A Disjunctive Interpretation of the Mathematical Aesthetic

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Abstract

Although mathematicians have long used aesthetic language to describe their work, characterizing theorems and proofs as “beautiful” or “elegant”, it is only in recent decades that a substantial literature on mathematical aesthetics has emerged in the philosophical community. In this paper, I argue that the aesthetic experience of mathematics cannot be reduced to epistemic considerations, provide examples of beautiful-yet-false theories in mathematics, and conclude that as a result, only a disjunctive view of the relationship between aesthetics and epistemology in mathematics, in which neither beauty nor truth is a necessary or sufficient condition for the other, is plausible. This is done while remaining agnostic as to the subjectivity or objectivity of mathematical beauty.

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Introduction

Carl Friedrich Gauss described mathematics as the “queen of the sciences”, and its apparent purity and *a priori* nature has led many to consider it the epitome of what can be achieved through the application of logic and human reason. Because of these uniquely rigorous and epistemologically sound foundations, the field has often been upheld as a valuable case study for philosophers (Shapiro, 2000), demanding explanation for how knowledge of abstract mathematical objects and structures is possible. Yet despite the general emphasis on the epistemic properties of mathematics, working mathematicians have also been long known to use aesthetic language to describe their work, characterizing theorems and proofs as “beautiful” or “elegant”.

For example, Hardy (1940) writes that

> The mathematician’s patterns, like the painter’s or the poet’s, must be beautiful; the ideas, like the colours or the words, must fit together in a harmonious way. Beauty is the first test: there is no permanent place in the world for ugly mathematics.

He also argues that the aesthetic value of a mathematical theorem depends on many properties, including beauty, seriousness, generality, depth, unexpectedness, inevitability, and economy (Hardy, 1940). His list, however, is certainly not comprehensive. A recent empirical survey (Johnson & Steinerberger, 2019) finds that mathematicians tend to consistently describe the aesthetic properties of mathematical arguments along dimensions of elegance, profundity, and clarity; moreover, Aristotle himself claimed that “the main species of beauty are orderly arrangement, proportion, and definiteness”, and that “these are especially manifested by the mathematical sciences” (Aristotle, n.d., VIII, 1078a).

In recent decades, a substantial literature on the mathematical aesthetic has emerged. Several open questions have featured prominently in the discussion, including: ‘What is the nature of mathematical beauty?’ (Blåsjö, 2012; Cellucci, 2015), ‘Is aesthetic value essential to mathematical structures and arguments, or projected onto them by observers?' (McAllister & Michele, 2005), and 'What is the relationship between aesthetic value and epistemic value in mathematics?' (Todd, 2008, 2018). Although I will discuss each of these questions as necessary to provide context, I am particularly interested in the aesthetic-epistemic dichotomy, and will focus on elucidating the distinction in this paper. In particular, I will argue for a disjunctive view of the relationship between aesthetics and epistemology in mathematics, in which neither beauty nor truth is a necessary or sufficient condition for the other. Before doing so, I will first discuss a framework for understanding the different types of aesthetic judgments that mathematicians make, and then consider the current challenges to the disjunctive view. Finally, I will argue that the disjunctive view is the most plausible model for the apparent coincidence of the standards for aesthetic and epistemic value in mathematics, despite leaving many open questions about the relationship between the two.

A Note on Empirical Observation

In considering the study of the philosophy of mathematics, Shapiro (2000) characterizes the view that philosophy "precedes" or "determines" practice as the *philosophy-first* paradigm. For example, those who subscribe to the view that modern mathematical discourse is incoherent due to unsound metaphysical or logical presuppositions, and that the field should therefore be revised to conform to first principles determined by philosophy, would be considered to be working within the philosophy-first paradigm. Many philosophers (if not most), and certainly most mathematicians, reject this paradigm. For the purposes of this paper, I also reject this view and instead hope to provide a coherent, post-hoc account of the aesthetic experience of mathematicians.

One interesting consequence of this approach is that empirical observations of mathematicians' aesthetic experiences become not only relevant to the discussion, but in many cases, necessary. This is especially so when considering the nature of the mathematical aesthetic, and how it relates to aesthetic experiences in other domains. Indeed, if we accept the Wittgensteinian perspective on semantics, we must inquire into how the aesthetic lexicon is actually used by mathematicians if we hope to understand the meaning that they are trying to convey. As a result, much of the essential empirical research on this topic comes not only from those interested in mathematics and aesthetics, but also mathematics education (Dreyfus & Eisenberg, 1986; Inglis & Aberdein, 2015, 2016, 2020; Sa, Alcock, Inglis, & Tanswell, 2023). I will refer to several of these important papers throughout this essay.

