7. Uncertainty propogation

Nothing can be measured with perfect accuracy, meaning that every measurement has some associated error. The measurement error might be caused by random noise in the measuring environment, or by mistakes made by the person doing the measuring. The measurement error might also be caused by limitations or imperfections associated with a measuring device. The device might be limited in its measurement precision, or perhaps it is biased in its measurements due to improper calibration, manufacture, or damage from previous use. All of this generates uncertainty with respect to individual measurements.

Recall from Chapter 6 the difference between precision and accuracy. We can evaluate the precision and accuracy of measurements in different ways. Measurement precision can be estimated by replicating a measurement (i.e., taking the same measurement over and over again). The more replicate measurements made, the more precisely a value can be estimated. For example, if we wanted to evaluate the precision with which the mass of an object is measured, then we might repeat the measurement with the same scale multiple times and see how much mass changes across different measurements. To evaluate measurement accuracy, we might need to measure a value in multiple different ways (e.g., with different measuring devices). For example, we might repeat the measurement of an object's mass with a different scale (i.e., a different physical scale used for measuring the mass of objects).

Sometimes it is necessary to combine different measured values. For example, we might measure the mass of 2 different bird eggs in a nest separately, then calculate the total mass of both the 2 eggs combined. The measurement of each egg will have its own error, and these errors will propagate to determine the error of the total egg mass for the nest. How this error propagates differs depending on if they are being added or subtracted, or if they are being multiplied or divided.

7.1. Adding or subtracting errors

In the case of our egg masses, we can assign the mass of the first egg to the variable X and the mass of the second egg to the variable Y. We can assign the total mass to the variable Z, where $Z = X + Y$. The errors associated with the variables X, Y, and Z can be indicated by E_X, E_Y, and E_Z, respectively. In general, if the variable Z is calculated by adding (or subtracting) 2 or more values together, then this is the formula for calculating E_Z,

$$E_Z = \sqrt{E_X^2 + E_Y^2}.$$

Hence, for the egg masses, the error of the combined masses (E_Z^2) equals the square root of the error associated with the mass of egg 1 squared (E_X^2) plus the error associated with the mass of egg 2 squared (E_Y^2). It often helps to provide a concrete example. If the error associated with the measurement of egg 1 is $E_X^2 = 2$, and the error associated with the measurement of egg 2 is $E_Y^2 = 3$, then we can calculate,

$$E_Z = \sqrt{2^2 + 3^2} \approx 3.61.$$

Note that the units of E_Z are the same as Z (e.g., grams).

7.2. Multiplying or dividing errors

Multiplying or dividing errors works a bit differently. As an example, suppose that we need to measure the total area of a rectangular field. If we measure the length (L) and width (W) of the field, then the total area is the product of these measurements, $A = L \times W$. Again, there is going to be error associated with the measurement of both length (E_L) and width (E_W). How the error of the total area (E_A) is propagated by E_L and E_W is determined by the formula,

$$E_A = A\sqrt{\left(\frac{E_L}{L}\right)^2 + \left(\frac{E_W}{W}\right)^2}.$$

Notice that just knowing the error of each measurement (E_L and E_W) is no longer sufficient to calculate the error associated with the measurement of the total area. We also need to know L, W, and A. If our field has a length of $L = 20$ m and width of $W = 10$ m, then $A = 20 \times 10 = 200\, m^2$. If length and width measurements have associated errors of $E_L = 2$ m and $E_W = 1$ m, then,

$$E_A = 200\sqrt{\left(\frac{2}{20}\right)^2 + \left(\frac{1}{10}\right)^2} \approx 28.3\, m^2.$$

Of course, not every set of measurements with errors to be multiplied will be lengths and widths (note, however, that the units of E_A are the same as A, m^2). To avoid confusion, the general formula for multiplying or dividing errors is below, with the variables L,

W, and A replaced with X, Y, and Z, respectively, to match the case of addition and subtraction explained above,

$$E_Z = Z\sqrt{\left(\frac{E_X}{X}\right)^2 + \left(\frac{E_Y}{Y}\right)^2}.$$

Note that the structure of the equation is the exact same, just with different letters used as variables. It is necessary to be able to apply these equations correctly to estimate combined error.

7.3. Applying formulas for combining errors

It is not necessary to understand why the equations for propagating different types of errors are different, but a derivation is provided in Appendix B for the curious.