Component response rate variation drives stability in large complex systems

Brad Duthie

**The stability of a complex system generally decreases with increasing system size, as is demonstrated by random matrix theory (May** [**1972**](#ref-May1972)**; Allesina and Tang** [**2012**](#ref-Allesina2012)**). This counter-intuitive result, first shown by May (May** [**1972**](#ref-May1972)**), is broadly relevant for understanding the dynamics and persistence of systems such as ecological (May** [**1972**](#ref-May1972)**; Allesina and Tang** [**2012**](#ref-Allesina2012)**), neurological (Gray and Robinson** [**2008**](#ref-Gray2008)**,** [**2009**](#ref-Gray2009)**), biochemical (Rosenfeld** [**2009**](#ref-Rosenfeld2009)**; MacArthur et al.** [**2010**](#ref-MacArthur2010)**) and socio-economic (Haldane and May** [**2011**](#ref-Haldane2011)**; Suweis and D’Odorico** [**2014**](#ref-Suweis2014)**; Bardoscia et al.** [**2017**](#ref-Bardoscia2017)**) networks. Much attention has especially been given to the stability of ecological communities such as food webs or mutualist networks, with recent work investigating how different community structures affect stability (Allesina and Tang** [**2012**](#ref-Allesina2012)**,** [**2015**](#ref-Allesina2015a)**; Mougi and Kondoh** [**2012**](#ref-Mougi2012)**; Gao et al.** [**2016**](#ref-Gao2016)**; Grilli et al.** [**2017**](#ref-Grilli2017)**; Patel et al.** [**2018**](#ref-Patel2018)**). But more broadly, stabilising mechanisms in complex systems remain under-developed, and the effect of variation in the response rate of individual system components remains an open problem (Allesina et al.** [**2015**](#ref-Allesina2015)**). Here I show that when components of a complex system respond to system dynamics at different rates (), the potential for system stability is markedly increased. Stability increases due to the clustering of some eigenvalues toward the centre of eigenvalue distributions despite the destabilising effect of higher variation among interaction strengths (). This effect of variation in becomes increasingly important as system size increases, to the extent that the largest stable complex systems would otherwise be unstable if not for . My results therefore reveal a previously unconsidered driver of system stability that is likely to be pervasive across all complex systems. Future research in complex systems should therefore account for the varying response rates of individual system components when assessing whole system stability.**

In 1972, May (May [1972](#ref-May1972)) first demonstrated that randomly assembled systems of sufficient complexity are almost inevitably unstable given infinitesimally small perturbations. Complexity in this case is defined by the size of the system (i.e., the number of interacting components; ), its inter-connectivity (i.e., the probability that one component will affect another; ), and the variance of interaction strengths ()(Allesina and Tang [2012](#ref-Allesina2012)). May’s finding that the probability of local stability falls to near zero given a sufficiently high threshold of has profound consequences across multiple disciplines, raising the question of how complex systems in, e.g., ecology(Mougi and Kondoh [2012](#ref-Mougi2012); Allesina and Tang [2012](#ref-Allesina2012); Allesina et al. [2015](#ref-Allesina2015); Grilli et al. [2017](#ref-Grilli2017)) or banking (May et al. [2008](#ref-May2008); Haldane and May [2011](#ref-Haldane2011); Bardoscia et al. [2017](#ref-Bardoscia2017)) are predicted to persist or change.

Randomly assembled complex systems can be represented as large square matrices () with components (e.g., species(Allesina and Tang [2012](#ref-Allesina2012)) or banks(Haldane and May [2011](#ref-Haldane2011))). One element of such a matrix defines how component affects component in the system at a point of equilibrium(Allesina and Tang [2012](#ref-Allesina2012)). Off-diagonal elements () therefore define interactions between components, while diagonal elements () define component self-regulation (e.g., carrying capacity in ecological communities). Traditionally, values of off-diagonal elements are assigned non-zero values with a probability , which are sampled from a distribution with variance ; diagonal elements are set to -1(May [1972](#ref-May1972); Allesina and Tang [2012](#ref-Allesina2012); Allesina et al. [2015](#ref-Allesina2015)). Local system stability is assessed using eigenanalysis, with the system being stable if the real parts of all eigenvalues () of are negative ()(May [1972](#ref-May1972); Allesina and Tang [2012](#ref-Allesina2012)). In a large system (high ), eigenvalues are distributed uniformly(Tao and Vu [2010](#ref-Tao2010)) within a circle centred at (the mean value of diagonal elements) and , with a radius of (May [1972](#ref-May1972); Allesina and Tang [2012](#ref-Allesina2012); Allesina et al. [2015](#ref-Allesina2015)) (Figs 1a and 2a). Local stability of randomly assembled systems therefore becomes increasingly unlikely as , , and increase.

