> (2.2.10) $\frac{s}{\kappa} - \delta = n + g$

is satisfied. That is true if and only if:

> (3.1.1) $\kappa^* = \frac{s}{n+g+\delta}$

If the capital-intensity $\kappa = \kappa^*$, then it is constant. The economy is then in balanced growth.

From (2.1.2) we see that—as capital-intensity κ is then constant at the value κ^*—the proportional growth rate g_Y of total income and production in the economy Y is then equal to n+g, the sum of the growth rate n of the labor force. From (2.1.3) we see that—as capital-intensity κ is then constant at the value κ^*—the proportional growth rate g_y of output per worker is than equal to teh proportional growth rate g of the efficiency of labor. From (2.1.1) we see that the proportional growth rate g_K of the economy's total capital stock is then the same $n + g$ as the growth rate g_Y of income and production Y.

3.1.2. Calculating the Balanced-Growth Equilbrium Path

We can then—if we know the parameter values of the model, the initial values L_0 and E_0 of the labor force and labor efficiency at some time we index equal to 0, and that the economy is on its balanced-growth equilibrium path—calculate what all variables of interest in the economy will be at any time whatsoever:

Total income and prdouction will be:

> (3.1.2) $Y^*_t = \left(\kappa^* \right)^\theta E_t L_t$ $= \left(\kappa^* \right)^\theta e^{gt}E_0 e^{nt}L_0$ $= \left(s/(n+g+\delta) \right)^\theta e^{gt}E_0 e^{nt}L_0$

Income and production per worker will be:

> (3.1.3) $y^*_t = \left(\kappa^* \right)^\theta E_t$ $= \left(\kappa^* \right)^\theta e^{gt}E_0$ $= \left(s/(n+g+\delta) \right)^\theta e^{gt}E_0$

The capital stock will be:

> (3.1.4) $K^*_t = \kappa^* Y^*_t$ $= \left(s/(n+g+\delta) \right)^{(1+\theta)} e^{gt}E_0 e^{nt}L_0$

The labor force will be:

> (3.1.5) $L^*_t = e^{nt}L_0$

And labor efficiency will be:

> (3.1.6) $E^*_t = e^{gt}E_0$

3.2. Converging to the Balanced-Growth Equilibrium Path

3.2.1. The Dynamics of Capital Intensity

But what if $\kappa \neq \kappa^*$? What happens then? Since $s = \kappa^*(n+g+\delta)$, we can multiply (2.2.9) by κ and then rewrite it in terms of the equilibrium capital-intensity κ^* as:

> (3.2.1) $\frac{d\kappa}{dt} = s/(1+\theta) - (n+g+\delta)\kappa/(1+\theta)$
>
> (3.2.2) $\frac{d\kappa}{dt} = (n+g+\delta)\kappa^*/(1+\theta) - (n+g+\delta)\kappa/(1+\theta)$

> (3.2.3) $\frac{d\kappa}{dt} = - \frac{n+g+\delta}{1+\theta} (\kappa-\kappa^*)$

This is of the form of the very first differential equations one encounters in mathematics: it is the exponential equation $dx/dt = k x$, with the constant k here equal to $-(n+g+\delta)/(1+\theta)$. This equation has the solution, if the value of capital-intensity κ is known at some time $t=0$ to be κ_0:

> (3.2.4) $\kappa = \kappa^* + e^{-((n+g+\delta)/(1+\theta))t}(\kappa_0 - \kappa^*)$

(3.2.3) holds always, for that moment's values of n, g, δ, θ , and s, whatever they may be.

(3.2.4) holds only while n, g, δ, θ , and s are constant. If any of them change, you then have to recalibrate and recompute, with a new initial value of κ_0, equal to its value when the model's parameters jumped, and a new and different value of κ^*.

If n, g, δ, θ, and s are constant or near-constant, then (3.2.4) is a very powerful tool: it tells us that the economy's capital-intensity κ follows over time a path of exponential convergence. It is, at time zero, equal to its initial condition κ_0. It then converges towards its asymptote κ^*, reducing the gap between its value and κ^* at any time t to a fraction $1/e$ of its previous value as of time $t + \Delta_{1/e}t$, where this $1/e$ convergence time is:

> (3.2.5) $\Delta_{1/e}t = (n+g+\delta)/(1+\theta)$.

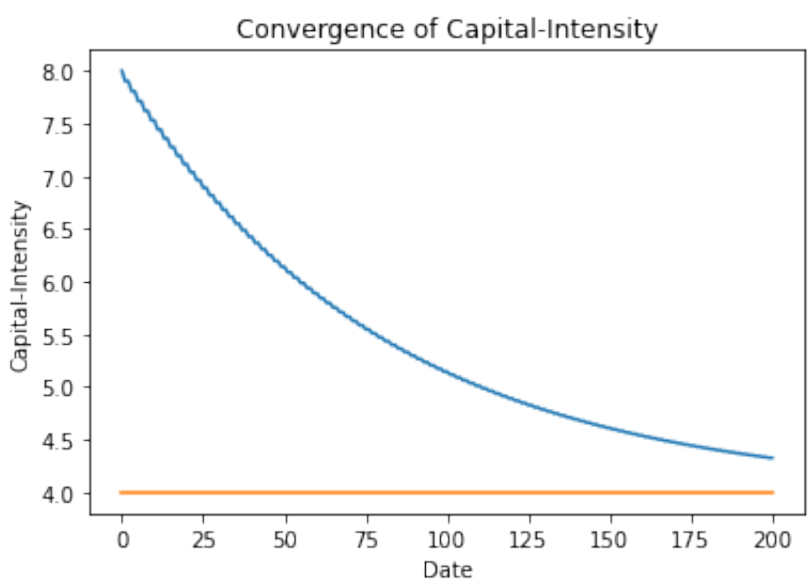
3.2.2. Gaining Intuition About the Convergence of Capital-Intensity to $\kappa^* $

Immediately below this paragraph are some Python code cells to help you gain some intuition with respect to the dynamics by which the capital-intensity $ \kappa $ of an economy following the Solow growth model converges to its balanced-growth equilibrium value $ \kappa^* $. Once again, play with the code cells—it is the only way to make the algebra real:

In [1]:

```
# CODE CELL 3.2.2.a. CONVERGENCE OF κ TO κ^*
#
# INTRO
#
# this code cell plots how the the economy's
# capital-intensity κ converges to its steady-
# state balanced-growth level κ^* no matter what
# the initial condition κ_0.
#
# either accept the values given below in the block for
# the parameters s, n, g, δ, and θ, and the intitial
# condition on capital-intensity κ_0, plus the time T
# you wish to calculate convergence for, or substitute
# your own preferred values in the relevant code lines
# in the block. then execute this code cell, and see
# what results:
#
# ==========================================================================
# BEGIN MODIFICATION BLOCK                                                 =
#                                                                          =
T = 200     #                                                              =
#                                                                          =
κ_0 = 8     #                                                              =
s = 0.20    #                                                              =
n = 0.01    #                                                              =
g = 0.015   #                                                              =
δ = 0.025   #                                                              =
θ = 1       #                                                              =
#                                                                          =
# END MODIFICATION BLOCK                                                   =
# ==========================================================================

# CODE (DON'T TOUCH THIS BELOW!)
#
import numpy as np
import pandas as pd
import matplotlib.pyplot as plt

κ_star = s/(n+g+δ)
```

```
κ_max = 2*κ_star
κ_min = 0.5

κ_star_series = [κ_star]
κ_series = [κ_0]

for t in range(T):
    κ_star_series = κ_star_series + [κ_star]
    κ_series = κ_series + [κ_star + (κ_series[t-1] - κ_star)*np.exp(-(n+g+δ)/(


κ_convergence_df = pd.DataFrame()
κ_convergence_df['steady_state_capital_intensity'] = κ_star_series
κ_convergence_df['capital_intensity'] = κ_series

ax = plt.gca()

κ_convergence_df.capital_intensity.plot(ax=ax)
κ_convergence_df.steady_state_capital_intensity.plot(ax=ax,
                    title = 'Convergence of Capital-Intensity')

ax.set_xlabel("Date")
ax.set_ylabel("Capital-Intensity")

plt.show()
```



In [2]:

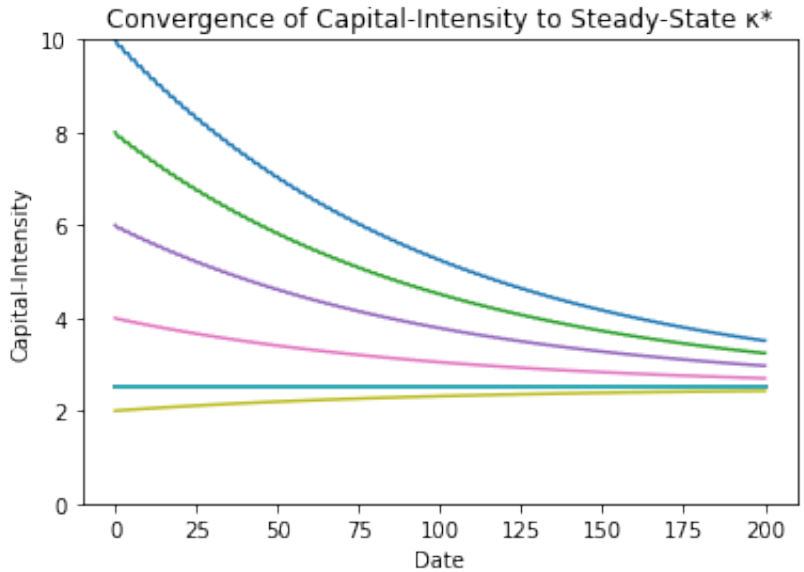
```
# CODE CELL 3.2.2.b. DEFINITION OF CLASS κ_convergence_graph
# DO NOT TOUCH THIS!!!!!
#
# this is a reference copy of the κ_convergence_graph
# Python class. it will be kept in the delong_classes
# local file and accessed as:
#
#     delong_classes.κ_convergence_graph
```

```
#
# use this class to plot how the the economy's
# capital-intensity κ converges to its steady-
# state balanced-growth level κ^* no matter what
# the initial condition κ_0:

import numpy as np
import pandas as pd
import matplotlib.pyplot as plt

class κ_convergence_graph:

    def __init__(self, κ_0 = 3,
                     s = 0.20,
                     n = 0.01,
                     g = 0.015,
                     δ = 0.025,
                     θ = 1,
                     T = 200):
        self.κ_0, self.s, self.n, self.g, self.δ, self.θ, self.T = κ_0, s, n,

    def draw(self):
        "Draw the convergence graph"
        κ_0, s, n, g, δ, θ, T = self.κ_0, self.s, self.n, self.g, self.δ, self
        κ_star = s/(n+g+δ)
        κ_max = 2*κ_star
        κ_min = 0.5

        κ_star_series = [κ_star]
        κ_series = [κ_0]

        for t in range(T):
            κ_star_series = κ_star_series + [κ_star]
            κ_series = κ_series + [κ_star + (κ_series[t-1] - κ_star)*np.exp(-(


        κ_convergence_df = pd.DataFrame()
        κ_convergence_df['steady_state_capital_intensity'] = κ_star_series
        κ_convergence_df['capital_intensity'] = κ_series

        ax = plt.gca()

        κ_convergence_df.capital_intensity.plot(ax=ax)
        κ_convergence_df.steady_state_capital_intensity.plot(ax=ax,
                  title = 'Convergence of Capital-Intensity to Steady-State K*

        ax.set_xlabel("Date")
        ax.set_ylabel("Capital-Intensity")
```

In [3]:

```
# CODE CELL 3.2.2.c. USING CLASS κ_convergence_graph TO EXAMINE
# THE CONVERGENCE OF A SOLOW GROWTH MODEL'S CAPITAL-
# INTENSITY κ TO ITS BALANCED-GROWTH VALUE:
```

```
#
# INTRO
#
#     κ^* = s/(n+g+δ)
#
# either accept the values given below in the block for
# the parameters s, n, g, δ, and θ, and the intitial
# capital-intensity conditions κ_max and κ_reduce, plus
# the time T you wish to calculate convergence for,
# or, alternatively, you can substitute your own preferred
# values in the relevant code lines found inside of
# the code block. then execute this code cell, and see
# what results:
#
#
# ========================================================================
# BEGIN MODIFICATION BLOCK                                               =
#                                                                        =
κ_max = 10    #                                                          =
κ_reduce = 2  #                                                          =
s = 0.15      #                                                          =
n = 0.02      #                                                          =
g = 0.015     #                                                          =
δ = 0.025     #                                                          =
θ = 2         #                                                          =
T = 200       #                                                          =
#                                                                        =
#                                                                        =
# END MODIFICATION BLOCK                                                 =
# ========================================================================
#


# CODE (DON'T THOUCH THIS BELOW!)

import delong_classes

plt.cla()

κ = κ_max
for i in range(5):
    cg = delong_classes.κ_convergence_graph(κ_0=κ, s = s, n = n,
         g = g, δ = δ, θ = θ, T = T)
    cg.draw()
    κ = κ-κ_reduce

plt.ylim(0, κ_max)
plt.show()
```



3.2.3. The Dynamics of the Other Variables in the Economy

Given:

> (3.2.4) $ \kappa = \kappa^* + e^{-((n+g+\delta)/(1+\theta))t}(\kappa_0 - \kappa^*) $

knowledge of the initial values of capital-intensity, the labor force , and the efficiency-of-labor $ \kappa_0, L_0,$ and $ E_0 $; and knowledge of the parameters $ s, n, g, \delta$; and $\theta $; we can then immediately calculate the values from time $ t = 0 $, until some parameter shifts, of all the other variables in the economy from ths equations:

> (3.2.6) $ Y_t = \left(\kappa_t \right)^\theta E_t L_t = \left(\kappa_t \right)^\theta e^{gt}E_0 e^{nt}L_0$
>
> (3.2.7) $ y_t = \left(\kappa_t \right)^\theta E_t = \left(\kappa_t \right)^\theta e^{gt}E_0 $
>
> (3.2.8) $ K_t = \kappa_t Y_t $
>
> (3.2.9) $ L_t = e^{nt}L_0 $
>
> (3.2.10) $ E_t = e^{gt}E_0 $

3.2.4. Gaining Intuition About Convergence and Shocks

Immediately below this paragraph are some Python code cells to help you gain some intuition with respect to the dynamics by which an economy following the Solow growth model converges to and then follows along its balanced-growth path. The first cell below contains a reference copy of the delong_classes.solow Python class. The second cell below plots a six-panel figure showing the behavior of variables of interest for T periods from an arbitrary starting point. This second cell graphs both a baseline scenario, and an alternative scenario showing what happened after a discontinuous jump in parameters, or what would have happened alternatively had parameters jumped. Once again, play with the code cells—it is the only way to make the algebra real:

In [4]:

```
# CODE CELL 3.2.4.a
# DEFINING THE delong_classes.solow PYTHON CLASS
# DONT TOUCH THIS CODE CELL!
#
# # this is a reference copy of the κ_convergence_graph
# Python class. it will be kept in the delong_classes
# local file and accessed as:
#
#     delong_classes.solow
#
# use this class to model the dynamic behavior of
# an economy well-described by the Solow growth model:

import numpy as np
import matplotlib.pyplot as plt
%matplotlib inline

class solow:

    """
    Implements the Solow growth model calculation of the
    capital-output ratio κ and other model variables
    using the update rule:

    κ_{t+1} = κ_t + ( 1 - α) ( s - (n+g+δ)κ_t )

    Built upon and modified from Stachurski-Sargeant
    <https://quantecon.org> class **Solow**
    <https://lectures.quantecon.org/py/python_oop.html>
    """

    def __init__(self, n=0.01,                # population growth rate
                       s=0.20,                # savings rate
                       δ=0.03,                # depreciation rate
```

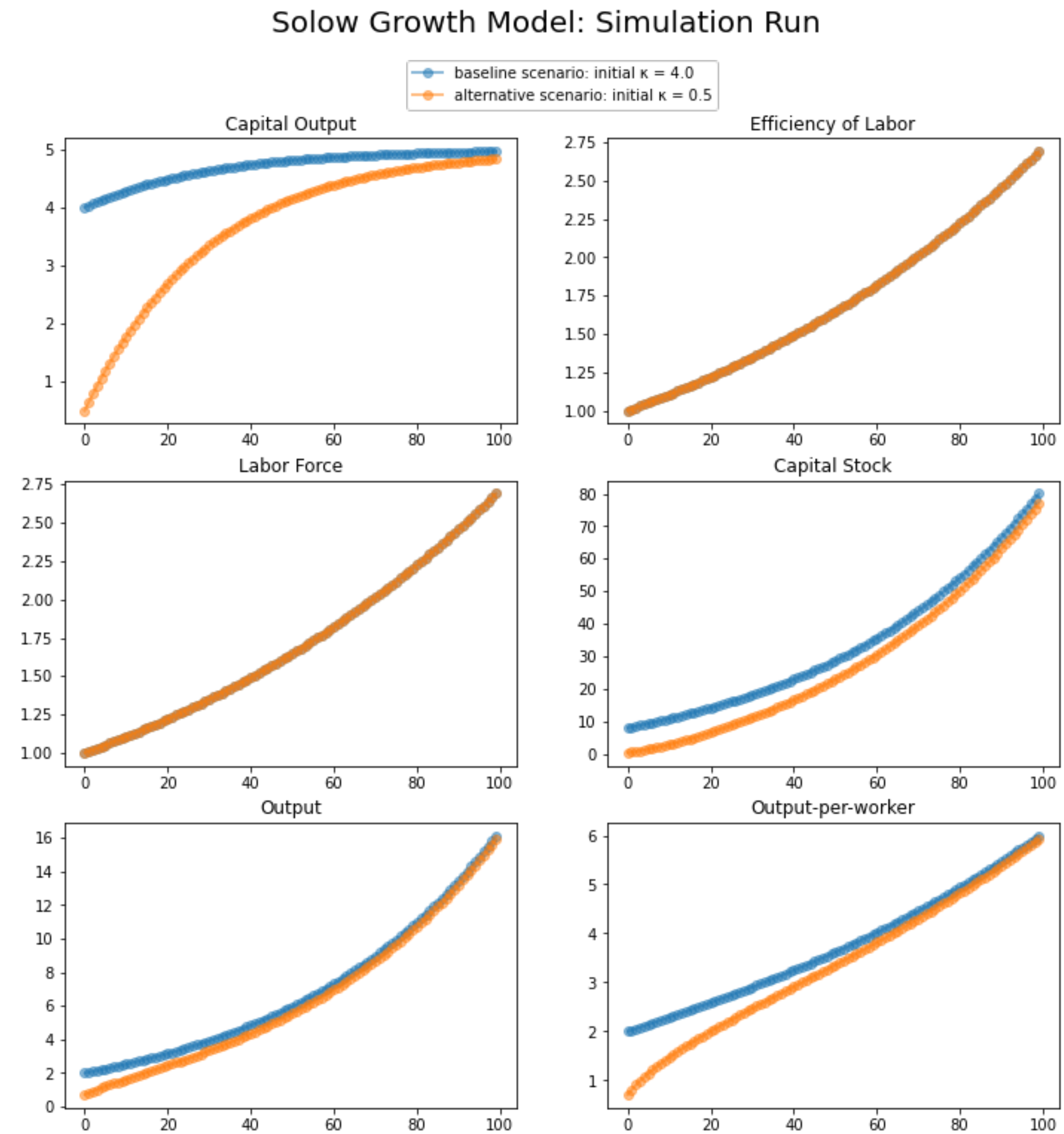
```
                 α=1/3,             # share of capital
                 g=0.01,            # productivity
                 κ=0.2/(.01+.01+.03), # current capital-labor ratio
                 E=1.0,             # current efficiency of labor
                 L=1.0):            # current labor force

        self.n, self.s, self.δ, self.α, self.g = n, s, δ, α, g
        self.κ, self.E, self.L = κ, E, L
        self.Y = self.κ**(self.α/(1-self.α))*self.E*self.L
        self.K = self.κ * self.Y
        self.y = self.Y/self.L
        self.α1 = 1-((1-np.exp((self.α-1)*(self.n+self.g+self.δ)))/(self.n+se
        self.initdata = vars(self).copy()

    def calc_next_period_kappa(self):
        "Calculate the next period capital-output ratio."
        # Unpack parameters (get rid of self to simplify notation)
        n, s, δ, α1, g, κ= self.n, self.s, self.δ, self.α1, self.g, self.κ
        # Apply the update rule
        return (κ + (1 - α1)*( s - (n+g+δ)*κ ))

    def calc_next_period_E(self):
        "Calculate the next period efficiency of labor."
        # Unpack parameters (get rid of self to simplify notation)
        E, g = self.E, self.g
        # Apply the update rule
        return (E * np.exp(g))

    def calc_next_period_L(self):
        "Calculate the next period labor force."
        # Unpack parameters (get rid of self to simplify notation)
        n, L = self.n, self.L
        # Apply the update rule
        return (L*np.exp(n))

    def update(self):
        "Update the current state."
        self.κ =  self.calc_next_period_kappa()
        self.E =  self.calc_next_period_E()
        self.L =  self.calc_next_period_L()
        self.Y = self.κ**(self.α/(1-self.α))*self.E*self.L
        self.K = self.κ * self.Y
        self.y = self.Y/self.L

    def steady_state(self):
        "Compute the steady state value of the capital-output ratio."
        # Unpack parameters (get rid of self to simplify notation)
        n, s, δ, g = self.n, self.s, self.δ, self.g
        # Compute and return steady state
        return (s /(n + g + δ))

    def generate_sequence(self, T, var = 'κ', init = True):
        "Generate and return time series of selected variable. Variable is κ 
```

```
        path = []

        # initialize data
        if init == True:
            for para in self.initdata:
                setattr(self, para, self.initdata[para])

        for i in range(T):
            path.append(vars(self)[var])
            self.update()
        return path
```

In [8]:

```
# CODE CELL 3.2.4.b. SIX-PANEL BASIC SOLOW GROWTH MODEL FIGURE
#
# use this code cell to create a six-panel time-series
# figure of the most interesting economic variables in
# the basic Solow growth model for T periods from an
# arbitrary starting point.
#
# this cell graphs both a baseline scenario and an
# alternative scenario showing what happened after
# a discontinuous jump in parameters, or what would
# have happened alternatively had parameters jumped.
#
# assign different parameters than the defaults to your
# baseline and alternative scenarios by including them
# as comma-separated parameter=value pairs in the parentheses
# at the end of the lines beginning with 's_base =' and
# 's_alt ='

import numpy as np
import matplotlib.pyplot as plt
%matplotlib inline
import delong_classes


# ===========================================================================
# BEGIN MODIFICATION BLOCK                                                  =
#                                                                           =
T = 100                                    # time length of the simulation=
#                                                                           =
s_base = delong_classes.solow(κ=4.0,s=.25) # baseline scenario parameters =
s_alt = delong_classes.solow(κ=0.5,s=.25)  # alternative parameters       =
#                                                                           =
# END MODIFICATION BLOCK                                                    =
# ===========================================================================
#
#
# CODE (DON'T TOUCH THIS BELOW!)

s_base.scenario = "baseline scenario"
s_alt.scenario = "alternative scenario"
```

```
figcontents = {
        (0,0):('K','Capital Output'),
        (0,1):('E','Efficiency of Labor'),
        (1,0):('L','Labor Force'),
        (1,1):('K','Capital Stock'),
        (2,0):('Y','Output'),
        (2,1):('y','Output-per-worker')
       }

num_rows, num_cols = 3,2
fig, axes = plt.subplots(num_rows, num_cols, figsize=(12, 12))
for i in range(num_rows):
    for j in range(num_cols):
        for s in s_base, s_alt:
            lb = f'{s.scenario}: initial K = {s.initdata["K"]}'
            axes[i,j].plot(s.generate_sequence(T, var = figcontents[i,j][0]),
            axes[i,j].set(title=figcontents[i,j][1])

#   global legend
axes[(0,0)].legend(loc='upper center', bbox_to_anchor=(1.1,1.3))
plt.suptitle('Solow Growth Model: Simulation Run', size = 20)
plt.show()
```


Solow Growth Model: Simulation Run
baseline scenario: initial κ = 4.0
alternative scenario: initial κ = 0.5
Capital Output
Efficiency of Labor
Labor Force
Capital Stock
Output
Output-per-worker

4. Using the Solow Growth Model

4.1. Convergence to the Balanced-Growth Path

4.1.1. The Example of Post-WWII West Germany

Economies do converge to and then remain on their balanced-growth paths. The West German economy after World War II is a case in point.

