0 Work In Progress

0.1 To Do

CRLB Read up on the Cramer Rao Lower Bound and see how it applies to our estimator. How close is our estimator to the CRLB?

Equivalent Reduce parallel/series subsystems to a single component that has the same probability of working. Match the mean. Reading from book borrowed from Dr. Weber.

Generalize AV/PAV This document contains our work on PAV in terms of Bernoulli random variables. Work out for the single link case with exponential random variables and use that to help generalize our work in terms of a generic function $h(\cdot)$

Model Justification Reread the tatikonda paper. Find physical justification for our setup.

0.2 Notes

The difficulty with system reliability determination is in enumerating the paths from source to sink, which grows as the number of components (edges) increases. For determining this we can use simulation.

Monte Carlo We represent the system as a graph $G(\mathcal{V}, \mathcal{E})$, with an edge for each component. Two edges are incident with each other if the corresponding components are connected. Each component either works or it doesn't. Generate a sequence of uniform random variables, one for each edge and assume independent failure of components. We construct a subgraph by taking edges that are on $(U_i < p_i)$. If the resulting subgraph is connected, then the system works.

$$\frac{1}{n}\sum_{i=1}^{n} \mathbb{I}[\exists (s-t) \text{ path in } G_i]$$

This estimator is unbiased

$$\mathbb{E}[\hat{r}] = \mathbb{P}[\exists (s-t) \text{ path in } G] = r$$

Antithetic Variables Same idea as with Monte Carlo, except after forming a subgraph based on U_i's, we form the compliment subgraph based on $1 - U_i$'s.

$$\hat{r}_{AV} = \frac{1}{n}\sum_{i=1}^{n/2} \mathbb{I}[\exists (s-t) \text{ path in } G_i] + \mathbb{I}[\exists (s-t) \text{ path in } G_i^c]$$

This estimator is unbiased as well

$$\mathbb{E}[\hat{r}] = \frac{1}{2}\left[\mathbb{P}[\exists (s-t) \text{ path in } G] + \mathbb{P}[\exists (s-t) \text{ path in } G]\right] = r$$

We know that $\mathbb{I}[\exists (s-t) \text{ path in } G]$ can be written as a function of the underlying U_i's $(h(U_1, \ldots, U_k))$ and because U and $1-U$ are identically distributed we know that $\mathbb{P}[\exists (s-t) \text{ path in } G] = \mathbb{P}[\exists (s-t) \text{ path in } G^c]$.

1 Single Link

We consider the case where a single source is transmitting a sequence of packets to a single destination (Figure 1) and the source records the outcome ($X_1^{(n)}$ of the packet. Examples of outcomes are loss information or delay information; we consider loss based tomography. In this case, the outomes are Bernoulli random variables, with $X_1^{(n)} = 1$ with probability α denoting successful reception of the packet. We are interested in developing an unbiased estimator for the link success probability α with a low variance.

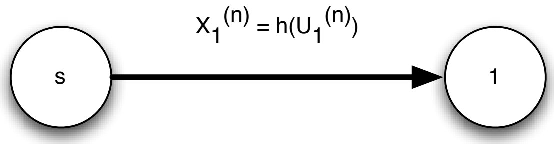


Figure 1: Single Link

1.1 Monte Carlo Estimation

The simplest estimation technique is Monte Carlo. We generate a sequence of unifrom random variables $U_1^{(n)}$ and transforming them to Bernoulli random variables $X_1^{(n)} = h(U_1^{(n)}) = \mathbb{I}_{U_1^{(n)} < \alpha}$. The Monte Carlo estimator is then

$$\hat{\alpha}_{MC} = \frac{1}{n} \sum_{i=1}^{n} X_1^{(i)} \tag{1}$$

This is clearly unbiased and has a normalized variance of

$$n \operatorname{Var}(\hat{\alpha}_{MC}) = \alpha(1 - \alpha) \tag{2}$$

The variance as a function of α is shown in Figure 2. The variance (or mean squared error) is maximum at $\alpha = \frac{1}{2}$.

