Component response rate variation drives stability in large complex systems

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The stability of a complex system generally decreases with increasing system size, as is demonstrated by random matrix theory[1,2]. This counter-intuitive result, first shown by May[1], is broadly relevant for understanding the dynamics and persistence of systems such as ecological[1,2], neurological[3,4], biochemical[5,6] and socio-economic[7–9] networks. Much attention has especially been given to the stability of ecological communities such as food webs or mutualist networks, with recent work investigating how different community structures affect stability[2,10–14]. But more broadly, stabilising mechanisms in complex systems remain under-developed, and the effect of variation in the response rate of individual system components remains an open problem[15]. Here I show that when components of a complex system respond to system perturbation at different rates (γ), the potential for system stability is markedly increased. Stability is caused by the clustering of some eigenvalues toward the centre of eigenvalue distributions despite the destabilising effect of higher interaction strength variation (σ^2). This effect of variation in γ becomes increasingly important as system size increases, to the extent that the largest stable complex systems would otherwise be unstable if not for $Var(\gamma)$. My results therefore reveal a previously unconsidered driver of system stability that is likely to be pervasive across all complex systems. Future research in complex systems should therefore account for the varying response rates of individual system components when assessing whole system stability.

In 1972, May[1] first demonstrated that randomly assembled systems of sufficient complexity are almost inevitably unstable given infinitesimally small perturbations. Complexity in this case is defined by the size of the system (i.e., the number of potentially interacting components; S), its connectance (i.e., the probability that one component will interact with another; C), and the variance of interaction strengths (σ^2)[2]. May's finding that the probability of local stability falls to near zero given a sufficiently high threshold of $\sigma\sqrt{SC}$ has profound consequences across multiple disciplines, raising the question of how complex systems in, e.g., ecology[2,10,13,15] or banking[7,9,16] are predicted to persist or change.

Randomly assembled complex systems can be represented as large square matrices (M) with S components (e.g., species[2] or banks[7]). One element of such a matrix, M_{ij}, defines how component j affects component i in the system at a point of equilibrium[2]. Off-diagonal elements ($i \neq j$) therefore define interactions between components, while diagonal elements ($i = j$) define component self-regulation (e.g., carrying capacity in ecological communities). Traditionally, off-diagonal elements are assigned non-zero values with a probability C, which are sampled from a distribution with variance σ^2; diagonal elements are set to -1[1,2,15]. Local system stability is assessed using eigenanalysis, with the system being stable if the real parts of all eigenvalues (λ) of M are negative $(\max(\Re(\lambda)) < 0)$[1,2]. In a large system (high S), eigenvalues are distributed uniformly[17] within a circle centred at $\Re = -1$ (the mean value of diagonal elements) and $\Im = 0$, with a radius of $\sigma\sqrt{SC}$[1,2,15] (Figs 1a and 2a). Local stability of randomly assembled systems therefore becomes increasingly unlikely as S, C, and σ^2 increase.

May's[1,2] stability criterion $\sigma\sqrt{SC} < 1$ assumes that individual components respond to perturbations of the system at the same rate (γ), but this is highly unlikely in any complex system. In ecological communities, for example, the rate at which population density changes following perturbation will depend on the generation time of individuals, which might vary by orders of magnitude among species. Species with short generation times will respond quickly (high γ) to perturbations relative to species with long generation times (low γ). Similarly, the speed at which individual banks respond to perturbations in financial networks, or individuals or institutions respond to perturbations in complex social networks, is likely to vary. The effect of such variance has not been investigated in complex systems theory. Intuitively, variation in γ might be expected

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⁴⁹ to decrease system stability by introducing a new source of variation into the system and thereby increasing

⁵⁰ σ. Here I show why, despite higher σ, complex systems in which γ varies are actually more likely to be stable,

⁵¹ especially when S is high.

