Types of Errors

Suggested Reading:
Barlow *SLUO Lecture 5– systematic errors*
Barlow: *systematic errors: facts and fictions*

Statistical errors — Uncertainty due to known statistical effects, such as: i) photon counting statistics (e.g., Poisson/shot), ii) detector or measurement noise (e.g., thermal noise), or iii) other sources of variance.

Systematic errors — Uncertainty not due to statistical uncertainty. Typically related to some unquantified measurement bias, or instrumental effect. These typically are multiplicative or additive corrections to the signal, and do not follow Gaussian statistics.

Systematic and statistical errors are usually presented as separate quantities

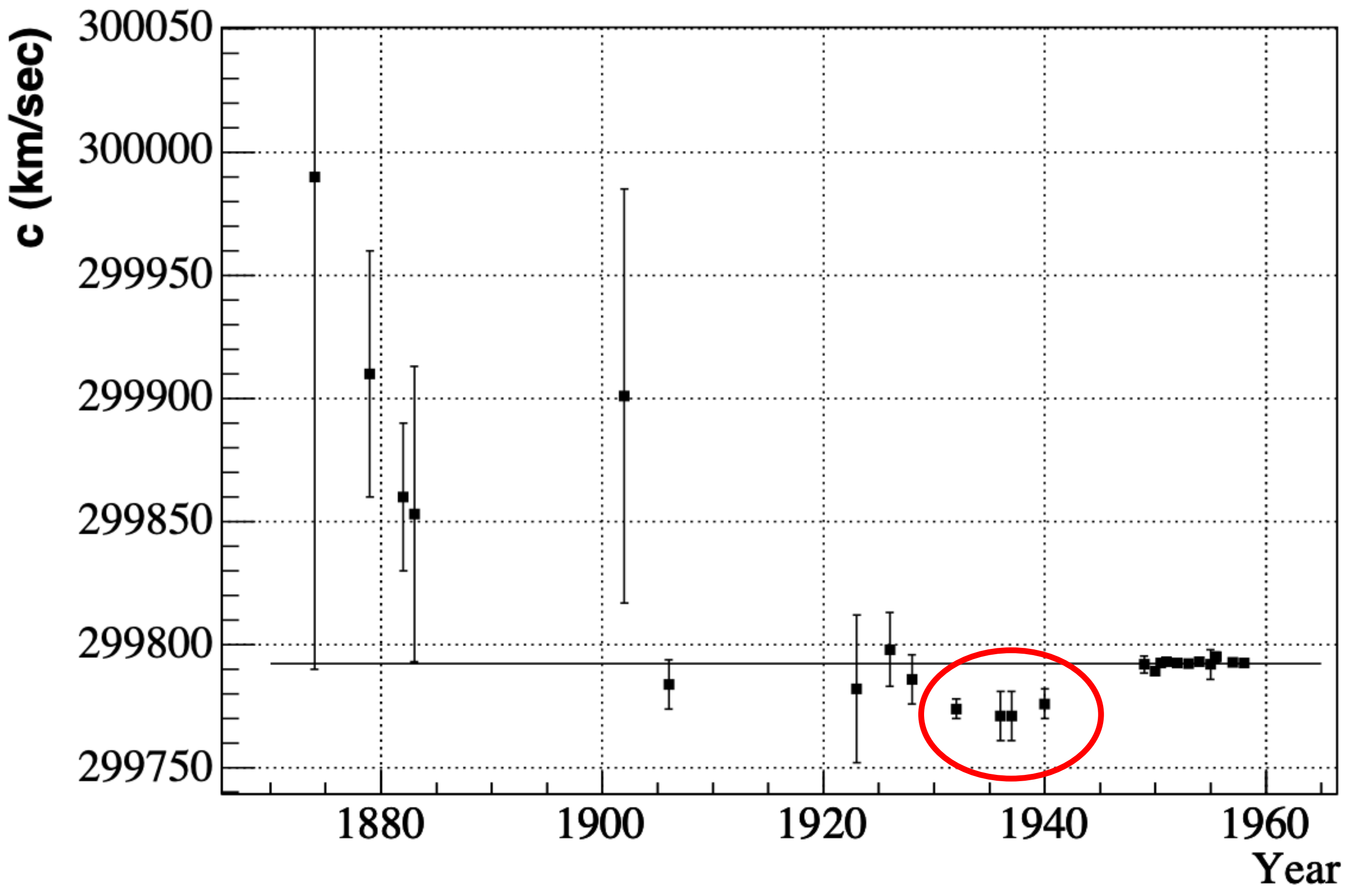
Systematic and statistical errors are quoted separately