1.2 Antithetic Variables

Our first step towards improving over the simple Monte Carlo estimation is the use of antithetic variables[**?**]. The idea here is that instead of generating a sequence with n random variables, we generate a sequence of length $\frac{n}{2}$ and form the estimator

$$\hat{\alpha}_{AV} = \frac{1}{n} \sum_{i=1}^{n/2} h(U_1^{(i)}) + h(1 - U_1^{(i)}) \tag{3}$$

Because expectation is linear and because $h(U_1^{(n)})$ and $h(1 - U_1^{(i)})$ are identically distributed, it is easy enough to verify that the estimator is unbiased. The normalized variance is given as

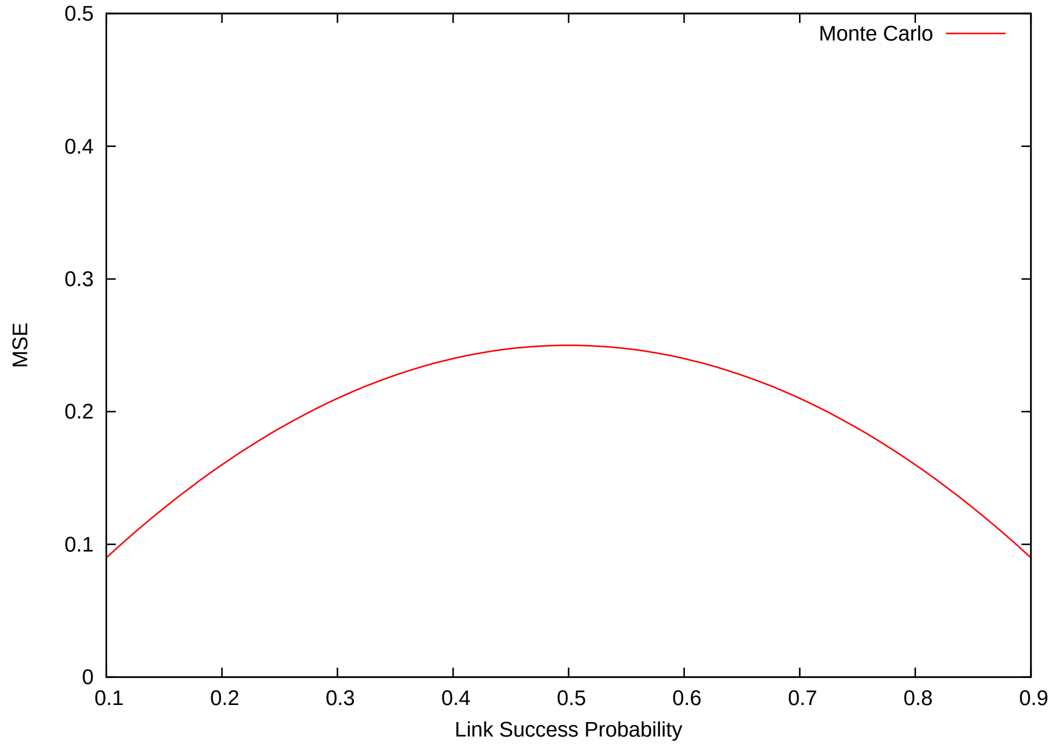


Figure 2: Normalized variance for Monte Carlo as a function of link success probability, α

$$
\begin{aligned}
n \operatorname{Var}(\hat{\alpha}_{AV}) &= \frac{1}{n} \sum_{i=1}^{n/2} \operatorname{Var}\left(h(U_1^{(i)}) + h(1-U_1^{(i)})\right) \\
&= \frac{\operatorname{Var}\left(h(U_1^{(i)})\right) + \operatorname{Var}\left(h(1-U_1^{(i)})\right) + 2\operatorname{Cov}\left(h(U_1^{(i)}), h(1-U_1^{(i)})\right)}{2} \\
&= \operatorname{Var}\left(h(U_1^{(i)})\right) + \operatorname{Cov}\left(h(U_1^{(i)}), h(1-U_1^{(i)})\right) \\
&= \mathbb{E}\left[h(U_1^{(i)})^2\right] + \mathbb{E}\left[h(U_1^{(i)})h(1-U_1^{(i)})\right] - 2\mathbb{E}\left[h(U_1^{(i)})\right]^2 \\
&= \alpha + 2\mathbb{I}_{\alpha > \frac{1}{2}}(\alpha - \frac{1}{2}) - 2\alpha^2
\end{aligned}
$$

The normalized variance as a function of α is shown in Figure 3. Note that the normalized variance is maximum at $\alpha = \frac{1}{4}$ and $\alpha = \frac{3}{4}$. From this figure we observe:

Proposition 1.

$$\operatorname{Var}(\hat{\alpha}_{AV}) < \operatorname{Var}(\hat{\alpha}_{MC}), \forall \alpha \in (0,1)$$

