Component response rate variation underlies the stability of complex systems (revision notes)

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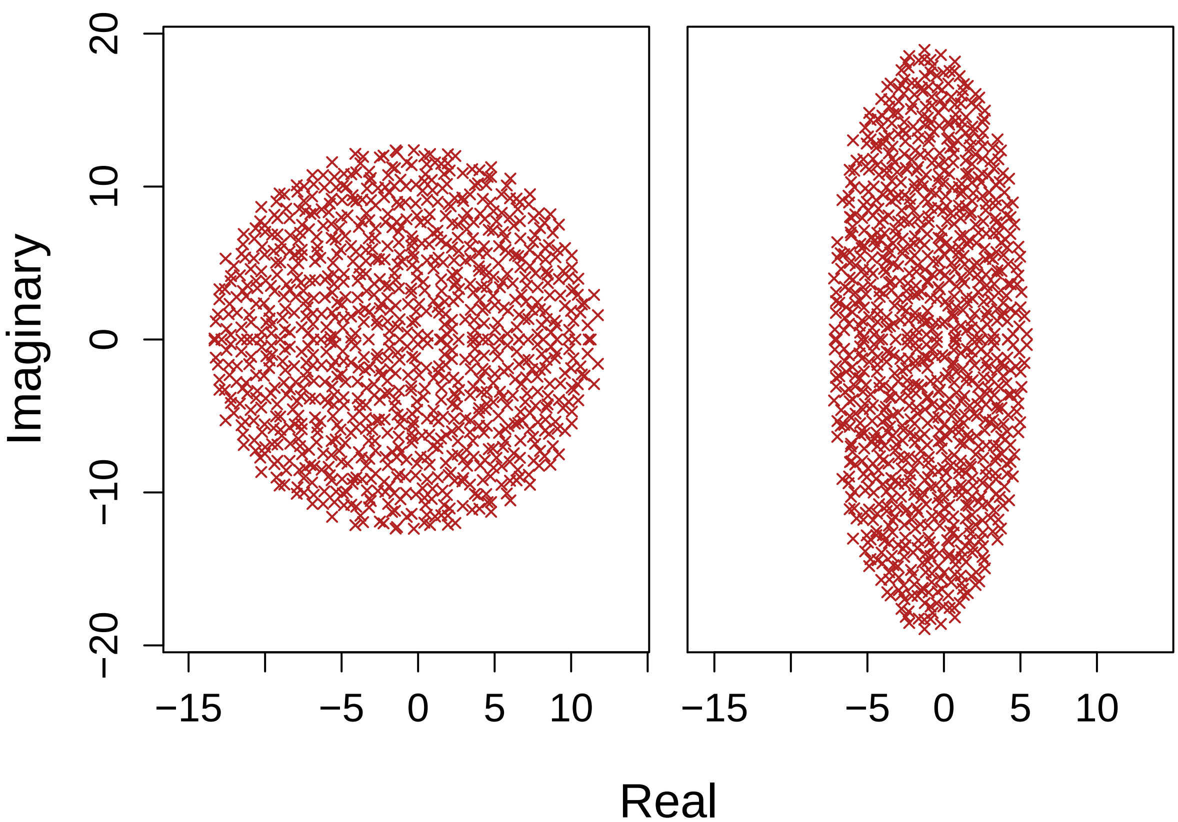
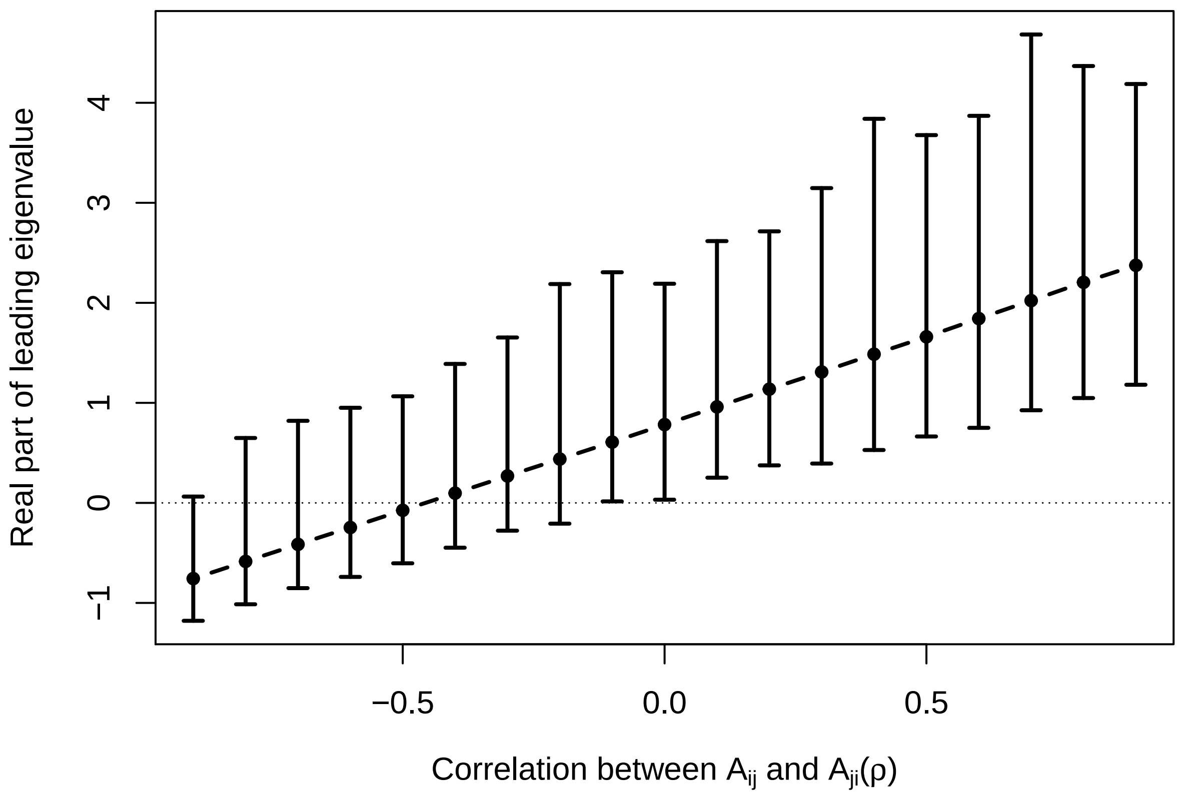
Role of correlated matrices in stabilisation