Types of Aesthetic Judgments

Although it is tempting to treat the aesthetic experience of mathematics as a monolithic phenomenon, mathematicians in fact make several types of aesthetic judgments in various different contexts. Sinclair (2004, 2011) characterizes three different roles that aesthetic judgments play in mathematics: *evaluative*, *generative*, and *motivational*. Evaluative judgments are those that concern the significance or elegance of particular mathematical results. These are the types of judgments made by publishers and reviewers of academic journals in deeming whether particular results or theorems are worthy of publication, and the judgments according to which mathematicians determine how best to present their work for others to read. Evaluative judgments are perhaps the most easily recognizable of the different types of judgments, as they are the ones for which aesthetic language is most often used and discussed. Generative judgments, on the other hand, are often made implicitly by mathematicians in the process of developing theories and attempting to prove new results—they concern which strategies should be used and which avenues should be pursued to most effectively solve a problem or provide a proof. Finally, motivational judgments are those that determine which problems are most worthy of study in the first place and which areas of mathematics should be explored.

As Ivanova (2017) points out, there is a similar feel to the distinction between these types of judgments and the distinction drawn by Reichenbach (Glymour &

Eberhardt, 2022) between the *context of discovery* and the *context of justification* in the philosophy of science. In the traditional interpretation of these two contexts, the former occurs when a mathematician or scientist is exploring the space of possible explanations and theories (i.e., making motivational judgments), while the latter occurs when they are attempting to justify the truth of a particular theory or explanation (i.e., making generative judgments while developing a proof). Although the context of justification is generally considered to be a strictly rational process (and in mathematics, logical consistency certainly reigns above all other considerations), Cellucci (2015) argues that aesthetic generative judgments also play a role in this context. Even this process, however, primarily entails selecting from among possible options; the differences is that the selection occurs on the level of possible lemmas, methods of computation, and other aspects of detailed argumentation rather than occurring on the level of theory as a whole.

There now seems to be a consensus that the types of judgments which we have traditionally deemed 'aesthetic' judgments, and which use a subset of the same aesthetic vocabulary with which we are familiar, truly do occur in both the context of discovery and the context of justification, as well as the evaluative judgments made after-the-fact. The question of whether these judgments are truly aesthetic in nature is a separate one, and I will address it in the next section.

Is the Mathematical Aesthetic Really Aesthetic?

The concept of the aesthetic has always been notoriously difficult to define, and has been linked to the concept of taste since the days of Hume. For many early theorists, aesthetic judgments were taken to be defined by immediacy and disinterest. By being immediate, it was meant that judgments of beauty were not based on reason or rational deliberation, but were instead as direct and instinctive as sense perception. By being disinterested, it was meant that aesthetic judgments were not based on any particular desire or interest, but were instead made for their own sake. Hume in particular argued that although aesthetic judgments, though immediate, were based on general principles about the world, an improved capacity for making these judgments (also known as taste) could be cultivated through experience (Shelley, 2022). This simplified picture of the early aesthetic tradition is, of course, much more complex and controversial in reality, but it does provide some important background for our discussion.

In the realm of mathematics, the Humean position would be that as people develop expertise in the field, their capacity for making aesthetic judgments about mathematical objects and arguments improves because they develop a more refined mathematical taste. Indeed, this does appear to reflect reality: Dreyfus and Eisenberg (1986) found that "an aesthetic appreciation for mathematics can be nurtured"; and according to Zeki, Romaya, Benincasa, and Atiyah (2014), while most mathematicians relate to perceiving a result as elegant or beautiful, nine out of twelve of the non-mathematicians in a survey conducted by Zeki et al. (2014) indicated that they do not "experience an emotional response" when they encounter "beautiful equations".

Another difficult question in aesthetics is whether aesthetic judgments are objective or subjective. Objective aesthetic judgments are those that hold universally, independent of the observer, while subjective aesthetic judgments are those that may vary from observer to observer because at their root, they are projected onto their objects by the subject. A notion of intersubjective agreement is also often invoked, in which aesthetic judgments tend to be shared by a particular group or society. It is with this conception of intersubjective agreement that McAllister and Michele (2005) put forth one of the strongest explanations of the relationship between aesthetic and epistemic value in mathematics. They present a theory of "aesthetic induction", in which the aesthetic value that the mathematical (or scientific) community place on a particular result or theory is a result of the state of the community's current "aesthetic canon", and the degree to which the result accords with that canon. For example, the results of denying the parallel postulate were thought to be ugly and "repugnant" by the mathematical community up until the 19th century, when such non-Euclidean geometries were shown to be consistent models of the other axioms of Euclidean geometry. Since that time, the aesthetic canon of the mathematical community has shifted to de-emphasize graphical representations and visual intuition, and to instead emphasize logical coherence and consistency. This shift in the aesthetic canon has resulted in the acceptance of non-Euclidean geometries as very beautiful and elegant.