You will often see results like: $\sigma = 45 \pm 4 \pm 1mb$

- The **first** number is the **central value**
- The **second** number is the **statistical error**
- The **third** number is the **systematic error**

This facilitates combining errors, but also shows the relative importance of the two factors. For example, if the systematics dominate, taking more data is not helpful.

Systematic errors: The Speed of Light



Summary of speed of light measurements from 1870-1960:

- One striking feature is the 17 km/sec shift between the series of experiments from 1930-1940 and later determinations.
- A post-mortem of the systematic uncertainties in these experiments concluded this was due to ***"observer bias"***, i.e., experimenters didn't want to get a result "inconsistent" with the accepted value
  - Beware "observer bias" in your Hubble constant lab!

Calculating the "Chi-squared": χ^2

Definition: $$\chi^2 = \sum_{i=1}^{N} \frac{(x_i - \mu)^2}{\sigma_i^2}$$

Where there are N measurements, x_i,
μ is the mean or model prediction,
σ_i is the measurements uncertainty

The expectation value $$E\left((x-\mu)^2\right) = \sigma^2$$

We expect $$\chi^2 = N_{dof} = N - N_{model}$$

Where N_{dof} is the number of degrees of freedom.

This enables several tricks:

(1) If we don't know the errors, we can compute the chi-squared and set it equal to N to estimate the errors.

(2) We can test how good the model is. We interpret this through the integrated probability of the chi-squared distribution also known as the ***probability to exceed (PTE)***. The PTE depends on chi-squared and the number of degrees of freedom.

Probability To Exceed (PTE) (the Pearson Test)

A chi-squared result should follow a chi-squared probability distribution ***f(χ²)*** with N degrees of freedom

You expect your measured chi2 to be approximately equal to the number of data points minus the number of model parameters.

The "PTE" tells you how likely you would have been to get your result,. e.g.,

The PTE on average should be 0.5, and there should be only a 2% chance that you find a PTE < 0.01 or > 0.99