In complex systems represented by large random matrices, correlation between matrix elements A_{ij} and A_{ji} affects the distribution of eigenvalues and therefore local stability. As the correlation between matrix elements (ρ) decreases, the eigenvalue spectra changes such that more variation falls along the imaginary axis. The figure panels below compare a random matrix in which $\rho = 0$ (left) to one in which $\rho =$ -0.5 (right). In both panels, complex systems include $S = 1000$ components, with diagonal elements of -1 and off-diagonal elements drawn from a normal distribution with a mean of $\mu = 0$ and standard deviation of $\sigma = 0.4$.

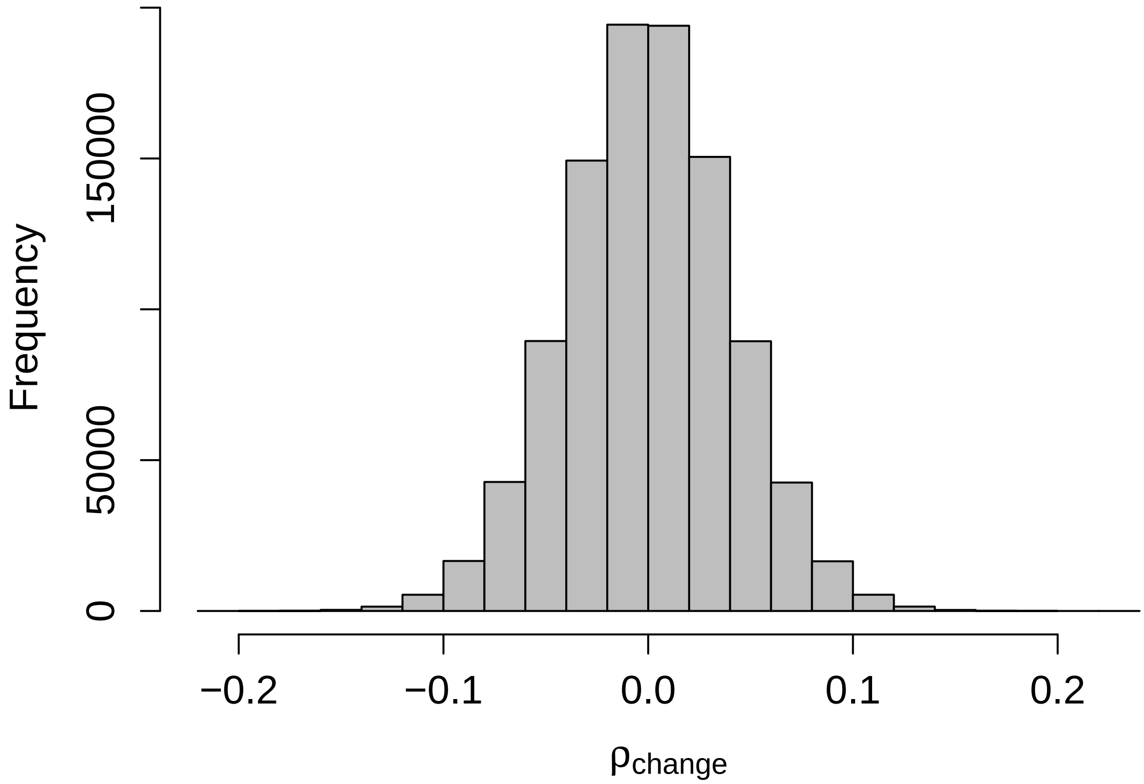


Because of this effect of ρ on the eigenvalue spectra, decreasing values of ρ will also tend to decrease the rightmost eigenvalue of the matrix $\mathbf{A}$. This makes it more likely that the complex system represented by $\mathbf{A}$ is locally stable, as stability occurs when all real parts of eigenvalues are negative. Note that this elongation along the imaginary axis is also characteristic of predator-prey communities (in which, by definition A_{ij} and A_{ji} have opposing signs). Also note that as ρ increases such that $\rho > 0$, the same elongation happens along the real axis, making random complex systems less likely to be stable.