We can see such convergence in action in many places and times. For example, consider the post-World War II history of West Germany. The defeat of the Nazis left the German economy at the end of World War II in ruins. Output per worker was less than one-third of its prewar level. The economy's capital stock had been wrecked and devastated by three years of American and British bombing and then by the ground campaigns of the last six months of the war. But in the years immediately after the war, the West German economy's capital-output ratio rapidly grew and converged back to its prewar value. Within 12 years the West German economy had closed half the gap back to its pre-World War II growth path. And within 30 years the West German economy had effectively closed the entire gap between where it had started at the end of World War II and its balanced-growth path.

In [1]:

```
# bring in libraries needed for python

import pandas as pd
import numpy as np
import matplotlib.pyplot as plt

# load data from the penn world table's estimates
# of growth among the g-7 economies
#
# (these estimates have problems; but all estimates
# have problems, and a lot of work has gone into
# making these)

pwt91_df = pd.read_csv('https://delong.typepad.com/files/pwt91-data.csv')
```

In [2]:

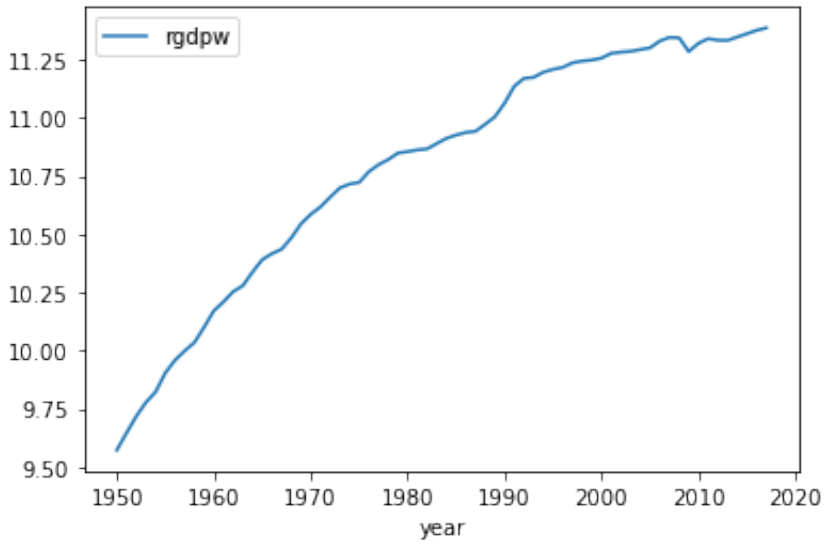
```
# pull america out of the loaded data table

is_America = pwt91_df['countrycode'] == 'USA'
America_df = pwt91_df[is_America]
America_gdp_df = America_df[['year', 'rgdpna', 'emp']]
America_gdp_df['rgdpw'] = America_gdp_df.rgdpna/America_gdp_df.emp
America_pwg_ser = America_gdp_df[['year', 'rgdpw']]
America_pwg_ser.set_index('year',  inplace=True)
```

```
/Users/braddelong/opt/anaconda3/lib/python3.7/site-packages/ipykernel_launcher
.py:6: SettingWithCopyWarning:
A value is trying to be set on a copy of a slice from a DataFrame.
Try using .loc[row_indexer,col_indexer] = value instead

See the caveats in the documentation: https://pandas.pydata.org/pandas-docs/st
able/user_guide/indexing.html#returning-a-view-versus-a-copy
```

In [3]:

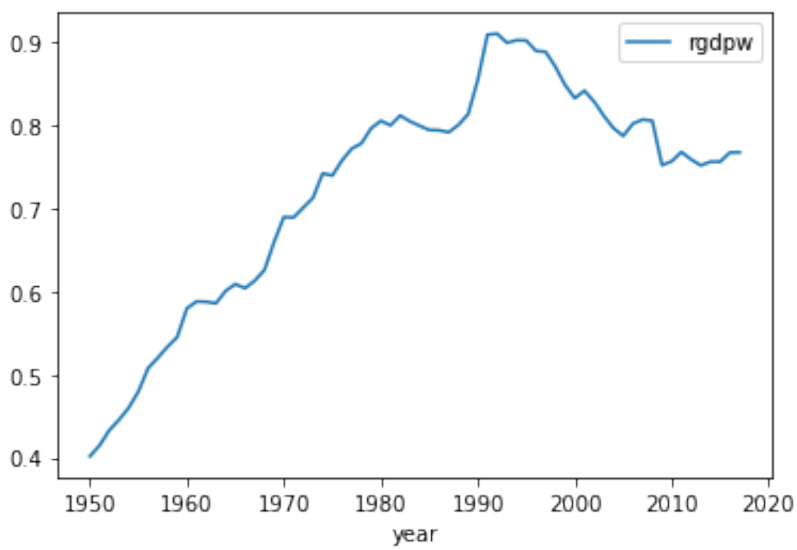
```
# pull germany out of the loaded data table, and
# plot growth

is_Germany = pwt91_df['countrycode'] == 'DEU'
Germany_df = pwt91_df[is_Germany]
Germany_gdp_df = Germany_df[['year', 'rgdpna', 'emp']]
Germany_gdp_df['rgdpw'] = Germany_gdp_df.rgdpna/Germany_gdp_df.emp
Germany_pwg_ser = Germany_gdp_df[['year', 'rgdpw']]
Germany_pwg_ser.set_index('year',  inplace=True)
np.log(Germany_pwg_ser).plot()
plt.show()
```

```
/Users/braddelong/opt/anaconda3/lib/python3.7/site-packages/ipykernel_launcher
.py:7: SettingWithCopyWarning:
A value is trying to be set on a copy of a slice from a DataFrame.
Try using .loc[row_indexer,col_indexer] = value instead

See the caveats in the documentation: https://pandas.pydata.org/pandas-docs/st
able/user_guide/indexing.html#returning-a-view-versus-a-copy
  import sys
```



In [4]:

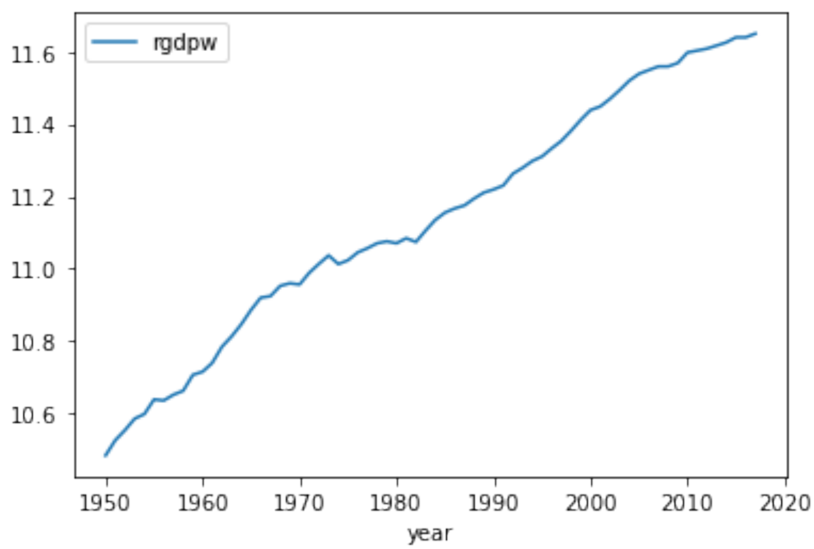
```
# compare german real national income per worker
# with american

Germany_ratio_ser = Germany_pwg_ser/America_pwg_ser
Germany_ratio_ser.plot()
plt.show()
```



The two figures above show, respectively, the natural logarithm of absolute real national income per worker for the German economy and real national income per worker relative to the U.S. value, both since 1950. By 1980 the German economy had converged: its period of rapid recovery growth was over, and national income per capita then grew at the same rate as that in the U.S., which had not suffered wartime destruction pushing it off and below its steady-state balanced-growth path. Then in 1990, at least according to this set of estimates, the absorption of the formerly communist East German state into the *Bundesrepublik* was an enormous benefit: the expanded division of labor and return of the market economy allowed productivity in the German east to more than double almost overnight. Thereafter the German economy has lost some ground relative to the U.S. as the U.S.'s leading information technology hardware and software sectors have been much stronger leading sectors than Germany's precision machinery and manufacturing sectors.

By comparison, the United States shows no analogous period of rapid growth catching up to a steady-state balanced-growth path. (There is, however, a marked boom in the 1960s, and then a return to early trends in the late 1970s and 1980s, followed by a return to previous normal growth in the 1990s and then a fall-off in growth after 2007.)

In [5]:
```
# plot american growth since 1950

np.log(America_pwg_ser).plot()
plt.show()
```



4.1.2. The Example of Post-WWII Japan

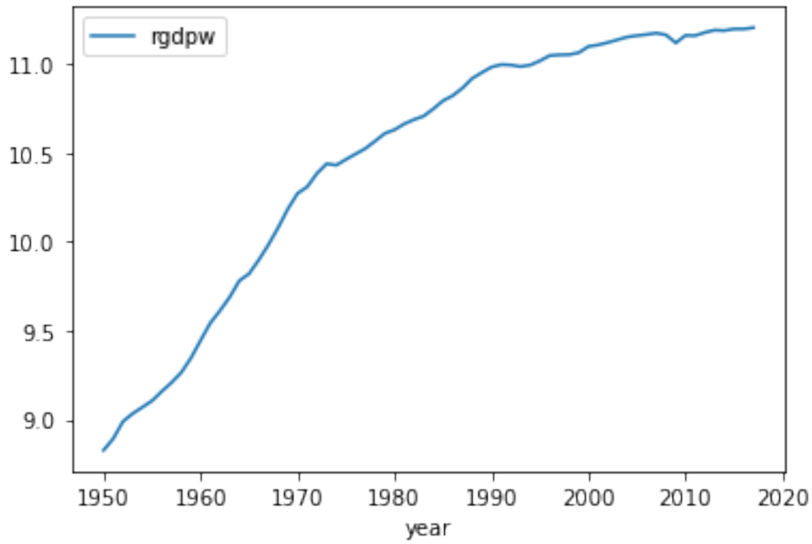
The same story holds in an even stronger form for the other defeated fascist power that surrendered unconditionally to the U.S. at the end of World War II.

In 1950, largely as a result of Curtis LeMay's B-29s, Japan is only half as productive as Germany, and only one-fifth as productive as the United States. Once again, it converges rapidly. After 1990 Japan no longer grows faster than and catches up to the United States. Indeed, like Germany it thereafter loses ground as its world class manufacturing sectors are also less powerful leading sectors than the United States's information technology hardware and software complexes.

In [6]:

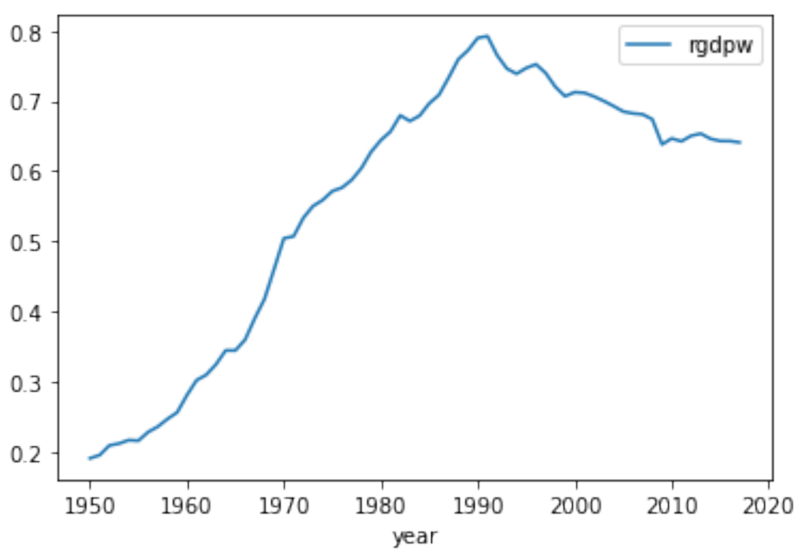
```
# pull japan out of the loaded data table, and
# plot growth

is_Japan = pwt91_df['countrycode'] == 'JPN'
Japan_df = pwt91_df[is_Japan]
Japan_gdp_df = Japan_df[['year', 'rgdpna', 'emp']]
Japan_gdp_df['rgdpw'] = Japan_gdp_df.rgdpna/Japan_gdp_df.emp
Japan_pwg_ser = Japan_gdp_df[['year', 'rgdpw']]
Japan_pwg_ser.set_index('year',  inplace=True)
np.log(Japan_pwg_ser).plot()
plt.show()
```

```
/Users/braddelong/opt/anaconda3/lib/python3.7/site-packages/ipykernel_launcher
.py:7: SettingWithCopyWarning:
A value is trying to be set on a copy of a slice from a DataFrame.
Try using .loc[row_indexer,col_indexer] = value instead

See the caveats in the documentation: https://pandas.pydata.org/pandas-docs/st
able/user_guide/indexing.html#returning-a-view-versus-a-copy
  import sys
```



In [7]:

```
# compare japan to america

Japan_ratio_ser = Japan_pwg_ser/America_pwg_ser
Japan_ratio_ser.plot()
plt.show()
```



4.1.3. The Post-WWII G-7

The same story holds for the other members of the G-7 group of large advanced industrial economies as well.

In [8]:

```
# pull britain out of the loaded data table

is_Britain = pwt91_df['countrycode'] == 'GBR'
Britain_df = pwt91_df[is_Britain]
Britain_gdp_df = Britain_df[['year', 'rgdpna', 'emp']]
Britain_gdp_df['rgdpw'] = Britain_gdp_df.rgdpna/Britain_gdp_df.emp
Britain_pwg_ser = Britain_gdp_df[['year', 'rgdpw']]
Britain_ratio_ser = Britain_pwg_ser/America_pwg_ser
Britain_pwg_ser.set_index('year',  inplace=True)
```

```
/Users/braddelong/opt/anaconda3/lib/python3.7/site-packages/ipykernel_launcher
.py:6: SettingWithCopyWarning:
A value is trying to be set on a copy of a slice from a DataFrame.
Try using .loc[row_indexer,col_indexer] = value instead

See the caveats in the documentation: https://pandas.pydata.org/pandas-docs/st
able/user_guide/indexing.html#returning-a-view-versus-a-copy
```

In [9]:

```
# pull italy out of the loaded data table

is_Italy = pwt91_df['countrycode'] == 'ITA'
Italy_df = pwt91_df[is_Italy]
Italy_gdp_df = Italy_df[['year', 'rgdpna', 'emp']]
Italy_gdp_df['rgdpw'] = Italy_gdp_df.rgdpna/Italy_gdp_df.emp
Italy_pwg_ser = Italy_gdp_df[['year', 'rgdpw']]
Italy_ratio_ser = Italy_pwg_ser/America_pwg_ser
Italy_pwg_ser.set_index('year',  inplace=True)
```

```
/Users/braddelong/opt/anaconda3/lib/python3.7/site-packages/ipykernel_launcher
.py:6: SettingWithCopyWarning:
A value is trying to be set on a copy of a slice from a DataFrame.
Try using .loc[row_indexer,col_indexer] = value instead

See the caveats in the documentation: https://pandas.pydata.org/pandas-docs/st
able/user_guide/indexing.html#returning-a-view-versus-a-copy
```

In [10]:

```
# pull canada out of the loaded data table

is_Canada = pwt91_df['countrycode'] == 'CAN'
Canada_df = pwt91_df[is_Canada]
Canada_gdp_df = Canada_df[['year', 'rgdpna', 'emp']]
Canada_gdp_df['rgdpw'] = Canada_gdp_df.rgdpna/Canada_gdp_df.emp
Canada_pwg_ser = Canada_gdp_df[['year', 'rgdpw']]
Canada_ratio_ser = Canada_pwg_ser/America_pwg_ser
Canada_pwg_ser.set_index('year',  inplace=True)
```

```
/Users/braddelong/opt/anaconda3/lib/python3.7/site-packages/ipykernel_launcher
.py:6: SettingWithCopyWarning:
A value is trying to be set on a copy of a slice from a DataFrame.
Try using .loc[row_indexer,col_indexer] = value instead

See the caveats in the documentation: https://pandas.pydata.org/pandas-docs/st
able/user_guide/indexing.html#returning-a-view-versus-a-copy
```

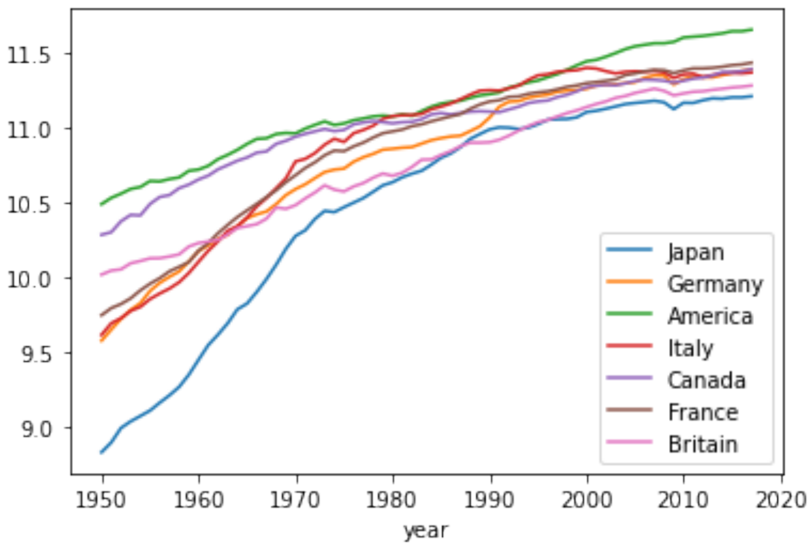
In [11]:

```
# pull italy out of the loaded data table

is_France = pwt91_df['countrycode'] == 'FRA'
France_df = pwt91_df[is_France]
France_gdp_df = France_df[['year', 'rgdpna', 'emp']]
France_gdp_df['rgdpw'] = France_gdp_df.rgdpna/France_gdp_df.emp
France_pwg_ser = France_gdp_df[['year', 'rgdpw']]
France_ratio_ser = France_pwg_ser/America_pwg_ser
France_pwg_ser.set_index('year',  inplace=True)
```

```
/Users/braddelong/opt/anaconda3/lib/python3.7/site-packages/ipykernel_launcher
.py:6: SettingWithCopyWarning:
A value is trying to be set on a copy of a slice from a DataFrame.
Try using .loc[row_indexer,col_indexer] = value instead

See the caveats in the documentation: https://pandas.pydata.org/pandas-docs/st
able/user_guide/indexing.html#returning-a-view-versus-a-copy
```