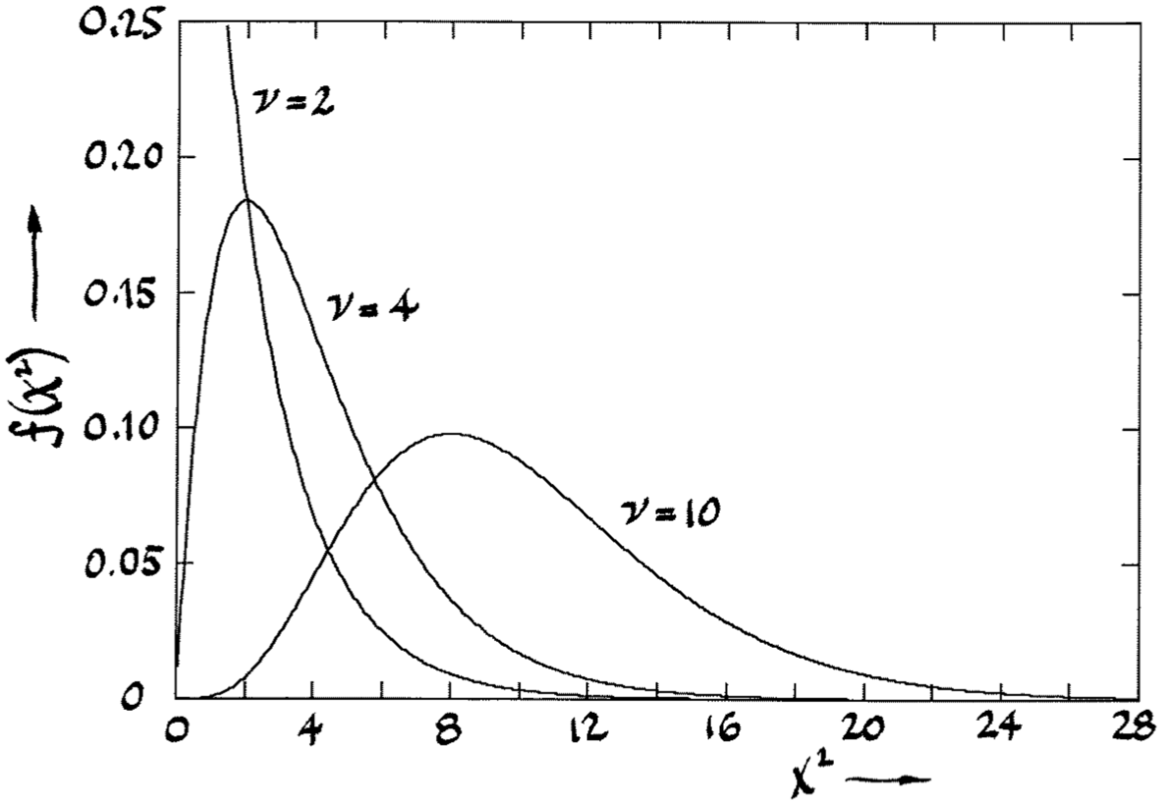


Figure 1 — The chi-square distribution for $\nu = 2$, 4, and 10.

e.g., Bevington (Data Reduction and Error Analysis for the Physical Sciences)

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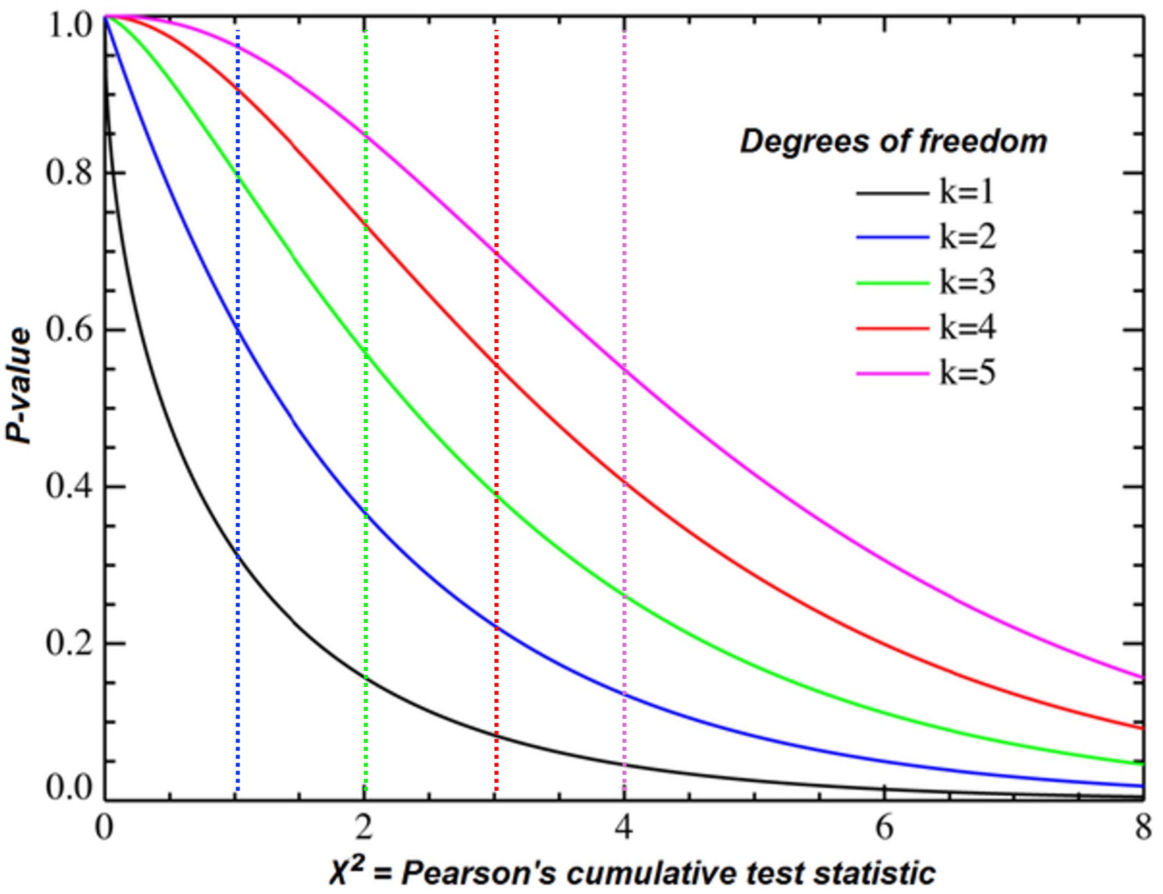