The above stability criterion assumes that individual components respond to perturbations of the system at the same rate (), but this is highly unlikely in any complex system. In ecological communities, for example, the rate at which population density changes following perturbation will depend on the generation time of individuals, which might vary by orders of magnitude among species. Species with short generation times will respond quickly (high ) to perturbations relative to species with long generation times (low ). Similarly, the speed at which individual banks respond to perturbations in financial networks, or individuals or institutions respond to perturbations in complex social networks, is likely to vary. The effect of such variance has not been investigated in complex systems theory. Intuitively, variation in might be expected to decrease system stability by introducing a new source of variation into the system and thereby increasing . Here I show why, despite higher , complex systems in which varies actually tend to be more stable, especially when is high.

Rows in define how a given component is affected by other components of the system, meaning that the rate of component response time can be modelled by multiplying all row elements by a scalar value (Patel et al. [2018](#ref-Patel2018)). The distribution of over components thereby models the distribution of component response rates. An instructive example compares one where for all in to the same when half of and half of . This models one system in which is invariant and one in which varies, but systems are otherwise identical (note in both cases). I assume , , and ; diagonal elements are set to and non-zero off-diagonal elements are drawn from . Rows are then multiplied by to generate . When , eigenvalues of are distributed uniformly within a circle centred at () with a radius of 1.265 (Fig. 1a). Hence, the real components of eigenvalues are highly unlikely to all be negative when all . But when values are separated into two groups, eigenvalues are no longer uniformly distributed (Fig. 1b). Instead, two distinct clusters of eigenvalues appear (grey circles in Fig. 1b), one centred at () and the other centred at (). The former has a large radius, but the real components have shifted to the left (in comparison to when ) and all . The latter cluster has real components that have shifted to the right, but has a smaller radius. Overall, for 1 million randomly assembled , this division between slow and fast component response rates results in more stable systems: 0 stable given versus 0 stable given .

Higher stability in systems with variation in can be observed by sampling values from various distributions. I now focus on a uniform distribution where (see Supporting Information for other distributions, which give similar results). As with the case of (Fig. 1b), when , allowing comparison of before and after variation in component response rate. Figure 2 shows a comparison of eigenvalue distributions given , , and . As expected (Tao and Vu [2010](#ref-Tao2010)), when , eigenvalues are distributed uniformly in a circle centred at () with a radius of 12.649. Uniform variation in leads to a non-uniform distribution of eigenvalues, some of which are clustered tightly around the centre of the distribution, but others of which are spread outside the former radius of 12.649 (red circle Fig 2b). This larger radius occurs because the addition of increases the realised of . The clustering and spreading of eigenvalues introduced by can destabilise previously stable systems or stabilise systems that are otherwise unstable. But where systems are otherwise too complex to be stable given , the effect of can often lead to stability above May’s(May [1972](#ref-May1972); Allesina and Tang [2012](#ref-Allesina2012)) threshold of .

To investigate the effect of on system stability, I simulated random matrices at and across ranging from (see Supporting Information for different values of and ). One million were simulated for each , and the stability of was assessed given versus (note that under these conditions, given when ). I found that the number of stable random systems was consistently higher given than when (Fig. 3), and that the difference between the probabilities of observing a stable system increased with an increase in ; i.e., the potential for to drive stability increased with system complexity. For the highest values of , nearly all systems that were stable given would not have been stable given , and the maximum observed for which a system was stable was given versus given (see Supporting Information for full results). This suggests that the stability of large systems might be dependent upon variation in the response rate of their individual components, meaning that factors such as generation time (in ecological networks), transaction speed (in economic networks), or communication speed (in social networks) needs to be considered when investigating the stability of complex systems.

Some care is needed when interpreting these results. First, I emphasise that is not stabilising per se; that is, adding variation in to a particular system does not necessarily increase the probability that the system will be stable (see Supporting Information). Rather, systems that are observed to be stable are more likely to vary in , and for this to be critical to their stability. This is caused by the shift in the distribution of eigenvalues that occurs by introducing (Fig. 1b, 2b), which can sometimes result in all but might also increase values.

**References**

Allesina, S., and S. Tang. 2012. Stability criteria for complex ecosystems. Nature 483:205–208. Nature Publishing Group.

Allesina, S., and S. Tang. 2015. The stability–complexity relationship at age 40: a random matrix perspective. Population Ecology 63–75.

Allesina, S., J. Grilli, G. Barabás, S. Tang, J. Aljadeff, and A. Maritan. 2015. Predicting the stability of large structured food webs. Nature Communications 6:7842.