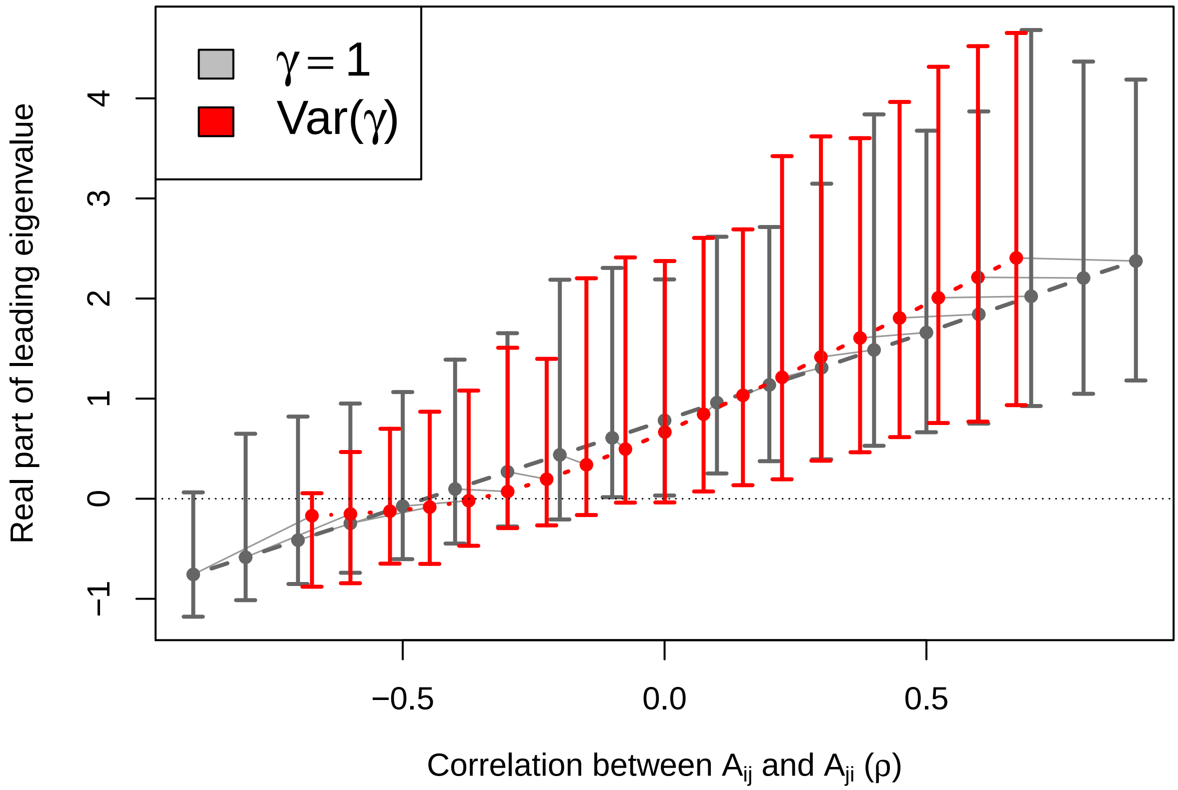
A simple numerical analysis illustrates the linear relationship between ρ and the expected value of the real part of the leading eigenvalue, $\max(\Re(\lambda))$. Below, I have run 10000 simulations across values of ρ from -0.9 to 0.9 for complex systems with $S = 25$ components. Error bars show 95% bootstrapped confidence intervals.



In the main text, I demonstrated that when S is finite but system complexity $\sigma\sqrt{SC}$ is high (C defines the connectance of $\mathbf{A}$, or the proportion of non-zero off-diagonal elements), variation in component response rate γ often underlies system stability. In other words, highly complex systems that are observed to be stable would typically not be if we removed the variation in their component response rates. Mathematically, this means that by multiplying $\mathbf{A}$ by a diagonal matrix γ with variable elements, the sign of $\max(\Re(\lambda))$ is sometimes flipped from positive to negative in these finite systems of high complexity. Interestingly, this increase in stability given $Var(\gamma) > 0$ is not necessarily caused by γ decreasing ρ. In fact, for $S = 25$, $\sigma = 0.4$, and $C = 1$, random complex systems that are stabilised by γ typically have increased ρ values. Note that for these parameter values, 1 million simulations found $\mathbf{M} = \gamma\mathbf{A}$ to be stable for 36 systems when all $\gamma_i = 1$, but 383 systems when $\gamma_i \sim \mathcal{U}(0, 2)$. Below shows the distribution of the difference in ρ between systems with versus without $var(\gamma)$ for 1000000 stabilised systems; that is, $\rho_{change} = \rho_{var(\gamma)} - \rho_{\gamma=1}$.