⁵² Rows in M define how a given component i is affected by other components of the system, meaning that

⁵³ the rate of component response time can be modelled by multiplying all row elements by a real scalar value

⁵⁴ γ_i[14] (see Supplementary Information for details). The distribution of γ over S components thereby models

⁵⁵ the distribution of component response rates. An instructive example compares one M where $\gamma_i = 1$ for all

⁵⁶ i in S to the same M when half of $\gamma_i = 1.95$ and half of $\gamma_i = 0.05$. This models one system in which γ is

⁵⁷ invariant and one in which γ varies, but systems are otherwise identical (note $E[\gamma_i] = 1$ in both cases). I

⁵⁸ assume $S = 200$, $C = 0.05$, and $\sigma = 0.4$; diagonal elements are set to -1 and non-zero off-diagonal elements

⁵⁹ are drawn randomly from $\mathcal{N}(0, \sigma^2)$. Rows are then multiplied by γ_i to generate M. When $\gamma_i = 1$, eigenvalues

⁶⁰ of M are distributed uniformly within a circle centred at $(-1, 0)$ with a radius of 1.265 (Fig. 1a). Hence, the

⁶¹ real components of eigenvalues are highly unlikely to all be negative when all $\gamma_i = 1$. But when γ_i values are

⁶² separated into two groups, eigenvalues are no longer uniformly distributed (Fig. 1b). Instead, two distinct

⁶³ clusters of eigenvalues appear (grey circles in Fig. 1b), one centred at $(-1.95, 0)$ and the other centred at

⁶⁴ $(-0.05, 0)$. The former has a large radius, but the real components have shifted to the left (in comparison to

⁶⁵ when $\gamma = 1$) and all $\Re(\lambda) < 0$. The latter cluster has real components that have shifted to the right, but has a

⁶⁶ smaller radius. Overall, for 1 million randomly assembled M, this division between slow and fast component

⁶⁷ response rates results in more stable systems: 1 stable given $\gamma = 1$ versus 32 stable given $\gamma = \{1.95, 0.5\}$.

⁶⁸ Higher stability in systems with variation in γ can be observed by sampling γ_i values from various distributions.

⁶⁹ I now focus on a uniform distribution where $\gamma \sim \mathcal{U}(0, 2)$ (see Supplementary Information for other distributions,

⁷⁰ which give similar results). As with the case of $\gamma = \{1.95, 0.5\}$ (Fig. 1b), $E[\gamma] = 1$ when $\gamma \sim \mathcal{U}(0, 2)$, allowing

⁷¹ comparison of M before and after variation in component response rate. Figure 2 shows a comparison of

⁷² eigenvalue distributions given $S = 1000$, $C = 1$, and $\sigma = 0.4$. As expected[17], when $\gamma = 1$, eigenvalues are

⁷³ distributed uniformly in a circle centred at $(-1, 0)$ with a radius of $\sigma\sqrt{SC} = 12.649$. Uniform variation in

⁷⁴ γ leads to a non-uniform distribution of eigenvalues, some of which are clustered tightly around the centre

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75 of the distribution, but others of which are spread outside the former radius of 12.649 (red circle Fig 2b).

76 This larger radius occurs because the addition of $Var(\gamma)$ increases the realised σ of M. The clustering and

77 spreading of eigenvalues introduced by $Var(\gamma)$ can destabilise previously stable systems or stabilise systems

78 that are otherwise unstable. But where systems are otherwise too complex to be stable given $\gamma = 1$, the effect

79 of $Var(\gamma)$ can often lead to stability above May's[1,2] threshold of $\sigma\sqrt{SC} > 1$.

80 To investigate the effect of $Var(\gamma)$ on system stability, I simulated random M matrices at $\sigma = 0.4$ and $C = 1$

81 across S ranging from 2-32. One million M were simulated for each S, and the stability of M was assessed

82 given $\gamma = 1$ versus $\gamma \sim \mathcal{U}(0, 2)$. I found that the number of stable random systems was consistently higher

83 given $Var(\gamma)$ than when $\gamma = 1$ (Fig. 3), and that the difference between the probabilities of observing a stable

84 system increased with an increase in S; i.e., the potential for $Var(\gamma)$ to drive stability increased with system

85 complexity. For the highest values of S, nearly all systems that were stable given $Var(\gamma)$ would not have been

86 stable given $\gamma = 1$ (see Supplementary Information for full results). This suggests that the stability of large

87 systems might be dependent upon variation in the response rate of their individual components, meaning

88 that factors such as generation time (in ecological networks), transaction speed (in economic networks), or

89 communication speed (in social networks) needs to be considered when investigating the stability of complex

90 systems.