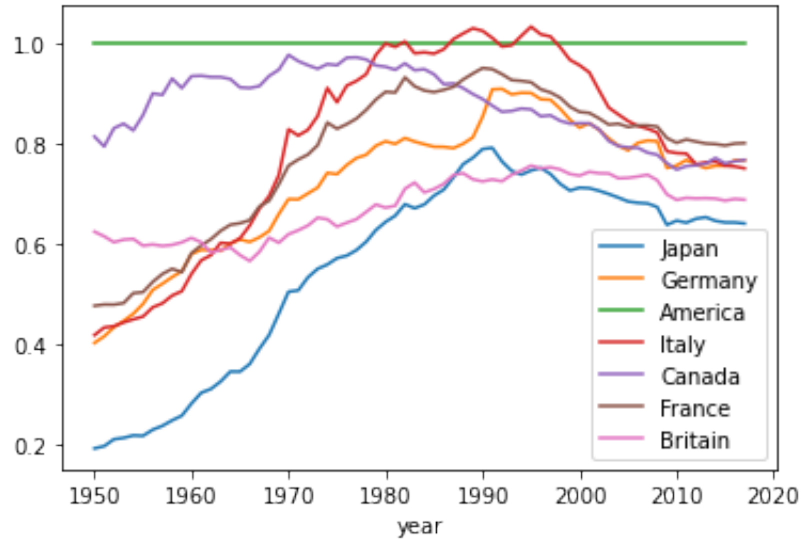
In [12]:

```
# plot g-7 natural logarithm levels of national
# income per worker

g7_df = pd.DataFrame()
g7_df['Japan'] = np.log(Japan_pwg_ser['rgdpw'])
g7_df['Germany'] = np.log(Germany_pwg_ser['rgdpw'])
g7_df['America'] = np.log(America_pwg_ser['rgdpw'])
g7_df['Italy'] = np.log(Italy_pwg_ser['rgdpw'])
g7_df['Canada'] = np.log(Canada_pwg_ser['rgdpw'])
g7_df['France'] = np.log(France_pwg_ser['rgdpw'])
g7_df['Britain'] = np.log(Britain_pwg_ser['rgdpw'])
g7_df.plot()
plt.show()
```



In [13]:

```
# calculate and plot g-7 levels of national income
# per worker as a proportion of american

g7_ratio_df = pd.DataFrame()
g7_ratio_df['Japan'] = Japan_pwg_ser['rgdpw']/America_pwg_ser['rgdpw']
g7_ratio_df['Germany'] = Germany_pwg_ser['rgdpw']/America_pwg_ser['rgdpw']
g7_ratio_df['America'] = America_pwg_ser['rgdpw']/America_pwg_ser['rgdpw']
g7_ratio_df['Italy'] = Italy_pwg_ser['rgdpw']/America_pwg_ser['rgdpw']
g7_ratio_df['Canada'] = Canada_pwg_ser['rgdpw']/America_pwg_ser['rgdpw']
g7_ratio_df['France'] = France_pwg_ser['rgdpw']/America_pwg_ser['rgdpw']
g7_ratio_df['Britain'] = Britain_pwg_ser['rgdpw']/America_pwg_ser['rgdpw']
g7_ratio_df.plot()
plt.show()
```



The idea—derived from the Solow model—that economies pushed off and below their steady-state balanced-growth paths by the destruction and chaos of war thereafter experience a period of supergrowth that ebbs as they approach their steady-state balanced-growth paths from below story holds for the other members of the G-7 group of large advanced industrial economies as well. In increasing order of the magnitude of their shortfall vis-a-vis the U.S. and the speed of recovery supergrowth, we have: France, Italy, Germany, and Japan. The three economies that escaped wartime chaos and destruction—the U.S., Britain, and Canada—do not exhibit supergrowth until catchup to their steady-state balanced-growth paths.

There is a lot more going on in the post-WWII history of the G-7 economies than just catchup to their steady-state balanced-growth paths after the destruction of World War II: Why do the other economies lose ground vis-a-vis the U.S. after 1990? Why does the U.S. exhibit a small speedup, slowdown, speedup, and then renewed slowdown again? What is it with Britain's steady-state balanced-growth path having so much lower productivity than the other Europeans? Why is Japan the most different from its G-7 partners? And what is it with Italy's attaining U.S. worker productivity levels in 1980, and then its post-2000 relative collapse? (The post-2000 collapse in Italian growth is real; the estimate that it was as productive as the U.S. from 1980-2000 is a data construction error.)

4.2. Analyzing Jumps in Parameter Values

What if one or more of the parameters in the Solow growth model were to suddenly and substantially shift? What if the labor-force growth rate were to rise, or the rate of technological progress to fall?

One principal use of the Solow growth model is to analyze questions like these: how changes in the economic environment and in economic policy will affect an economy's long-run levels and growth path of output per worker Y/L.

Let's consider, as examples, several such shifts: an increase in the growth rate of the labor force n, a change in the economy's saving-investment rate s, and a change in the growth rate of labor efficiency g. All of these will have effects on the balanced- growth path level of output per worker. But only one—the change in the growth rate of labor efficiency—will permanently affect the growth rate of the economy.

We will assume that the economy starts on its balanced growth path—the old balanced growth path, the pre-shift balanced growth path. Then we will have one (or more) of the parameters—the savings-investment rate s, the labor force growth rate n, the labor efficiency growth rate g—jump discontinuously, and then remain at its new level indefinitely. The jump will shift the balanced growth path. But the level of output per worker will not immediately jump. Instead, the economy's variables will then, starting from their old balanced growth path values, begin to converge to the new balanced growth path—and converge in the standard way.

Remind yourselves of the key equations for understanding the model:

The level of output per worker is:

> (4.1) $\frac{Y}{L} = \left(\frac{K}{Y} \right)^{\theta}E$

The balanced-growth path level of output per worker is:

> (4.2) $\left(\frac{Y}{L} \right)^* = \left(\frac{s}{n+g+\delta} \right)^{\theta}E$

The speed of convergence of the capital-output ratio to its balanced-growth path value is:

> (4.3) $\frac{d(K/Y)}{dt} = -(1-\alpha)(n+g+\delta) \left[\frac{K}{Y} - \frac{s}{(n+g+\delta)} \right]$

where, you recall $\theta = \alpha/(1-\alpha)$ and $\alpha = \theta/(1+\theta)$

4.2.1. A Shift in the Labor-Force Growth Rate

Real-world economies exhibit profound shifts in labor-force growth. The average woman in India today has only half the number of children that the average woman in India had only half a century ago. The U.S. labor force in the early eighteenth century grew at nearly 3 percent per year, doubling every 24 years. Today the U.S. labor force grows at 1 percent per year. Changes in the level of prosperity, changes in the freedom of migration, changes in the status of women that open up new categories of jobs to them (Supreme Court Justice Sandra Day O'Connor could not get a private-sector legal job in San Francisco when she graduated from Stanford Law School even with her amazingly high class rank), changes in the average age of marriage or the availability of birth control that change fertility—all of these have powerful effects on economies' rates of labor-force growth.

What effects do such changes have on output per worker Y/L—on our mea sure of material prosperity? The faster the growth rate of the labor force n, the lower will be the economy's balanced-growth capital-output ratio $s/(n + g - \delta)$. Why? Because each new worker who joins the labor force must be equipped with enough capital to be productive and to, on average, match the productivity of his or her peers. The faster the rate of growth of the labor force, the larger the share of current investment that must go to equip new members of the labor force with the capital they need to be productive. Thus the lower will be the amount of invest ment that can be devoted to building up the average ratio of capital to output.

A sudden and permanent increase in the rate of growth of the labor force will lower the level of output per worker on the balanced-growth path. How large will the long-run change in the level of output be, relative to what would have happened had labor-force growth not increased? It is straightforward to calculate if we know the other parameter values, as is shown in the example below.

4.2.1.1. An Example: An Increase in the Labor Force Growth Rate: Consider an economy in which the parameter α is 1/2, the efficiency of labor growth rate g is 1.5 percent per year, the depreciation rate δ is 3.5 percent per year, and the saving rate s is 21 percent. Suppose that the labor-force growth rate suddenly and permanently increases from 1 to 2 percent per year.

Before the increase in the labor-force growth rate, in the initial steady-state, the balanced-growth equilibrium capital-output ratio was:

> (4.4) $ \left(\frac{K_{in}}{Y_{in}} \right)^* = \frac{s_{in}}

> {(n_{in}+g_{in}+δ_{in})} = \frac{0.21}{(0.01 + 0.015 + 0.035)} = \frac{0.21}{0.06} = 3.5 $

(with subscripts "in" for "initial).

After the increase in the labor-force growth rate, in the alternative steady state, the new balanced-growth equilibrium capital-output ratio will be:

> (4.5) $ \left(\frac{K_{alt}}{Y_{alt}} \right)^* = \frac{s_{alt}} {(n_{alt}+g_{alt}+δ_{alt})} = \frac{0.21}{(0.02 + 0.015 + 0.035)} = \frac{0.21} {0.07} = 3 $

(with subscripts "alt" for "alternative").

Before the increase in labor-force growth, the level of output per worker along the balanced-growth path was equal to:

> (4.6) $ \left(\frac{Y_{t, in}}{L_{t, in}} \right)^* = \left(\frac{s_{in}} {(n_{in}+g_{in}+δ_{in})} \right)^{α/(1–α)} E_{t, in} = 3.5 E_{t, in} $

After the increase in labor-force growth, the level of output per worker along the balanced-growth path will be equal to:

> (4.7) $ \left(\frac{Y_{t, alt}}{L_{t, alt}} \right)^* = \left(\frac{s_{alt}} {(n_{alt}+g_{alt}+δ_{alt})} \right)^{α/(1–α)} E_{t, alt} = 3 E_{t, alt} $

This fall in the balanced-growth path level of output per worker means that in the long run—after the economy has converged to its new balanced-growth path—one-seventh of its per worker economic prosperity has been lost because of the increase in the rate of labor-force growth.

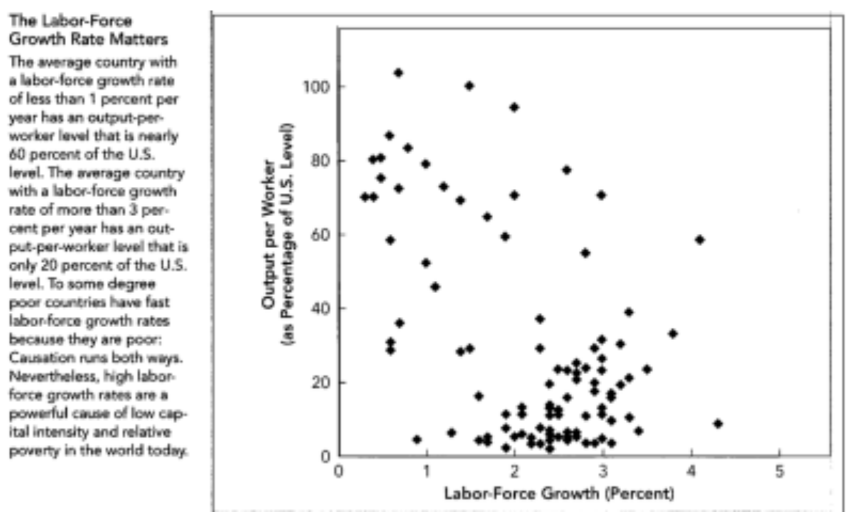
In the short run of a year or two, however, such an increase in the labor-force growth rate has little effect on output per worker. In the months and years after labor-force growth increases, the increased rate of labor-force growth has had no time to affect the economy's capital-output ratio. But over decades and generations, the capital-output ratio will fall as it converges to its new balanced-growth equilibrium level.

A sudden and permanent change in the rate of growth of the labor force will immediately and substantially change the level of output per worker along the economy's balanced-growth path: It will shift the balanced-growth path for output per worker up (if labor-force growth falls) or down (if labor-force growth rises). But there is no corresponding immediate jump in the actual level of output per worker in the economy. Output per worker doesn't immediately jump—it is just that the shift in the balanced-growth path means that the economy is no longer in its Solow growth model long-run equilibrium.

4.2.1.2. Empirics: The Labor-Force Growth Rate Matters: The average country with a labor-force growth rate of less than 1 percent per year has an output-per-worker level that is nearly 60 percent of the U.S. level. The average country with a labor-force growth rate of more than 3 percent per year has an output-per-worker level that is only 20 percent of the U.S. level.

To some degree poor countries have fast labor-force growth rates because they are poor: Causation runs both ways. Nevertheless, high labor-force growth rates are a powerful cause of low capital intensity and relative poverty in the world today.

Figure 4.2.1. The Labor Force Growth Rate Matters: Output per Worker and Labor Force Growth



How important is all this in the real world? Does a high rate of labor-force growth play a role in making countries relatively poor not just in economists' models but in reality? It turns out that it is important. Of the 22 countries in the world in 2000 with output-per-worker levels at least half of the U.S. level, 18 had labor-force growth rates of less than 2 percent per year, and 12 had labor-force growth rates of less than 1 percent per year. The additional investment requirements imposed by rapid labor-force growth are a powerful reducer of capital intensity and a powerful obstacle to rapid economic growth.

It takes time, decades and generations, for the economy to converge to its new balanced-growth path equilibrium, and thus for the shift in labor-force growth to affect average prosperity and living standards. But the time needed is reason for governments that value their countries' long-run prosperity to take steps now (or even sooner) to start assisting the demographic transition to low levels of population growth. Female education, social changes that provide women with more opportunities than being a housewife, inexpensive birth control—all these pay large long-run dividends as far as national prosperity levels are concerned.

U.S. President John F Kennedy used to tell a story of a retired French general, Marshal Lyautey, "who once asked his gardener to plant a tree. The gardener objected that the tree was slow-growing and would not reach maturity for a hundred years. The Marshal replied, 'In that case, there is no time to lose, plant it this afternoon.'"

4.2.2. The Algebra of a Higher Labor Force Growth Rate

But rather than calculating example by example, set of parameter values by set of parameter values, we can gain some insight by resorting to algebra, and consider in generality the effect on capital-output ratios and output per worker levels of an increase Δn in the labor force growth rate, following an old math convention of using "Δ" to stand for a sudden and discrete change.

Assume the economy has its Solow growth parameters, and its initial balanced-growth path capital-output ratio

> (4.8) $\left(\frac{K_{in}}{Y_{in}} \right)^* = \frac{s_{in}}{(n_{in}+g_{in}+\delta_{in})}$

with "in" standing for "initial".

And now let us consider an alternative scenario, with "alt" standing for "alternative", in which things had been different for a long time:

> (4.9) $\left(\frac{K_{alt}}{Y_{alt}} \right)^* = \frac{s_{alt}}{(n_{alt}+g_{alt}+\delta_{alt})}$

For the g and δ parameters, their initial values are their alternative values. And for the labor force growth rate:

> (4.10) $n_{alt} = n_{in} + \Delta n$

So we can then rewrite:

> (4.11) $\left(\frac{K_{alt}}{Y_{alt}} \right)^* = \frac{s_{in}}{(n_{in}+g_{in}+\delta_{in})} \frac{(n_{in}+g_{in}+\delta_{in})}{(n_{in} + \Delta n +g_{in}+\delta_{in})} = \frac{s_{in}}{(n_{in}+g_{in}+\delta_{in})} \left[\frac{1}{1+\frac{\Delta n}{(n_{in}+g_{in}+\delta_{in})}} \right]$

The first term on the right hand side is just the initial capital-output ratio, and we know that $1/(1+x)$ is approximately $1–x$ for small values of x, so we can make an approximation:

> (4.12) $\left(\frac{K_{alt}}{Y_{alt}} \right)^* = \left(\frac{K_{in}}{Y_{in}} \right)^* \left[1 - \frac{\Delta n}{(n_{in}+g_{in}+\delta_{in})} \right]$

Take the proportional change in the denominator $(n+g+\delta)$ of the expression for the balanced-growth capital-output ratio. Multiply that proportional change by the initial balanced-growth capital-output ratio. That is the differential we are looking for.

And by amplifying or damping that change by raising to the $\alpha/(1–\alpha)$ power, we get the differential for output per worker.

4.2.3. A Shift in the Growth Rate of the Efficiency of Labor

4.2.3.1. Efficiency of Labor the Master Key to Long Run Growth: By far the most important impact on an economy's balanced-growth path values of output per worker, however, is from shifts in the growth rate of the efhciency of labor g. We already know that growth in the efhciency of labor is absolutely essential for sustained growth in output per worker and that changes in g are the only things that cause permanent changes in growth rates that cumulate indehnitely.

Recall yet one more time the capital-output ratio form of the production function:

> (4.13) $\frac{Y}{L} = \left(\frac{K}{Y} \right)^{\theta} E$

Consider what this tells us. We know that a Solow growth model economy converges to a balanced-growth path. We know that the capital-output ratio K/Y is constant along the balanced-growth path. We know that the returns-to-investment parameter α is constant. And so the balanced-growth path level of output per worker Y/L grows only if, and grows only as fast as, the efficiency of labor E grows.

4.2.3.2. Efficiency of Labor Growth and the Capital-Output Ratio: Yet when we took a look at the math of an economy on its balanced growth path:

> (4.14) $\left(\frac{Y}{L} \right)^* = \left(\frac{s}{n+g+\delta} \right)^{\theta} E$

we also see that an increase in g raises the denominator of the first term on the right hand side—and so pushes the balanced-growth capital output ratio down. That implies that the balanced-growth path level of output per worker associated with any level of the efficiency of labor down as well.

It is indeed the case that—just as in the case of an increased labor force growth rate n—an increased efficiency-of-labor growth rate g reduces the economy's balanced-growth capital-output ratio $s/(n + g - \delta)$. Why? Because, analogously with an increase in the labor force, increases in the efficiency of labor allow each worker to do the work of more, but they need the machines and buildings to do them. The faster the rate of growth of the efficiency of la or, the larger the share of current investment that must go to keep up with the rising efficiency of old members of the labor force and supply them with the capital they need to be productive. Thus the lower will be the amount of investment that can be devoted to building up or maintaining the average ratio of capital to output.

4.2.4. The Algebra of Shifting the Efficiency-of-Labor Growth Rate

The arithmetic and algebra are, for the beginning and the middle, the same as they were for an increase in the rate of labor force growth:

Assume the economy has its Solow growth parameters, and its initial balanced-growth path capital-output ratio:

> (4.15) $ \left(\frac{K_{in}}{Y_{in}} \right)^* = \frac{s}{(n_+g_{in}+δ)} $

(with "in" standing for "initial"). Also consider an alternative scenario, with "alt" standing for "alternative", in which things had been different for a long time, with a higher efficiency-of-labor growth rate g+Δg since some time t=0 now far in the past:

> (4.16) $ \left(\frac{K_{alt}}{Y_{alt}} \right)^* = \frac{s}{(n+g+Δg+δ)} $

We can rewrite this as:

> (4.17) $ \left(\frac{K_{alt}}{Y_{alt}} \right)^* = $ $ \frac{s}{(n+g_{in}+δ)} \frac{(n+g_{in}+δ)}{(n +g_{in}+Δg+δ)} = $ $ \frac{s}{(n+g_{in}+δ)} \left[\frac{1}{1+\frac{Δg}{(n+g_{in}+δ)}} \right] $

Once again, the first term on the right hand side is just the initial capital-output ratio, and we know that 1/1+x is approximately 1–x for small values of x, so we can make an approximation:

> (4.18) $ \left(\frac{K_{alt}}{Y_{alt}} \right)^* = \left(\frac{K_{in}}{Y_{in}} \right)^* \left[1 - \frac{Δg}{(n+g_{in}+δ)} \right] $

Take the proportional change in the denominator of the expression for the balanced-growth capital output ratio. Multiply that proportional change by the initial balanced-growth capital-output ratio. That is the differential in the balanced-growth capital-output ratio that we are looking for.

But how do we translate that into a differential for output per worker? In the case of an increase in the labor force growth rate, it was simply by amplifying or damping the change in the balanced-growth capital-output ratio by raising it to the power $ \theta = (α/(1–α))$ in order to get the differential for output per worker. We could do that because the efficiency-of-labor at every time t $ E_t $ was the same in both the initial and the alternative scenarios.

That is not the case here.

Here, the efficiency of labor was the same in the initial and alternative scenarios back at time 0, now long ago. Since then E has been growing at its rate g in the initial scenario, and at its rate g+Δg in the alternative scenario, and so the time subscripts will be important. Thus for the alternative scenario:

> (4.19) $ \left(\frac{Y_{t, alt}}{L_{t, alt}} \right)^* $ $ \left(\frac{s}{(n+g_{in} + \Delta g + \delta)} \right) ^{\theta)}(1+(g_{in}+ \Delta g))^t E_0 $

while for the initial scenario:

> (4.20) $ \left(\frac{Y_{t, ini}}{L_{t, ini}} \right)^* $ $ \left(\frac{s}{(n+g_{in} + \delta)} \right) ^{\theta} $ $ (1+g_{in})^t E_0 $

Now divide to get the ratio of output per worker under the alternative and initial scenarios:

>(4.21) $ \left(\frac{Y*{t, alt}/L*{t, alt}}{Y*{t, ini}/L*{t, ini}} \right)^*

\left(\frac{n+g*{in}+|delta}{(n+g*{in}+\Delta g + \delta)} \right)^{\theta} (1+ \Delta g)^t $

Thus we see that in the long run, as the second term on the right hand side compounds as t grows, balanced-growth path output per worker under the alternative becomes first larger and then immensely larger than output per worker under the initial scenario. Yes, the balanced-growth path capital-output ratio is lower. But the efficiency of labor at any time t is higher, and then vastly higher if Δgt has had a chance to mount up and thus $(1+\Delta g)^t$ has had a chance to compound.