McAllister and Michele actually spend more time discussing the “aesthetic induction” in scientific theories than in mathematical theories (giving quantum mechanics as a particular case study), but the analogy holds up quite well regardless. In essence, they argue that the aesthetic canon of a society at a given time is ultimately determined by the empirical success of existing theories and the properties they are known to have. For example, since symmetry and parsimony are known to be properties of many successful theories, they are considered to be beautiful—but only for as long as they continue to be consistently associated with successful theories. This theory holds up well in the case of one recent development in mathematics—the rise of the computer-assisted proof, particularly to prove the four-color theorem. Since at no time in history had proofs of mathematical theorems been provided which could not, at least in theory, be checked by holding the entire argument in one’s head, the idea of a proof that could not be verified by a human was disconcerting to many mathematicians. Indeed, Rota (1997) explains that

> Mathematicians have been ambivalent about such a verification. On the one hand, every mathematician professes to be satisfied to learn that the conjecture has been settled. On the other hand, the behavior of the community of mathematicians belies such a feeling of satisfaction.

This is an excellent example of the aesthetic canon of the mathematical community being challenged by a new development in the field. The computer-assisted proof of the four-color theorem was not, and continues to not be accepted as beautiful by the community, but McAllister and Michele predict that computer-assisted proofs can become more beautiful overtime as the standards of the community change due to the intersubjective nature of the aesthetic (McAllister & Michele, 2005).

However, their position leads to some problems of its own, including the possibility that the aesthetic judgments of mathematicians are really just epistemic judgments in disguise. Todd (2008) is perhaps the most prominent example of someone sympathetic to this view. While he does not fully take the position of aesthetic-epistemic identity himself, he provides strong grounds for suspecting that the distinction between the aesthetic and epistemic in mathematics is, in fact, illusory, and therefore argues that we should assume this to be the case until we can demonstrate otherwise. He first claims that the vocabulary used to describe mathematical beauty is much weaker than that used to define beauty in other domains. A scientist, engineer, or data analyst may exclaim "Beautiful!" upon making a discovery or successfully processing a dataset, but we generally interpret this exclamation as expressing only excitement, intellectual pleasure, or satisfaction rather than appreciation of beauty. The argument is that a similar situation occurs in mathematics, and that the paucity of expressive power held by the aesthetic vocabulary in mathematics is due to the fact that, at their core, the words used are not even expressing aesthetic judgments, but some other type of judgment. The second claim is that, when forced to explain their evaluation of an argument as aesthetically valuable, the criteria that mathematicians fall back on are almost always epistemically valuable as well. This is precisely the claim that McAllister and Michele (2005) make, and what lends credibility to the theory of aesthetic induction. If all the elements of *elegant* proofs happen to coincide with those of *valid* proofs, what evidence do we have to say that there is a distinction between the two types of value?

I find the first claim rather unconvincing. While it is true that the range of adjectives used by mathematicians to describe the aesthetic qualities of their work is limited, this is not by itself evidence that they are not having an affective experience. Indeed, in comparison to the size and scope of the art world (which forms the obvious basis of comparison), the mathematical community is both much smaller and much less accessible to outsiders. Mathematicians constantly create and define new technical terms as they develop new mathematical structures and theories, but while most of humanity has, at some point, taken time to appreciate paintings, music, and literature, only a small fraction of the population has taken the opportunity to appreciate mathematical proofs, and it is therefore not surprising that the vocabulary used to describe the aesthetic experience of mathematics is less fully developed than in the arts.

The second claim is more nuanced and is worth considering in more detail. While it is possible that a more thorough empirical investigation into the reasons mathematicians give for describing proofs as beautiful could reveal instances where the criteria for aesthetic and epistemic value diverge, I do not here object to the current empirical claim that the criteria often coincide. Instead, I would like to further investigate the implications of this coincidence to determine whether this precludes a distinction between the two types of value. In particular, I would like to consider whether the putative conflation of the aesthetic and epistemic occurs in the same way for evaluative, generative, and motivational judgments, or if there are differences between the three types.