Proof. First, consider when $\alpha < \frac{1}{2}$

$$\alpha - 2\alpha^2 < \alpha(1-\alpha)$$

$$\alpha - 2\alpha^2 < \alpha - \alpha^2$$

$$0 < \alpha^2$$

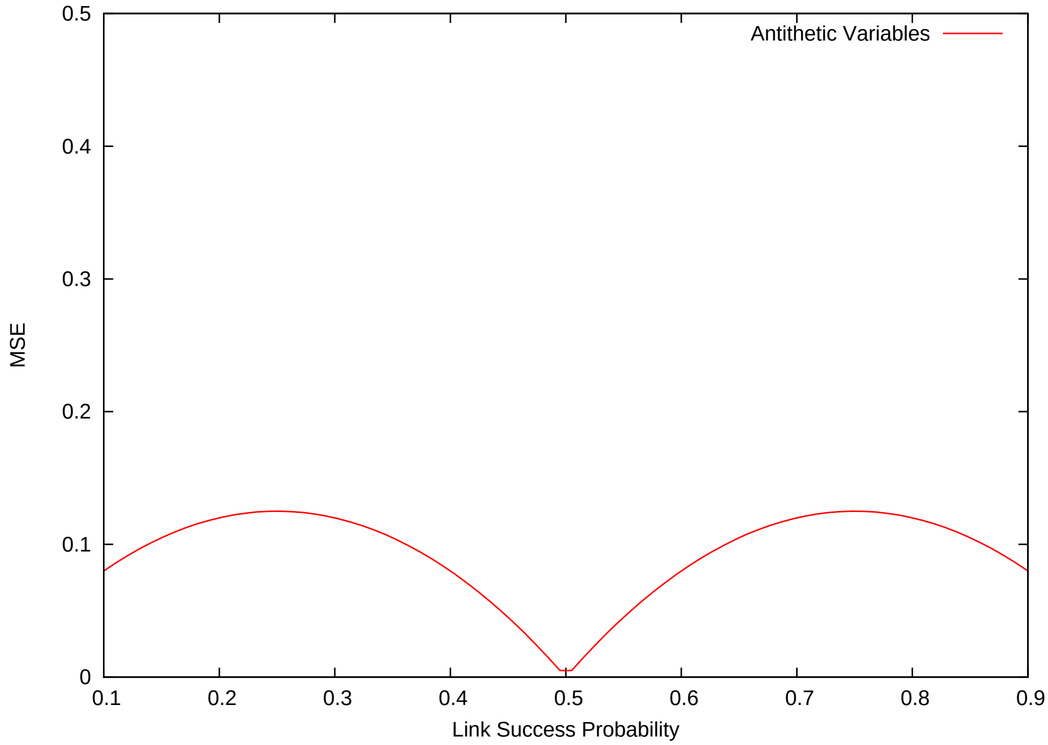


Figure 3: Normalized variance for antithetic variables as a function of link success probability, α

Next consider when $\alpha > \frac{1}{2}$

$$3\alpha - 1 - 2\alpha^2 < \alpha - \alpha^2$$
$$0 < \alpha^2 - 2\alpha + 1$$
$$0 < (\alpha - 1)^2$$

Finally, when $\alpha = \frac{1}{2}$ we have $\mathrm{Var}\left(\hat{\alpha}_{AV}\right) = 0$ and $\mathrm{Var}\left(\hat{\alpha}_{MC}\right) = \frac{1}{4}$ □

Using antithetic variables, we have the greatest variance reduction when the variance of the Monte Carlo estimator is at its maximum. The closer α is to either 0 or 1, there is less variance reduction.

1.3 Parameterized Antithetic Variables

We wish to develop an estimator with a tunable parameter β such that the variance reduction is maximized when $\beta = \alpha$, but which is an unbiased estimator for all α and all β.

Consider α fixed and unknown and to be estimated, and $\beta \in [0, 1]$ chosen based on some prior assumptions about α. Generate $U_1^{(n)} \sim \mathrm{Uni}(0, \beta)$ and from there set

$$\bar{U}_1^{(n)} = \frac{1-\beta}{\beta} U_1 + \beta. \tag{4}$$

Note that $\bar{U}_1^{(n)} \sim \mathrm{Uni}(\beta, 1)$. We consider an estimator of the form

$$\hat{\alpha}_{PAV} = \frac{1}{n} \sum_{i=1}^{n/2} \left(\beta \left(\mathbf{1}_{U_1^{(i)} \leq \alpha} + \mathbf{1}_{\beta - U_1^{(i)} \leq \alpha} \right) + (1-\beta) \left(\mathbf{1}_{\bar{U}_1^{(i)} \leq \alpha} + \mathbf{1}_{1 - \bar{U}_1^{(i)} + \beta < \alpha} \right) \right) \tag{5}$$