Yes, a positive in the efficiency of labor growth g does reduce the economy's balanced-growth path capital-output ratio. But these effects are overwhelmed by the more direct effect ofa larger g on output per worker. It is the economy with a high rate of efficiency of labor force growth g that becomes by far the richest over time. This is our most important conclusion. In the very longest run, the growth rate of the standard of living—of output per worker—can change if and only if the growth rate of labor efficiency changes. Other factors —a higher saving-investment rate, lower labor-force growth rate, or lower depreciation rate—can and down. But their effects are short and medium effects: They do not permanently change the growth rate of output per worker, because after the economy has converged to its balanced growth path the only determinant of the growth rate of output per worker is the growth rate of labor efficiency: both are equal to g.

Thus, if we are to increase the rate of growth of the standard of living permanently, we must pursue policies that increase the rate at which labor efficiency grows—policies that enhance technological and organizational progress, improve worker skills, and add to worker education.

4.2.4.1. An Example: Shifting the Growth Rate of the Efficiency of Labor: What are the effects of an increase in the rate of growth of the efficiency of labor? Let's work through an example:

Suppose we have, at some moment we will label time 0, t=0, an economy on its balanced growth path with a savings rate s of 20% per year, a labor force growth rate n or 1% per year, a depreciation rate δ of 3% per year, an efficiency-of-labor growth rate g of 1% per year, and a production function curvature parameter α of 1/2 and thus a $ \theta = 1 $. Suppose that at that moment t=0 the labor force $ L_0 $ is 150 million, and the efficiency of labor $ E_0 $ is 35000.

It is straightforward to calculate the economy at that time 0. Because the economy is on its balanced growth path, its capital-output ratio K/Y is equal to the balanced-growth path capital-output ratio (K/Y)*:

> (4.22) $\frac{K_0}{Y_0} = \left(\frac{K}{Y} \right)^* = \frac{s}{n+g+\delta} = \frac{0.2}{0.01 + 0.01 + 0.03} = 4$

And with an efficiency of labor value $E_0=70000$, output per worker at time zero is:

> (4.23) $\frac{Y_0}{L_0} = \left(\frac{K_0}{Y_0} \right)^{\theta} = 4^1 (35000) = 140000$

Since the economy is on its balanced growth path, the rate of growth of output per worker is equal to the rate of growth of efficiency per worker. Since the efficiency of labor is growing at 1% per year, we can calculate what output per worker would be at any future time t should the parameters describing the economy remain the same:

> (4.24) $\left(\frac{Y_t}{L_t}\right)_ini = (140000)e^{0.01t}$

where the subscript "ini" tells us that this value belongs to an economy that retains its initial parameter values into the future. Thus 69 years into the future, at t=69:

> (4.25) $\left(\frac{Y_69}{L_69}\right)_ini = (140000)e^{(0.01)(69)} = (140000)(1.9937) = 279120$

Now let us consider an alternative scenario in which output per worker is the same in year 0 but in which the efficiency of labor growth rate g is a higher rate. Suppose $g_{alt} = g_{ini} + \Delta g$, with the subscript "alt" reminding us that this parameter or variable belongs to the alternative scenario just as "ini" reminds us of the initial scenario or set of values. How do we forecast the growth of the economy in an alternative scenario—in this case, in an alternative scenario in which $\Delta g=0.02$

The first thing to do is to calculate the balanced growth path steady-state capital-output ratio in this alternative scenario. Thus we calculate:

> (4.26) $\left(\frac{K}{Y} \right)_{alt}^* = \frac{s}{n + g_{ini} + \Delta g + \delta} = \frac{0.20}{0.01 + 0.01 + 0.02 + 0. 03} = \frac{0.20}{0.07} = 2.857$

The steady-state balanced growth path capital-output ratio is much lower in the alternative scenario than it was in the initial scenario: 2.857 rather than 4. The capital-output ratio, of course, does not drop instantly to its new steady-state value. It takes time for the transition to occur.

While the transition is occurring, the efficiency of labor in the alternative scenario is growing at not 1% but 3% per year. We can thus calculate the alternative scenario balanced growth path value of output per worker as:

> (4.27) $ \left(\frac{Y_t}{L_t}\right)_{alt}^* = \left(\frac{K}{Y} \right)_{alt}^{*\theta} E_0 e^{(0.01+0.02)t} $

And in the 69th year this will be:

> (4.28) $ \left(\frac{Y_69}{L_69}\right)_{alt}^* = (2.857)(35000) e^{(0.03)(69)} = 792443 $

How good would this balanced growth path value be as an estimate of the actual behavior of the economy? We know that a Solow growth model economy closes a fraction $(1-\alpha)(n+g+\delta)$ of the gap between its current position and its steady-state balanced growth path capital-output ratio each period. For our parameter values $(1-\alpha)(n+g+\delta)=0.035$. That gives us about 20 years as the period needed to converge halfway to the balanced growth path. 69 years is thus about 3.5 such halvings of the gap—meaning that the economy will close 9/10 of the way. Thus assuming the economy is on its alternative scenario balanced growth path in year 69 is not a bad assumption.

But if we want to calculate the estimate exactly? 820752.

The takeaways are three:

For these parameter values, 69 years are definitely long enough for you to make the assumption that the economy has converged to its Solow model balanced growth path. One year no. Ten years no. Sixty-nine years, yes.

Shifts in the growth rate g of the efficiency of labor do, over time, deliver enormous differentials in output per worker across scenarios.

The higher efficiency of labor economy is, in a sense, a less capital intensive economy: only 2.959 years' worth of current production is committed to and tied up in the economy's capital stock in the alternative scenario, while 4 years' worth was tied up in the initial scenario. But the reduction in output per worker generated by a lower capital-output ratio is absolutely swamped by the faster growth of the efficiency of labor, and thus the much greater value of the efficiency of labor in the alternative scenario comes the 69th year.

4.2.5. Shifts in the Saving Rate s

4.2.5.1. The Most Common Policy and Environment Shock: Shifts in labor force growth rates do happen: changes in immigration policy, the coming of cheap and easy contraception (or, earlier, widespread female literacy), or increased prosperity and expected prosperity that trigger "baby booms" can all have powerful and persistent effects on labor force growth down the pike. Shifts in the growth of labor efficiency growth happen as well: economic policy disasters and triumphs, countless forecasted "new economies" and "secular stagnations", and the huge economic shocks that were the first and second Industrial Revolutions—the latter inaugurating that global era of previously unimagined increasing prosperity we call modern economic growth—push an economy's labor efficiency growth rate g up or down and keep it there.

Nevertheless, the most frequent sources of shifts in the parameters of the Solow growth model are shifts in the economy's saving-investment rate. The rise of politicians eager to promise goodies—whether new spending programs or tax cuts — to voters induces large government budget deficits, which can be a persistent drag on an economy's saving rate and its rate of capital accumulation. Foreigners become alternately overoptimistic and overpessimistic about the value of investing in our country, and so either foreign saving adds to or foreign capital flight reduces our own saving- investment rate. Changes in households' fears of future economic disaster, in households' access to credit, or in any of numerous other factors change the share of household income that is saved and invested. Changes in government tax policy may push after-tax returns up enough to call forth additional savings, or down enough to make savings seem next to pointless. Plus rational or irrational changes in optimism or pessimism—what John Maynard Keynes labelled the "animal spirits" of individual entrepreneurs, individual financiers, or bureaucratic committees in firms or banks or funds all can and do push an economy's savings-investment rate up and down.

4.2.5.2. Analyzing a Shift in the Saving Rate s: What effects do changes in saving rates have on the balanced-growth path levels of Y/L?

The higher the share of national product devoted to saving and gross investment—the higher is s—the higher will be the economy's balanced-growth capital-output ratio $s/(n + g + \delta)$. Why? Because more investment increases the amount of new capital that can be devoted to building up the average ratio of cap ital to output. Double the share of national product spent on gross investment, and you will find that you have doubled the economy's capital intensity, or its average ratio of capital to output.

As before, the equilibrium will be that point at which the economy's savings effort and its investment requirements are in balance so that the capital stock and output grow at the same rate, and so the capital-output ratio is constant. The savings effort of society is simply sY, the amount of total output devoted to saving and investment. The investment requirements are the amount of new capital needed to replace depreciated and worn-out machines and buildings, plus the amount needed to equip new workers who increase the labor force, plus the amount needed to keep the stock of tools and machines at the disposal of more efficient workers increasing at the same rate as the efficiency of their labor.

> (4.29) $sY = (n+g+\delta)K $

And so an increase in the savings rate s will, holding output Y constant, call forth a proportional increase in the capital stock at which savings effort and investment requirements are in balance: increase the saving-investment rate, and you double the balanced-growth path capital-output ratio:

> (4.30) $ \frac{K}{Y}_{ini}^* = \frac{s_{ini}}{n+g+\delta} $
>
> (4.31) $ \frac{K}{Y}_{alt}^* = \frac{s_{ini}+\Delta s}{n+g+\delta} $ KY∗alt=s+Δsn+g+δ
>
> (4.32) $ \frac{K}{Y}*{alt}^* - |frac{K}{Y}*{ini}^* = \frac{Δs}{n+g+δ}

with, once again, balanced growth path output per worker amplified or damped by the dependence of output per worker on the capital-output ratio:

> (4.33) $ \frac{Y}{L}^* = \frac{K}{Y}^* E $

4.2.5.3. Analyzing a Shift in the Saving-Investment Rate: An Example: To see how an increase in the economy's saving rate s changes the balanced-growth path for output per worker, consider an economy in which the parameter $\theta = 2$ (and $α = 2/3$}, the rate of labor-force growth n is 1 percent per year, the rate of labor efficiency growth g is 1.5 percent per year, and the depreciation rate δ is 3.5 percent per year.

Suppose that the saving rate s, which had been 18 percent, suddenly and permanently jumped to 24 percent of output.

Before the increase in the saving rate, when s was 18 percent, the balanced-growth equilibrium capital-output ratio was:

> (4.34) $\frac{K}{Y}_{ini}^* = \frac{s_{ini}}{n+g+δ} = \frac{0.18}{0.06} = 3$

After the increase in the saving rate, the new balanced-growth equilibrium capital- output ratio will be:

> (4.35) $\frac{K}{Y}_{alt}^* = \frac{s_{ini} + {\Delta}s}{n+g+δ} = \frac{0.24}{0.06} = 4$

We can see, with a value of $\theta = 2$, that balanced-growth path output per worker after the jump in the saving rate is higher by a factor of $(4/3)^2 = 16/9$, or fully 78 percent higher.

Just after the increase in saving has taken place, the economy is still on its old, balanced-growth path. But as decades and generations pass the economy converges to its new balanced-growth path, where output per worker is not 9 but 16 times the efficiency of labor. The jump in capital intensity makes an enormous differ ence for the economy's relative prosperity.

Note that this example has been constructed to make the effects of capital intensity on relative prosperity large: The high value for θ means that differences in capital intensity have large and powerful effects on output-per-worker levels.

But even here, the shift in saving and investment does not permanently raise the economy's growth rate. After the economy has settled onto its new balanced-growth path, the growth rate of output per worker returns to the same 1.5 percent per year that is g, the growth rate of the effciency of labor.

5. Pre-Industrial Economies

5.1. The Solow-Malthus Model

Two major changes to the Solow model are needed in order to make it useful for making sense of the pre-industrial past. The first is to make labor efficiency depend on the scarcity of resources. The second is to make the rate of population and labor force growth depend on the economy's prosperity. We call the changed model that results from these changes the "Solow-Malthus" model.

5.1.1. Population, Resource Scarcity, and the Efficiency of Labor

Thus we first need to make efficiency of labor a function of available natural resources per worker. We do this by setting the rate of efficiency of labor growth g equal to the difference between the rate h at which economically useful ideas are generated, and the rate of population and labor force growth n divided by an effect-of-resource scarcity parameter γ, because a higher population makes natural resources per capita increasingly scarce. Therefore:

> (5.1) $\frac{dE/dt}{E} = \frac{d\ln(E)}{dt} = g = h - \frac{n}{\gamma}$

Thus:

> (5.2) $\frac{d}{dt} \left(\frac{Y}{L}\right)^* = 0$ whenever $h - \frac{n}{\gamma} = 0$
>
> (5.3) $n^{*mal} = \gamma h$

is the population growth rate at which $\frac{d}{dt} \left(\frac{Y}{L}\right)^* = 0$

When population is growing at the rate n^{*mal}, the efficiency of labor—and thus the steady-state growth-path level of production per worker Y/L—is constant. This captures the idea that even though human technology was advancing over the ten millennia before the Industrial Revolution, living standards were not because the potential benefits from technology and organization for productivity were offset by the productivity-diminishing effects of smaller farm sizes and more costly other natural resources to feed and provide for the growing population.

5.1.2. Determinants of Population and Labor Force Growth

We also need to make the rate of growth of the population and labor force depend on the level of prosperity $y = Y/L$; on the "subsistence" standard of living for necessities y^{sub}; and also on the fraction $1/\phi$ of production that is devoted to necessities, not conveniences and luxuries, and thus enters into reproductive and survival fitness. The higher the resources devoted to fueling reproductive and survival fitness, the faster will be the rate of population growth:

> (5.4) $\frac{dL/dt}{L} = \frac{d\ln(L)}{dt} = n = \beta \left(\frac{y}{\phi y^{sub}}-1 \right)$

Then for population to be growing at its Malthusian rate:

> (5.5) $\gamma h = \beta \left(\frac{1}{\phi}\right) \left(\frac{y}{y^{sub}}- \phi \right)$
>
> (5.6) $y^{*mal} = \phi y^{sub} \left(1 + \frac{ n^{*mal}}{\beta}\right) = \phi y^{sub} \left(1 + \frac{ \gamma h}{\beta}\right)$

Note that (5.4) only holds for poor populations. When populations grow rich and literate enough—and when women acquire enough social power—human societies undergo the demographic transition: women limit their pregnancies to the number of children they desire, confident that they will pretty much all survive to outlive them. Beyond a certain income level, equation (5.4) no longer holds. But it did hold up until well after the start of the Industrial Revolution.

5.1.3. The Full Malthusian Equilibrium

Then with (5.1) and (5.4) added to our Solow growth model to turn it into the Solow-Malthus model, we can calculate the full Malthusian equilibrium for a pre-industrial economy. We can determine the log level E of the efficiency of labor:

> (5.7) $\ln(E) = \ln(H) - \frac{\ln(L)}{\gamma}$

Then since:

> (5.8) $y^{*mal} = \left(\frac{s}{\gamma h +\delta} \right)^\theta E$
>
> (5.9) $\ln(\phi) + \ln\left(y^{sub} \right) + \ln\left(1 + \frac{\gamma h}{\beta} \right) = \theta \ln(s) - \theta \ln(\gamma h +\delta) + \ln(E)$

The population and labor force in the full Malthusian equilibrium will be:

> (5.10) $\ln(L_t^{*mal}) = \gamma \left[\ln(H_t) - \ln(y^{sub}) \right] + \gamma \theta \left(\ln(s) - \ln(\delta) \right) - \gamma \ln(\phi) + \left(- \gamma \theta \ln(1 + \gamma h/\delta)$
>
> - $\gamma \ln\left(1 + \frac{\gamma h}{\beta} \right) \right)$

Or:

> (5.11) $L_t^{*mal} = \left[\left(\frac{H_t}{y^{sub}} \right) \left(\frac{s}{\delta} \right)^\theta \left(\frac{1}{\phi} \right) \left[\frac{1}{(1+\gamma h/\delta)^\theta} \frac{1}{(1+\gamma h/\beta)} \right] \right]^\gamma$

5.2. Interpretation and Analysis

5.2.1. Understanding the Malthusian Equilibrium

Thus to analyze the pre-industrial Malthusian economy, at least in its equilibrium configuration:

- Start with the rate h at which new economically-useful ideas are being generated and with the responsiveness β of population growth to increased prosperity.
- From those derive the Malthusian rate of population growth: $n^{*mal} = \gamma h$
- Then the Malthusian standard of living is: $ y^{*mal} = \phi y^{sub} \left(1 + \frac{

\gamma h}{\beta}\right) $

- And the Malthusian population is:$ L_t^{*mal} = \left[\left(\frac{H_t}{y^{sub}} \right) \left(\frac{s}{\delta} \right)^\theta \left(\frac{1}{\phi} \right) \left[\frac{1}{(1+\gamma h/\delta)^\theta} \frac{1}{(1+\gamma h/\beta)} \right] \right]^\gamma $

Thus at any date t, the Malthusian-equilibrium population is:

1. the current level $ H_t $ of the valuable ideas stock divided by the (sociologically determined, by, for example western European delayed female marriage patterns, or lineage-family control of reproduction by clan heads) Malthusian-subsistence income level $ y^{sub} $ consistent with a stable population on average, times

2. the ratio between the savings-investment rate $ s $ and the depreciation rate $ \delta $, raised to the parameter $ \theta $ which governs how much an increase in the capital-output ratio raises income—with a higher $ \theta $ the rule of law, imperial peace, and a culture of thrift and invetment matter more, and can generate "efflorescences"—times

3. one over the conveniences-and-luxuries parameter $ \phi $—it drives a wedge between prosperity and subsistence as spending is diverted categories that do not affect reproduction, such as middle-class luxuries, upper-class luxuries, but also the "luxury" of having an upper class, and the additional conveniences of living in cities and having trade networks that can spread plagues—times

4. two nuisance terms near zero, which depend on how much the level of population must fall below the true subsistence level at which population growth averages zero to generate the (small) average population growth rate that produces growing resource scarcity that offsets the (small) rate of growth of useful ideas. all this

5. all raised to the power $ \gamma $ that describes how much more important ideas are than resources in generating human income and production.

(1) is the level of the stock of *useful ideas* relative to the requirements for subsistence. (2) depends on how the rule of law and the rewards to thrift and entrepeneurship drive savings and investment, and thus the divisio of labor. (3) depends on how society diverts itself from nutrition and related activities that aim at boosting reproductive fitness and, instead, devotes itself to conveniences and luxuries—including the "luxury" of having an upper class, and all the conveniences of urban life. (4) are constant, and are small. And (5) governs how productive potential is translated into resource scarcity-generating population under Malthusian conditions.

And recall the full Malthusian equilibrium standard of living:

> (5.12) $y^{*mal} = \phi y^{sub} \left(1 + \frac{ \gamma h}{\beta}\right)$

The level of income is:

1. The luxuries-and-conveniences parameter ϕ, times
2. The level of subsistence y^{sub}, times
3. The (small and constant) nuisance parameter $1 + \gamma h/\beta$ needed to generate average population growth $n^{*mal} = \gamma h$.

5.2.2 Implications for Understanding Pre-Industrial Civilizations

Production per worker and thus prosperity are thus primarily determined by (a) true subsistence, (b) the wedge between prosperity and reproductive fitness produced by spending on conveniences and luxuries that do not impact reproductive success, plus a minor contribution by (c) the wedge above subsistence needed to generate population growth consonant with the advance of knowledge and population pressure's generation of resource scarcity.

With this model, we can investigate broader questions about the Malthusian Economy—or at least about the Malthusian model, with respect to its equilibrium:

- How much does the system compromise productivity, both static and dynamic, to generate inequality?
- How would one rise in this world—or avoid losing status relative to your ancestors?
- How does the system react to shocks?:
    - like a sudden major plague—like the Antonine plague of 165, the St. Cyprian plague of 249, or the Justinian plague of 542—that suddenly and discontinuously pushes population down sharply...
    - like the rise of a civilization that carries with it norms of property and law and commerce, and thus a rise in the savings-investment rate s...
    - like the rise of an empire that both creates an imperial peace, and thus a rise in the savings-investment rate s, and that also creates a rise in the taste for luxuries ϕ (and possibly reduces biological subsistence y^{sub}...
    - like the fall of an empire that destroys imperial peace, and thus a fall in the savings-investment rate s, and in the taste for luxuries ϕ (and possibly raises biological subsistence y^{sub} as looting barbarians stalk the land...
    - a shift in the rate of ideas growth...
    - a shift in sociology that alters subsistence...

The fall of an empire, for example, would see a sharp decline in the savings-investment share $ s $, as the imperial peace collapsed, a fall in the "luxuries" parameter $ \phi $, as the taste for urbanization and the ability to maintain gross inequality declined, and possibly a rise in $ y^{sub} $, if barbarian invasions and wars significantly raised mortality from violent death.

This model provides an adequate framework—or I at least, think it is an adequate framework—for thinking about the post-Neolithic Revolution pre-Industrial Revolution economy.