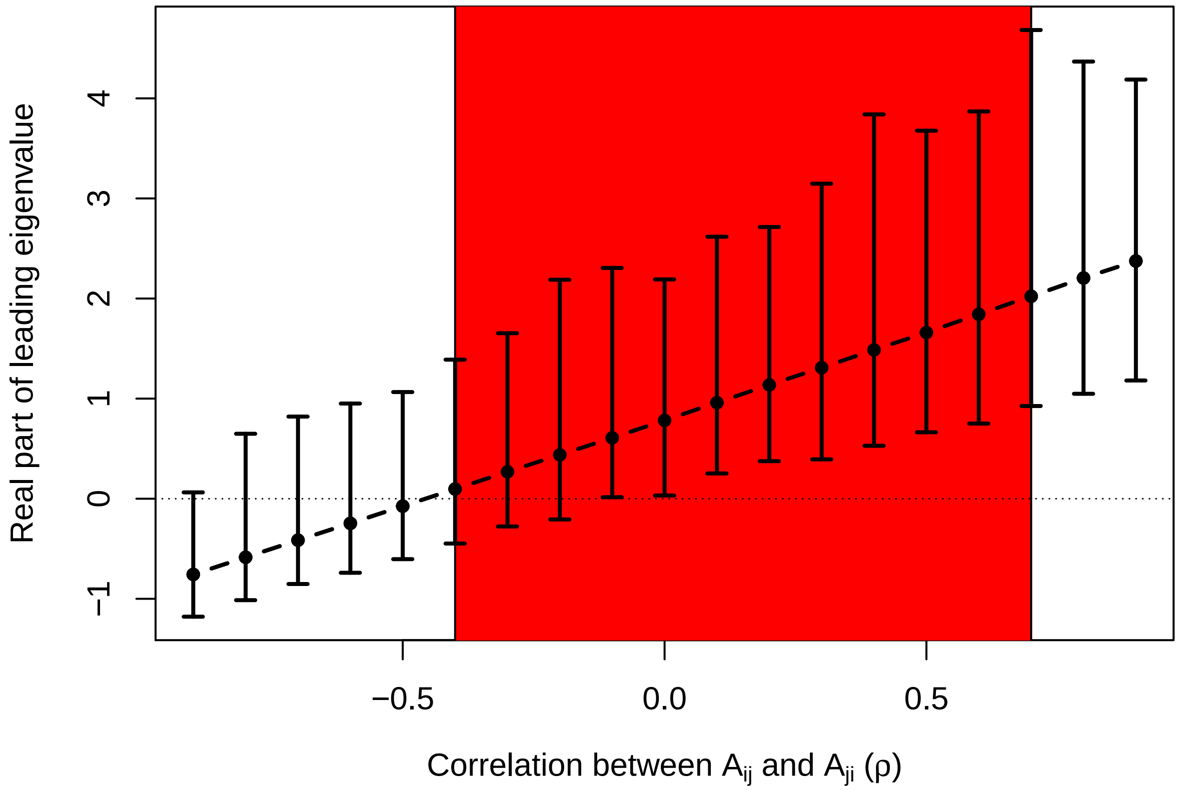


When $\mathbf{A}$ was stabilised by $Var(\gamma)$, the change in ρ was normally distributed around 4.8394472×10^{-6} (95% CIs: $-7.3124139 \times 10^{-5}$, 8.7487968×10^{-5}). Hence, decreasing the correlation between $\mathbf{A}_{ij}$ and $\mathbf{A}_{ji}$ was not by itself the cause of stability. For a clearer picture of the effect of γ, it is useful to show the relationship between ρ and $\max(\Re(\lambda))$ again as above, but this time also for how the relationship changes given $Var(\gamma)$. That is, given $\mathbf{M} = \mathbf{A}$ in addition to $\mathbf{M} = \gamma\mathbf{A}$.