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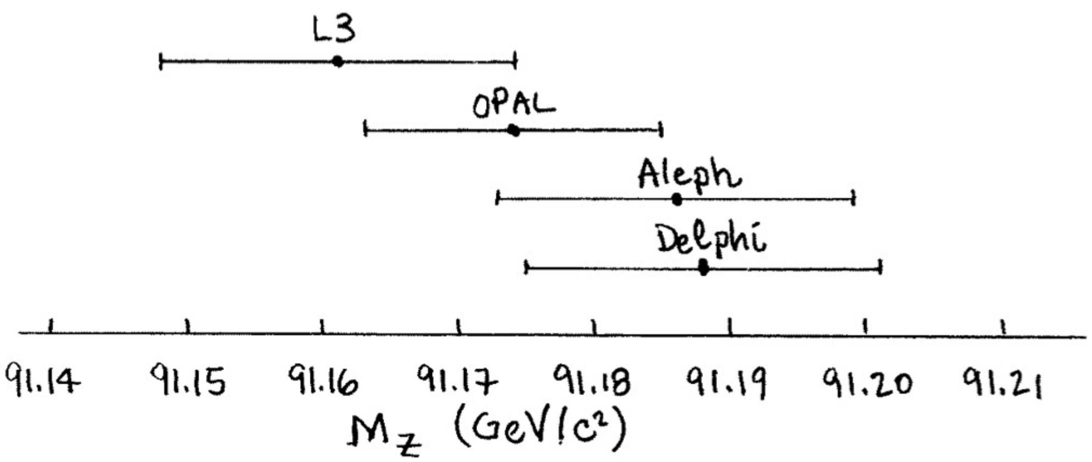
PTE Example: Four measurements of Z-Boson mass

Detector	Mass in GeV/c^2
L3	91.161 ± 0.013
OPAL	91.174 ± 0.011
Aleph	91.186 ± 0.013
Delphi	91.188 ± 0.013



Measurements of the Z^0 boson.

Calculate a ***weighted average (M)*** and ***uncertainty (σ)*** from the 4x data points:

$$\overline{M}_Z = \frac{\sum M_i/\sigma_i^2}{\sum 1/\sigma_i^2} \qquad \sigma^2_{\overline{M}_Z} = \frac{1}{\sum 1/\sigma_i^2}$$

Each data point will contribute more, or less, weight based on the inverse of its uncertainty squared ($1/\sigma_i^2$), e.g., 10x more uncertainty means 100x less weight.

$$\overline{M}_Z \pm \sigma_{\overline{M}_Z} = 91.177 \pm 0.006$$

We have 4x measurements with similar uncertainty, so expect the weighted average's uncertainty to be ~ sqrt(4) = 2x less

PTE Example: Are the data consistent?