In [1]:

```
# DEFINING CLASS MALTHUSIAN
#
# kept in delong_classes
#
# in general use:
#    import delong_classes
#
#    m = delong_classes.malthusian
#
#    .__init__ :: initialize
#    .update :: calculate the next year's values
#    .gen_seq :: return time series of selected variable
#    .steady_state ::calculate the steady state

import matplotlib.pyplot as plt
%matplotlib inline

import numpy as np

class malthusian:

    """
    Implements the Malthusian Model with:

    1. population growth
        n =  β*(y/(ϕ ysub)-1)

    2. growth of efficiency-of-labor
        g = h-n/γ
    """
    def __init__(self,
                 L = 1,                 # initial labor force
                 E = 1/3,               # initial efficiency of labor
                 K = 3.0,               # initial capital stock

                                 # determinants of n (population growth):
                 β = 0.025,             # responsiveness of population growth t
                 ϕ = 1,                 # luxuries parameter
                 ysub = 1,              # subsistence level
```

```
                             # determinants of g(efficiency-of-labor growth
                 h = 0,              # rate at which useful ideas are genera
                 γ = 2.0,            # effect-of-resource scarcity parameter

                 s = 0.15,           # savings-investment rate
                 α = 0.5,            # orientation-of-growth-toward-capital 
                 δ = 0.05,           # deprecation rate on capital parameter
                ):
        self.L, self.E, self.K, self.h, self.γ, self.s, self.α, self.δ = L, E
        self.β, self.ϕ, self.ysub = β, ϕ, ysub

        # production (or output)
        self.Y = self.K**self.α*(self.E*self.L)**(1-self.α)
        self.y = self.Y/self.L

        # capital-output ratio
        self.κ = self.K/self.Y

        # population growth
        self.n = self.β*((self.y/(self.ϕ*self.ysub)) - 1)

        # growth rate of efficiency-of-labor
        self.g = self.h-self.n/self.γ

        # store initial data
        self.initdata = vars(self).copy()

    def update(self):
        # unpack parameters
        K, s, Y, δ, L, n, E, g, α =self.K, self.s, self.Y, self.δ, self.L, se
        β, ϕ, ysub, h, γ = self.β, self.ϕ, self.ysub, self.h, self.γ

        #update variables
        K = s*Y + (1-δ)*K
        L = L*np.exp(n)
        E = E*np.exp(g)
        Y = K**α*(E*L)**(1-α)
        y = Y/L
        κ = K/Y
        n = β*(y/(ϕ*ysub)-1)
        g = h-n/γ

        #store variables
        self.K, self.s, self.Y, self.δ, self.L, self.n, self.E, self.g, self.
        self.κ, self.y = κ, y

    def gen_seq(self, t, var = 'κ', init = True, log = False):
        "Generate and return time series of selected variable. Variable is κ 

        path = []

        # initialize data
```

```
        if init == True:
            for para in self.initdata:
                 setattr(self, para, self.initdata[para])

        for i in range(t):
            path.append(vars(self)[var])
            self.update()

        if log == False:
            return path
        else:
            return np.log(np.asarray(path))

    def steady_state(self, disp = True):
        "Calculate variable values in the steady state"
        #unpack parameters
        s, γ, h, δ, ϕ, ysub, β, α= self.s, self.γ, self.h, self.δ, self.ϕ, se

        self.mal_κ = s/(γ*h+δ)
        # malthusian rate of population growth
        self.mal_n = γ*h
        # malthusian standard of living
        self.mal_y = ϕ*(ysub+γ*h/β)
        self.mal_E = self.mal_y*((γ*h+δ)/s)**(α/(1-α))

        if display == True:
            return(f'steady-state capital-output ratio κ: {self.mal_κ:.2f}')
            return(f'Malthusian rate of population growth n: {self.mal_n: .2f
            return(f'Malthusian standard of living y: {self.mal_y:.2f}')
            return(f'steady-state efficiency-of-labor E: {self.mal_E:.2f}')
        else:
            return(self.mal_κ,self.mal_n,self.mal_y,self.mal_E)
```

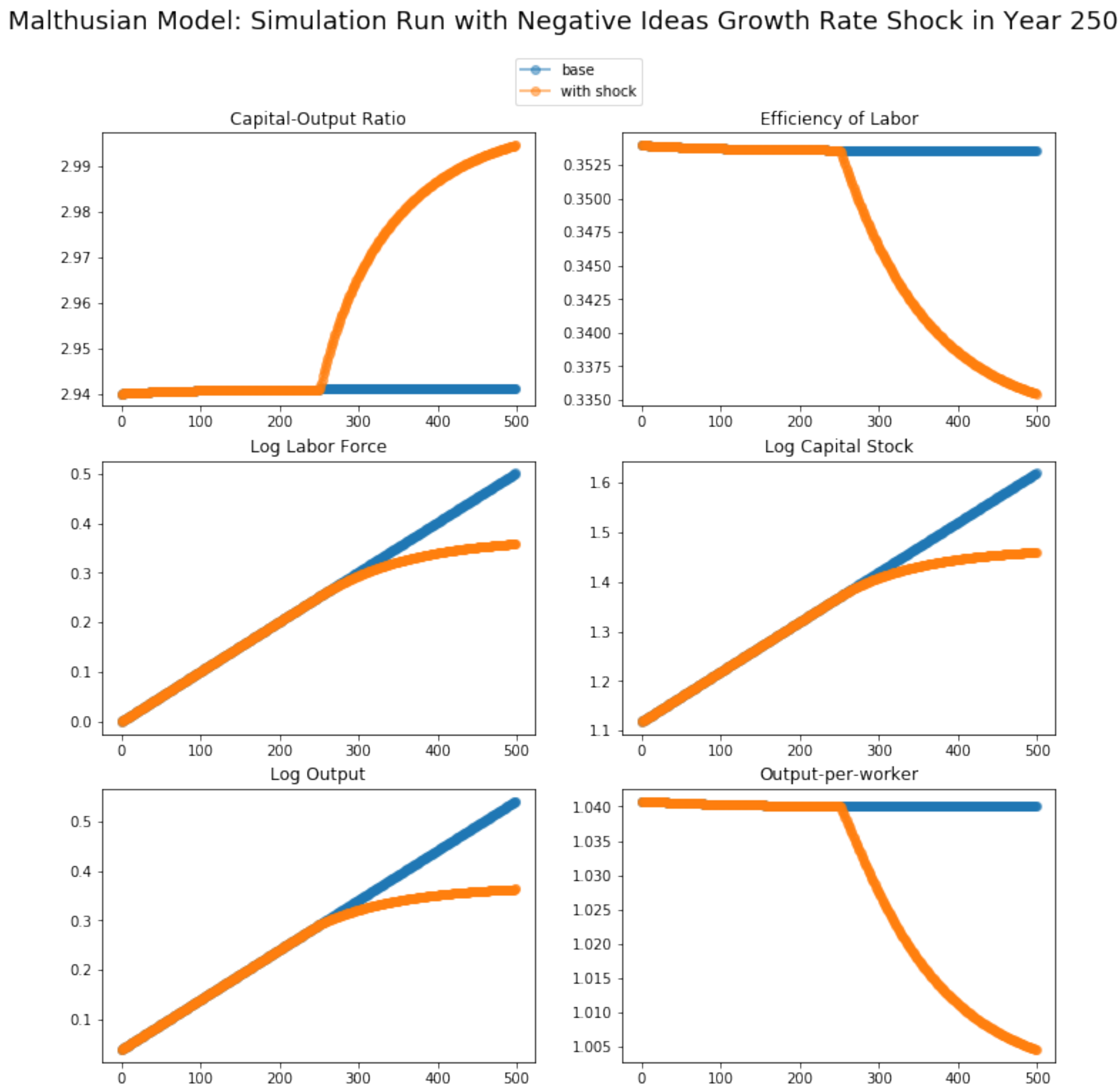
5.3. Using the Malthusian Model

5.3.1. Parameter-Value Jumps, the Malthusian Model, & Ancient Economic History

We can use the same trick we used before to analyze the *convergence* of the Malthusian model: how an economy in equilibrium then adapts and converges to a new equilibrium path if circumstances change. To get an idea of how this works, let us take an economy originally in Malthusian equilibrium with a slow rate of economic ideas growth: $ h = 0.05%/year $. And let us track it from year 0 to year 500, shocking the sysem by supposing that useful ideas stop being developed in the year 250. We can then use our class.malthusian to trace the evolution of the economy.

In [4]:

```
# USEFUL IDEAS GROWTH STOPS IN YEAR 250
#
# h=0.0005 initially, with economy in Malthusian
# steady state
#
#

import numpy as np
import matplotlib.pyplot as plt
%matplotlib inline

import delong_classes

m  = delong_classes.malthusian(h=0.0005, E=0.354, K=3.06)

# generate and store sequences before the change:
T1 = 250 # time before change
T2 = 250 # time after change

figcontents = {
        (0,0):('K','Capital-Output Ratio', False),
        (0,1):('E','Efficiency of Labor', False),
        (1,0):('L','Log Labor Force', True),
        (1,1):('K','Log Capital Stock', True),
        (2,0):('Y','Log Output', True),
        (2,1):('y','Output-per-worker', False)
       }

num_rows, num_cols = 3,2
fig, axes = plt.subplots(num_rows, num_cols, figsize=(12, 12))
for i in range(num_rows):
    for j in range(num_cols):
        for scenario in {'base', 'with shock'}:
            seq = m.gen_seq(T1, var = figcontents[i,j][0], log = figcontents[
            lb = f'{scenario}'
            if scenario == 'with shock':
                m.h = 0
            seq = np.append(seq, m.gen_seq(T2, var = figcontents[i,j][0], log
            axes[i,j].plot(seq,'o-', lw=2, alpha=0.5, label=lb)
            axes[i,j].set(title=figcontents[i,j][1])

axes[(0,0)].legend(loc='upper center', bbox_to_anchor=(1.1,1.3))
plt.suptitle('Malthusian Model: Simulation Run with Negative Ideas Growth Rat
plt.show()
```

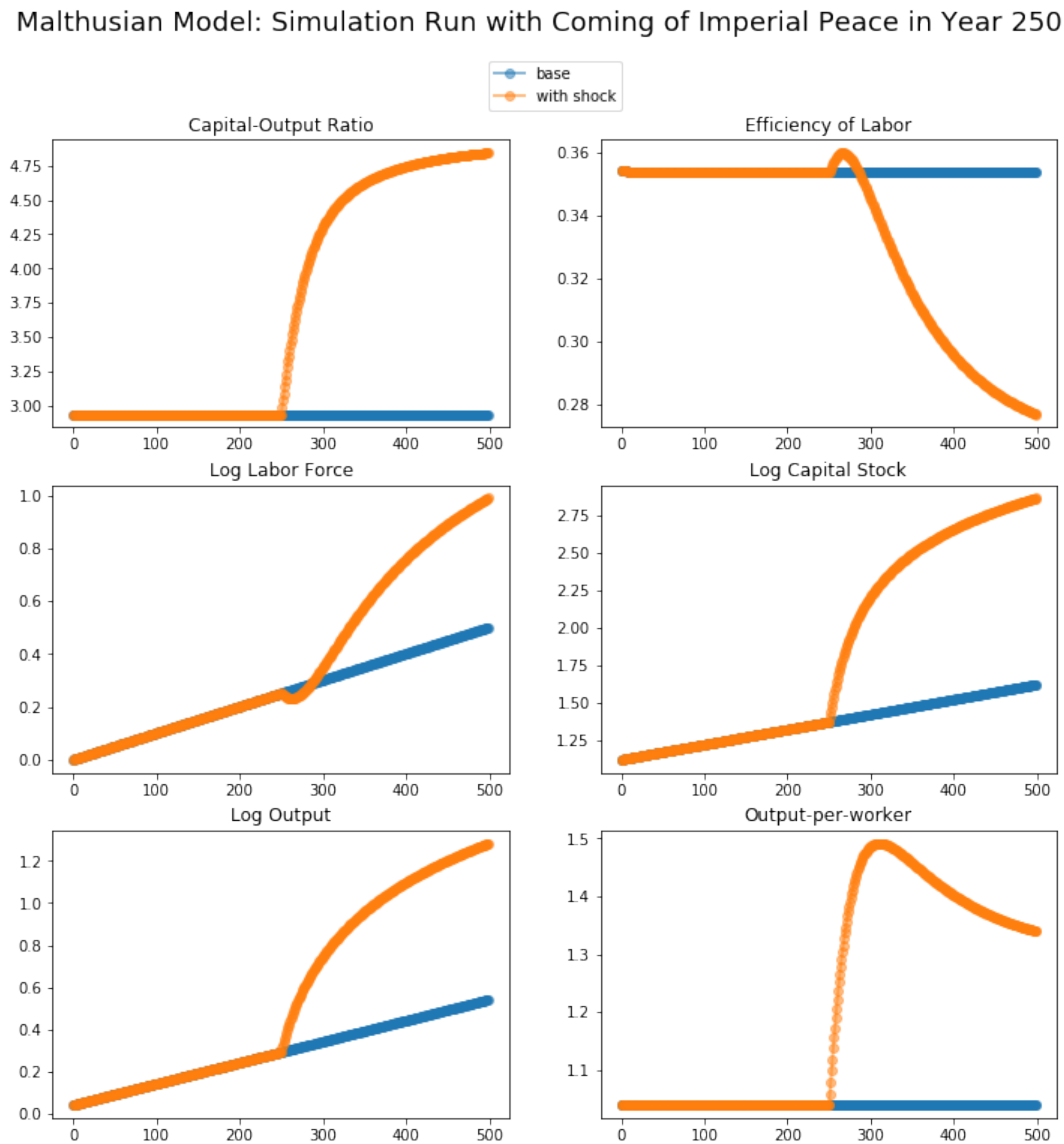


What do we find? The pre-shock equilibrium

In [5]:

```
# THE COMING OF AN IMPERIAL PEACE IN YEAR 250
#
# an imperial peace raises consumption of luxuries and of
# urbananization, so parameter φ jumps: φ = 1 --> 1.25
# simultaneously, law and order boost savoings-investment,
# so the parameter s jumps: s = 0.15 --> 0.25

import numpy as np
import matplotlib.pyplot as plt

import delong_classes
```

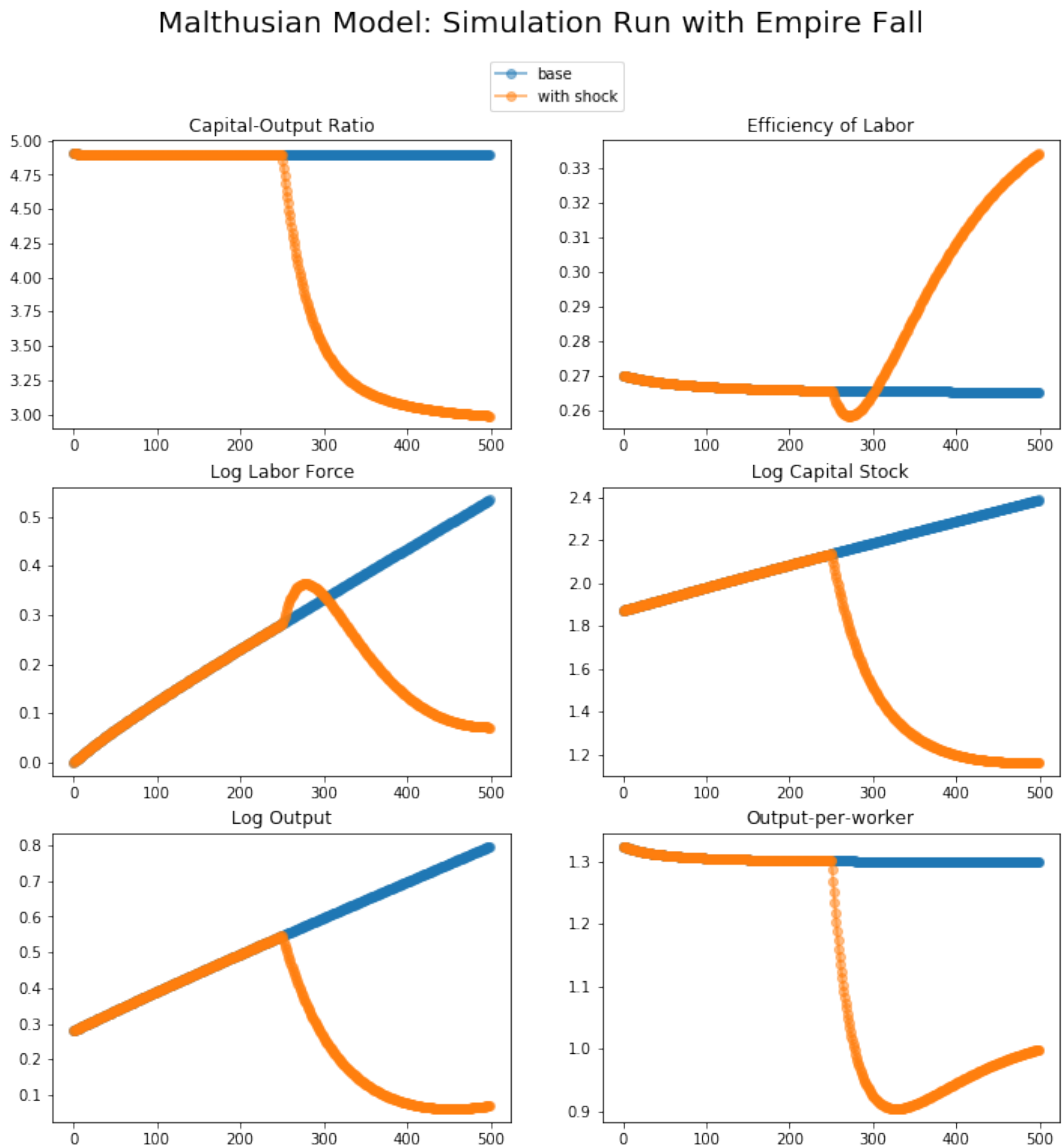
```
m  = delong_classes.malthusian(h=0.0005, E=0.354, K=3.06)

# generate and store sequences before the change:
T1 = 250 # time before change
T2 = 250 # time after change

figcontents = {
        (0,0):('K','Capital-Output Ratio', False),
        (0,1):('E','Efficiency of Labor', False),
        (1,0):('L','Log Labor Force', True),
        (1,1):('K','Log Capital Stock', True),
        (2,0):('Y','Log Output', True),
        (2,1):('y','Output-per-worker', False)
       }

num_rows, num_cols = 3,2
fig, axes = plt.subplots(num_rows, num_cols, figsize=(12, 12))
for i in range(num_rows):
    for j in range(num_cols):
        for scenario in {'base', 'with shock'}:
            seq = m.gen_seq(T1, var = figcontents[i,j][0], log = figcontents[
            lb = f'{scenario}'
            if scenario == 'with shock':
                m.φ = 1.25
                m.s = 0.25
            seq = np.append(seq, m.gen_seq(T2, var = figcontents[i,j][0], log
            axes[i,j].plot(seq,'o-', lw=2, alpha=0.5, label=lb)
            axes[i,j].set(title=figcontents[i,j][1])

axes[(0,0)].legend(loc='upper center', bbox_to_anchor=(1.1,1.3))
plt.suptitle('Malthusian Model: Simulation Run with Coming of Imperial Peace
plt.show()

# greater consumption of luxuries and of urban life initally
# puts downward pressure on the population, but in less than
# a generation the productivity and division-of-labor benefits
# from higher savings-investment more than compensate
#
# the economy heads for an imperial peace steady state with
# higher capital-intensity κ = 3 --> 4.9; a lower efficiency
# of labor due to heightened resource scarcity E = 0.353 --> 0.27,
# and a higher labor force
#
# output per capita rises, overshoots its new steady-state value,
# and peaks about two generations after the coming of the imperial
# peace:
```



In [6]:

```
# THE COLLAPSE OF AN EMPIRE IN YEAR 250
#
# an imperial collapse reduces consumption of luxuries
# and of urbananization, so parameter φ falls: φ = 1.25 --> 1;
# simultaneously, the anarchy lowers savoings-investment, so
# the parameter s falls: s = 0.25 --> 0.15


import numpy as np
import matplotlib.pyplot as plt
```

```
import delong_classes

m  = delong_classes.malthusian(h=0.0005, E=0.27, K=6.5, φ = 1.25, s = 0.25)

# generate and store sequences before the change:
T1 = 250 # time before change
T2 = 250 # time after change

figcontents = {
        (0,0):('K','Capital-Output Ratio', False),
        (0,1):('E','Efficiency of Labor', False),
        (1,0):('L','Log Labor Force', True),
        (1,1):('K','Log Capital Stock', True),
        (2,0):('Y','Log Output', True),
        (2,1):('y','Output-per-worker', False)
       }

num_rows, num_cols = 3,2
fig, axes = plt.subplots(num_rows, num_cols, figsize=(12, 12))
for i in range(num_rows):
    for j in range(num_cols):
        for scenario in {'base', 'with shock'}:
            seq = m.gen_seq(T1, var = figcontents[i,j][0], log = figcontents[
            lb = f'{scenario}'
            if scenario == 'with shock':
                m.φ = 1.0
                m.s = 0.15
            seq = np.append(seq, m.gen_seq(T2, var = figcontents[i,j][0], log
            axes[i,j].plot(seq,'o-', lw=2, alpha=0.5, label=lb)
            axes[i,j].set(title=figcontents[i,j][1])

axes[(0,0)].legend(loc='upper center', bbox_to_anchor=(1.1,1.3))
plt.suptitle('Malthusian Model: Simulation Run with Empire Fall', size = 20)
plt.show()

# the collapse in demand for luxuries and for urban life initally
# allows an upward jump in the population, but in less than
# a generation the productivity and division-of-labor costs
# from lower savings-investment more than compensate
#
# the economy heads for a post-collapse steady state with
# lower capital-intensity κ = 4.9 --> 3 ; a higher efficiency
# of labor due to reduced resource scarcity E = 0.27 --> 0.353,
# and a lower labor force
#
# output per capita falls, undershoots its new steady-state value,
# and troughs out about two generations after the collapse of the
# empire:
```