The Aesthetic-Epistemic Dichotomy

Consider the experience of the mathematician in the context of discovery. In deciding what types of research questions to ask and what types of problems to work on, the mathematician is guided by a number of factors, including the perceived difficulty of the problem, the perceived importance of the problem, and the perceived likelihood of success. While there are certainly many other complex and nuanced factors, I argue that epistemic values alone cannot guide these decisions. For mathematicians and logicians, epistemic properties are binary—a proof is either valid or invalid, a statement is either true or false, a theory is either consistent or inconsistent, and a proposition is either provable or unprovable. When working within algebraic topology, analytic number theory, or any other subfield, the mathematician has infinitely many possible questions to ask and problems to work on, and the vast majority of them are either trivially true or trivially false (Poincaré, 1910, p. 324-325). These propositions are not interesting, although it doesn't appear that they lack any more epistemic value than deeper propositions. To preference one above another and actually make a decision, the mathematician must appeal to some other type of value, and I argue that in the absence of evidence to the contrary, it is reasonable to assume that this value is aesthetic. Hardy describes *important* problems have the essentially aesthetic property of being *serious*, and says that "the seriousness of a mathematical theorem lies, not in its practical consequences, which are usually negligible, but in the significance of the mathematical ideas which it connects" (Hardy, 1940, p. 16). Ivanova (2017) also claims that "aesthetic values often drive the preference of one theory over another in the case of underdetermination". Even if this process of selecting problems to work on is not guided by aesthetic value, I argue that the truth-functional nature of epistemic value makes it insufficient to explain the process.

Once the mathematician has chosen a problem to work on, she must then decide what approach to take to provide a proof or solution. She is again faced with infinitely many possible approaches, but I can admit here that in practice, only a vanishingly small number of these is likely to be successful, and in truth only a finite number of these is even considered. Poincaré (1910) discusses this generative process in greater detail, comparing the way in which the unconscious mind of an experienced mathematician does not bother presenting the conscious mind with large numbers of false leads to the way in which instructors of graduate courses do not bother assessing principles that were learned in high school. A natural filtering process occurs which depends on the mathematician's prior experience and knowledge of the subject. With concrete examples, Poincaré and other mathematicians have described this process as being guided chiefly by aesthetic factors, and to be emotionally affected by the intermediate results that are discovered along the way (Sinclair, 2004).

One particularly interesting example comes from Papert (1978), who analyzed whether these aesthetic generative judgments could also be experienced by non-mathematicians. They described that when given minimal instruction and guidance on how to prove that $\sqrt{2}$ is irrational, and only told to begin with the initial equation $\sqrt{2} = p/q$, the vast majority of participants became excited and confident when they reached the form $p^2 = 2q^2$, despite understanding what next steps in the proof would need to be, or even understanding the idea of a proof by contradiction in general. Papert claimed that the pleasure experienced by the participants indicated that an aesthetic judgment had been made when they judged that form of the equation to be insightful, since the participants had no other way of evaluating their progress from an epistemic or utilitarian perspective.

Finally, once the mathematician has found a potential proof (i.e., has entered the context of justification), she must evaluate it. Ultimately, all relevant epistemic considerations are subsumed in whether the term “proof” is a valid description for the set of symbols and manipulations that have been written down. Those symbols constitute a proof if and only if they are epistemically and logically sound. Moreover, the truth-functional nature of logic means that one proof of a theorem is exactly as valid as any other. And yet for some reason, mathematicians continue to prefer certain proofs of results over others, and to search for proofs that are more elegant, more insightful, or more beautiful. Why go about proving the same theorem differently, if an aesthetic judgment is not being made?

In tribute to what Paul Erdős called “The Book” (a hypothetical book in which God had written the most beautiful proofs of mathematical theorems), mathematicians Martin Aigner and Günter Ziegler recently published a volume titled *Proofs from THE BOOK*, which contains a collection of proofs that are particularly elegant, insightful, or beautiful. The book does not contain any novel theorems, definitions, or results; everything within was previously known to the world. It also is not of particular value from a pedagogical perspective; without an understanding of basic calculus, linear algebra, and number theory, a reader will likely not be able to understand or appreciate many of the proofs, and with a strong understanding of these topics, the book will not provide much new insight. The book is not a reference text, and it is not a textbook. The book instead claims to be a collection of proofs that are beautiful, offered only for readers’ enjoyment. If the book does not have aesthetic value as judged from an evaluative perspective, then it likely has no value at all (Aigner & Ziegler, 1999).

In summary, the mathematician claims that many factors affecting their work are aesthetic in nature, but Todd (2008) contends that we should be suspicious of this claim and that we should not simply take their statements at face value. Given that at each of the levels discussed above, something beyond the epistemic seems to be in play, where does this leave us now?