91 It is important to point out that $Var(\gamma)$ is not stabilising per se; that is, adding variation in γ to a particular

92 system M does not necessarily increase the probability that the system will be stable (see Supplementary

93 Information). Rather, systems that are observed to be stable are more likely to vary in γ, and for this $Var(\gamma)$

94 to be critical to their stability. This is caused by the shift in the distribution of eigenvalues that occurs by

95 introducing $Var(\gamma)$ (Fig. 1b, 2b), which can sometimes result in all $\Re(\lambda) < 0$ but might also increase $\Re(\lambda)$

96 values.

97 To further investigate the potential of $Var(\gamma)$ to be stabilising, I used a genetic algorithm (the space of

98 possible γ values was too large to search exhaustively[18]; see Supplementary Information). For each of 40000

99 random M ($\sigma = 0.4$, $C = 1$), the genetic algorithm initialised 1000 different sets of $\gamma \sim \mathcal{U}(0, 2)$ values

100 of size S. Eigenanalysis was performed on M using each set of γ values, and the 20 sets with the lowest

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101 max $(\Re(\lambda))$ each produced 50 clonal offspring with subsequent mutation and crossover between the resulting

102 new population of 1000 γ sets. The genetic algorithm terminated if a stable M was found, 20 generations

103 occurred, or a convergence criteria of minimum fitness increase between generations was satisfied. Across

104 $S = \{2, 3, ..., 39, 40\}$, sets of γ values were found that resulted in stable systems with probabilities that were

105 orders of magnitude higher than when $\gamma = 1$ (see Supplementary Information), meaning that stability could

106 often be achieved by manipulating S γ values rather than $S \times S$ M elements. Hence, managing the response

107 rates of system components in a targetted way can potentially facilitate the stabilisation of complex systems

108 through a reduction in dimensionality.

109 I have focused broadly on random complex systems, but it is also worthwhile to consider more restricted

110 interactions such as those of specific ecological networks[2]. These include systems in which all interactions

111 are negative (competitive networks), positive (mutualist networks), or i and j pairs have opposing signs

112 (predator-prey networks). In general, competitive and mutualist networks tend to be destabilising, and

113 predator-prey network tend to be stabilising[19]. When $Var(\gamma)$ is applied to each, the proportion of stable

114 competitive and predator-prey networks increases, but the proportion of stable mutualist networks does

115 not (see Supplementary Information). Additionally, when each component of M is interpreted as a unique

116 species and given a random intrinsic growth rate[20], feasibility is not increased by $Var(\gamma)$, suggesting that

117 variation in species generation time might be unlikely to drive stability in purely multi-species networks (see

118 Supplementary Information).

119 My results show that complex systems are more likely to be stable when the response rates of system

120 components vary. These results are broadly applicable to complex biological and social networks.