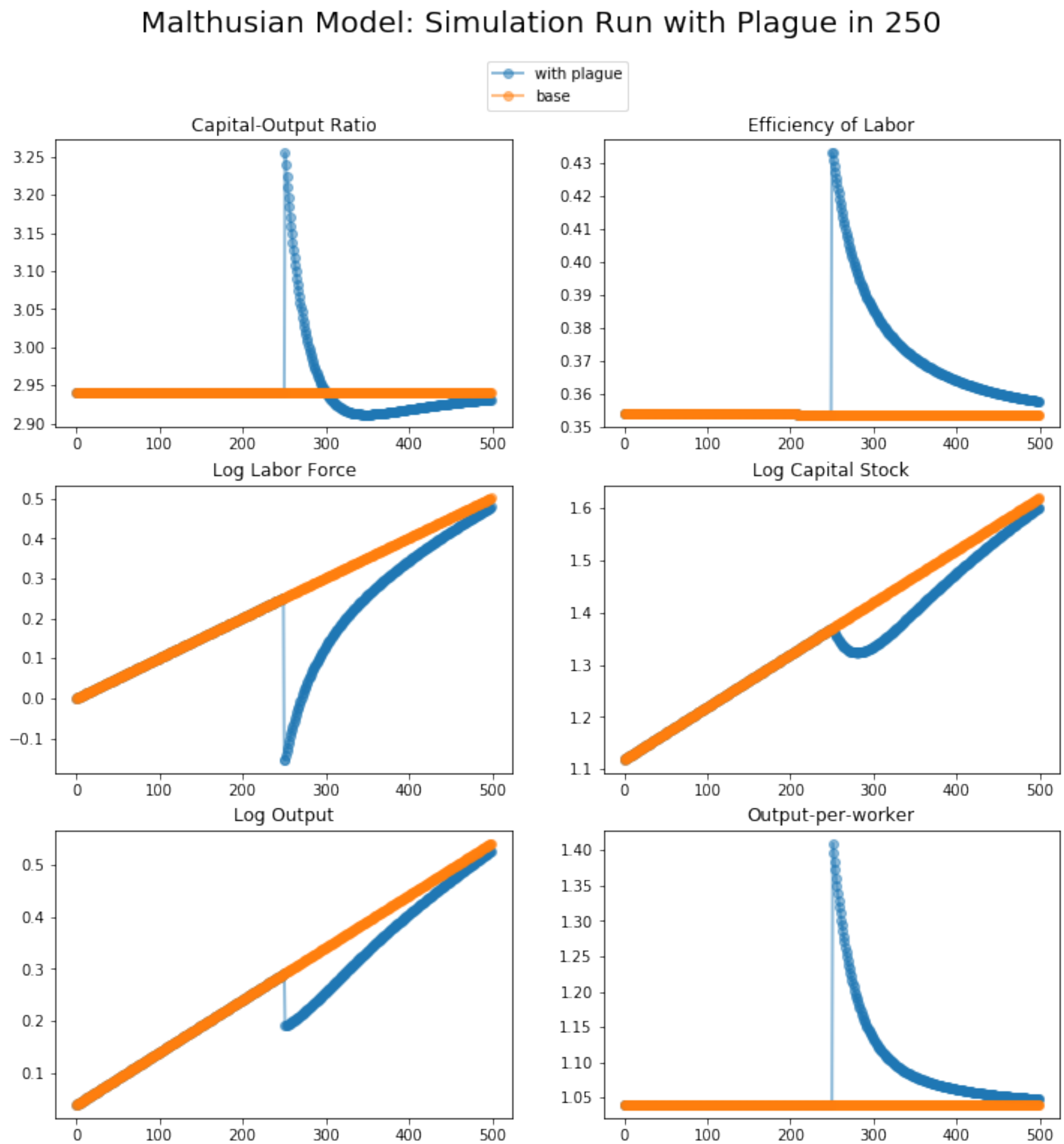


Consider a plague in 250:

The plague decreased the total population (and thus labor force) by 1/3.

In [7]:
```
# IMPACT OF A PLAGUE IN YEAR 250
#
# the plague carries off 1/3 of the population
#
# the efficiency of labor immediately jumps up
```

```
# because of reduced resource scarcity
#
#

import numpy as np
import matplotlib.pyplot as plt

import delong_classes

m  = delong_classes.malthusian(h=0.0005, E=0.354, K=3.06)

# generate and store sequences before the change:
T1 = 250 # time before change
T2 = 250 # time after change

figcontents = {
        (0,0):('K','Capital-Output Ratio', False),
        (0,1):('E','Efficiency of Labor', False),
        (1,0):('L','Log Labor Force', True),
        (1,1):('K','Log Capital Stock', True),
        (2,0):('Y','Log Output', True),
        (2,1):('y','Output-per-worker', False)
       }

num_rows, num_cols = 3,2
fig, axes = plt.subplots(num_rows, num_cols, figsize=(12, 12))
for i in range(num_rows):
    for j in range(num_cols):
        for scenario in {'base', 'with plague'}:
            seq = m.gen_seq(T1, var = figcontents[i,j][0], log = figcontents[
            lb = f'{scenario}'
            if scenario == 'with plague':
                m.L = 2/3*m.L
                m.E = m.E*(2/3)**(-1/m.γ)
            seq = np.append(seq, m.gen_seq(T2, var = figcontents[i,j][0], log
            axes[i,j].plot(seq,'o-', lw=2, alpha=0.5, label=lb)
            axes[i,j].set(title=figcontents[i,j][1])

axes[(0,0)].legend(loc='upper center', bbox_to_anchor=(1.1,1.3))
plt.suptitle('Malthusian Model: Simulation Run with Plague in 250', size = 20
plt.show()

# the population recvoeres about halfway back to steady-state
# in each generation
#
# the dynamics of the capital-intensity show a little bit of
# cyclicality: the capital-output ratio undershoots its steady-
# state value by a bit, and then climbs back up:
```



To summarize, the shock on idea development will change the steady state and the development path, while a plague would not change the steady state.

Shocks to the System...

- A plague, pushing $ L_t $ down substantially below $ L_t^{*mal} $...
- Fall (or rise) of empire:
  - A breakdown (or buildup) of law-and-order, raising incentives to save and invest, and so raising $ s $...
  - A greater "taste" for inequality, luxuries, and urbanization, raising $ \phi $...
- Invaders and other raiders, raising $ \delta $...
- A speed-up or slowdown in innovation, raising or lowering $ h $...
- Large-scale destruction of the societal web and thus the societal division of labor, lowering $ H $...
- An increased rate of death from disease or violence, raising or lowering $ Y^{sub} $...

5.A. Convergence to the Malthusian Equilibrium

Recall:

> (1.7) $ \frac{dL/dt}{L} = \frac{d\ln(L)}{dt} = n = \beta \left(\frac{y}{\phi y^{sub}}-1 \right) $
>
> (2.14) $ \ln(y) = \theta\ln(\kappa) + \ln(H) -\ln(L)/\gamma $

Substitute:

> (4.1) $ \frac{1}{L}\frac{dL}{dt} = \frac{d\ln(L)}{dt} = n = \beta \left(\frac{\kappa^\theta H L^{-\gamma}}{\phi y^{sub}}-1 \right) $
>
> (4.2) $ \frac{d\kappa}{dt} = -(1-\alpha)(h + (1-1/\gamma)n +\delta)\kappa + (1-\alpha)s $

Define ideas-adjusted-for-population $ I $:

> (4.3) $ I = H L^{-1/\gamma} $
>
> (4.4) $ i = h - n/\gamma $
>
> (4.5) $ \frac{d\kappa}{dt} = -(1-\alpha)(\gamma h - (\gamma-1)i +\delta)\kappa + (1-\alpha)s $
>
> (4.6) $ \frac{d\kappa}{dt} = (1-\alpha)s -(1-\alpha)(\gamma h +\delta)\kappa + (1-\alpha) (\gamma-1)i\kappa $

> (4.7) $ \frac{1}{I}\frac{dI}{dt} = i = h - n/\gamma = h - \frac{\beta}{\gamma} \left(\frac{\kappa^\theta I}{\phi y^{sub}}-1 \right) $

Then we have two state variables—capital-intensity $ \kappa $, the capital-output ratio, and ideas-adjusted-for-population $ I $. We have two dynamic equations: The rate of change of ideas-adjusted-for-population $ I $ is a function of the capital-output ratio and itself. And the rate of change of capital-intensity $ \kappa $ is a function of itself and of the rate of change of ideas-adjusted-for-population $ I $.

The steady state is then:

> (4.8) $ I^{*mal} = \frac{H}{L^{1/\gamma}} = \phi y^{sub}\left(\frac{\delta}{s}\right)^{\theta}\left(1+ \frac{\gamma h}{\delta}\right)^{\theta}\left(1 + \gamma h/\beta \right) $
>
> (4.9) $ \kappa^{*mal} = \frac{s}{\gamma h + \delta} $

Define:

> (4.10) $ I = (1 + \xi) I^{*mal} $
>
> (4.11) $ \kappa = (1 + k) \kappa^{*mal} = (1 + k) (s/(\delta + \gamma h)) $
>
> (4.12) $ \frac{1}{1 + \xi}\frac{d\xi}{dt} = h - \frac{\beta}{\gamma} \left(\frac{(1+k)^\theta\left(\kappa^{*mal}\right)^\theta (1+\xi) I^{*mal}}{\phi y^{sub}}-1 \right) $
>
> (4.13) $ \frac{1}{1+\xi}\frac{d\xi}{dt} = i = h - \frac{\beta}{\gamma} \left((1 + \gamma h/\beta)(1+k)^\theta (1+\xi) - 1 \right) $
>
> (4.14) $ \frac{1}{1+\xi}\frac{d\xi}{dt} = i = h - \left((h + \frac{\beta}{\gamma}) (1+k)^\theta (1+\xi) - \frac{\beta}{\gamma} \right) $

Using the approximation:

> $ 1 + \theta k = (1+k)^{\theta} $
>
> (4.15) $ \frac{1}{1+\xi}\frac{d\xi}{dt} = h - h - \frac{\beta}{\gamma} - h \theta k - \frac{\theta \beta}{\gamma}k - h \xi - \frac{ \beta}{\gamma}\xi + \frac{\beta}{\gamma} $
>
> (4.16) $ \frac{d\xi}{dt} = \left[h - h - \frac{\beta}{\gamma} - h \theta k - \frac{\theta \beta}{\gamma}k - h \xi - \frac{ \beta}{\gamma}\xi + \frac{\beta}{\gamma} \right](1+\xi) $

> (4.17) $ \frac{d\xi}{dt} = -(h \theta + \theta \beta / \gamma)k - (h + \beta/\gamma)\xi $

This is our linearized exponential-convergence equation for the deviation of ideas-adjusted-for-the-population $ \xi $.

Now on to the capital-instensity. Recall:

from our definition of $ k $ we get:

> (4.18) $ \frac{d\kappa}{dt} = \frac{dk}{dt}\kappa^{*mal} $
>
> (4.19) $ \kappa^{*mal}\frac{dk}{dt} = (1-\alpha)s -(1-\alpha)(\gamma h +\delta)(1+k)\kappa^{*mal} + (1-\alpha) (\gamma-1)i(1+k)\kappa^{*mal} $
>
> (4.20) $ \kappa^{*mal}\frac{dk}{dt} = (1-\alpha)s -(1-\alpha)(\gamma h +\delta)\kappa^{*mal} -(1-\alpha)(\gamma h +\delta)k\kappa^{*mal} + (1-\alpha) (\gamma-1)i(1+k)\kappa^{*mal} $
>
> (4.21) $ \kappa^{*mal}\frac{dk}{dt} = -(1-\alpha)(\gamma h +\delta)k\kappa^{*mal} + (1-\alpha) (\gamma-1)i(1+k)\kappa^{*mal} $
>
> (4.22) $ \kappa^{*mal}\frac{dk}{dt} = -(1-\alpha)sk - (1-\alpha) (\gamma -1)(h\theta + \theta \beta / \gamma)k + (h + \beta/\gamma)\xi)\kappa^{*mal} $
>
> (4.23) $ \frac{dk}{dt} = -(1-\alpha)(\delta + \gamma h)k - (1-\alpha) (\gamma-1)(h\theta + \theta \beta / \gamma)k - (1-\alpha) (\gamma-1)(h + \beta/\gamma)\xi$
>
> (4.24) $ \frac{dk}{dt} = -(1-\alpha)\left[\delta + \gamma h + (\gamma-1)(h\theta + \theta \beta / \gamma) \right]k - (1-\alpha) (\gamma-1)(h + \beta/\gamma)\xi$

Thus our linearized exponential-convergence system for the deviation of ideas-adjusted-for-the-population $ \xi $ and the deviation of capital-intensity $ k $ from steady-state Malthusian equilibrium is:

> (4.24) $ \frac{dk}{dt} = -(1-\alpha)\left[\delta + \gamma h + (\gamma-1)(h\theta + \theta \beta / \gamma) \right]k - (1-\alpha) (\gamma-1)(h + \beta/\gamma)\xi$
>
> (4.17) $ \frac{d\xi}{dt} = -(h \theta + \theta \beta / \gamma)k - (h + \beta/\gamma)\xi $

6. What Economists Have to Say About Innovation, and the Rate of Ideas Growth

6.1. The Depressing Bottom Line

The rate of economic growth—first in population, and more recently in average living standards and productivity growth rates—hinges on the proportional rate of increase h in the human stock of useful ideas H. And this rate has been extraordinarily variable in the long sweep of human history. And if the trends of the past century and a half ere to continue for trhe next three, we would look forward to truly cray-cray levels of abundant human wealth:

Longest-Run Global Economic Growth (2019)

Date	ideas Level H	Total Real World Income Y (billions)	Average Real Income per Capita y (per year)	Total Human Population L (millions)		Rate of Population and Labor Force Growth n	Rate of Efficiency-of-Labor Growth g	Rate of Ideas-Stock Growth h
-68000	1.0	$0	$1,200	0.1				
-8000	5.0	$3	$1,200	2.5		0.005%	0.000%	0.003%
-6000	6.3	$6	$900	7		0.051%	-0.014%	0.011%
-3000	9.2	$14	$900	15		0.025%	0.000%	0.013%
-1000	16.8	$45	$900	50		0.060%	0.000%	0.030%
0	30.9	$153	$900	170		0.122%	0.000%	0.061%
800	41.1	$270	$900	300		0.071%	0.000%	0.035%
1500	53.0	$450	$900	500		0.073%	0.000%	0.036%
1770	79.4	$825	$1,100	750		0.150%	0.074%	0.149%
1870	123.5	$1,690	$1,300	1300		0.550%	0.167%	0.442%
2020	2720.5	$90,000	$11,842	7600		1.177%	1.473%	2.061%
2100	13474.9	$485,096	$53,900	9000	?	0.211%	1.894%	2.000%
2200	99566.8	$3,584,405	$398,267	9000	?	0.000%	2.000%	2.000%
2500	40168118.9	$1,446,052,279	$160,672,475	9000	?	0.000%	2.000%	2.000%

Moreover, right now the divergences across national economies are as great as they have ever been, and orders of magnitude greater than they were three centuries ago or even one century ago. What insights can economists offer into these phenomena?

Unfortunately, the bottom line is that economists have little that is terribly useful to say about the proportional rate $ h $ at which the human stock $ H $ of useful and valuable ideas about technology and organization increases. It would not be too much a parody to say that economists know only four things:

1. People learn by doing: trying to produce, and succesfully producing, brings with it knowledge about how to produce more efficiently and effectively.
2. People learn by investing: a great deal of knowledge is embodied in the particular capital goods themselves produced and deployed; if you do not invest, a great deal of your knowledge remains theoretical.
3. People learn by researching and developing: focused attention on the process of developing technology can be very effective—much more so than simply relying on the side-effects of those whose major focus is on production itself
4. Knowledge is non-rival: once it is generated, it can and should be spread as widely as possible, for there is no downside for society as a whole from sharing.

But that—especially that without sound and solid quantitative estimates of the size and importance of these effects and channels—is rather thin gruel given that the growth and diffusion of useful knowledge about production and organization is the big enchilada in the process of economic growth.

Here we will focus on (3) and (4), leaving (1) and (2) for later "applications" sections of this course:

6.2. Knowledge Is Non-Rival

6.2.1. Logical Implications

Useful ideas about technology and organization are non-rival: one person's work in adding to $ H $ can rapidly benefit all—if it is allowed to spread. And attempts to keep it from spreading—to limit knowledge's distribution by somehow charging those using it a price—must violate the optimality condition that the costs imposed on people for making use of commodities reflect and match the burden that their withdrawal of the commodities from the common stock imposes on the rest of the community, for with non-rival commodities there is no such withdrawal. The insights that knowledge is key and that knowledge is non-rival are now nearly two centuries old. We can find them in in 1843: **Friedrich Engels** (1843): *Outlines of a Critique of Political Economy* https://www.marxists.org/archive/marx/works/1844/df-jahrbucher/outlines.htm:

> According to the economists, the production costs of a commodity consist of three elements: the rent for the piece of land required to produce the raw material; the capital with its profit, and the wages for the labour required for production and manufacture.... [Since] capital is “stored-up labour”... two sides–the natural, objective side, land; and the human, subjective side, labour, which includes capital and, besides capital, a third factor which the economist does not think about–I mean the mental element of invention, of thought, alongside the physical element of sheer labour.
>
> What has the economist to do with inventiveness? Have not all inventions fallen into his lap without any effort on his part? Has one of them cost him anything? Why then should he bother about them in the calculation of production costs? Land, capital and labour are for him the conditions of wealth, and he requires nothing else. Science is no concern of his.
>
> What does it matter to him that he has received its gifts through Berthollet, Davy, Liebig, Watt, Cartwright, etc.–gifts which have benefited him and his production immeasurably? He does not know how to calculate such things; the advances of science go beyond his figures. But in a rational order which has gone beyond the division of interests as it is found with the economist, the mental element certainly belongs among the elements of production and will find its place, too, in economics among the costs of production.
>
> And here it is certainly gratifying to know that the promotion of science also brings its material reward; to know that a single achievement of science like James Watt’s steam-engine has brought in more for the world in the first fifty years of its existence than the world has spent on the promotion of science since the beginning of time...

And yet indeed it was the case that mainstream economists, for generations, paid remarkably little of their attention to "inventiveness". Engels was right—at least about mainstream economists' strange neglect. (I take no stance on whether Engels was right in his belief that mainstream economists cannot see the world as it is but only illusions that are caused by our particular institutional framework and convenient to those whom our current institutional framework serves most fulsomely.) Engels was right so much so that Paul Romer received the Nobel Prize in 2018 for his attempts to bring "inventiveness" back to the center. When **Robert Solow** (1987): *Growth Theory and After* https://www.nobelprize.org/prizes/economic-sciences/1987/solow/lecture/ gave his Nobel lecture on the occasion of the earlier "economic growth" Nobel Prize, awarded in 1987, he noted that his theory had little to say about “technical change in the broadest sense” and that this was a huge flaw:

> The “neoclassical model of economic growth” started a small industry... stimulated hundreds of theoretical and empirical articles... very quickly found its way into textbooks... is what allows me to think that I am a respectable person to be giving this lecture today.... Gross output per hour of work in the U. S. economy doubled between 1909 and 1949; and some seven-eighths of that increase could be attributed to “technical change in the broadest sense” and only the remaining eight could be attributed to conventional increase in capital intensity.... I had expected to find a larger role for straightforward capital formation...

Solow goes on to write that his attempts to say something meaningful and important about the determinants of $ h $ failed:

> "embodiment", the fact that much technological progress, maybe most of it, could find its way into actual production only with the use of new and different capital equipment... [and so] a policy to increase investment would thus lead... also to a faster transfer of new technology into actual production, which would [matter much].... That idea seemed to correspond to common sense, and it still does.... If common sense was right, the embodiment model should have fit the facts significantly better than the earlier one. But it did not...

There is a literature, springing from Paul Romer's work in the 1980s, focusing on the implications of non-rivalry in the use of ideas: that one person's work in adding to $ H $ can rapidly benefit all. The first conclusion is that production must in some sense be subject to increasing returns. We know that if all material "inputs" were to double then, since the new inputs could just do the same things as the old ones, production would at least double. In knowledge production, however, pursuing the same lines of inquiry and thus making all the same discoveries twice would be silly. Doubled material inputs with doubled effort devoted to knowledge creation should therefore more than double output. An economy in which non-rival knowledge is important will therefore exhibit *scale effects*: size matters, and in a good way for growth.

The useful literature can be seen as building on this first conclusion.

6.2.2. Fitting the Entire Span of Human History: Michael Kremer on Growth since One Million B.C.

Theory (1993): Non-rivalry in the use of and non-crowding in the production of useful ideas about technology and organization are the fundamental underlying assumptions of the first milestone to visit on our path through the literature: Michael Kremer: **Michael Kremer** (1993) *Population Growth and Technological Change: One Million B.C. to 1990* https://delong.typepad.com/files/kremer-million.pdf:

> The long-run history of population growth and technological change is consistent with the population implications of models of endogenous technological change... a highly stylized model in which... the growth rate of technology is proportional to total population... the Malthusian assumption that population is limited by the available technology, so that the growth rate of population is proportional to the growth rate of technology. Combining these assumptions implies that the growth rate of population is proportional to the level of population.... The prediction that the population growth rate will be proportional to the level of population is broadly consistent with the data... until recently.... If population grows at finite speed when income is above its steady state... per capita income will rise over time. If population growth declines in income at high levels of income, as is consistent with a variety of theoretical models and with the empirical evidence, this gradual increase in income will eventually lead to a decline in population growth.... As the model predicts, the growth rate of population has been propor- tional to its level over most of history.... Among technologically separate societies, those with higher initial population had faster growth rates of technology and population...