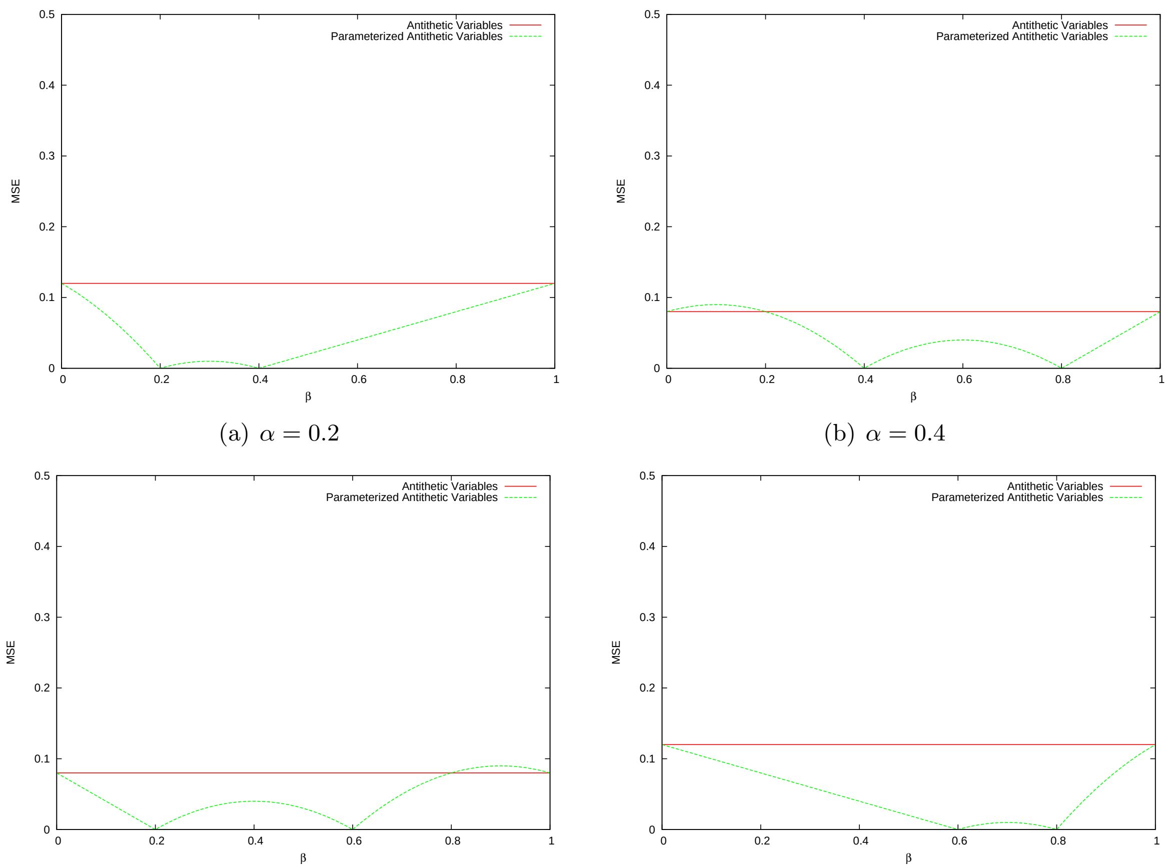
Then it is straightforward to show

$$
\begin{aligned}
\mathbb{E}[\hat{\alpha}_{PAV}] &= \frac{1}{2}\left[\beta\left(\mathbb{P}(U_1 \le \alpha) + 1 - \mathbb{P}(U_1 \le \beta - \alpha)\right) + (1-\beta)\left(\mathbb{P}(\bar{U}_1 \le \alpha) + 1 - \mathbb{P}(\bar{U}_1 \le 1 + \beta - \alpha)\right)\right] \\
&= \begin{cases} \frac{1}{2}\left[\beta\left(\frac{\alpha}{\beta} + 1 - \frac{\beta-\alpha}{\beta}\right) + (1-\beta)\left(0 + 1 - 1\right)\right], & \alpha \le \beta \\ \frac{1}{2}\left[\beta\left(1 + 1 - 0\right) + (1-\beta)\left(\frac{\alpha-\beta}{1-\beta} + 1 - \frac{1+\beta-\alpha-\beta}{1-\beta}\right)\right], & \alpha > \beta \end{cases} \\
&= \alpha
\end{aligned} \tag{6}
$$