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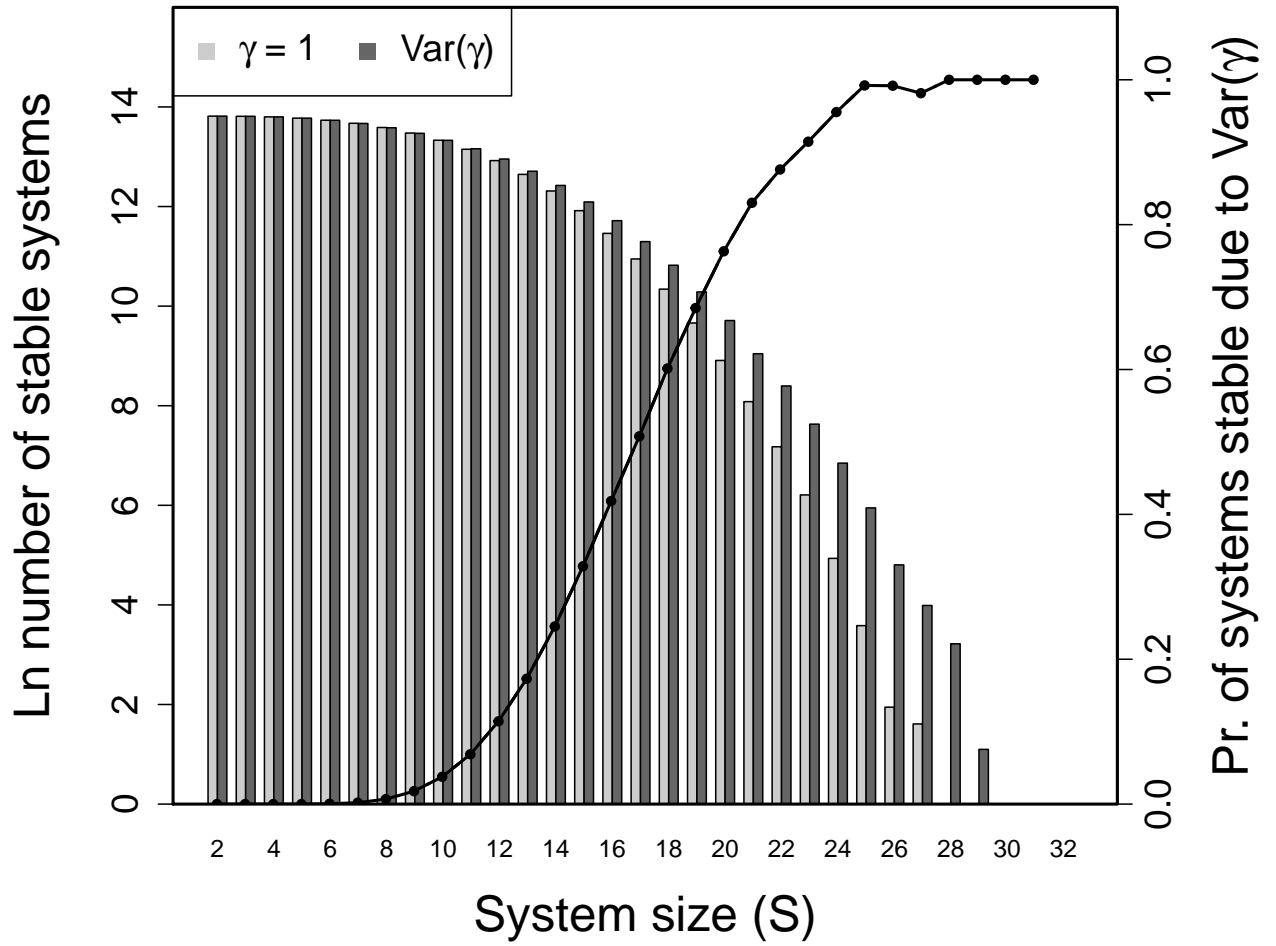
Figure 1: Example distribution of eigenvalues before (a) and after (b) separating a randomly generated complex system into fast ($\gamma = 1.95$) and slow ($\gamma = 0.05$) component response rates. Each panel shows the same system where $S = 200$, $C = 0.05$, and $\sigma = 0.4$, and in each case $E[\gamma] = 1$ (i.e., only the distribution of γ differs between panels). **a.** Eigenvalues plotted when all $\gamma = 1$; distributions of points are uniformly distributed within the grey circle with a radius of $\sigma\sqrt{SC} = 1.238$ centred at -1 on the real axis. **b.** Eigenvalues plotted when half $\gamma = 1.95$ and half $\gamma = 0.05$; distributions of points can be partitioned into one large circle of $\sigma\sqrt{SC} = 1.718$ centred at $\gamma = -1.95$ and one small circle of $\sigma\sqrt{SC} = 0.044$ centred at $\gamma = -0.05$. In a, the maximum real eigenvalue $\max(\Re(\lambda)) = 0.2344871$, while in b $\max(\Re(\lambda)) =$ -0.0002273135, meaning that the complex system in b but not a is stable because in b $\max(\Re(\lambda)) < 0$. In 1 million randomly generated complex systems under the same parameter values, 1 was stable when $\gamma = 1$ while 32 were stable when $\gamma = \{1.95, 0.05\}$. Overall, complex systems that are separated into fast versus slow components tend to be more stable than otherwise identical systems with identical component response rates.
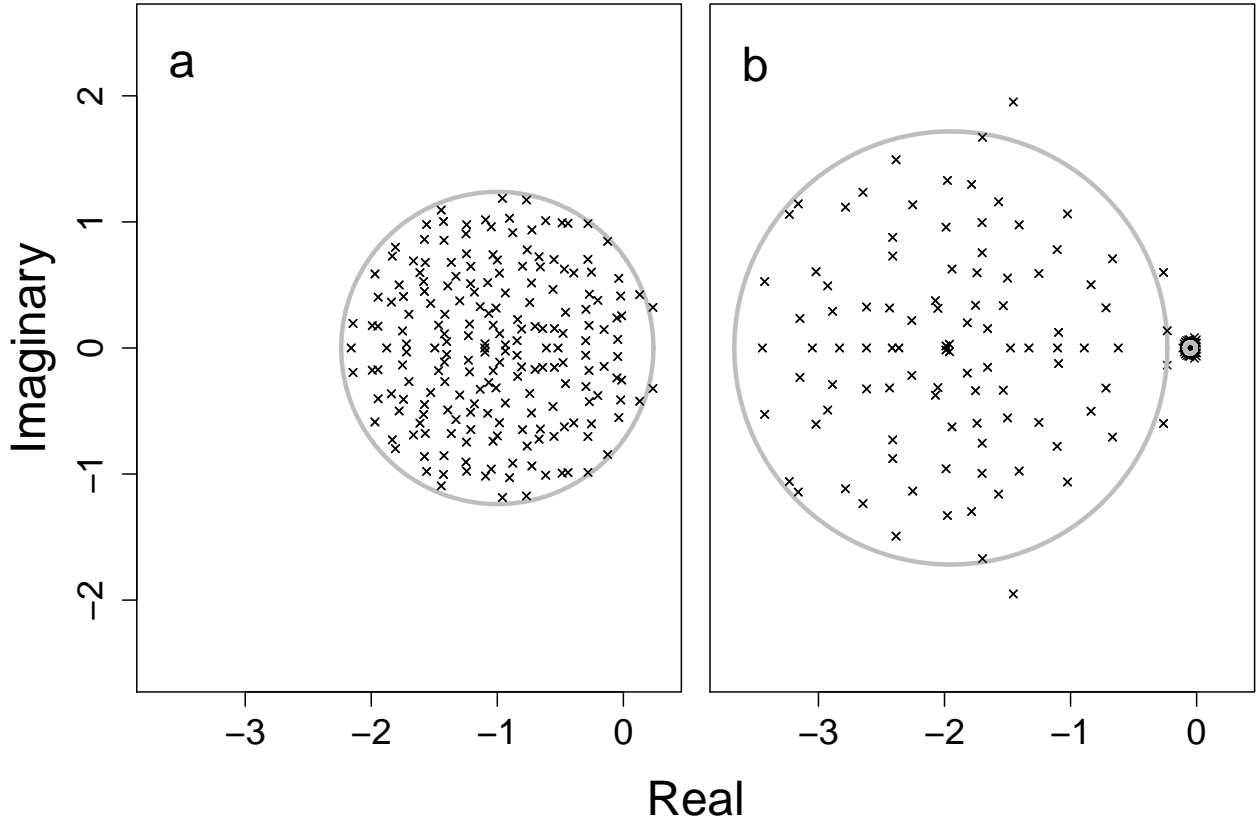
**Figure 2: Distributions of eigenvalues before (a) and after (b) introducing variation in com-
ponent response rate (γ) in complex systems.** Each panel show the same system where $S = 1000$,
$C = 1$, and $\sigma = 0.4$. **a.** Eigenvalues plotted in the absence of $Var(\gamma)$ where $E[\gamma] = 1$, versus **b.** eigenvalues
plotted given $\gamma \sim \mathcal{U}(0, 2)$, which increases the variance of interaction strengths (σ^2) but clusters eigenvalues
toward the distribution's centre (-1, 0). Black and red elipses in both panels show the circle centred on the
distribution in panels a and b, respectively, which have a radius of $\sigma\sqrt{SC}$. Proportions of $\Re(\lambda) < 0$ are 0.549
and 0.557 for a and b, respectively.