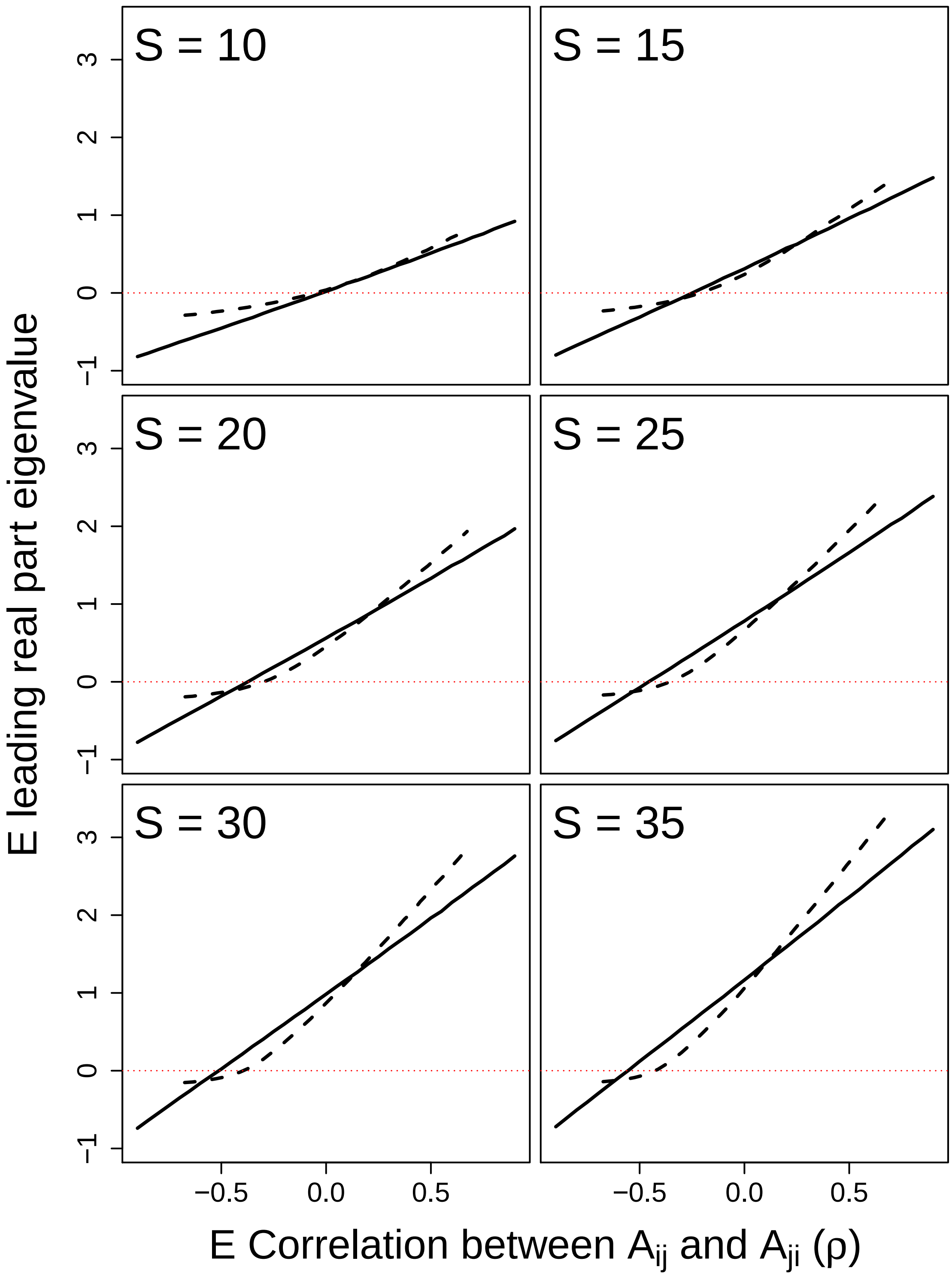
Including $Var(\gamma)$ introduces a nonlinear relationship between ρ and $\max(\Re(\lambda))$. Points along the x-axis are spaced more closely together given $Var(\gamma)$ because $Var(\gamma)$ tends to decrease the absolute magnitude of ρ. Grey and red points centred on $\rho = 0$ represent the same simulations, before (grey) and after (red) including $Var(\gamma)$. Grey and red points to the left and right show decreasing and increasing simulated ρ values, respectively. Faint grey lines connect points for the same set of simulations, and where the red point is lower than the black point, the expected $\max(\Re(\lambda))$ was lower given $Var(\gamma)$.

The region of ρ values for $\mathbf{A}$ that would result in increased stability given $Var(\gamma)$ is highlighted with red shading below.



In this red shaded region above, $\max(\Re(\lambda))$ is decreased by $Var(\gamma)$. The shaded region encompasses values of ρ between -0.4 and 0.7. But 1000000 simulated $\mathbf{A}$ had values that ranged between -0.2642846 and 0.2668648, meaning that $\max(\Re(\lambda))$ was always expected to decrease given $Var(\gamma)$ even if $Var(\gamma)$ caused ρ to increase.

The curvature of the relationship between ρ and $\max(\Re(\lambda))$ is consistent across different values of S, as shown below. Including γ always results in a concave upward relationship between ρ and $\max(\Re(\lambda))$.