Bardoscia, M., S. Battiston, F. Caccioli, and G. Caldarelli. 2017. Pathways towards instability in financial networks. Nature Communications 8:1–7. Nature Publishing Group.

Gao, J., B. Barzel, and A. L. Barabási. 2016. Universal resilience patterns in complex networks. Nature 530:307–312. Nature Publishing Group.

Gray, R. T., and P. A. Robinson. 2008. Stability and synchronization of random brain networks with a distribution of connection strengths. Neurocomputing 71:1373–1387.

Gray, R. T., and P. A. Robinson. 2009. Stability of random brain networks with excitatory and inhibitory connections. Neurocomputing 72:1849–1858.

Grilli, J., M. Adorisio, S. Suweis, G. Barabás, J. R. Banavar, S. Allesina, and A. Maritan. 2017. Feasibility and coexistence of large ecological communities. Nature Communications 8.

Haldane, A. G., and R. M. May. 2011. Systemic risk in banking ecosystems. Nature 469:351–355. Nature Publishing Group.

MacArthur, B. D., R. J. Sanchez-Garcia, and A. Ma’ayan. 2010. Microdynamics and criticality of adaptive regulatory networks. Physics Review Letters 104:168701.

May, R. M. 1972. Will a large complex system be stable? Nature 238:413–414.

May, R. M., S. A. Levin, and G. Sugihara. 2008. Complex systems: Ecology for bankers. Nature 451:893–895.

Mougi, A., and M. Kondoh. 2012. Diversity of interaction types and ecological community stability. Science 337:349–351.

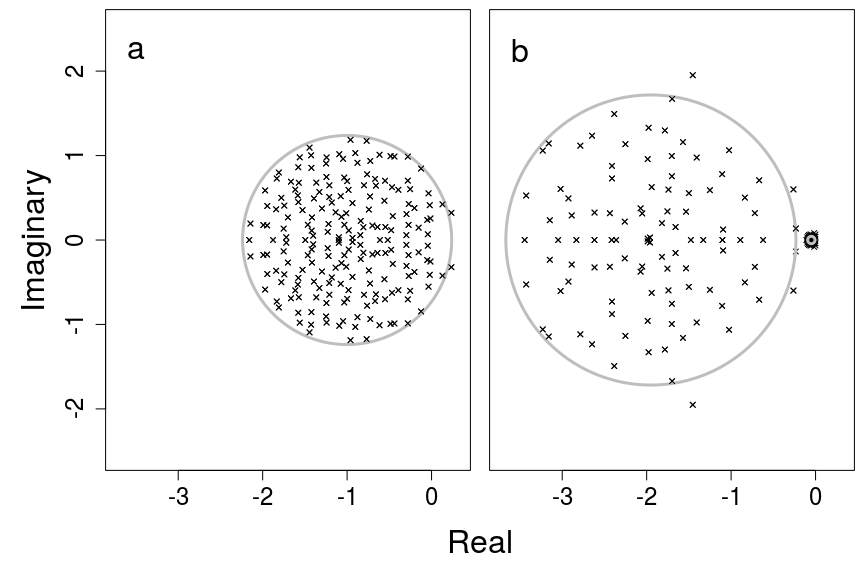
Patel, S., M. H. Cortez, and S. J. Schreiber. 2018. Partitioning the effects of eco-evolutionary feedbacks on community stability. American Naturalist 191:1–29.

Rosenfeld, S. 2009. Patterns of stochastic behavior in dynamically unstable high-dimensional biochemical networks. Gene Regulation and Systems Biology 3:1–10.

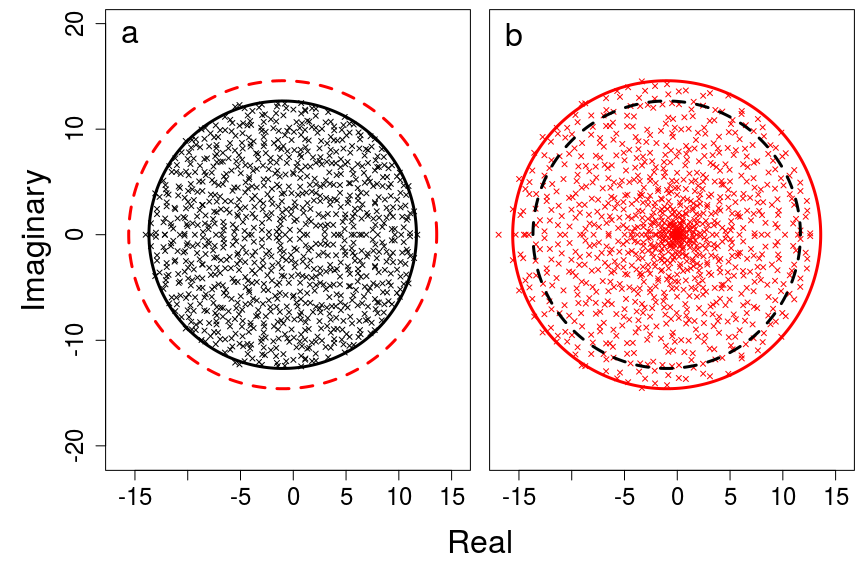
Suweis, S., and P. D’Odorico. 2014. Early warning signs in social-ecological networks. PLoS ONE 9.