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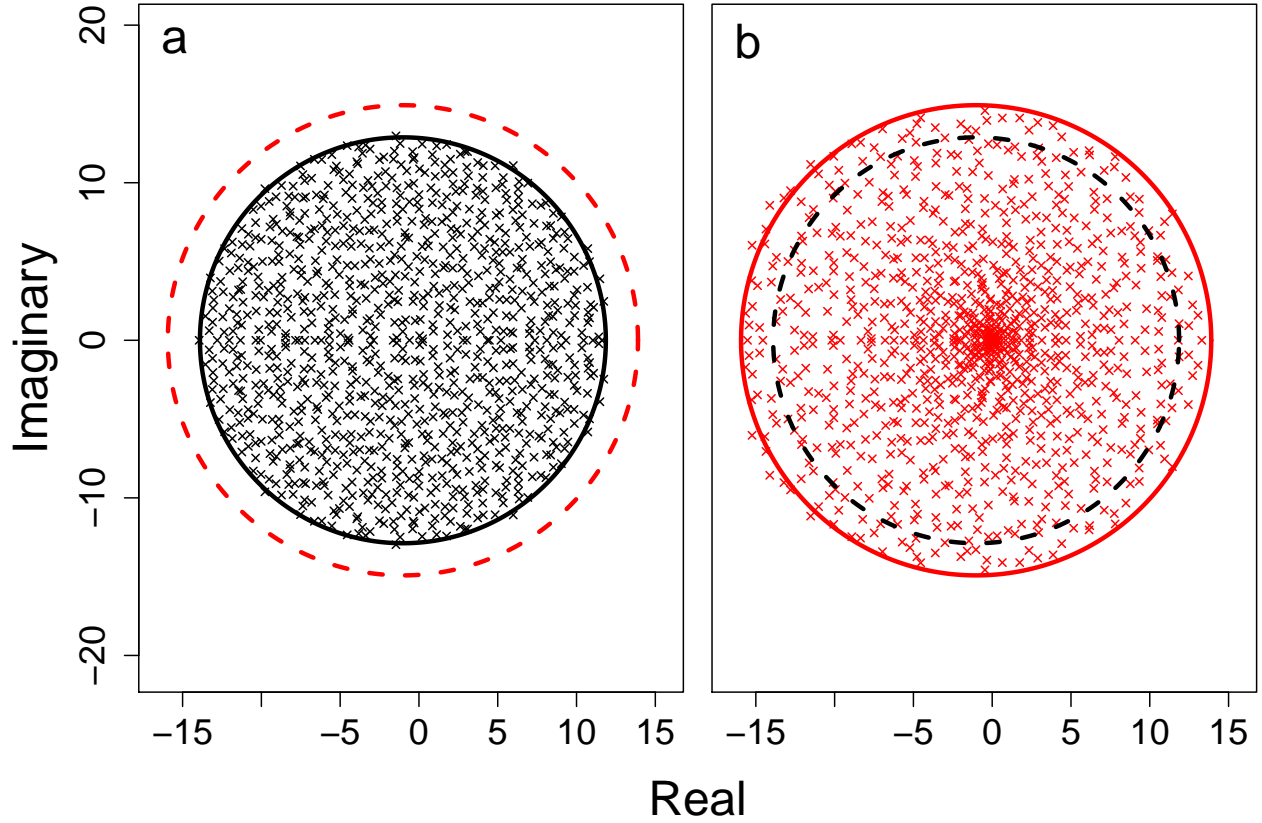
Figure 3: Stability of large complex systems with and without variation in component response rate(γ). The ln number of systems that are stable across different system sizes (S) given $C = 1$, and the proportion of systems in which variation in γ is critical for system stability. For each S, 1 million complex systems are randomly generated. Stability of each complex system is tested given variation in γ by randomly sampling $\gamma \sim \mathcal{U}(0, 2)$. Stability given $Var(\gamma)$ is then compared to stability in an otherwise identical system in which $\gamma = E[\mathcal{U}(0, 2)]$ for all components. Light and dark grey bars show the number of stable systems in the absence and presence of $Var(\gamma)$, respectively. The black line shows the proportion of systems that are stable when $Var(\gamma) > 0$, but would be unstable if $Var(\gamma) = 0$.