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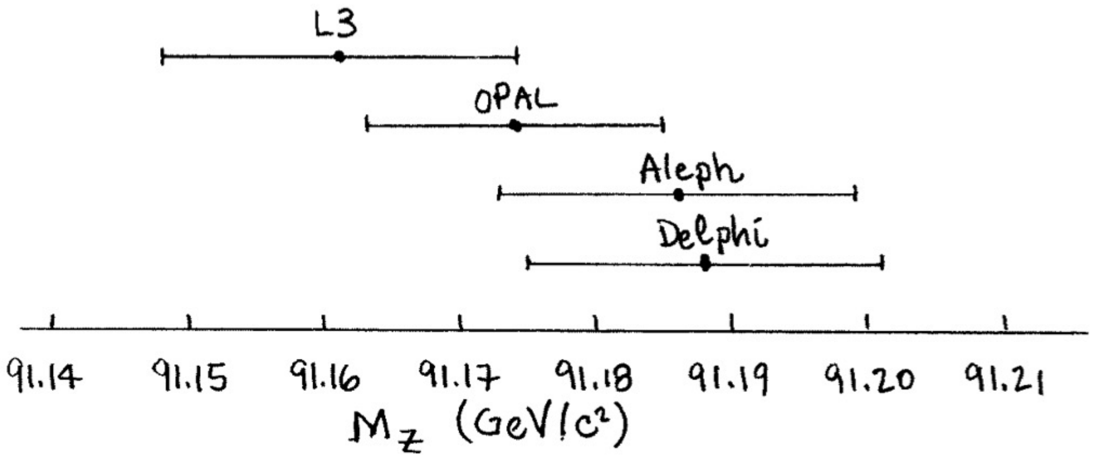

Measurements of the Z^0 boson.

Calculate via the chi-squared/chi2:

$$\chi^2 = \sum_{i=1}^{4} \frac{(M_i - \overline{M}_Z)^2}{\sigma_i^2} \approx 2.78$$

We expect this chi2 to be drawn from a chi-squared distribution with 3 degrees of freedom (DOF) (i.e., 4x data points – 1x model parameter, the mean value)

The integrated probability ***f(χ^2)*** to get a chi2 of 2.78 or higher with 3 DOF is 0.42 (42%)

Estimating the Likelihood given the χ^2

- Assuming Gaussian errors, the χ^2 gives you Likelihood (L) of the model parameters

$$L = \left(\frac{1}{\sqrt{2\pi\sigma^2}}\right)^N e^{-\frac{1}{2}\boxed{\sum_{i=1}^{N} \frac{\left(x_i - \mu\right)^2}{\sigma_i^2}}}$$

$$\log L = -\frac{1}{2}\underbrace{\boxed{\sum_{i=1}^{N} \frac{\left(x_i - \mu\right)^2}{\sigma_i^2}}}_{= \chi^2} + C$$

The boxed quantity is the chi-squared (χ^2)

The "best-fit" (or most-likely) model is defined as when the chi2 is minimized (or the likelihood is maximized).

Note: Generalizable to multidimensional models μ(x|a,b,c) and correlated noise (C_{ij}) between data points (x_i, x_j)

- The covariance matrix, Cij, tells you the level of correlated noise between data points x_i and x_j
- In the presence of data correlations:

$$\chi^2 = \sum_{i=1}^{N}\sum_{j=1}^{N} \frac{(x_i - \mu)^2}{\sigma_{ij}} = (x-\mu)^T \left[C_{ij}^{-1}\right](x-\mu)$$

- And the gaussian likelihood is (still) given by:

$$L = Ne^{-\chi^2/2}$$

Where N is a normalization constant, such that the integral of the Likelihood = 1

Correlated noise example:
https://astrobites.org/2014/07/01/beyond-chi-squared-an-introduction-to-correlated-noise/