Conjunctive and Disjunctive Perspectives

Given the previous analysis, it does not seem possible to reduce the aesthetic judgments of mathematicians to an epistemic basis, which seems to address Todd's concerns about the potential for aesthetic induction theory. However, the original problems which motivated his initial inquiry have not, at their heart, been resolved (Todd, 2008). Are the aesthetic and epistemic really connected on a fundamental level, even if the aesthetic is not completely reducible to the epistemic? Or is there a fundamental distinction between the two types of value?

Todd (2008) differentiates between the conjunctive position, in which there is a fundamental dependency relation between beauty and truth, and the disjunctive position, in which there is not. In the disjunctive view, there should be instances of beautiful-and-true theorems, beautiful-yet-false theorems, ugly-yet-true theorems, and ugly-and-false theorems, while in the conjunctive view, one of these categories should be empty. Mathematicians can provide plenty of examples for the beautiful-and-true (Aigner & Ziegler, 1999) and ugly-and-false theorems, and as noted earlier, the existence of the ugly-yet-true is what motivates mathematicians to search for more elegant arguments and alternative proofs to results that have already been proven. But what about the beautiful-yet-false? Are there any examples of arguments, conjectures, or other theories that were widely considered to be beautiful, but which were later shown to be false?

In the absence of valid examples, the disjunctive view is not necessarily disproven, but it is certainly weakened—as it is reasonable to assume that the aesthetic and epistemic are at least somewhat connected, given how many mathematicians extol beauty as evidence of truth. I would like to present two possible examples of beautiful-yet-false concepts in mathematics. I find both of these examples to be compelling, but I have not yet conducted any empirical research into the views of other mathematicians on these topics. While such research would be essential under an objective or intersubjective view of beauty, and although there is conflicting evidence concerning how often mathematicians agree on what is beautiful (Inglis & Aberdein, 2016, 2020; Sa et al., 2023), I believe that even under a subjective perspective, it would be useful to know whether other mathematicians find these examples to be beautiful in order to determine whether my opinion is, in this case, idiosyncratic.

The first example is the Hilbert Program (Zach, 2023). The Hilbert Program was a project proposed by David Hilbert in the early 20th century to prove the consistency of arithmetic—and therefore provide a solid logical foundation for all mathematics—by reducing arithmetic to a finite set of axioms and rules of inference. The program was widely regarded and well-known within the mathematics and logic communities, and because of the beautiful idea that all of mathematics could be proven to be consistent, it was widely believed to be true (yet unproven) until Kurt Gödel published his incompleteness theorems in 1931, demonstrating that the program was impossible to complete.

The second example is the conjecture, believed by many mathematicians for thousands of years, that Euclid's fifth postulate (also known as the *parallel postulate*), given at the beginning of his *Elements*, could be proven from the other four. This postulate states that given a line ℓ_1 and a point P not on that line, there is exactly one line ℓ_2 through P that is parallel to ℓ_1. Euclid's other four postulates are all relatively simple, and were thus accepted unproblematically by mathematicians as axioms for geometry. Since the parallel postulate was so unintuitive by comparison, geometers believed that its seeming necessity was ugly and inelegant, and that its truth should be proven rather than assumed. This belief was so widespread that it was not until the 19th century that mathematicians began to seriously consider the possibility that the parallel postulate was truly independent, resulting in counterexamples to its necessity and the birth of the consistent mathematical subfields of non-Euclidean geometry (Gray

& Ferreirós, 2021).

Interestingly, I would also argue that both Gödel's incompleteness theorems and non-Euclidean geometry are beautiful as well. While truth must satisfy the principle of non-contradiction for epistemic claims to have meaning, the same is not generally true for beauty, and this is, I believe, a reason for which the beautiful-yet-false is possible. The apparent existence of beautiful-yet-false concepts in mathematics lends further credence to the disjunctive view of the relationship between the aesthetic and the epistemic. But it makes the apparent relationship between beauty and truth, felt by many mathematicians, even more mysterious.

Conclusion

I have no particularly compelling explanation as for why truth seems to be so closely related to beauty in mathematics, nor do I understand the nature of the mathematical sublime itself. The claims that mathematicians and scientists make in trusting their aesthetic judgments to lead to epistemic truth seem unjustified, and the question of whether aesthetic properties are essential to mathematical objects and arguments, or projected onto them by mathematicians, remains an open one. Nevertheless, many of these questions about the nature of the mathematical aesthetic are common to aesthetics in general. While further investigation into the relationship between truth and beauty is necessary, the aesthetic and epistemic are not reducible to one another, and the nature of the mathematical aesthetic must be a disjunctive one.

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