In short, two heads are better than one:

Assume that output is given by

> $$Y = Ap^\alpha R^{1-\alpha}$$

where A is the level of technology, p is population, and $R = 1$ is land, henceforth normalized to one. Per capita income y therefore equals $Ap^{\alpha - 1}$. opulation increases above the steady-state equilibrium level of per capita income y^* and decreases below it. Diminishing returns to labor imply that a unique level of population, p^* generates income y^*:

> $$p^* = \left(\frac{A}{y^*} \right)^{(1/(1-\alpha))}$$

In a larger population there will be proportionally more people lucky or smart enough to come up with new ideas. If research productivity per person is independent of population and if A affects research output the same way it affects output of goods (linearly, by definition), then the rate of change of technology will be:

> $$\frac{dA}{dt} = \pi Ap$$

Take the log derivative of the population determination equation:

> $$\frac{dln(p)}{dt} = \left(\frac{1}{1-\alpha} \right) \frac{dln(A)}{dt}$$

and substitute in the expression for the growth rate of technology:

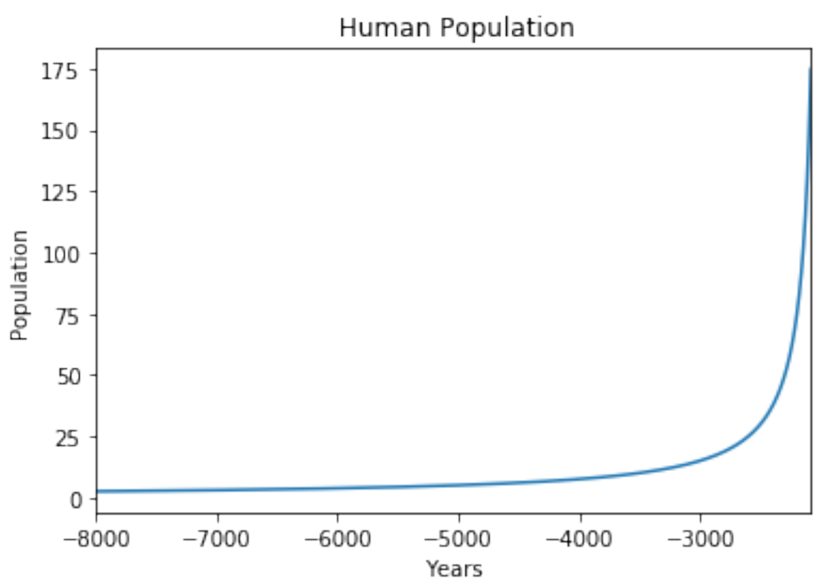
> $$\frac{dp}{dt} = \frac{\pi p^2}{1-\alpha}$$

to get superexponential growth of population (and total income)—as long as the Malthusian régime lasts, there is no demographic transition, .

To get an idea of what this means, let us run a computational experiment. There were 2.5 million people 10,000 years ago, at the invention of agriculture. There were 15 million people 5,000 years ago, at the invention of writing. There were 170 million people in the year 1. Let's calibrate this model to 2.5 million people in the year -8000 and 15 million people in the year -3000: a value of $\pi/(1-\alpha) = 0.00006666$ serves. But that value predicts that human population would cross 170 million heading upwards not in the year 1 but in the year -2080: early in the Bronze Age.

In [59]:

```
# fitting the Kremer model to historical population
# for years -8000 and -3000:

import numpy as np
import pandas as pd
import matplotlib.pyplot as plt
%matplotlib inline

Year = [-8000]
P = [2.5]
α = 0.5
π = 0.00003333

for t in range(-7999, -2080):
    Year = Year + [t]
    P = P + [P[t+7999] + π*P[t+7999]**2/(1-α)]

List = [Year, P]
P_df = pd.DataFrame(List).transpose()
P_df['Log Population'] = np.log(P)
nlabel = ['Year', 'Population', 'Log_Population']
P_df.columns=nlabel
P_df.set_index('Year', inplace = True)

P_df.Population.plot()

plt.ylabel('Population')
plt.xlabel('Years')
plt.title(' Human Population')
plt.show()
```



If we want to fit our three pre-1 benchmarks, we cannot have two heads being fully as good as one. So, instead, let us assume not that a 1% increase in the STEM workforce raises the rate of technological progress by 1% but rather by λ% for some parameter λ. So the dynamics for population then become:

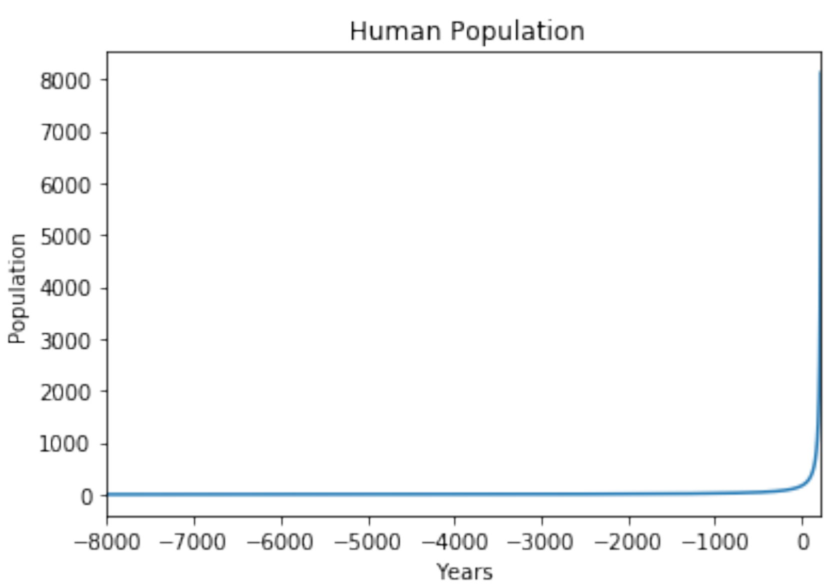
> $\frac{dp}{dt} = \frac{\pi p^{1+\lambda}}{1-\alpha}$

$\alpha = 0.5$, $\pi = 0.00003264$, $\lambda = 0.8529$ fit the pre-1 benchmarks well. But those benchmarks predict that the human population would have exploded in the following two centuries, and crossed rthe world's current population of 7.6 billion in the year 221.

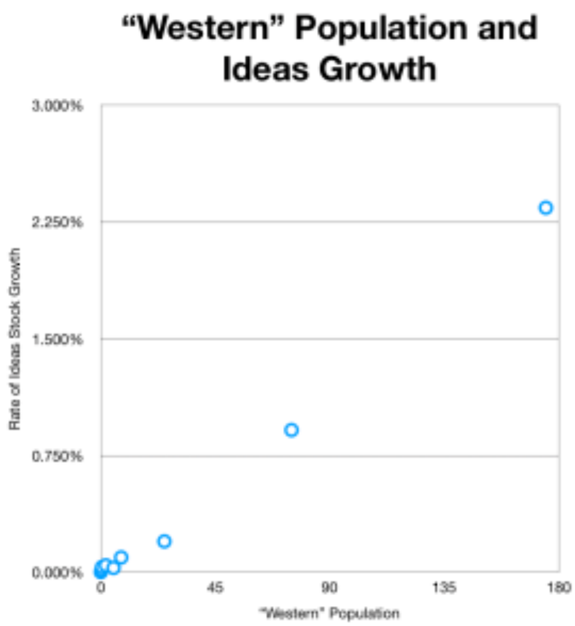
Even if two heads are not quite as good as one—are only 1.85 times as good as one—there need to be other sources of drag in order to have kept the world from an Industrial Revolution-class breakthrough late in the Later Han, and under the late Antonine and Severian dynasties:

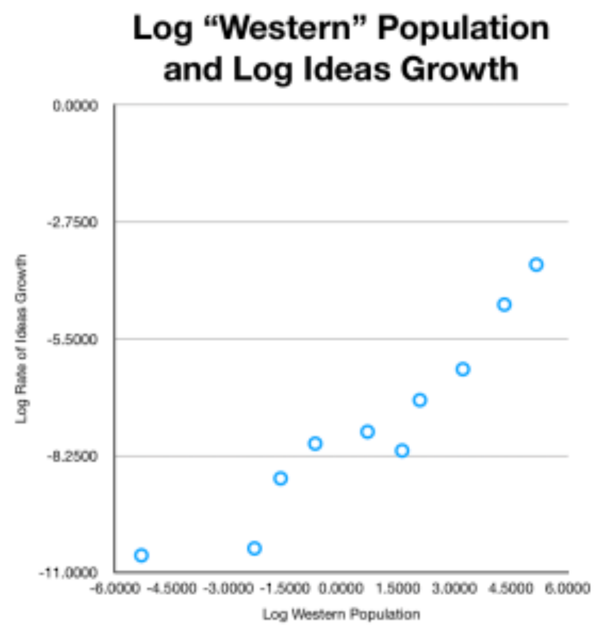
In [2]:

```
# fitting the Kremer model to historical population
# for years -8000, -3000, and 1:

import numpy as np
import pandas as pd
import matplotlib.pyplot as plt
%matplotlib inline

Year = [-8000]
P = [2.5]
α = 0.5
π = 0.00003264
λ = 0.8529


for t in range(-7999, 221):
    Year = Year + [t]
    P = P + [P[t+7999] + π*P[t+7999]**(1+λ)/(1-α)]

List = [Year, P]
P_df = pd.DataFrame(List).transpose()
P_df['Log Population'] = np.log(P)
nlabel = ['Year', 'Population', 'Log_Population']
P_df.columns=nlabel
P_df.set_index('Year', inplace = True)


P_df.Population.plot()

plt.ylabel('Population')
plt.xlabel('Years')
plt.title(' Human Population')
plt.show()
```



How Well Does This Fit Human History?: Still, all in all, Michael Kremer says: it does not fit badly badly. And, indeed, up to 1900 the rate of change of the human population is indeed roughly proportional to the square of the population—as long as we start not with the invention of agriculture but with the invention of writing, and with a hiccup as the Roman and Han empires collapsed in the second third of the first millennium. After 1900 things fall apart: increasing populations and the increasing ability of people to use technology and wealth to help their investigations do not pay dividends, either in further accelerating population growth or in increasing the rate of growth of global incomes.





Nevertheless, two heads are better than one. Or maybe not: surely the effective STEM labor force depends on means of knowledge recording and communication. And it is not foolish to expect *ex ante* that there would be some diminishing returns from exhaustion of low-hanging fruit at some point. We do seem to see a jump up in growth with the invention of writing, and cities. Shouldn't we also see a jump up with the alphabet? Shouldn't we also see a jump up with the invention of printing? Perhaps the effects of the picking of the low-hanging fruit in exhausting opportunities and slowing growth civilization-wide are visible in the slowdown after the year one. Perhaps the effective STEM labor force gets big bumps up with the alphabet and with printing that together, in the large, offset this exhaustion.

Clearly, however, two heads are better than one will not suffice to understand the relative constancy of global economic growth rates since the coming of modern economic growth around 1870, or even the failure of Roman and Han civilization to usher in an industrial revolution.

6.2.3. Chad Jones on R & D-Based Models of Economic Growth

Can we preserve the insights that ideas are non-rival and that technology is the ballgame and still understand why growth did not accelerate faster and bring us an Industrial Revolution early in the first millennium, and, in fact, has not further accelerated since the late 1800s? Chad Jones believes we can, and he lays out his case in **Charles I. Jones** (1995): *R&D-Based Models of Economic Growth* https://delong.typepad.com/files/jones-r--d.pdf:

> The prediction of permanent scale effects on growth from the R&D equation means that the models of Romer/Grossman-Helpman/Aghion-Howitt and others are all easily rejected.... However, the R&D-based models [remain] intuitively very appealing.... [Is there] a way to maintain the basic structure of these models while eliminating the prediction of [permanent] scale effects [on the rate of growth?]...

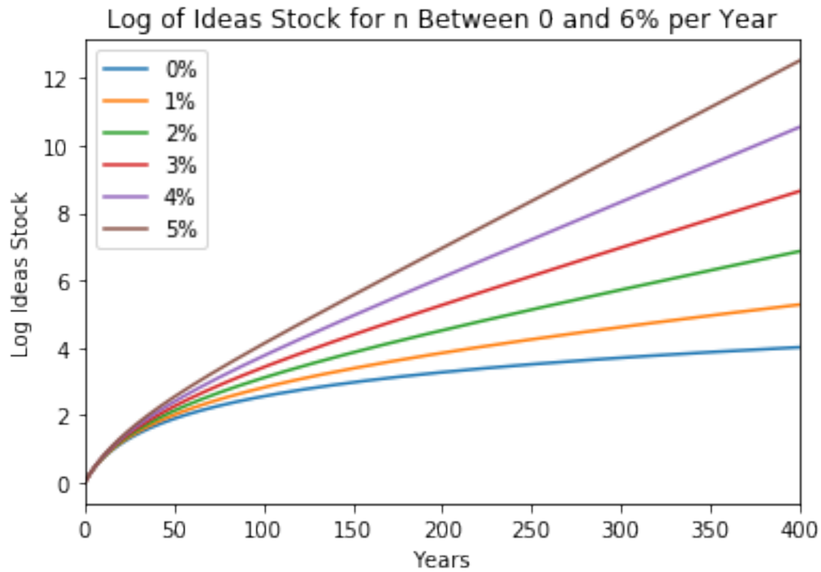
Jones's answer is "yes". Jones accomplishes this by building a basic model that has both (a) an "as the low-hanging technological fruit is picked, maintaining the same proportional growth rate for the ideas stock H becomes harder" effect (the parameter $\phi < 1$); and (b) an "as the STEM workforce increases, researchers tend to step on each others' toes and get in each others' way" effect (the parameter $\lambda < 1$):

> $\frac{dH}{dt} = \pi L_{stem}^{\lambda} H^{\phi}$
>
> $\frac{dH/dt}{H} = \pi L_{stem}^{\lambda} H^{\phi - 1}$

To gain some intuition, let's consider six different economies in which the rate of growth n of the STEM labor force varies from 0 to 6% per year, in which the initial levels of both the ideas stock H_0 and the STEM labor force L_0 are set at 1, and let us set the R&D crowding parameter $\lambda = 0.5$ and, just to get striking results, the exhaustion of low hanging fruit parameter at the very low level of $\phi = 0.1$. Looking out at the evolution of the log of the ideas stock for 400 years:

In [3]:

```
# watching how H evolves for different values
# of STEM labor force growth, from 0 to 5%
# per year

import numpy as np
import pandas as pd
import matplotlib.pyplot as plt
%matplotlib inline

T = 400
δ = 0.1
H_0 = 1
n = [0, 0.01, 0.02, 0.03, 0.04, 0.05]
nlabel = ['0%', '1%', '2%', '3%', '4%', '5%']
φ = 0.1
λ = 0.5
L_0 = 1
J = []


for i in range(6):
    H = [H_0]
    for t in range(T):
        H = H + [H[t]+(δ*H[t]**φ)*(L_0**λ)*np.exp(λ*n[i]*t)]
    J = J + [H]

J_df = pd.DataFrame(J)

Jones_df = J_df.transpose()

Jones_df.columns=nlabel

np.log(Jones_df).plot()

plt.ylabel('Log Ideas Stock')
plt.xlabel('Years')
plt.title('Log of Ideas Stock for n Between 0 and 6% per Year')
plt.show()
```



What is going on here? We can see from the constancy of the slopes on the right hand side of this log grapch that the ideas stock $ H $ is heading for some steady-state growth rate that is higher the higher is the rate of growth of the STEM labor force. And for that convergence to a constant growth rate to happen, in the long run the increase in the effective STEM labor force $ L_{STEM}^{\lambda} $ has to be exactly offset by diminishing returns to innovative effort $ \delta H^{\phi - 1} $. Thus along the ideas-stock steady-state balanced-growth path:

> $ \lambda \frac{1}{L_{stem}} \frac{dL_{stem}}{dt} = (1 - \phi) \frac{dH/dt}{H} $
>
> $ \lambda n_{stem} = (1 - \phi) h^* $

The level of ideas $ H^* $ at which that growth rate $ h^* $ would be attained is characterized by:

> $ \frac{\lambda n_{stem}}{1 - \phi} = \pi L_{stem}^{\lambda} (H^*)^{\phi-1} $

So $ H $ grows more rapidly than $ h^* $ until it closes in on the value of $ H^* $ that is on the steady-state balanced-growth path:

> $ H^* = $ $ \left(\frac{\pi (1 - \phi)}{\lambda} \right)^{1/(1-\phi)} $ $ \left(\frac{1}{n_{stem}} \right)^{1/(1-\phi)} $ $ L_{stem}^{\lambda/(1-\phi)} $

And then the growth rate in the steady-state is characterized by:

> $ h^* = \left(\frac{\lambda}{1 - \phi} \right) n_{stem} $

Thus the rate of growth of the ideas stock along the steady-state balanced-growth growth path will be proportional to the rate of growth of the STEM labor force with constant of proportionality $ \lambda / (1 - \phi) $ (the degree to which more researchers step on one anothers' toes divided by how important it is that the low-hanging innovation fruit has already been picked), and the level of the ideas stock along the steady-state growth path will vary inversely with the rate of growth $ n$ of the STEM labor force raised to the power $ {1/(1-\phi)} $, and directly with the level $ L_{stem} $ of the STEM labor force raised to the power $ {\lambda/(1-\phi)} $.

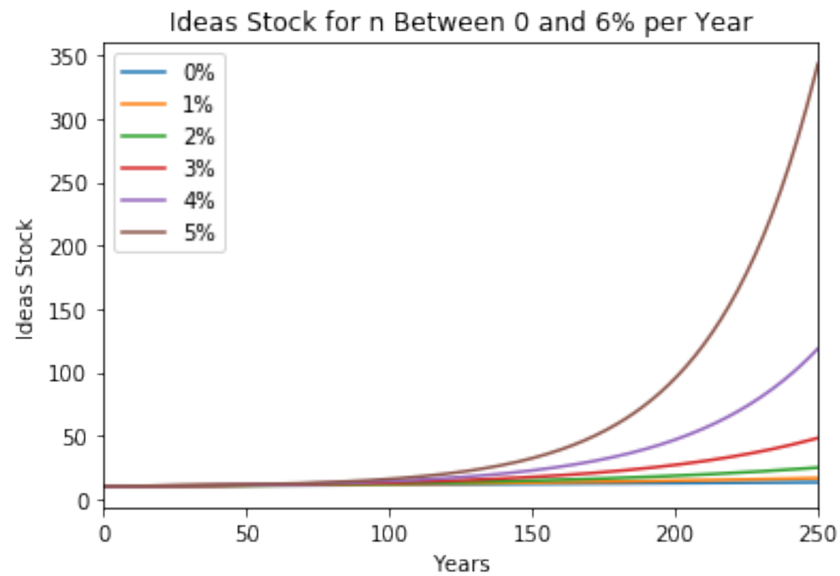
If $ H < H^* $, then $ H $ is growing faster than $ h^* $, and the scale variable $ H L_{stem}^{-\lambda/(1-\phi)} $ is rising. When $ H = H^* $, then $ h = h^* $, and the scale variable is then constant:

> $ H L_{stem}^{-\lambda/(1-\phi)} = \left(\pi (1 - \phi)/\lambda \right)^{1/(1-\phi)} n_{stem}^{-1/(1-\phi)} $

And, indeed, looking at the levels of the ideas stock over 150 years reveals, first, initial superexponential growth; that growth rate then declines until the growth rate asymptotes (for $ n > 0 $) to merely exponential growth at the rate $ h^* = \lambda n_{stem} / (1 - \phi) $:

In [51]:

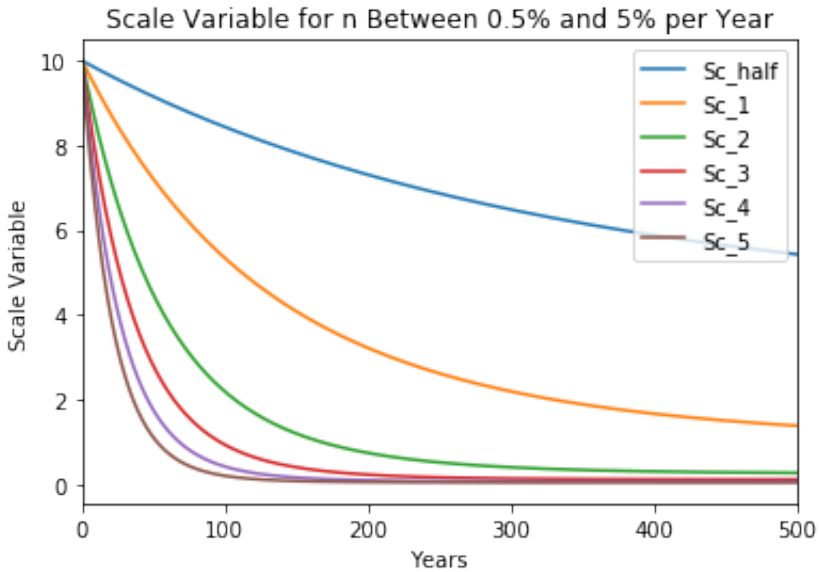
```
# watching how H evolves for different values
# of STEM labor force growth, from 0 to 5%
# per year

import numpy as np
import pandas as pd
import matplotlib.pyplot as plt
%matplotlib inline

T = 250
π = 0.01
H_0 = 10
n_stem = [0, 0.01, 0.02, 0.03, 0.04, 0.05]
nlabel = ['0%', '1%', '2%', '3%', '4%', '5%']
φ = 0.1
λ = 0.5
L_0 = 1
J = []


for i in range(6):
    H = [H_0]
    for t in range(T):
        H = H + [H[t]+(π*H[t]**φ)*(L_0**λ)*np.exp(λ*n_{stem}[i]*t)]
    J = J + [H]

J_df = pd.DataFrame(J)

Jones_df = J_df.transpose()

Jones_df.columns=nlabel

Jones_df.plot()

plt.ylabel('Ideas Stock')
plt.xlabel('Years')
plt.title('Ideas Stock for n Between 0 and 6% per Year')
plt.show()
```