We can then find the normalized variance as

$$
n\,\mathrm{Var}(\hat{\alpha}_{PAV}) = \begin{cases} (\alpha-\beta)(\beta-(2\alpha-1)), & \alpha \le \frac{1}{2} \quad \beta \le \alpha \\ (\beta-\alpha)(2\alpha-\beta), & \qquad\quad \alpha < \beta \le 2\alpha \\ \alpha(\beta-2\alpha), & \qquad\quad \beta > 2\alpha \\ (1-\alpha)((2\alpha-1)-\beta), & \alpha > \frac{1}{2} \quad \beta \le \alpha \quad \beta \le 2\alpha - 1 \\ (\alpha-\beta)(\beta-(2\alpha-1)), & \qquad\quad \beta \le \alpha \quad 2\alpha - 1 < \beta \\ (\beta-\alpha)(2\alpha-\beta), & \qquad\quad \beta > \alpha \end{cases} \tag{7}
$$

The normalized variance for both antithetic variables and parameterized antithetic variables is shown in Figure 4.



(a) $\alpha = 0.2$ (b) $\alpha = 0.4$

(c) $\alpha = 0.6$ (d) $\alpha = 0.8$

Figure 4: Normalized variance for parameterized antithetic variables: (a) $\alpha = 0.2$, (b) $\alpha = 0.4$, (c) $\alpha = 0.6$, and (d) $\alpha = 0.8$

Proposition 2.

$$\mathrm{Var}\left(\hat{\alpha}_{PAV}\right) < \mathrm{Var}\left(\hat{\alpha}_{MC}\right), \forall \alpha, \beta \in (0,1)$$

Proof. If $\alpha < \frac{1}{3}$, then $\max_{\beta} \mathrm{Var}\left(\hat{\alpha}_{PAV}\right) = \alpha - 2\alpha^2$; we are doing no worse than antithetic variables. We proved in Proposition 1 that antithetic variables is always better than Monte Carlo. If $\frac{1}{3} < \alpha < \frac{1}{2}$, then $\max_{\beta} \mathrm{Var}\left(\hat{\alpha}_{PAV}\right) = \dfrac{(1-\alpha)^2}{4}$.

$$\frac{(1-\alpha)^2}{4} < \alpha(1-\alpha)$$
$$\frac{(1-\alpha)}{4} < \alpha$$
$$1 - \alpha < 4\alpha < 4\frac{1}{2}$$
$$1 - \alpha < 2$$

Because $\alpha \in [0,1]$, the above is true and we conclude that $\mathrm{Var}\left(\hat{\alpha}_{PAV}\right) < \mathrm{Var}\left(\hat{\alpha}_{AV}\right)$ for $\alpha < \frac{1}{2}$.

If $\alpha > \frac{2}{3}$, then $\max_{\beta} \mathrm{Var}\left(\hat{\alpha}_{PAV}\right) = \alpha + 2(\alpha - \dfrac{1}{2}) - 2\alpha^2$; again we are doing no worse than antithetic variables. If $\frac{1}{3} < \alpha < \frac{1}{2}$, then $\max_{\beta} \mathrm{Var}\left(\hat{\alpha}_{PAV}\right) = \dfrac{\alpha^2}{4}$.

$$\frac{\alpha^2}{4} < \alpha(1-\alpha)$$
$$\frac{\alpha}{4} < 1 - \alpha$$
$$\alpha < 4(1-\alpha) < 4\frac{1}{2}$$
$$\alpha < 2$$

Because $\alpha \in [0,1]$, the above is true and we conclude that $\mathrm{Var}\left(\hat{\alpha}_{PAV}\right) < \mathrm{Var}\left(\hat{\alpha}_{AV}\right)$ for $\alpha > \frac{1}{2}$.

$\square$

Corollary 1. *If* $\alpha < \frac{1}{3}$ *or* $\alpha > \frac{2}{3}$*, then*

$$\mathrm{Var}\left(\hat{\alpha}_{PAV}\right) < \mathrm{Var}\left(\hat{\alpha}_{AV}\right), \forall \beta \in (0,1)$$

Proof. See the proof of Proposition 2. $\square$

I still need to type up the propositions for the ranges of beta when alpha is in between 1/3 and 2/3 and the proposition to quantify the maximum worst case choice of beta.

2 Two Links