In all panels above, the solid line shows the relationship between the expected ρ between $\mathbf{A}$ elements A_{ij} and A_{ji} given no variation in component response rates (i.e., the diagonal matrix equals the identity matrix, $\gamma = \mathbf{I}$, so $\mathbf{M} = \mathbf{A}$). The dotted line shows the same relationship given variation in component response rates (i.e., the diagonal matrix contains elements drawn from a random uniform distribution between 0 and 2, so $\mathbf{M} = \gamma\mathbf{A}$).

Trying to think analytically

Trying to think analytically, note that the correlation ρ adjusts the criteria for stability as follows[1],

$$\sigma\sqrt{SC}\,(1+\rho) < 1.$$

Given that diagonal values of $\mathbf{M}$ are set to -1, We can know the relationship $S = 1600$, $C = 1$, and $\sigma = 1/40$. Under these parameter values, given $\rho = 0$, we know that $\sigma\sqrt{SC}\,(1+\rho) = 1$. Hence, the expected value of $\max(\Re(\lambda)) = 0$

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## Warning in xy.coords(x, y, xlabel, ylabel, log): imaginary parts discarded
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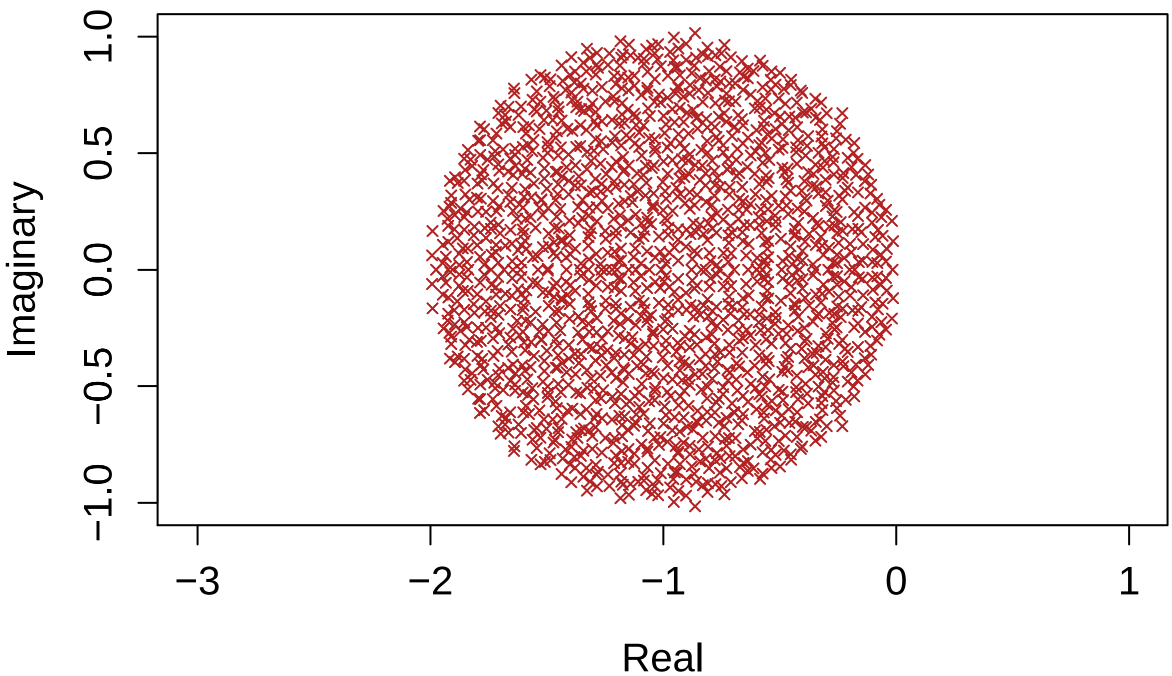

References

1. Allesina, S. & Tang, S. The stability–complexity relationship at age 40: a random matrix perspective. *Population Ecology* 63–75 (2015). doi:10.1007/s10144-014-0471-0