Tao, T., and V. Vu. 2010. Random matrices: Universality of ESDs and the circular law. Annals of Probability 38:2023–2065.

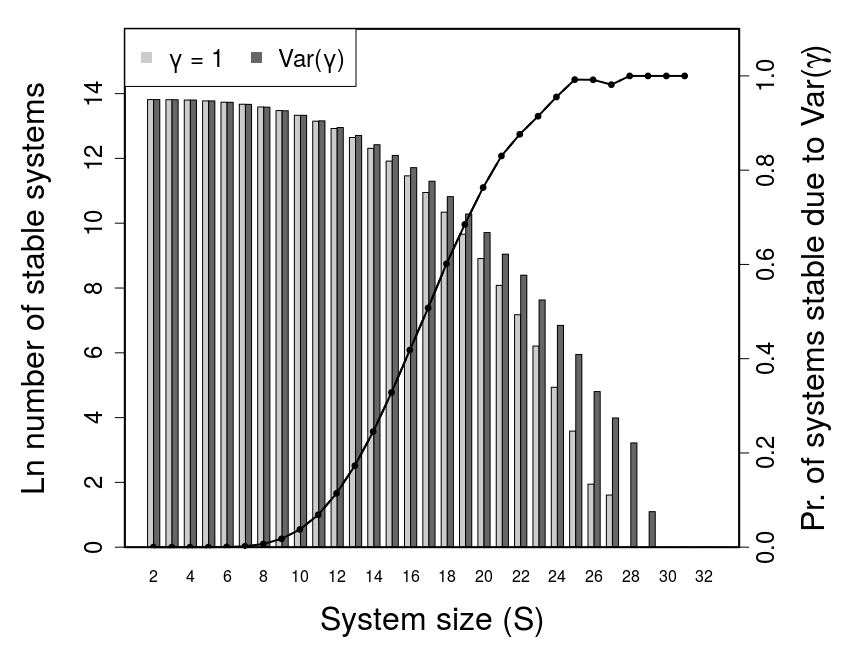
**Figure 1: Example distribution of eigenvalues before (a) and after (b) separating a randomly generated complex system into fast () and slow () component response rates.** Each panel shows the same system where , , and , and in each case (i.e., only the distribution of differs between panels). **a.** Eigenvalues plotted when all ; distributions of points are uniformly distributed within the grey circle with a radius of 1.238 centred at -1 on the real axis. **b.** Eigenvalues plotted when half and half ; distributions of points can be partitioned into one large circle of 1.718 centred at and one small circle of 0.044 centred at . In a, the maximum real eigenvalaue 0.2344871, while in b -0.0002273135, meaning that the complex system in b but not a is stable because in b . In 1 million randomly generated complex systems under the same parameter values, 0 were stable when while 0 were stable when . Overall, complex systems that are separated into fast versus slow components tend to be more stable than otherwise identical systems with identical component response rates.



**Figure 2: Distributions of eigenvalues before (a) and after (b) introducing variation in component response rate () in complex systems.** Each panel show the same system where , , and . **a.** Eigenvalues plotted in the absence of where , versus **b.** eigenvalues plotted given , which increases the variance of interaction strengths () but clusters eigenvalues toward the distribution’s centre (-1, 0). Black and red elipses in both panels show the circle centred on the distribution in panels a and b, respectively, which have a radius of . Proportions of are 0.546 and 0.556 for a and b, respectively.



**Figure 3: Stability of large complex systems with and without variation in component response rate ().** The number of systems that are stable across different system sizes () given , and the proportion of systems in which variation in is critical for system stability. For each , 1 million complex systems are randomly generated. Stability of each complex system is tested given variation in by randomly sampling . Stability given is then compared to stability in an otherwise identical system in which for all components. Light and dark grey bars show the number of stable systems in the absence and presence of variance in , respectively. The black line shows the proportion of systems that are stable when , but would be unstable if .



**Figure 4: Distributions of variation in component response time evolved to generate stable systems**