How fast does this Jones model converge to its steady-state balanced-growth path with its constant rate of increase h^* in the ideas stock? To understand this, we need to look at our scale variable $H L_{stem}^{-\lambda/(1-\phi)}$:

In [53]:

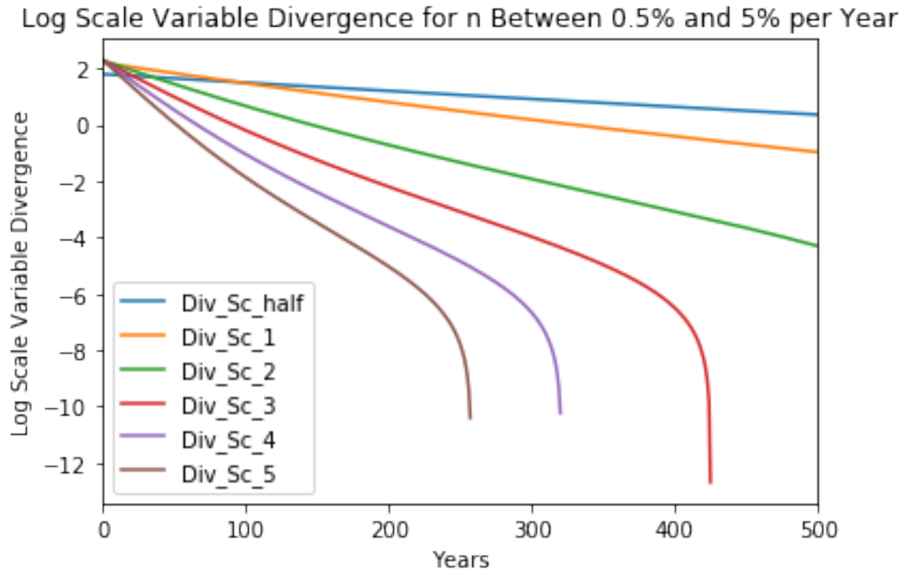
```
# watching how the scale variable evolves for different values
# of STEM labor force growth, from 0 to 5% per year

import numpy as np
import pandas as pd
import matplotlib.pyplot as plt
%matplotlib inline

T = 500
π = 0.01
H_0 = 10
n_{stem} = [0.005, 0.01, 0.02, 0.03, 0.04, 0.05]
nlabel = ['H_half', 'L_half', 'H_1', 'L_1', 'H_2', 'L_2', 'H_3', 'L_3', 'H_4'
φ = 0.5
λ = 0.5
L_0 = 1
J = []


for i in range(6):
    H = [H_0]
    L = [L_0]
    for t in range(T):
        H = H + [H[t]+(π*H[t]**φ*(L_0**λ)*(np.exp(λ*n_{stem}[i]*t)))]
        L = L +[L[t]*np.exp(n[i])]
    J = J + [H]
    J = J + [L]

J_df = pd.DataFrame(J)

Jones_df = J_df.transpose()

Jones_df.columns=nlabel

Jones_df['Sc_half'] = Jones_df.H_half*(Jones_df.L_half**(-λ/(1-φ)))
Jones_df['Sc_1'] = Jones_df.H_1*(Jones_df.L_1**(-λ/(1-φ)))
Jones_df['Sc_2'] = Jones_df.H_2*(Jones_df.L_2**(-λ/(1-φ)))
Jones_df['Sc_3'] = Jones_df.H_3*(Jones_df.L_3**(-λ/(1-φ)))
Jones_df['Sc_4'] = Jones_df.H_4*(Jones_df.L_4**(-λ/(1-φ)))
Jones_df['Sc_5'] = Jones_df.H_5*(Jones_df.L_5**(-λ/(1-φ)))

sclabel = ['Sc_half', 'Sc_1', 'Sc_2', 'Sc_3', 'Sc_4', 'Sc_5']

Jones_df[sclabel].plot()

plt.ylabel('Scale Variable')
plt.xlabel('Years')
plt.title('Scale Variable for n Between 0.5% and 5% per Year')
plt.show()

plt.show()
```



In [55]:

```
Jones_df['Div_Sc_half'] = Jones_df['Sc_half'] - 4
Jones_df['Div_Sc_1'] = Jones_df['Sc_1'] - 1
Jones_df['Div_Sc_2'] = Jones_df['Sc_2'] - 1/4
Jones_df['Div_Sc_3'] = Jones_df['Sc_3'] - 1/9
Jones_df['Div_Sc_4'] = Jones_df['Sc_4'] - 1/16
Jones_df['Div_Sc_5'] = Jones_df['Sc_5'] - 1/25

asymplabel = ['Div_Sc_half', 'Div_Sc_1', 'Div_Sc_2', 'Div_Sc_3', 'Div_Sc_4',

np.log(Jones_df[asymplabel]).plot()

plt.ylabel('Log Scale Variable Divergence')
plt.xlabel('Years')
plt.title('Log Scale Variable Divergence for n Between 0.5% and 5% per Year')

plt.show()
```

```
/Users/delong1/opt/anaconda3/lib/python3.7/site-packages/ipykernel_launcher.py
:10: RuntimeWarning: invalid value encountered in log
  # Remove the CWD from sys.path while we load stuff.
```



MEMO: Start with the Jones Rather than the Kremer Equation

We can also start with the Jones model dynamic equations:

> $ \frac{dH_t}{dt} = \pi H_t^{\phi} L^{\lambda} $

And consider how they would operate back in the Malthusian régime, in which the workforce is near subsistence and increases with ideas as:

> $ L = c_1 H^\gamma $
>
> $ \frac{dH}{dt} = \pi_{mal} c_1 H^{\phi + \gamma \lambda} $

After the Malthusian régime ends, if the population were to then become constant, we would find a shift to:

> $ \frac{dH}{dt} = \pi_{post} c_2 H^{\phi} $

Notes and Musings: Growth of the STEM Labor Force:

25 bachelor's degrees per 1000 23 year olds in 1900...

300 bachelor's degrees per 1000 23 year olds today...

60-fold multiplication in college graduates in the U.S....

20-fold multiplication in h since 1870

> $ \frac{\lambda}{1-\phi} = \frac{1}{3} $

45,000,000

Inflection points in the effective STEM workforce

- writing
- printing
- formal education

In [63]:

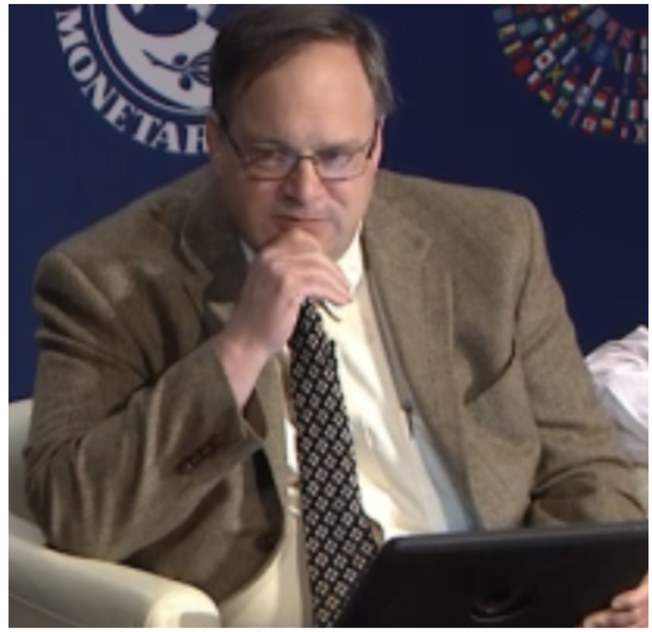
```
# MEMO: dependence of H* on φ
#

φ = 0.0
λ = 0.5
n = 0.01
L_STEM = 1
π = 0.01
H_star_list = []
φ_list = []

for i in range (9):
    φ = φ + 0.1
    H_star = (π*(1-φ)/(λ*n))**(1/(1-φ))
    H_star_list = H_star_list + [H_star]
    φ_list = φ_list + [φ]

D = [φ_list,H_star_list]
Dep_df  = pd.DataFrame(D).transpose()
deplabel = ['φ', 'H*']

Dep_df.columns=deplabel

Dep_df.set_index('φ', inplace = True)

Dep_df
```

Out[63]:

Φ	H*
0.1	1.921481e+00
0.2	1.799492e+00
0.3	1.617165e+00
0.4	1.355092e+00
0.5	1.000000e+00
0.6	5.724334e-01
0.7	1.821815e-01
0.8	1.024000e-02
0.9	1.024000e-07

Lecture Notes: What Economists Have to Say About the Rate of Ideas Growth

- Ask me two questions...
- Make two comments...
- Further reading...



<https://tinyurl.com/20181029a-delong>

---

weblog support: https://github.com/braddelong/long-form-drafts/blob/master/solow-model-6-innovation.ipynb
nbViewer: https://nbviewer.jupyter.org/github/braddelong/long-form-drafts/blob/master/solow-model-6-innovation.ipynb
datahub: http://datahub.berkeley.edu/user-redirect/interact?account=braddelong&repo=long-form-drafts&branch=master&path=solow-model-6-innovation.ipynb

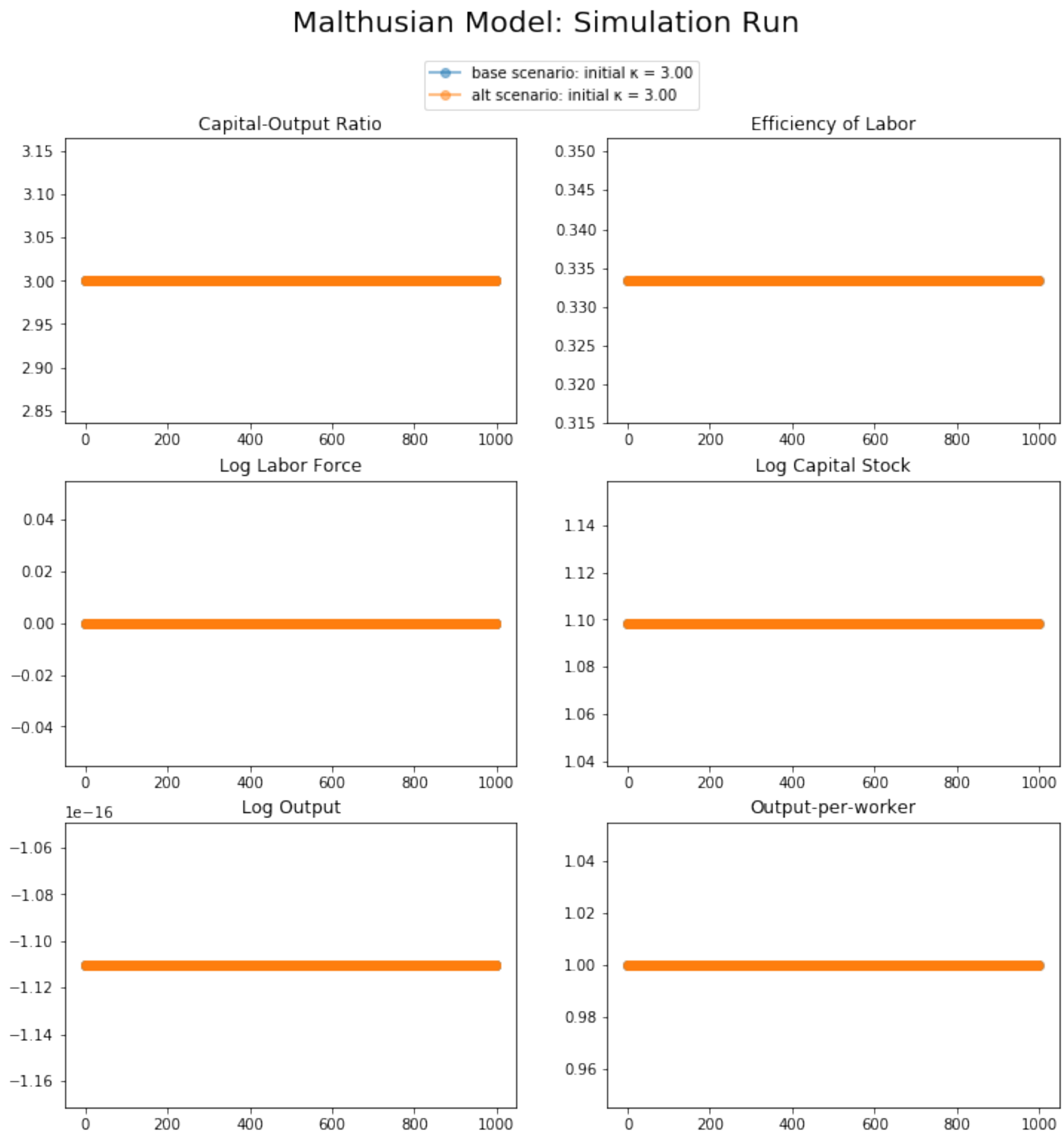
---

Memo:

https://nbviewer.jupyter.org/github/braddelong/long-form-drafts/blob/master/solow-model.ipynb?flush_cache=true
https://nbviewer.jupyter.org/github/braddelong/long-form-drafts/blob/master/solow-model-2-basics.ipynb?flush_cache=true
https://nbviewer.jupyter.org/github/braddelong/long-form-drafts/blob/master/solow-model-3-growing.ipynb?flush_cache=true
https://nbviewer.jupyter.org/github/braddelong/long-form-drafts/blob/master/solow-model-4-using.ipynb?flush_cache=true
https://nbviewer.jupyter.org/github/braddelong/long-form-drafts/blob/master/solow-model-5-pre-industrial.ipynb?flush_cache=true
https://nbviewer.jupyter.org/github/braddelong/long-form-drafts/blob/master/solow-model-6-innovation.ipynb?flush_cache=true

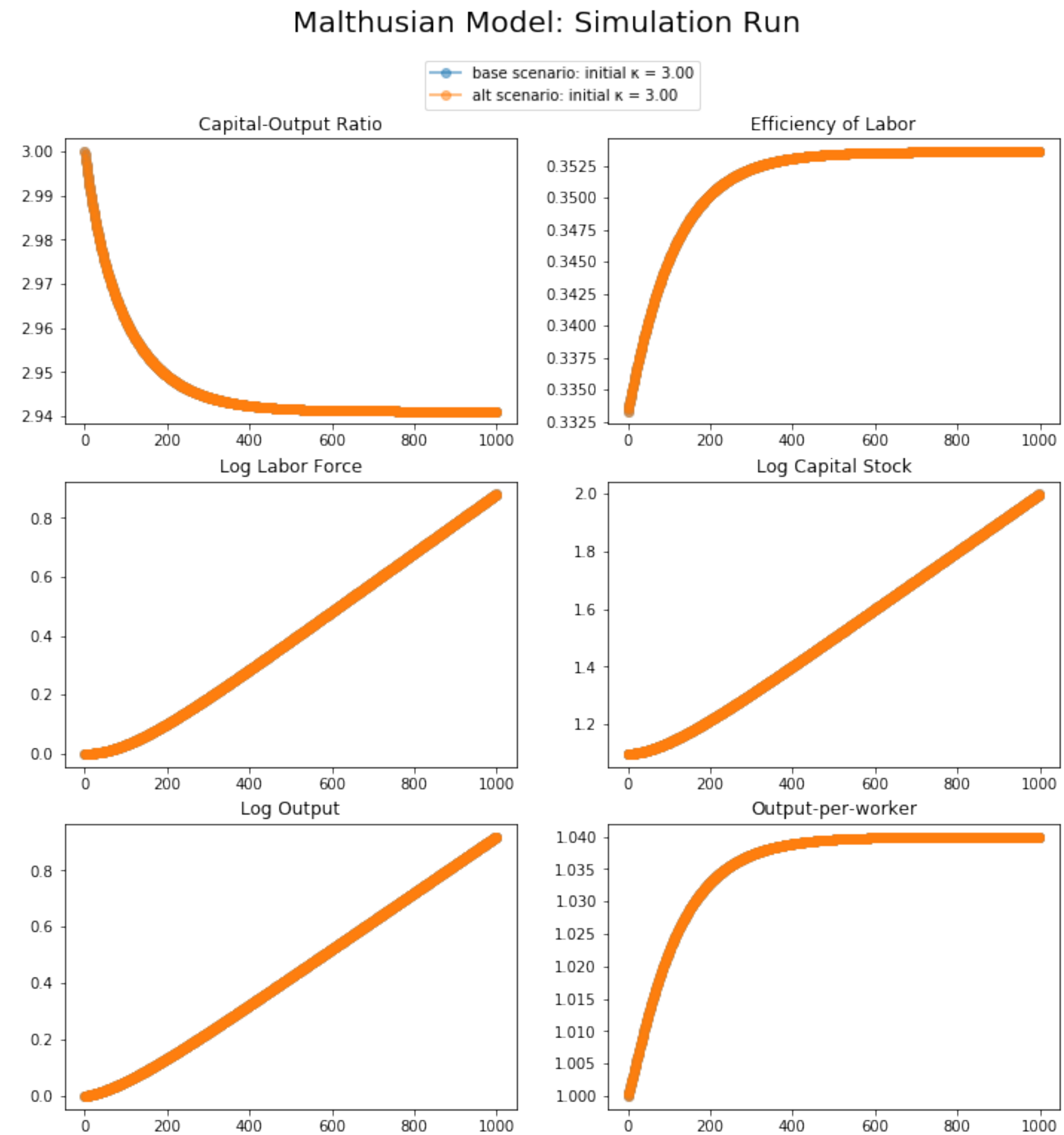
In [2]:

```
# TESTING THE MALTHUSIAN CLASS
#
# No idea stock growth (h = 0)

import numpy as np
import matplotlib.pyplot as plt
%matplotlib inline

import delong_classes

T = 1000

m_base = delong_classes.malthusian(K=3.0)
m_base.scenario = "base scenario"
m_alt = delong_classes.malthusian(K=3.0)
m_alt.scenario = "alt scenario"

figcontents = {
        (0,0):('K','Capital-Output Ratio', False),
        (0,1):('E','Efficiency of Labor', False),
        (1,0):('L','Log Labor Force', True),
        (1,1):('K','Log Capital Stock', True),
        (2,0):('Y','Log Output', True),
        (2,1):('y','Output-per-worker', False)
       }

num_rows, num_cols = 3,2
fig, axes = plt.subplots(num_rows, num_cols, figsize=(12, 12))
for i in range(num_rows):
    for j in range(num_cols):
        for m in m_base, m_alt:
            lb = f'{m.scenario}: initial K = {m.initdata["K"]:.2f}'
            axes[i,j].plot(m.gen_seq(T, var = figcontents[i,j][0], log = figc
            axes[i,j].set(title=figcontents[i,j][1])

#   global legend
axes[(0,0)].legend(loc='upper center', bbox_to_anchor=(1.1,1.3))
plt.suptitle('Malthusian Model: Simulation Run', size = 20)
plt.show()
```


Malthusian Model: Simulation Run
base scenario: initial κ = 3.00
alt scenario: initial κ = 3.00
Capital-Output Ratio
Efficiency of Labor
Log Labor Force
Log Capital Stock
Log Output
1e−16
Output-per-worker

In [3]:

```
# TESTING THE MALTHUSIAN CLASS
#
# Adding idea stock growth fast enough that
# (with salience of ideas parameter γ=2)
# population will double every 700 years
# (h=0.0005)
#
# Starting the economy at the h=0 steady-state...

import numpy as np
import matplotlib.pyplot as plt
%matplotlib inline

import delong_classes

T = 1000

m_base = delong_classes.malthusian(h=.0005)
m_base.scenario = "base scenario"
m_alt = delong_classes.malthusian(h=.0005)
m_alt.scenario = "alt scenario"

figcontents = {
        (0,0):('κ','Capital-Output Ratio', False),
        (0,1):('E','Efficiency of Labor', False),
        (1,0):('L','Log Labor Force', True),
        (1,1):('K','Log Capital Stock', True),
        (2,0):('Y','Log Output', True),
        (2,1):('y','Output-per-worker', False)
       }

num_rows, num_cols = 3,2
fig, axes = plt.subplots(num_rows, num_cols, figsize=(12, 12))
for i in range(num_rows):
    for j in range(num_cols):
        for m in m_base, m_alt:
            lb = f'{m.scenario}: initial κ = {m.initdata["κ"]:.2f}'
            axes[i,j].plot(m.gen_seq(T, var = figcontents[i,j][0], log = figc
            axes[i,j].set(title=figcontents[i,j][1])

#   global legend
axes[(0,0)].legend(loc='upper center', bbox_to_anchor=(1.1,1.3))
plt.suptitle('Malthusian Model: Simulation Run', size = 20)
plt.show()
```


Malthusian Model: Simulation Run
base scenario: initial κ = 3.00
alt scenario: initial κ = 3.00
Capital-Output Ratio
Efficiency of Labor
Log Labor Force
Log Capital Stock
Log Output
Output-per-worker